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ABSTRACT

The second volume of the proceedings of 21st annual meeting of the International Group for the Psychology of Mathematics Education contains the following papers: (1) "The Dilemma of Transparency: Seeing and Seeing through Talk in the Mathematics Classroom" (J. Adler); (2) "Abstraction is Hard in Computer-Science Too" (D. Aharoni and U. Leron); (3) "Constructing Purpose in Mathematical Activity" (J. Ainley); (4) "Effective Teachers of Numeracy in UK Primary Schools: Teachers' Beliefs, Practices and Pupils' Learning" (M. Askew, M. Brown, V. Rhodes, D. Wiliam and D. Johnson); (5) "Can the Average Student Learn Analysis?" (R. R. Baldino, A. Buttner Ciani and A. Carolina Leal); (6) "Cognitive Units, Connections and Mathematical Proof" (T. Barnard and D. Tall); (7) "Subjective Elements in Children's Comparison of Probabilities" (M.J. Canizares, C. Batanero, L. Serrano and J.J. Ortiz); (8) "Reunitizing Hundredths: Prototypic and Nonprototypic Representations" (A.R. Baturo, and J. Cooper); (9) "Students' Perceptions of the Purposes of Mathematical Activities" (A. Bell, R. Phillips, A. Shannon, and M. Swan); (10) "Stereotypes of Literal Symbol Use in Senior School Algebra" (L. Bills); (11) "Approaching Theoretical Knowledge through Voices and Echoes: A Vygotskian Perspective" (P. Boero, B. Pedemonte, and E. Robotti); (12) "The Transition from Arithmetic To Algebra: Initial Understanding of Equals, Operations and Variable" (T.J. Cooper, G. M. Boulton-Lewis, B. Atweh, H. Pillay, L. Wilss & S. Y. Mutch); (13) "Exploring Imagery in P, M and E" (C. Breen); (14) "Teachers' Framework for Understanding Children's Mathematical Thinking" (G.W. Bright, A.H. Bowman and N.N. Vacc); (15) "The Story of Sarah: Seeing the General in the Particular?" (L. Brown, and A. Coles); (16) "Effective Teachers of Numeracy in UK Primary School: Teachers' Content Knowledge and Pupils' Learning" (M. Brown, M. Askew, V. Rhodes, D. Wiliam and D. Johnson); (17) "Metaphorical Thinking and Applied Problem Solving: Implications for Mathematics Learning" (S. Carreira); (18) "Algebra as Language in Use: A Study with 11-12 Year Olds using Graphic Calculators" (T.E.A. Cedillo); (19) "Emergence of Novel Problem Solving Activity" (V. Cifarelli); (20) "NESB Migrant Students Studying Mathematics: Vietnamese Students in Melbourne and Sydney" (P.C. Clarkson and L. Dawe); (21) "Young Children's Concepts of Shape" (D.H. Clements, J. Sarama

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and S. Swaminathan); (22) "Learning from Distributed Theories of Intelligence" (P. Cobb); (23) "Australian and Indonesian Student Teacher Beliefs about Mathematics and Performance on a Classic Ratio Task" (J. Conroy and B. Perry); (24) "The Transition from Arithmetic To Algebra: A Cognitive Perspective" (G.M. Boulton-Lewis, T.J. Cooper, B. Atweh, H. Pillay, L. Wilss and Mutch); (25) "A New Approach for Intelligent Tutoring Systems: An Example for Statistical Activities" (G. Bueno and C.A. Cuevas); (26) "Early Development of Algebraic Representation Among 6-13 year-old Children: The Importance of Didactic Contract" (L.A.P. Brito and F. Jorge-Tarcisio Da Rocha); (27) "The Affective Domain in Mathematical Problem-Solving" (V.A. DeBellis and G.A. Goldin); (28) "Creating a Shared Context: The Use of a Multimedia Case in a Teacher Development Course" (H.M. Doerr, K. McClain, and J. Bowers); (29) "Triple Approach: A Theoretical Frame To Interpret Students' Activity in Algebra" (T.-P Drouhard and C. Sackur); (30) "When Does a Way of Working Become a Methodology" (J. Duffin and A. Simpson); (31) "Development of Seventh-Grade Students' Problem Posing" (L.D. English); (32) "A Close Look at the Use of Mathematics-Classroom-Situation Cases in Teacher Education" (R. Even and Z. Markovits); (33) "Action-Based Strategies in Advanced Algebraic Problem Solving" (P.L. Ferrari); (34) "Tacit Mechanism of Combinatorial Intuitions" (E. Fischbein and A. Grossman); (35) "Shadows on Proof" (F. Furinghetti and D. Paola); (36) "Perpendicular Lines--What is the Problem? Pre-service Teachers' Lack of Knowledge on How To Cope with Students' Difficulties" (H. Gal & S. Vinner); (37) "Actions and Invariant Schemata in Linear Generalizing Problems" (J.A. Garcia-Cruz and A. Martinon); (38) "A Classroom Discussion and an Historical Dialogue: A Case Study" (R. Garuti); (39) "The Importance of Social Structure in Developing a Critical Social Psychology of Mathematics Education" (P. Gates); (40) "Meaning of Proofs in Mathematics Education" (J.D. Godino and A.M. Recio); and (41) "A Visual Presentation of a Longitudinal Study: Design and Analysis" (F.V. Gomes and C. Hoyles). (ASK)

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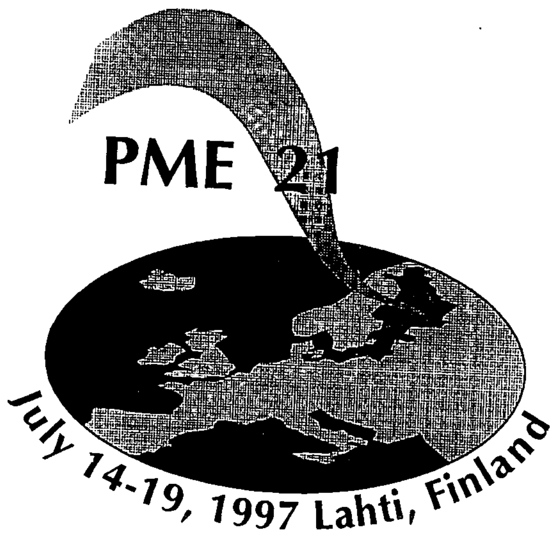

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Research Reports

RESEARCH REPORTS

THE DILEMMA OF TRANSPARENCY: SEEING AND SEEING THROUGH TALK IN THE MATHEMATICS CLASSROOM

Jill Adler, Witwatersrand University

In this paper, talk is understood as a tool and resource for mathematical learning in school. As a resource it needs to be seen (be visible) to be used, and as a tool it needs to be seen through (to be invisible) to provide access to mathematical learning. This paper argues that the dual function of visibility and invisibility of talk in mathematics classrooms creates dilemmas for teachers. An analytic narrative vignette drawn from a secondary mathematics classroom in South Africa illustrates the 'dilemma of transparency' that mathematics teachers face, particularly if they are teaching multilingual classes.

INTRODUCTION

The paper draws from a study of South African secondary mathematics teachers' knowledge of their practices in their multilingual classrooms (Adler, 1996a). In initial interviews, English-speaking teachers whose 'whites only' classrooms had recently and rapidly become racially integrated argued the benefit to all learners of explicit mathematics language teaching (Adler, 1995). This implies that language itself, and particularly talk, becomes the object of attention in the mathematics class and a resource in the teaching-learning process. Now that their classes included pupils whose main language was not English, it became obvious to these teachers that they needed to be more explicit about instructions for tasks, as well as mathematical terms and the expression of ideas.

In follow-up workshops in the study, Helen specifically problematised the issue of explicit language teaching. She has tried to develop mathematical language teaching as part of her practice in her multilingual classroom. However, as she sees and reflects on her teaching she begins to question what this means in practice and whether and how explicit mathematics language teaching actually helps. And we are alerted to a dilemma: *There is always the problem in explicit language teaching of 'going on too long', of focusing too much on what is said and how it is said. Yet explicit mathematics language teaching appears to be a primary condition for access to mathematics, particularly for those pupils whose main language is not English or for those pupils less familiar with educated discourse.*

This paper argues that Lave and Wenger's idea that access to a practice requires its resources to be 'transparent', while not usually applied to language as a resource, nor to learning in school, is useful and illuminating here. Explicit mathematics language teaching, where teachers attend to pupils' verbal expressions as a public resource for whole class teaching, offers possibilities for enhancing access to mathematics, especially in multilingual classrooms. However, such practices easily slip into possibilities for alienation through a shift of attention off the mathematical problem and onto language per se. Teachers' decision-making at critical moments, while always a reflection of both their personal identity and their teaching context, requires the ability to shift focus off and then back onto the mathematical problem. The challenge, of course, is when and how such shifts are best for whom and for what. These assertions will be instantiated and illuminated through an analytic narrative vignette (Erickson, 1986) based on an episode in Helen's multilingual Std 9 (Grade 11) trigonometry class.

SOME BACKGROUND AND THEORETICAL COMMENT

The wider study from which this paper is drawn is framed by a sociocultural theory of mind where consciousness is constituted in and constitutive of activity in social, cultural and historical contexts (Lave and Wenger, 1991; Vygotsky, 1978, 1986; Mercer, 1995). For Lave and Wenger (1991), becoming knowledgeable about a practice, like mathematics, is the fashioning of identity in, and as part of, a community of practice (pp. 50-51). Becoming knowledgeable means becoming a *full participant* in the practice, and this involves, in part, learning *to* talk in the manner of the practice. Furthermore, becoming knowledgeable in a practice entails having access to a wide range of ongoing activity in the practice - access to old-timers, other members, to information, resources and opportunities for participation. Such access hinges on the concept of *transparency*.

> The significance of artifacts in the full complexity of their relations with the practice can be more or less *transparent* to learners. Transparency in its simplest form may imply that the inner workings of an artifact are available for the learner's inspection ... transparency refers to the way in which using artifacts and understanding their significance interact to become one learning process (pp. 102-3).

Becoming a full participant means engaging with the technologies of everyday practices in the community, as well as participating in its social relations. Thus, access to artifacts in the community through their use and understanding of their significance is crucial. Often material tools, artifacts - technologies - are treated as given. Yet, they embody inner workings tied with the history and development of the practice and which are hidden - these need to be made available. Lave and Wenger elaborate 'transparency' as involving the dual characteristics of *invisibility* and *visibility*:

> ... invisibility in the form of unproblematic interpretation and integration (of the artifact) into activity, and visibility in the form of extended access to information. This is not a simple dichotomous distinction, since these two crucial characteristics are in a complex interplay (p. 102).

Access to a practice relates to the dual visibility and invisibility of its resources. In other words, the invisibility of mediating technologies in a practice is necessary for focus on and supporting the visibility of the subject matter in the practice. Meira's (1995) analysis of tool use in mathematics classrooms is illustrative here: he distinguishes 'fields of invisibility' which enable smooth entry into a practice, and 'fields of visibility' which extend information by making the world visible.

Managing this duality of visibility and invisibility of resources in classrooms can create dilemmas for teachers. The example of pupil discussion of a mathematical task is illuminating if one understands talk as a resource in the practice of school mathematics. Discussion of a task should enable the mathematical learning and so be invisible. It is the window through which the mathematics then can be seen. At the same time, the specificity of mathematical discourse inevitably enters such discussion and can require explicit attention, that is, needing to be visible. It is possible then that the discussion itself becomes the focus of attention, rather than a means to the mathematics. Here it obscures access to mathematics, by becoming too visible itself. This possibility might well be exaggerated in multilingual situations where learners bring a number of different main languages.

Lave and Wenger's concept of transparency is developed in contexts of apprenticeships where there is a situated and continuous movement from peripheral to full participation in

a social practice (p. 53). As I have argued elsewhere (Adler, 1996b) the school is a very different context from those of apprenticeships. Lave and Wenger recognise this, but by their own admission (pp. 39-41) they do not address what, for example, could be distinct about the visibility and invisibility of resources for mathematics learning in school. Limited space precludes rehearsing the argument here. Suffice it to say that sociocultural theory, particularly as it is proposed by Vygotsky (1986, 1979) and elaborated by Mercer (1995) provides the conceptual tools to comprehend and explain the special nature of classroom learning and hence mathematical knowledge produced in the context of schooling. In particular Mercer's distinction between *educational discourse* - the discourse of teaching and learning in the classroom - and *educated discourse* - new ways of using language, 'ways with words' and the importance of access to both for success in school is crucial.

> Teachers are expected to help their students develop ways of talking, writing and thinking which will enable them to travel on wider intellectual journeys, understanding and being understood by other members of wider communities of educational discourse: but they have to start from where learners are, to use what they already know, and help them go back and forth across the bridge from 'everyday discourse' into 'educated discourse' (Mercer, 1995, p. 83).

Thus, in relation to talk as a teaching-learning resource and needing to be both visible and invisible for access to school mathematics, Mercer's argument suggests a bridge and mediational roles for teachers in moving between talk as the invisible window through which mathematics can be seen, and, in Helen's terms, more explicit mathematical language teaching.

From this sociocultural perspective, the teaching and learning of mathematics in multilingual contexts needs to be understood as three-dimensional. It is not simply about access to the language of learning (in this case English). It is also about access to the language of mathematics (educated discourse and scientific concepts) and access to classroom cultural processes (educational discourse). To find out how teachers manage their complex practices, in-depth initial interviews, classroom observations, reflective interviews, and workshops were conducted. These provided the empirical base for a qualitative study with a purposive, theoretical and opportunity sample of six qualified and experienced mathematics teachers, two from each of three different multilingual contexts in South Africa. Helen was one of these teachers.

In the wider study, the notion of a 'teaching dilemma' (Berlak & Berlak, 1981; Lampert, 1985) was the key to unlocking teachers' knowledge of teaching and learning mathematics in complex multilingual settings. The wider study revealed that teachers in different multilingual contexts face different dilemmas in their teaching, thus supporting the notion of teaching as a contextualised social practice (Adler, 1995). Of course, what teachers reflect on and talk about is only part of what they know. What happens in practice? In particular, how does Helen work with the dilemma of transparency, with explicit mathematics language teaching and the need for both visibility and invisibility of talk in her class?

THE CONTEXT

Helen is white and English-speaking with six years of secondary experience. Helen teaches in a girls-only, historically white state school. This school deracialised faster than most other similar schools, and at the time of the research, fewer than 50% of the pupils were white. The school is well-resourced. The class where observation and videoing were carried out

was a 'mixed ability' class of 30 pupils. English, Sesotho and Zulu, all now offical languages, were some of the main languages in this class. The language of instruction in the school is English, and all public interaction in Helen's classes is in English.

Helen's classes, while largely teacher-directed, are interactive and task-based. She introduced trigonometry to the particular class (in their Grade 10/ Std 8 year) with an outdoor activity investigating shadow length caused by the sun at different times of the day. This was followed by activities where groups of pupils measured and compared the ratios of sides of a right-angled triangle with one angle of 40 degrees. Working on the reports that groups presented, she built their understanding of constant ratios.

During the research workshops Helen invited participating teachers to reflect with her on her own videos and on whether or not explicit language teaching actually helps, on whether and how working on pupils' ability 'to' talk mathematics is a good thing and 'saying it' is indicative of understanding, of knowing. That the dilemma of transparency is particularly strong for Helen is not surprising considering her view of mathematics as language, of language as a crucial resource in the practices in her classroom and of a strong relationship between language and learning. In short, Helen appears to share Lave and Wenger's notion that becoming knowledgeable means learning *to* talk, learning mathematical discourse. In her words: ... *if they start to describe something to me in accurate mathematical language it does seem to reflect some kind of mastery ...*

The dilemma of transparency is illustrated by what Helen brings to the second workshop as a result of her action research with this same class in the following year.

A VIGNETTE - A CLASSROOM EPISODE

As mentioned, the episode below takes place in the first trigonometry lesson of Standard 9, the year following video-taping Helen's teaching trigonometry to her Std 8 class. Helen asks pupils in groups of four to discuss what 'trigonometry' means to them, and then to report back their meanings to the rest of the class in a 'maximum of two minutes per group ... using key words and putting across your main ideas'. Most of the presentations related trigonometry to determining 'the size and sides of the angles', in right-angled triangles and that 'there are six ratios'. Specifically, two groups' explanations, based on similar triangles, included the following expressions: '... *uh we said the ratio of two angles is independent to the size of the angle in the other two triangles...'* and '*We came to the same thing that the ratio of two sides is independent to the size of the tri, of the angle in two triangles.'* After all presentations, Helen moved to the front of the class. She drew the class' attention to various aspects of the reports, and then focused explicitly on the expressions italicised above:

H: *Say that to me slowly, the*

S6: *(H writes as pupil talks) The ratios of the two sides () is independent to the size of the angles () in the two triangles ...*

H: *Is independent to ...?*

S6: *The two tri.., is independent, no, the two sides is independent ...*

H: *The ratio of the two sides is independent to?*

S6: *The size of the angles in the two triangles (and H finishes writing).*

H: *Let's look at that statement carefully ... What does that statement mean to, uh, to anyone?*

S6: *It means that, uh, whether the angles () when you've got two triangles, and the angles come up to the same degree, you, uh, it*

doesn't matter how long or short the triangle is, your angles, as long as your angles are equal (inaudible)
H: *Now listen to what you said: how long or short the triangles are?*
S6: *The length, the length of the triangle.*
H: *Triangle is a shape.*
S's: *(Mumbling) The length of the sides.*
H: *The length of the sides of the triangle. OK. You know. Let's just look at this word "independent". OK. Now I know when I teach this, I use the word independent and then you think, well that's a nice fancy word to use. If I just repeat it nicely in the right sentence then she'll be very impressed. But, when you use the word independent you've got to know what it means. What does it mean? Phindiwe?*
Phin: *(some mumbling) It stands on its own.*

After distinguishing 'length of sides' from 'sides of triangle' Helen pulls the word independent out on its own, and attends to its meaning. She then returns to focus on the sentence in which it is placed.

H: *OK. All right. Is that statement true?*
S's: *No//Yes. ()*
H: *Must I put a true or a false at the end of it?*
S's: *True//false*
H: *OK. Who says it's true?*
S6: *(Puts her hand up)*
H: *S6 says its true 'cause she said it.*
S's: *(laugh)*
H: *OK, who says its false?*
S's: *(laugh)*
H: *What do you think?*
Phin: *I don't know, I don't understand the sentence.*
H: *OK, let's try and sort out the sentence. The ratios of two sides, that's a true part of the line, uh, of the sentence . Does that make sense?*
S's: *Yes*
H: *OK. ... So the ratio is independent from what? Size of the angle in the two triangles? () It's true, who says it's true? Why?*
S7: *Because, mam, um, I think it means that, no, uh, if if you, if you have, uh, one big triangle and you have one small triangle and you have the same angle in both of them, uh, the the size of the angles is equal, then the ratio of the, of the sides won't change.*
H: *Now listen to what you're saying. You're saying you've got (), you said to me (and H links the bold words below to related words on the board as she speaks) you've got the* ***size*** *of two* ***triangles*** *and then you said that the* ***angle inside them*** *is the same, OK. So if we want to, is what she said different to what is on the board at the moment.*
S's: *No//yes ()*
H: *she said to me the ratio of the two sides is independent of the* ***SIZE*** *of the triangle,* ***WHEN*** *you've got the same angle in all of them. So is* ***NOT*** *true to say that the ratios are independent of the size of the* ***ANGLE****. The size of the angle is* ***EXACTLY*** *what makes the* ***FUNDAMENTAL DIFFERENCE****. Because if I've got two triangles, these two beautiful triangles over here, 40, 40 (and she fills in 40 degrees into two similar triangles on the board),*

and these two over here, 20, 20 (and again fills in these angle sizes onto another set of similar triangles on the board). () Would I get if I say spoke about () sin here and sin here? OK? Will I get the same answer?

S's: *No*

H: *No! I'll get two different answers. So it is not true to say to me it is independent of the size of the angle - because the angle if it is 40, makes the difference to 20, right. It's the size of the **TRIANGLE** that makes the difference. () Does that make sense to you?*

S's: *No*

H: *What doesn't make sense?*

S2: *Mam?*

H: *Ja*

S2: *It makes a difference to what?*

H: *It makes a difference ...to ...*

S's: *(laugh)*

H: *Where was I starting off? ... um, let me start again...*

(Helen then recaps by drawing attention to diagrams on the board, to how two different right-angled triangles each with 40 degree angles will have the same ratios between their sides, as will two different right-angled triangles each with 20 degree angles. But the two sets of ratios will be different precisely because the angles across the triangle pairs are different. And then she asks the pupil who first articulated the sentence to tell the class what she understands in her own words.)

HELEN'S REFLECTIONS

Opening the second workshop, before showing the extract above, she says:

> *Jill and I talked about the part where a child put forward what she thinks is going on in relation and it is a question of even though her language is not clear is there understanding amongst the rest of the students? ... it seems like the rest do understand even though she is using incorrect language. So we can watch and think around that.*

She then plays the video from the point where the student says: *the ratio of the two sides is independent to the size of the angles in the two triangles* and she is writing what is being said word for word on the board for the class to think about. She reflects:

> *Just after the sentence is written on the board and I ask: 'What do you understand by this statement?', one child puts forward a perfect explanation. She talks about the angle being the same in both triangles and I pick up on that ... and then this child (getting to the place on the video where a second pupil is responding) now does it absolutely perfectly. So, that is two very good expressions of what is going on. And yet when you ask the class: 'Is this sentence correct?' (Pointing to the sentence she has written verbatim from the first student on the board), there is this complete silence. So the question for me is: even in the minds of those two children who put forward such consistent*

explanations, what's going on with them? () that they cannot ... um ... pick up incorrectness in the sentence?

She then revisits her question in the first workshop: 'if they can say it, do they know it?' and finally, she poses a central question on verbalisation and the dilemma of transparency:

*... in retrospect, when I look at that lesson, I went on **but much too long** (laughter) on and on and on and I keep saying the same thing and I repeat myself, on and on ... But the thing is then if you have a sense that there is a shared meaning amongst the group can you go with it? um ... when the sentence is completely wrong? ... Can you let it go? Can a teacher use a sense of shared meaning to move on?*

Helen's working assumptions of a strong relationship between language and thought are seriously challenged as she experiences and observes pupils expressing clear and correct mathematical thinking but not being able to discern problematic expression in/of others; and of pupils saying things 'wrong' but creating a sense that they have some grasp of the mathematics in play. She also sees how through her explicit attention to their use of dependent and independent, the pupils lost their focus on the mathematical and trigonometric problem from which this use arose.

DISCUSSION

Through the episode in Helen's class and her reflections we see what we know only too well: that some mathematics is difficult for pupils to say precisely and with meaning. She provides opportunity for pupils, amongst themselves, to elaborate and then share their meanings of 'trigonometry'. This elicitation of pupils' thinking suggests to her that there is confusion and she moves to clarify this through a particular scaffolding process where she questions, bringing into focus the incorrect use of the concept and term 'independent', and finally reformulates and recaps emphasising what she sees as most significant in the description of trigonometry that has emerged from the pupils. But this explicit language teaching is a struggle here.

Helen's knowledge helps us identify a fundamental pedagogic tension in explicit practices with respect to language issues, and particularly talk, in her multilingual mathematics class. She harnesses talk as a resource in her classroom. As a resource in the practice, its transparency, i.e. its enabling use by learners, is related to both its visibility and invisibility. Helen attends to pupils' expression as a shared public resource for class teaching. This is a characteristic of classrooms that is not shared by many other speech settings (Pimm, 1996). The language itself becomes visible and the explicit focus of attention. It is no longer the medium of expression, but the message itself - that to which the pupils now attend.

On reflection, Helen feels that her attempt to enable access to mathematical (educated) discourse brings the problem of 'going on too long'. In making mathematical language visible, it becomes opaque, obscuring the mathematical problem. The dilemma of transparency arises: of whether (and when) to make mathematical language explicit. And there are both political and educational dimensions to this dilemma. If Helen 'goes on too long', she diminishes pupils' opportunities to use educational discourse and inadvertently obscures the mathematics at play. If she leaves too much implicit then she runs the risk of losing or alienating those who most need opportunity for access to educated discourse. She wonders about the possible effects of leaving in play a shared sense of

trigonometric ratios but a public display of incorrect mathematical language: 'if they don't say it right, can I let it go?'.

CONCLUSION

Through Helen, we see that explicit mathematics language teaching, while beneficial, is not a straight forward 'good thing'. It brings a language-related dilemma of transparency with its dual characteristics of visibility and invisibility. It is not simply a matter of 'going on too long' but of managing the shift of focus between mathematical language and the mathematical problem (and of course these are intertwined). Lave and Wenger's notion of transparency illuminates classroom processes. Transparency involves both visibility and invisibility, just as with a window. Resources need to be seen to be used. As tools, they also need to be invisible to illuminate aspects of practice. So too with talk as a resource for mathematics learning in school. Mathematics learners need to harness talk as a resource, focus on it when necessary, but then render it invisible and as a means for building mathematical knowledge. This is the specificity of talk as a resource in the school context. There is no resolution to the dilemma of transparency for mathematics teachers, only its management through careful mediational moves when making talk visible in moments of practice.

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ABSTRACTION IS HARD IN COMPUTER-SCIENCE TOO

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Abstract

Research in computer science (CS) education, as compared to that in math education, is still in its infancy. We show that methods and theoretical frameworks used in research on mathematical thinking can be extended to CS education. This paper presents some results from an ongoing research on undergraduate students' conceptions of data structures. The analysis of students' thinking highlights similarities and differences between methods and mental processes in CS and in mathematics. Some similarities are the process-object duality, fragility of knowledge and difficulties in attaining an abstract disposition. In contrast, There are subtle differences in the meaning and use of abstraction in the two disciplines, centered around the important CS concept of abstraction barriers.

1. Abstraction in mathematics and in computer-science

It is a well documented theme in the math education research literature that students experience much difficulty in negotiating abstraction levels. Moreover, there is a substantial body of theoretical discussion which strives to postulate various mental processes that might account for these difficulties. For example, starting from Piaget and continuing to present-day theorists, the difficulties of passing from process to object conception of various mathematical entities (such as function or group) has been documented and analyzed (e.g., Breidenbach et al, 1992; Sfard & Linchevski, 1994).

In the present report we wish to extend the scope of that discussion to a different discipline and different population, namely, to computer science (CS) majors in a major Israeli University. This extension is particularly noteworthy due to two special features of the CS department: First, unlike the math department, enrolling in the CS department involves fierce competition and elaborate filtering; as a result, we are dealing here with students who must rank very high on their university entrance score. Secondly, abstraction is a programming methodology of central importance in modern CS, and as such is taught explicitly and emphatically in all courses involving methodology. (In mathematics, in contrast, instructors *use* abstraction all the time, but they don't *talk* about it and do not consider it as part of the subject matter of the course proper.) It might be expected, therefore, that CS students would be more disposed to using the abstract tools taught in their classes.

It should be pointed out that there are subtle differences in the way the term *abstraction* is used in math vs. CS. These differences are elaborated in Leron (1987); for the present discussion, it will suffice to mention the difference in what is taken to be the *opposite* of "abstract" in the two disciplines. In mathematics, a common answer is, "the opposite of abstract is concrete". Thus, if students in an abstract algebra course complain (as they frequently do) that the stuff is too abstract, a standard response would be to give a "concrete" example. In CS, in contrast, the opposite of "abstract" usually means "dealing with the details of implementation in a particular machine or in a particular programming language." An abstract approach to data structures — which is one of the central topics investigated here — would stipulate the organization of the particular data structure and the operations it admits. For example, an abstract definition of the data structure "linear array" would be: a linear array is a set of ordered pairs (index, value), where all the indices are distinct, together with the operations **Insert** (inserting a new pair into the array) and **Get** (returning the value at a specified index). For example, a linear array can be used to represent our weekly entertainment schedule as follows:

Entertainment = {(Mon, movie), (Tue, home), (Wed, concert),...}

As a centrally important methodology in the design of complex software systems, students are urged to use *abstraction barriers*, in order to keep their thinking on a given problem relatively free from the intrusion of "low-level" constraints of a particular programming language (Abelson & Sussman, 1985).

2. Students conceptions of abstract data structures

For the research, we held semi-structured interviewes with 9 CS majors during their study of the course "data structures". The interview questions covered the following topics: data structures in general, arrays, stacks, queues, linked lists, and the construction of a data structure to fit the requirements of a given problem. The questions covered declarative formulations ("what is an array?"), operative formulations ("what is required from a data structure in order to be called 'an array'?"), operations on data structures ("how can a circle in a linked list be found?"), and more general questions for probing into the student's thinking ("is 'variable' a data

structure?"). In addition to the interviews, classes dealing with data structures were observed and documented.

The data analysis is still going on. For the analysis we use methods and theoretical frameworks from research on mathematical thinking. Previous research in CS education has mainly documented and analyzed programming difficulties (e.g. Lee & Lehrer, 1987; Sharma, 1986-87), but there is hardly any research on mental processes involved in thinking on CS concepts.

So far we have identified several mental (cognitive and affective) processes, which we list here by labels only, due to space limitations. Cognitive processes: programming-oriented thinking, conflicting mental structures for the same data structure, constraint-oriented thinking, extrinsic view of data structures, restricted prototypes for data structures categories. Affective processes: avoiding algorithms with heavy (machine) computational demands, avoiding algorithm detail and manual check.

In the remainder of this paper, we elaborate on 3 of the above-mentioned processes.

2.1. Programming-oriented thinking

The question "what is an array?" has been asked by the authors many times, not only during the interviews, but also in incidental discussions. The question was posed to undergraduate students and to expert computer-scientists. Only in few cases, an abstract definition of an array was given. Most of the answers were similar to the one in following interview excerpt:

```
I:   Can you tell me what is an array?
Dan: An array is a continuous area in the memory [of the computer],
     which holds elements of the same type.
```

We emphasize the following phenomena:

- Dan's thinking is programming oriented: he refers to an array as being held in the computer memory, namely, as implemented in some (as yet unspecified) programming environment.
- Moreover, Dan's thinking is programming-*language* oriented, i.e. tied to a specific programming language: he talks about continuous area in the computer memory, which is how an array is implemented in the programming language C, but not in all

languages. Programming-language oriented thinking is on a still lower abstraction level than programming oriented thinking.

- Dan sees an array as containing elements of the same type, again — a property which holds in C but not necessarily in other languages such as APL or LISP. Again, we see a programming-language oriented thinking.

The next excerpt from an interview with Guy, emphasizes further this phenomenon; the singly-underlined parts refer to programming, and the doubly-underlined ones refer specifically to programming in C:

> `OK, what is an array? An array is a sequence… it is a continuous memory segment […] and one can get to it using a key which is a continuous key. Actually, […] it is some segment which is allocated at the beginning of the program, it is allocated by the declaration, and is inhibited from being used for other purposes by other entities. […] I define its size by what is declared inside the brackets, and the program allocates a continuous area in memory to which I can get using a certain key […] 0 to, hmm, n-1, hmm, which is the size I declared it with.`

As can be seen, Guy's answer in general is strongly based on programming, and in parts even on programming in a specific language (C).

In another case, Ron was asked to solve a problem by presenting a general algorithm. He solved it using stacks[1]. During his work, Ron talked about emptying the stack by repeating the POP operation:

```
Ron:  Hmm… this means… hmm… what I'll do is… at the beginning, it
      is a check of the stack […] hmm… if it is full […] I'll do
      POP till the beginning of the stack.

I:    How do you know that you got to the beginning?

Ron:  I know the beginning address, right?
```

[1] A *stack* is a data structure of the LIFO (Last In First Out) kind: it behaves like a vertical stack of books on a library desk: the books are added ("PUSHed") onto the stack one after the other, and when one draws ("POPs") books from the stack, the first to be drawn is the last to have been entered.

The student talks about the *beginning address* of the stack. This answer is correct as far as stack implementation in C is concerned; however, it doesn't refer to the abstract data structure "stack", as indicated the problem formulation.

The above examples and others which are not presented here, indicate that the students identified data structures with their implementation in the computer's memory; this, in fact, is the lowest abstraction level of a data structure. The students did not solve problems in a general manner using abstract data structures, but kept referring to a particular "concrete" implementation. This is analogous to students in the math classroom, who often can work successfully with specific *representations* of functions (such as a table, a formula or a graph), but not with the concept of function as such.

Referring back to the process-object duality, one interpretation might be that a process conception of a particular data structure would be tied to its implementation in a particular machine or language, but that an object conception would be required in order to work with it abstractly. In this interpretation, we might say that the students have acquired a process conception of the data structure in question, but not an object conception. An object conception might also be required for effective use of abstraction barriers (Leron, 1987), which would allow students to ignore implementation details, while concentrating on the problem structure.

It is important to emphasize that, looking at the situation from the student's own perspective, the students were actually solving correctly most of the given problems, hence might not be motivated to work harder in order to achieve a more abstract solution. Similarly, it may be the case that working within the C language they could solve successfully most of the problems given in the course, and so didn't feel the need to work harder to achieve a more abstract conception. However, as in mathematics, the abstract approach is a powerful and desirable habit of mind, which becomes indispensable in advanced courses and projects. Instructors who want to cultivate this approach in their students, need to look for problems which will make this a powerful tool *for the students*. Our research shows that just *telling* them about it is not enough: as long as the less abstract (and apparently easier) approach is good enough to solve the required problems, they will not make the extra effort needed to climb the abstraction ladder.

In fact, as the following section demonstrates, the very same students can detach themselves from the implementation details when the problem forces them to do it.

2.2. Constraints-oriented thinking

In solving non-trivial problems in any discipline, we need to negotiate between the requirements and the constraints of the problem. In CS, in addition to the constraints inherent in the given problem, there are also the "low-level" constraints imposed by the particular hardware and software environment. The standard approach taught to present-day CS students is to suppress low-level detail by erecting an appropriate abstraction barrier: first, solve the problem assuming an ideal (relative to the given problem) software environment, and only at a later stage worry about how to implement this ideal environment on top of the one you actually have (Abelson & Sussman, 1985). We call this style of problem solving *requirements-oriented thinking.* If this approach is not heeded, the low-level details make it harder to solve the problem in the first place, and the solution tends to be messy-looking if one is eventually obtained; we call this approach — *constraints-oriented thinking.*

Some of the questions posed to the students intentionally contained implementation constraints. For example:

```
You are to computerize a certain restaurant so that the waiters
will enter into the computer each customer's order in turn, and
the chef will draw from the computer one order at a time. The
owner of the restaurant has purchased a programming environment
which has only stacks, since it was very cheap. How will you go
about doing the job?
```

A requirements-oriented approach to this problem would go roughly as follows: "Taking into account the requirements of the problem, what we need is a *queue*[2]. In our problem, the waiters enter each order at the end of the queue, and the chef draws from its head. The algorithms for the operations are such and such. Now, since we have a constraint — we only have stacks — we now have another (lower order) problem: how to implement a queue using stacks."

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[2] A queue is a data structure of the FIFO (First In First Out) kind; it behaves like a queue for a bus: a new element is inserted at the queue's end, and an element may be drawn from the queue's head.

Most of the students didn't use this approach, but rather opted for constraints-oriented thinking (We are not presenting here the actual data, due to space limitations): they "played" with the stacks in many ways, tried to enter the data (customers' orders) into one stack, then into two stacks; they tried transferring data between the stacks, adding auxiliary stacks, and so on. All the while, they seemed to be "groping in the dark". Even in cases where they eventually menaged to implement a queue using stacks, they didn't separate this lower-order problem from the main problem, but rather continued to work with the stacks at the level of the waiters' and chef's actions.

The problem, again, is lack of awareness of abstraction levels and, as a result, missing abstraction barriers. Similar phenomena have been observed in the work of math students (Leron, 1987).

2.3. Conflicting mental structures for the same concept

Let us go back to the "array" concept. After answering that an array is a continuous area in the memory of the computer, the students were asked the following question:

```
Suppose there are two separate segments – 20Kbytes and 30Kbytes
– available in the memory. Is it possible to implement an array
of 40Kbytes?
```

All the students answered affirmatively, and described various ways of doing so. This answer is clearly in conflict with their earlier continuous-segment answer. The two answers indicate the existence of two conflicting mental structures for the "array" concept: The first is programming oriented, or even a programming-language oriented, referring to the array *by its implementation*. The second refers to the array *by its properties* — the very heart of abstract data structures. These conflicting mental structures apparently co-exist in the student's mind, each being called upon in a different situation, according to needs. This is reminiscent of the phenomenon, widely discussed in math education research literature, of the *fragility of knowledge*, or *knowledge in pieces* (Brousseau & Otte, 1991; diSessa, 1988; Smith, diSessa & Rochelle, 1993).

3. Conclusion

The first conclusion that can be drawn from this report is that, despite differences in subject matter and population, methods and results of research in math education can

in many cases be extended to CS education. A second conclusion is that abstraction is difficult, even when the problem is relatively elementary and the students are relatively advanced. Perhaps a better way of putting it is that thinking at certain levels of abstraction is not a *natural* thing for students to do: they will mostly work on the lowest abstraction level that still enables them (albeit sometimes at great effort) to get a working solution. It follows that if we want our students to develop a disposition towards using more abstract tools, we need to work harder at finding problem situations that would make it worthwhile *for them* to use these tools.

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CONSTRUCTING PURPOSE IN MATHEMATICAL ACTIVITY

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This paper offers a theoretical discussion of how the ways in which the purpose of mathematical activity in primary classrooms is constructed by the participants may affect the learning and teaching of mathematical ideas. The argument draws on areas of my own writing, but is offered here as a starting point for directions of future research.

In this paper I offer the outline of an argument exploring the role of *purpose* in the learning and teaching of mathematics. I offer it here as a signpost to the directions of future work, since it contains many conjectures which need to be explored through further research, but I believe it may also offer a novel way of looking at some of the issues currently under consideration in the field of mathematics education.

In social, political and even educational arenas, mathematics is commonly portrayed as a subject whose importance is based on its utility in employment and daily life.

> *Mathematics is only 'useful' to the extent to which it can be applied to a particular situation.* (Cockcroft (1982), para 249)

This justification of mathematics continues in the face of evidence that not only does adult life require knowledge of a relatively small subset of the mathematics taught in schools, but that the mathematical skills learned in school are frequently rejected in favour of alternative methods in contexts where their use could be of practical value.

Even if learning mathematics could be justified in terms of utility, the concerns of the adult world are generally far removed from the experiences of young children. Thus there is a gap between the experiences of children learning mathematics and the purposes that are perceived by adults for that learning. I feel that little attention has been paid by curriculum developers, or by researchers, to the questions of what sense children make of the experience of learning mathematics, and *why* they think they are learning it. I have become increasingly convinced that the ways in which children construct the purpose of mathematical activity in the classroom may have significant effects on their learning, and have important implications for both teaching and curriculum development.

Contextualising mathematics

Attempts to give purpose to mathematics by contextualising the abstract content of the curriculum, particularly through the use of 'real-world' or 'everyday' problems,

has a long history. However, there seems to be an equally long history within research in mathematics education of studies of the difficulties which children have in combining mathematical and 'real-world' knowledge productively in these contexts (see for example Boaler (1993), Verschaffel *et al.* (1996)).

One outcome of the recent interest in research which explores the uses of mathematics in different areas of everyday life and employment (see for example Lave (1988), Nunes *et al.* (1993), Schliemann (1995)) has been the recognition that, far from being an inferior form of mathematical activity, 'street mathematics' has characteristics which may be of value in formal education. On this basis, many developments in 'realistic mathematics education' have explored ways of bringing features of street mathematics into the classroom via '*naturally occurring or meaningfully imagined situations*' (Nunes *et al.* (1993) p. 154).

One feature of street mathematics which has been discussed by a number of researchers as potentially transferable to the classroom is the notion of an apprenticeship model of teaching and learning (see for example Lave (1988); Masingila (1993)). In analysing the advantages of an apprenticeship model, Masingila identifies three key features:

> *(a) an apprenticeship model enables mathematical knowledge to be developed within a context, (b) cognitive development can occur as students work co-operatively with their teacher, and (c) a mathematics culture is developed within the classroom and students are initiated into this mathematics community.* (p. 21)

This analysis overlooks a crucial difference between the classroom and out-of-school contexts: that of purpose. When an apprentice learns carpet laying, fishing or carpentry by working alongside a master, both are essentially engaged in *the same* purposeful task, although they may perform different aspects of it. The master's agenda includes initiating the apprentice, and the apprentice knows that she is there to learn, but overlaying this is the value and purpose of the task which is being performed. Master and apprentice share an understanding of the overall task, and the purpose for the individual skills and techniques that are require to complete it. For both of them, there is a clear pay-off in performing the task well.

In the classroom, even if situations can be created where children's interest is engaged in purposeful or meaningfully imagined tasks, and in which they can work co-operatively with their teacher, the purposes of the tasks of master and apprentices will not be the same. The teacher's purpose is not to create the Logo program, build a puppet theatre, explore the mathematics within an investigation or to win the game; it is to teach. What is more, both teacher and pupils know this, and any pretence on the teacher's part that things are otherwise will be recognised as such. Thus, even though the apprenticeship model offers much that is of value in thinking about creating

meaningful mathematical experiences in the classroom, I feel it is important to be realistic about its limitations.

Lave's (1988) notion of learning as *situated* within a particular context offers a useful framework within which to explore aspects of children's behaviour in mathematics classrooms. Lave sees the context in which learning takes place as shaping the cognition, whilst at the same time being shaped, in the learner's perception, by the cognition. This notion of situated cognition has proved valuable in providing ways of looking at cognition in 'out-of-school' contexts. I want to turn the focus back into the classroom, and look at school mathematics as situated within the complex environment of the classroom. In particular, I see the individual's ways of constructing purpose within an activity as a key feature on the context. The classroom is, to a considerable extent, the 'real world' of young children.

In this discussion it is difficult to avoid the expressions 'real', 'reality', 'real world'. I find these words both problematic and unavoidable, and so I would like to be explicit about the ways in which I shall use them. First I want to detach the notion of *reality* from contexts, and attach it instead to the perceptions of individuals. So, a problem involving the lengths of curtains in relation to particular windows is a *real* context for me as an adult with an interest in interior decorating, but is not real for most primary school children, or for a colleague who finds the subject of curtains unexciting.

Secondly, I want to detach 'real' from 'real world'. The quality of an individual's engagement with a problem which makes it 'real' for them does not lie solely in its utility or application, nor in its physical existence. For young children, the boundaries between fact and fantasy are often drawn differently from those of adults, but adults can also become highly engaged with problems which are set in fantasy contexts.

Finally, I want to extend the notion of 'real-ness' being a quality of how an individual perceives and engages with a problem and detach 'real' from the opposite of 'abstract'. Abstract problems can be very real in terms of the interest and engagement they arouse. We risk denying children access to huge areas of mathematical culture if we make the decision on their behalf that only what belongs to the 'real world' can be interesting.

In much of my research into the views of children and their teachers about mathematical activities, I have found a number of discontinuities in their perceptions of the nature of the activity they are engaged in, and of the purpose of school mathematics (see for example Ainley (1988, 1991)). If the purpose which is routinely offered for learning mathematics is its utility in contexts outside the classroom, which are not real for the children either in terms of their familiarity or of their intrinsic interest, then it is unsurprising that children will invent other ways of constructing the purposes of mathematical activity within the classroom.

The student voice which I hear through much of my reading and my own research seems to build a cumulative picture of an elaborate, ritualised game being played out by children in response to their constructions of the behaviour of teachers within the classroom context. These constructions permeate much of their experience of school, and I believe that we need to see their learning of mathematics as firmly situated within this context if we are to appreciate and understand some of their behaviour.

I offer one example from my own recent research in illustration here. In reporting on early stages of the Primary Laptop Project (Ainley and Pratt (1995)), Pratt and I described one example of a behaviour we saw in many children when they were first introduced to the graphing facilities of a spreadsheet. Children were interested to explore the range of graphs they could produce, and the graphic effects offered by the software. When it came to selecting a graph to print out for inclusion in their project folders, many children made choices which surprised us. They seem to be guided solely by the visual appearance of the graph, and paid no attention to whether or not the chosen graph displayed the data appropriately.

Our first interpretation of this behaviour was to feel impatient with children who seemed to be 'playing' with the software, rather than paying attention to the mathematics. When we questioned them about their graphs however, we began to hear a different construction of the purpose of the activity. The criteria some children used for choosing their graphs tended to be aesthetic rather than mathematical. Their preference was for ones which looked complex and/or unusual. Questions about the meaning of their graphs were often met with incomprehension. It began to emerge that the children did not see graphs as meaningful, or as ways of communicating information. Their construction of the purpose of graphs, based on their previous experiences within school, was that graphs were essentially decorative, used to brighten up classroom displays.

One way of looking at this behaviour is to see this as analogous to the activity often described as 'emergent writing'. Young children typically begin to imitate the behaviour of adults writing long before they develop the skills required for 'real' writing. In doing so they imitate both the form and the purpose of the activity: they don't just write, they write letters, shopping lists, menus. Through engaging in this activity they learn important lessons about what writing is for.

I would like to describe the activity many children engaged in with the spreadsheet as 'emergent graphing'. The power of the technology allowed them to play at producing the sorts of graphs they had seen in the adult world. The strategy we decided to adopt within the project was to accept these graphs, and to encourage children to work with them in ways that we worked with other graphs, for example by reading back information which they contained. Alongside this, we tried to design activities in

which children produced and worked with graphs - and importantly in which they saw us as teachers working with graphs - in more directed ways. Gradually, we felt that the children's understanding of the purpose of graphing developed as they enlarged their range of skills in using them (Ainley (1995), Pratt (1995)).

A different perspective on mathematical activity

My research into children's perceptions of the purpose of teachers' questions (Ainley (1988)) led to my first notion of the ways in which children's experience of mathematical activity are shaped by the school context. It seems to me now that the same shift in perspective may offer an alternative account of why attempts to contextualise mathematics in the classroom are often ineffective. If children construct the majority of teachers' spoken questions as designed to *test* their understanding, it seems probable that they will interpret written questions, such as word problems, in the same way, even if this is not the purpose for which teachers use them. Indeed there seems to be a number of purposes which teachers may offer for setting mathematical ideas and techniques in context. Three possibilities are:

- to support children's understanding of the mathematics;
- to support children in transferring their knowledge to situations outside the classroom by showing them what it is useful for;
- to test the children's understanding by requiring them to apply their knowledge.

I conjecture that many primary/elementary school teachers would offer explanations which cover or combine the first two of these, but may not recognise the third as a distinct category, even though the purpose here is radically different. (It may be that their colleagues teaching in secondary schools would take different views.) In contrast, children's experience of word problems in textbooks is that they frequently form the last section on the page, following more straightforward examples of the 'sums' on their own. As the contents of the page generally progress in difficulty, it is natural to see this last section as the hardest, designed to extend your thinking, or to catch you out, depending on your point of view. Indeed, it is difficult to imagine any other purpose: if the contextualised problems were designed to help children's understanding of the mathematics, surely they would be offered first.

I am led by this analysis to conjecture that many children will construct the use of contextualised problems in school mathematics as a hurdle to be overcome, rather than as an aid to their learning. The problems are there to make it more difficult to recognise the calculation which has to be carried out to arrive at the right answer, which is, after all, what the school game is all about. If this is how children construct the purpose of the activity, then a sensible strategy to adopt is to pay no attention to the context, which may distract from this goal. It may be that children are not *unable*

to interpret word problems or to transfer knowledge from one situation to another: in the classroom situation, they simply may not see this as the purpose of the activity.

I see a large part of children's experience of mathematics as an activity situated in classrooms, and shaped by their perceptions of the purposes of schooling. I believe that the underlying reason why most attempts to contextualise mathematics fail to enable children to apply their knowledge in other situations, is because of a failure to pay attention to how the purposes of mathematical activity are understood by the participants. Teachers and curriculum developers may use real world contexts with the purpose of showing pupils how a particular piece of mathematics can be useful. But if children construct the purpose of the activity - and indeed of all school mathematics - as 'getting the right answers', they will be unable to appreciate what the teacher's purpose is. Indeed they may fail to appreciate the more fundamental idea that mathematical knowledge *is* useful, because the classroom context shapes their perceptions of mathematical activity so strongly.

The role of purpose

I see the notion of *purpose* as central both to interpreting mathematical activity in the classroom, and to the quality of children's mathematical thinking. For me the notion of purpose is clearly distinct from that of motivation. Children may be motivated by their enjoyment in carrying out a task, or by the novelty of a situation, but still see little purpose in what they are doing. The difference in the quality of attention which comes from engaging in a purposeful task in very marked.

From my work with Logo, I have seen repeatedly the effects of a clear end-product in generating a powerful sense of purpose for children (and for adults). There seems to be something very distinctive about the ways in which mathematical ideas are addressed and understood when they are met within the context of a Logo project. The child's ownership of the project also has significant effects on the interactions between teacher and pupil. I would not wish to claim that this kind of purposeful activity is unique to Logo: similar observations may be made about children's work in a range of other product-oriented activities. (This idea is developed as *constructionism* by Harel and Papert (1991) and others.) However there are particular features of computer environments which seem to both generate and sustain this sense of purpose: rapid feedback from the computer, and the ability to adjust and correct ideas with ease, encourage children to engage in purposeful activities. My observations of children's work in computer environments have focused my attention further on the significance of the children's perception of purpose and how this may relate to the end-product of the activity. This has led to the second meaning within my deliberately ambiguous choice of title: the exploration of ways of constructing mathematical activities in which the purposes of teachers and of children can be brought in line with each other.

I have found that in many classrooms, and for much of the time, children have different perceptions of the purposes of mathematical activities from those of their teachers. This affects the ways in which they see mathematical tasks, and the ways in which they interpret teachers' behaviour. As a result, teachers and pupils may be working at cross purposes, and teachers may see children's responses as demonstrating a lack of understanding, or of attention, or even as deliberate subversion of the objective of the lesson. However, it also seems clear to me that children work hard at making sense of mathematical activity, even when they are given little basis on which to do this. They construct purposes for their activities within the context of their experience of the classroom and the school, even though they often fail to appreciate the wider purposes which teachers and curriculum developers intend to convey in the ways in which tasks are contextualised. Often these mis-matches arise because children, and sometimes their teachers, are not able to distinguish those aspects of mathematics which are matters of convention from more significant mathematical concepts.

In designing activities for children within the Primary Laptop Project, we have often used the model of Logo projects; aiming for tasks within which children can be given the freedom to explore and make decisions, and which the children themselves will see as purposeful. However it has become apparent that these conditions are not sufficient to produce activities in which children will engage with the mathematical ideas which are part of *our* purpose. In working on these ideas with Dave Pratt we have come to distinguish the overall purpose of the activity from the *utility* of the mathematical ideas used within it. I offer two brief examples here which I hope will serve to illustrate the distinction.

One activity we have used with many groups of children involves trying to design a good paper 'helicopter' (aspects of this activity are discussed in Pratt (1995)). Children needed to test their designs by timing how long the helicopter flew, but quickly realised that their timings were inaccurate. Within the activity we were able to offer them the facility on the spreadsheet to find the (mean) average of a set of results as a way of balancing out the inaccuracies of their measurements. At this point the children used the computer to generate the average value which they then used to plot a graph. They did not learn how to calculate the average, but they did learn something about the way in which this value might be used, within the context of answering a question which they found both real and intriguing.

A second example may be found in a study of an activity involving the maximisation of the area of a sheep pen (Ainley (1996)). Within this activity they made use of a graph of their results to try to identify the maximum value, and then translated their method for calculating the dimensions of the pen into a spreadsheet formula to generate more (and more accurate) results, which in turn produced a more useful

graph. The boys' attention was primarily on solving the problem, which, despite its rather contrived setting, became real for them through being sufficiently intriguing (and to some extend also through the interest which we as teachers were taking in their solution). They are able to appreciate the utility of both the graph, and the formula which would allow them generate data which would draw a 'better' graph.

I believe that appreciating the utility of a concept or procedure through being able to apply it in a purposeful context is an extremely powerful way of learning mathematics. The quality children's work, and the mathematical levels that we have been able to reach within the Primary Laptop Project, using activities which have been designed in this way, strongly supports this view. Moreover, it seems that children who learn about the utility of mathematical ideas in this way, also have the opportunity to learn that mathematics *is* useful, not only in the adult world, but in their world as well. The mathematics classroom seems to be the most appropriate and convenient context in which to locate school mathematics, and attention to issues of purpose and utility offer the possibility of constructing learning environments which support the application of what is learned there to the world beyond the classroom.

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EFFECTIVE TEACHERS OF NUMERACY IN UK PRIMARY SCHOOLS: TEACHERS' BELIEFS, PRACTICES AND PUPILS' LEARNING:

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Abstract
This paper reports on part of a study examining the links between teachers' practices, beliefs and knowledge and pupil learning outcomes in the development of numeracy with pupils aged five to eleven. From a sample of 90 teachers and 2000 pupils, we developed detailed case studies of 18 teachers. As part of these case studies we explored the teachers' beliefs about what it means to be numerate, how pupils become numerate and the roles of the teachers. From the data three sets of belief orientations were identified: connectionist, transmission and discovery. Results from pupil assessments suggest that there was a connection between teachers demonstrating strong orientation to one of these sets of beliefs and pupil numeracy gains.

1 Aims of the study

The aims of the study Effective Teachers of Numeracy, funded by the UK's Teacher Training Agency (TTA) were to:

1 identify what it is that teachers of five to eleven year olds know, understand and do which enables them to teach numeracy effectively;

2 suggest how the factors identified can be more widely applied.

The working definition of numeracy used by the project was a broad one:

> Numeracy is the ability to process, communicate and interpret numerical information in a variety of contexts.

Evidence was gathered from a sample of 90 teachers and over 2000 pupils on what the teachers knew, understood and did and outcomes in terms of pupil learning.

Studies have pointed to the importance of establishing of a particular classroom culture (Cobb, 1986), raising the issue of teachers' belief systems about mathematical knowledge, how it is perceived as generated and learnt, and the impact upon pupils' learning. It may be that beliefs about the nature of the subject are more influential than mathematical subject knowledge per se (Lerman, 1990; Thompson, 1984).

Many studies, particularly in the USA, focus on effective classroom practice and routines (Berliner, 1986) but research demonstrates the difficulty that teacher experience in adopting new practices without an appreciation of and belief in the underlying principles (Alexander, 1992). Further, teachers may have adopted the rhetoric of 'good' practice in teaching mathematics without changes to their actual practices (Desforges & Cockburn, 1987). While teachers' classroom practices and subject knowledge were also foci of this research, this paper concentrates on the findings related to teachers' belief systems. (For full details of the research see Askew et al., 1997)

2 Identifying effective teachers of numeracy

Careful identification of teachers believed to be effective in teaching numeracy was crucial to this study. The idea that effective teachers are those who bring about identified learning outcomes was our starting point for the project. **We decided that as far as possible the identification of effective teachers of numeracy would be based on rigorous evidence of increases in pupil attainment, not on presumptions of 'good practice'.**

From an initial sample size of all the primary schools in three local education authorities (some 587 schools), together with Independent (private) schools, we selected eleven schools, providing a sample of 90 teachers. We selected the majority of these eleven schools on the basis of available evidence (national test scores, IQ data, reading test scores and baseline entry assessments) suggesting that the teaching of mathematics in these schools was already effective.

A specially designed test ('tiered' for different age ranges) of numeracy was administered to the classes of these 90 teachers, first towards the beginning of the autumn term 1995, and again at the end of the spring term 1996 (classes of five year olds were only assessed the second time). Average gains were calculated for each class, providing an indicator of 'teacher effectiveness' for the teachers in our sample.

In order to broadly classify the relative gains, the teachers were grouped into three categories of *highly effective*, *effective*, or *moderately effective*. This classification was made by putting the classes in rank order within year groups according to the average gains made (adjusted to take into account the fact that it was harder for pupils to make high gains if their initial test score was high). The cut-off points between high, medium and low gains were made on pragmatic grounds, so that classes in each year group fell into three roughly equal groups but avoiding any situation where classes with nearly equal adjusted gains were allocated to different groups. The groups were not based on any predetermined quantitative differences between the classes based on expectations of what a 'medium' gain should be.

3 Teacher case study data

Research on the links between knowledge, beliefs and practice suggested a mix of techniques to elicit teachers' knowledge and understanding backed up by classroom observation to examine actual practices. From the sample of 90 teachers we worked closely with 18 teachers who formed our case study teachers providing data over two terms on classroom practices together with data on teacher beliefs about, and knowledge of, mathematics, pupils and teaching. These teachers were identified in advance of the second round of pupil assessment, and chosen through discussion with head teachers and, where appropriate, with advice from the LEA inspectors and advisors. While the emphasis was on identifying effective teachers, the group of 18 were chosen so that their pupils were evenly distributed across ages 5 to 11 (year groups 1-6).

3.1 Classroom observations

In total, 54 lessons were observed, three for each of the case study teachers. Data gathered included a focus on:

- organisational and management strategies - how time on task is maximised, catering for collective and individual needs, coping with range of attainment
- teaching styles - intervention strategies, questioning styles, quality of explanations, assessment of attainment and understanding, handling pupil errors
- teaching resources - sources of activities, range of tasks, resources available, expected outcomes
- pupil responses - ways of working, evidence of understanding.

3.2 Case study teacher interviews

Fifty-four interviews were conducted, three for each case study teacher:

- background interview: providing evidence on training and experience as well as information on beliefs, knowledge and practices in teaching numeracy; teachers own perceptions of what has made them successful teachers of numeracy, and reasons for factors identified
- 'concept mapping' interview: this interview was based around a task that explored the teachers understanding of aspects of mathematics related to teaching numeracy.
- 'personal construct' interview: this interview was structured around a task that focused on the particular group of pupils that the teacher was currently teaching in order to explore the beliefs and knowledge about pupils and how they came to be numerate.

The data were analysed using qualitative coding methods and the constant comparative method to build up models of belief systems (Lincoln & Guba, 1985; Miles & Huberman, 1984; Strauss & Corbin, 1990)

4 Orientations in teachers beliefs.

From the analysis of the case study data three models of sets of beliefs that emerged as important in understanding the approaches teachers took towards the teaching of numeracy:

- connectionist - beliefs based around both valuing pupils' methods and teaching strategies with an emphasis on establishing connections within mathematics;
- transmission - beliefs based around the primacy of teaching and a view of mathematics as a collection of separate routines and procedures;
- discovery- beliefs clustered around the primacy of learning and a view of mathematics as being discovered by pupils.

	connectionist	transmission	discovery
Beliefs about what it is to be a numerate pupil	Being numerate involves: • using both efficient and effective methods of calculation ; • confidence and ability in mental methods; • selecting a method of calculation on the basis of both the operation and the numbers involved; • awareness of the links between aspects of the mathematics curriculum; • reasoning, justifying and, eventually, proving, results about number.	Being numerate involves: • the ability to perform set procedures or routines; • confidence and ability in paper and pencil methods; • selecting a method of calculation primarily on the basis of the operation involved; • confidence in separate aspects of the mathematics curriculum; • able to 'decode' context problems to identify a particular routine or technique.	Being numerate involves: • finding the answer to a calculation by any method; • confidence and ability in practical methods; • selecting a method of calculation primarily on the basis of the operation involved; • confidence in separate aspects of the mathematics curriculum; • being able to use and apply mathematics using practical apparatus.
Beliefs about pupils and how they learn to become numerate	Becoming numerate is a social activity based on interactions with others. • Pupils learn through being challenged and struggling to overcome difficulties. • Most pupils are able to become numerate. • Pupils have calculating strategies but the teacher has responsibility for helping them refine their methods.	Becoming numerate is a individual activity based on following instructions. • Pupils learn through being introduced to one mathematical routine at a time and remembering it. • Pupils vary in their ability to become numerate. • Pupils' strategies for calculating are of little importance - they need to learn standard procedures.	Becoming numerate is a individual activity based on actions on objects. • Pupils need to be 'ready' before they can learn certain mathematical ideas. • Pupils vary in the rate at which their numeracy develops. • Pupils' own strategies are the most important: understanding is based on working things out yourself

	connectionist	transmission	discovery
	• Misunderstandings need to be recognised, made explicit and worked on.	• Misunderstandings are the result of failure to 'grasp' what was being taught and need to be remedied by reinforcement of the 'correct' method.	• Misunderstandings are the result of pupils not being 'ready' to learn the ideas.
Beliefs about how best to teach pupils to become numerate	• Teaching and learning are seen as complementary. • Numeracy teaching is based on *dialogue* between teacher and pupils to explore each others' understandings. • Learning about mathematical concepts and the ability to apply these concepts are learned alongside each other. • Connections joining mathematical ideas needs to be acknowledged in teaching. • Application is best approached through challenges that need to be reasoned about.	• Teaching is seen as taking priority over learning. • Numeracy teaching is based on *verbal explanations* so that pupils understand teachers' methods. • Learning about mathematical concepts precedes the ability to apply these concepts • Mathematical ideas need to be introduced in discrete packages. • Application is best approached through 'word' problems: contexts for calculating routines	• Learning is seen as taking priority over teaching. • Numeracy teaching is based on *practical activities* so that pupils discover methods for themselves. • Learning about mathematical concepts precedes the ability to apply these concepts • Mathematical ideas need to be introduced in discrete packages. • Application is best approached through using practical equipment

Table 1: Key distinctions between connectionist, transmission and discovery orientations towards teaching numeracy.

These orientations are "ideal types". No one teacher is likely to fit exactly within the framework of beliefs of any one of the three orientations. Many will combine characteristics of two or more.

However, it was clear that those teachers with a strong *connectionist* orientation were more likely to have classes that made greater gains over the two terms than those classes of teachers with strong *discovery* or *transmission* orientations.

Analysis of the data revealed that some teachers were more predisposed to talk and behave in ways that fitted with one orientation over the others. In particular, Anne, Alan, Barbara, Carole, Claire, Faith (the teacher initial matches the school code, so Anne and Alan are from same school), all displayed characteristics indicating a high level of orientation towards the *connectionist* view. On the other hand, Beth and David both displayed strong *discovery* orientations, while Elizabeth and Cath were both clearly characterised as *transmission* orientated teachers.

Other case study teachers displayed less distinct allegiance to one or other of the three orientations. They held sets of beliefs that drew in part from one or more of the orientations. For example, one teacher had strong *connectionist* beliefs about the nature of being a numerate pupil but in practice displayed a *transmission* orientation towards beliefs about how best to teach pupils to become numerate.

	Highly effective	**Effective**	**Moderately effective**
Strongly Connectionist	Anne, Alan Barbara Carole, Faith		
Strongly transmission			Cath Elizabeth
Strongly discovery			Beth David
No strong orientation	Alice	Danielle, Eva Dorothy, Fay	Brian Erica

Table 2. The relation between orientation and effectiveness

The connection between these three orientations and the classification of the teachers into having relatively high, medium or low mean class gain scores suggests that there may be a relationship between pupil learning outcomes and teacher orientations.

5 Links between orientation and practice

5.1 Orientation and the role and nature of mental strategies in pupils becoming numerate

All the teachers, whether leaning towards a *connectionist*, *transmission* or *discovery* orientation saw some aspects of mental mathematics as important. Knowing basic number bonds and multiplication facts provided a baseline of expectations within all three orientations.

However, the *connectionist orientated* teachers viewed mental mathematics as going beyond this recall of number facts. Mental mathematics did not involve simply knowing number bonds but having a conscious awareness of connections and relationships to develop mental agility.

This mental agility meant that for the *connectionist* teachers mental mathematics also involved the development of flexible mental strategies to handle efficiently number

calculations. Working on mental strategies, they believed, laid foundations that extended the pupils' levels of competency. Developing confidence in flexible mental methods meant that pupils would be able to tackle calculations for which methods had not been taught.

5.2 Orientation and teacher expectations

The *connectionist* orientated teachers placed strong emphasis on challenging all pupils. They believed that pupils of all levels of attainment had to be challenged in mathematics. Being stretched was not something that was not restricted to the more capable pupils. They had high levels of expectations for all pupils irrespective of ability. Intelligence was not seen as static and all pupils were regarded as having the potential to succeed.

In contrast the *transmission* and *discovery* orientated teachers may provide challenge for the higher attaining pupils but structured the mathematics curriculum differently for lower attaining pupils.

5.3 Orientation and style of interaction

The *connectionist* teachers' lessons were generally characterised by a high degree of focused discussion between teacher and whole class, teacher and groups of pupils, teacher and individual pupils and between pupils themselves. The teachers displayed the skills necessary to manage effectively these discussions. The teachers kept pupils focused and on task by organising these discussions around problems to solve, or sharing methods of carrying out calculations.

In school A, one of the *most effective* schools, there was a consistent approach to interacting with pupils throughout the years. Right from age five pupils were expected to be able to explain their thinking processes. Because the pupils were explaining, rather than simply providing answers to questions that the teacher already knew the answer to, the lessons were characterised by dialogue. In this discussion both parties, teacher and pupils, were having to listen carefully to what was being said by others. The result was pupils who, by eleven, were confident and practised in sharing their thinking and challenging the assumptions of others.

5.4 Orientation and the role of mathematical application

For the *discovery* or *transmission* orientated teachers, application of knowledge involved pupils putting what they had previously learnt into context. Problems presented 'puzzles' where the pupils already have the required knowledge and the challenge is only to sort out which bit to use. Alternatively, problems were a means of demonstrating to pupils the value of what they are learning.

The *connectionist* orientated teachers also recognised the importance of being able to apply computational skills. But over and above this they did not see it as a necessary pre-requisite that pupils should have learnt a skill in advance of being able to apply it. Indeed, the challenge of an application could result in learning.

6 Discussion

The importance of these orientations lies in how practices, while appearing similar may have different purposes and outcomes depending upon differences in intentions behind these practices.

We would suggest that these orientations towards teaching mathematics need to be explicitly examined in order to understand why practices that have surface similarities may result in different learner outcomes. While the interplay between beliefs and practices is complex, these orientations provide some insight into the mathematical and pedagogical purposes behind particular classroom practices and may be as important as the practices themselves in determining effectiveness.

Other teachers may find it helpful to examine their belief systems and think about where they stand in relation to these three orientations. In a sense the *connectionist* approach is not a complete contrast to the other two but embodies the best of both them in its acknowledgement of the role of both the teacher and the pupils in lessons. Teachers may therefore need to address different issues according to their beliefs: the transmission orientated teacher may want to consider the attention given to pupil understandings, while the discovery orientated teacher may need to examine beliefs about the role of the teacher.

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CAN THE AVERAGE STUDENT LEARN ANALYSIS?

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Abstract

A case-study of one-year efforts of three students trying to learn mathematical analysis is reported. Concepts of concept image, concept definition, procept *and* encapsulation *are used to support the adopted didactical strategy consisting in emphasizing the propositional calculus with explicit applications of the four rules of inference in such a way as to submit the concept image to the control of the concept definition, aiming at the encapsulation of the ε–δ discourse. A detailed example is provided. Effects of the learning efforts on the students and on the faculty are discussed in terms of affective energy.*

The research question

This paper reports a case study jointly developed by one teacher, two undergraduate students in a teacher training program and one graduate student in Mathematics Education program. The word *analysis* refers basically to the definition of limit and the construction of the real numbers. The expression *learn analysis* refers to the encapsulation of a particular process as an object. The word *average* refers to the students' self evaluation: they ranked themselves in the second quarter of their classes and in the second group described by Pinto & Gray [1995, p. 2-25]. Among equally ranked peers they detected widespread rote learning. The directive research question emerged naturally from their dissatisfaction and desires: *can the average student like us learn analysis?* Or is this subject reserved only to the so-called "gifted" ones?

Methodology

The group met once-a-week for three hours during 1996. The activity was considered as part of a honors fellowship project for one of the undergraduate students, a chance to improve learning for the other and an opportunity to rebuild the mathematical basis for the graduate student. In the first meeting, methodological directive lines on subject-matter, didactical strategy, meta-cognition and evaluation were established. Negotiation proceeded along the year.

The subject matter was dictated by the syllabus and homework of a regular one-year mathematical analysis course that the undergraduate students were taking from another teacher. In the second semester the group decided to concentrate on a single subject: the construction of real numbers. This subject had come up several times in the first semester. The teacher suggested to take the Cauchy sequence approach in order to boost opportunities to work with epsilons and deltas. The only available Portuguese language source that describes the construction in detail happens to contain a mistake in the proof of the fundamental theorem on the completeness of the real numbers. A task was proposed to the group: *in this chapter there is a mistake; find it, give a counter example and produce a correct proof.*

A didactical strategy was chosen: instead of looking for a smooth transition from the intuitive to the formal level, a radicalization of the cut between *concept image* and *concept*

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definition should be tried, by training the students on semi-formal treatment of propositional calculus. The four rules of inference: universal and existential particularization and generalization should be spelled out and systematically used.

Discussions on meta-learning and meta-teaching were carried out at the end of each session. They concerned the difficulties and progresses of each student and the overall evaluation of the day's work. Some session of the first and all sessions of the second semester were videotaped. Some videos were viewed and discussed by the students. The way the teacher conducted the sessions, the opportunity, aim and effects of his interventions were analyzed and adjusted along the year.

As for evaluation, the group agreed that a final research report should be submitted to PME-21 and partial results should be presented in local meetings during the year. In the beginning of the second semester the honors undergraduate student was scheduled to present a purely mathematical report to her peer group and their program's advisors at the end of the year. The performance of the undergraduate students in exams of the regular analysis course were also to be observed.

The theoretical framework

It was agreed that the theoretical framework to interpret data should be the conceptualization developed by the Advanced Mathematical Thinking group of PME: *concept image, concept definition, conflict factor* [Tall & Vinner, 1981], *process, concept, procept, ambiguity process-product, encapsulation* [Gray and Tall, 1994]. It turned out that the research subject matched that of Pinto & Gray [1995] and Pinto & Tall [1996], namely, students' misconceptions about limits, rational and real numbers and the use of formal definitions. The difference is that these authors seek to investigate the existing students' state of knowledge and institutional conditions, while the present research tries to produce a change in the state of knowledge and to investigate the outcomes and possibility conditions of such an attempt. It should be qualified as a *case study* under an *action-research* approach.

The didactical strategy: "1/n XPTO 0"

In this paragraph the necessity of a didactical strategy stressing the discontinuity between concept image and concept definition will be justified. Next, a fairly detailed description of the particular *elementary procept* that supports this strategy will be presented. Finally a certain ambiguity of process-concept will be held as the expression of the advanced mathematical thinking in analysis and will be described in terms of *encapsulation*.

The didactical strategy of continuity. The undergraduate students had been exposed to the "intuitive definition" of limit in calculus courses and the graduate student had also been exposed to the "formal definition" in an analysis course like the one that the undergraduate students were taking at that moment. In the students' own opinion, they "attempted to learn definitions by rote but in the main failed to understand the underlying concepts" [Pinto & Gray, 1995, p. 2-18]. Work with them a year before [Leal et al, 1996] had produced the evidence that they shared most of the "observed errors" about limits pointed out by Davis & Vinner [1986, p. 294]. These authors formulate a major question about misconceptions: "Is there a way to teach these concepts so that misleading images will not be formed? Or are these "naive" images unavoidable and will be formed no matter how the concept is taught?" [p. 285]. We add: what to do if they are already formed? According to the authors, *influence of language* is one of the sources of misconceptions about limits [p. 298]. Words

such as "limit" have undue connotations, either inside or outside mathematics. In order to avoid them, Davis & Vinner report to have tried unsuccessfully, or at lest without clear success, to postpone the introduction, not only of the concept definition but, also, of the very word "limit". "The word *limit* was not introduced until after the correct mathematical concept was seemingly well established" [Davis & Winner, 1986, p. 299]. Postponing the concept definition until a reliable concept image can be formed is the same strategy pointed out by Tall & Vinner [1981] in the SMSP:

"(...) in the SMSP (...) the concept images of limits and continuity are carefully built up over the two years of the course with fairly formal concept definitions only being given at the very end. In this way the concept image is intended to lead naturally to the concept definition" [Tall & Vinner, 1981, p. 155].

We shall call such attempts the *didactical strategy of continuity*. It consists in seeking a natural transition form the concept image to the concept definition of limit by painstakingly expanding and adjusting the concept image so that it can take in the concept definition.

<u>Difficulties with the didactical strategy of continuity</u>. Since the continuity strategies are dominant in almost any textbook on calculus or analysis, we may trust that they are associated with, if it is not the cause of:

"(...) the almost insignificant effect that a course on analysis had in changing the quality of mathematical thinking of a group of students (...). (...) despite their extensive work with real numbers, their concept image **had not expanded to take in the concept definition**" [Pinto & Gray, 1995, p. 2-18, our emphasis].

In the first meeting, the students expressed their understanding about the formal definition of limit by the following phrase: "*For any epsilon there is an N, starting from which the sequence converges*". The teacher asked: "*Do your mean that before this N the sequence might diverge?*". Along the discussion the students ran into several contradictions but the game "someone gives you an epsilon and you have to find an N such that" appeared to them as an arbitrary caprice of the teacher. The persistence of the above phrase indicated that the students were trying to graft the concept definition into the concept image. They were calculating limits correctly and propositions such as the limit of the product of a bounded sequence by a sequence converging to zero is zero, seemed completely obvious to them. When asked to produce a formal proof, they mixed phrases from their concept images with phrases from the concept definition. They soon started referring to bounded variables outside the formulas where they had been introduced. Whenever they referred to "this epsilon" in a formula such that $\forall\varepsilon\ P(\varepsilon)$ the teacher replied: "*I see no epsilon on this black-board*", and replaced the epsilon by another symbol, attempting to show that the meaning of the proposition remained unchanged. This produced some astonishment among the students but no positive effects. The situation is well described as a *potential conflict factor* in Tall & Vinner [1981]:

"A more serious type of potential conflict factor is one in the concept image which is at variance not with another part of the concept image but with the formal concept itself. Such factor can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors **unless the formal concept definition develops a concept image** which can then yield a cognitive conflict. Students having such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous" [Tall & Vinner, 1981, p. 154, our emphasis].

The teacher made an effort to emphasize the role of definitions in mathematics but his attempt was rebuffed. The students manifested their conception of "definition" as a "*complete description*" of an object. For them the definition of limit was simply intended to make the idea of limit "*more precise*". Asked to choose a couple of similar notions among *definition, theorem,* and *axiom,* they did not hesitate in uniting definition with, either axiom or theorem. "The everyday life thought habits take over and the respondent is unaware of the need to consult the formal definition. Needless to say that, in most cases, the reference to the concept image cell will be quite successful. This fact does not encourage people to refer to the concept definition cell" [Vinner, 1991, p. 73]. The teacher tried to emphasize the arbitrary character of definitions: "*Definitions are arbitrary*. Definitions are "man made". Defining in mathematics is **giving a name**" [Vinner, 1991, p. 66, our emphasis]. However, the comparison of definition to the ritual of baptism made the students laugh a lot. (They later discussed the video.)

Rupture of concept image and concept definition. It seems that looking for a continuous transition such that the concept image would be progressively adjusted and would terminate by incorporating the concept definition, leads to difficulties already recognized by Vinner [1991]:

"Only non routine problems, in which incomplete concept images might be misleading, can encourage people to refer to the concept definition. Such problems are rare and when given to students considered as unfair. Thus, **there is no apparent force** which can change the common thought habits which are, in principle, inappropriate for technical contexts" [Vinner, 1991, p. 73, our emphasis].

If there is "no apparent force", how to unbalance students' notions? The answer to this question may be found in a previous paper of the same author: "(...) unless the formal concept definition develops a concept image which can then yield a cognitive conflict" [Tall & Vinner, 1981, p. 154]. At this point the notion of *concept definition image* comes in: "For each individual a concept definition generates its *own* concept image (...) which might (...) be called the "concept definition image" [Tall & Vinner, 1981, p. 153]. The question now becomes: how to make the *concept definition image* strong enough so that it acquires the power of redressing the whole concept image? The answer provided in this paper is: by stressing precise rules to manipulate the concept definition until an object is formed and simultaneously submitting the concept image to the control of the concept definition. This implies attributing an independent statute to the concept definition and introducing a rupture between concept image and concept definition.

The new didactical strategy. Gray & Tall [1994] characterize the advanced mathematical thinking as the possibility of ambiguous use of *process* and *product* evoked by the same symbol. As for limits, the *process* is the *tendency* towards the limit and the *product* is the value of the limit:

"The notation $\lim_{x \to a} f(x)$ represents both the **process of tending to a limit** and the **concept of the value of the limit**, as does $\lim_{n \to \infty} s_n$ (...)" [Gray & Tall, 1994, p. 120, our emphasis].

"We conjecture that the dual use of notation as process and concept enables the more able to "tame the process of mathematics into a state of subjection"; instead of having to cope consciously with the duality of concept and process, the good mathematician thinks **ambiguously** about the symbolism for **product** and **process**" [Gray & Tall, 1994, p. 121, our emphasis].

The new didactical strategy consists in redefining *process* and *product* in the situation of limits, consequently aiming at another form of the ambiguity. It starts recalling

that the *concept definition* is a *verbal form*: "We shall regard the *concept definition* to be a form of words used to specify that concept" [Tall & Vinner, 1981, p. 152]. The *process* is then redefined as the *sequence of inferences* necessary to deal with the form of words used to specify the concept of limit (propositional calculus). The *product* is redefined as the demonstration, that is, the *effect of truth* of the discourse supported by such inferences. This means a shift of emphasis towards language, while keeping the same basic conceptualization of Advanced Mathematical Thinking.

Precisely, according to the old ambiguity, the use of the symbol "lim1/n=0" meant either a *tendency process* or a *final value*. The new ambiguity consists in using this symbol to mean, either that for every epsilon we can find an N (the process), or that the proposition "lim1/n=0" is true, that is, it can be sustained (by an epsilontic discourse) in the forum of mathematical community (product). Indeed, whenever a mathematician claims that something is trivial, as they like to do, s/he is not thinking on the "cognitive complexity process-concept" but s/he is exercising *this specific form* of process-product ambiguity: s/he is ready to sustain a discourse in terms of a chain of propositions. The process of (epsilontic) discourse has been encapsulated as an object (claim). In order to be realized, such a strategy should provide the formation of an *elementary procept* leading to the construction of this specific object.

"An *elementary procept* is the amalgam of three components: a *process* that produces a mathematical *object*, and a *symbol* that represents either the process or the object" [Gray & Tall, 1994, p. 121. authors' emphasis].

Having identified the process as the ε–δ discourse framed by the propositional calculus, the object became the referent produced by the discourse. Thus the didactical strategy of rupture aimed at attaining the limit *procept* from the side of the *concept definition*. However, one point was missing. In order to complete the construction of the elementary procept a *symbol* was necessary. The experience was that the old symbol $\lim a_n = L$ inevitably drew the students' attention towards the concept image: for them, "lim" was the signifier attached to the idea of tendency, "lim" was *the name* of the concept image. It was necessary to adopt a *name for the concept definition*. A neutral signifier was chosen to play a temporary role: XPTO. So a definition was made and an exercise was proposed:

"a_n XPTO L means $\forall\varepsilon\ \exists N\ \forall n\left(n>N \rightarrow |a_n - L| < \varepsilon\right)$. Show that $\frac{1}{n}$ XPTO 0"

It is necessary to stress that XPTO is not a new symbol for the limit; it is a new symbol for the definition, a name for the definition, not a name for the limit. It is a temporary signifier to be used, not while the concept image is not well established, but while the concept definition is not strong enough to rule the concept image. The effects of the brute force declaration of traditional analysis courses: "from now on "$\lim_{n\to\infty} a_n = L$ means this epsilontic definition", that is, the old name now also means something else, have been negative on students. Of course, this is the desired form of the final ambiguity, but it cannot be attained by overt imposition[4].

"*This has nothing to do with getting closer*", explained the teacher. "*That N that you have found was just a sketch. The proof starts now*". He meant that the concept image had to be fully controlled and redressed in terms of the concept definition. An adaptation of Rosser [1953] allowed to take full advantage of the propositional calculus without losing

[4] Because in Lacan's terms, it implies a reorganization of the subject's *jouissnce*.

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sight of the mathematical meaning of the propositions. The four inference rules were made explicit and connected to language models such as the classical syllogism. The students were required to shape every homework exercise of their analysis course into this final form. All proofs had first to be "sketched" and then "written down". Image and definition were connected but each domain had its independent validity criteria. What had to be proved was put as a question and surrounded by question-marks. This allowed the proof to proceed simultaneously, progressing from the hypothesis and regressing from the thesis, allowing a step-by-step control of what remained to be proved. Concept image was evoked precisely at the moment of exhibiting a constant to answer a question introduced by the existential quantifier. Once the last question had been answered the proof was complete. There was no need to rewrite it in affirmative terms. This strategy will be exemplified below, as it was presented by the students in a poster session of a work - shop in May.

	Example of the XPTO strategy	
	Convention: ε is a positive real variable, n e N are positive integer variables. Bars over letters introduce new variables, maintaining their respective restrictions.[i]	
1	Hypothesis: $\begin{cases} a_n \ \text{XPTO} \ 0 \\ \forall n \ \lvert b_n \rvert \leq K \end{cases}$	
2	Thesis: $a_n b_n$ XPTO 0	
3	**Proof:**	**Clarifying the reader**
4	? $a_n b_n$ XPTO 0?	From the definition we have to show that:
5	? $\forall\varepsilon \ \exists N \ \forall n > N \ \lvert a_n b_n \rvert < \varepsilon$?	Once this question is answered, the proof is finished.
6	Take any $\bar{\varepsilon}$? $\exists N \ \forall n > N \ \lvert a_n b_n \rvert < \bar{\varepsilon}$?	In order to show that $\forall\varepsilon \ P(\varepsilon)$ it suffices to take an arbitrary $\bar{\varepsilon}$ and show that $P(\bar{\varepsilon})$. This rule is called *existential generalization.*
7	$\forall\varepsilon \ \exists N \ \forall n > N \ \lvert a_n \rvert < \varepsilon$	From the hypothesis, by definition.
8	$\exists N \ \forall n > N \ \lvert a_n \rvert < \dfrac{\bar{\varepsilon}}{K}$	Since $\forall\varepsilon \ P(\varepsilon)$ holds and since $\frac{\bar{\varepsilon}}{K} > 0$, in particular $P(\frac{\bar{\varepsilon}}{K})$ also holds. This rule is called *universal particularization*, it is the form of the classical syllogism every man is mortal, Socrates is a man, hence Socrates is mortal.
9	Let $\overline{N}$ be such that $\forall n > \overline{N}, \ \lvert a_n \rvert < \dfrac{\bar{\varepsilon}}{K}$	Since $\exists N \ P(N)$ holds, we can count on a particular $\overline{N}$ such that $P(\overline{N})$. This rule is called *existential particularization.*
10	? $\forall n > \overline{N} \ \lvert a_n b_n \rvert < \bar{\varepsilon}$?	Since, for such $\overline{N}$ we have $P(\overline{N})$, we may conclude that $\exists N \ P(N)$, answering the last question. This rule is called *existential generalization.*
11	Take any $\bar{n} > \overline{N}$? $\lvert a_{\bar{n}} b_{\bar{n}} \rvert < \bar{\varepsilon}$?	By universal generalization, it suffices to answer the question for this $\bar{n}$.
12	$\lvert a_{\bar{n}} b_{\bar{n}} \rvert = \lvert a_{\bar{n}} \rvert \lvert b_{\bar{n}} \rvert < \dfrac{\bar{\varepsilon}}{K} K = \bar{\varepsilon}$	From (9) and from the hypothesis, by universal particularization.

Results and discussion

The first question that should be asked is the following: did it work? The undergraduate students passed their analysis course, but this is not a reliable parameter; many who apparently ranked bellow them also passed. However the honors student made a mathematics-style exposition to another teacher in the mathematics department about the completeness of the real numbers defined in terms of equivalence classes of Cauchy sequences, which is a fairly involved ε–δ subject. "*She was self-confident on that epsilontic stuff*", he reported. On another occasion the students reported: "*Now we know in which formula to enter with $\varepsilon/3$ and where to pick the δ from. When the teacher does it, we can follow her, but when she doesn't we can't avoid filling in the gaps.*" When the students were writing the final mathematical report to the honors program they reported: "*We had trouble in refraining ourselves from applying the inference rules at every instance of the resumes of previous results that did not form part of the main body of the paper, otherwise we would never end it.*" From such reports, it seem that they are playing with the ε–δ discourse as a new toy. They still cannot take it for granted and move on, but the encapsulation of the ε–δ discourse seems at its final phase. They only have to say "this is trivial", as mathematicians do.

This is the final stage of a long process. The teacher led the students to complete some formal proofs of exercises that they had done in the analysis courses. They immediately recognized the power of the method and tried to imitate it. However, at the beginning the students tried to use the inference rules prematurely, before the sketch had sufficiently been worked. In the meetings, several times it happened that at the very end of the formal proof the students lost sight of the sketch, and the whole story had to bee retaken. Some sessions lasted for more than three hours. At a certain moment, in June, the teacher requested: "*Forget about the formal proofs for the next three weeks and concentrate on the sketches*". At that moment it was not clear that the strategy would work.

Of course, it can be argued that if the same time and effort had been dedicated to the classical continuity strategy, the same result would have been attained. However the story of this case shows that such an strategy had failed before and it would have been difficult for the students to find *affective energy* to engage in it. On the other hand the XPTO worked not only as a symbol for the ε–δ definition but also as a brand for the group. When the students first showed the strategy in a poster session of a work-shop intended for students and faculty, despite their efforts in contrary, some faculty members received the XPTO as an unnecessary new symbol for the limit. A concealed similar point of view was also expressed by some of their colleagues. This made them angry. They believed in what they were doing and they wanted to show it to people. They felt as the pioneers of the new strategy, not as the underdogs of the old one. This was the *affective energy* that drove them along the year.

The students evaluated the attitude of such faculty members. "*They looked irritated at the XPTO. It seems they do not want to take into account that students my have difficulties in analysis*" one of them said. Later in the year a video of one of the sessions was shown to the teacher of the analysis course. Her first reaction was: "*But this cannot be done in a regular classroom*". The students connected this episode with the first and concluded: "*If our strategy works, they seem to feel obliged to use to it. This is a threaten to their old habits*".

Actually up to the end of October the encapsulation of the inference rules into a single object had not occurred. The existential particularization had simply been abandoned in several proofs. The connection of the rules with everyday language situations had been lost. The concept image was getting loose and recovering control over the concept definition. At this moment the teacher calmly reminded the students: "*Next month you are going to expose this to the faculty. They will certainly ask you about the apologetic poster session of last May when you claimed that these rules were so important. What ere you going to answer?*" He suggested: "*Perhaps you should tell them that our strategy did not work and make a traditional mathematical exposition as they like you to do*".

This remark had a decisive effect. The students started scheduling appointments among themselves in order to prepare for the exposition. The fact that they could not trust the book but, on the contrary, had to find a mistake in it, made them to become independent from the teacher. They assumed that the fight for understanding and making themselves understood was theirs. The *demand* produced by this kind of situation is well known to everyone who has learned mathematics. So it can certainly be argued that all that the XPTO strategy did, was to install a certain pressure. We agree. But, was there any other way to do it?

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[i] **Theorem VI.4.2.** If $P_1, P_2, \dots, P_n, Q$, are statements, not necessarily distinct, and x is a variable which has no free occurrences in any of $P_1, P_2, \dots, P_n$, and if $P_1, P_2, \dots, P_n, \vdash Q$, then $P_2, \dots, P_n, \vdash (x)\,Q$ [Rosser, 1953, p. 106].**Theorem VI.6.8.** Let x and y be variables and P and Q be statements. Let Q be the result of replacing all free occurrences of x in P by occurrences of y and P be the result of replacing all free occurrences of y in Q by occurrences of x. Then: $\vdash (x)\,F(x) \equiv (y)\,F(y)$ [Rosser, 1953, p. 121].

Cognitive Units, Connections and Mathematical Proof

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Mathematical proof seems attractive to some, yet impenetrable to others. In this paper a theory is suggested involving "cognitive units" which can be the conscious focus of attention at a given time and connections in the individual's cognitive structure that allow deductive proof to be formulated. Whilst elementary mathematics often involves sequential algorithms where each step cues the next, proof also requires a selection and synthesis of alternative paths to make deductions. The theory is illustrated by considering the standard proof of the irrationality of $\sqrt{2}$ and its generalisation to the irrationality of $\sqrt{3}$.

Cognitive units and connections

The logic of proof is handled by the biological structure of the human brain. As a multi-processing system, complex decision-making is reduced to manageable levels by suppressing inessential detail and focusing attention on important information. A piece of cognitive structure that can be held in the focus of attention all at one time will be called a ***cognitive unit***. This might be a symbol, a specific fact such as "3+4 is 7", a general fact such as "the sum of two even numbers is even", a relationship, a step in an argument, a theorem such as "a continuous function on a closed interval is bounded and attains its bounds", and so on. It should be noted that what is a cognitive unit for one individual may not be a cognitive unit for another. The ability to conceive and manipulate cognitive units is a vital facility for mathematical thinking. We hypothesise that two complementary factors are important in building a powerful thinking structure:

1) the ability to *compress* information to fit into cognitive units,
2) the ability to *make connections* between cognitive units so that relevant information can be pulled in and out of the focus of attention at will.

Compression is performed in various ways, including the use of words and symbols as tokens for complex ideas ("signifiers" for something "signified"). These may sometimes be "chunked" by grouping into sub-units using internal connections. A more powerful method in mathematics uses symbols such as 2+3 as a pivot to cue either a mental process (in this case addition) or a concept (the sum). This has become a seminal construct in process-object theories (Dubinsky, 1991; Sfard, 1991). The combination of *pro*cess and con*cept* which can be evoked by the same symbol is called a *procept* (Gray & Tall, 1994). However, the notion of procept is not the only instance of compression in mathematics:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston 1990, p. 847)

The connections link cognitive units in the focus of attention to other cognitive structures which will, as a whole, be termed the *intermediate working memory*. As different items are brought into the focus of attention, the intermediate working memory changes dynamically, opening up new connections and shutting off others. As a consequence, different external prompts may lead to the making of different connections.

Dynamic sequences of links are routinised as action schemas and performed in the background, taking up little focus of attention. This generates procedural ability to carry out familiar processes. The greater power of flexible thinking arises from using the links in a collection of connected cognitive units—processes, sentences, objects, properties, sequences of logical deduction, etc—to conceive it as a single entity that can be both manipulated as a concept and unpacked as a schema. This idea has been formulated many times in different ways (eg the "varifocal theory" of Skemp (1979) in which a concept may be unpacked as a schema and a schema viewed as a concept, or the encapsulation of a schema as an object (Cotrill *et al*; in press)). More than just saving mental space as a shorthand in place of a collection of items, it carries with it, just beneath the surface, the structure of the collection and is *operative* in the sense that the live connections within the structure are able to guide the manipulation of the compressed entity. These may then become new units in new cognitive structures, building a hierarchical network spanning several layers. Used successfully, this offers a manageable level of complexity in which the thought processes can concentrate on a small number of powerful cognitive units at a time, yet link them or unpack them in supportive ways whenever necessary.

Mathematical proof introduces a form of linkage different from the familiar routines of elementary arithmetic and algebra. In addition to carrying out *sequential* procedures in which each mathematical action cues the next, mathematical proof often requires the synthesis of several cognitive links to derive a new *synthetic* connection. In the proof of the irrationality of $\sqrt{2}$, for instance, having written $\sqrt{2}=(a/b)$ as a fraction in lowest terms, the step from "$\sqrt{2}=(a/b)$" to "$a^2= 2b^2$" is a sequence of algebraic operations, but the step from this to "a is even" requires a synthesis of other cognitive units, for instance "a is either even or odd" and "if a were odd, then a^2 would be odd." We hypothesise that synthetic links constitute an essential difference between procedural manipulations in arithmetic and algebra and the more sophisticated thinking processes in mathematical proof.

Data collection

To investigate the role of synthetic links in proof, clinical interviews were used focusing on the proof of the irrationality of $\sqrt{2}$ and $\sqrt{3}$. Eighteen students were selected at three different stages in the mathematics curriculum: 15/16 year olds in a mixed comprehensive school taking mathematics GCSE, 16/17 year olds in the sixth form of a boys' independent school taking A-level mathematics, and first year university mathematics students. It would be unlikely that a student would be able to produce a proof of the irrationality of $\sqrt{2}$ without prior experience, so each was invited to participate in a two-person dialogue, attempting to make sense of a proof presented as a sequence of steps. At each stage he or she was asked to explain the given step and perhaps suggest a strategy for moving on:

(i) *Suppose $\surd 2$ is not irrational.*
(ii) *Then $\surd 2$ is of the form a/b, where a, b are whole numbers with no common factors.*
(iii) *This implies that $a^2 = 2b^2$,*
(iv) *and hence that a^2 is even.*
(v) *Therefore a is even.*
(vi) *Thus $a = 2c$, for some integer c,*
(vii) *It follows that $b^2 = 2c^2$,*
(viii) *giving that b^2,*
(ix) *and hence also b, is even.*
(x) *The conclusion that a and b are both even contradicts the initial assumption that a and b have no common factors.*
(xi) *Therefore $\surd 2$ is irrational.*

After this each student was asked to suggest a proof for the irrationality of $\surd 3$.

Analysis of responses

(i) the notion of proof by contradiction

Before being shown the proof, the idea of supposing that $\surd 2$ was *not* irrational and looking for a contradiction was not suggested by *any* students who had not met the proof before. At this stage they are used to manipulating symbols through sequential action schemas to produce a "solution". They are unfamiliar with the possibility of proving something true by initially supposing it to be false—a conflict likely to provoke cognitive tension and insecurity.

(ii) translation from verbal to algebraic

Students with no previous experience of the proof found the idea of writing a fraction in its lowest terms a familiar concept, but the idea of writing this in the algebraic form "$\surd 2=a/b$" proved less obvious, but acceptable.

(iii) a routinised algebraic manipulation

Having agreed to suppose that $\surd 2$ is equal to the fraction a/b, where a and b are whole numbers, students were usually successful in showing that this implies $a^2=2b^2$ using routine algebraic manipulation. However, some students who had seen the proof before and resorted to attempting to memorise it did not always handle the algebra securely. For instance, university student S began by stating the general strategy for the proof by contradiction, yet could not deal with many details. Instead of constructing the proof himself, he recalled that the lecturer "did some fancy algebra which I couldn't actually reproduce." When asked to do so, he wrote "$\left(\frac{a}{b}\right)^2 = 2$", followed by "$a^2=4b^2$", saying, "I think that's what he did, but he did it in one step whereas normally I would've taken two." When asked to fill in the details, he obtained the correct result $a^2=2b^2$. Similarly, student M said, "I remember him saying to prove that a is even", but could not remember how. In contrast, Student L compressed the whole operation in a single step, but was able to give further details on request.

(iv) a link from algebra to verbal representation

None of the students new to the proof spontaneously linked "$a^2=2b^2$" to "$a^2$is even", although they all readily accepted its truth. (The link loses information, saying "a^2 is twice a whole number" rather than "a^2 is twice *the square of* a whole number". Students may feel instinctively uneasy losing information, without articulating their concern.)

(v) Synthesising a non-procedural step

The step from "a^2 is even" to "a is even" requires a more subtle synthesis of links with other cognitive units. Students offered a number of different strategies, including:

(a) *Correct justification,* involving a sequence of appropriate connections, usually along the lines "a is either even or odd", but "a odd implies a^2 is odd", and as "a^2 is *not* odd", this implies "a must be even."

(b) *Strong conviction but without justification,* such as, "an even number square has got to have a square root that is even" and "well, it just sort of *is* [even]."

(c) *Empirical verification,* trying some numeric cases and asserting that there are no exceptions.

(d) *Inconclusive reasoning,* offering related statements, justified or otherwise, which did not help further the argument, such as, "If you could say that a^2 had a factor of 4, then that [a even] would definitely be true."

(e) *False reasoning,* using inappropriate links, such as the claim which occurred more than once that if a^2 is an integer multiple of 2, then a is an integer multiple of $\sqrt{2}$.

(f) *Unable to respond without help.*

The correct justification was not evoked initially by most students new to the proof or by some of those who sought to remember the proof by rote. The cognitive units "a is even" and "a^2 is even" can coexist in the focus of attention so they may be seen as happening at the same time rather than one implying the other. "a^2 is even" seems to have a stronger natural link to "a is even" than to "a is odd", thus failing to evoke the alternative hypothesis.

Some students responded in several categories. For instance, Student S began with response (b) quoting the authority of the lecturer, saying, "the root of an even number is even—he just assumed it." When challenged, he reasoned inconclusively, then tried specific cases:

Interviewer:	So the root of six is even.
Student S:	Good point. [five seconds pause]
Interviewer:	If a number is not even, what is it?
Student S:	It's odd.
Interviewer:	So you've got a choice of odd or even, does that help you?
Student S:	Yeh, I see, it's got to be rational, I think, so … a rational root is either … odd or even and if the square is even, then the rational root is even. Is that clear?
Interviewer:	Uh, well …
Student S:	So what I'm thinking is the root of 4, 4's even and 2's even, root of 16 equals 4, … 's even. I can't remember any other simple squares in my head that are even …

Students who could not proceed (category (f)) were given a prompt referring to the odd-even dichotomy. This often led to a response of type (a), (d) or (e) above. For example, the prompt "Every integer is even or odd" was often followed by the response, "An odd number squared is odd". The thought of considering concrete examples (category (c)) was rarely evoked by this cue.

(vi) From "a is even" to "a=2c for a whole number c"

The translation from the verbal statement "a is even" to the algebraic statement "$a=2c$" was usually straightforward, but again students such as university student M—who admitted trying to memorise proofs—had a faulty recollection of what to do:

Interviewer: If you know that a is even, how can you write a? How do you write down that a is an even number ?

Student M: If you put a 2, ... you put an a in front of it, like $2a$ ... I don't know, I'm sorry. I can't remember.

(vii)–(ix) The chance to repeat earlier arguments

Having concluded that "$a=2c$ for a whole number c", the next steps of the proof often evoked earlier ideas. No student had any difficulty with the procedural steps substituting "$a=2c$" into "$a^2=2b^2$" and simplifying "$4c^2 = 2b^2$" to get $b^2= 2c^2$. Students invariably saw that this situation was similar to the earlier case for a, and asserted that b is also even.

(x)–(xi) establishing the contradiction

Some students new to the proof did not recall that a/b was assumed in lowest terms, so did not see that "a and b both even" gives a contradiction. Student C was silent for 45 seconds until reminded: "we cancelled out until we had no common factors," immediately replying:

> "Oh, right, ... that can't be the case because if they are both even numbers, then they will have common factors, like two."

Those who had seen the proof before in school or at university immediately grasped the contradiction, including those who had misremembered the detail of earlier steps.

Generalising the proof to the irrationality of √3

When proving the irrationality of √3, all students began by supposing that √3 was equal to a fraction a/b in its lowest terms, a typical remark being, "I presume you start in the same way." On translating this to $a^2=3b^2$, *all* of them evoked the link with a being "even or odd" and were unable to proceed further. (Just one student wondered whether the "eveness" might relate to the 2 under the square root sign.) A suggestion that "$a^2=3b^2$" tells something different from " evenness or oddness of a" usually evoked divisibility by 3, but then *none* of the students could show unaided that "a^2 is divisible by 3" implies "a is divisible by 3". In particular none considered the algebraic argument squaring the three cases $a=3n$, $3n+1$ or $3n+2$ (a synthetic connection requiring coordination of three different possibilities).

A further suggestion focusing on factorisation into primes was sufficient to help all the university students and some sixth formers to produce suitable arguments although often expressed in an idiosyncratic manner. Student T, for instance, said:

" ... the (square) root of a^2, I mean a, that doesn't involve the factor 3. Therefore you've still got a factor 3 which you can divide into a."

She seems to be saying that if 3 does not divide one of the a-factors of $a \times a$, then it must divide the other a.

Student J in the youngest group also imagined a^2 as a product of two a factors saying:

" that has got repeated factors of that, so you can't get [ten seconds pause] ... just imagining how many factors of things. ... They're going to have the same factors. So yes, 3 would have to divide a."

Discussion

Figure 1 is a representation of some of the typical linkages that may occur in an initial proof that √2 is irrational, omitting idiosyncratic links (which occur widely in individual cases). It is a collage of difficulties encountered by students where links denoted by $\rightsquigarrow$ often prove more difficult than those denoted by $\longrightarrow$ and those in grey scale are intermediate links which may or may not be evoked in detail.

Figure 2 displays a compressed proof structure available to many students who had experienced the proof before; this may be compressed further as an overall strategy in Figure 3. Even Student S, who remembered little detail and used loose terminology to describe his ideas was able to say,

"I'd take the case where I assumed it was a rational and fiddle around with the numbers, squaring, and try to show that ... if it was rational then you'd get the two ratios a and b both being even so they could be subdivided further, which we'd assumed earlier on couldn't be true so our assumption it was rational can't be true."

A number of themes arose highlighting difficulties experienced by this sample of students:

(a) The overall notion of proof by contradiction (which becomes less problematic with familiarity).

(b) Translation between familiar terms "odd and even" and algebraic representations are acceptable, but not always initially evoked.

(c) The step "a^2 even implies a even" is initially not easy to synthesise and remains so for those in the sample who attempted to remember the proof by rote. For some the cognitive units "a^2 even" and "a even" coexist and the direction of implication is not relevant; for others the idea "a^2 even" is more strongly linked to "a even" rather than to the operative alternative "a odd".

(d) In contrast to other difficulties, most students readily evoked the recent argument for "a is even" to assert directly that "b is even".

(e) The assumption "a/b is in lowest terms" was not always recalled by students new to the proof, but became part of the long-term global strategy.

(f) The link in the proof of "√2 irrational" to the colloquial terms "even-odd" was more powerful than the link to "divisible or not by 2", thus blocking a natural extension to the corresponding proof for irrationality of √3.

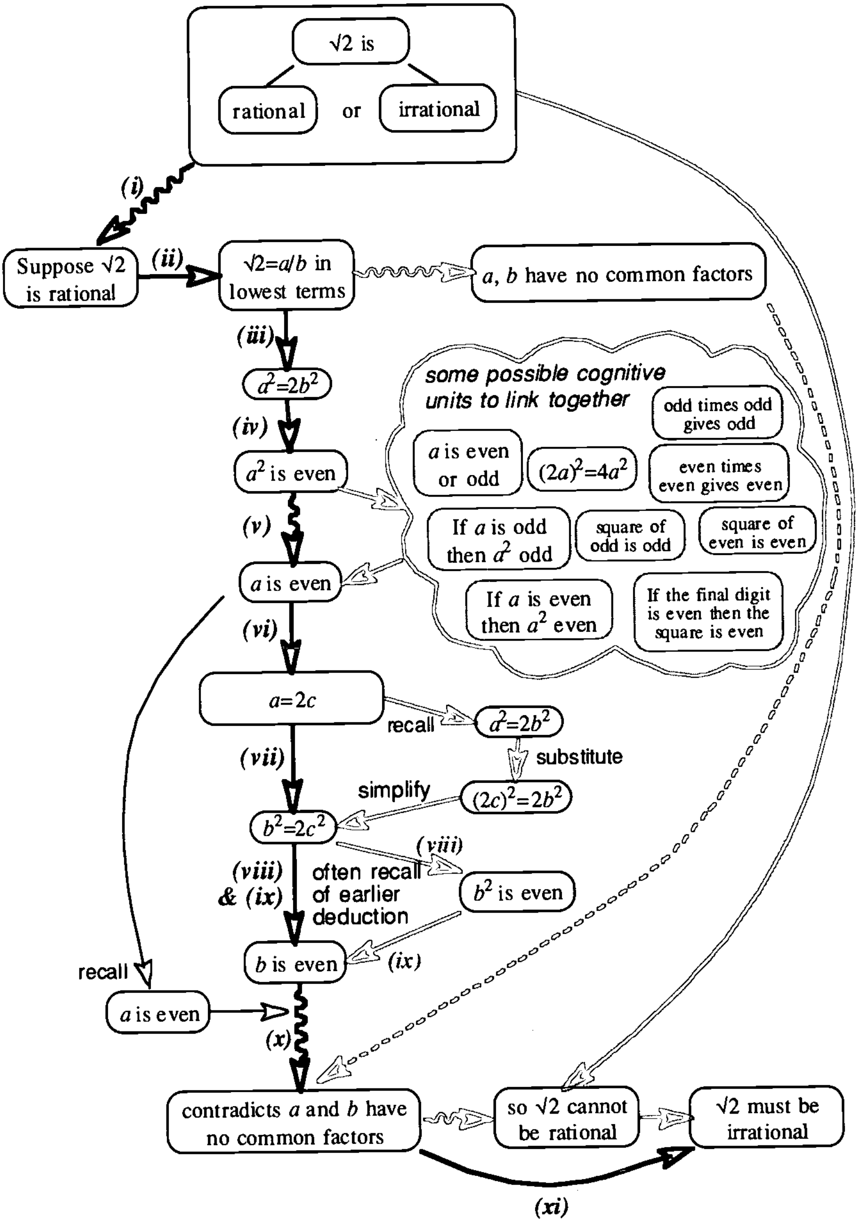

Figure 1: Observed cognitive units and connections in an initial proof of √2 irrational

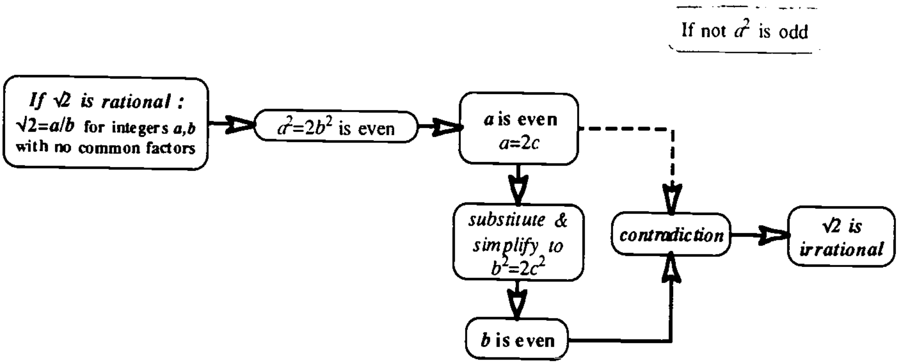

Figure 2: a compressed proof that √2 is irrational by deriving a contradiction

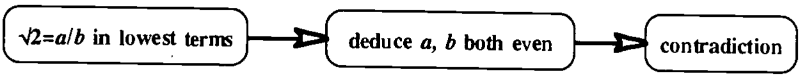

Figure 3: a compressed strategy for the proof

Summarising the broad development of the proof of the irrationality of √2 and √3, we see that there are several initial difficulties that make it a formidable challenge for the uninitiated. Some become less problematic with familiarity, but there is sufficient difficulty to cause a bifurcation in understanding. Some students make meaningful links that allow them to compress the information into richly connected cognitive units. Others remember some of the ideas they were told—even the overall strategy of the proof—yet may rely on the authority of their teacher rather than building their own meaningful links which might help reconstruct the subtle detail.

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SUBJECTIVE ELEMENTS IN CHILDREN'S COMPARISON OF PROBABILITIES

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SUMMARY

In this research work we study the comparison of probabilities by 10-14 year-old pupils. We consider the different levels described in research about these tasks, though we incorporate subjective distractors, which change the predicted difficulty of some items. Analysis of students' arguments serves to determine their strategies, amongst which we identify the "equiprobability bias" and the "outcome approach". Analysis of response patterns by the same pupil serves to show that the coincidence between the difficulty level of probabilistic and proportional tasks is not complete and points to the existence of different types of probabilistic reasoning for the same proportional reasoning level.

Over the last few years, new curricula for compulsory levels in different countries have introduced the study of randomness and probability at very early ages and suggest an active and exploratory teaching methodology. The success of this proposal depends, however, on the adequate choice of tasks, according to students' capacities. In particular, proportional reasoning should be taken into account, as it is essential for estimating and comparing probabilities.

Background

Research into children's capacity to compare two probabilities started with Piaget and Inhelder (1951), who investigated children's reasoning according to the different stages described in Piaget's theory. Their results indicate that children at level I only solve the cases of double impossibility, double certainty or certainty - impossibility; at level IB, problems depending on only one variable are solved; level IIA is characterized by the success in problems that can be solved through additive comparisons; level IIB is characterized by a progressive empirical solution of proportionality problems and, finally at stage III, a general solution is found.

Following Piaget and Inhelder, other researchers, such as Yost et al. (1962), Goldberg (1966), Davies (1965), Hoeman and Ross (1971), Falk et al. (1980), and most recently Truran (1994) have undertaken the study of childrens' abilities to compare probabilities. The work by Fischbein et al. (1970s), who compared the reasoning of groups of children with and without specific instruction, has particular interest for education.

Since comparing probabilities entails the comparison of two fractions, the work by Piaget in the field of probability created a great deal of interest in

proportional reasoning (e.g.; Karplus et al., 1983; Behr et al., 1992). Noelting (1980 a and b) extended the categories of proportional comparison problems considered by Piaget and Inhelder (1951) and determined different levels in these problems and in the associated strategies, according to Piaget's development stages.

DESCRIPTION OF THE RESEARCH

An important difference between comparing fractions and comparing probabilities is that the result of a proportional problem refers to a certain event, while the result of a probability problem implies a degree of uncertainty. On the other hand, the subjects sometimes consider subjective elements to assign probabilities.

In this work we continue our previous study of the influence of these subjective elements (Godino et al., 1994), analyzing childrens' strategies when comparing probabilities in tasks that contain these elements and their difficulty level, as compared to problems without subjective distractors. To achieve this aim, we applied a written questionnaire (complemented by individual interviews with some pupils) to a sample of 144 pupils from 10 to 14 years of age, during the course 1995-96. Below we describe the questionnaire and the results obtained.

Questionnaire

The questionnaire was composed of 8 items. The statement of item 1, 2, 3, 6 and 7, taken from Green (1983) is similar to the following item 1, varying the composition of the urns and the order of distractors:

Item 1.- Two boxes have in them some white balls and some black balls. You must pick a black ball to win a prize. The boxes are shaken up and you cannot see inside.
Box A has 3 black balls and 1 white ball; Box B has 2 black balls and 1 white ball.
Which box gives a better chance of picking a black ball?
(A) Box A ....___
(B) Box B..___
(C) Same chance..................................___
(D) Don't know...................................___
¿Why? ....

Item 4. Gilla is 10 years old. In her box, there are 40 white marbles and 20 black ones. Ronit is 8 years old. In her box there are 30 white marbles and 15 black ones. Each of them draws one marble from her own box, without looking. Ronit claims that Gilla has a greater chance of extracting a white marble because she is the older one, and therefore she is the cleverest of both of them. What is your opinion about this?

Item 5. Uri has, in his box, 10 white marbles and 20 black ones. Guy has in his box 30 white marbles and 60 black ones. They play a game of chance. The winner is the child who pulls out a white marble first. If both take out simultaneously a white marble no one is the winner and the game has to go on. Uri claims that the game is not fair because in Guy's box there are more white marbles than in his box. What is your opinion about this?

In addition, we use items 4 and 5, taken from Fischbein and Gazit (1984). In these two items subjective elements were introduced. In item 4 we used a causal factor (the age of the child that takes out the ball may affect the result) to study the belief of some children in the possibility of controling random phenomena (Fischbein, et al., 1991). In item 5, the belief that, in spite of having equal proportions of possible and favourable cases, the absolute number of favourable cases represents an advantage was introduced.

Since the problem implies the comparison of fractions, the different difficulty levels identified by Noelting (1980 a) and b) were used, as is indicated in Table.1, where the average age found by Noelting to reach this proportional level is also shown.

Table 1.- Classification of items according to Noelting's levels

Item	Fractions	Level (Noelting)	Other	Average age
1	(3,1); (2,1)	IA comparison of 1st term		3.6
2	(5,2); (5,3)	IB; comparison of 2nd term		6.4
3	(2,2); (4,4)	IIA; unit equivalence class		8.1
4	(40,20); (30,15)	IIB; any equivalence class	Subjective factors	10.5
5	(10,20); (30,60)	IIB;any equivalence class and two proportional terms	Subjetive factors	10.5
6	(12,4); (20,10)	IIIA; integer ratio in the fraction terms		12.2
7	(7,5); (5,3)	IIIB; any fraction		15.10

RESULTS AND DISCUSSION

In Table 2 we present the percentage of correct solutions, according to age and mathematical ability, which was measured by pupils' average score in the previous academic year. We also include the percentage of correct answers in the total sample (Total) and in the sample of pupils who gave a consistent and complete explanation of their strategy in the problem (Total corrected). The results show that comparing probabilities is not easy, not even for the older pupils, who are able to operate with fractions.The percentage of correct responses in items 4 and 6 (subjective distractors) that belong to category C2 in Fischbein et al. (1970) research is lower than that found by these authors in children of the same age without instruction in this type of problem. However, as these authors used a different experimental task, this point need further research.

We also point out to the inversion in the order of difficulty predicted by Noelting's classification in item 4, where we introduce subjective distractors and in items 2 and 3, probably because in a random situation, the attention is centered on the favorable cases, more than on the unfavorables ones. As a rule, there is an

improvement with age and general reasoning level, though not in every grade or in every item.

Table 2 : Percentage of correct responses in the items

Item	Age of pupils 10-11 (n=36)	11-12 (n=37)	12-13 (n=38)	13-14 (n=32)	Mathematics score Low (n=43)	Middle (n=58)	High (n=42)	Total (n=143)	Total corrected
1	75.0	70.3	86.8	87.5	81.4	67.2	95.2	79.7	80.8
2	52.8	67.6	65.8	56.2	62.8	56.9	64.3	60.8	62.6
3	47.2	54.1	81.6	73.6	65.1	60.3	78.6	63.6	66.9
4	6.0	27.0	23.6	23.8	9.5	22.4	28.5	20.0	25.7
5	13.9	32.4	39.5	43.7	25.7	26.6	47.6	32.5	37.0
6	30.6	27.0	34.2	21.9	27.9	25.9	33.3	28.7	30.4
7	19.4	5.41	5.3	6.2	11.6	6.9	9.5	9.1	9.8

STUDENTS' STRATEGIES

We analyzed the arguments provided by the pupils to justify their response, which were classified according to the strategies described below.

A) Single variable strategies: Comparing the number of possible cases; comparing the number of favorable cases and comparing the number of unfavorable cases.

B) Two variables strategies: Additive strategies, correspondence and multiplicative strategies.

These strategies were taken from Noelting's (1980b) classification, though these and the following type C strategies have also been described by other researchers in the field of probability (e.g., Fischbein et al., 1970; Green, 1983 and Truran, 1994).

C) Other types: Based on luck, using either "equiprobability bias" (Lecoutre, 1992) or "outcome approach" (Konold. 1989); taking the decision depending on the arrangement of marbles or other irrelevant aspects in the task.

Table 3: Percentage of different strategies in the items

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Item 7
Posible cases	4.9	1.4	5.6	1.4	0.7	7.0	7.7
Favorable cases	51.7	29.4	16.8	25.0	42.7	27.3	13.3
Unfavorable cases	2.8	34.3	1.4	0.7	4.9	11.2	2.1
Additive	7.7	4.9	15.4	2.9	4.2	21.0	39.9
Correspondence	13.3	11.9	36.4	15.0	26.6	6.3	1.4
Multiplicative	1.4	0.0	0.7	0.0	0.0	0.7	1.4
Luck	11.2	4.2	4.2	27.1	4.9	8.4	7.7
Other	5.6	11.2	14.7	5.0	4.2	12.6	18.9
No answer or inconsistent	1.4	2.8	4.9	22.1	12.1	5.6	7.7

Though, in general, the strategy of comparing favorable cases prevails, we can see in Table 3 that the pupils changed their strategy according to the level of difficulty of the problems. In the simplest problems they use single variable strategies, resorting to additive or correspondence strategies in more complex problems.

RESPONSE PATTERNS

In Table 4, we present the patterns of answers to the different items. We have ordered the items according to their difficulty (percentage of success) and each pupil's pattern is represented by a vector with 7 components. For example, if a pupil has the pattern 0100010 he has failed all the items, except 3 (second place in difficulty) and 4 (sixth place).

Table 4: Response patterns in comparing probabilities

Response pattern Item: 1	3	2	5	6	4	7	N.correct	Frequency
1	1	1	1	1	1	1	7	2
1	1	1	1	1	1	0	6	5
1	1	1	1	0	1	0	5	6
1	1	1	1	1	0	0	5	7
Other pattern							5	3
1	1	1	0	1	0	0	4	13
1	1	1	1	0	0	0	4	14
Other pattern							4	14
1	1	1	0	0	0	0	3	14
Other pattern							3	13
1	1	0	0	0	0	0	2	5
Other pattern							2	10
1	0	0	0	0	0	0	1	19
Other pattern							1	10
0	0	0	0	0	0	0	0	9

This representation is used in Gutman's scalogram to evaluate whether the different items in a test can be described by a linear scale. Then it is assumed that a pupils with *n* correct asnwers in the test. would have more probability of succeeding in the *n* easiest items. This does not happen in our test, where only 66 cases of 144 students follow the pattern assumed by the Gutman's scalogram.We believe that this result also confirms the difference between proportional and probabilistic reasoning, since the items used by Noelting, with the same difficulty level as that used in our research, followed Gutman's pattern. Below we comment on the pattern found.

Pupils with 7 correct answers (Level IIIB): only 2 students who consistently used multiplicative strategies (level IIIB) and were not affected by the distractors in items 4 and 5.

Pupils with 6 correct answers (Levels IIIA and IIB): All of them failed item 7, though only 2, who would be classified at level IIIA used a relevant strategy to solve item 6. The others used the difference between possible and favorable cases and would be included at level IIB.

Pupils with 5 correct answers (Level IIIA and IIB): There are two main patterns: a) Pupils who failed items 6 and 7 (6 cases), generally employing additive strategies to solve these two problems, though in items 2 and 3 some of them employed correspondence (level IIB); b) Pupils who failed either item 4 or 5 (level IIB), with subjective distractors) and correctly solved item 6 (level IIIA) (9 cases).

Pupils with 4 correct answers (Levels IIIA, IIB, IIA and IB). a) Pupils succeeding in the four easiest items (15 cases). These pupils begin to solve the simplest problems in level IIB. Item 4 was failed because their reasoning followed the "outcome approach"; b) Pupils succeeding in problems 1, 2 and 3, failing problems 4 and 5 (level IIB) and solving item 6 correctly (level IIIA; 13 pupils). Failure in items 4 and 5 was due to giving a greater probability to the urn with the greater number of favorable cases, in spite of having the same proportion. c) Other non systematic patterns (15 cases). Success in difficult items was due to strategies valid for this problem, but not for the general case. Failure in easy items was due to choosing a strategy not adapted to the problem. The level of proportional reasoning amongst these children varied between IB and IIA .

Pupils with 3 correct answers (Levels IIA, IB, IA) a) Pupils who correctly answered the first three items (13 cases;level IIA); b) Pupils who correctly answered items 1 and 2 and failed item 3, through not applying the correspondence strategy (7 cases; level IB). Their success in another problem was due to a mistaken strategy that was productive for that particular problem; c) The rest of the cases (5) had no systematic pattern. All of them solved problem 1 correctly (level IA).

Pupils with 2 correct answers. (Level IA, IB, IIA): a) correct answers to items 1 and 3 (5 cases). Generally item 3 was solved with an additive strategy, that, though wrong, provided a correct answer to this problem. They were not able to compare unfavorable cases in item 2, and therefore gave a wrong solution to this problem (level IA). b) Pupils with a correct solution to item 2, through additive comparison, who would be located in the level IB. c) One pupil correctly solved 2 and 3 using correpondence (IIA), in spite of having failed item 1, probably through not paying attention The remaining pupils (8) do not show identifiable

patterns, although some of them systematically reasoned according to the outcome approach.

Pupils with 1 correct response (IIA, IB, I or inferior). a) Pupils who only solved item 1, generally with additive strategies (level I; 19 cases). b) Pupils who solved item 2 (level IB), though failed item 1 through reasoning according to the outcome approach (level IB, 2 cases) c) The remainder (8 cases) only solved item 3 correctly. They systematically gave a response based on either the "outcome approach" or the "equiprobability bias" and the correct solution to item 3 was due to the data in this problem. These children had a very poor level of proportional reasoning, though some ideas about probability idea- even though incorrect- were observed.

Pupils who failed all the items (Absence of proportional reasoning).These pupils (9 cases) can be divided into two types: either they only compared favorable cases, or they systematically reasoned according to the "outcome approach" or the "equiprobability bias". They are the pupils in which proportional reasoning had not yet started to develop or who maintain incorrect belief about probability.

CONCLUSIONS

With the same level of proportional reasoning the success in comparing probabilities was very varied, with the exception of levels IIIB (7 responses) and IIIA (6 responses). Consequently, though these reasoning are related, there was no total coincidence. The lack of fit was due to the following causes:

a) Factors of the problem that induce the assignment of subjective probabilities, in the two items taken from Fischbein and Gazit (1984).

b) Reasoning according to either the outcome approach (Konold, 1989) or to equiprobability bias (Lecoutre, 1992).

c) Greater attention to favorable cases, even in problems that must be solved by comparing unfavorable cases, possibly reasoning according to the availability heuristics.

These three mechanisms are not relevant when comparing proportions, while they may arise in a probabilistic problem. Consequently, the teacher must consider these factors, in addition to proportional reasoning, when approaching the teaching of the probability to children. Proportional reasoning level was low, in general, in our sample. This might be an obstacle for learning probability, though, also the teaching of probability could well be a rich context for improving the development of proportional reasoning in these pupils.

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REUNITISING HUNDREDTHS: PROTOTYPIC AND NONPROTOTYPIC REPRESENTATIONS

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This paper reports on a study in which 29 Year 6 students (selected from the top 30% of 176 Year 6 students) were individually interviewed to explore their ability to reunitise hundredths as tenths (Behr, Harel, Post & Lesh, 1992) when represented by prototypic (PRO) and nonprototypic (NPRO) models. The results showed that 55.2% of the students were able to unitise both models and that reunitising was more successful with the PRO model. The interviews revealed that many of these students had incomplete, fragmented or non-existent structural knowledge of the reunitising process and often relied on syntactic clues to complete the tasks. The implication for teaching is that instruction should not be limited to PRO representations of the part/whole notion of fraction and that the basic structures (equal parts, link between name and number of equal parts) of the part/whole notion needs to be revisited often.

The notion of a unit underlies the decimal number system. However, Steffe (1986) has identified four different ways of thinking about a unit, namely, *counting* (or *singleton) units*, *composite units*, *unit-of-units* and *measure unit,* with each type apparently representing an increasing level of abstraction. When considering whole numbers, *singleton units, composite units* and *unit-of-units* need to be considered (see Figure 1) whereas with decimal fractions, the *measure unit* needs to be invoked (Behr, Harel, Post, & Lesh, 1992). (See Figure 2.) There is a consensus in the literature (Behr et al. 1992, Harel & Confrey, 1994; Hiebert & Behr, 1988, Lamon, 1996) that the cognitive complexity involved in connecting referents, symbols and operations can be attributed mainly to the changes in the nature of the unit.

Partitioning, unitising and *reunitising* are important to the development of rational number concepts but are often the source of young students' conceptual and perceptual difficulties in interpreting rational-number representations (Baturo, 1996; Behr et al, 1992; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983). In particular, reunitising, the ability to change one's perception of the unit, requires a flexibility of thinking that may be beyond young children. This has importance for hundredths which need to be thought of as a number of hundredths sometimes and as a number of tenths at other times. Similarly, tenths need to be thought of as a number of tenths or as a number of hundredths.

The cognitive complexity required to process the unit-of-units notion has major implications for acquiring an understanding of the decimal number system. For example, each place needs to be reunitised in terms of the unit/one for a complete understanding of the place-value relationships to be known. Figure 1

shows the ways in which 5 tens (represented by 5 base-10 blocks) can be unitised in terms of singleton and composite units and composite unit-of-units.

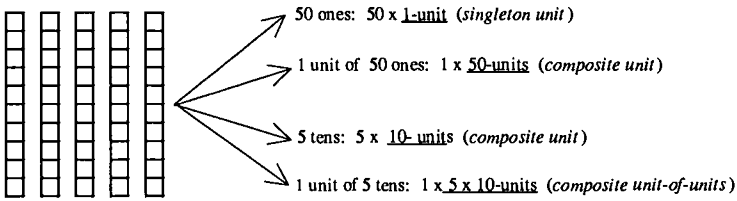


Figure 1. Various notions of a unit applied to tens and ones.

Figure 2 shows that similar thinking is required to process a number such as 0.20. However, the extra dimension of the *unit measure* needs to be invoked (Behr et al., 1992) to relate the part to the whole. To transform the units in the different ways and to keep track of these transformations with respect to the shaded parts requires a great deal of flexible thinking and would most likely place a strain on cognitive loading.

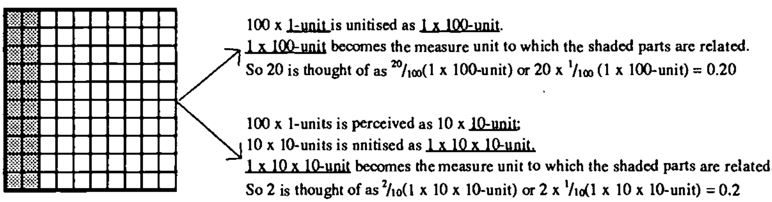


Figure 2. Units-of-units notion applied to tenths and hundredths.

When a whole is partitioned into tenths only, students need only unitise once (i.e., the 10 × 1-unit is unitised as 1 × 10-unit) and therefore there is only one measure unit to be invoked. Similarly, if hundredths only are to be considered. However, when hundredths need to be perceived as both tenths and hundredths, as they are for recording purposes and for renaming from one place to the other (equivalence), then the cognition required becomes much more complex.

THE STUDY

One hundred and seventy-six students from two schools (low-middle and middle-high socioeconomic backgrounds) were administered a diagnostic instrument that was developed to assess the students' understanding of the numeration processes (i.e., number identification, place value, regrouping, ordering, and estimating) related to tenths and hundredths. The students were classified in terms of their overall mean for the test and 29 students were selected from the top 30% for interviewing. This group of students comprised

12 high-performing students (HP – ≥ 90%), 11 medium-performing students (MP – 80-90%) and 8 low performing students (LP – 70-80%).

Semistructured individual interviews were undertaken and incorporated a set of tasks (presented in the same order) designed to probe the students' structural knowledge with respect to reunitisng hundredths for both PRO and NPRO area representations. Figure 3 shows the two tasks on which this paper reports. The full study was reported in Baturo (1996).

TASK 1 (prototypic) TASK 2 (nonprototypic)

Shade 0.6 of the shape below. Shade 0.2 of the shape below.

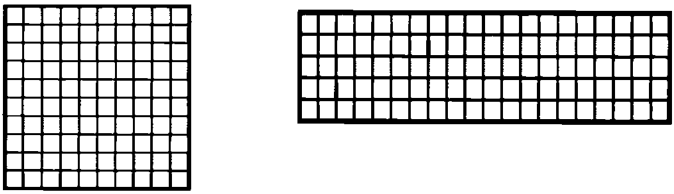


Figure 3. The reunitising tasks.

The interviews were conducted at the students' schools and took approximately 30 minutes to complete. They were video-taped, transcribed into protocols and then analysed for commonalities in achievement and strategy use within and between the performance categories (HP, MP, LP).

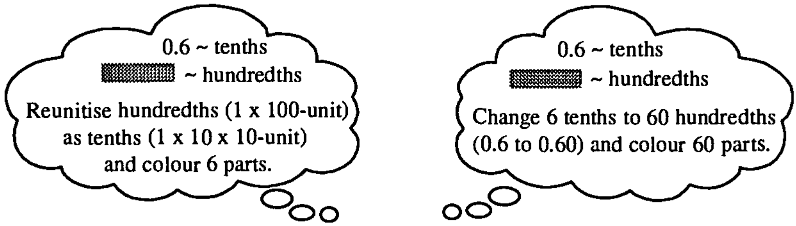
RESULTS

Task 1

Twenty-one (10 HP, 8 MP, 3 LP) of the 29 students were correct, shading either 6 rows or 6 columns. The remaining 8 students(2 HP, 3 MP, 3 LP) all coloured 6 hundredths. No student mentioned that they counted the number of parts in order to <u>unitise</u> the shape as 1 × 100-units; rather, they seemed to have the expectation that there were 100 equal parts, an expectation that could be attributed to the overuse of the PRO model. When asked to read how much had to be shaded, 4 of the 8 incorrect students (1 HP, 1 MP, 2 LP) immediately realised their error (e.g., *I should have shaded 6 strips* – MP8) and shaded the correct amount. Three of the remaining 4 students (1 HP, 2 MP) were able to identify and rectify their incorrect response only after they had been focused on unitising the shape. The remaining student (LP4), whose protocol is provided, appeared to be so bewildered by her original answer that she seemed to lose all ability to unitise.

LP4 [I: How much did you have to shade here?] *A six – I don't know really.* [I: What's this number (pointing to the 0.6 again because she seemed to be looking at what she had coloured)?] *Six* (after a pause). [I: Six what?] *Is it one sixth?* [I: That's (writing $^1/_6$) 1 sixth. What's this number (the 0.2 she had read correctly in an earlier task)?] *One second or something.*

Two different strategies could be identified from the students' responses to the question: *How did you work out how much to shade*? These were classified as *reunitising* (RU) in which the 1 × 100-unit of the given diagram was reunitised as 1 × 10 × 10-units (either rows or columns) or as *equivalence* (EQ) in which the number, 0.6, was reunitised as 0.60, and 60 hundredths were shaded. Figure 4 shows the difference in thinking required by the reunitisation and equivalence strategies.



A. Reunitisation strategy B. Equivalence strategy

Figure 4. Cognitive differences in reunitisation and equivalence.

Both strategies required an understanding of equivalence between tenths and hundredths (i.e., 10 h = 1 t) in order to be applied successfully and this notion was often explicated by students. A third category, *prototypic* was suspected because some students referred to tenths as "strips" or "lines" which may have been the result of prototypic thinking and not as a consequence of having equivalence. That is, the 10 × 10 PRO model always has tenths arranged in rows or columns and therefore they can be perceived without requiring the cognition of equivalence (10 h = 1 t) or reunitisation (1 × 100-unit can be reunitised as 1 × 10 × 10-units). However, this strategy was too subtle to distinguish from the reunitisation strategy so students who were suspected of employing a prototypic strategy were given the benefit of the doubt and classified as using the reunitisation strategy.

The EQ strategy appeared to be used by 10 students (4 HP, 5 MP, 1 LP) and was identified in protocols such as the following. (No student shaded 60 hundredths at random; rather, each student shaded groups of 10.)

HP3: *Because 6 tenths is the same as 60 hundredths and it* (indicating the diagram) *was divided into hundredths so I just shaded 60.* [I: Show me the 6 tenths parts.] *The whole rows* (indicating).

HP10: I *just see these* (hundredths) *as ones and so I colour 60.*

MP12: *It* (diagram) *was divided up into hundredths so you had to colour 60.* [I: Did you change that (0.6) in your mind to 60 hundredths?] *Yes.*

LP2: *Six tenths is the same as 60 hundredths so I thought of zero on the end* (of 0.6) *and just coloured 60.*

Nineteen students (8 HP, 6 MP, 5 LP) appeared to use the RU strategy as they made reference to restructuring the hundredths in the diagram. The following protocols show the variety of thinking that was used in reunitising hundredths as tenths.

HP4: *Cos 60 hundredths also makes 6 tenths, what I did I thought that these* (his shaded columns) *could also be these* (indicating the tenths in an earlier task in which the PRO model had been partitioned into 10 equal columns) *and shaded 6.*

HP6: *There were 100 pieces and if 10 were 1 tenth then I'd need to colour in 6* (indicating her shaded columns). [I: So can you see that (the whole shape) as 100 little parts and as 10 of something else?] *Yes.* [I: When you divide it in your mind in 10 parts, what does that 10 part look like?] *Like that* (indicating a tenth in an earlier task). *Or if I had a 100 of those little cube things* (possibly referring to MAB ones), *I could divide them into 10 groups evenly* (indicating separate groups with her hands).

MP1: *I shaded just one – I guess I took them – the vertical ones* (partitions) *– out of my mind and just shaded it in* (his shaded 6 rows). [I: You blocked the little bits from your mind so you could see these rows going across?] *Yes* [I: So you saw them as 10 rows of 10 then?] *Yes.*

MP5: *I just did 6* (indicating the shaded columns) *because there's 6 there* (0.6) *and forgot about the boxes.*

MP7: *Well, I saw the little squares and there* (0.6) *it says to show 6 tenths in hundredths so I coloured 6 of these* (indicating the rows).

The following protocols provide examples of what was suspected of being prototypic reasoning.

HP11: *Well you just – you know that six take away ten is four so you miss four columns and you just colour in the rest.* [I: So how did you see the tenths? Do the tenths just go across?] *Well, you just know that that's tenths* (pointing to the rows).

MP8: *I should have coloured strips.* (She had shaded 6 hundredths.)

Task 2

Nineteen (8 HP, 7 MP, 4 LP) of the 29 students correctly shaded 1 row, 2 half-rows or 4 columns of the NPRO shape. Of the 10 incorrect students, 1 (LP6) had not attempted the task, 1 (MP12) had shaded half the shape whilst the remaining 8 students had shaded 2 hundredths, 2 rows or 2 columns. Shading 2 parts was thought to be the most naive strategy because no attempt had been made to ratify the numerical amount with the pictorial representation. Shading 2 rows or columns was thought to be less naive because an attempt to ratify the symbolic and pictorial representations had been made but prototypic reasoning (strips, rows, columns) had been used to reunitise the hundredths as tenths.

With respect to unitising, no student mentioned counting the parts, in Task 1, in order to unitise the model as 1 × 100-unit and this behaviour had been attributed to the expectation of 100 equal parts that is generated by the overuse of the PRO pictorial representation of hundredths. In this task, 8 students (6 HP, 1 MP, 1

LP), all of whom shaded the correct amount, mentioned counting the parts to establish how may there were in order to unitise the shape as 1 × 100-unit. However, when asked to read the number and then say whether the shape represented tenths all but one student (MP7) immediately recognised their error and made the appropriate changes. MP7 (who had shaded 2 columns of 5) revealed that he had a problem in unitising the shape as hundredths as shown by his protocol.

I: *Now how do we know whether that's* (his shading) *right or wrong*?

S: *Count up here* (top row) *and see how many altogether. Well, there's 20 in each row* (after counting) *so 20, 40 60, 80, 100* (pointing to the end of each row as he counted). [I: So what would 1 tenth of that be?] *It would be just one of these* (indicating a small square). [I: No, that's 1 hundredth. What about 1 tenth?] (No response) [I: You said before that that (indicating the first column he had shaded) was 1 tenth. Do you still think that's 1 tenth of the whole thing?] *Yes*.

With respect to reunitising, the protocols revealed the same types of strategies that were revealed in Task 1, namely, the RU strategy (used by 21 students – 9 HP, 7 LP, 5 LP) and the EQ strategy (used by 7 students – 3 HP, 3 MP, 1 LP).

Results across the tasks

Table 1 provides the students' initial and amended solutions for both reunitisation tasks and shows that 5 students (2 HP, 2 MP, 1 LP) who had shaded the correct amount in Task 1 did not shade the correct amount in Task 2. This behaviour supports the belief that reunitisation is not established until it can be applied to both PRO and NPRO representations.

Table 1 also shows that 5 (2 HP, 2 MP, 1 LP) of the 8 students (2 HP, 3 MP, 3 LP) who were incorrect in Task 1 were also incorrect for Task 2 and, with the exception of the LP student who was unable to provide a solution, made the same error, namely, coloured the numbers given (i.e., 6 and 2) irrespective of the pictorial representation. The behaviour (i.e., incorrect in the first task but correct in the second task) of the remaining 3 students (1 MP, 2 LP) could probably be attributed to the NPRO model. For example, the model was different from the model usually given to represent hundredths and therefore this oddity acted as a metacognitive "trigger", alerting the students to examine the task more closely.

The 8 students who self-corrected their response revealed that they had the appropriate reunitising knowledge available but had not accessed it at the time of the test. Failure to access the knowledge could have been due to external environmental factors (one student said she couldn't think because the teacher was talking), to internal personal factors such as tiredness, illness, early closure, or to task novelty clashing with task expectations (for example, being asked to shade hundredths only when the diagram represents hundredths and to shade tenths only when the diagram is partitioned into tenths). On the other hand, the interview probably had had some teaching effects because of the probes

regarding the whole, the equality of the parts and the number of equal parts that comprise the whole.

Table 1
Students' responses and solution strategies accessed in the reunitisation tasks.

	Task 1		Task 2			Task 1		Task 2	
	Shading	Strategy	Shading	Strategy		Shading	Strategy	Shading	Strategy
HP1	6 C	RU	4 C	RU	MP1	6 R	RU	4 C	RU
HP2	6 C	RU	4 C	RU	MP2	6 C	EQ	4 C	RU
HP3	6 R	EQ	1 R	RU	MP3	6 h; 6 R	EQ	2 h; 4 C	EQ
HP4	6 C	RU	4 C	RU	MP4	6 h; 6 R	RU	1 R	EQ
HP5	6 C	RU	1 R	RU	MP5	6 C	EQ	2 h; 4 C	RU
HP6	6 C	RU	4 C	RU	MP6	6 C	RU	2 × ½ R	RU
HP7	6 h; 6R	RU	2 h; 1 R	EQ	MP7	6 R	RU	2 C	—
HP8	6 C	EQ	4 C	EQ	MP8	6 h; 6 R	RU	2 h; 2 × ½ R	RU
HP9	6 C	RU	4 C	RU	MP9	6 R	EQ	4 C	RU
HP10	6 C	EQ	2 R; 2 × ½ R	EQ	MP10	6 R	RU	4 C	RU
HP11	6 R	RU	½; 4 C	RU	MP11	6 C	EQ	4 C	EQ
HP12	6 h; 6 R	EQ	2 h; 1 R	RU					
LP1	6 h; 6 R	RU	1 R	RU					
LP2	6 C	EQ	4 C	RU					
LP3	6 C	RU	4 C	RU					
LP4	6 h	RU	4 C	EQ					
LP5	6 C	RU	2 h; 1 R	RU					
LP6	6 h; 6 R	RU	—; 1 R	RU					

Table 1 also reveals that 9 students (3 HP, 4 MP, 2 LP) did not maintain their strategy across the two tasks. Six students (2 HP, 3 MP, 1 LP) changed from the EQ to the RU strategy whilst 3 students (1 HP, 1 MP, 1 LP) changed from the RU to the EQ strategy.

CONCLUSIONS

Table 2 provides the correct solutions (based on initial responses) in terms of the performance categories. It shows that, with respect to performance overall, the students were able to reunitise the PRO representation (Task 1) more easily than the NPRO representation (Task 2).

Tale 2
Correct initial responses to both tasks in terms of the performance categories.

	Performance categories			Overall
	HP (n = 12)	MP (n = 11)	LP (n = 6)	All (n = 29)
Task 1	10 (83.3%)	8 (72.7%)	3 (50.0%_	21 (72.4%)
Task 2	8 (66.7%)	7 (63.7%)	4 (66.7%)	19 (65.5%)
Both correct	8 (66.7%)	6 (54.5%)	2 (33.3%)	16 (55.2%

With respect to the performance categories, Table 2 shows that differential exists between the categories in Task 1 but not in Task 2. Within the categories, differential between tasks was exhibited by the LP group. The deviant behaviour of the LP students on Task 2 was attributed to the teaching effects of the interview in Task 1.

With respect to identifying students who understand tenths and hundredths, this study revealed that performance alone is not a sound indicator. However, it also revealed that, even when the student's strategy is probed, it is sometimes difficult to know whether syntactic features are used as a crutch or whether they are the end-product of structural knowledge which has been integrated and simplified. The interviews also revealed that high-performing students are not necessarily sound in all aspects of fraction knowledge. For example, some may have a sound understanding of the notion of fraction but cannot reunitise tenths as hundredths whilst others exhibit a sound understanding of the concept and the unitising, reunitising and partitioning processes when PRO representations are provided but cannot extend this understanding to NPRO representations. Moreover, some LP students who had performed poorly on the test performed quite well in the interview, indicating that they had the available knowledge but could not access this knowledge at the time of the test.

There seems to be evidence, however, that: (a) the fraction concept and the unitising, reunitising and partitioning processes are essential for performing in decimal fractions with competence; (b) each of these components needs to be connected if a student is to be labelled as having an understanding of decimal fractions; and (c) instruction must include PRO and NPRO representations.

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Students' perceptions of the purposes of mathematical activities

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Introduction

The study to be described in this paper formed part of a larger project entitled Pupils' Awareness of Learning in Mathematics. The aims of this project were to investigate the metacognitive skills and concepts possessed by students of secondary school age in some typical mathematical learning environments, to explore the feasibility of raising the levels of their awareness by appropriate interventions, and to study the effects of such enhancement on the students' mathematical attainments.

The outcomes of the project are described in a Summary Report; a Teachers' Handbook containing the set of suggested enhancement activities, trialled and including examples of students' work; an Evaluation report, containing the evaluative instruments, partially developed but needing further improvement, together with the results obtained; and a set of Case Studies of the seven classes during the main experimental year.

Background

Our interest in students' awareness of their learning arose from our work in a previous project *Diagnostic Teaching in Mathematics* (ESRC 8491/1) (Bell et al, 1985). In this, a teaching methodology based on identification of students' concepts and misconceptions and resolution of the latter by exposure, cognitive conflict and discussion, proved to be strikingly more effective than more usual methods, particularly for longer term retention. What became clear was that such teaching methods demanded a radical change in the students' conceptions of what was appropriate activity in a mathematics lesson. An orientation towards obtaining correct answers had to give place to a recognition that the aim was to acquire correct, well knit concepts and methods, and that this involved being willing to expose ones own ideas and approaches, even if wrong, and to look for personal satisfaction in the enlightenment provided by participation in a focused discussion. This in turn depended on an awareness of the nature of this type of learning and its distinction from memorisation and fluency practice. This led us to consider the possibility of achieving improved learning across the whole mathematics curriculum by increasing students' awareness of learning methods and their purposes.

Metacognition has several aspects. Flavell (1976) defined it as

> "knowledge concerning one's own cognitive processes and products or anything related to them" (p.232)

but he states that it also refers to

> "the active monitoring and consequent regulation and orchestration of these processes." (p.232)

In the field of mathematical education, most metacognitive research has focused on the learning of general problem solving strategies (e.g. Schoenfeld, 1982,1985; Garofalo et al, 1985; Lester, 1988; Siemon, 1992). A notable exception is the work of Slife et al (1985), who showed that it was possible to distinguish certain aspects of metacognitive ability from general ability and from mathematical attainment. These aspects were the pupils' abilities to predict their likely success rate on a given set of computations, and afterwards to identify their correct and incorrect solutions.

A substantial amount of experimentation in the encouragement of metacognitive activity in school and teacher education settings has been built around the PEEL project, based in Melbourne, Victoria, Australia (Baird and Mitchell, 1986; Baird and Northfield, 1992). In this project, a substantial number of teachers at a particular school worked concertedly at

developing methods by which the students (aged 15-16) took greater control of their own learning.

In this project, lessons were rated as achieving Involvement, Awareness or Control on the part of pupils; awareness was of the aims and objectives of the lesson, and control referred to their participation in the determining of these aims. Substantial and far-reaching changes in the approach to learning were achieved by some, though not all, teachers and classes; a notable obstacle was the resistance generated by the severe conflicts with students' existing concepts of learning.

An experiment somewhat similar to our own, but with a single class of primary school (year 6) children, was conducted by Herrington (1992). His one-year programme sought to improve learning strategy awareness, mathematical achievement and confidence towards learning mathematics; it used some 70 short interventions involving concept mapping, a Think Board, self-questioning and writing. Significantly better gains than those of a control group were shown on learning strategy awareness, and non-significant improvements in confidence and mathematical attainment.

Biggs (1987) categorised older students' motive and strategies as Surface, Deep and Achieving, depending on whether they embodied instrumental motives (e.g. to meet assessment requirements nominally) or intrinsic, meaning-oriented strategies. He noted that deep approaches and outcomes were associated with metacognitive skills, and, in a large scale survey of Year 11 and tertiary students, showed correlation between these measures and student self-rated performance. This is one of the few studies connecting metacognition directly with a performance measure (albeit an imperfect one).

Our own project has focused on enhancing reflective activities and on providing lesson experiences through which students may acquire specific knowledge about learning tasks and processes; and this in real classroom settings.

The project had three phases, first, a preliminary exploration of students'beliefs and perceptions; second, development of evaluative and intervention materials; third, a pre-and post-tested experimental year. This involved 25 classes, of which 7 were fully supported and observed regularly by the research team.

The present study

This was a small-scale experimental study conducted in the summer term 1992, towards the end of the observational period. It is distinct from the pre-post evaluative written tests in that it is concerned with understanding students' perceptions of the purposes of mathematical activities *in which they have been participating*. Data are reported from four classes, two of which were also taking part in the main study.

The government enquiry into the teaching of mathematics in schools in the UK, like the NCTM Standards in the USA, has encouraged a much broader range of learning activities than is currently in evidence.The UK report cites exposition, discussion, practical work, practice, problem solving and investigational work as particular elements that should be present . This advice partly stems from a review of research which identified four elements needed in mathematical instruction facts, skills, conceptual structures, general strategies (Bell et al, 1983). The development of conceptual structures, those richly interconnecting bodies of knowledge and understanding which underpin performance, requires considerable reflection and discussion by the learner. Facts and skills are brought up to a level of recall or fluency through regular practice. General strategies require tasks in which pupils make decisions as to which skills or knowledge to deploy or which approach to take.

The introduction of such activities may, however, prove to be necessary but insufficient if pupils are unaware of their purpose. This lack of awareness is likely to lead to students paying undue attention to unimportant or superficial aspects of the task. In our early interviews with pupils, we found considerable anecdotal evidence for this. In particular, our observations noted

that students often perceive their task as to "get work done" rather than to gain insight or understanding.

This study was designed to provide experimental data on pupils' perceptions of the purpose behind five different classroom activities, and to see how these differed from the purposes perceived by mathematics teachers.

The experimental design

Five lessons were taught to four different classes drawn from different comprehensive schools (F, C, M and O). Classes O and M had been exposed to a substantial number of awareness-raising interventions as part of our main study; class F had experienced just a few and class C had experienced only one intervention.
The twenty lessons were on the general theme of multiplication and division with decimal numbers. They were led by the same teacher who standardised her approach as far as possible. Pupils were randomly allocated to groups at the start of each lesson, although not all of the tasks were suitable for group work. The teacher did not articulate the intended purpose of the lesson at any stage, as it was the purpose of this study to discover how well the pupils could deduce this from the activities themselves. At the end of each lesson, students were asked to describe the purpose of the lesson in their own words. They were also asked to rate each of the following purposes 0, 1 or 2 according to whether they felt that this was "not a purpose", was "helped a bit", or was "the main purpose" of the lesson:

> *What do you think are the purposes of this lesson?*
> *Below is a list of possible purposes.*
> *Think about these and then write 2, 1 or 0 next to each one.*
>
> *2 - means that this was one of the main purposes of this lesson.*
> *1 - means that this was not one of the main purposes but it may have helped a bit.*
> *0 - means that this was not a purpose of the lesson at all.*
>
> *Remember: you can write as many 2's, 1's and 0's as you like,*
> *but make sure that you read each statement carefully.*
>
> *This lesson was to help you:*
>
> *(a) to get better at discussing and explaining.*
> *(b) to practise multiplying quickly and accurately.*
> *(c) to practise measuring and drawing accurately.*
> *(d) to learn how to plan and organise.*
> *(e) to learn when multiplying is the right thing to do.*
> *(f) to find the largest answer.*
> *(g) to get better at writing explanations.*
> *(h) to help you understand what multiplication really means.*

The five lessons were as follows:

Lesson 1: Concept discussion : "Believe it or not?"

Students were randomly allocated to groups. Each group was handed six statements such as

- To multiply by 10 you just add a nought on the end.
- Multiplying makes numbers bigger.
- It doesn't matter which way round you do a multiplication; ie a x b = b x a.....

For each statement, pupils were asked to discuss whether it is always, never or sometimes true. They were also asked to produce examples to illustrate their reasoning. A class discussion was then held. This lesson resulted in much animated discussion considering the meaning and

effects of multiplication and division. It exposed many common misconceptions and invalid or only partially valid generalisations.

Lesson 2: *Practical construction: "Maximising the volume of a box"*

Students were each given a 17cm by 17cm printed square. They were instructed to make a box by cutting a 1cm by 1cm square from each corner and folding the resulting shape up into a shallow tray. The teacher asked the class to calculate the volume of this tray.
Each group was given a second sheet of paper and asked to draw another square of side 17cm and to make a different sized box. A particular dimension for the square to be cut from each corner was allocated to each group. The dimensions and the resulting volume of each box were collated on the board, and the students then considered the question of maximising the volume.

This was a highly structured practical lesson. All organisational decisions were made by the teacher. The students merely employed the length x breadth x height algorithm using a calculator. The main focus of the activity was therefore in drawing, cutting out and making the boxes, in a quiet and busy atmosphere. The achieved purpose was therefore to give students practice at measuring and drawing accurately.

Lesson 3. *A calculator investigation:* *"Maximising a product"*

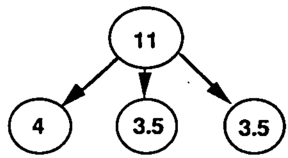
Groups were given a calculator and a copy of the following problem

Split 11 into several pieces.

You can choose the number of pieces and the size of each piece.

Now multiply the pieces together.

What is the largest answer you can make?



4 x 3.5 x 3.5 = 49

Can you beat 49?

Students were encouraged to work on this problem in any way they wished.
The intended purpose of this task is to make pupils realise that a systematic approach is required if the problem is to be tackled effectively. The task may also be used to develop concepts of decimal place value, and a feeling for estimation

Lesson 4. *Skills practice:* *"Crossnumber"*

In this short lesson, pupils were randomly allocated to groups as before, then given a copy of a "crossnumber" puzzle and were asked to complete it without the use of a calculator.
This lesson was aimed at improving fluency with the multiplication and division algorithms. Pupils worked quietly and individually, although many found the task demanding.

Lesson 5 *Recognising the operation:* *"Do not solve it"*

Groups were given twenty-four word problems, and were asked to write down the calculations that would have to be performed for their solution. They were told not to evaluate these expressions. Typical problems were:

1. How many bootlaces, each 8 metres long, can be made from a 4 metre string?
5. A watch costs £6. If I pay for it over a 12 week period, how much must I pay each week?

Pupils were thus expected to write down, but not evaluate, the calculations 4÷0.8 and 6÷12, respectively. The intention here was to encourage pupils to focus on recognising the structure

of multiplication and division problems (including partition and quotition types) and thus identify the correct operation to perform. Pupils worked quietly and individually on this task.

Students' responses

At the end of each lesson, students were asked to write down a free response to the question: "What do you think was the purpose of this activity?" Typically, they wrote down one or two sentences. These were analysed and grouped according to key words or phrases. The most common categories of answers are described below.

Concept lesson: Believe it or not? (n = 91)

Free responses	
To learn to work in a group	51%
To practise explaining	51%
To discuss	45%
To improve understanding of x and +	27%
To practise x and +	22%

Students appreciated that a major purpose of this lesson was to improve their ability to discuss and explain. This was in close agreement with the teacher's view. In addition, students felt that the social aspect of learning to work as part of a group was also important. Students rated the comprehension and written communication purposes rather lower than did their teacher.

Practical construction: Max Box (n = 102)

Free responses	
To learn about areas and/ or volumes	53%
To learn to work in a group	40%
To measure accurately	26%
To find a pattern in the results	16%

The questionnaire response shows that the students and their teacher are in close agreement that the princip purpose is to practise measuring and drawing accurately. This rates rather less significantly on the free responses, where most students perceive the task as being primarily concerned with learning about volumes and areas. The teacher was not concerned with this; in fact during the lesson, she had simply stated the algorithm for calculating volumes without justification. The students again rated the importance of the social aspects of this lesson more highly than the teacher.

Calculator investigation: Splitting 11 (n = 104)

Free responses	
To learn to use a calculator	31%
To develop the ability to x and +	21%
To think about/ understand decimals	13%
To use decimal numbers	16%
To problem solve/ investigate	9%

Students seemed to find it much less easy to identify a clear purpose in this activity than in any other we observed. Their answers were much more diverse. It is noticeable that only two students mentioned an aim related to "planning and organising", the major intended purpose on the part of the teacher. In nearly one half of the free responses, students merely described the activity without analysing its purpose at all. Students were attracted towards the more obvious surface objective: "to use a calculator" and "to find the largest answer". These again are barely more than describing the activity. A considerable number also chose "to practise multiplying quickly and accurately" both in the free and in the questionnaire responses. Presumably they meant "on a calculator", as there was no practice at written algorithms in the lesson.

Recognising the operation: Do not solve it (n = 107)

Free responses	
To recognise when to x or +	29%
To describe the method for problems	19%
To practise multiplying/dividing	17%
To solve everyday problems	10%

Students' perceptions in their free responses were again diverse. Many simply described the activity with little or no analysis of its purpose. It is noticeable that a number (17%) seemed to believe that the activity would help them to improve in their computational facility, although nowhere in the activity were they expected to perform a calculation. This was again shown on the questionnaire responses, although these agreed more closely with their teacher's perceptions.

Skills practice: Crossnumber **(n = 106)**

Free responses

To practise multiplying/ dividing	73%
To practise working quickly/accurately	44%
To practise working without a calculator	38%
To develop mental fluency	17%

This task produced the closest agreement between the teacher and the students. The questionnaire, however, shows that a number of students believe that in some way the performance of the calculation helps to increase understanding of the meaning of multiplication. Possibly this is true if the calculation is done mentally, as seems to be the case for some students, but there may be still some confusion between learning how to multiply and learning when to multiply.

Students' Ratings of Purposes

Students ratings of the ten statements in the questionnaire were also analysed. In this section we focus on a selection of these items, particularly those where there is an interesting mismatch between the perceptions of the teacher and the students.

(d) ... to learn how to plan and organise

Means of Students' Ratings (2 = main purpose, 0 = not a purpose)

Concept discussion - Believe it or not?	1.05	
Practical work - Max Box	0.63	
Calculator investigation - Split 11	0.59	**
Recognising the operation - Do not solve it	0.40	
Skills practice - Cross number	0.14	

Lessons $F(4,12) = 8.55$, $p<.01$
Schools $F(3,12) = 2.61$, ns

Here there is a clear disparity. The teacher considered that the calculator investigation is the only one which involves an appreciable amount of planning and organising, while the students rated two other lessons higher than this. Potentially, the calculator lesson was the most open activity which gave the maximum freedom to students to tackle the problem in their own way. Perhaps students were unaware of the scope for action they were being offered.

(h) ... to help us understand what multiplication really means

Means of Students' Ratings (2 = main purpose, 0 = not a purpose)

Concept discussion - Believe it or not?	1.20	**
Skills practice - Cross number	0.98	
Recognising the operation - Do not solve it	0.94	*
Calculator investigation - Split 11	0.74	
Practical work - Max Box	0.33	

Lessons $F(4,12) = 27.8$, $p<.001$
Schools $F(3,12) = 2.12$, ns

Both the students and the teacher agreed that the 'Believe it or not?' lesson was most relevant here, but the students also gave a high rating to the Cross Number lesson which simply offers repetitive practice of multiplication.

A comparison of items (b) and (e) shows that at least some students can distinguish the idea of practising muliplication from the skill of selecting it as the correct operation, but on this item these different aspects of multiplication seem more confused. It could be that some students do not understand the phrase 'really mean', or it could be that they choose to interpret it in a number of quite different ways.

(i) ... to see how to use mathematics in our everyday lives

Means of Students' Ratings (2 = main purpose, 0 = not a purpose)

Recognising operation - Do not solve it	0.78	**
Skills practice - Cross number	0.44	
Concept discussion - Believe it or not?	0.37	
Calculator investigation - Split 11	0.29	
Practical work - Max Box	0.20	*

Lessons $F(4,12) = 10.6$, $p<.001$
Schools $F(3,12) = 1.58$, ns

The lesson 'Do not solve it' uses story problems that make simple connections to real events, and both the teacher and the students rated this as most relevant to mathematics in everyday life. But the teacher also rated the practical work lesson as relevant while the students saw this as least relevant.

Perhaps the teacher saw possibilities in the practical work lesson which the majority of students missed. For the teacher, the task belongs to a class of realistic optimising tasks (e.g. wall papering a room as cheaply as possible, finding the shortest route from A to B), whereas the students appear to see the exercise as less relevant than even the three 'pure' mathematics lessons. Though this is one of the more widely applicable types of problem in the world of work, the students did not perceive it as having relevance to their own everyday lives.

Mismatch scores

A score was devised to measure the degree of mismatch between the teacher's perception of a lesson and a student's perception. This was calculated for each student in each lesson by sub tracting the teacher's rating of an item on the questionnaire from the student's rating, and adding up the absolute difference for all ten items. This yields a score where 0 indicates complete agreement, and an increasingly positive score indicates an increasing disparity between the teacher's and the students' perceptions.

Below are the means of the mismatch score for each lesson and for each school.

Calculator investigation - Splitting 11	6.36
Practical work - Max Box	6.09
Concept discussion - Believe it or not?	6.05
Recognising operation - Do not solve it	5.18
Skills practice - Cross number	3.64

School F	5.98
School C	5.78
School O	5.15
School M	4.94

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Lessons $F(4,12) = 19.6$, $p<.001$
Schools $F(3,12) = 4.87$, $p<.05$

There are statistically significant differences across both lessons and schools.
The mismatch becomes more pronounced as lessons become more open with an increasingly process (rather than product) oriented agenda. The lowest score is with the skills practice lesson where the students' task is very familiar and well defined, and the purposes of the lesson are apparent to nearly everybody. The highest score is with the calculator investigation, where students' opportunity for freedom of action is greatest.

The two classes which had been exposed through project activities to a substantial number of interventions designed to raise their awareness of mathematical processes (M & O) had lower mismatch scores than the two which had only experienced a few (F & C); which suggests that the awareness raising activities have increased understanding of the purpose of many types of lessons. The effect is not confined to the more open or more unusual styles of lesson.

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STEREOTYPES OF LITERAL SYMBOL USE IN SENIOR SCHOOL ALGEBRA

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I make the case that conventionally in algebra certain roles are associated with some letters. This is particularly true of x and y, which achieve a special status through their conventional usage. Through examples from my classroom I explore the implications of these conventions for students and the tensions implied for the teacher.

Spend a few moments working on this problem:

Find the equation of a straight line which passes through the point (m, c)

If you had your mathematical education in a setting where m was conventionally used to stand for the gradient of a straight line, you might have experienced a discomfort in being required to use m in the same context but in a different role. In Bills (1997) I describe the responses of some colleagues to this problem. Most of them had been schooled in England and had been used to seeing m and c in the context of the equation $y = mx + c$, which is almost universally used here as a 'general equation of a straight line'. Many reported insecurity, discomfort or strangeness. They coped with this by using adapted versions of the 'template' equations which they would familiarly use in this kind of problem. For example, one used $\frac{y - y_1}{x - x_1} = k$ in place of $\frac{y - y_1}{x - x_1} = m$. They described the new template as a 'translation' and spoke of having to 'hang on to' the familiar procedure.

I offer this problem as a vehicle for experiencing the special roles played by some letters in our mathematical culture and the added layer of difficulty created by forcing the use of different letters. I will use the word 'stereotyping' to describe our expectation that certain letters fulfil certain roles.

Literature on the effect of stereotyping on students' learning is sparse. Wagner (1979) points out that a change of placeholder (from x to y) makes no difference to the mathematical meaning but that in the case of a verbal placeholder there is a difference, say between *he* and *she*. He makes the point that the choice of letter to represent a mathematical variable makes no difference to the relationships between quantities it is used to describe. However the psychological difference to the reader can be immense, as demonstrated in my first example.

Vieta was one of the first to make psychological use of a stereotyped expectation of role. He was the first mathematician known to use letters to stand for known but unspecified quantities (in some sense parameters). He used vowels for unknown-but-to-be-found variables and consonants for unknown-but-to-be-given quantities, thus producing an expectation of role by his choice of symbol (van der Waerden 1985).

Furinghetti and Paolo (1994) observed the effects of stereotyping in students' responses to one of their questionnaire items. The item presented the expression $x^2 + y^2 + c^2 + 2xy + 2yc + 2xc$ and asked what was the role of each letter, in the student's opinion. The options offered were 'variable', 'constant', 'parameter' and 'unknown' in each case. The greatest number chose 'unknown' for x and for y, whilst 'parameter' was the most popular choice for c, even though the expression is symmetrical with respect to the three quantities, and was given without any context.

Booth (1985) reports an aspect of stereotyping amongst younger students. Asked 'what is the "y"?' a student volunteered a yacht, yoghurt or yam. Booth suggests that the idea that the letter used to represent the object is the first letter of the name of the object may be reinforced by algebra work schemes which use algebraic initialization as a memory aid.

Taken together these references suggest that, although the choice of letter in a certain role is in some sense arbitrary, for the student there are consequences of the choice.

As part of a wider research project, which examined students' experiences of working with more than one variable, I recorded some classroom incidents which highlighted aspects of stereotyping. In this paper I will concentrate on the stereotyped roles of x and y.

What is a?

In January 1994 I recorded the following about a lesson with some sixteen year old students in an English school. Peter is the teacher and Tommy one of the students.

> After some work on the remainder and factor theorems and on division of polynomials, Peter asks the class
>
> Factorise
>
> (1) $x^3 - 1$ (2) $x^3 - 8$ (3) $x^3 - a^3$
>
> We both walk around the class looking at students' work. I go to the table where Tommy is working. He is about to begin (3). He asks for my help, saying 'I don't know what a is'.

In this lesson, which was focused on methods of factorising polynomials and solving polynomial equations, Peter wanted to take the opportunity to introduce the students to the factorisations of the difference between two cubes and the sum of two cubes, commonly expressed as $a^3 - b^3$ and $a^3 + b^3$. All the polynomials they had factorised so far were in x and had numerical coefficients. Tommy's statement ('I don't know what a is') betrays that the unfamiliarity of this situation had thrown him back into the state of wanting to evaluate the letter. ('Letter evaluation' was identified as one form of response to variables by Küchemann (1981)). Notice that he was not concerned that he did not know what x was. The

role of x as a variable, that is as a quantity which can take any value and takes no particular value, was well-established. It was the social practice in this school as in many others to write expressions in one variable in terms of x. In this task (factorising $x^3 - a^3$) the roles of x and a are, in a sense, the same. I could argue that this task is exactly equivalent to factorising $a^3 - b^3$. However the very fact of using x and a instead of a and b relocates the task into a different context, that is the context of polynomials in x, with its attendant connotations of functions, graphs and equations. For Tommy, in the context of having just worked on factorising $x^3 - 1$ and $x^3 - 8$, the roles of x and a are very different. By the end of my conversation with Tommy he still was not comfortable with the presence of a. I suggest that his comfort with x and discomfort with a are explained partly by the immediate context and partly by his familiarity with the use of x.

This example, then, highlights the stereotypical role played by x in many algebraic contexts. It is the generic unknown in equations to be solved and the generic variable in functional expressions, as well as being the independent variable in the equation of a curve and first co-ordinate of the general point on a curve.

Locus

In an interview at the end of a year's course in pure maths Lorne (aged 17) selected this question to work on

The point (a, b) is equidistant from the x-axis and the point (1, 2). Find an equation linking a and b.

He worked through it unaided by me except that I corrected one or two errors in algebraic manipulation as they arose. He used x and y throughout to stand for the co-ordinates of the point referred to as (a, b) in the question.

Lorne: Then $y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{5}{4}$. (*writes this*) .. That's the answer to it (*writes ANS by this last equation*).

Liz: Okay. Umm .. what was this question about then?

Lorne: Umm the equal distance in, the equal distance between the one point and the other point - locus.

Liz: Right. You said this was like a question that we've done. (*We had done two questions involving equations of parabolae the previous day.*)

Lorne: Uhmhm.

Liz: What's erh similar and what's different about it?

Lorne: Umm .... well it's the same, what I'm doing here is working out the equation of a, of the actual line, but the question says find an equation linking well linking a and b.

Liz: Uhmhm.

Lorne: Which is the same thing isn't it? Or is it?

Liz: You tell me.

Lorne: Umm ... yes.

Liz: So you haven't strictly speaking answered their question, have you?

Lorne: No, not quite, just an equation.

Liz: So if I was being umm pedantic and saying 'can I have an answer to the question please?'

Lorne: then umm that's not the answer (*he scribbles out the ANS which he had written next to his equation*).

Liz: (*laughter*)

Lorne: It's, it's

Liz: It was dangerous to write 'answer' next to something.

Lorne: Yes. It's similar to what we've done but what you've got to do is find an equation linking a and b, and umm .... linking a and b, I mean, ... I don't know actually umm .. an equation linking a and b. I, I don't, I don't quite understand what finding the equation linking a and b really means.

Liz: Uhmhm. Well what erh what part does a and b play in this question?

Lorne: It means it's any point on the parabola which is this same length between the point (1, 2) and the x-axis.

Liz: Right so it's any point on that parabola that you've sketched.

Lorne: Yes.

Liz: Umm when you wrote this equation down you were referring to a point on the parabola.

Lorne: Yes.

Liz: What did you call it?

Lorne: .. I called it, ... what do you mean what did I call, I mean I

Liz: Well you were talking about this point here weren't you?

Lorne: Yes.

Liz: What are the co-ordinates of that point?

Lorne: The co-ordinates of that point is umm (a, b) ,(y, x) umm (p, q).

Liz: Yes, quite. (a, b) or $(x, y.)$

Lorne: Anything ..... yes.

Liz: What, you used x and y.

Lorne: Ah ha. So what I could do is put erh, y is b, so $b = \frac{1}{4}a^2$ could I?

Liz: b equals a quarter what?

Lorne: $\frac{1}{4}a^2$ +. So, I'm not, what I'm answering is the equation linking x and y instead of a and b. So it's $b = \frac{1}{4}a^2 - \frac{1}{2}a + \frac{5}{4}$. (*writes this*) Which is the answer. (*writes ANS by his last equation*)

In his working on this question Lorne used x and y to stand for the co-ordinates of a general point on the parabola almost unconsciously. His answer to my first question on this issue ('.. I called it, ... what do you mean what did I call') suggests that he had not recognised my description of his choice of letters. My question was based on the perception that choosing letters to stand for the co-ordinates of the general point is equivalent to naming that point. A later answer ('The co-ordinates of that point is umm (a, b) ,(y, x) umm (p, q)') suggests that he had not recognised that he had made any choice. He did not see that his writing of the first equation $y = \sqrt{(1-x)^2 + (2-y)^2}$ implicitly made such a choice. His familiarity with x and y in this role made the choice automatic, that is un-noticed. In fact his utterance, 'what I'm doing here is working out the equation of a, of the actual line, but the question says find an equation linking well linking a and b', suggests that, in his eyes, for an equation to represent a curve (the <u>actual</u> line) it must be expressed in terms of x and y.

Test question

The same class which I referred to above were set a test in January 1994. Among the questions was the following:

Problem Q A circle has centre (2, 4) and passes through the point (–1, 5). The point (p, q) lies on the tangent which touches the circle at (–1, 5). Find an equation linking p and q. Hence write down the equation of the tangent.

Of the students who made any substantial attempt at the question, all but one worked with x and y rather than p and q. Some obtained an equation in terms of x and y and then substituted p and q into it. Some did not include p and q in their answer at all.

The students' attention, I suggest, was on finding the equation of the tangent and the steps on the way to that aim (finding the gradient of the radius and hence of the tangent, obtaining the equation of a straight line with this gradient and passing through (–1, 5)). In order to focus on these steps they lost sight of the specific detail of this question and its reference to the point (p, q). They used the letters they were familiar with using for a general point, that is x and y.

These students, along with Lorne, demonstrate that their use of x and y as co-ordinates of a general point on a curve is almost unconscious. Mention of other

letters in the text of a question was insufficient to bring the issue to the surface. When Lorne was challenged about his choice of letters, his recognition of what he needed to do to satisfy the needs of the question was not immediate. His use of x and y was so automatic that it took some discussion before he noticed it.

As an experienced mathematician I am aware of my ability to choose a letter when some expression of generality is required. Students may be much more restricted in their awareness of this choice. One of the factors which restricts their awareness of choice is the conventional use of certain letters in certain roles. For example, in answering 'Find the general equation of a line which passes through the point (3, 2)' students may use the letter m to stand for the gradient without experiencing any choice. I will describe choices dictated by common usage in this way as *culturally determined*. In answer to the question 'Find the equation of a line with gradient m which passes through the point (3, 2)' the choice of m is *mathematically necessary*. For students these two questions and the choices implied may be indistinguishable.

A general linear equation

A class of adult initial teacher education students was discussing the solution of linear equations. I asked them to give me 'a general form for a linear equation' and $ax + by = c$ was offered by Gill, a member of the group. Another member offered $ax + b = c$ and then $ax + b = 0$ to general approval. I asked

> 'These equations (*the ones we had looked at so far*) have had xs in them - they haven't had any other letter in them. Now the equation that Gill's brought up here ($ax + by = c$) has got a, b, c, x and y in it and some people are objecting to the y. Why are you objecting to the y and not the a, b and c?'

After a few moments' pause there were two replies to this question:

> 'You're assuming that a, b and c are just ordinary numbers and x and y are the variables'

> 'a and b are used to stand for numbers that you know and x and y are numbers that you don't know'

Although it was not something to which they could recall having previously given any conscious thought, these students were in no doubt that a, b and c played different roles from x and y. In the ensuing discussion they described this as 'conditioning'. Some of them expressed surprise that they accepted this difference between roles without any good reason or conscious acknowledgement.

The role of x and y as the co-ordinates of a general point on a curve combines with the roles of x as the unknown in equations and as the argument of functions to set them apart from all other letters. These students' explanations of the differences are not entirely coherent but they are deeply felt.

Discussion

Each of the examples above points to some aspect of the unique roles played by x and y in our mathematical culture (by this I mean, in particular, the culture represented by teachers and examiners of 'A' level mathematics in England and Wales, and into which pupils need to be, to some degree, inducted. Many of the features of this culture are common to other groups). The very strong cultural pressure to use x and y in the circumstances exemplified above makes it almost a mathematical necessity. Consider for instance, the task

'Find the equation of a straight line which passes through the point (x, y)'

Responses of the form $y = mx + c$ hold on to the conventional roles of x and y whilst making their new roles, suggested by the question, as unknown particulars, untenable. Responses of the form $Y = m(X - x) + c$ relinquish the expected roles of x and y in order to have them adopt others.

Whilst it may be true in the strict mathematical sense that, in mathematics, the choice of letter does not necessarily convey information about the quantity for which it stands, it is by no means the case from the cultural point of view. I have already made a distinction between mathematically necessary and culturally determined choices. The examples I have given show that choice of literal symbol can convey a great deal about the role of the quantity that it represents. In particular the letters x and y carry with them a great many contexts, meanings and metonymic triggers.

These messages conveyed by use of letter can be useful or obstructive for the student. A tension exists for the teacher between establishing the conventions of mathematical society and exposing them as culturally but not mathematically necessary.

On the one hand, my practice of cultural conventions in the use of letters allows me to automatise procedures. I can perform a procedure without placing my attention on that procedure. The role of each quantity in the procedure is captured by the name, that is the letter I use. I do not need to ask myself (for example) 'why was I trying to calculate c?'. I know that the value I have calculated is the value of the y-intercept. My attention is not on the meaning of c and can therefore be on some other aspect of the problem. These conventions can also assist students in dealing with what Adda (1982) refers to as 'homonymy', that is the different roles of letters within the same equation. In her example, '$ax^2 + bx + c = 0$', the roles of a, b and c are in fact separated from that of x by conventional usage so that distinguishing between them is not an apparent difficulty for students.

On the other hand, the repeated use of convention in symbol choice makes the cultural nature of the conventions invisible. It removes from view the choice of letter, so that the distinction between convention and mathematical necessity is blurred. The automatisation of procedures is useful precisely because it removes

attention from that procedure. The drive to automatise through rehearsal may remove attention too soon from where it is needed.

Conventional use of letters is a means of control for the expert user. These users can free their attention from the routine to place it on the unfamiliar. They also have the option of not using the conventional letters if they wish. The novice, by contrast, is controlled by the choice of letters. Their ability to perform a task may depend crucially on its being expressed in terms of the conventional literal symbols or on its being possible to perform the task by using the familiar notation.

Summary

In the mathematical culture of school and beyond, some letters carry particular meanings as variables because of conventional usage. Unconventional use can cause discomfort for the problem solver and may even prevent their reaching a solution. Whilst the experienced mathematician can exercise control by distinguishing between cultural determination and mathematical necessity, an inability to make this distinction may cause problems for the novice.

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APPROACHING THEORETICAL KNOWLEDGE THROUGH VOICES AND ECHOES: A VYGOTSKIAN PERSPECTIVE

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This report deals with the ongoing construction of an innovative theoretical framework designed to organise and analyse early student approach to theoretical knowledge in compulsory education, the aim being to overcome the limits of traditional learning and constructivist hypothesis. Referring to Vygoskian analysis of the distinction between everyday and scientific concepts and the Bachtinian construct of 'voice', and drawing on previous teaching experiments, we hypothesise that the introduction in the classroom of 'voices' from the history of mathematics and science might (by means of suitable tasks) develop into a 'voices and echoes game' suitable for the mediation of some important elements of theoretical knowledge.

1. Introduction.

How to approach basic elements in modern-day scientific culture represents a serious problem in the compulsory education system. In this report we shall refer in particular to theorems, to algebraic language and to the mathematical modeling of natural and social phenomena; henceforth we shall use the term 'theoretical knowledge' to cover the above elements of mathematics. On the one hand, these are relevant for orienting and preparing students for the later study, as well as transmitting important aspects of the human cultural heritage to new generations (Boero, 1989a). On the other hand, the most common educational strategies (either traditional or not) to approach theoretical knowledge appear to be unproductive for most students, even in upper-secondary and tertiary education. In Italy as in other countries, mathematics and science theories are 'explained' by the teacher to students as from the 10th grade; the students' job is to understand them, to repeat them in verbal or written tests and to apply them in easy problem situations. The results are well known: for most students, theories are only tools for solving school exercises and do not influence their deep conceptions and ways of reasoning.

Constructivism too presents limits as regards the approach to theoretical knowledge: see Newman, Griffin & Cole (1989). We have noticed profound gaps in the aspects of mathematics mentioned at the beginning, gaps which are difficult to bridge even with the teacher's help. These are between the expressive forms of students' everyday knowledge and the expressive forms of theoretical knowledge; between the students' spontaneous way of getting knowledge through facts and theoretical deduction; and between students' intuitions and the counterintuitive content of some theories.

The ongoing research study, which is partially reported in this paper, aims to give useful elements for interpreting and overcoming the above difficulties in the approach to theoretical knowledge.

2. A Vygotskian (and Bachtinian) Perspective.

The difficulties encountered in the traditional and constructivist approaches pose a series of questions. We shall try to describe the route we have taken to reach the definitions and hypotheses presented in Section 3.

What constitutes the gap between spontaneous and theoretical thinking? To address this issue we have considered the distinction proposed by Vygotskij, between everyday and scientific concepts (Vygotskij, 1992, chap. VI). It is common knowledge that this is one of the most controversial aspects of Vygotskij's work. It has often been considered outdated as it contains a systematic critique of the position taken by Piaget in the twenties, a position later revised by Piaget himself. On the other hand, the most significant examples Vygotskij uses to develop his arguments concern language and social sciences, with some generalisation to mathematics and natural sciences that are not always pertinent. In Vygotskij's school, Davydov himself has pointed out several weak and even contradictory points (Davydov, 1972). In addition, Vygotskij claims it is possible to 'teach' scientific concepts and theories to the point where they are 'internalised'; yet his hypothesis does not succeed in overcoming the learning paradox: 'How can a structure generate another structure more complex than itself?' and, more particularly, 'How does internalisation take place?' (see the discussion of Bereiter's paradox in Engeström, 1991). All the above objections have lead to underestimation of other aspects of Vygotskian analysis, such as the following: the systematic character of theoretical knowledge (versus the a-systematic nature of everyday knowledge); and the transition of scientific concepts from words to facts, versus the transition of everyday concepts from facts to word. Only recently have some researchers (e.g. John Steiner, 1995) called attention to the significance of these aspects of Vygotskian analysis. We notice that they shed light on the gap between students' everyday knowledge and theoretical knowledge, and offer a single perspective on a variety of different aspects of mathematics, such as those indicated at the beginning of this report.

Why is constructivist approach unable to bridge the gap between everyday and theoretical knowledge? On the basis of his distinction between everyday and scientific concepts, Vygotskij hypotesises that, in children's intellectual growth, their everyday knowledge has to be developed towards theoretical knowledge by establishing links with theoretical knowledge and that theoretical knowledge has to be connected with facts by establishing links with children's everyday knowledge. Yet, according to Vygotskij, the development of everyday concepts is not spontaneous: the child cannot be left alone to pursue this process because theoretical knowledge has been socially constructed in the long term of cultural history and cannot be reconstructed in the short term of the individual learning process. In short, 'exposure' to theoretical knowledge is necessary, and must be provided together with explicit links to children's knowledge.

Which aspects of theoretical knowledge are to be chosen? In our view, cultural meaning and student motivation are the most important criteria. Therefore, priority should be given to leaps forward in the cultural history of mankind, even if, for the abovementioned reasons, these are the most difficult areas for school study. The sorts of topics we are referring to include, for instance, the theory of the fall of bodies of Galilei and Newton, Mendel's probabilistic model of the transmission of hereditary traits, mathematical proof and algebraic language - all aspects with a counterintuitive character. These are 'scientific revolutions' related to historical figures from the history of science (Galilei, Newton, Mendel, Euclid, Viete). In many cases, scientific revolutions have been accomplished by overcoming epistemological obstacles (Bachelard, 1938) which were a crucial part of previous

knowledge. The same obstacles are often found in individual history as well (Brousseau, 1983).

How are the leading ideas of scientific revolutions expressed? Bartolini Bussi (1995) has suggested referring to the Bachtinian construct of *'voice'* to describe some crucial elements of the turning points in scientific thinking. Bachtin's seminal work centers on literature, but some researchers in general and mathematics education have found several interesting elements therein (Bosch, 1994; Seeger, 1991; Wertsch, 1991). As far as the approach to theories is concerned, we draw on some aspects of Bachtin's work:

- the idea that human experience does not speak by itself but needs original voices that interpret it; the voices are produced in a social situation and gradually recognised by society until they become the shared way of speaking of the human experience;
- the idea that such voices act as voices belonging to real people with whom an imaginary dialogue can be conducted beyond time and space. The voices are continuosly regenerated in response to changing situations (they are not mummified voices to be listened to passively, but living tools for interpreting changing human experience).

How can students be 'exposed' to the leading ideas of scientific revolutions? If we transpose these ideas to the fields of science and mathematics (intended as a 'field of experience': Boero & al. 1995) we gain a useful perspective for our purposes: teachers can become mediators of 'voices' (of 'historical voices' in particular), which embody those scientific revolutions whose sense is to be conveyed to new generations. This process must take place in a social situation where the voices are renewed in accordance with changing cultural perspectives.

3. Towards a Theoretical Framework for the "Voices and echoes game".

Retrospective analysis of some teaching experiments (performed several years ago in the Genoa Group classes) confirmed the idea that scientists' voices may be exploited to approach theoretical knowledge and provided us with hints for further operational activity. As an example, let us consider the teaching experiment reported in Boero & Garuti, 1994. Students were asked to produce a brief, general statement about the relationships between heights of objects and the length of sunshadows they cast; they were asked subsequently to compare their statements with official statements of the so-called 'Thales theorem'. Analysis of the students' texts revealed an interesting phenomenon: many students had tried to rephrase their statements in order to make it resemble to the official statement, or to rephrase the official statement in order to make it to resemble their own. This was a constructive effort of a quite different nature from the production of an original statement; in fact it was an effort to 'echo' proposed 'voices'! A similar phenomenon is reported in Bartolini Bussi (1996), where the 'voice' of Piero della Francesca is exploited during a primary school perspective drawing activity.

Taking into account these experiences and the reflections summarized in the preceding section, we have undertaken the construction of a theoretical framework for a new methodological approach to theoretical knowledge. We have defined the 'voices and echoes game' and elaborated a general hypothesis concerning the effectiveness of this game in approaching theoretical knowledge (see 3.2.).

Consequently, we have planned a teaching experiment, which was performed in five 8th-grade classes (see 3.3.).

Analysis of the teaching experiment allowed us to elaborate a language (see 3.4.) that we consider useful for describing, classifying and interpreting student behaviour during the 'voices and echoes' game, and which is also helpful in recognising and conveniently managing that behaviour.

We think that the research work performed so far makes it possible to plan further teaching experiments aimed at understanding better the mechanisms of individual and social cognition that allow the 'voices and echoes game' to work well; another aim would be to detect the control variables for classroom work. In summary, we consider that we have built an initial theoretical framework for a 'didactical engineering'(Artigue,1992) considered as a tool for developing research.

3.1. The 'Voices and echoes game'

Some verbal and non-verbal expressions (especially those produced by scientists of the past - but also contemporary expressions) represent in a dense and communicative way important leaps in the evolution of mathematics and science. Each of these expressions conveys a content, an organization of the discourse and the cultural horizon of the historical leap. Referring to Bachtin, we call these expressions *'voices'*.

Performing suitable tasks proposed by the teacher, the student may try to make connections between the voice and his/her own conceptions, experiences and *personal senses* (Leont'ev), and produce an *'echo'*, i.e. a link with the voice made explicit through a discourse. The 'echo' is an original idea, intended to develop our new educational methodology.

What will henceforth be called the *'voices and echoes game'* is a particular educational situation aimed at activating the production of echoes by students. To this end, specific tasks may be proposed: *'How.... might have interpreted the fact that...'*, or: *'Through what experiences ... might have supported his hypothesis'*; or: *'What analogies and differences can you find between what your classmate said and what you read...'*, etc. The echoes produced may become objects for classroom discussion. Some may be transformed (given appropriate stimuli and praise from the teacher) into voices which renew those introduced by the teacher and equated to the students' outlook and specific experiences.

We note that the object of the 'voices and echoes game' is not to construct a concept or an original solution to a problem, nor is it to validate a student product. Rather, the point is to compare a text (generally not produced by the student who make the comparison) with another text or with some data from everyday experience in order to detect congruences or contradictions. In this way the transition of students' thought to a theoretical level can be enhanced. Our general hypothesis on this issue is that the 'voices and echoes game' may allow the classroom's cultural horizon to embrace some elements which are difficult to construct in a constructivist approach to theoretical knowledge and difficult to mediate through a traditional approach:

- contents (especially, counter-intuitive conceptions) which are difficult to construct individually or socially;
- methods (for instance, mental experiments) far beyond the students' cultural horizon;

- kinds of organization of scientific discourse (for instance, scientific dialogue; argumentation structured into a deductive chain) which are not a natural part of students' speech.

In the case of important and counter-intuitive theories (such as Galilei's and Newton's theory of falling bodies, which was the object of our teaching experiment), we think that the transition towards the revolutionary theory should be made by overturning the contrasting theory that preceded it. Consequently, the 'voices and echoes game' should start with historical voices that give a theoretical representation of students' intuitions and interpretations. There are a number of different reason for this approach: cognitive and didactic reasons (students need to take on board epistemological obstacles - see Brousseau, 1983 and, from a different perspective, Fischbein,1994); historical and cultural reasons (important scientific changes do not happen in a cultural vacuum, but occur when new theories substitute old ones); reasons related to student transition to a theoretical dimension (a theoretical dimension may be more accessible if it initially concerns theories which resemble students' conceptions about natural phenomena or mathematical entities).

3.2. A Teaching Experiment

A teaching experiment involving the 'voices and echoes game' was performed in five 8th-grade classes, of different level, belonging to different environments, and with partially different school background. Bearing this diversity in mind, management of classroom work differed from one class to another, although the succession of voices and the tasks for the production of echoes was similar in all the classrooms.

The theories chosen for our teaching experiment concerned falling bodies. Preceding classroom experiences perfomed by the Genoa Group had shown that 8th-grade students' spontaneous knowledge about this phenomenon is limited to perceptual data, with scarce cultural elaboration. Our hypothesis was that through the 'voices and echoes game' some historic voices (Aristotle) might encapsulate student perceptions in a meaningful and precise theoretical way, while other voices (Galilei) might lead them to overturn Aristotle's theory.

Each voice was read in the classroom under the guidance of the teacher, who provided paraphrases, explanations of words, and information concerning the general cultural framework of the voice. Following each voice there were tasks that called for the production of echoes, as well as classroom discussion of some of the echoes produced .

For each class, the teaching experiment lasted from 12 to 16 hours.

Recordings of classroom discussions and individual texts were collected.

This teaching experiment produced learning results which were much better and more extensive than those usually achieved when 8th-grade students approach theoretical knowledge. The following positive aspects were common to all the classrooms (although varying in continuity and extension from class to class):

- students acquired contents, methods and ways of organizing discourse contained in the theoretical knowledge proposed to them through the voices;
- high quality scientific debate was attained at particular moments, which differed from class to class. The importance of this lies not so much in the discoveries made (in most cases they were inherent in the voices proposed by the teacher), but in the fact that ancient scientific debate was revived and related to the present cultural and

expressive horizon. It can reasonably be hoped that, once constructed and experienced in the classroom, this approach may be applied to other aspects of theoretical knowledge both in later studies and in daily life. This is in line with Bachtin' s hypothesis about literature, in which the reader starts to refer the read text to her/his personal and contemporary collective experience.

In the 'voices and echoes game' situations performed in our teaching experiment, the productivity of the different phases varied from class to class, for reasons which, althought not fully clarified yet, appear to depend not only on the classroom background, but also on the peculiar dynamic evolution of the situation and the particular didactic choices of teachers. Notwithstanding the differing productivity, interesting patterns in student behaviour were observed; these allowed us to create a classification of student behaviour related to the general aims of the 'voices and echoes game' (see 3.1.)

3.3. Description and Classification of Student Behaviour

Students may produce echoes of different types (depending on tasks and personal reactions to them). First of all, we need to distinguish *individual echoes* and *collective echoes* (these are produced during a classroom discussion which may concern some of the individual echoes selected by the teacher).

Individual echoes can be classified as follows:

- *superficial echoes:* these are produced in an effort to perform a task requiring echo, but do not succeed in understanding the voice. These can be recognized in inappropriate use of terms and expressions deriving from the voice, contradictions, confusion between students' conceptions and those inherent in the voice, etc.
- *mechanical echoes:* precise paraphrasing of a verbal voice or the correct solution of a standard drill exercise. The student does not go beyond the level of 'mechanical echo' if she/he is incapable of exploiting the content and/or the method conveyed by the voice in order to solve a problem which differs to some extent from the situation inherent in the voice;
- *assimilation echoes:* these can be detected when the student is capable of transferring the content and/or method conveyed by the voice to other problem situations proposed by the teacher that are only partly similar to that inherent in the voice (see Matteo, Annexe). The student does not go beyond the level of the assimilation echo if his/her manner of considering natural phenomena or mathematical entities does not take the voice into account when faced with destabilising problem situations;
- *resonances:* beyond the level of assimilation, the situation of resonance is the most interesting of all. In this case the student appropriates the voice as a way of reconsidering and representing his/her experience; the distinctive sign of this situation is the ability to change linguistic register by seeking to select and investigate pertinent elements *('deepening')*, and finding examples, situations, etc. which actualize and multiply the voice appropriately *('multiplication')* (see Enzo, Annexe);
- *dissonances* (similar to resonance, but with opposition to the content and/or method conveyed by the voice).

The echoes which develop at the collective level may consist of series of individual echoes of the voice at the center of discussion *('source voice')*; these occur one after the other irrespective of classmates echoes. At the other extreme,

there may be a high level of connection between successive echoes. In particular, both the examples related to the 'source voice' and the expressions and expressive registers may undergo rapid and intensive enrichment. In other words, collective echoes may reveal phenomena of multiplication and deepening, by exploiting both the 'source voices' and classmates' echoes. We call this phenomenon *'multiple echo'*.

In a 'multiple echo' situation, *'classroom voices'* can be generated: these renew the 'source voices' proposed by the teacher in terms of expression and cultural references. The multiplication and deepening phenomena, stimulated by students' examples and continuously enriched by new expressions and experiences, may make it possible not only to express the content and methodological structure of the source voice using the students' own language but also to refer these to the students' cultural horizon.

We believe that the 'multiple echo' and the production of 'classroom voices' are the conditions which allowed some meaningful experiences of true scientific debate to take place during our teaching experiment.

3. Ongoing research

The above theoretical framework remains limited to the level of description and classification of student behaviour. The available data do not allow exact interpretation of the cognitive processes involved, nor do they provide reliable indications for reproducing 'voices and echoes game' situations. Further experiments currently being planned should allow us to progress from detecting the described behaviours (through the indicators quoted in the preceding section) to interpreting them, and in particular to identifying variables involved. This research should focus on the following, interrelated questions:

- when students are engaged in tasks requiring echo production, what are the mechanisms of individual and social cognition through which they appropriate the level of theoretical organisation of discourse inherent in the voices? As we saw in the introduction, this point represents one of the main elements forming the gap between student thought and theoretical thought. Considering this point, and the importance attributed by Vygotskij (1978) to imitation, we need to pay special attention to the functions of the mechanical echo (which can be easily 'forced' through suitable tasks);
- What are the cognitive and affective mechanisms through which the historical personality 'takes part' (when his voice is introduced by the teacher in the classroom) as an interlocutor in classroom debate? The effectiveness of the 'voices and echoes game' seems to depend on this imaginary 'participation' in the game (see also Bartolini Bussi, 1996);
- What are the variables (class background, kinds of tasks, suggested sign systems, etc.) which the productive development of the 'voices and echoes game' depends on, particularly in the production of resonances and the phenomenon of 'multiple echo'? Observations made so far suggest that available or suggested sign systems strongly influence multiplication and deepening phenomena at an individual level. As to 'multiple echo', we think that familiarity with collective discussion (as the place where students carry out the social construction of knowledge) is a necessary condition but is not in itself sufficient for generating this type of echo.

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Annexe

Excerpts from one of Aristotle's *voices*, which was selected for the 'voices and echoes game': *Each body moves towards its place, if it was removed with force [...]. According to its nature, fire moves upwards, earth moves downwards [...]. The reason for he heaviness or the lightness of the other bodies (i.e. compound bodies) is the difference between the simple bodies (earth, water, air, fire) which are their components. Bodies may be light, or heavy depending on the greater or lesser quantity of this or that simple body they contain.*

Task: *If you were Aristotle, what would you tell a young student of yours in order to explain why smoke moves upwards?*

Matteo's echo: *Because smoke derives from fire and does not contain earth, it tends to move upwards, due to its affinity with fire*

Enzo's echo: *Smoke is produced by fire and fire is absolutely light, but it is also produced by wood, which is heavy but is also light, so fire prevails because wood is heaviness - lightness and fire is only lightness; consequently smoke moves upwards, but not so much as fire, because it is kept downwards by the residual part of wood.*

THE TRANSITION FROM ARITHMETIC TO ALGEBRA: INITIAL UNDERSTANDING OF EQUALS, OPERATIONS AND VARIABLE

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This paper discusses understanding of operations, equals, operational laws and variable in relation to understanding of algebra. It proposes a two-path model for developing complex algebra. It reports on a study in which 51 grade 7 students were interviewed and their knowledge of the components of this model categorised (binary arithmetic, binary algebra and complex arithmetic). It describes the students' responses which indicated difficulties with equals, division, commutativity, order conventions, and multiples of unknowns.

The literature has stressed the link between arithmetic and algebra (e.g., Cortes, Vergnaud & Kavafian, 1990; Linchevski, 1995; MacGregor and Stacey, 1995), has identified a gap in this transition (e.g., Herscovics & Linchevski, 1994), and proposed a pre-algebra level to cover this gap (e.g., Filloy & Rojano, 1989). This paper reports on the first stage of a three-year longitudinal study instigated to follow students from the beginning of early algebra instruction, grade 7, to the completion of initial algebra instruction, grade 9. It focuses on the initial interviews undertaken before any algebra instruction and reports on students' understanding of: (a) two aspects of arithmetic that appear to continue into algebra, *equals* and the *operational laws*; and (b) an aspect of algebra new to arithmetic students, *variable*. The purpose is to explore students' readiness for algebra instruction and linear equations in terms of prerequisite knowledge.

Equals. The presence of an equals sign means that both sides of an equation are equivalent and that information can be processed from either direction in a symmetrical fashion (e.g., Kieran, 1992; Linchevski, 1995). However, research indicates that students have a persistent idea that the equals sign is either a *syntactic indicator*, a symbol indicating where the answer should be written, or an *operator sign*, as a stimulus to action or 'to do something' (e.g., Behr, Erlwanger & Nichols, 1980; Denmark, Barco & Voran, 1976; Filloy & Rojano, 1989). The research also indicates that a restricted understanding of equals appears to persist through primary school (e.g., Baroody & Ginsburg, 1983), continue into secondary and tertiary education (e.g., Behr, Erlwanger & Nichols, 1980), and affect mathematics learning at these levels. As well, there appears to be a lack of attention to the two different ways equality can be approached: (a) in static terms as '*balance*', for example, 2+3 balances 5; and (b) in dynamic terms as '*change*' or 'transformation', for example, 2+3 changes 2 to 5 by adding 3 (Cooper & Baturo, 1992).

Operational laws. A sound understanding of operational laws is essential for generalisation and recognition of patterns between numbers that is a basis of the transition from arithmetic to algebra (Bell, 1995) and for solving of algebraic equations (Demana and Leitzet, 1988). The operational laws are what enables the

numbers and the operations of addition and multiplication to form the mathematical construct called a Field. The properties of operations that have significance for algebra are the *commutative*, *associative* and *distributive* laws, *inverse* and the *order convention* (e.g., Bell, 1995; Cooper & Baturo, 1992; Demana and Leitzel, 1988). Misconceptions with respect to these properties may lead to a "conceptual obstacle" in algebraic understanding (Bell, 1995; Herscovics & Linchevski, 1994).

Variable. According to Usiskin (1988), variable can be conceived in the following four ways with the first three conceptions to be replaced by the fourth when expertise is gained: as a *generalisation* of arithmetic; as an *unknown* in procedures for solving certain types of problems; as a *relationship* among quantities; and as a member of an *abstract system*. However, some educators (e.g., Chalouh & Herscovics, 1988) have argued that unknown is not an appropriate conception for variable as it does not represent multiple meanings. Hence, Sfard and Linchevski (1994) proposed variables as generalisations and relationships as more advanced conceptions than variable as unknown. In this they reflected Kucheman (1981) who proposed six different levels for students' understanding of variable: as a number, without meaning, as an object, as a specific unknown, as a generalised number, and as an abstract variable.

Students have difficulty with the concept of variable (Booth, 1988) and this difficulty can be basic to a lack of success in algebra (Demana & Leitzel, 1988). It is difficult to move from arithmetic to algebra; students' conceptions of operations performed on numbers have to change in order that the concept of operating on variables may be developed (Filloy & Rojano, 1989). Common student misconceptions include believing variable only has meaning when its value is known and thinking a variable represents objects instead of numbers (e.g., Booth, 1988; McGregor, 1991). Kucheman (1981) found that students generally operate at his first three levels.

METHOD

Data was gathered by a structured clinical interview. All students were given the same tasks in the same order. Interesting responses were probed for cause.

Sample. The sample consisted of 51 grade 7 students (12 years of age) from four state schools in Brisbane Australia before they undertook any algebra instruction.

Tasks. The interview tasks were developed as a result of a content analysis of algebraic equations such as 3x+7=22 which hypothesised that such algebraic equations were complex in relation to their use of operations and that they were the end product of a two-path sequence of topics that included binary arithmetic (24+37 and 35x29), complex arithmetic (more than 2 operations - 24+35x29), and binary algebra ($3x$, x+5). Figure 1 briefly outlines the model. A fuller description of the model is in Boulton-Lewis et al (1997).

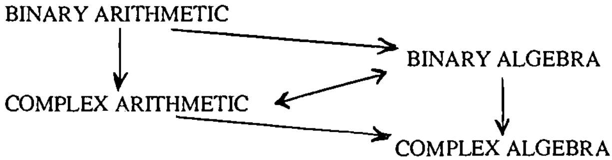

Figure 1. *Two path model for developing complex algebra (Boulton-Lewis et al, 1997)*

The tasks took account of the potential influence of number size and the order in which numbers and letters are presented and covered equals, operational laws and variable. For equals, the students were asked for the difference between 28/7+20 and 28/7+20= and what the equals sign meant in 28/7+20= and 28/7+20=60-36. For the operational laws, the students were asked: (a) for the operations to complete 35??76=76??35 (commutative law); (b) how 60+18 and 42+36 could assist with 6x13 (distributive law); (c) to relate 5x71=355 and 355?5=71 and 64-29=35 and 35?29=64 (inverse); and (d) would the answer change if the operations in 3x6-2 were interchanged and how they would solve 32+(12x8)/3 (order convention). For variable, the students were asked what the boxes and letters meant in expressions and equations, []+5, []+5=9, $3x$ and x+7=16.

Along with these tasks, the meaning of the operations were checked by asking the students for their understanding of the four operations, and the use of concrete material to represent a box or a letter in a linear equation was checked with counters and cups.

Procedure. The students were removed from class and interviewed for approximately 20 minutes. The expressions and equations used in the tasks were placed on cards. The interviews were videotaped.

RESULTS

Analysis. The videotaped interviews were transcribed into protocols which were analysed using software Non-numerical Unstructured Data Indexing Searching and Theory-building (NUDIST, 1994). Initial analysis of the data was used to identify key ideas which formed categories and subcategories. NUDIST was used to classify the protocols under these categories/subcategories and to develop explanations for the students' responses. Overall, the NUDIST analysis categorised responses as satisfactory or unsatisfactory, and as arithmetical (using arithmetic based approaches), algebraical (using algebraically based approaches) and no idea (unable to determine the basis of the approach used). However in this paper, responses are only given in their subcategories.

Student responses. The responses of the students for the tasks are summarised in Table 1. The responses with regard to meaning of operations and using concrete materials in linear equations follow the task results.

Table 1

Student-response categories for equals, operational laws and variable (n=51).

TASKS	RESPONSE CATEGORY	NUMBER
Equals		
28÷7+20 =	Answer (e.g., *It's asking for the answer*)	41
	Outcome (e.g., *In total, what is the actual outcome of the sum*)	8
	Equal it all (e.g., *That you have to equal it all up together*)	2
8÷7+20 = 60-36	No idea (e.g., *I'm not sure what the equals sign means*)	2
	Another sum (e.g., *Gives answer and starts onto another sum*)	5
	Answer (e.g., *Put down the answer of the sum before it*)	7
	Answer is 60 (e.g., *Equals 60 and then they've taken away 36*)	11
	Answer is 60 - 36 (e.g., *The answer is 60 take-away 36*)	9
	Answers the same (e.g., *28 divide 7 plus 20 same as 60 take 36*)	10
	Both sides equal (e.g., *Both sides are 24, so it's equivalent*)	7
Operations		
Commutative		
35??76 = 76??35	No idea (e.g., *I can't think of anything*)	11
	Variety of signs (e.g., *You could probably use all of them*)	14
	Answer is after = (e.g., *35 'any operation' 76 wouldn't equal 76*)	4
	Either way (e.g., *Plus/times can be either way; divide/minus can't*)	14
	Same answer (e.g., *They were same numbers with plus/multiply*)	5
	Equivalence (e.g., *Equal the same on each side of equals sign*)	3
Distributive		
6x13 60+18	No idea (e.g., *I don't see what they've done*)	17
	Other (e.g., *Put a zero after the 6 and add a 5 onto the 13*)	14
	Connection (e.g., *Timesed 6 by 10 to get 60 & 6 by 3 ... 18*)	20
6x13 42+36	No idea (e.g., *You could choose any numbers*)	24
	Other (e.g., *6 times 7 to get 42 and 13 times 3 I think to get 36*)	13
	Connection (e.g., *13 was 6 & 7. Six 6's are 36 and 6 7's are 42*)	14
Inverse		
5x71=355; 355??5=71	No idea (e.g., *You use the same numbers*)	4
	Calculation (e.g., *just work out this one, 5 x 71 = 355*)	9
	Connection (e.g., *Because that's times so that would be divide*)	14
	Reverse (e.g., *It's the reverse - divide because five 71's is 355*)	9
	Opposite (e.g., *Divide because it is the opposite of times*)	15
64-29=35; 35?29=64	No connection (e.g., *knew 35+29=64, didn't even look at other*)	2
	No idea (e.g., *Subtraction, because it's the same as the other one*)	3
	Calculation (e.g., *Add, not times because work into more digits*)	8
	Guess & check (e.g., *With adding it would probably get close*)	1
	Connection (e.g., *Minus and plus are kinda like partners*)	14
	Reverse (e.g., *Well if you reverse plus sum, you get minus sum*)	7
	Opposite (e.g., *Plus and minus are the opposite to each other*)	16
Order convention		
effect of swapping x and - in 3x6-2	No idea (e.g., *I don't know - Maybe*)	1
	Yes (e.g., *Yes. It would just be the same*)	17
	No (e.g., *No. Do times first, because it's a higher number*)	5
	Different answer (e.g., *Multiplying is a bigger value than minus*)	8
	Left to right (e.g., *Should go left to right unless it has brackets*)	8
	Correct use of convention (e.g., *Use that BOMDAS thing*)	12

32+(12x8)/3	No idea (e.g., *First I would do the sum 32+12x8*)	16
	Left to right (e.g., +, *x then / ... because numbers are there first*)	8
	Incorrect order (e.g., *12 times 8 plus 32, and divide it by 3*)	15
	Incomplete (e.g., *12 on top, times 8, divide 3 and then add 2*)	2
	Correct order (e.g., *12 times 8, divide by 3, and then add 32*)	10
Variable		
[]+5 (expression)	No idea (e.g., *I don't know what the square means*)	2
	Other (e.g., *Nothing plus 5, square plus 5*)	6
	Answer (e.g., *Can't work out answer, don't know what it is*)	4
	Like *x* or *y* (e.g., *It's an unknown, like x or y value, like algebra*)	2
	Unknown number (e.g., *Missing number - you put number there*)	29
	Any number (e.g., *It can be any number*)	8
[]+5=9 (equation)	Other (e.g., *Nothing plus 5 equals 9*)	6
	Solved (e.g., *That equals 4, because a square plus 5 equals 9*)	22
	Can be solved (e.g., *It means something plus 5 equals*)	5
	Like *x* or *y* (e.g., *Well it can be x or y*)	1
	Unknown number (e.g., *It means that a number plus 5 equals 9*)	17
3x (expression)	No idea (e.g., *I wouldn't have a clue*)	11
	Times (e.g., *Three x - I think of 3 times itself*)	32
	Like *x* or *y* (e.g., *The number 3 and it might be x like in the letter*)	3
	A number (e.g., *It could be like another number, like 31*)	5
x+7=16 (equation)	No idea (e.g., *I'm not sure what the x is*)	3
	Times (e.g., *Times then plus the 7 to make the 16*)	13
	Solved (e.g., *It's a mystery number which is 9, so 9 plus 7 = 16*)	7
	Like *x* or *y* (e.g., *Like the square we had, probably put a y there*)	6
	Unknown number (e.g., *x is the unknown, have to work it out*)	22

Equals. For the first series of operations, all but one student stated, in various ways, that '=' meant that you had to find an answer. One student said that it sometimes means equivalent. For the second task, 9 students explained that 60-36 was the answer and 17 that both sides of the equation were the same. The other half of the sample could not explain adequately what '=' meant in that context.

Operational laws. Commutativity for addition and multiplication was explained satisfactorily by 43% of the sample, and the inverse operations of addition/subtraction and multiplication/division were satisfactorily explained by 74% and 76% of the sample. Only 23% of the sample could satisfactorily explain the correct order of operations for the first task and 19% for the second one (which was not presented sequentially).

Variable. In the first expression, 76% of the students could adequately explain that a box meant an unknown. In the equation, 6 students could not explain the box satisfactorily, 27% either solved the equation or said it could be solved, and 35% said that it was like x or y or that it meant an unknown number that needed to be found. When asked to explain x in the linear equation $x+7=16$, 70% either solved the equation, said it was like a box or the x and y's in algebra, or said that it represented an unknown number. However, when asked about x in the expression $3x$, only 16% had an intuitive idea about the meaning of the x. Because of this difference in performance, the $3x$ was placed in an equation and the students again asked what the x meant. Again, only a low percent (17%) had any understanding.

Meaning of operations. The majority of students had a sound knowledge of the symbols and meanings of operations for addition (82%), subtraction (98%), multiplication (86%) and division (61%). Most of the students explained addition and adding or plus, although 9 students still thought of it as counting on. Subtraction was mainly explained as 'take off' or 'take away', although one student said it meant to count backwards. Satisfactory descriptions of multiplication were 'multiply' and 'lots or arrays', whilst 7 students explained it as repeated addition. Division was explained satisfactorily as 'divide' or in terms of 'times', 2 students did not know what it was, and 35% thought of it in terms of repeated subtraction.

Concrete materials. Most of the students (75%) just replaced the box in the linear equation with counters and gave an arithmetical explanation of this representation. A few students (10%) were able to give an algebraic explanation for the equation but they did not use any materials. When the box was replaced in the equation by x, approximately 50% of the students used counters and gave an arithmetical explanation, whilst, as before, another 10% gave an algebraic explanation. However, this time they represented the x by a space or a cup.

DISCUSSION

Almost 100% of the students believed that the equals sign in an unfinished equation with a series of operations meant find the answer, and only 50% of the students could say that the equals sign in a completed equation meant that both sides of the equation were the same (similar to, e.g., Behr, Erlwanger & Nichols, 1980). This means that, in subsequent learning of algebra, many students would initially want to find the answer after the sign and at least half of them would need to learn the concept of equivalence. As well, there was little evidence of equals being perceived in terms of 'either direction' or 'change' (Cooper & Baturo, 1992; Kieran, 1992)

Most of the students had sufficient understanding of the basic binary operations or sequences of binary operations for subtraction, multiplication and addition, in that order, to be able use them as a basis for algebra. Some students did not have sufficient understanding of division. Two thirds of the group had sufficient understanding of the inverses of multiplication/division and addition/subtraction. About 50% of the sample did not understand commutativity. A significant minority did not understand distributivity and, to a lesser extent, inverse. Only about 20-25% of the sample had sufficient understanding of the correct order of arithmetical operations to allow them to apply this satisfactorily to learning linear equations. There was some evidence that students were having difficulty with the distributive law as it applied to division in the last order convention task. According to the literature, any inadequacies in arithmetic will cause difficulties in algebra (e.g. Bell, 1995; Herscovics & Linchevski, 1994).

More than half the sample could solve an equation with a box as an unknown number or knew intuitively that it was like an x or y despite having no explicit

instruction in variables. The majority of students understood what x meant in a linear equation but less than a fifth of them had a satisfactory concept of multiples of x. However, their understanding focused on variable as 'unknown', which is more a prealgebra than an algebra understanding (e.g., Chalouh & Herscovics, 1988; Sfard & Linchevski, 1994). When asked to use concrete materials, most of the students used them to illustrate their arithmetical solutions. A few had an intuitive idea of algebra and did not need materials. Most of the students who used materials and gave an arithmetical answer really did not need the materials either as evidenced by their explanations for variables.

CONCLUSIONS

At this stage, the developmental sequence for the sample appears to fit well with the two-path model in Figure 1, that is, that complex algebra develops from binary arithmetic via both binary algebra and complex arithmetic (arithmetic with a sequence of operations). With respect to the components of the model, the students appeared to have some difficulties in all parts.

With respect to binary arithmetic, the students knowledge was, for the most part, satisfactory. For instance, most students appeared able to use binary operations in linear equations. However, the students studied need better understanding of division and equals. With respect to binary algebra, the students had difficulty with multiples of the unknown. They need careful and explicit instruction in the meaning of x as a variable rather than the unknown and then in the meaning of multiples of x (perhaps with the use of cups to represent variable). With respect to complex arithmetic, the students need better understanding of the order convention for a sequence of operations and to learn the equivalence meaning of equals in an equation (perhaps with the use of a balance beam analogy). They interpreted equals narrowly, as calling for an answer. Some students also need instruction on the commutative and distributive laws and inverse. As well, students' understanding of division distributivity and inverse as it applies to a sequence of operations should be foci of further research.

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EXPLORING IMAGERY IN P, M and E.

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This paper represents an exploratory formulation of theoretical aspects of the use of imagery as a tool for aiding learning and draws on the author's work with pre-service mathematics education students. The use of images in teaching particular mathematical content topics is described and then contrasted with their use in the domains of didactics and psychology. Issues of universality as opposed to personal preference in respect to the choice of images are discussed as well as possible methodological and therapeutic consequences.

1. Introduction

> Time and Space are Real Beings.
> Time is a man, Space is a Woman.
>
> William Blake

> Algebra and geometry can be contrasted by describing the paths by which we arrive at each. So we get sound, repetition and time on the way to algebra - and sight, imagery and space on the way to geometry.
>
> Dick Tahta 1995

> In general it is visual imagery that is used. But the dynamics of the mind when formalised produces all the conceivable algebras. Algebra differs from geometry in that the first describes mental dynamics while the other uses mental content, imagery.
>
> Caleb Gattegno, 1965:38

Hilbert (1952) identified two tendencies which he said illuminated the dual nature of mathematics. The one was the tendency towards abstraction, which 'seeks to crystallise the logical relations inherent in the maze of material ... in a systematic and orderly manner'. The other was the tendency towards intuitive understanding which stressed processes of visualisation and imagery. Generally schools have mainly concentrated on the former and a consequence of this has led to the claim that 'a vast majority of students do not like thinking in terms of pictures' (Eisenberg 1994). This view has been challenged by the research of Wheatley and Brown (1994) which shows that, far from being reluctant to visualise, many students use their visualisations as a tool for meaning-making in mathematics. Presmeg's (1985) study makes the point that, since much of this visualisation is of a private nature, students' imagery may not be apparent in written protocols. My own experience has been that images provide an important tool for learning.

Over the course of the past twelve years, I have been offering method of mathematics courses for preservice mathematics teachers, and I have become aware of the enormous potential for using images as a powerful starting point for offering rich learning situations. While these images initially had a strong focus on

mathematical topics and were mainly of a visual nature, I have increasingly used different types of imagery as a learning tool for exploring concepts that range widely over the varied dimensions that impact on the life of a mathematics teacher. In the following sections of this paper I will attempt to describe and reflect on this experience in order to mark out some of the considerations and issues for selection and use of images in the teaching situation.

2. Mathematical Images?

> ***Vignette 1***. *The room is darkened and the class watches a screen where a scene is unfolding. A horizontal line segment first appears and then two intersecting lines appear above the line and then drop down so that each touches one end of the line segment. The intersecting lines move around and their point of intersection traces out a circle. The film is over in less than three minutes and when the lights are on, the teacher asks the class to close their eyes and try to recapture some of the images from the film. Over the period of the next week's lessons the class discusses and debates what they saw and then begins to work on the agreed-on images.*

This lesson is using one of the films developed by Jean-Louis Nicolet, a Swiss mathematician. The methodology described is one that has been used by the author but the use of these films has been more fully described by Tahta (1981) and Gattegno (1981). The power of using the films is that the original image is simple - there is no sound track to accompany the film. The students are particularly challenged in working on what has obviously been a neglected skill - the use of imagery - but as they talk about what they saw they enter into debate and listen. The teacher's main task is to maintain their focus on what they saw. The challenge to act as script writer to give the necessary instructions to ensure that the intersecting lines trace out a circle encourages them to look for the what remains constant - the rigid framework established by the fixed angle between the intersecting lines. As they script the film, students investigate the same special cases that they will later find in their school Euclidean Geometry syllabus. Each time that I have used this film's set of images I have been struck by the enormous returns this initial investment of time and energy gives the students.

> ***Vignette 2***. *The class is invited to work on the image of a point travelling at a uniform speed around the circumference of a circle on which a horizontal diameter has been drawn. Their attention is drawn to the relationship between the moving point and a point on the diameter which is found by dropping a perpendicular from the moving point to the diameter. The teacher works with the whole class and gets participants to work on their images and to describe what they see. Necessary information is provided at appropriate stages to allow the class to move further.*

The image given here is simpler and even clearer than in the previous example, and as the class works on the image they inevitably find themselves coming face to face with the core concepts of trigonometry.

The basic images used to introduce the area to be investigated in both these vignettes seem to be powerful and universal, and getting students to work on them can only lead them to the essence of a particular mathematical topic. They are essentially economical images that give direct access to the mathematical concept and have the added advantage of seeming to be timeless and context-free, so they should be suitable for exploration by any group of people throughout history. The teacher controls and focuses discussion to ensure that each student is able to reach an inevitable conclusion. It is a powerful teaching method and a rewarding task would be to form a collection of similar 'canonical' (relating to a specific rule or concept) images.

It was my experience of the successful introduction of these images into my mathematics classes that led me to explore the use of images that had generated powerful personal insights into the dynamics of teaching into my method course material. The next section will explore two of these images and look for similarities and differences between these and the canonical images described above.

3. Educational Images?

> ***Vignette 3.*** *The students have been involved in a role play situation that concerns an incident of poor discipline in the classroom. The teacher's authority has been directly challenged by one of the class and the students have begun discussing various options that would be open to them as teacher. At this point the lecturer gets the students to break out of role and offers them an image taken from Aikido. The students get into pairs and stand facing each other. They push against each other by making contact with the fists of their outstretched right arms. They act out three possibilities. In the first they keep pushing against each other as hard as they each can and the strongest wins after a lengthy struggle. In the second, the teacher stops pushing suddenly and the student's fist strikes the teacher on the chest with force. In the third option the teacher steps aside quickly and, maintaining contact with the student's fist, allows the student's energy to pass him by. The students change roles and re-enact all three possibilities again. They discuss the implications of these options, and then return to role to discuss the teacher's options in the discipline problem.*

What sort of image is this? The image of the teacher choosing not to engage at the moment and to allow the student's negative energy to pass him by in a controlled and guided way is certainly a personally powerful image. The image also seems to travel well in that it was also used in an English university setting and the students seemed to have no difficulty relating to the image. This suggests a universality in the image. Furthermore, an unsolicited comment after the session where a student said that he had learned more from the activity than he had from the full term's lectures on classroom management testifies that at least for him the image was economical.

But does the image lead inevitably to a specific rule in such a way that it can be called canonical?

> ***Vignette 4.*** *The class is organised into pairs with partners facing each other. To the accompaniment of gentle music the students are asked to assign one partner to be the leader. The follower has to mirror the movements of the teacher. After a while the roles of leader and follower are reversed and the 'mirror dancing' activity continues to music. The next step is to continue the task, but the challenge now is to create a smooth 'dance' where the lead changes from one to the other without jumps. Again the activity and its implications are discussed in a non-focused way. The pairs are then given a remedial teaching situation to role play where the teacher tries to find the point of concept block by asking questions. They are asked to hold the last mirror dance image in mind as they work, so that, once the teacher asks the question she passes the leadership role over to the student so that the student can try to teach the teacher about his perceptions and understandings.*

This is not an easy exercise to describe but the aim is to follow Kierkegaard's (1938) advice where 'to be the teacher you have to be a learner and understand what it is that he understands and in the way that he understands it'. The image of changing roles from follower to leader has been powerful for the class particularly when the person holding the pen is described as the 'leader'.

It seems that one of the differences between the images that were offered in the previous section and the above two images is that the images were such that students in the former cases reached the same endpoint - geometric and trigonometric insights that could be shared and agreed upon. In the latter two cases presented in this section, although the images may be universal and easily accessible to the students, the interpretation of these images and conclusions drawn may be open to dispute. For example, a method student made the following comments in his course journal:

> The punching/blocking exercise also gave me problems. Is letting it flow past you really helping the other person? I prefer open confrontation between two people, not necessarily in public though. We turn the other cheek so as not to give confrontation to those who seek it, but if we love someone we will take care to instruct them in what is right. I aim to go the extra mile with them in love so as to dissolve their desire for conflict with me.

Different interpretations such as this do not seem to have diminished the impact of the image. There is a universality in these images that has provided students over the years important access to the concepts addressed. The role of the methodology used in presenting these images has been to increase possibilities for individual action rather than formulate specific 'rules' for action. A great deal is left open for personal interpretation and further reflection is encouraged, but not forced or mediated, through the use of journals. So the earlier use of the word canonical to describe the

images in the mathematical section also refers to the outcome of working on the image. The images used in this section are personal in the sense that they have been selected from the repertoire of images that have had a significant impact on the lecturer. The extent of their universality can only be determined through their use in different contexts, and the existence of alternative and possibly more appropriate images depends on the contributions and offerings of the recipients. What is also clear is that the use of an open-ended methodology with these images allows students to impose their own personal belief systems onto their acceptance or rejection of the implications of the interpretation of the images in a way that was not possible with the simpler and more directed canonical image from the mathematics section. For example, one student wrote the following in his journal.

> Love will cause you to be patient with the pupil, not to threaten and intimidate them. Love will make you strain to find more imaginative and effective Ways of teaching the children. Love will make you humble and willing to learn yourself. God is Love and yet God is a leader. Therefore I reject a leaderless state as being more desirable.

This is much more a comment on the inner world of the student himself than on the insights that the presented image was intended to provide about classroom methodology, and leads directly to the third strand of this paper.

4. Psychological Images?

> As lines so loves oblique may well
> Themselves in every angle greet.
> But ours so truly parallel,
> Though infinite can never meet.
>
> Andrew Marvell, from: *Definition of Love*

This extract from one of the metaphysical poets linking the realm of the psyche to mathematical ideas is intended to signal a challenge to the neat summary and categorisation that I have attempted to draw in the previous two sections. Part of the problem with the use of images in education that I have described above, is that they move into the messy business of life and hence the teacher approaches them more circuitously and leaves room for differing interpretations, while the earlier canonical mathematical images appear purer and allow the teacher to maintain a directed focus on the outcome of the exploration. There are various suggestions in the literature that it might not be so easy.

> ***Vignette 5****. A one-to-one situation: A girl is asked to close her eyes and to picture a screen in front of her eyes. She then pictures a circle on that screen as well as a green line which initially stands outside the circle. She is then asked to picture the green line moving from one side of the circle to the other, and is told that when the line crosses the circle it becomes red. After she has spent some time describing her images, she is asked what*

colour the line is when it touches the circle? She replies "Oh dear, it's a helpless man".

This scene, described by Tahta (1970), allows us to enter the 'real world' of a pupil and to see the way in which the line has taken on its own identity. This takes us into the world of psychoanalysis, where symbols can act as condensations for a variety of other initially hidden meanings. These thoughts, associations and feelings will inevitably vary from person to person. Thus symbols are often charged with some personal meaning that is not apparent from the initial form of presentation of the symbol. In the above case the mathematical symbols being invoked triggered such a charge in the girl.

I had a similar experience during a course at Exeter in the seventies when we were shown a film based on Poincare's work on n-dimensional space, where various differently coloured lines were moving around with increasing rapidity and chaos - only to reach some stability when a new colour (dimension) was discovered. When asked to give an account of the film, I did not immediately relate the film's contents to mathematics, but rather identified it with the frenetic and apparently hopeless struggle of the people of South Africa for freedom from apartheid. I can remember feeling rather hesitant at giving this interpretation to peers in a mathematics class, and, after it received a very subdued reception, I hurriedly moved on to try to find some more appropriate mathematical insights.

In attempting to address the same Poincare's question as to why so many people fail at mathematics, several writers have begun to explore the possibility that, in some cases, this failure may have psychological roots. Tahta (1994) quotes the work of Weyl-Kailey, a teacher and therapist, who describes a depressed adolescent for whom the answer to $5 - 2$ was always 2. When he was asked to display this problem by using his fingers, he would be unable to sustain an answer of three and had to fold another finger down. Weyl-Kailey suggests that this was a way in which he kept out of the family conflict (since 3 is associated with the family triple - mother, father and child). Pimm (1994) describes the startling evidence obtained by Melanie Klein from her comprehensive survey of the role of the school in the libidinal development of children. In analysing a particular case, Klein is led to conclude that the tendency to overcome the fear of castration seems in general to form one of the roots from which counting and arithmetic have evolved. Similarly, Maher (1994) believes that the experience of actually doing geometry is the quintessence of mathematical activity and that the affective power of geometry comes from the mirror phase of personal development (Lacan: the child's first gaze at its whole self in the mirror; and Winnicott: the child's reflection in the eyes of its mother gazing at it).

This brief sampling from literature, together with the two examples given earlier, suggests that in some cases mathematical symbols provide a trigger that may uncover hidden condensations that may be connected with disturbed incidents or unresolved conflicts. It also appears that there might be a reciprocal relationship between the symbol and its condensation.

> For Lacan, mathematics is not disembodied knowledge. It is constantly in touch with its roots in the unconscious. This contact has two consequences: first, that mathematical creativity draws on the unconscious, and second, that mathematics repays its debt by giving us a window back to the unconscious..[...] so that doing mathematics, like dreaming, can, if properly understood, give us access to what is normally hidden from us.
>
> Turkle 1992, p.240

The possibility exists that working more dynamically with geometric symbols as images, and at mathematics in general, might contribute to the development of the inner self through the resolution of psychic conflicts by symbolic means.

5. Reflecting on Canonical and Universal Images

This brings me back to the canonical mathematical images that were described in section 2. As presently set up, the students are being asked to direct a narrow and focused gaze on the symbols and to exclude all hidden meanings. The methodology of presentation specifically excludes the possibility of straying from this path. In contrast, in presenting the images described in section 3, the task is left open-ended enough for students to explore whatever response the image triggers. In the situations described this is likely to impact on their outer and inner lives. Section 4 presents some evidence that suggests the possibility that uncovered conflicts may be at the root of a student's fear of mathematics and also that work with mathematical images may enhance the development of the student's inner self. The question that needs to be asked is whether teachers need to take responsibility for possible charges that are triggered by the mathematical symbols by allowing the nature of the questions asked to be sufficiently open to allow this possibility of exploration to be regarded as valid.

In addressing this question it is important to realise that the examples in section 4 came from one-to-one situations and especially that the psychoanalytic situations were conducted by a trained professional. It is unlikely that teachers will have the training or the time to work on an individual basis with any of their students, and should thus not be expected to work explicitly on acting in a therapeutic way. However there seems to be some advantage in opening the possibilities of methodology to allow differing interpretations of images (such as in my reported Exeter experience) to be accepted as valid for contribution to class situations. In doing this, teachers will be allowing students to work on their outer and inner realities as they work with mathematical symbols. Present evidence suggests that in working this way, students may well be engaged in an implicit sense-making and healing process.

Finally, it seems as if it might be useful to re-look at the benefits of the more focused methods used in working with the canonical mathematical images when working with the educational images. It appears that there is too much freedom given for personal interpretation and there may well be canonical possibilities that need to be

more directly explored. The following student's comments on the leading/following activity stands in contrast to the earlier 'God is Love' extract at the end of section 3, and in this case the contradicting interpretation does not seem to stem from a differing belief system. It seems as if a clearer and more directed methodology would help clear up ambivalences about intended interpretations.

> The point was not to promote a leaderless state, but to demonstrate/experience a situation where one can move freely and intentionally between the leading and following states. I believe that until you are able to do this your Sunday school children will not feel comfortable in freely participating in your lessons.

Further reflection and research on the use of canonical, personal and universal images as a teaching tool is obviously called for.

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TEACHERS' FRAMEWORKS FOR UNDERSTANDING CHILDREN'S MATHEMATICAL THINKING

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This study identified frameworks in teachers' analyses of children's solutions to mathematical problems and monitored changes in framework use across the first year of implementing cognitively guided instruction. In May 1995 and June 1996, participants (22 female teachers in grades K-3) completed transcript analyses of a dialogue between a 1st-grade teacher and 3 students. Responses were categorized by 5 frameworks: developmental, taxonomic, problem solving, curriculum, deficiency. The curriculum framework was used most often, followed by the problem solving and deficiency frameworks. Across the two administrations, use of the curriculum framework increased, and use of the problem solving framework decreased.

In implementing cognitively guided instruction (CGI), teachers learn to assess students' thinking (primarily through listening to students explain solutions to mathematics problems) and then use that knowledge to plan instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Fennema, Carpenter, & Peterson, 1989). The information gathered by teachers about students would seem to be highly influenced by a teacher's frameworks for human development, curriculum, and mathematics, since these frameworks are filters for deciding what aspects of students' explanations to attend to and what aspects to ignore.

As part of a five-year inservice project to help teachers learn to use CGI, we are gathering a variety of data on teachers' beliefs, interpretations of children's solutions to mathematics problems, and instructional decision making. In this paper, we analyze the frameworks that seemed to have been used by project participants as they analyzed a transcript of interactions between a first grade teacher and three of her students. In the transcript the students explained their

solutions to word problems. In particular, we focused on changes in these frameworks from the beginning of the project to the end of the first year of implementing CGI. Earlier we reported a detailed analysis of participants' initial interpretations of the transcript (Bowman, Bright, & Vacc, 1996). In this paper we extend that analysis by identifying more explicitly the frameworks that participants seemed to bring to the task of interpreting students' mathematical thinking and by monitoring use of frameworks during implementation of CGI.

CGI has been repeatedly described and evaluated (Carpenter, et al., 1989; Fennema, et al., 1989; Fennema, Franke, Carpenter, & Carey, 1993; Fennema, et al., 1996; Peterson, Fennema, Carpenter, & Loef, 1989). Briefly, CGI is an approach to teaching mathematics in which knowledge of children's thinking is central to instructional decision-making. Teachers use research-based knowledge about children's mathematical thinking to help them learn specifics about individual students and then to adjust instruction (e.g., sequencing of types of mathematical problems, kinds of numbers used in problems) to match students' performance.

Method

Participants and Instrument

The study was conducted during the first year of a five-year teacher enhancement project (NSF Grant ESI-09450518) in which primary-grade teachers are being given opportunities to learn to use CGI as a basis of mathematics instruction. Teachers and mathematics educators from different regions in North Carolina formed five teams; each team is composed of 2 teacher educators (i.e., team co-leaders) and 6 primary-grade teachers. The data reported here come only from the primary-grade teachers.

All project participants completed a transcript analysis instrument. The instrument contains three teacher-and-student dialogues (Mac, Tom, and Sue) that occurred while a group of 23 first-grade students worked individually on 5 written problems. The teacher interacted with Mac after he had completed the problem: *If frog's sandwiches cost 10 cents, and he had 15 sandwiches, how much did frog's sandwiches cost altogether?* As the teacher moved to Tom's desk, Tom was working on the same problem. The teacher's interaction with Sue occurred as she was working on a different problem: *Frog had 15 sandwiches. If each sandwich cost 5 cents, how much do all the sandwiches cost altogether?* After reading the dialogues, participants were asked to state their conclusions about the three children's (a) levels of thinking and (b) mathematical

understanding. Participants were also asked to identify specific evidence from the transcript that was important to them in making those conclusions. No definition for the phrases "levels of thinking" or "mathematical understanding" were asked for or provided during the administration of this instrument.

Transcript analyses were completed during a morning session on the first day of the project's introductory workshop in May 1995 and again on the first day of the second summer's workshop in June 1996. The instrument was administered in a whole group setting, but participants worked individually, without discussions. Complete data were available for 22 teachers: 5 kindergarten teachers, 7 first-grade teachers, 4 second-grade teachers, and 6 third-grade teachers. All teachers were female.

Between the two administrations of the instrument, teachers participated in two formal workshops (3 days in May 1995 and 10 days in July 1995) and began CGI implementation during the 1995-96 school year. During 1995-96 each team met after school approximately once a month to discuss their progress, each teacher was visited approximately once a month during mathematics instruction by one of her team's co-leaders, and each teacher was visited during mathematics instruction once each semester by project staff. The purpose of the visits was to support teachers as they struggled with implementing CGI; visits were never used to "evaluate" teachers.

Analysis of Responses

First, content analysis on verbatim written responses was completed manually. Responses were carefully dissected, fragments grouped by content, and category labels were identified for clusters of comments. Seven categories of responses were created (Bowman, et al., 1996). Second, a variety of frameworks that might reflect the ways that teachers analyzed the transcript were considered. Evidence for each framework was discussed by the authors, until agreement was reached on the nature of evidence that would be accepted for categorizing responses according to these frameworks. Five frameworks emerged, and teachers' responses were then re-categorized according to these frameworks.

The five frameworks, along with brief quotes from teachers' responses to suggest the use of each framework, are presented below:

1. developmental (e.g., Piaget): "Mac ... solved the problem in an abstract way ... Sue needs to work on a very concrete level."
2. taxonomic (e.g., Bloom): "[Sue] has a grasp of some mathematical concepts but she is unable to apply the skills."

3. problem solving (e.g., Polya): "Mac and Tom understood what the problem was asking.... Sue does not know what she needs to find out."
4. curriculum (e.g., grade-appropriate content knowledge): "[Mac] knew to add a 0 at the end of the 15 because he was counting by 10's." "[Mac's] understanding of math goes beyond what one would expect of a first grader."
5. deficiency (e.g., lack of mathematical prerequisites): "Sue has not mastered the concept of skip counting."

In North Carolina the State Board of Education has specified mathematics objectives for each grade level. Elementary teachers are held responsible for teaching the mathematics objectives assigned to their grade levels; teachers are usually very cognizant of the objectives for their grades and for the grade that immediately follows. Consequently, the curriculum framework was expected to be commonly used in interpreting children's thinking.

Results

Teachers' Rankings of Children's Thinking

The teachers almost universally agreed that Mac and Tom exhibited higher levels of thinking and better mathematical understanding than Sue. Teachers did not always explicitly order the three students by their levels of thinking, but the comments universally referenced greater understanding by Mac and Tom than by Sue. Often, teachers suggested that Mac exhibited higher levels of thinking than Tom, though sometimes teachers' comments left the impression that they thought that Mac and Tom were at about the same level of thinking.

Teachers often defined "levels of thinking" either relatively as higher level thinker versus lower level thinker or advanced thinker versus less advanced thinker or absolutely in terms of a developmental framework. Less precise descriptions included "good thinker" and "independent thinker." In the first administration of the transcript analysis, there was consensus among the teachers that use of concrete objects necessarily indicated lower level thinking while use of mental math and visualization showed higher level thinking.

Teachers' Frameworks

The categorizations of teachers' apparent frameworks for interpretation of students' responses are given in Table 1. On the two administrations, about half of the teachers were classified as using only one framework (9 and 11 teachers, respectively) and about half were classified as using multiple frameworks (11 and 10 teachers, respectively). In each administration one teacher's response was too

brief to be classified, though the two instances were for different teachers. Because of the classification of multiple frameworks for some responses, the sum of the percentages for some grade levels is more than 100%. Interestingly, all of the grade K teachers were classified as using only one framework on each of the administrations, while all other teachers were classified as using more than one framework on either the first or the second administrations (or both).

Table 1. Percentages of Teachers (First Administration/Second Administration) Categorized for Each Framework

	Grade				
	K	1	2	3	Total
Framework	(n=5)	(n=7)	(n=4)	(n=6)	(n=22)
Developmental	00 / 00	29 / 14	00 / 00	00 / 17	09 / 09
Taxonomic	00 / 00	00 / 00	25 / 25	17 / 00	09 / 05
Problem Solving	60 / 20	43 / 29	50 / 25	17 / 17	41 / 23
Curriculum	20 / 40	57 / 86	75 / 75	67 / 83	55 / 73
Deficiency	20 / 40	29 / 43	25 / 50	67 / 17	36 / 36

In both administrations, the curriculum framework seemed to be the one most often used by teachers, with the problem solving and deficiency frameworks next most often used. The general "psychological" frameworks were not frequently used. From the first to second administrations, the curriculum framework was more frequently used, and the problem solving framework was less frequently used. The deficiency framework was used about equally often in the two administrations. In each of grades K, 1, and 2 the teachers collectively exhibited use of the same frameworks in both administrations.

Twelve of the teachers were classified as using a common framework in both administrations. In 9 instances the common framework was curriculum (one at grade K, three at grade 1, two at grade 2, and three at grade 3), in 2 instances the common framework was problem solving (one at grade K and one at grade 1), and in 1 instance the common framework was developmental (grade 1).

From the first to second administrations, most teachers either "gave up" or "took on" one or more frameworks; that is, for each teacher, a framework identified in the first administration was not identified in the second administration, or a framework identified in the second administration was not identified in the first administration. It must be remembered that because of the

identification of multiple frameworks, some teachers were classified as giving up or taking on several different frameworks. In particular, 1 teacher gave up the developmental framework and 1 teacher took it on; 2 teachers gave up the taxonomic framework and 1 teacher took it on; 7 teachers gave up the problem solving framework and 3 teachers took it on; 3 teachers gave up the curriculum framework and 5 teachers took it on; and 8 teachers gave up the deficiency framework and 8 teachers took it on. Particularly striking is the fact that none of the teachers who used the deficiency framework on the first administration used it on the second administration.

Discussion

The categorization of frameworks undoubtedly underestimates the actual use of multiple frameworks by teachers. The classifications were made only on the basis of what the teachers wrote; their thinking is certainly more complex than their writing. We are unsure how to interpret the fact that the only teachers who were classified as using a single framework on both administrations were exactly the teachers in grade K. Perhaps this reflects the limited amount of content specified for grade K in the North Carolina curriculum. Further investigation is warranted on the relationship between use of frameworks and teachers' knowledge about curriculum.

Not surprisingly, teachers focused most frequently on the curriculum framework; this is the framework that is most likely to be familiar to all teachers in North Carolina. The "stability" of the use of the curriculum framework across administrations (i.e., 9 of the 12 teachers who used this framework on the first administration also used it on the second administration) may reflect the deep familiarity of the teachers with the state curriculum. Further study is needed about the stability of use of frameworks and teachers' understanding of the knowledge underlying each framework.

The problem solving framework can, at some level, be thought of as being related to teachers' view of curriculum, at least as instantiated in textbooks. Over the past 10-20 years, many elementary school textbooks have used a set of "stages" for problem solving derived from Polya's (1945) four steps for problem solving: understand the problem, devise a plan, carry out the plan, look back. It is not surprising, therefore, that teachers would think at least initially about these stages in trying to interpret students' solutions to word problems.

It is encouraging, however, that across the first year of implementing CGI, teachers would put more emphasis on students' demonstration of understanding of

specific content (e.g., as instantiated in the state curriculum) rather than on generic stages of problem solving. Teachers in the project seem to be trying to understand the specifics of students' solutions rather than generic thinking skills. It would be interesting to study the relationship between teachers' knowledge of their students and the teachers' use of particular frameworks.

The use of the deficiency framework might also be interpreted as supporting the development of teachers' skills at understanding specific mathematical thinking of students rather than generic thinking. But we are not able to interpret the fact that the teachers who used the deficiency framework on the first and second administrations are totally different teachers.

In light of the small numbers of teachers in the grade-level groups, the differences in collective use of frameworks across these groups are probably not important. It is possible, however, that there may be some influence of the increasingly broader range of content objectives from grades K to 3.

On average, each teacher either gave up or took on almost two frameworks; the range in number of changes was from 0 to 4. This seems like a substantial amount of change and might reflect the changes in philosophy of mathematics teaching that project participants are likely to have made during their struggles to implement CGI (e.g., Fennema et al., 1996). It would be interesting to investigate whether similar changes in use of frameworks might occur either independent of inservice programs or along with other kinds of inservice programs.

We will continue to follow the project teachers' interpretations of the classroom dialogue. One pattern that we have noticed in their explanations is that in the second administration there were a variety of "prescriptive" comments, some related to the children and one related to the teacher.

> "[Tom] got the right answer but could use more experiences with money."
>
> "Sue needs more practice with simpler problems using 10's or 5's."
>
> "Teacher should have waited to see what she [Sue] was going to do."

These kinds of comments were completely absent in the first administration. We expect that as the project progresses, more and more often teachers will "spontaneously" move from assessing children's thinking to prescribing needed changes in instruction. Further, we expect that teachers will increasingly base their assessment on identification of specific mathematical strengths and weaknesses of the children. This suggests that there might also be an increase in their use of the curriculum and deficiency frameworks.

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THE STORY OF SARAH:

SEEING THE GENERAL IN THE PARTICULAR?

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Using an enactivist methodology (Reid, 1996) one classroom incident, Sarah's story, observed through our work as teacher-researchers, is repeatedly analysed. The paper tells the story, through three critical stages, of the developing complexity of our theories-in-action (Schön, 1991) over a period of 18 months. These theories-in-action are related to the way in which teacher and student purposes (Brown and Coles, 1996) act as organising foci through which intuitive ways of knowing (Bruner 1974, Fischbein 1982, Gattegno 1987) are accessed. The parallels between our learning, as teacher-educator and teacher, and the learning of our students are marked. As 'narrative authors' (Clandinin and Connelly, 1991) we aim to share our particular re-tellings of experience; the general can be found through the active participation of the reader through resonance and re-working of these stories in their own practice.

In this paper we will use one classroom incident, 'Sarah's story', as a vehicle to explore our developing theories-in-action (Schön, 1991) as we, a pair of teacher-researchers, worked together and kept returning to this incident in our reflections over a period of 18 months. We work within what Bruner (1990) called a 'culturally sensitive psychology':

> '(which) is and must be based not only upon what people actually do but what they say they do and what they say caused them to do what they did. It is also concerned with what people say others did and why ... how curious that there are so few studies that (ask): how does what one does reveal what one thinks and believes.' (p16-17)

As teachers in a classroom, so much of our behaviours in response to what the children say and do are seemingly automatic and yet deal with a complex space. Our work looks at the detail of practice, what we do, using the strategy of giving 'accounts of' (Mason, 1994) significant incidents for us and reflecting on, or accounting for those incidents to probe our motivations and implicit beliefs and theories (Claxton, 1996). Here, the paper tells the story of our developing theoretical frames which help us not only to articulate our interpretations of events but also to work on our practice. Mason (1996) stresses our sense of the transformative aspects of our work:

> 'the overt product of research is some supported assertion(s). A covert product of research is a transformation in the perspective and thinking of the researcher. Undoubtedly, one of the most significant effects of any piece of research in education is the change that takes place in the researcher.' (p58)

It is with these changes that the story which we tell in this paper is concerned. Alf Coles is a teacher of mathematics to 11-18 year olds in the UK and Laurinda Brown a mathematics teacher-educator. We work together in Alf's classroom, each with our own questions and agendas to which we are true and use an enactivist methodology where:

> 'we work with a common collection of data, about which we each reach conclusions related to our own interests and theories ... The analysis of data in enactivist research can also be seen as a process of coevolution of ideas. Theory and data coemerge in the medium of the researcher. The necessity of theory to account for data results in a dialogue between theory and data, with each one affecting the other. As enactivist researchers we attempt to make use of this interaction to transform the analysis of data into a continual process of change and encourage this process as the mechanism of our own continuing learning.' (Reid, 1996)

Laurinda's questions are concerned with ways of working with teachers in training and those new to the profession to develop effective practice. Alf's questions are concerned with developing his own practice as a teacher and the learning of mathematics of the students in his classroom. Inevitably there are many overlaps and parallels in our concerns and developing theories. These commonalities interest and motivate us. We use, for ease of the reader of this paper, the first person singular where classroom incidents are described and where there are strong resonances in our work.

From looking at one instance of a mathematical pattern it is possible to see the general structure but perhaps not be aware of how to generalise through an awareness of sequences and patterns. As teachers we recognise that we have more connections available to us within the mathematics of the classroom than our students. We see our task as increasing the complexity of the mathematics available to the students within the problems on which they work. It is important therefore to be aware of what is *not* available to them and work with that. In the same way we offer our interpretations, in increasing complexity, of one classroom incident. These interpretations have developed through time in relation to awarenesses of patterns within our teaching and are therefore general for us even though we have chosen not to share a range of other supporting stories. We are exploring the possibilities inherent in the assertion (Brown, S, quoted in Pimm, 1987) that:

> 'One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children. Our preoccupation with replicability and generalisability frequently dulls our senses to what we may see in the unique unanticipated event, that has never occurred before and may never happen again.' (p194)

What follows, after a brief background to the incident, is an account of what we call 'Sarah's story'. We then give three interpretations of this story to tell another story, through three critical stages, of our developing theories-in-action, our 'transformations in perspective' over 18 months. This is how we talk about our practice. As a reader, what of this is true for you? Certainly take some time before reading our interpretations to work on where the resonances lie for you in reading the story and what you are reminded of from your own experiences and practice. What ideas stay with you from the interpretations? What generality can you see in these particulars?

Background

The mixed ability class of 11 and 12 year olds were in the middle of an investigation related to perimeter and area. They had all started with the problem of finding the rectangle with the largest area, given a perimeter of 12cm. Having solved this starter problem students were encouraged to try other perimeters, begin looking for patterns and start generating and working on their own questions. At the beginning of one lesson we shared what they had found out so far:

- The largest area for any different perimeter is a square.
- To draw a rectangle with a perimeter of an odd number you must use halves.
- Odd sides means an odd area.
- Even sides implies an even area.
- Divide the perimeter by 4, then times by itself, what you get is biggest area.

These statements were written on the board as they came and no explanations were asked for or given. When the list was completed, I added that I had been thinking about the first question and wondered whether it was possible to find the perimeter of a square when the area was 50cm^2. There was an invitation to stay with what they were working on or incorporate any of these ideas into their investigations. The class continued to work individually or in small groups.

Sarah's story

After some time Sarah, who had been working on the 50cm^2 problem, came up to me stating that the perimeter must be 4cm. I drew a square with area 1cm^2 on the board with 1s marked around the perimeter (see Fig 1) and waited for some response. When none came I asked her how she had worked the 4cm out. Sarah talked about 'reversing the rule divide the perimeter by 4 then times by itself to get the biggest area'. I started writing the flow diagram for this as she spoke and Sarah reversed 'times by itself' to 'divide into itself' and 'divide by 4' to 'times by 4'. This gave 50 / 50 x 4 = 4.

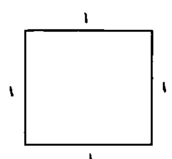

Fig 1

I said, 'OK, times by itself' and wrote: 4 $\rightarrow$

She quickly replied '16' having had experience of writing functions in this way, and I wrote 16 next to the arrow, followed by: 6 $\rightarrow$

The answer '36' again came quickly. I offered: $\leftarrow$ 49

She responded with 7 immediately. We agreed that this had not been 'divide into itself' but what was it? She went off to work on this.

Interpretation 1: It's not the answer that is wrong!

Why did Sarah come up to me with her offer of 4cm for the perimeter of the square of area $50cm^2$? The question was posed by me and perhaps Sarah was wanting to show that she had a solution? Or perhaps she was aware that the perimeter could not be 4cm and she was wanting to sort this out? We cannot know. We are not concerned with Sarah's possible motivations. What we keep coming back to is what our actions as teachers say about what we think and believe.

Reflecting on what happened I did not dismiss Sarah's offer of 4cm as being wrong, nor tell her what the answer should be, nor immediately try to understand her thought processes. I offered a square with perimeter 4cm (Fig 1) which was not responded to. I also did not try to find out why Sarah did not respond. It would be possible to tell stories about why Sarah said nothing. Perhaps she was already aware of this conflict? Perhaps she could not connect my image with her way of getting to the answer of 4cm? I let this go since even a directed question to try to get Sarah to work out the area of the square with perimeter 4cm seems to be taking Sarah further away from her own thoughts.

Why not ask straight away how she had got 4cm? The initial offer had come from sharing my own understandings of what a square of perimeter 4cm would look like. This was creating a conflict for me in that how could a square of perimeter 4cm have an area so much greater than one? My prime interest is not in how Sarah got 4cm at this stage. In working with students interactively, if I recognise that I do not accept what has been said I share what seems to be creating my problem. Given that this image did not take us any further I needed some more information so that another offer might be possible. I now asked how she had got 4cm. This provoked an energised statement from her 'reversing the rule *divide the perimeter by 4 then times by itself* to get the biggest area' which seemed to be saying 'This has to be true - even though it feels odd'.

Here the student is convinced of their world. I have something to offer again which comes from my awareness that times by itself and divide into itself are not inverse operations. The offer comes in a form where I am not telling. I am offering objects in the world for her to adapt to. She has a strong awareness of inverse (doing and undoing) but, as it turned out, needed to extend those ideas to cope with squared functions on the domain of positive integers forced by the context. At a later stage she might meet a context where ←49 might demand a different response.

We have developed a practice of never commenting on answers as being wrong since they have arisen from the student making sense of their world. We work with the process offering more complexity for the student to adapt to in some way which can include our images which are in conflict in some way with theirs.

Interpretation 2: 'It's not the answer that is wrong!' as a purpose

What is motivating Sarah to come and interact with me? How do I know what to respond?

No two events or responses are ever quite the same in the classroom. In contrast, when I begin to work on a new piece for the piano I first attend to the fingering in detail and practise difficult transitions. Each time I play the piece the fingering will be the same; eventually the fingering is automatic. There is little in the detail of our practice of the teaching and learning of mathematics which is exactly repeatable in this way; no one has come up to me with Sarah's question before.

What does seem repeatable is on the level of what we have come to call 'purposes' (Brown and Coles, 1996). For instance, at the start of a topic or theme how can we find out what the individual students in our class know and where they find problems so that we can make decisions about what to offer? We, as teachers, would have a whole variety of possible strategies which we could choose to adopt to carry through such a purpose. Which one we use would depend on the individual circumstances of the class.

In Sarah's story we recognise our intention of not commenting on answers as being wrong and the consequent actions as a purpose. We are not simply telling what is right from our own viewpoint but are moving away from the right/wrong dichotomy into something richer and more complex. Purposes help us to deal with the decision-making necessary in the face of the complexity of the classroom. I did not know that these particular circumstances would arise, but, working through a filter of 'it's not the answer that is wrong!' allowed me to be aware in the moment of my behaviours in the face of Sarah's statements. The purpose is the distillation of a complex web of intentions, thoughts, past experiences and actions which inform my practice. In the moment I am staying with what Sarah is saying and responding to that.

In preparation for the lesson in which the Sarah incident occurred we had been working with the purpose of 'sharing responses'. Alf, as a teacher, was interested in extending his repertoire of strategies to explore the richness of responses among the students. Laurinda was working with how a model for teacher development, related to purposes, functioned in allowing Alf to work on this. As the students in the story worked on their own questions, or ones which had been written on the board, they too were making decisions about where to take their investigations. They were exploring an increasingly complex field of ideas about area and perimeter.

These parallels were striking for us. Sarah was working with energy on this mathematics. She was 'mathematising' (Wheeler, 1975). Her purposes seemed to be related to sorting out relationships between area and perimeter at a global level but, as the story proceeds, within that purpose she is identifying other aspects to focus on such as 'what's the perimeter of a square with area $50cm^2$?'

and 'what's the opposite of times by itself?'. What could be motivating Sarah is the fact that these are her questions which she has chosen to engage with within the wider purpose set to the class by the teacher.

There is a parallel between our ability to move between purposes such as 'sharing responses' or 'it's not the answer that is wrong!' related to the range of behaviours to carry through those purposes in action and Sarah's ability to move between her purposes within mathematics and accessing the techniques which she needs. What seems important for all the purposes for teachers and students is the way they motivate learning. They are removed in some way from the current action but provide an organising strand, often over a long time, for learning through experience and also support the decision making necessary for the individual to act in the moment.

Interpretation 3: 7 is instinctual

How can students and teachers in these new situations make decisions so fast? Sarah responded with 7 immediately. I offered the square (Fig 1) and moved into working with the squaring function immediately. When we first started to talk about Sarah's story we referred to the response of 7 as being instinctual. What allows us to act so quickly in the face of complexity? There is time for neither analysis nor reflection.

We kept returning to thinking about Sarah's response of 7 whilst reading about intuition and analysis. We recognise two uses of intuition as discussed by Bruner (1974) and Fischbein (1982). One use develops through previous learning experiences (including analysis) where frequency of exposure educates a seemingly automatic response. We have come to call examples of this use 'educated intuitions'. For instance: The shop assistant asked for 5p to help give change. I reached into my pocket and took out, by feel, a coin and gave it to the man who registered surprise. I then noticed that I had given him a new 10p coin which was similar to a 5p coin which had gone out of circulation over ten years earlier! Those experiences of handling the old 5p coin had created an educated intuition which was still available, inappropriately in this case because of a change in the environment. Another example would be *finding myself* dropping a perpendicular to a line from a point when solving a problem. This automatic behaviour has arisen through recognition, over time, of its importance in a range of contexts and a gradual condensation (Fischbein, 1982) through use.

Sarah's 7 is not an educated intuition. There is still work to be done. She had not said 7 in response to 49 by using 'divide into itself'. It was not a condensation of past experience, but seems to have been provoked by the experiences immediately preceding its articulation. The response was, nonetheless, immediate. She might have some educated intuitions related to multiplication facts which were used in the moment, but she now has something to work on in relation to questioning 'divide into itself'. We recognise this as an example of the second use of

intuition which provides motivational energy, whether in the sense of Bruner's (1974) 'invitation to go further' or Fischbein's (1982) 'global perspective' or our preferred working definition of intuition as a way of knowing from Gattegno (1987):

> 'needed when encountering complexity, and one wants to respect it, to maintain it ... Once we become aware that we can function as an intuitive person, we find that all (other ways of knowing) are renewed and capable of serving us as they never have before.' (p79)

It was as if Sarah surprised herself by her response and then went away to work through what she had done using analysis and other ways of knowing. She knew something she did not know she knew.

My response to Sarah's first statement that the perimeter of the square was 4cm was immediate and in the moment. In parallel with the story above there was no time for reflection or analysis. I knew that which I do not know that I knew. The purpose of 'It's not the answer that is wrong!' acts as a mechanism through which the intuitions, both holistically and through educated intuitions, can operate. My attention being on looking for that which I do not understand or which makes me uncomfortable when talking with a student is a necessary precondition to noticing what is there and then acting. Over time the behaviours happen seemingly automatically. The decision-making is simplified whilst I am respecting the complexity of the situation, trusting that I will act. I become an intuitive practitioner both acting in the experience and learning from it since I am allowing the world to change me as much as I change the world.

Damasio (1994) (thanks to David A Reid for pointing me in this direction through an unpublished paper) writes on how complex decision-making is facilitated by feedback from the brain's emotional centre. We are beginning to explore links with this work and the relationships between intuitive and analytical ways of knowing. The crucial elements here are the speed at which the decisions happen and the richness of possible responses developed through experience. We currently have an image of purposes supporting our decision-making as intuitive practitioners in the classroom allowing us to work with and in the complexity through adaptation to and from experience.

Conclusion

What can be seen in general from this particular will depend on your resonances. We have offered our interpretations, but the 'Story of Sarah' and the metastory of our developing theories-in-action now have an independent existence. You will, perhaps, notice and work with different things to where our own attention lies and we hope you will find your own stories.

We have developed an image of an intuitive practitioner who is able to subordinate their own (teaching) purposes to the learning of the students. It is this subordination that allows the

accumulation of experience which can condense into educated intuitions and is a continuous process of learning for us as teachers. The research process of interacting theory and data with the re-tellings of the stories from our practice over time is essential to our work since in itself the creation of narrative forms part of that holistic sense which drives us forward and allows us to act.

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EFFECTIVE TEACHERS OF NUMERACY IN UK PRIMARY SCHOOLS: TEACHERS' CONTENT KNOWLEDGE AND PUPILS' LEARNING

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Abstract

This paper reports on part of a study examining the links between teachers' practices, beliefs and knowledge and learning gains in numeracy with pupils aged 5 to 11. From a sample of 90 teachers and 2000 pupils, 18 teachers.were selected for case study. Data on teachers' content knowledge was gathered through concept mapping interviews, lesson observation and questionnaires. Most teachers had an adequate level of subject knowledge but the most effective teachers provided explanations which were both conceptual, rather than procedural, and connected with more diverse meanings and representations.There was a negative relation between effectiveness and level of formal mathematical qualification, and a positive relation with extended professional development in mathematics teaching.

1. Aims of the study

This paper reports one area of the results of a 16-month study *Effective Teachers of Numeracy*, funded during 1995/6 by the UK Teacher Training Agency.

The aims of the study were:

1 to identify what it is that primary teachers know, understand and do which enables them to teach numeracy effectively;

2 to suggest how the factors identified can be more widely applied.

The area of the work reported in this paper is that concerned with teachers' mathematical content knowledge. Other aspects of the work are reported by Askew *et al.* (1997).

2. Related Theory and Research

A broad definition of numeracy (or 'number-sense') was used:

> *Numeracy is the ability to process, communicate, and interpret numerical information in a variety of contexts.*

Shulman (1987) included subject content knowledge as one aspect of teachers' understanding and knowledge that impact on practice. Many of the findings in this area present a deficit model of teacher knowledge. For example, in terms of mathematical content knowledge, research shows that many teachers' own mathematical understandings are limited (Wragg, Bennett &Carré, 1989; Kennedy, 1991; Bennett & Turner-Bisset (1993); Aubrey, 1994). On the basis of such findings it has been argued that improving teachers' own mathematical knowledge base will lead to better teaching (Alexander, Rose, & Woodhead, 1992).

While this may be a logical conclusion of such research, there appears to be little research to support this conclusion in practice: research may demonstrate that teachers with limited mathematical knowledge are not very effective, but there is scant evidence that teachers with sound mathematical knowledge are actually more effective. Where evidence for the importance of mathematical subject knowledge is presented it tends to be based on the effect on classroom practice rather than pupil outcomes (e.g.Bennett & Turner-Bisset, 1993). However Leinhardt, Putnam, Stein, & Baxter (1991) in their analysis of good and poor mathematics teaching concluded that subject knowledge impacted in several ways. Teacher's mental plans for lessons were dependent upon their familiarity with the content to be taught (c.f. Borko, Livingston, McCaleb, & Mauro, 1988) and the questions asked and explanations offered to pupils reflected the teachers subject knowledge.

Other research has suggested that it may be more important to have a sound grasp of pedagogical content knowledge than subject content knowledge (Carpenter, Fennema, Peterson, & Carey, 1988), or that beliefs about the nature of the subject are more influential than mathematical subject knowledge per se (Thompson, 1984; Cobb, 1986; Lerman, 1990).

3. Sample

It was important in selecting teachers who were believed to be effective teachers of numeracy to use criteria which as far as possible were based on rigorous evidence of increases in pupil performance.

From an initial sample size of all the primary schools in three local education authorities covering areas with different socio-economic characteristics(some 587 schools), together with Independent (private) schools, we selected eleven schools, providing a sample of 90 teachers. We selected the majority of these eleven schools on the basis of available evidence (national test scores, IQ data, reading test scores and baseline entry assessments) suggesting that the teaching of mathematics in these schools was already effective.

However other schools were selected to provide a contrast, some being indicated by the data to be of average or weak effectiveness.

18 teachers were selected as *case-study* teachers, and detailed data was collected about these teachers by interview and observation. It was intended that the *case-study* teachers would be selected as the most effective teachers. However because of time constraints the selection of *case-study* teachers had to be done before the final pupil data was available, and therefore on the basis of recommendations from the headteacher and local advisory staff. When the pupil data across both sets of schools was analysed it became clear that the *case study* sample contained both more and less effective teachers of numeracy.

4. Methods of data collection and analysis

Pupil Attainment The identification of effective teachers of numeracy was based on rigorous evidence of increases in pupil attainment.

The pupil assessment instruments had to be both appropriate for a wide range of attainment and age, and take a broad view of numeracy. A series of three 'tiered' assessments were developed: many items were common to two tiers and some to all three to enable comparisons across year-groups. All the tests were designed to be verbally administered to the whole class by the class teacher.The specially devised tests were based on earlier work at King's College (Denvir & Brown, 1987).

More than 2000 pupils in the classes of the 90 teachers were tested twice, with an interval of 6 months between the testing. Marking, coding and analysis was conducted centrally by trained personnel. The mean gains for each class, adjusted to compensate for ceiling effects, allowed the relative effectiveness of teachers of each year-group to be compared.

Teacher content knowledge An understanding of the teachers' numeracy subject knowledge was built up from data from three sources

- questionnaire data from the full sample of 90 teachers;
- profiles of mathematical subject knowledge for the 18 *case study* teachers from *focus* schools, arising from *concept mapping* interviews, and to a more limited extent from other interviews;
- observations of 54 mathematics lessons with the 18 *case study* teachers.

Each of the 90 teachers provided information about their level of qualification in mathematics as part of a background questionnaire. All primary teachers trained over the last 20 years have been required to attain a grade C or more at GCSE (taken at age 16). In addition some teachers had specialised in mathematics to age 18 (Advanced Level), some had specialised in mathematics as part of a degree in education, and some had obtained mathematics or science degrees prior to teacher training.

Some teachers had also completed in-service diplomas or certificates as a result of extended courses of professional development (the equivalent of 10 days or more) in mathematics education.

While qualifications can give some broad indication of mathematical competence, for many teachers this is not a reliable measure as it is not a valid assessment of their understanding of basic numeracy concepts, and in any case often reflects achievements gained many years previously.

It did not seem either appropriate or helpful to give teachers a 'test' on numeracy; since what was of interest was less their formal ('decontextualised') knowledge than their 'craft' ('contextualised') knowledge i.e. how they were able to deploy content knowledge in planning and in teaching numeracy.

It was also felt that any test would have unfairly discriminated against the teachers of younger pupils; teachers of older pupils preparing for national tests would be much more familiar with the type of 'test item' normally used in numeracy. It would also have been only too easy to set off a 'panic' reaction.

Observation provided information which was comparable between teachers in the sense that it related to what they were doing with pupils. This data was analysed using ethnographic methods.

Nevertheless it was thought that it would be valuable to have some data on teachers' more global understanding and knowledge of numeracy that was consistent in covering the same areas with all teachers. It was therefore decided that the most appropriate method would be an interview which would allow teachers to talk informally about how they understood numeracy. This was called the *concept mapping* interview.

During the *concept mapping* interviews, using a method similar to that of Leinhardt (1990), the 18 case-study teachers from the focus schools were asked to propose mathematical ideas which they considered to be important in numeracy (e.g. fractions, multiplication, estimating areas), writing each one on a card. They were then asked to draw a diagram showing how these concepts (supplemented where necessary by some suggested by the researcher) linked together, and also to explain the nature of the links. Interviews were taped and transcribed.

The method of analysis was partly quantitative and partly qualitative. It emerged that two distinct aspects of numeracy subject knowledge needed to be given attention:

- knowledge of *content* - knowledge of facts, skills and concepts of the numeracy curriculum, e.g. knowing what a median is and how to calculate it;

- knowledge of *relationships* - knowledge of how different aspects of the mathematics content relate to each other, for example, the relationship between decimals and fractions.

From the list of ideas that the teachers produced and the way they grouped them together, two measures of knowledge of the *content* of mathematics related to the teaching of numeracy were developed:

- *fluency* - the number of valid numeracy concepts suggested (the range given by teachers varied between 12 and 22);

- *scope* - the breadth of the teacher's vision of numeracy, measured by the number of broad aspects of numeracy touched on, e.g. whether the concepts given cover aspects such as the meaning of operations, methods of calculation, estimation, measurement, etc. (teachers volunteered concepts from between 5 and 10 different aspects of numeracy).

Three measures were used for the way teachers identified *relationships* between aspects of numeracy subject knowledge:

- *links*- the number of legitimate links proposed between concepts, for example merely indicating that there is a link between fractions and decimals (between 12 and 23 links were noted by different teachers)
- *explanation* - the percentage of links that were at least to some extent explained; for example stating that both decimals and fractions are 'just ways of demonstrating parts of a whole' but choosing not to elaborate further
- *understanding* - the percentage of links well-explained. Typically this included a key relationship (e.g. inverse operations) and/or a number of aspects which were relevant
- *depth* - the percentage of links which are explained in conceptual terms rather than being only procedural (rule-based).

5. Results and Discussion

A teacher's own subject knowledge is clearly an important aspect of a primary teachers' competence in teaching numeracy. However exactly what aspects of a teachers' knowledge made a significant difference in terms of pupil gains was much harder to identify than was anticipated. Certainly it turned out to be nothing as straightforward as the level of qualifications, or the fluency with which teachers could list ideas which contributed to numeracy.

(a). In terms of *adequate* understanding of mathematical concepts there was little to distinguish between the case-study teachers. Some teachers were uncertain in regard to specific items of numeracy knowledge, but this was either at levels they were not teaching or in non-fundamental areas; either way there was little evidence that this would do clear damage to children's numeracy standards.

There was very little relationship between the effectiveness of teachers in terms of the mean gains made by their pupils and the teachers' performance on the concept mapping interview. Teachers were generally able to give a reasonably comprehensive list of the key ideas in numeracy, to relate them appropriately and to provide at least some element of correct explanation of these relationships.Only in relation to the *depth* variable was there any distinction (see next section).

Nor was there a strong relationship overall between any of these variables and the year groups taught by the teachers. Although some of the best performances were by some of the teachers of younger pupils, there was some tendency for early years teachers to do less well on the *explanation* and *understanding* variables i.e. the number and quality of the explanations for the links were less strong. This was almost always because of weaker contributions in areas where the ideas were more advanced than those they were currently teaching.

In none of the 84 lessons were there any significant mathematical errors made by teachers, and in only two were there occasions when teachers found themselves to be clearly limited by their knowledge. Although these examples did demonstrate

gaps in subject knowledge, neither would seem to be especially damaging or difficult to retrieve. Most teachers admitted that there were sometimes questions to which they did not know the answer, but that they had people whom they could ask, or books they could refer to. The more confident teachers were unashamed about losing face, and made a positive learning opportunity of it, encouraging pupils to find out the answer before they did.

Similarly in the interviews, no mistakes were made but two of the 18 teachers (both teachers of 7-year-olds) confessed that they could not immediately remember how to convert 1/7 to a decimal. However despite the panic it caused them, they both felt confident that they would be able to recall or work out a method given more time.

(b) Effective case-study teachers whose pupils made large gains tended to demonstrate deeper understanding of the links between different numeracy concepts than other teachers in that they both gave a high proportion of conceptual explanations and gave explanations which connected with more alternative meanings and representations.

The group of most effective teachers were almost all classified as *connectionist* teachers in terms of their beliefs and practices (see Askew *et al.*, 1997, for details). In their classroom practice, and in their justification of this in interviews, they tended to prioritise pupils' ability to relate and select from different mathematical ideas, and different representations of each idea. Similarly in the concept mapping interview they were happy to elaborate on their explanations of why they had linked different ideas, trying to identify a variety of meanings, although their explanations were often rather incoherent and not necessarily mathematically sounder than those of other teachers.

Teachers who were least effective in our case-study sample could classified as belonging to one of two styles in terms of their beliefs and practices, *transmission* and *discovery*. *Transmission* teachers gave a higher proportion of superficial procedural explanations than conceptual explanations in relation to other teachers, which was unsurprising in relation to their procedural priorities in the classroom. *Discovery* teachers did not differ significantly from *connectionist* teachers on the overall quality of their responses as indicated by the variables described, but tended to give less elaborated responses in terms of alternative meanings and representations.

The multi-faceted nature of the meanings and uses of concepts in numeracy are what makes the teaching of numeracy challenging, and it is the knowledge and awareness of these that appears to distinguish between the most effective and the least effective teachers.

c) There was a negative correlation between the level of teachers' formal mathematical qualifications and their effectiveness in terms of pupil gains. However there was a strong positive relation between attendance at a course

of extended professional development in mathematics education and teacher effectiveness.

One reason why higher mathematics qualifications did not appear to improve performance was that what the teachers had learnt at higher levels of mathematics was irrelevant, in that it was too far removed from what they had to understand to teach effectively. Some of those with high qualifications demonstrated a lack of ability to explain connections between low level concepts. Some also had negative attitudes to the subject which they attributed to very procedural approaches at more advanced levels of mathematics.

In contrast, teachers who had been on extended professional development courses spoke about their realisations during these courses about different strategies and representations used by pupils, and the enthusiasm such courses had given them for mathematics.

Informal conversations with more knowledgeable contacts also appeared to improve effectiveness.

6. Implications

Lack of evidence of any positive association between formal mathematical qualifications and pupil gains should not be interpreted as suggesting that mathematical subject knowledge is not important. What would appear to matter is not the level of formal qualification but the nature of the knowledge about the subject that teachers have.

Ball (1991) argues that correctness, meaning and connectedness are requirements of teachers' mathematical subject content knowledge for teaching mathematics for understanding. Although not corresponding exactly to our categories there would appear to be some similarities.

One implication of this is that teachers do not need additional mathematical knowledge. More is not necessarily better in terms of helping pupils understand mathematics. Rather, primary schools teachers may need to develop a fuller, deeper and more connected understanding of the number system in order to effectively teach numeracy. This would include the multifaceted nature of meanings and applications of mathematical elements and operations and their many representations, and in particular the use of different representations of the same concept and the same representation used with different meanings.

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METAPHORICAL THINKING AND APPLIED PROBLEM SOLVING: IMPLICATIONS FOR MATHEMATICS LEARNING

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Abstract

The notion of metaphor is at least as remote as the works of Aristotle and since then it never ceased to generate controversial judgements about its nature. Today those who place metaphors at the core of our conceptual systems are still fighting the idea that metaphors are ornamental devices of little or no importance. From an educational perspective it is undeniable that any attempt to escape from metaphors is condemned to failure. This is also the case in mathematics education.

This paper addresses the value, the nature, and ultimately the need for metaphor in a kind of pedagogical scenario where mathematization and applied problem solving stimulate the production of mathematical meanings. Drawing on a modelling situation from micro-economics, we intend to stress the fact that connecting mathematics with real phenomena involves the creation of nets of metaphors. Mathematical models, we argue, are just the formal surface of metaphorical matrixes.

Introduction

Among the many semantic aspects involved in mathematics teaching and learning, the concept of metaphor is becoming a strong pole of attraction for those who strive to understand the ways in which people rely on their conceptual systems to come up with meanings for mathematical ideas and processes.

It has been long since metaphor found a place of its own as an object of study and critique in a number of different subjects but its power of informing and feeding the philosophical and scientific paradigms only recently has started to be appreciated (Santos, 1994).

The contemporary views about the role and importance of metaphor, particularly those that are embraced by some new approaches in the cognitive sciences, can not deny the fact that they have inherited a long and rich history that goes back to the times of Aristotle.

This is why we often find in contemporary works on the nature of metaphor a clear preoccupation in overcoming the rhetorical and aesthetic aspects of metaphor, thus denouncing a need to stand aside from a certain tradition where metaphor is depreciated. After all, there are good reasons for associating metaphor with the fields of poetics, literary ornaments and linguistic refinements. Metaphor has been pushed to the realms of Poetics and Rhetoric by Aristotle himself and for centuries the use of the word metaphoric was tied to the opposite of truth and rational argumentation.

Such a long standing tradition in western philosophical thinking has contributed to some of the present suspicions about the value of metaphor. In fact, according to Santos (1994), the contemporary rehabilitation of metaphor has not yet been set free from some Cartesian reminiscences where concepts are supposed to have a clear and universal content, which is built apart from all experience and history.

To be aware of this intellectual heritage should then places us on a better position to

capture the new insights proposed by those who are suggesting a crucial role for metaphor in creating meanings to new concepts and ideas. In particular, it may illuminate our attempts to interpret the kind of metaphorical thinking that takes place in mathematical classes as an intrinsic part of the ongoing mathematical activity.

For the discussion of the role of metaphor in mathematics learning some valuable trails have already been open by the works of others like Pimm (1987; 1995), Nolder (1991), Sfard (1994), Sierpinska (1994), and Lopez-Real (1989; 1990), and their contributions on the pedagogical status of metaphor have decisively informed our current theoretical point of view.

In our present approach we are specifically addressing the field of mathematical modelling and applied problem solving which has been in the centre of our work for the last five years, through the development of research projects and teaching experiments. Therefore, in this paper we wish to stimulate a discussion over the pedagogical and cognitive role of metaphor in real world problem solving, where the process of mathematical modelling and metaphorical thinking are regarded as being fundamentally interwoven.

The consequences of this structural relationship between models and metaphors in the activity of mathematization are then approached from a mathematics education point of view.

Conceptual Metaphors as Cognitive Tools

In his paper on the contemporary theory of metaphor Lakoff (1993) stressed one of the key points about the true nature of metaphor in our conceptual systems. There is more to the notion of metaphor than the usual conception of a figure of speech. We have to look deeper into our conventional and common forms of thought in our everyday experiences to unfold what is the essential place and function of metaphor. We don't just talk or write metaphorically, we actually think and act on the basis of powerful and pervasive metaphors. If evidence corroborates the ubiquitous character of metaphor in our ways of conceiving and understanding the world, from the more mundane to the more abstract domains, it is because metaphor is one of our tools for learning and coming to know.

The centrality of metaphor in the development of concepts entangles a particular epistemological view — ideas and knowledge are not concealed in some sort of closed and hermetic conceptual compartments. Ideas have a capacity of crossing boundaries, of travelling between systems and semantic domains and of finding new grounds where to germinate in new creative forms.

The cognitive power of metaphor results from its capacity to generate and transform the relationships between topics that tend to be seen as separate or even independent. To think metaphorically is to put in action the latent connections between different concepts and domains thus enabling new ways of perceiving reality or, as Black (1993) puts it, to understand how things really are.

Before going into the analysis of the value of metaphor in mathematics learning and understanding we shall take a closer look at its mechanism by focusing on two crucial processes that are embedded in the way metaphors work — those of *projection* and *interaction*.

Every account of a conceptual metaphor must start with the idea that two conceptual domains, which are called the source domain and the target domain, are brought into play. The metaphor allows one to look at the target domain in terms of the source domain, by projecting knowledge of the source domain onto the target domain. In this process, the metaphor sheds a new light over the target domain. To be more precise, we would say that a certain complex of implications inherent to the source domain is projected onto the target domain where it produces a parallel complex of implications (Black, 1993).

It seems however that this projective quality of metaphors should be extended in such a way that projection is not reduced to a one-way path. Interaction comes as a necessary attribute of metaphors when we realise that some of the properties and features of the source domain are reinterpreted when we produce a metaphor. In other words, metaphor has a mediating role in bringing together two conceptual domains. It is a way of eliciting meanings for the concepts of both the target domain and the source domain.

Whenever a metaphor is created, some things are chosen to be the relevant aspects to which attention is drawn and simultaneously there are things that become ignored. In this point it comes as a good example the metaphor of the complex plane, suggested by Pimm (1987). This is likely to be an adequate metaphor if one intends to make salient some of the properties of the complex numbers. It takes us across the isomorphism between the set of complex numbers and the set of points in the Euclidean plane, it suggests a representation for the number a+bi as an ordered pair (a,b), it invites us to see a complex number as a vector and the set of complex numbers as a vector space and it may even show us a complex number as a location in the plane determined by a direction and a distance: $r\text{cis}\,\theta$.

It is quite true, as Pimm notices, that this projection of geometrical concepts onto the notion of complex numbers does not state what complex numbers are; it is far from being a definition of the complex numbers.

To understand a concept with the help of a metaphor is something very different of summing up the concept under a definition or description that would encapsulate some pre-fixed meaning. The metaphor induces similarities and multiple meanings, elicits some aspects of the concept while neglecting others and above all it creates connections between concepts. Nothing is given but the mediating tool to approximate different domains and that is the basis upon which an opportunity to make sense of a certain concept is created. The metaphor does not deliver the right meaning, it provokes the production of meanings inasmuch as it stimulates an act of understanding (Sierpinska, 1994).

Consequences of Metaphors for Learning

According to Petrie and Oshlag (1993), there are two general ways of considering the role of metaphor in educational settings: one that seeks the purity and rigour of literal meanings and leads to the conception of metaphor as an obstacle to learning, and the other that finds in metaphor a valuable resource to the understanding of new ideas.

The first position is described by Aspin (1984) as the purists' demand for pure and clear communication of meanings between language users who hold that "metaphor is simply a confusing or emotive use of language and, when not actually meaningless, quite unsuited to the rigors of scientific or philosophical discourse" (p. 23).

The second one is well represented in the words of Holton (1984) who argues that "in the work of the active scientist there are not merely *occasions* for using metaphor, but *necessities* for doing so, as when trying to remove an unbearable gap or monstrous fault" (p. 98). That's why, as a result of those necessities, "we speak of families of radioactive isotopes, consisting of a parent, daughters, grand-daughters, etc. We constantly tell stories of evolution and devolution, of birth, adventure and death on the atomic, molar, or cosmic scale." (p. 101). Another good instance of this view is offered by Sfard (1994) in her account of the metaphorical forms of reasoning of mathematicians. The revealing statement "First of all I have to get a metaphor...." is one of a series of testimonies showing how metaphorical thinking is quite alive in even the most abstract areas of mathematics.

In what concerns the pedagogical value of metaphors in education, and in spite of a certain enthusiastic wave on the study of the cognitive potentials of metaphors one can easily agree on the fact that metaphors are not neutral. This means that there are subjective and idiosyncratic forces operating in the understanding of a metaphor, presumably comparable to the understanding of a joke, as Aspin (1984) describes it: "metaphors are like jokes — or lies: they meet, or fail to meet, with 'uptake' in terms of the hearer; and, in accordance with the hearer's knowledge of language, receptivity and imagination, so their utterance is more or less 'happy'" (p. 33).

As Pimm (1987) and Nolder (1991) recall us, metaphors in mathematics learning and teaching are not immune to risks and pitfalls. However, being aware of those risks should not inhibit us of underlining the idea expressed by Sierpinska (1994) that building a metaphor is often a good sign of an act of understanding.

Therefore, while keeping in mind that words' uses and meanings are plastic and plurivocal, we should reflect on the words of Aspin (1984) for whom there is no language without metaphor — "metaphor is a basic feature of language and we strive in vain to avoid it" (p. 29), — and on Black's arguments (1993) in sustaining that metaphor is the only way of actually knowing how things really are.

Having accepted the bottom line that metaphors can not be dismissed from language in school scenarios, our investigation about the role of metaphor in mathematics learning will then be directed to the metaphorical nature of applied problem solving situations.

In the next section we intend to illustrate the metaphorical side of mathematical

modelling in a problem from micro-economics, where we expect to exhibit how mathematical concepts are projected onto real world concepts.

An Example: Measuring the Utility of Consumption

Economists define consumption as the last stage of production and the ultimate end of all economical activity. Therefore, the final aim of production is the satisfaction of the consumer needs and desires, that is, the production of *utility*.

Under this assumption, the production of a certain good must be adapted to the characteristics of the market. For instance, if we take the production of bread, it is not difficult to imagine how relevant it can be to know how to manage the production of different kinds of bread in quantity and variety. Thus economists are obviously interested in finding a mathematical model to describe (and control) the consumer's utility with the consumption of bread, which means to come up with a process for measuring and quantifying utility.

A first approach to this economical problem started with the hypothesis that the consumer could be thought of as a kind of *machine* that received an input of goods and services and produced an output of satisfaction or utility. Under this conception, the utility was supposed to be measured as a cardinal quantity, that is, a cardinal value, expressing the exact number of units of utility corresponding to the psychological output of the consumption of a good. Thinking in terms of a cardinal quantity one would be induced to imagine the creation of a special instrument — something to be called an *utilmeter* — that would be plugged to the consumer and register the effective utility he would get out of the consuming.

However this first metaphorical approach to the mathematical modelling of utility could not prevail since nobody has been able to come up with such a device that would provide a direct measurement of an ideal quantity, which is, in fact, a subjective result of consumption.

A second approach was then attempted, this time based on the idea that the consumer is usually confronted with a choice between several products offered (as it happens, for instance, with the various kinds of bread available in bakeries and supermarkets). In this case, the aim was to look for an ordinal model of utility.

For the sake of simplicity, one can restrict the mathematical modelling to the consumption of two kinds of loaves of bread: white bread and brown bread. The modelling process begins with the notion that a consumer has a daily purchase of these two types of bread. We assume that the quantity of white bread, x, and the quantity of brown bread, y, purchased by the consumer may be represented by an ordered pair (x,y). Each ordered pair is metaphorically considered as a basket of the two goods. Thus we have the consumer deciding what is the basket that he or she prefers and, projecting this situation onto mathematics ideas, we have each basket being represented by a point on the plane XOY. The set of all possible pairs (of all the baskets) will be a set of points in the plane and it will be called the consumption space.

At this point, what must be determined are the preferences of the consumer between several different baskets. In particular, it is expected that some choices will be indifferent to the consumer. This set of indifferent choices (a set of points) will define a curve on the plane — the so called indifference curve. And this is how mathematics is found to be a suitable domain to obtain a representation of the consumers' choices. If we ascribe a real number to each of these indifference curves a new progress is made; since the set of real numbers is an ordered set, it will induce via metaphor an order structure on the consumer's relative preferences. In fact, the higher the number attached to the curve the higher is the preference of the consumer for the correspondent set of baskets. The introduction of an index number is clearly a convenient way of summing up the range of preferences and it facilitates the use of mathematics through the application of the notion of indifference curves. By asking the consumer to reveal his or her choices between more and more baskets of bread, one can identify more and more points lying in successive indifference curves. This way we can gradually arrive at a theoretical description of the consumer's preferences, which gives us his or her map of indifference. It provides us a metaphorical (a mathematical) picture of his range of preferences.

In possession of the map of indifference, we can use mathematics to find the best continuous function of two variables, u(x,y), that fits the level curves referred above as the indifference curves. The mathematical model thus obtained is called an ordinal model of utility. It should be noted that the two-variable function u(x,y) was looked for only after the map of indifference has been constructed. So the fundamental data are to be found in the indifference curves and not on the values of the function u, which makes a fundamental difference from the cardinal model.

We have just seen how the mathematical knowledge about representation of curves on the plane and about the relationship between a family of indexed curves and a two-variable function can provide the means to quantify the utility, by resorting to the discovery of the consumer's preferences between several baskets of bread.

We want to claim that this process illustrates many instances of metaphorical thinking within the mathematical modelling process; some examples are the following:

- the different baskets of bread are points on the plane;
- the set of baskets which are indifferent to the consumer are curves on the plane;
- the consumer's preferences are ordered numbers;
- the consumer's range of preferences is a map of curves;
- the consumer's relative preference for a particular basket is the output of a certain two-variable function.

Throughout the modelling process what economists do is to think about the problem of utility *in terms* of mathematical ideas and concepts. So they choose a particular angle of vision — one from which utility is seen as something indirectly derived from the consumer's preferences between several alternatives. What the utility function really gives is an information about the ordering of the consumer's choices.

From Metaphors to Models: Implications for Mathematical Meaning

The example from economics is aimed to foster some reflection upon the general notion of mathematical modelling. There is a widely accepted view according to which a mathematical model of a real phenomenon or situation consists of a triple made of a certain part of the real world, R, of a certain piece of mathematics, M, and of a certain correspondence, f, that is established between them (Niss, 1989). Some authors (like Skovsmose, 1989) contend over the need to integrate a human element among these entities based on the argument that the modelling subject carries with him objectives, motives, beliefs, theoretical backgrounds, and different sociocultural constraints. This surely represents a strong point to the recognition that the modelling activity, and in particular mathematical modelling, is not a kind of straightforward process where one is supposed to isolate the real world variables and to find the right mathematics to translate their behaviour in a way that matches reality.

One fundamental question still persists and it concerns the *how's* of mathematical modelling. More precisely, what remains disturbing about mathematical modelling is — how do we create that correspondence f between reality and mathematics? — or in other words — how do we make reality and mathematics fit together?

We are offering a possible answer to this question in saying that each model is a result of a conceptual metaphor. To create a mathematical model of a certain aspect or phenomenon of our experience we have to find some way of articulating two different conceptual domains. We claim that the connection to be found between them demands the production of a conceptual metaphor. Therefore, looking for a metaphor is the first real step towards any attempt of mathematical modelling. Built in that metaphor there are the mechanisms of projecting inferences from one domain to the other. That is the place where mathematical models are to be found, those are the operational elements that make possible to look at the real problem in terms of mathematical ideas. From this perspective, behind any possible model there must be a metaphor and so without a metaphor no modelling can be successfully achieved.

To stress our main point, we would say that metaphors are the mediating key elements without which no process of mathematization is ever conceivable. More precisely, the relevance of this or that piece of mathematics can only be judged in the frame of a metaphorical approach to the problem.

Therefore the role of metaphor in mathematical modelling is one of mediation between looking at phenomena and explaining them. Thus a mathematical model is only formal and abstract in the surface since its holds in it a sometimes complex metaphorical matrix. In a way, a mathematical model is just a shadow of a hidden metaphor, where the sensible and imaginative elements were covered up by the formal and the symbolic.

To look at applied problem solving from this perspective raises a new range of possibilities to develop meanings for mathematical ideas and concepts. To shift our attention from the translation process — that is from the naive view of modelling where a one-to-one mapping between reality and mathematics is suggested (Pimm, 1995) — to

the investigation of the nets of metaphors involved in a real world problem, will foster the conception of a many-to-many mapping and enlarge the semantic potential of applied problem solving.

As it was suggested with the economics problem of conceptualising utility, the metaphoric nets provide the sources for understanding and explaining both the real economical problem and the mathematics that is being activated as a result of metaphorical thinking, through projection and interaction.

To invest in the creation and unveiling of the metaphorical matrixes makes the activity of applied problem solving a privileged scenario for interpretation in mathematics classrooms. If students are invited to see mathematics concepts in terms of something else (even in terms of other mathematical objects), as it is common in poetry or literature when other relevant semantic fields are evoked and connections are found, they will experience the need to work on the meanings of the concepts involved.

Finally, we feel that metaphors have usually such an elasticity that its power in generating new mathematical ideas and even new problems is quite surprising. In saying so, we are also remarking that the same real world problem has the potential to be extended in different new ways if the metaphorical thinking is at the heart of the discussion in mathematics teaching and learning. In this perspective we consider equally valid and desirable the opportunities for students to encounter and explore metaphors suggested by others or to invent and expose their owns.

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ALGEBRA AS A LANGUAGE IN USE: A STUDY WITH 11-12 YEAR OLDS USING GRAPHIC CALCULATORS.

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Abstract

A research carried out with twenty three 11-12 year olds who have not had algebra instruction is reported. The classroom environment was arranged so as to take advantage of the symbolic facilities offered by the graphic calculator to introduce children to learn the algebraic code through using it, in a similar way in which we learn the mother tongue. Analysis of individual interviews and children's written work provided promising results that show that after twelve 50 minute sessions, where children were using the calculator code to describe number patterns, most of them were able to extend this experience to confront algebra problem situations. The results also showed that children developed informal notions and strategies that allowed them to sort out tasks involving algebraic equivalence and inverting linear functions. The report discusses the theoretical and methodological issues that provide a rationale for such encouraging results.

Background and theoretical issues

The major aim of this research was to explore the learning of algebra within the pragmatic paradigm of language acquisition. The report centres only on one of its research aims: *investigate the extent to which the use of the calculator language as a means of expressing general rules governing number patterns, helps children grasp that the algebraic code can be used as a tool for coping with problem situations*. This pragmatic view implies conceiving a teaching approach in which the learning environment mirrors, as much as possible, those social circumstances which frame the acquisition of the mother tongue. Accordingly, such an approach must be different from both a syntactic or semantic teaching-oriented approach.

A *syntactic*-oriented approach is conceived here as a teaching position in which the pupil plays the role of 'consumer of linguistic input', more specifically, a consumer of those rules governing the use of algebraic code. A good deal of text books exemplify this approach: first definitions and rules, then a list of exercises and problems to be solved.

A *semantic*-oriented approach relies on supporting the introduction of algebraic syntax by providing pupils with 'meanings' for the symbolic system. The teacher is the most active person in the classroom and tries to offer as many different approaches to problem solving as possible intending to help pupils induce general properties or rules from a limited number of examples.

A *pragmatic*-based approach must allow pupils to enter into algebra by using its code, this principle marks the main difference with the other approaches. This approach is not based on syntactic rules or definitions (which characterise a syntactic approach) nor on rich examples for children to be followed and later on induce generalisations (which characterise a semantic-based approach). The pragmatic approach is founded on a tight relation between context and language use, so that the use of language can always be checked upon context itself. Though it seems paradoxical to propose starting to use a formal symbolic language before we know at least some definitions about it, there is a good example: children learn their native tongue without any previous knowledge of grammar rules or definitions. In principle, both natural language and school algebra deal with learning to use a sign system.

One of the most overt differences between acquiring these sign systems is that natural language is learnt within the rich environment provided by adult-child interaction, it embodies a learning process which is hugely aided by what Bruner (1983) calls a Language Acquisition Support System (LASS). The present research proposes that, following Bruner's concept of LASS, the school setting can be artificially arranged to create an *Algebra Acquisition Support System* (AASS), a system in which the teacher's expertise in using algebraic code is strengthened by incorporating a technological component (graphic calculator) that allows him/her to achieve a milieu where children encounter the algebraic code as language-in-use for expressing and negotiating mathematical ideas.

The theoretical referent adopted in this study mainly relies on Bruner's research on the acquisition of the mother tongue (1980, 1982, 1983, 1990) [1]. Some outcomes and principles drawn from Bruner's work were recast to provide support for the design of a classroom environment within which the teaching of algebra could be approached attempting to simulate the ways in which children learn the rudiments of natural language. The adoption of children's language acquisition as a theoretical referent was inspired by the characteristics of the symbolic facilities offered by the graphic calculator. The graphic calculator code offers a tight link between numerical facts and the algebraic language that allows us to put 11-12 year olds in the position of using the calculator's language without having previous instruction about its structure and syntax rules. It is hypothesised that if the learning environment is suitably arranged, such a link may provide the children with a referent that helps them deal with the algebraic sign system being supported by their previous arithmetic knowledge.

Methodological issues

The calculator's role.

The calculators used in this study allow three ways of representing functional relationships[2]: the analytic expression, used to type a program (figure 1); the tabular representation, obtained on the calculator's screen by inputting a range of values to the program's variable (figure 2), and the graphic representation The tabular representation was used as an arithmetic referent for the analytic expressions. These characteristics of the machine's operation were exploited to create a mathematical environment where the calculator's formal code is available to anyone with basic arithmetic skills.

The activity consists of a game-like task in which the children 'guess' someone else's program. Pupils must recognise the numeric pattern shown in a table and program the calculator to produce this table. The interaction occurs on two levels: student-machine and student-teacher. The underlying hypothesis is that pupils, *through use*, create meanings for the calculator's sign system, somehow emulating the process through which we acquire the basics of our native

Fig. 1

$? \rightarrow A$:	$2 \times A+5$
Declaring variables	Programming expression

Fig. 2

1	7
3	11
6	17
7	19

[1] A detailed discussion of this theoretical approach can be found in Cedillo, 1996.

[2] Graphics resources were not used.

tongue. When they engage in these activities the children are using the programming code as the language that the calculator 'understands'. Arithmetic plays the role of context that helps them set up and verify conjectures which they express through the calculator's language.

Subjects

Eight[3] children were chosen to be observed during the experimental phase using a case-study methodology. They were selected according to their mathematical attainment prior to the experimental phase. This was done as follows: (i) a boy and girl of below average attainment, (ii) two boys and two girls of average attainment, and, (iii) a boy and girl of above average attainment.

Tasks

Activities were introduced in worksheets (a total of 55), this way of presenting the tasks was intended to respect (as much as possible) each child's pace, which in fact is the way in which language acquisition occurs. In order to do this the following working routine was set up: at the beginning of the class each child was delivered an envelope containing a format's sheets without being told how many should be completed. They returned it at the end of the class. In the next class they collected their envelopes, finding their work marked by the teacher along with the sheets they had not completed. Activities were organised into six groups called *formats*. Format 1 contains the 'raw material' on which formats 2 to 6 elaborate. In this format, expressions containing letters are introduced as the mathematical language that allows children to control the calculator. For example, running the program $2\times A+1$ for A=2, 5, 9 outputs the table shown in figure 3[4]. They are then asked to (i) Find how the input is operated on to get the output, and express that in natural language, (ii) program the calculator to reproduce the worksheet's table, and (iii) Complete another table given with the same program.

Fig. 3

? 2	5
? 5	11
? 9	19

This game contains the basic elements used to constitute the communication platform on which increasingly complex activities were designed. This structure was intended to help children gain self confidence in using the new mathematical 'words' involved in the calculator's code and start making sense of the new formal code in-use. For example, expressions of the form $ax+b$ were like 'new words' for children; the use of these expressions imply leaving some calculations in suspense which is something that they seldom confronted when working arithmetically. This routine was used to softly introduce new elements which were intended to keep children's interested in doing the tasks as they gained experience in dealing with the new code to cope with different mathematical tasks. Each worksheet included a new element, be it numerical, with a sign or decimal point, or structural, like 'two step' rules, for example $3\times D$ is a 'one step' rule and $3\times D+1$ is a 'two step' rule. Below, each format used in this study is described as is the sequence in which they were introduced.

Format 1

This format consists of 15 worksheets that are aimed at introducing the use of the calculator programming code. Here, the children were supposed to learn 'how to say' to the calculator

[3] The below average boy got sick at the middle of the study and was out of school for two months.

[4] Although the calculator recognises expressions like 2A+1, the arithmetical notation formerly known by the children was respected.

the rules of linear functional relationships which were presented in its tabular form (5 sessions of 50 minute each).

Format 2

This format consists of five worksheets. The rules the children constructed were used to create a table which they gave as a clue for a fellow pupil to guess what program was being used to produce such a table (one 50 minutes session).

Format 3

This format consists of 10 worksheets aimed at introducing the notion of equivalence between algebraic expressions. The tasks are presented as follows: firstly, children are asked to program the calculator so that it duplicates a given table. Then pupils are required to construct at least four more programs which must display the same table (three sessions of 50 minutes each).

Format 4

This format consisted of 10 worksheets, its content is based on finding rules of decreasing functions (expressions of the form b-ax). As well, the child is confronted with story-based problems which require the pupil to symbolise part-whole relationships, for example, arbitrarily cutting in two parts a piece of wire with length 16 cm, if one of these parts is called x, the other should be called $16-x$ (Three sessions).

Format 5

The tasks were aimed at introducing the notion of 'inverse programs' (inverse functions) and were delivered as follows: for a given table, pupils were asked to find a program that outputs it, then a program that outputs the inverse table (two sessions).

Format 6

This format consists of 10 worksheets aimed at observing the extent to which children can extend their experience in Formats 1-5 to negotiating problem solutions. A succinct description of the tasks is made in what follows. Worksheets 46-48 deal with sequences presented by geometrical patterns. The pupils are asked to program the calculator so that it helps them to obtain any specific member of the sequence. Worksheets 49-51 and 54-55 require the children to cope with word-based problem situations (like calculating the perimeter or the area of rectangular shapes where the length is 30 meters larger than twice the width, or calculating the length and width where a relationship between them and the perimeter are given). Worksheets 51-53 concern problem situations which involve the notion of percentage.

Data gathering

The main sources of data were (i) children's written work throughout the fieldwork, (ii) individual interviews (each of the case-study children was interviewed three times, twice during the study, and once at the end), and (iii) notes taken by the researcher after each classroom session during the fieldwork addressing relevant children's interventions.

Results[5]

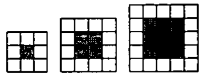
The children's algebraic attainment throughout the study provides empirical evidence for the approach to learning a new sign system by using it, and for the potential of the graphic calculator as a fundamental support in the fulfilment of this enterprise. With different level

[5] Due to space constraints the results are discussed only around the tasks in Format 6.

of attainment, all the case-study children were able to use the calculator code to cope with algebra word problems. Some examples of children's work are discussed below.

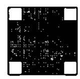
A relevant issue raised when children confronted problem solving is how they resorted to using their incipient notions and strategies as tools for negotiating solutions, particularly those notions about algebraic equivalence and inverting a given function. For example, Jennifer and Jimena, using different strategies, looked for an equivalent expression to obtain fresh information to face a problem situation. The following extracts illustrate these claims.

Fig. 1



Jenny made the program $(A+2)^2-(A\times A)$ to obtain the number of white squares in any member of the sequence shown in figure 1. She then became engaged, on her own, in inverting the program to complete the table for the cases where outputs were given. The complexity of the expression did not allow her to do it but she finally found another way of interpreting the number pattern and produced the equivalent program $A\times4+4$. She then built the "inverse program" to complete the task: $(A-4)\div4$.

Fig. 2

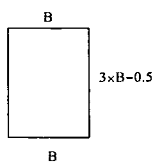


Jenny explained that she made the program $(A+2)^2-(A\times A)$ "*taking away the area of the grey square from the area of the whole square ... The "A" is the length of the grey square* (fig. 1) *... I found a different program when I saw the shape as a cross*" (fig. 2). This allowed her to count the number of squares surrounding the grey square, then added the four squares on the corners: $A\times4+4$.

Jimena's work provides another interesting example. It illustrates how context provides support for children's insights when children are ready to face new problem situations. Jimena became engaged in inverting a complex expression to complete a table where the *outputs* were given. Her attempt led her to "uncover" the distributive law. She had made the program $(A\times2+A\times3\times2)\times53$ to compute the cost of any window wooden frame which "they all are three times as high as they are wide and the price per metre is \$ 53.00" (worksheet 49). When working out the inverse function she found that $(A\times2+A\times3\times2)\times53=106\times A+318\times A$. She explained it as follows "*if I had two sides which cost 53 each, altogether should cost 106 times the length of one ... I did the same with the other two sides of the window ... I then checked it with the calculator and saw it works*".

The following episode with Erandi exemplifies another children's strategy. The problem situation was the following:

> "In the sculptures parlour of a certain Art Gallery, the windows have the following features: Their sizes vary, but, in all of them the height is 50 cm less than three times the width. The material used to build the frames costs \$62 per metre. Can you program the calculator so that it helps you compute the cost of any window frame? " (worksheet 50).



It is relevant that Erandi, being supported by context, used her incipient knowledge about algebraic simplification in producing an expression that properly describes these relationships. She built the program $\mathbf{((B+B)+(B\times6-1))\times62}$ to compute the cost of any window's frame. To explain how she obtained this program Erandi sketched a diagram like the one on the right to explain: "*The width is B ... there is another B on the top ... the height is 50 cm less than three times the width, that is ... 3×B−0.5 ... the*

opposite side is the same ... Then I computed the perimeter, that's B plus B plus the other two ... they are six times B but one metre less (pointing at 0.5)... *all this multiplied by 62 gives the cost*".

The final illustrative situation to analyse here is Jimena's solution to the following problem: **Find the length and width that gives the maximum area for the "three sides" rectangle with perimeter 100 metres".**

Jimena made the program (100−A)÷2×A. She wrote the following explanation: "*(100−A)÷2 is going to give the short side, if I multiply it by "A", which is to be the large side, I will get the area*".

Since the children were not given examples that directly relate problem solving with their previous experience (describing number patterns), their achievements document the potential of putting them in the position of learning a language, not in the role of spectator, but through use.

The experience of describing number patterns using the calculator language helped pupils make sense of traditional algebra word problems and provided pupils with a formal code to negotiate problem solutions. This result strongly contrasts with outcomes obtained in studies that have investigated the effects of introducing school algebra through describing number patterns (Stacey, 1989; Herscovics, 1989; Arzarello, 1991; MacGregor and Stacey, 1993; Stacey and MacGregor, 1996). These studies reported students' difficulties in producing algebraic rules from patterns and tables. MacGregor and Stacey (1996) concluded that "a patterns-based approach does not automatically lead to better understanding; the way students are taught and the practice exercises that they do may promote the learning of a routine procedure without understanding" (p. 3). They reported that students were able to recognise and describe the involved quantitative relationships, but their approach was rather a rhetorical description (in the sense of Harper, 1987) which leave children far from describing the problem algebraically.

There are various factors that may explain the strong contrast observed in the findings of the present study and previous research outcomes. What seems the most immediate explanation is that the students reported by MacGregor and Stacey worked within a paper and pencil environment. MacGregor and Stacey (1996) found that most of the students guided their procedures by natural language descriptions. They conclude that this approach hardly helps them structure an algebraic expression to properly describe the relationships between two variables. This contrasts with the fact that the calculator programming language is situated within the computing environment, this feature places the children within a milieu where algebraic formulation becomes an inherent part of the problem situation to be solved. The use of the calculator language leads children to <u>describe the relationships in a problem situation operationally</u>, <u>even if they make this description in natural language</u>. When working with the calculator the children do not look for the relationship between the "x" and "y" variables to find out the underlying pattern (which was the question used by MacGregor and Stacey); the calculator environment allows us to make the same question so that the children are led to think of what operations they can make with the input in order to produce the correspondent output. The data obtained from the present study provide evidence for this assertion: when the children were asked to use natural language to describe the relationship in-

volved in a number pattern, they used expressions which always include an operative description, for example, "*I multiplied by 2*" which they expressed as A×2 to program the calculator. When the rules were more sophisticated, they ignored the constraint of using natural language and directly used calculator language, for example, 3×A+2, "*because the calculator language makes it easier to explain this*" (Diego, Format 1). This use of the calculator code allowed the children to focus on the operational structure of the calculator expressions, whether describing number patterns or describing the relationships involved in story-based problem situations. This operational approach does not necessarily occur when the children work within a paper and pencil environment, where natural language is the immediate means of communication, this situation seems to lead children to see the use of algebraic code as a sophisticated teacher's imposition.

The mathematical content and the sequence of the tasks used during the study provide another source of explanation for pupils' achievements in problem solving. The tasks addressed the following issues: expressing generality (formats 1 and 2), algebraic equivalence (Format 3), inversion (Format 4), decreasing linear functions (Format 5), and problem solving (Format 6). A close look at the tasks provides an explanatory framework for how the children developed such notions and strategies which finally they exhibited when coping with negotiating problem solutions. This review is intended to provide support for the conclusion that these tasks shaped a didactic 'route to algebra problem solving'.

The tasks in Formats 1 and 2 allowed the introduction of calculator language as a language-in-use. The main feature of these tasks was to place children in the position of using the calculator code to fulfil their communicative intention. This guided the children to gain awareness of the inherent generality of the algebraic expressions they were using from the beginning of the study. The tasks in these formats also introduced children to the use of parentheses and the idea of inverse function (finding the input when the output was given). The tasks in Format 2 introduced children to the notion of algebraic equivalence. During individual interviews they showed they were able to operate with algebraic expressions when the task was changed to that of transforming an algebraic expression to make it equivalent to a target expression, for example, transform the program B×7 so that it produces the same as the program B×9. The work carried out by the pupils in Format 6, where they produced expressions as ((A×3)×2+(A×2))×53, suggests that the experience of transforming algebraic expressions was a key point in helping children gain awareness of the feasibility of using expressions of the form ax+bx+c. The tasks in Format 4 required the children to deal with inverting linear functions. The children's responses to questions about number sequences show how their previous experience with inverting linear functions helped them cope with problem situations which required them to apply their incipient notion of inverse relationships. Finally, the tasks in Format 5 introduced the children to new number patterns generated by linear decreasing functions. The children's responses to worksheet 55, where they produced expressions like ((100−A)÷2×A, provide evidence of the extent to which their experience in producing decreasing functions influenced the ways in which they used the algebraic code to negotiate solutions.

Final remarks

The data drawn from this study shows that the approach to algebra as a language in use helped children use the calculator language to negotiate solutions for algebra word problems, and confront tasks involving algebraic manipulation, such as simplifying similar terms, transforming an algebraic expressions to make it equivalent to another, and inverting linear functions. The study suggests that the pupils have reached a promising starting point to confront more traditional school algebra. Nevertheless, there are still many aspects of algebra which the children did not encounter within this study. A number of research questions should be faced in order to refine/consolidate the results of the existing study. Among the major issues leading to further research are the following:

In which sense may the pragmatic approach to teaching and learning algebra help/obstruct children's learning:

a) of formal syntactic rules for algebraic manipulation?
b) when confronting algebra problem solving which involves using equations?
c) of more formal methods for establishing algebraic equivalence?
d) of a more formal approach to the notion of function?
e) of graphs as another way of representing number relationships?
f) when confronting that a conjecture about number relationships cannot be validated on the basis of the results obtained from specific cases?
g) of the value of counterexamples as a means of proof/refuse mathematical conjectures?

The above questions tell us about the potentialities and limitations of the present study. About its potential because these questions give an account of the wide range of algebraic topics that children experienced during a relatively short school time (about 18 hours). As well, these questions tell us of the limitations of the present study because they bring to light issues that still have to be investigated before setting up stronger claims about the potential of the approach to learning and teaching of algebra as a language in use, and the support provided by the symbolic capabilities of the graphic calculator to fulfil such an enterprise.

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EMERGENCE OF NOVEL PROBLEM SOLVING ACTIVITY

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This paper examines the novel problem solving actions of a college student. The analysis highlights the role of the solver's inferential processes (abductions, deductions, inductions) as structuring resources that contribute to both the solver's understanding of the problem and the emerging novelty that constitutes viable solution activity.

Introduction

The philosopher and logician Charles Saunders Peirce (1839-1914) asserted that there occurs in science and everyday life a pattern of reasoning wherein explanatory hypotheses are constructed to account for unexplained data or facts. Peirce called this kind of reasoning ***abduction***, distinguishing the process from the two traditionally recognized inferential types of reasoning, induction and deduction. Specifically, abduction furnishes the reasoner with a novel hypothesis to account for surprising facts; it is the initial proposal of a plausible hypothesis on probation to account for the facts, whereas deduction explicates hypotheses, deducing from them the necessary consequences, which may be tested inductively. According to Peirce, abduction is the only logical operation which introduces any new ideas, "for induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis" (as quoted in Fann, 1970, p. 10).

Since Peirce argued that abduction covers "all the operations by which theories and conceptions are engendered" (as quoted in Fann, 1970, p. 8), it appears that abduction may play a prominent role in the mathematical knowledge that learners construct while in the process of solving a problem. Of particular interest here is the role of abduction as a sense-making process which aids solvers in "getting a handle on" or developing understanding about the problems they face. In this context, the solver's hypotheses may include novel ideas about the problem, that pave a way for them to make conjectures about both potential courses of action to carry out as well as the result(s) of those actions.

The work of Polya (1945) is based on ideas consistent with the view that problem solvers reason hypothetically in the course of solving a problem. Specifically, Polya identified heuristic reasoning as "reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem" (Polya, 1945, p. 113). Further, Polya cited the usefulness of varying the problem when solvers fail to achieve progress towards their goals because the solvers' consideration of new questions serves to "unfold untried possibilities of contact with our previous knowledge" (Polya, 1945, p. 210). Hence, solvers who reason hypothetically are (1) cautious in their reflections about

appropriate courses of action to carry out; (2) always looking to monitor the usefulness of the activity they plan to carry out; and, (3) willing to adopt a new perspective of the problem situation when their progress is impeded.

The importance of such reflective activity has been emphasized by Burton (1984), identifying the process of making conjectures as a component of mathematical thinking through which "a sense of any underlying pattern is explored" (Burton, 1984, p. 38). More recently, Mason (1995) has remarked on the importance of examining where learners' conjectures come from, suggesting that a fresh examination of the abduction process is warranted. Nevertheless, the researcher agrees with Anderson's (1995) contention that the process of abduction is transitory and slippery, difficult to foster, impossible to teach, and probably easy to discourage.

Objectives

The purpose of the study was to clarify the processes by which learners construct new knowledge in mathematical problem solving situations, with particular focus on instances where the learner's emerging abductions or hypotheses help to facilitate novel solution activity. The perspective taken here is that problem solving situations are self-generated by solvers, arising from their interpretations of the tasks given to them. Their interpretations may suggest to them questions and uncertainties, the consideration of which helps them construct goals for purposeful action. Successful completion of the task may involve many such constructions, all generated in the course of on-going activity and each monitored for its usefulness by the solver, as well as having the potential to re-organize their evolving goals and purposes. In this way, problem solving can be viewed as a form of hypothetical reasoning, where solvers try out viable strategies to relieve cognitive tension, involving no less than their ability to form conceptions of, transform, and elaborate the problematic situations they face.

In an earlier study (Cifarelli and Sáenz-Ludlow, 1996), examples of hypothetical reasoning activity were discussed, highlighting its mediating role and its transformational influence in the mathematical activity of learners. The current study sought to extend these results by specifying more precisely the ways that learners' self-generated hypotheses serve to organize and transform (or re-organize) their mathematical actions while resolving problematic situations.

Methodology

Twelve graduate students enrolled in a Linear Algebra class taught by the researcher participated in the study. The students were interviewed on 3 occasions throughout the course. These interviews took the form of problem solving sessions, where students solved a variety of algebraic and non-algebraic word problems while "thinking aloud". All interviews were videotaped for subsequent analysis. In addition to the video protocols, written transcripts of the subjects' verbal responses as well as their paper-and-pencil activity were used in the analysis.

Based on the analysis of the verbal and written protocols, a case study was prepared for each solver. The solvers' protocols were examined to identify episodes where they faced genuinely problematic situations. Previous studies conducted by the researcher characterized the conceptual knowledge of solvers in terms of their ability to build mental structures from their solution activity (Cifarelli, in press). For example, solvers were inferred as having constructed re-presentations when their solution activity suggested they could combine mathematical relationships in thought and mentally act on them (e.g., they could reflect on their solution activity as a unified whole, mentally "run through" proposed solution activity, and anticipate the results without resorting to pencil-and-paper actions). The current study examined more thoroughly the novel actions of solvers, with particular focus on the role that hypothetical reasoning played as a structuring resource for solvers.

Analysis

Subjects of the study solved both algebraic and non-algebraic tasks. One of the non-algebraic tasks involved exploring an array of letters that could be used to spell out the palindrome **WAS IT A CAT I SAW** (see below). The use of such non-algebraic tasks in the study enabled the researcher to observe solvers grappling with problems which were unfamiliar to them, requiring novel solution activity.

WAS IT A CAT I SAW

An early edition of Alice in Wonderland included the spatial diagram shown here.

See how many mathematical problems you can make up and solve using the array.

```
            W
          W A W
        W A S A W
      W A S I S A W
    W A S I T I S A W
  W A S I T A T I S A W
W A S I T A C A T I S A W
  W A S I T A T I S A W
    W A S I T I S A W
      W A S I S A W
        W A S A W
          W A W
            W
```

The following section describes the problem solving activity of John as he solves the palindrome task.

John's Inferential Processes. John was an aspiring secondary mathematics teacher and proved to be a strong mathematics student in the Linear Algebra class, achieving high scores on all exams and assignments throughout the course. He demonstrated strong problem solving activity throughout the interviews, as indicated by the novelty of his actions in completing the tasks.

Upon reading the instructions for the palindrome task, John interpreted that "his problem" was to make up and solve mathematical problems. As he began to formulate problems to solve, John remarked on how the task differed from other problem solving tasks he had encountered:

John: *Well, this is unusual. I think of problem solving as usually here is one problem, there is a problem you solve it. This is like you make a problem, it's like... orders of abstraction because I get, the problem is to make up problems. It's a little difficult.*

Despite his comment about the difficulty of the task, John routinely generated five problems, all of which had to do with counting letters and words:

Table 1: John's Problem Solving - Part 1

1. How many of each type of letter ?

2. How many times does a particular word show up ?

3. How many words total ?

4. How many letters in the array ?

5. How many different patterns of counting the letters ?

As he was working, John remarked on the superficial quality of the problems he had constructed. For example, in formulating problems #2 and #5, John commented:

John: *How many mathematical problems can you make up and solve using the array? ... Hm,you could ask how many times a particular word shows up* ***(writes it on problem sheet as problem #2***[1]*). The answer will depend on the word. Like, 'WAS' is going to show up a lot of times. 'IT' shows up fewer times. ... They're problems, but they're so, I don't know, so unsatisfying. I'd like to find something interesting.*

John: *You might want to come up with how many different ways to count the letters. You could move around ... how many different... sort of basically different patterns of counting can you establish* ***(writes it on problem sheet as problem #5)****? You could also do rows and 1, 3, 5, 7 and you would be adding odds duplicate 2 ones, 2 threes, 2 five's, 2 sevens, 2 nines, 2 eleven's, 2 thirteen's, do it that way. It reminds me in a way of Pascal's Triangle, but I don't think you can do anything with that because they're not offset the right way. Let's see ... that's about all.*

The episodes above are noteworthy for the following reasons. First, John's comments indicate a genuine lack of interest on his part regarding his newly constructed problems: he sees his made-up problems as "unsatisfying", and that he "would like to find something interesting". Second, John's comment about Pascal's Triangle appears to be a reference to an idea that he sees as both interesting and potentially useful. (His idea about Pascal's Triangle will re-appear later in the interview when he formulated a new problem.) Third, after generating problem #5, he has given up his quest to pose additional problems and looks to the interviewer for direction. The interviewer then prompts John:

Interviewer: *Can I give you one?*

John: Yeah.

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[1] **Comments in boldface describe the non-verbal actions of the solver.**

Interviewer: *Okay, the palindrome Was It A Cat I Saw, can you give me some ways that you could spell it in the array?*

John: *Oh, to spell out the whole thing? Oh, okay, well, I mean you obviously got the 2... you could... all diagonals.* ***(solver traces the palindrome along vertical and horizontal paths, followed by many seconds of reflection)*** *..... Interesting!*

With the comments above, John initiates a shift in his reasoning activity. He has a new problem to solve, his curiosity has been aroused, and he begins to become more engaged in the problem situation. He then adopts a pattern of hypothetical reasoning activity, generating provisional explanations of what his new problem might be about. He proceeds to explore the array of letters in a more focused manner, with a view towards learning about the different ways he could spell out the palindrome. For example, after tracing several paths through the array, each spelling out the palindrome, John generates a hypothesis:

John: *If you come in anywhere, you take a turn and finish it* ***(traces several zigzag paths on the array).*** *From below, you turn up. At every place you got ... options of switching. It's like you've got a network type problem or something like that. You get little nodes and you're going through nodes or something.*

John's hypothesis about the array of letters constituting a network of switching options helped organize and direct his further explorations. Further, his hypothesis, while providing him with only an initial idea about the different paths, served as a source from which he was able to elaborate and derive more sophisticated properties about the different ways of spelling out the palindrome. For example, after traversing several more paths through the array, John hypothesized a property that all appropriate paths must possess:

John: *Hm, ... Was It A Cat I Saw. I could finish here, that will get down to the C but I could come up or finish that way, or I could come up and finish that way, or finish that way, or finish that way, or that one. So,* ***(several seconds of reflection)*** *it's all of them have to go through here because that's your only C. And if you're going to get the palindrome, you've got to go through the C, so they all have to go through the center.*

John then summarizes his ideas, using the following analogy:

John: *Kind of reminds me ... of Chinese Checkers or something... it's like regardless of where you start, you've got to diagonally move your way in ... and ...in some way or another... work your way out.*

John elaborates on his analogy as he further explores the array:

John: *You can ... you can make a move down from this position ,you know, down or right, down or right, down or right ...* ***(solver traces a zigzag path through the array)*** *... Here, you can't go right anymore. You've got to go down either way as you work you way through here. It looks like kind of like a bus. You know, the kind of problems where they talk city blocks and how many they could get from here to here, you know, I get what 2 choices here, then you get, then here you got 2 choices, so to get here, you got 4 choices to get here ... 8 choices to get here, but then these 2, those are straight in, these you got 2 choices ... get 1, 2, ... It is just basically, I mean, it's like 8 paths to get in here from here ... Because you've 2, 2, 2 ... I don't know, I'm thinking out loud.*

From this elaboration of his original analogy, John proceeds to look for a pattern to help him to count the paths:

John: *I don't know...if I play with a little more it turns a pattern 1, 2... may be its more than 8... 1,2,... Yes, it has to be more than 8. But that's just from this position !!*

John's sudden surprise was a realization that there might be many more paths to spell out the palindrome than he initially anticipated. He reflects on this unexpected result, and continues to press forward towards a solution:

John: *And you can come in from all these other positions. But I mean this has the most choices* ***(points to middle W on the boundary).*** *If you start here* ***(points to horizontal and vertical paths)****, you've got to come straight in, and you could either go straight out ... or straight out if you come here you get one choice. At some point you've got to branch over to main channels, sort of speak, sooner or later you get a branch over the main channel... two possibilities... it seems you could work something out with that. I mean in terms of like literally how many paths could you come up with the form of the palindrome ... a lot !!*

John then generates a hypothesis about a possible solution to counting the paths:

John: *I feel it has something to do with powers of two because you've got choices of 2 in each of these nodes. Even if you can work your way up from here, still you've got 2 choices of each node, but you're basically working your way in, and then you've got to work your way out. Let's see, ...* ***(many seconds of reflection)*** *... actually working your way out you get like 3 choices from here but then once you've got to one of those ...* ***(more reflection)*** *... you could, ... its kind of mindboggling!!*

John continues to explore, now looking to use "powers of two" as part of his solution.

John: *But it does remind me like, you know, city streets or patterns or even like probability in it like from here to here there is one path, from here to here, there is, you know, 1, 2, 3, 4, 5, you could count may be, I don't know, or it might have to do with is how many letters there are in 8 paths. I'm not sure.* ***(many seconds of reflection here)*** *1, 2, 3, 4, 5, 6 ... something like that and count from here, same thing, you probably find a pattern.* ***(more reflection here before exclaiming)*** *Oh, this would be would be Pascal's Triangle, 1, 6, 10, 15, 10, 6, 1*[2] *...* ***(writes sequence on boundary of array);***

John's sudden and emphatic statement relating Pascal's Triangle is noteworthy because: (1) it emerged as a result of his persistence to find a pattern for counting the paths (i.e., came out of a genuine problem solving opportunity that he had initiated and sustained); and, (2) signaled a newfound confidence for him, placing a high level of certainty on both his prior as well as future actions. John continued his activity of counting the paths, using his knowledge of Pascal's Triangle to organize his actions.

John: *I'm trying to figure out Pascal's Triangle in my head. ... it reminds me of a problem I had in class one time. It's like you leave your house, go to Pizza Hut or Taco Bell, you have to make a turn at every corner. But yeah, it's like this, there's more paths from here than from anywhere else* ***(points to middle W on the boundary)****, and I can just do Pascal's Triangle, you know, I'll double check my arithmetic's, OOPS, I'm off. So, 1, 6, 15, 20, 15, 6, 1, so I'd say, ..., you know, there's*

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[2] **while the solver's conjecture was viable, these numbers do not correspond to a row of numbers in Pascal's Triangle; he appeared to make an error in recalling the rows of Pascal's Triangle, an error he later identifies and corrects.**

that many ways to get into the center once you pick where you're going to start, and then **(traces a path and appears surprised when he gets to the center C)** *to finish is like, do you let yourself repeat? I mean can you come back* <u>*this way*</u> **(points to the same quadrant in which he started)** *to finish palindrome? or you have to go out to each sector, or you have to kind of spell it out or it's like... What if you come back out in same sector? Just with in here this way. So you'd have this many ways to get in, but then from here out* **(several seconds of reflection here)** *... it'd probably be the sum of this* **(points to sequence of numbers on boundary)***, 2 to the 6^{th} ... 64, so I'm guessing basically there would be 64 ways to come back out. No, it is not just a guess, it's an* <u>*intuition*</u> *because there is basically 64 ways to come in, spread out in these different positions, so they reach 64 ways to go out because once you get to the center you have more choice in finishing the letter.*

In asserting his "intuition", John has placed some certainty on his reasoning, confident that he is close to a solution to the problem. John continues to explore and elaborate his hypothesis:

John: *To start it you're kind of locked into a particular place to start. So, I guess what you could come in to the C as I'm figuring 64 different ways, and you could come out 64 different ways. So just within the sector, there are 64^2 different ways you come out. And if you start, I mean multiply here by 4... another 16, something to get the whole thing. If you get to go in here then you get 4 choices coming out here. 4 times 4 I think 16 times, that'll give you like total ways.*

John: *That's a good question!* **(solver points to the 5 prior problems he made up and had written down)** *These really didn't satisfy me so much.*

Discussion

The results characterize John as an assertive, aggressive sense-maker, continuously looking to make sense of the situations he finds himself in, and at the same time, aggressively projecting results of his problem solving actions towards the solution of other questions and problems. It was inferred that John's hypotheses, while providing ideas that contributed to his solution activity, also served to create new questions or problems for him to address, which were then actualized in the form of particular explorations. In this way, his evolving hypotheses concerning what the problem was about, went hand-in-hand with his continuously changing goals of what he was trying to achieve through his actions. In other words, as John solves his problems, new problems arise for him that need to be addressed.

Table 2 summarizes the researcher's inferences about the relationship between John's goals and purposes (his problems) and hypotheses that contributed to his knowledge about his problems (his solved problems).

Table 2: John's Problem Solving - Part 2

Goals and Purposes: His Problems	John's Hypotheses: His Solved Problems
explore some ways to get the palindrome (what constitutes a path?)	H_1 : need to get onto a diagonal to spell out the palindrome
explore properties of the paths (what properties are common to all paths ?)	H_2 : all paths need to go through the center C and back out again to spell the palindrome
looking for some efficient way to count paths (what is the pattern ?)	H_3 : number of ways to spell out the palindrome appears related to Pascal's Triangle
looking to make a generalization (is going out the same as going in ?)	H_4 : the number of ways into the center C is the same as the number of ways out to the boundary

From Table 2 it is clear that John's hypotheses evolved continuously in the course of his actions as he determined how many ways there were to spell the palindrome. With each hypothesis, John solved a problem, the result of which fueled his understanding of the overall situation. The linear appearance of Table 2 is not meant to suggest a linear progression of problem formulation followed by hypothesis generation; rather, the researcher posits a relationship where the solver's problems and problem solving activity continually feed and nourish each other, each providing sources of action.

In more theoretical terms, John's problem solving performance constituted a confluence or flowing together of his evolving hypotheses, deductions and inductions. Specifically, John hypotheses, while serving to answer questions that arose for him in the course of his on-going solution activity, also served as conceptual springboards to (1) provide structure for his potential actions (i.e., by structure I mean he could organize his potential activity in ways that were compatible with his goals), and (2) actualize hypothetical relationships in solution activity (i.e., self-generate particular trials that could feedback to his conjectures). For example, he started with the relatively primitive question of what constitutes a path, generating a hypothesis ($\mathbf{H_1}$) that enabled him to inductively generate and examine several actual paths. Results of these inductive trials provided him with feedback that enabled him to abduce more sophisticated properties about appropriate paths through the array ($\mathbf{H_2}$-$\mathbf{H_4}$), with each successive hypothesis suggesting new questions to explore.[3]

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NESB MIGRANT STUDENTS STUDYING MATHEMATICS: VIETNAMESE STUDENTS IN MELBOURNE AND SYDNEY[1]

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This paper describes one part of a project which is working with migrant bilingual children who are learning mathematics in Australia. We are particularly interested in students who choose to switch between their languages when processing mathematical problems. In this paper data collected in both Sydney and Melbourne from grade 4 students will be discussed. Particular emphasis is given to the Vietnamese students' responses. Comparisons between the two cities which have some different approaches and conditions in their schools will be noted. Comment on Cummins threshold hypothesis will be made, as will some reflections on the implications for teachers and curriculum developers.

Australia is a land of many languages. Although the official and by far the most dominant language is Australian English, many other languages are used everyday as citizens go about their daily lives. One way this has been recognized is through the continuing policy of 'multiculturalism'. This policy is not just a recognition that Australia has drawn citizens from many lands, some of which do not have English as their first language, but is a policy that actively recognizes the multicultural background of its citizen as they live together in one nation.

In most urban schools in Australia's major cities there are many languages represented. It is common for a class grouping of 25 to 30 children to come from families representing five or six countries. In some parts of Sydney and Melbourne this can rise to ten or more. What then is the teacher to do who is probably a monolingual English speaker? For the most part English is used as the language of the classroom, although other languages represented in the classroom are recognized in some way. Such a response is normally justified on social grounds such as it gives access and status to each student's particular cultural background which is held to be important. The authors are totally in favor of this response. However they suggest that other cognitive grounds are just as important and are often not recognized by teachers.

The interplay between language and mathematics learning is now recognized as being a critical factor for the mathematics classroom (see Ellerton & Clarkson, 1996 for a comprehensive review). Earlier research by the authors has contributed to this debate in the area of bilingualism; Clarkson with primary students in Papua New Guinea and Dawe with secondary students in England (see for example

Clarkson, 1992; Dawe, 1983). This earlier independent work drew on the theoretical position of Cummins (Cummins & Swain, 1986). Cummins hypothesized that the level of competence that a bilingual child achieved in both her/his languages was critical to academic performance.

In working with teachers in Australian schools there was not always a recognition of this cognitive feature of bilingualism. Often a naive position was taken that the first language (L1) was some what irrelevant, although competence in the language of learning, in this context English (L2), may be important. Many non bilingual teachers, the majority in our systems, were not really aware that their bilingual students would indeed swap languages while thinking about their classwork. If this was conceded by some teachers, then such a possibility may happen for cultural or language based work, but not for mathematics. Of course when you are teaching children from such a diverse background of languages, in the pressures of 'keeping the classroom going', it is not always easy to see such common threads. This is a very different teaching situation for an English speaking teacher who might have a classroom of bilingual children, but virtually all of them are from say a Spanish speaking background, which may happen in the southern states of the USA

The Present Project

The authors have reported on this project elsewhere and the details will not be repeated here (Clarkson, 1996; Clarkson & Dawe, 1994). This paper reports on preliminary analysis of a data base of some 850 year 4 (age 9-10 years) students studying mathematics in Melbourne or Sydney. Eighteen schools were involved with parents coming from 42 identified country plus others, 34 of which were predominately at least non English speaking. For this paper, 252 cases were drawn from this data base representing those classes which had a high proportion of Vietnamese students. But still 24 identified countries were represented with 18 different non English speaking backgrounds.

In this paper the following group tests will be referred to:

- an English language competency test with a maximum score of 20,
- a Vietnamese language test with a maximum score of 8,
- a mathematics test composed of symbolic items with no words in an alternate answer format, the raw score of which can be converted to Byte Scores, a measure of cognitive level,
- a mathematics test which was composed of short extended answer word problems (Mathematical Word Problems Test) with a maximum score of 10,

a mathematical test which was composed of open ended items in that there was more than one correct answer (Mathematical Novel Problems Test). This test gave rise to a raw score (the number of items for which one answer was correct giving a

maximum score of 10) and a 'novel' score (one point for each correct answer, with up to three answers scored per item, giving a maximum score of 30).

Using L1 for Mathematics?

The first notion we were interested in exploring was whether the children did use their two language when attempting mathematical problems. Tables 1 and 2 show that indeed an important percentage of this sample of students did so for each of the mathematics tests. There seemed to be a consistent higher proportion of Sydney students who used their L1 in the solution process. Based on observations of the schools in which the authors were working, there did seemed to be a higher support for the maintenance of the students' L1 in Sydney schools with the employment of full-time bilingual teachers for this specific task. In turn this seemed to be a function of the larger size of the Sydney schools and hence their ability to direct a larger amount of money to this task, even though the proportion of funds in both Melbourne and Sydney individual schools may be similar. However it seems clear that the first message for teachers and curriculum developers is that a high proportion of students will be using their L1 for at least some of their mathematical thinking.

It is one thing to know that students are using their L1 in the solution process. However whether it makes a difference to the academic performance of students is another question. Table 3 suggests that the use of L1 does have some effect, but it is not consistent across mathematical context, and different schooling experiences may also play some role.

TABLE 1: Number of items for which the following percentage of students chose to use their L1 for at least part of the solution process on the Symbols Test (Sydney N=57; Melbourne N=85)

No. of items chosen	Sydney	Melbourne	No. of items chosen	Sydney	Melbourne
0	47	62	11	0	2
1	2	0	12	2	1
2	0	4	13	4	2
3	4	1	14	0	2
4	5	2	15	0	1
5	0	1	16	4	0
6	4	0	17	2	0
7	0	1	18	0	1
8	2	0	19	0	2
9	2	2	20	19	11
10	5	2			

TABLE 2: Number of items for which the following percentage of students chose to use their L1 for at least part of the solution process on the Word Problems Test and on the Novel Problems Test (Sydney N=57; Melbourne N=85)

WORD PROBLEMS			NOVEL PROBLEMS		
No. of items chosen	Sydney	Melbourne	No. of items chosen	Sydney	Melbourne
0	49	49	0	47	62
1	4	5	1	2	2
2	5	6	2	5	5
3	5	5	3	7	1
4	2	6	4	2	6
5	4	5	5	5	4
6	5	6	6	2	4
7	4	2	7	4	4
8	4	7	8	3	4
9	2	0	9	4	1
10	18	9	10	19	8

TABLE 3: Correlations between scores on the three tests and the number of items for which students chose to use their L1 for at least part of the solution process (Sydney N=57; Melbourne N=85)

Type of test mathematics test	Sydney		Melbourne	
	N	r	N	r
Symbols	20	- 0.16	21	- 0.04
Words	23	- 0.53*	30	- 0.18
Novel - raw score	30	- 0.26	32	0.30*
- novel score	30	- 0.20	32	0.30*

The Effects of Bilingualism

Table 4 shows the means and standard deviations for the two language tests and two mathematics tests. It is seen that there appears to be little difference between the performance of the groups.

To have some measure of the bilingual students' language competencies, a process similar to that employed by the authors and others in earlier studies was used. The frequency of scores on the English Language Test for the sample of English speakers was analyzed. Cut off scores which divided the group into thirds were determined. These cut off scores were then used to partition the sample of Vietnamese students into three groups. The frequency of scores on the Vietnamese Language Test for the Vietnamese students was analyzed. The group was partitioned into two groups using the median score. Hence the Vietnamese students were partitioned into six cells. In this way it was able to identify Vietnamese students who had relatively high competence in both their languages, students who had relatively low competence in both their languages, and students who had high competence in one of their languages termed 'one dominant' students. There were also some students in this sampling process who were dropped out of the sample as they were

TABLE 4: The means and standard deviations of four group instruments for the total sample of students, the Vietnamese speaking, and the English speaking students.

GROUP	INSTRUMENT	MEAN	STAN. DEV.
Total Sample (N=252)			
	Language (Eng) (max. 20)	12.6	4.9
	Math Words (max. 10)	6.5	2.2
	Novel (R.S.) (max. 10)	4.7	4.4
	Novel (N.S.) (max. 30)	9.0	4.9
Vietnamese speakers (N=93)			
	Language (Eng) (max. 20)	12.0	4.9
	Math Words (max. 10)	6.7	2.2
	Novel (R.S.) (max. 10)	4.5	2.2
	Novel (N.S.) (max. 30)	9.2	5.4
English speakers (N=48)			
	Language (Eng) (max. 20)	12.7	5.0
	Math Words (max. 10)	6.6	2.1
	Novel (R.S.) (max. 10)	4.9	1.9
	Novel (N.S.) (max. 30)	10.3	5.3

deemed to have a medium competency in English. Hence a new variable 'Bilingual Language' with three categories was defined.

Three analyses of variance were computed to investigate the effect of the level of bilingual competence on the raw scores of the Mathematical Word Problems Test, the Mathematical Novel Problem Test, and on the 'novel' score on the latter test. In each analysis the Byte Score, which is a measure of their cognitive level, for each student was incorporated (Tables 5, 6 and 7).

Each analysis showed that the students included as having high competence in both their languages outperformed the other two groups, although Scheffe tests indicated that in each case the differences between this group and the 'one dominant' group were not statistically different. Both these groups however were significantly different to the students who were categorized as having low competency in both their languages. Although this is a small sample and more detailed analysis is still to be completed, it is interesting to note that these results are in line with Cummins' threshold hypotheses, and are similar to previous results that the authors have found working with very different groups of students in other countries.

Why do Students Swap?

The analyses show so far that the bilingual students do swap between their languages when doing mathematics, a fact not always recognized by teachers and curriculum developers. Further there is some suggestion that this is influenced by mathematical context and schooling. Not only this, the competencies that students have in both their languages may well be another important factor in their learning of mathematics.

TABLE 5: Analysis of variance using the score on the Mathematical Word Problems Test as the dependent variable, a Bilingual Language (LL) variable as an independent variable, with the Byte Score (MSBS), a measure of cognitive level as a covariant.

Source of Variation	Sum of Squares	DF	Mean Square	F	Sig of F
Covariates	42.81	1	42.81	18.74	.00
MSBS					
Main Effects	56.87	2	28.43	12.44	.00
LL					
Explained	99.68	3	33.23	14.54	.00
Residual	127.97	56	2.29		
Total	227.65	59	3.86		

Multiple classification analysis

Grand Mean = 6.65

Variable + Category	N	Unadjusted Dev'n	Eta	Adjusted for Independents + Covariates Dev'n	Beta
LL					
1	10	-2.15		-1.90	
2	42	0.18		0.14	
3	8	1.72		1.62	
			0.56		0.51
Multiple R Squared					0.44
Multiple					0.66

TABLE 6: Analysis of variance using the raw score on the Mathematical Novels Problem Test as the dependent variable, a Bilingual Language (LL) variable as an independent variable, with the Byte Score (MSBS), a measure of cognitive level as a covariant.

Source of Variation	Sum of Squares	DF	Mean Square	F	Sig of F
Covariates	7.65	1	7.65	2.06	.16
MSBS					
Main Effects	47.24	2	23.62	6.36	.00
LL					
Explained	54.89	3	18.30	4.93	.00
Residual	211.67	57	3.71		
Total	266.56	60	4.44		

Multiple classification analysis

Grand Mean = 4.61

Variable + Category	N	Unadjusted Dev'n	Eta	Adjusted for Independents + Covariates Dev'n	Beta
LL					
1	11	-1.79		-1.71	
2	42	0.20		0.19	
3	8	1.39		1.36	
			0.44		0.43
Multiple R Squared					0.21
Multiple R					0.45

TABLE 7: Analysis of variance using the 'novel' score on the Mathematical Novel Problem Test as the dependent variable, a Bilingual Language (LL) variable as an independent variable, with the Byte Score (MSBS), a measure of cognitive level as a covariant.

Source of Variation	Sum of Squares	DF	Mean Square	F	Sig of F
Covariates	65.20	1	65.20	3.12	.08
MSBS					
Main Effects	177.31	2	88.65	4.24	.02
LL					
Explained	242.50	3	80.83	3.87	.01
Residual	1191.44	57	20.90		
Total	1433.93	60	23.90		

Multiple classification analysis

Grand Mean = 9.03

Variable + Category	N	Unadjusted Dev'n	Eta	Adjusted for Independents + Covariates Dev'n	Beta
LL					
1	11	-3.85		-3.58	
2	42	0.63		0.58	
3	8	1.97		1.85	
			0.38		0.36
Multiple R Squared					0.17
Multiple R					0.41

However there is still the intriguing question of why do students swap languages, in learning environments that do not actively encouraged students to do so, nor recognize that this strategy is used. A sample of students from each school was interviewed on how they completed three or four mathematical problems, in particular what language did they used in the process, and why. Some speculation on this has been entered into elsewhere (Clarkson & Dawe, 1994) and some preliminary comments have also been recorded (Clarkson, 1996). Further analysis of the interview data is progressing and future papers will focus specifically on these matters. It seems to us that some interlocking factors are present here, with probably more than one having an influence in any one situation. Difficulty is certainly one dimension. If the student feels the item is difficult because of meanings not being clear when reading or comprehending the problem, then a switch may occur. However an affective response may also be playing some part in that students simply like to use this or that language. Memory also plays a role in that some students may recognize a problem, or an aspect of a problem, which is similar in some way to a previous one for which they obtained help from a significant other. If that help was given in their L1, then this may prompt a switch as the student enters fully into that memory situation.

We can not stress enough the complexity of the process in which the students are involved. Any solution process is involved, if it is more than rote application of a well known routine for the student. With bilinguals there are clearly added fields of complexity. We feel we are dealing with 'messy' data all the time, not because we

have inadequate methodological approaches, but because the very nature of the phenomena is complex and 'messy'. However we also believe it is giving rich insights into how children learn mathematics.

Conclusion

In 'The Age', Melbourne's leading broadsheet newspaper, a recent article on page three was headed 'Schools are failing immigrants: study' (Milburn, 1996). The gist of the argument was that students from non English speaking backgrounds at school needed more help with learning English as a second language. We have no problem with this position, as far as it goes. The recent government cutbacks at both Federal and State level do not help. But we would contend that this is only part of the story. Such students also need help with maintaining competence in their L1. Just as teachers recognize the importance of different ways of thinking about mathematics, be they analytic, visual, etc., and attune their methods of teaching to support these thinking strategies, so they, and curriculum writers, also need to be far more aware of the role that L1 plays across the whole of the school curriculum including mathematics, and plan to use the advantages that this can bring.

Notes

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YOUNG CHILDREN'S CONCEPTS OF SHAPE

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We investigated children's early conceptions of geometric shapes with the goal of creating detailed descriptions of these ideas. Data were collected through individual clinical interviews of 97 children ages 3.5 to 6.9 emphasizing identification and descriptions of shapes and reasons for these identifications. Young children initially formed schemas based on feature analysis of visual forms. As these schemas develop, children continue to rely primarily on visual matching to distinguish shapes. They are, however, also capable of, and show signs of, recognizing components and simple properties of familiar shapes. Thus, evidence supports previous claims (Clements & Battista, 1992) that a pre-representational level exists before van Hiele level 1 ("visual level") and that level 1 should be reconceptualized as syncretic rather than visual (Clements, 1992).

Extensive evaluations of mathematics learning indicate that elementary students are failing to learn basic geometric concepts and geometric problem solving, especially when compared to students from other nations (Clements & Battista, 1992). Apparently, much learning of geometric concepts has been by rote. Students frequently do not recognize components, properties, and relationships. One tenet of teaching for understanding, accepting a constructivist view of learning, is building on a child's existing ideas. We investigated children's early conceptions of geometric shapes to provide detailed descriptions of these ideas.

Theoretical Framework

Previous studies of children's geometric conceptions have provided useful foundations, but have also left gaps that are critical to the development of curriculum and the improvement of teaching. Three dominant lines of inquiry have been based on the theories of Piaget, the van Hieles, and cognitive psychology (Clements & Battista, 1992). Piagetian and cognitive psychology research have illuminated children's conceptions, but have not been grounded in educational concerns. Also, much of Piagetian research investigated the topological primacy hypothesis, which has not received strong support (Clements & Battista, 1992). In contrast, van Hielian research was grounded in educational concerns; however, the

original theory (van Hiele, 1959; van Hiele, 1986; van Hiele-Geldof, 1984), and most of the subsequent research studied students in middle school and beyond. Indeed, Clements & Battista (1992) postulated that a combination of the Piagetian and van Hielian perspectives is necessary, as the latter theory does not adequately describe young children's conceptions.

This study was designed to investigate the geometric concepts formed by young children. The goal was to answer the following questions. What criteria do preschool children use to distinguish one shape from another? Do they use criteria in a consistent manner? Is the content, complexity, and stability of these criteria related to age or gender?

Method

Participants were 97 predominantly middle class children, 48 boys and 49 girls from two preschools and an elementary school with two kindergartens. All children whose parents completed permission forms were involved. The children were aged 3.5 to 6.9 years and were divided into 3 groups according to age. Children younger than 4.5 years at the time of the study were grouped as the 4-year-olds (n=25), children between 4.5 and 5.5 years were grouped as the 5-year-olds (n=30), and those above 5.5 were grouped as the 6-year-olds (n=42).

Data were primarily collected through clinical interviews of the 97 children by researchers in an one-on-one setting. The focus of the interview was the children's responses as they performed shape selection tasks. These were pencil and paper tasks with the children marking all the circles on a page of separate geometric figures, similarly for squares, triangles, and rectangles (most of the distractors were visually similar to the goal shape), and ending with circles and squares in a complex configuration of overlapping forms (see Appendix). Each interview, lasting about 20 minutes, was videotaped. The responses were scored and coded, and the data analyzed to determine patterns and trends in the children's understanding of geometric concepts.

The first data set was created by scoring children's selections for correctness. The second was an analysis of children's verbalizations, both spontaneous and in response to the interviewer's questions. The interviewer asked these open-ended questions to clarify the criteria the children were using in making the selections (e.g., "How did you know that was a rectangle?"). Children's responses were coded into one of 22 response categories, based on the van Hiele theory and previous research. Each response category was classified in one of two superordinate categories, "visual" or "property." A visual response was coded for any reference to a form looking like an object and for descriptive such as "pointy", "round" and "skinny". A property response was one in which the child referred to the geometric components or properties of the form, such as "four sides the same length." In cases of multiple responses about a single figure, the dominate response was coded when possible; if no response was dominant, code 20 (more than one response in the visual category) or code 30 (more than one response in the property category) was used. Multiple responses spanning both categories were coded in the property category.

Findings

Developmental and gender differences were assessed with analyses of variance for each task. There were no significant difference between boys and girls on the overall scores of any shape selection task. To analyze children's verbal responses, we calculated the percentage of visual- and property-based responses children gave for each shape. To look at specific responses, we categorized each of the shapes into two groups, examples and non-examples of the shape class, and calculated percentages of each verbal response based on the total number possible for each group. For example, 84% of the 25 4-year-old children responded "I don't know" to one or more of the 9 examples of circles shown.

The circle selection task was the easiest for the children with a mean score of 14.5 out of a possible 15. A significant developmental difference was found in this task with the 6-year-olds performing significantly better than the younger children ($F = 5.54, p < .005$). Among the distractors, the ellipse (shape 11) was the most distracting with 12% of the children marking it as a circle; most of these children were either 4 or 5 years old. In addition, 20% of the 4-year olds identified the curved shape (shape 10) as a circle. Of the 57% of children who responded about the reasons for selections 56% were visual responses. Children, particularly the 4- and 5-year-old, gave more verbal responses for the non-examples than examples of circles. For the examples of the circles, the most dominant response was "round, curved, no straight sides, no corners." No one used a property response to justify a circle. Further, there was no significant correlation between children's property-based responses and their correct selection of the circle.

The mean score on the square selection task was 11.2 out of a possible 13, suggesting that the children were quite able to discriminate a square from the other forms on the page. Though there was no significant overall developmental difference, 28% of the 4-year olds and 13% of the 5-year olds, compared to only 5% of the 6-year-olds, identified the rhombus (shape 3) as a square. In contrast, 63% of the 4- and 5-year-olds and 68% of the 6-year-olds accepted squares with no side horizontal (shapes 5, 11, and 13) as squares. Of the 59% of responses giving reasons for selections, 44% were visual responses, generally "It just looks like a square." This was predominantly seen among the 4- and 5-year-old children who used more verbal responses for the non-examples rather than the examples of squares. The 15% of property responses referring to the number of sides and corners suggested that some children at this age are beginning to discriminate squares by components or properties. The statistically significant correlation (r=0.32) between property-based responses and children's correct selection of the square suggests that children are more likely to be accurate in their square identification should their reasoning be based on the shapes' attributes.

The triangle selection task was more difficult for the children; the mean score was 8.2 out of a possible 14. There was no significant difference between the age groups. Of the 70% of selection responses, 52% were in the visual category and 18% were in the property category, indicating young children's limited ability to recognize (or perhaps verbalize) geometric components and properties of this shape

category. Again, the children (particularly the 4- and 5-year-old) provided more verbal responses for the non-examples rather than the examples of the triangles. Of the visual responses, referring to another shape on the same page or alluding to "It looks like" appeared to be more popular. Of the property-based responses, the children referred to the number of sides and corners most frequently. One developmental pattern among the three age groups and their selection of particular figures emerged. The 5-year-olds were more likely than either the 4- or 6-year-olds to correctly identify the examples of triangles (shapes 1, 6, 8, 10, 11, 12), thereby indicating an inverse U-shaped trend in growth. However, the 5-year-olds were also more likely than both the 4- and 6-year-olds to accept curved sides, either convex or concave (shapes 3, 5, 7, 14).

The total mean score on the rectangle selection task was 8.01 out of a possible 15. Again the responses were mainly visual (52% of the total 61% of responses) confirming the children's apparent reliance on comparison to a visual image when distinguishing between forms. Again, more verbal responses were used for non-examples than examples. Of the visual responses, children responded "It looks like" more frequently. Of the few property-based responses, the 5-year-old children (more so than the others) referred to the number of sides and corners. The 4-year-olds were more likely to accept the squares as rectangles. Shape 2 was selected by 28% of the 4-year-olds, compared to 17% and 10% of the 5- and 6-year-olds, respectively. Shape 7 was selected by 16% of the 4-year-olds, in contrast to 3% and 7% of the 5- and 6-year-olds respectively accepting it. The children tended to accept "long" parallelograms or trapezoids (shapes 3, 6, 10, and 14) as rectangles; they were less likely to choose the shorter and nonparallelogram forms as rectangles. Children referred to properties less frequently for rectangles than they did for triangles and squares.

Finally, on the circle/square complex configuration, the mean score was 16.77 out of a total of 28. A significant difference was found between the age groups with the 6-year-olds children scoring better than those younger (F=13.526, p<.0001). The younger children identified two of the ellipses, shapes 6 and 14 (50% and 42% respectively), as circles. A fewer number of children so identified ellipses as circles in the circle selection task. In this task, some of the circles and squares were embedded within each other; children were less likely to identify these embedded shapes. For instance, shape 8 (a square) has shape 9 (another square) embedded within it and is also divided into four quarters to create four more squares (shapes 10, 11, 12 and 13). While 32% of the children selected shape 8, only 17% selected shape 9 and an even fewer number selected shapes 10, 11, 12 and 13. Few of the 4-year-olds, in particular, selected these embedded squares. The same was true with the circles. While 76% of the children identified shape 26 (an outer circle), only 17% identified the circle embedded within this shape (shape 27). Also, only 35% of the children marked the square inside the circle (shape 23), whereas 87% of the children selected the circle itself (shape 22). Overall, scores were lower on this embedded figures task. Few children provided verbal responses for these shapes, with hardly any property-based responses. A caveat is that some children showed

signs of tiring on this task and may have been giving less attention to their selections.

Several patterns emerged across the various shapes. On the square, triangle and rectangle tasks the children sometimes appeared not to distinguish the concepts of "side" and "corner" (or "point"). A child would say that a form had four "sides" and then, when asked to count them, would count the corners. This was particularly prevalent among preschool children and needs to be considered in further research.

Discussion

We investigated young children's conceptions of geometric shapes. Children identified circles with a high degree of accuracy. Six-year-olds performed significantly better than the younger children, who more frequently chose the ellipse and curved shape. Most children described circles visually, if at all. In sum, the circle was easily recognized but difficult to describe for these children. Evidence indicates that they matched the proffered shapes to a visual prototype.

Children were only slightly less accurate in identifying squares. Younger children were less accurate classifying nonsquare rhombi but no less accurate classifying squares without horizontal sides. A minority of the children's reasons for selections referred to properties, but there was a significant relationship between such responses and correct selections.

Children were less accurate recognizing triangles and rectangles. Property responses were again present but infrequent (18% for triangles) There was an inverse-U pattern in which 5-year-olds were more likely than younger or older children to accept both non-standard triangles and those with curved sides.

Children identified slightly more than half of the rectangles correctly. The 4-year-olds were more likely to accept the squares as rectangles. All children tended to accept "long" quadrilaterals with at least one pair of parallel sides as rectangles. They referred to properties less frequently for rectangles than for triangles and squares.

Children's accuracy was lowest on the circle/square complex configuration; in addition, the 6-year-olds were significantly more accurate than the younger children. More children identified ellipses as circles in this complex and embedded configuration.

These results have two theoretical implications for the realm of children's geometric understanding. First, the data support previous claims (Clements & Battista, 1992) that a pre-representational level exists before van Hiele level 1 ("visual level"). Children who cannot reliably distinguish circles or triangles from squares should be classified as pre-representational; those that are learning to do so should be considered in transition to, rather than "at," the visual level. We propose that children at this level are just starting to form schemas (networks of relationships connecting geometric concepts and processes in specific patterns) for the shapes. These early, unconscious schemas perform pattern matching through feature analysis (even though the objects form undifferentiated, cohesive units in

children's experience, c.f. Smith, 1989). For example, nascent schemas may ascertain the presence of a closed, "rounded" shape to match circles, four near-equal sides to match squares, and parallelism of opposite "long" sides to match rectangles. Later, these schemas incorporate other visual elements, such as the right angles of square, and thus can produce traditional prototypes. These prototypes may be over- or undergeneralized compared to mathematical categorization, of course, depending on the exemplars and nonexemplars children experience.

Second, the results support a reconceptualization of van Hiele level 1. The high proportion of visual responses were in line with theoretical predictions. However, there is also evidence among these young children of a recognition of some components and properties of shapes, though these may not be clearly defined (e.g., sides and corners). Some children appear to use both matching to a visual prototype (via feature analysis) and reasoning about components and properties to solve these selection tasks. Thus, this study provides evidence that Level 1 of geometric thought as proposed by the van Hieles is more syncretic than visual, as Clements (1992) suggested. That is, this level is a synthesis of verbal declarative and imagistic knowledge, each interacting with and enhancing the other. Thus, the name syncretic level, rather than the visual level, signifying a global combination without analysis (e.g., analysis of the properties of figures). At this level, children express the declarative knowledge more easily to explain why a particular figure is not a member of a class, because the contrast between the figure and the visual prototype provokes descriptions of differences. Children making the transition to the next level sometimes experience conflict between the two parts of the combination (prototype matching vs. component and property analysis), leading to incorrect and inconsistent task performance.

For example, in some cases, as children made increasingly more references to attributes during the categorization tasks, they also made fewer correct selections. These errors were made because, instead of relying solely on comparison to a mental prototype, these older children began to rely on attributes that they had determined as defining the category. For the square, many younger children interviewed categorized a form as a square because it "just looked like one". However, some children attended to the integral attributes that for the young child are "four sides the same and four points". The fact that a square also is defined by four right angles is not yet understood as integral by the young child and, thus, the acceptance of a rhombus as a square. The young child's reliance on nonintegral attributes or inattention to integral attributes thus leads to categorization errors. Mervis and Rosch (1981) theorized that generalizations based on similarity to highly representative exemplars will be the most accurate. This would account for the higher number of correct categorizations by those children who appeared to be making categorization decisions based on comparison to a visual prototype without attention to nonintegral features. Finally, strong feature-based schemas and integrated declarative knowledge, along with other visual skills, may be necessary for high performance, especially in complex, embedded configurations. To form useful declarative knowledge, especially robust knowledge supporting transition to

level 2, children must construct and consciously attend to the components and properties of geometric shapes as cognitive objects.

There were no significant difference between boys and girls on the overall scores of any shape selection task. Consistent with recent reviews, there is no data to support any hypothesis of gender difference in early geometric concept acquisition.

In summary, young children initially form schemas based on feature analysis of visual forms. These children can be classified as pre-representational (level 0). As these schemas develop, children continue to rely primarily on visual matching to distinguish shapes. They are, however, also capable of, and show signs of, recognizing components and simple properties of familiar shapes. The discrimination of shapes using a combination of visual schemas and an initial understanding of components and properties is characteristic of Clements' (1992) definition of a syncretic level of geometric understanding, a redefinition of the van Hieles' level 1, or visual level.

Descriptions of children's early conceptions of geometric shapes are important not only for theory, but also for teacher education (e.g., for cognitively-guided instruction models), and for developers of constructivist-oriented curricula. Too often, teachers and curriculum writers assume that students in early childhood classrooms have little or no such knowledge (Thomas, 1982). Obviously, this belief is incorrect; even preschool children exhibit substantial knowledge of simple geometric forms. Instruction should build on this knowledge and move beyond it. Students do not reach the descriptive level of geometry in part because they are not offered geometric problems in their early years (van Hiele, 1987). The "prolonged period of geometric inactivity" (Wirszup, 1976, p. 85) of the early grades leads to "geometrically deprived" children (Fuys, Geddes, & Tischler, 1988).

Evidence supporting the hypothesized syncretic level and an earlier pre-representational also provides useful information to researchers and to teachers of young children. We consider this level the syncretic level, rather than the visual level, signifying a global of combination of declarative and imagistic knowledge (without analysis). The question should not be whether geometric thinking is visual or not visual, but rather, whether imagery is limited to unanalyzed, global visual patterns or includes flexible, dynamic, abstract, manipulable imagistic knowledge. This latter type of knowledge, and the concurrent development of active and reflective visualization that acts on figures, not just on drawings, is a viable goal at all levels of thinking. So is robust, explicit knowledge of the components and properties of geometric shapes as cognitive objects.

The limited verbalizations of these young children, and the consequential ambiguity of the meaning of their utterances, imply that these results are suggestive rather than conclusive. We are presently conducting research using materials and methodologies to address remaining shortcomings in our knowledge of young children's geometric concepts of shapes.

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LEARNING FROM DISTRIBUTED THEORIES OF INTELLIGENCE

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The analysis reported in this paper focuses on the distributed view of intelligence developed by Pea and colleagues. I begin by identifying general areas of agreement and then discuss two points of contention. The first concerns the legitimacy of taking the individual as a unit of analysis, and here I argue that the distributed perspective implicitly accepts key tenets of mainstream American psychology's view of the individual even as it rejects it. The second point of contention concerns distributed intelligence's characterization of tool use. Drawing on a distinction made by Dewey, I argue that it is more useful for the purposes of instructional design to view activity that involves using an artifact as the tool, rather than the artifact per se.

As the title of this paper implies, I see much value in analyses of activity developed by Pea (1993) and his colleagues that stress the distributed nature of intelligence. I will discuss both the contributions of this theoretical orientation and the adaptations that I and my colleagues have found it necessary to make for our purposes as mathematics educators interested in instructional design.

Background

The practice of conducting longitudinal classroom teaching experiments in collaboration with teachers constitutes the background for the discussion. As part of the part of the process of preparing for an experiment, we clarify our overall instructional intent and outline provisional sequences of instructional activities. As Gravemeijer(1994) clarifies, the initial phase of this design process involves an anticipatory thought experiment in which the designer envisions how the instructional activities might be realized in interaction in the classroom, and how students' interpretations and solutions might evolve as they participate in them. In approaching design in this manner, the designer formulates conjectures about both the course of students' mathematical development and the means of supporting and organizing it. The domain-specific instructional theory that we draw on when conducting these initial thought experiments is that of Realistic Mathematics Education (Treffers, 1987). The issue of tool use and thus of the distributed nature

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of intelligence comes to the fore when we consider how students' mathematical learning is to be supported in this approach to instructional design.

A central heuristic of RME proposes that instructional sequences should involve settings in which students are explicitly encouraged to develop models of their initially informal mathematical activity. This modeling activity might involve acting with physical devices and computer-based tools, or it might involve making drawings, diagrams, or tables, or developing non-standard notations and using conventional mathematical notations. The conjecture underlying this heuristic is that, with the teacher's guidance, students' *models of* their informal activity will evolve into *models for* increasingly sophisticated mathematical reasoning. The designer therefore speculates that a shift will occur such that means of symbolizing initially developed as protocols of action (Dörfler, 1989) will subsequently take on a life of their own and become integral to mathematical reasoning in a range of settings. In this approach to design, tool use is viewed as central to the process by which students mathematize their activty. As a consequently, anticipating how students might act with particular tools and what they might learn as they do so is central to our attempts to support their mathematical development. I and my colleagues have followed this approach with some success in several teaching experiments, including one in which we designed computer microworlds as components of coherent instructional sequences. Vygotsky's (1987) claim that the tools with which people act profoundly influences the understandings they develop is therefore more than a theoretical conjecture for us. It instead describes the pedagogical reality in which we act when conducting teaching experiments. Given these considerations, it is readily apparent that theories that emphasize the distributed nature of intelligence are of great interest to us.

Distributed Intelligence

On my reading the distributed account of intelligence developed by Pea and his colleagues has evolved from mainstream American psychology and draws heavily on Vygotsky's treatment of semiotic mediation. A central assumption of this viewpoint in that intelligence is distributed "across minds, persons, and the symbolic and physical environments, both natural and artificial" (Pea, 1993, p. 47.) In Dorfler's (in press) formulation, this implies that "the whole system made up of the subject and the available cognitive tools and aids realizes the thinking process . . . Mathematical thinking for instance not only *uses* those cognitive tools as a separate means but they form a constitutive and systematic part of the thinking process." Consequently, tool use does not merely amplify human capabilities but is instead integral to the creation and reorganization of those capabilities.

One of the strengths of this view of intelligence is the manner in which it attempts to transcend the traditional philosophical dualism between the cognizing individual and the world about which he or she cognizes. In our work, I and my colleagues have also cast our analyses of students' mathematical activity non-dualist terms. In particular, we view students as participating in and contributing to the

development of classroom mathematical practices that involve reasoning with tools. By virtue of this participation, they are seen as acting in a taken-as-shared world of signification that constitutes what Lemke (in press) calls the semiotic ecology of the classroom. As a consequence, the relation between the students' activity and the world in which they act can be characterized as one of mutual constitution, a position consistent with Pea's viewpoint. As Whitson (in press) observes, theoretical approaches of this type that begin with activity in a world of signification simply bypass a number of philosophical issues including the classical problem of reference.

The Individual as a Unit of Analysis

The areas of consensus identified above serve to situate contrasts between the distributed perspective and the interpretive stance that has emerged in the course of our work in classrooms. The first of the two contrasts I will draw concerns Pea's critique of analyses that take the individual as a unit of analysis. In his view, the functional system consisting of the individual, tools, and social contexts is the appropriate unit of analysis. Pea's admonition contrasts sharply with the explicit attention that I and my colleagues give to individual students' interpretations and meanings. I will shortly suggest that the difference between Pea's position and ours hinges on the way in which the individual is characterized. First, however, I want to explain why I we find it essential to analyze individual students' mathematical interpretations as part of our practice in classrooms.

The lessons conducted in the course of a teaching experiment typically involve small group or individual activity followed by a teacher-orchestrated whole class discussion that focuses on the students' interpretations and solutions. During small group and individual activity, the teacher usually circulates around the classroom to gain a sense of the diverse ways in which the students are attempting to solve the tasks. For our part, I and a graduate research assistant each observe and interact with two students to document the process of their mathematical development on a daily basis. Towards the end of small group or individual work, the teacher, the graduate assistant, and I "huddle" in the classroom to discuss our observations and to plan for the subsequent whole class discussion. In these conversations, we routinely focus on individual students' qualitatively-different interpretations and meanings in order to develop conjectures about mathematically significant issues that might potentially emerge as topics of discussion. In this opportunistic approach, our intent is to capitalize on our prior observations by identifying specific students whose explanations might give rise to substantive mathematical discussions that will advance our pedagogical agenda. It is important to emphasize that our primary concern is with the quality of the discussion as a social event in which the students will participate. In particular, we conjecture that participating in discussions in which mathematical significant issues emerge constitutes supportive situations for the students' mathematical development. The teacher's role in these discussions is therefore not to persuade or cajole the students to accept one particular interpretation, but is instead to initiate and structure a conversation about issues judged to be mathematically significant issues per se.

It is important to emphasize that, as part of this planning process, we adopt a psychological constructivist perspective that focuses squarely on individual students' activity. To be sure, we are aware that we are analyzing individual students' activity as they participate in the mathematical practices of the classroom community. Nonetheless, a focus on individual students' diverse meanings is a central aspect of our classroom practice in that it enables us to be more effective in our collaborations with teachers. However, analyses of this type are prime examples of those disallowed by distributed theories of intelligence.

Characterizing the Individual

I have noted that distributed theories of intelligence evolved from mainstream American psychology. More precisely, this approach as formulated by Pea has developed in part by resisting key tenets of mainstream psychology. Foremost amongst these is the traditional separation between internal representations in the head and external representations in the world. However, distributed accounts of intelligence as formulated by Pea and his colleagues carry the vestiges of their development from mainstream psychology even as they react against it. This is particularly apparent in the debate between Pea (1993) and Solomon (1993) on the legitimacy of taking the individual as a unit of analysis.

Salomon contends that, in distributed accounts of intelligence, "the individual has been dismissed from theoretical consideration, possibly as an antithesis to the excessive emphasis on the individual by traditional psychology and educational approaches. But as a result the theory is truncated and conceptually unsatisfactory" (p. 111). Salomon goes on to argue that some competencies are not distributed but are instead solo achievements, and that the individual is the appropriate unit of analysis in such cases. Pea for his part counters that many tools and social networks are invisible, and that intelligence is distributed even in the case of apparently solo intelligence and purely mental thinking processes. Despite these differences in perspective, Pea and Salomon appear to agree on at least one point. The individual of whom they both speak is the disembodied creator of internal representations who inhabits the discourse of mainstream psychology. It is this theoretical individual who features in Pea's claim that intelligence is distributed across the individual, tools, and social context. In developing his viewpoint, Pea, in effect, attempts to equip this mainstream character with cultural tools and place it in social context. However, in doing so, he implicitly accepts the traditional characterization of the individual and preserves it as a component of tool-person systems even as he rejects it. In my view, however, the implicit assumption that mainstream psychology offers the only possible conception of the individual should itself be scrutinized.

As a starting point, I note that the psychological orientation that I and my colleagues take when analyzing individual students' activity is not part of the mainstream story, but is instead part of an alternative European tradition that draws on aspects of Piaget's (1970) genetic epistemology (Johnson, 1987; Winograd & Flores, 1986). In this tradition, there is no talk of processing information or

creating internal representations. Instead, intelligence is seen to be embodied, or to be located in activity. Further, rather representing a world, people are portrayed as individually and collectively enacting a taken-as-shared world of signification (Varela, Thompson, & Rosch, 1991). The goal of analyses conducted from this perspective is therefore not to specify cognitive mechanisms located in the head that intervene between input information from the environment and observed output responses. Instead, it is to infer the quality of individuals' experience in the world, and to account for developments in their ways of experiencing in terms of the reorganization of activity and of the world acted in.

Once this shift is made in the characterization of the individual, the dispute between Pea and Salomon dissipates. It no longer makes sense to talk of intelligence being stretched over individuals, tools, and social contexts. In particular, the physical devices and notations that people use are not considered to stand apart from the individual but are instead viewed as constituent part of their activity. As a consequence, students are described as reasoning with physical devices, computer-based tools, and notations. What, from the distributed intelligence perspective, is viewed as a student-tool system is, from the perspective I have outlined, characterized as an individual student engaging in mathematical activity of which the tool is constituent part. Thus, although the focus of this psychological viewpoint is explicitly on individual activity, its emphasis on tools is generally consistent with the notion of mediated action (John-Steiner, 1995; Meira, 1995).

With regard to the remaining component of the functional system posited by distributed intelligence, social context, I have already suggested that a students' activity can be viewed as an act of participating in the collective mathematical practices of the classroom community. As a consequence, we find it essential to coordinate psychological analyses of individual students' activity with an analysis of evolving the mathematical practices in which they participate (cf. Cobb, Gravemeijer, Yackel, et al., in press). This latter analysis of communal practices, it should be noted, simultaneously delineates the learning of the classroom community and the evolving social situation of the individual students' mathematical development. In such an approach, the basic relation between the communal practices and the activity of the students who participate in them is one of reflexivity. This is an extremely strong relation in that it does not merely mean that individual activity and communal practices are interdependent. Instead, it implies that one literally does not exist without the other. Cast in these term, both the process of individual students' mathematical development and its products, increasingly sophisticated ways of mathematical knowing, are seen to be social through and through. As a consequence, although psychological analyses are an essential part of our practice, they do not by themselves result in adequate accounts even of individual students' mathematical development. By the same token, an analysis that focuses only on communal practices is also inadequate *for our purposes*. Given our agenda, we find it necessary to focus on both the classroom practices in which students participate and the quality of their individual acts of participation.

In the analytical approach that I have sketched, individual activity is necessarily located in social context that is not assumed to exist apart from that activity. As a consequence of this reflexive relation, it does not make sense to talk of intelligence as being stretched over the individual and the social context. As is the case with tools, the issue of whether it is legitimate to take the individual as a unit of analysis arises only if one accepts mainstream psychology's characterization of the individual. The issue dissolves if social context is viewed as an integral aspect of individual activity. Given the alternative view of the individual, there is no need to equip individuals with tools or to place them in social context for the simple reason that individuals do not act apart from tools and contexts.

Tool Use

The second point where I and my colleagues find it necessary to depart from distributed accounts of activity concerns the way in which tool use is characterized. We have seen that, in distributed accounts, intelligence is said to be stretched over individual-tool-context systems. In this scheme, tools are treated in purely instrumental terms that separates ends from means. I can clarify this contention by referring to Dewey's (1977) discussion of two different ways of thinking about tool use. The particular example that Dewey considers is that of the role of scaffolding in the construction of a building. In one characterization, the scaffolding is viewed as an external piece of equipment, and in the other it is viewed as integral to the activity of building. He argues that "only in the former case can the scaffolding be considered a mere tool. In the latter case, the external scaffolding is *not* the instrumentality; the actual tool is the action of erecting the building, and this action involves the scaffolding as a constituent part of itself" (p. 362).

The view of the individual implicit in distributed accounts of intelligence leads to the first of these characterizations in which people are equipped with tools. Analyses cast in these terms provide compelling demonstrations that the introduction of a tool results in changes in forms of activity. For example, it has frequently been noted that students who are equipped with computers can "off load" computational processes and engage to a greater extent in planning and problem solving activities. Illustrations of this type clarify that tools are not mere amplifiers of activity. However, accounts based on the first of the two characterizations that Dewey identifies typically limit their focus to that of documenting the reorganizations that occur when people are equipped with tools by contrasting before and after snap shots. Although analyses of this type might be appropriate for many purposes, they do not address an issue central to my interests as a mathematics educator. This concerns the *process* by which mathematical activity evolves. When we analyze teaching experiment data, for example, it is not sufficient to demonstrate that the students' mathematical activity is qualitatively different than that of students who are not equipped with particular tools. Instead, when planning teaching experiments, we find it essential to anticipate the process by which ways of reasoning with tools might evolve. Further, when planning whole class discussions, we find it important to consider the various qualitatively-distinct ways in which individual students act with

those tools. In doing so, we adopt the second of Dewey's two characterizations of tool use by viewing the tool as a constituent part of the students' activity that is itself the instrumentality.

Bateson's (1973) example of a blind person using a stick provides perhaps the most well-known illustration of this second viewpoint on tool uses.

Suppose I am a blind man, and I use a stick. I go tap, tap, tap. Where do I start? Is my mental system bounded at the handle of the stick? Is it bounded by my skin? Does it start halfway up the stick? Does it start at the top of the stick? (p. 459)

For Bateson, the person-acting and the artifact-acted-with are inseparable. Significantly, in making this point, Bateson approaches activity from the inside rather than from the position of someone observing a blind person. He asks us to pretend that we are blind and to imagine the nature of our experience when using the stick. This actor's viewpoint stands in sharp contrast to the observer's orientation inherent in distributed accounts wherein an artifact and a person using it treated as separate components of a functional system. For the actor, however, the two are inseparable. In the case of Bateson's illustration, the tool is the act of tapping with the stick, not the stick per se. More generally, I and my colleagues preference for this actor's viewpoint is not restricted to the issue of tool use but is instead central to our activity as mathematics educators who co-participate in the learning-teaching process with teachers and their students. To co-participate is to engage in communicative interactions that involve a reciprocity of perspectives characteristic of the actor's viewpoint.

Conclusion

In this paper, I have delineated general areas of agreement with distributed theories of intelligence and have discussed two points of contention. In doing so, I have attempted to illustrate what I have learned as I have come to understand these theories, thereby acknowledging my debt to their developers. The challenges that these theories pose for those of us who see value in constructivist analyses of individual students' activity is particularly apparent in the case of tool use. As Walkerdine (1988) observes, semiotic processes in general and symbolizing in particular have often played little if any role in constructivist analyses of mathematical development. Instead, mathematical reasoning has sometimes been viewed as occurring apart from mediational means, with symbols serving as mere vehicles used to express its results. Much therefore remains to be learned from distributed analyses of mathematical activity. The challenge as I have framed it is to view mediational means as constituent parts of individual students' qualitatively distinct ways of participating in communal practices. The analyses that I and my colleagues have conducted of recent teaching experiments represent one attempt to move in this direction.

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AUSTRALIAN AND INDONESIAN STUDENT TEACHER BELIEFS ABOUT MATHEMATICS AND PERFORMANCE ON A CLASSIC RATIO TASK[1]

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Abstract

This paper reports on data from Indonesia and Australia which investigates first year primary (elementary) student teachers' performance on the classic student/professor ratio task and links this to their espoused beliefs about mathematics, mathematics learning and mathematics teaching. It forms part of a larger study covering five countries which was commenced in 1993. This bilateral investigation shows that there are significant differences to be found between the national cohorts in espoused beliefs and approach to the ratio task but not correctness of solutions. However, there appears to be little relationship between espoused beliefs, approach taken to the ratio task and correctness of solutions either within the two national cohorts or between them.

Introduction

It might be expected that there is some connection between the underlying beliefs about mathematics held by individuals and the nature of the performance of those individuals on a given piece of mathematics (Garofalo, 1989; Schoenfeld, 1985). In this paper, the beliefs about mathematics espoused by groups of beginning elementary student teachers in one university in Indonesia and two in Australia have been compared with the students' performance on a classic ratio task.

Beliefs and Mathematics

A belief can be defined as "*any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase: 'I believe that...'* " (Rokebach, 1968, p.2). "*...Everyone has a set of beliefs about how mathematics is learned. These beliefs have an influence on all aspects of teaching.*" (Baroody, 1987, p.5). Pajares (1992, p.237) suggests that beliefs are "*the single most important construct in educational research*". The complex interaction of the affective and cognitive domains in the learning and teaching of mathematics continues as an important area of investigation in mathematics education.

Proportional Reasoning

Proportional reasoning plays such a critical role in a student's mathematical development that it has been called a "*watershed concept, a cornerstone of higher*

[1] The authors are indebted to Mr Peter Howard, Australian Catholic University, Sydney, Australia for valuable assistance in collecting some of the Australian data.

mathematics, and a capstone of elementary concepts" (Lamon, 1993, p.41). How children and adults use proportional reasoning and solve proportion problems has been the focus of extensive research (see, for example: Behr, Harel, Post & Lesh, 1992; Conroy & Sutriyono, 1993; Conroy & Perry, 1996; Cramer & Post, 1993; Dube, 1990; Lawton, 1993; Perry, Foong & Conroy, 1996; Singer & Resnick, 1992; Tourniaire & Pulos, 1988).

Background studies

Dube (1990) investigated the performance of 240 high school students in Papua New Guinea in solving a one step ratio and proportion problem. She found that the solutions fell into two categories of approach which she called *holistic* and *analytic-synthetic*. "*The equation which they wrote down was the result either of a global perception of the entire problem as an integral whole, or of explicitly and carefully defined steps, first breaking up the problem into the given and the unknown, then writing down the required equation after using semantic and mathematical reasoning, algebraic manipulations, and arithmetical calculations. The first approach was termed* holistic; *the second* analytic-synthetic." (Dube, 1990, p.9). Roughly the same proportion of correct and incorrect solutions were produced by each approach (holistic: 40% correct; analytic-synthetic: 45% correct). However, the analytic-synthetic approach was by far the most frequently used (by 72% of all students). Dube (1990) analysed the solutions in terms of particular strategies used by students using the analytic-synthetic approach. This has been done also by the present authors but is beyond the scope of this paper.

The subsequent investigation by Conroy & Sutriyono (1993) of the performance of 140 Indonesian first year student teachers produced slightly different results. While the Dube (1990) classification of approach was applicable to the Indonesian group, the correctness of solution was found to depend on approach. As well, Conroy & Sutriyono (1993) investigated a possible connection between the Indonesian students' beliefs about mathematics and their performance on the problem-solving task. These data were discussed at the Sixth South East Asian Conference on Mathematics Education in Surabaya, 1993, and some interest was expressed in extending the study.

Further data were collected in Indonesia as well as in Australia, Germany, Singapore and Thailand. In this paper, only the new data from Indonesia and the Australian data have been considered.

The Sample

The Australian sample consisted of 178 primary student teachers from two universities (one Catholic and one secular) in Sydney, NSW. Both groups of students (46 and 132 respectively) were in the first semester of six-semester bachelor degree programs, each with its own curriculum.

The Indonesian cohort comprised 78 students in a non-government (Christian) university in Central Java. They were in the first semester of a four-semester diploma program preparing them as primary school teachers.

Methodology

The research questions for this study were:

A. Do beginning primary student teachers in Australia and Indonesia hold similar beliefs about the various aspects of mathematics?

B. Is the ratio problem equally difficult for beginning primary student teachers in Australia and Indonesia?

C. For beginning primary student teachers, is correctness of solution on the ratio task related to their beliefs about mathematics?

D. Do beginning primary student teachers who use similar approaches on the ratio task also hold similar beliefs about mathematics?

To gather data on their beliefs about mathematics, the students were presented with five incomplete sentences about mathematics, its learning and its teaching. They were asked to complete the sentences individually in whatever way they felt appropriate. To encourage the maximum openness of response, no verb was included in the incomplete sentences. The instructions were as follows:

Please complete the sentences given:

Question 1. In my opinion, mathematics . . .

Question 2. In my opinion, the process of obtaining mathematics knowledge . . .

Question 3. In my opinion, mathematics in schools . . .

Question 4. In my opinion, pupils involved in the process of obtaining mathematics knowledge . . .

Question 5. In my opinion, teaching mathematics in schools . . .

Students were given sufficient space after each statement to write their ideas fully.

The students were also presented with the following problem on a separate sheet of paper:

Please work the following problem as completely as possible:

'In a certain school there are 15 students for every teacher. If S is the number of students and T is the number of teachers, write down the equation which represents the given situation.'

Students were encouraged to write whatever explanation was necessary to support their answers. The ratio problem is identical with that used in the two previously cited studies (Dube, 1990; Conroy & Sutriyono, 1993) and is derived from the classic student/professor problem of Lochhead (1980). Students were not given a specific time limit for the two tasks but, in general, took approximately half an hour to complete both. The task was presented to Australian students in English and to the Indonesian students in Bahasa Indonesia. Back-translation was used to check the accuracy of translations and the compatibility of the different versions. In part, this explains some of the apparently awkward English used in the sentences (for example, '*In my opinion, pupils involved in the process of obtaining mathematics ...*'). The

coding of student teacher responses to both the 'belief sentences' and the ratio task was completed by one of the researchers who is proficient in both English and Bahasa Indonesia.

Results

A. *Beliefs*

Responses for the incomplete sentence: *In my opinion, mathematics . . .* were categorised into five main beliefs; namely, mathematics can be viewed as:

a. an affect (enjoyable, interesting, confusing, difficult etc);
b. being useful (important, necessary, beneficial in daily life etc);
c. a body of knowledge (related to other sciences, possessing broad content, explaining things in general etc);
d. an exact science (concerned with true results, calculation, formulae, technical terms etc); or
e. a way of thinking (needing rational thought, gaining confirmation through proof, concerned with how to know and define etc).

Sometimes responses combined two or more of these ideas or gave ideas that fell outside the main categories.

Almost equal percentages of the student teachers from Australia (35%) and Indonesia (37%) believed mathematics to be useful. However, 32% of the Australians viewed mathematics as an affect, against only 17% from Indonesia and only 6% of Australian students believed mathematics to be a body of knowledge, against 15% of the Indonesian student teachers. A chi-square analysis showed the differences between the cohorts from the two countries on their beliefs about mathematics were significant ($\chi^2(6, \underline{N} = 256) = 20.95$, $\underline{p}<0.005$).

Student teachers' responses to the incomplete sentence concerning their beliefs about *the process of obtaining mathematics knowledge* fell into three broad groups; namely, mathematics learning viewed as depending on:

a. some aspect of teaching (should be sequenced, requires patient teaching, provision of visual aids etc);
b. some aspect of the learning experiences (memorising formulae, concentrated effort, use of logic, life experiences etc); or
c. some aspect of the subject matter (difficult content, human creativity etc).

Again, responses also included combinations of these views or expressed other views.

Results show that for Australian students overall, opinions were fairly evenly divided among these three alternative beliefs about mathematics learning (19%, 26% and 26% respectively). However, Indonesian students were more likely to believe that mathematics learning was affected by either some aspect of teaching (28%) or some aspect of the learning experiences (35%). While these trends are clear, there was no statistical significance in these differences.

Responses to the sentence relating to *mathematics in schools* were grouped into six main categories; namely, beliefs that school mathematics can be seen as:

a. having utilitarian value;
b. affecting attitudes;
c. having broad cognitive implications (for example, it develops thinking);
d. depending on teaching for its quality;
e. needing to match the interests, abilities and understandings of students; or
f. depending on the quality of the curriculum.

Some students gave combinations of two or more of these views or gave other views.

Just as in the sentence concerning beliefs about mathematics, a greater percentage of student teachers from both Australia (27%) and Indonesia (32%) chose 'utility' in their response to this sentence than any other category. There were no statistically significant differences between the responses to this sentence from the two countries.

Completed sentences espousing student teachers' beliefs about *pupils involved in the process of obtaining mathematics knowledge* could be divided into four categories; namely, beliefs that children's learning of mathematics was influenced by:

a. affective factors (children's interest, motivation, enjoyment etc);
b. its activeness and relatedness to daily life;
c. cognitive and developmental factors (levels of ability, thinking skills etc); or
d. its reliance on memorisation and practice.

Some 32% of the Indonesian student teachers believed that mathematics learning was influenced by cognitive and developmental factors as against only 19% of the Australian cohort. Conversely, only 5% of the Indonesians believed that mathematics learning was influenced by its activeness and relatedness to daily life as against 24% of the Australians. A chi-square analysis showed the differences between the cohorts from the two countries on their beliefs about pupils obtaining mathematics knowledge were highly significant ($\chi^2(5, \underline{N} = 242) = 24.77$, $\underline{p}<0.0005$).

Student teachers' beliefs about *teaching mathematics in schools* fell into four categories; namely, beliefs that such teaching:

a. should relate mathematics to daily life and encourage student activity;
b. can influence attitudes;
c. can be an obstacle to learning; or
d. needs to relate to student learning.

Higher proportions of Australian student teachers than Indonesian student teachers held the beliefs that mathematics teaching needs to be relevant and active (26% and 18% respectively) and that such teaching can influence attitudes (28% and 15% respectively). On the other hand the Australians (4%) were less likely than the Indonesians (19%) to espouse the belief that teaching needs to relate to student learning. A chi-square analysis showed the differences between the cohorts from the two countries on their beliefs about teaching mathematics in schools were significant ($\chi^2(5, \underline{N} = 252) = 24.10$, $\underline{p}<0.0005$).

B. The Ratio Task

The one step Dube ratio task was a reasonably difficult mathematical problem for the beginning student teachers in both the Australian and Indonesian samples (27% and 35% correct, respectively). There was no statistically significant difference between the level of performance on the task by the two national groups.

C. Beliefs and Performance on Ratio Task

The data did not indicate any statistically significant relationships between the espoused beliefs of the student teachers surveyed and their performance on the Dube ratio task. This held true whether the data were analysed separately in the two national groups or as one overall set of data. What usually occurred was that the student teachers held one or two predominant beliefs concerning each of the five aspects raised by the open-ended sentences, and these were fairly evenly spread between students who achieved correct and incorrect solutions.

While not statistically significant, the only sentence which did give rise to some interesting trend data is '*In my opinion, teaching mathematics in schools ...*'. For the Australian student teachers, 29% obtaining an incorrect solution believed that teaching should relate mathematics to daily life and encourage activity against 18% who obtained a correct solution. Thirty-four percent of Australian student teachers obtaining a correct solution believed that teaching mathematics in schools can influence attitudes while 24% obtaining an incorrect solution felt this way. While the same trends could not be discerned as clearly in the Indonesian data, the Australian data was sufficiently strong to ensure that similar trends could be found in the analysis of the overall combined data set.

D. Beliefs and Approaches to the Ratio Task

There was a highly significant relationship between the country of origin of the student teachers in the sample and the approach they took to the Dube ratio task ($\chi^2(1, \underline{N} = 211) = 26.71$, $\underline{p}<0.00001$). Eighty-three percent of the Australian students who provided a solution to the problem used an holistic approach while the Indonesians were evenly split between holistic and analytic-synthetic approaches (49% and 51% respectively). This, coupled with the differences in beliefs between the cohorts of student teachers from the two countries, raised expectations of statistically significant relationships between the student teachers' espoused beliefs and their approaches to the ratio task. However, this occurred with reference to only one of the 'belief sentences'.

When the data were analysed in the two separate national groups, it was found that the approach used to the ratio task was independent of the beliefs espoused by the student teachers. When the analysis was done on the combined data set from both countries, the same was true except when dealing with the statement '*In my opinion, mathematics in schools ...*'. In this case, a statistically significant result ($\chi^2(7, \underline{N} = 210) = 19.02$, $\underline{p}<0.01$) was obtained with 38% of student teachers using an

analytic-synthetic approach seeing mathematics as having utilitarian value as against 25% of the 'holistic approach' student teachers. Further, higher proportions of student teachers using an holistic approach believed that mathematics in schools affects attitudes (18%) and depends on teaching (20%) than those using an analytic-synthetic approach (9% and 10%, respectively).

Discussion and Conclusion

It should be of no surprise that beginning primary student teachers from two countries as different in their education systems as Indonesia and Australia (at least New South Wales) should have developed beliefs about mathematics, mathematics learning and mathematics teaching which were different in many respects. Similarly, it was not surprising that these same student teachers should attack a ratio problem in markedly different ways. Perhaps the most surprising finding of this study was that these differences were not nearly so marked when relationships between the approach and performance on the ratio task and beliefs about mathematics were considered. Despite differences in language, culture, curriculum, school system, teaching methods and teacher education, there was much in common.

Cross-cultural studies are fraught with difficulty in that so many factors may be acting on the variables under consideration that meaningful interpretation of results can be tenuous. The value of such studies is that they can, nonetheless, identify what appear to be significant factors acting across or within cultures, and point the way to further research which controls for these factors and investigates their influence.

It would appear from this investigation that there are notable similarities and differences across the samples from Indonesia and Australia in terms of beliefs about mathematics, success rates on the ratio problem and the approaches used in the problem solution. Further research needs to be undertaken in an attempt to ascertain reasons for these differences. Possible topics for investigation include analysis of composition of mathematics curricula used in schools, approaches to learning and teaching mathematics in schools, the influence of language on problem-solving approaches, the nature of teacher education programs and the interaction of these with beliefs about mathematics. The authors hope to enlist further assistance from international colleagues to gather additional data in a wider study concerning the beliefs and performance of teacher education students.

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THE TRANSITION FROM ARITHMETIC TO ALGEBRA: A COGNITIVE PERSPECTIVE

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This paper discusses the transition from arithmetic to algebra from a cognitive perspective, proposes a two path model for learning algebra, and uses the results of two studies to illustrate the importance of cognitive load and appropriate sequencing through binary algebra and complex arithmetic to effective learning of early algebra.

Early algebra teaching and learning has been a major research area in mathematics education. Part of this research has focused on the transition from arithmetic to algebra. Other research has addressed specifically the difficulties and obstacles to developing algebraic concepts caused by what has been described as cognitive gaps (Booth, 1988; Herscovics and Linchevski, 1994) or didactic cuts (Filloy & Rojano, 1989) between arithmetic and algebra. Filloy and Rojano (1989) believe the cognitive gap is located between the knowledge required to solve arithmetical equations, by inverting or undoing, and the knowledge required to solve algebraic equations by operating on or with the unknown. They suggested that an operational level, one of "pre-algebraic knowledge", is needed between arithmetic and algebra. Herscovics and Linchevski (1994) argued similarly that while properties and conventions are crucial in algebra, they can be replaced in arithmetic with an operational approach.

SEQUENTIAL DEVELOPMENT OF ALGEBRA KNOWLEDGE

Linchevski and Herscovics (1996) found, in research into seventh graders' solutions for first degree equations in the unknown, that for equations with only one occurrence of the unknown, e.g. $ax+b=c$, $ax+b+c=d+e$, nearly all the students used inverse operations in the reverse order. When the unknown appeared as a subtrahend or a divisor (e.g., $37-n=18$), equations were solved arithmetically without any transformation of the original equation. With examples involving two occurrences of the unknown there was a fundamental shift in procedures with the majority of students using a process of systematic approximation based on numerical substitution. They concluded that students could not operate spontaneously on or with the unknown and that grouping algebraic terms is not a simple problem. They argued that algebraic expressions are intuitively viewed as computational processes (cf. Sfard & Linchevski, 1994) and that in teaching, instead of proceeding from the variable to the expression to the equation, arithmetical solution of linear equations might be more suitable initially for learning to operate on or with the variable.

Biggs and Collis (1982) described development of algebraic concepts in terms of the SOLO (Structure of Observed Learning Outcomes) Taxonomy and on the

basis of Collis' (1975) research. SOLO responses occur sequentially as follows; prestructural (incompetence, nothing is known about the area), unistructural (one relevant aspect is known); multistructural (several relevant independent aspects are known), relational (aspects of knowledge are integrated into a structure), extended abstract (knowledge is generalised to a new domain). These responses occur in cyclical fashion, for increasingly more formal modes of learning, from sensori-motor, iconic, concrete-symbolic, formal-1 to formal-2 modes. Students learning school algebra should respond at least in the concrete symbolic mode, that is relate their knowledge of operations to the symbols that represent them. Biggs and Collis found responses for understanding pronumerals as follows; unistructural, map the pronumeral directly into a specific number; multistructural, map the pronumeral into a few sets of numbers; relational, conceive of it as a generalised number to represent all conceivable numbers; extended abstract, think of it as a variable. For numbers, operations and closure sequential responses were; unistructural, success with arithmetical operations where one closure was required (they asserted that the working memory capacity required for such items was low and quick closure was achieved on the basis of a minimal use of data, p. 62); multistructural, success both with large numbers involving single operations and with a series of operations in sequence with small numbers (closure is made in sequence with series of small numbers and is not necessary with single operations on large numbers); relational, `generalized' elements, that is large numbers and x standing for particular numbers (the idea of operations was generalized sufficiently so that there was no longer the need to close each operation immediately); extended abstract, a new level of functioning, closure was not required, and problems with operations on variables were solved.

Halford (e.g. 1993) proposed a structure mapping theory of cognitive development where binary operations, that is three elements considered in each mapping decision (operations with one closure as described by Collis), were at the system mapping level, and compositions of operations, that is operations where four elements or more must be considered in each mapping decision (operations with more than one closure as described by Collis), were at the multiple system mapping level. A linear equation such as $x+5=13$ is a binary operation as is $8+5=13$ and both should make the same demand on capacity to process information. What makes them different is the extra knowledge, that of x representing the unknown, required to compute the first equation. Halford and Boulton-Lewis (1992) proposed a sequence of development of arithmetical and algebraic knowledge from the initial use of concrete materials to represent number, to arithmetic expressions, to interpretation and manipulation of operational symbols and parentheses in arithmetic, to recognition of the correspondence of these examples in arithmetic and in algebra. It was asserted that the tasks themselves would make increasing demands on processing capacity and that the recognition of the correspondence between the

arithmetic and algebraic equations would depend on requisite knowledge, that is, of unknowns and then variables.

Mathematical analysis of sequence for learning complex equations. Linear equations in algebra such as $2x+3=11$ include three crucial components: an equals sign, a series of more than one operation, and a variable 'x'. Therefore, these equations are described as complex as opposed to a single binary operations such as $x+5=6$. We propose that solution of a complex equation is the end product of a learning sequence of mathematical concepts that includes: (a) binary arithmetic operations; (b) complex arithmetic (a series of operations on numbers); and (c) binary algebraic operations. These considerations underlie the construction of a two path model for learning complex algebra where binary arithmetical operations and complex arithmetical operations are necessary components of one path and binary arithmetical operations and binary algebraic operations are necessary components of a second path. This means that understanding binary operations such as $2x$ and $x+3$ should be an important prerequisite to understanding $2x+3=11$ and that understanding of operational laws should also be applied to series of operations. This is because relating laws to more than one operation in a series is important to understanding inverse operations which require the order of the series to be reversed as well as the operations. The two path model assumes that learning linear equations will be facilitated by understanding of similar (isomorphic) structures in complex arithmetic. Hence, understanding of arithmetical structure becomes an important component of learning algebra.

On the basis of the discussions above, we are testing the two path model of the sequential development of prealgebraic and algebraic thinking in a longitudinal study. In addition to the mathematical analysis the basic assumptions supporting the model are that: (a) the developmental literature suggests acquisition of prealgebraic and algebraic concepts in the following order - one occurrence of the unknown in binary operations, a series of operations on and with numbers and the unknown, multiples of the unknown, acceptance of lack of closure and immediate solution with a series of operations on the unknown, and finally, relationships between two variables and operations on them; (b) the most accessible route to algebra is through arithmetical procedures for solving problems with one unknown; and (c) crossing the cognitive gap or the didactic cut requires knowledge that the equals sign represents equivalence and, at least, knowledge that letters represent the unknown in algebra.

Representations and strategies used in teaching early algebra. MacGregor and Stacey (1995) asserted that `many recommendations in the pedagogical literature [for teaching algebra] ... have no supporting research background' (p. 82). They found, for example, that the use of patterns in primary grades and beyond as a foundation for algebra did not lead to the understanding that might be expected. This section presents an explanation for the difficulties students

experience with algebra based on the relative cognitive demands of the representations and strategies used. There appears to be considerable confusion, in school based documents and in textbooks, about the use of strategies and representations in the introduction of prealgebra and algebra and also, about the conceptions or aspects of algebra to which they relate. A range of representations and strategies are often used for short periods with little regard to the extra demand they are likely to make on students' capacity to process information and with no apparent connection between them and the various algebraic concepts that they are used to facilitate.

Sowell (1989) compared the outcomes of mathematics instruction with and without concrete or pictorial materials and found the results of 60 studies were mixed. She concluded that mathematics achievement is increased through the long-term use of concrete instructional materials and that students' attitudes towards mathematics are improved when they have instruction with such materials by teachers who are knowledgeable about their use. Sweller (Ward & Sweller, 1990) proposed a cognitive load theory which predicts that tasks will be more difficult if there is redundancy in the information which must be processed or if attention must be split between two different sources of information. Concrete materials and other representations impose a demand, additional to the task, on capacity to process information unless these materials are well known (Halford & Boulton-Lewis, 1992), and all physical models contain intrinsic restrictions that can lead to cognitive difficulties (Bher et al, 1983). Hart (1989) stated that there is apparently little connection for children between a practical or material-based approach to mathematics and formal or symbolic mathematical language. She suggested that we need to think carefully about assumptions that we make concerning the transition from practical to formal work because it appears that the gap between the two types of experience is too large and suggested we need to find effective transitional experiences. Quinlan (Quinlan, Low, Sawyer & White, 1993) described the use of containers with small objects inside them to represent variables, and objects outside of the containers to represent constants, in linear expressions to introduce variables. He has demonstrated that this approach is successful if it is used consistently and if teachers help students to make explicit mappings from the concrete materials to the symbols. Thompson (1988) recommended different coloured counters to represent directed numbers and variables in equations. For his approach, not only must students keep track of the solution process of the equation, they must also keep in mind that counters, for example of one colour, cancel out another. This is likely to increase processing load and lead to confusion. Balance beams can be used, for example, to lead from arithmetic identities to algebraic expressions with occurrences and multiples of the unknown on both sides of the equation (Linchevski & Herscovics, 1996); however, the balance beam itself can increase cognitive load unless its function is well understood.

MacGregor and Stacey (1995) investigated the use of patterns and tables of variables to teach functional relationships between two variables, as an introduction to algebra. These included geometrical patterns and tables of related variables. It has been argued that introducing algebra in this way facilitates its use as a language for expressing relationships and that the patterns can be supported by concrete materials. These approaches require students to understand variables and the syntax of algebra, to take into account two operations at once and the relationship between them, and in some cases to relate this to concrete representations. It is obviously a more cognitively demanding task than the use of a linear equation with one unknown and it is no wonder that MacGregor and Stacey 1993) found that many students have difficulty generating algebraic rules from patterns and tables.

An equation can be conceived in terms of balance or in terms of a two-way reversible change. Variables can be conceived in terms of generalisations, unknowns and relationships and well as abstractly (Usiskin, 1988). If different representations are used to lead to different conceptions, and not explicitly related to those conceptions, then they will surely cause all but the best students to develop a very confused understanding of variables and equations. It seems clear that representations, if used, must be used consistently because if a particular representation or model is unfamiliar it will add to the processing load of the task. They must be used for a long enough period because students need time to think and use materials experimentally. And the different algebraic concepts to be derived from the materials must be made explicit and related to each other.

As we stated above, the most accessible route to algebra could be through solving simple linear equations arithmetically. This fits with Sfard's operational perspective of algebra (Sfard & Linchevski, 1994) and has the advantage that the transition can be made from arithmetic to algebra without the need for concrete representations except in the early stages of learning about numbers and arithmetic and perhaps to illustrate the concepts of the variable. This is in keeping with the developmental sequence proposed by Halford & Boulton-Lewis (1992).

CLASSROOM STUDIES

We have been undertaking a series of studies to investigate: (a) the knowledge that students acquire and the strategies they use in moving from arithmetic to algebra; and (b) the effectiveness of different representations in students' aquisition of algebraic knowledge and strategies. The first of these (Boulton-Lewis, Cooper, Atweh, Mutch & Wilss, 1995) was a one year study of one class of grade 8 who were being introduced to variable, equation and solution of linear equations by instruction which used concrete materials (containers and objects). The second is a three-year longitudinal study which will follow students from knowledge of arithmetic and prealgebra in grade 7 to the completion of initial algebra instruction in grade 9. In these two studies, it was expected that: (a) teachers would use a range of representations and strategies to assist students to

understand unknown and variable, expression and equation, and solution of equations and that these would be one focus of analysis; (b) teachers would be interviewed to determine what and how they are teaching; and (c) the students would be interviewed to determine their understanding of relevant arithmetic and algebra concepts and their use of strategies and representations with particular emphasis on linear equations.

In this paper, the results of the one year study and the initial interviews undertaken in the first year of the longitudinal study will be discussed in relation to the developmental sequence for algebra discussed earlier.

Effectiveness of concrete representations. The results in the one year study were unequivocal at one level - after instruction, at the post-interview, not one of the 21 students in the class used the procedures with concrete materials taught to them to solve linear equations. A minority (6) incorrectly interpreted the linear equation in terms of one operation, while the majority (14) used inverse operations in the reverse order (1 student used trial and error). In this, the study supported the findings of Linchevski and Herscovics (1996). Only one of the students could use containers and objects correctly to represent the equation. No students used the containers and objects voluntarily. Even when directly asked to use materials to solve an equation, only four of the 21 students were able to reproduce the techniques shown in instruction. When asked about materials, students gave only limited support to their usefulness.

This failure of direct instruction to influence students' approaches to solution is interesting on a number of fronts. First, the inverse operations strategy uses a different conception of equation (two-way change) from the taught materials approach (balance). Second, the students' responses reflected the findings of Hart (1989) in that a gap exists between concrete and symbolic representation. Third, the finding seems to reinforce the heavy cognitive load involved in using containers and objects. Fourth, the finding also seems to support Kieran's (1992) argument that algebra knowledge develops from procedural to structural as inverse operations is a procedural strategy while the containers and objects approach appears to have structural tendencies. However, this first study also highlighted the common difficulty that students have with multiples of a variable such as $3x$. Many students interpreted $3x$ as one variable and a three. It also highlighted the need for students to understand expressions with a sequence of operations if they are to understand linear equations in terms of operating on a variable. This and a further search of the literature led to the proposing of two path model for learning algebra.

Arithmetic to algebra. In the first year of the longitudinal study, interviews with the 51 grade 7 students provided information about knowledge of operations and operational laws, equals, pronumerals, variables, and solution strategies for linear equations. The students would have been taught about operations and equals as part of the curriculum in arithmetic, but any knowledge of

variables and linear equations could only have been derived intuitively from their knowledge of arithmetic

Most of the students had sufficient understanding of the basic binary operations or sequences of binary operations for subtraction, multiplication and addition, in that order, to be able use them as a basis for algebra. Some students did not have sufficient understanding of division. Two thirds of the group had sufficient understanding of the inverses of multiplication/division and addition/subtraction. About 50% of the sample did not understand commutativity. Only about 20-25% of the sample had sufficient understanding of the correct order of arithmetical operations to allow them to apply this satisfactorily to learning linear equations. With regard to the equals sign in an unfinished equation with a series of operations, almost 100% of the students believed it meant find the answer, and in the completed equation only half of the students could say that it meant that both sides of the equation were the same. This means that in subsequent learning of algebra most of them would initially want to find the answer after the sign and at least half of them would need to learn the concept of equivalence. More than half the sample could solve an equation with as an unknown number or knew intuitively that it was like an x or y despite having no explicit instruction in variables. The majority of students understood what x meant in a linear equation but less than a fifth of them had a satisfactory concept of multiples of x. When asked to use concrete materials most of the students used them to illustrate their arithmetical solutions. A few had an intuitive idea of algebra and did not need materials. Most of the students who used materials and gave an arithmetical answer really did not need these either as evidenced by their explanations for variables.

IMPLICATIONS

With regard to the knowledge that students acquire and the strategies they might use in moving from arithmetic to algebra; it would seem that most of the students studied would need better understanding of division and the order of operations in complex arithmetic. Most students appeared able to use binary operations in linear equations and interpret '=' as calling for an answer. They would need to learn the equivalence meaning of '=' in an equation (perhaps with the use of a balance beam). They would also need careful and explicit instruction in the meaning of x as a variable rather than the unknown and then in the meaning of multiples of x. Thus, for the most part, the students have acquired satisfactory knowledge of the binary arithmetic and binary algebra components of the two path model from an arithmetical perspective, with the exception of knowledge of multiples of the unknown. The developmental sequence for the sample fits well, at this stage, with the two path model, that is, that complex algebra develops from binary arithmetic via both binary algebra and complex aritimetic (arithmetic with a sequence of operations).

With regard to the effectiveness of different representations in students' aquisition of algebraic knowledge and strategies, it is evident from our studies that students did not want to use concrete representations themselves, preferring a mental inverse-operations approach. In fact, use of concrete representations by students seemed to be counterproductive due to difficulties with cognitive load. However, there does appear to a place for containers and objects to represent variable and multiples of variable, if used explicitly and unambiguously by the teacher. Overall, our research has found need for consistency in dealing with the conception of equals and variable when using representations.

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A NEW APPROACH FOR INTELLIGENT TUTORING SYSTEMS: AN EXAMPLE FOR STATISTICAL ACTIVITIES

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ABSTRACT

In the last 30 years several efforts with different aproaches have been done in order to use the potential of the computer for education; however, there is still a long way to go in order to obtain educational software really useful for a wide group of students in their goal of acquisition of knowledge. In this paper are presented a proposal for a new class of Intelligent Tutoring Systems (ITS) and an example of how statistical activities can be implemented in an ITS of this class.

INTRODUCTION

To obtain computer software for education is a very complex task as can be seen by the wide range of educational software developed in the last 30 years. This software was built following different epistemological positions and under many different degrees of refinement. In this paper, we present a new approach for building educational software that combines two very popular epistemological positions, the traditional and the constructivist. Our compromise lies in between the teacher direction and the autonomy of the student. The goal of this approach is to obtain ITS to function as partners of the teachers. The outline of the paper is the following: in the first section, we present a brief review of existing computer software for education, pointing out their epistemological position, and discussing their characteristics; in the second section we present the approach proposed, in the third section we present a few examples of how statistical activities can be implemented using this approach and, in the last section we give some concluding remarks.

1. COMPUTER SOFTWARE FOR EDUCATION

The idea of improving, individualizing, and making more flexible the teaching with with a computer have produced a lot of educational software. In this section we follow the classification of educational computer software given by Allison and Hammon (1990) which is based on the software's mode of operation: programmed instruction, learning support environments, and intelligent tutoring systems. This

classification is by no means exhaustive with respect to computers teaching aids, and there exits other kinds of scientific and professional software of general purpose, not specifically educational, that have been widely used in the teaching of sciences, particularly in engineering and statistics.

Programmed instruction. In their begginings, this kind of systems, called Computer Assisted Instruction systems (CAI) were strongly influenced by the conductism theory developed by Skinner (1986). Many of these systems utilized the computer as a delivery device to convey information or "knowledge" and were no more than electronic turn-pagers. Others, implemented the simplified idea of knowledge acquisition based on the stimulus-responce binomio defined for Skinner, to "teach" the student aspects of some specific domain. This type of software includes the works of Skinner himself, the works of Ayscough, and Palmer and Oldehoedft (O'Shea and Self, 1983).

Later, two very ambitious projects were launched in the USA with the aim of a massive use of the computer, the TICCIT and PLATO projects (ibid.). Most of the educational materials developed under these projects still were pretty rigid in their teaching strategy because of the agglutination of the pedagogical and the domain knowledge.

Recently, more flexible programmed instruction software for specific courses has been developed (Bishop et al., 1992).

The epistemological approach that underlies the programming instruction based systems is known as the traditional one. It sees the knowledge as a kind of package which can be delivered, and in order to "individualize" the teaching, the software are designed with branching capabilities which allow the student to direct their path through the system, and many of them are of the drill and practice type. Despite the fact that this kind of software has been widely critized because of its rigidity, most of the commercial educational software is of this type.

Learning support environments. The most known representative of this approach, known as microworlds, is the LOGO project developed by Feurzeig and Papert

(Papert, 1980) that has been widely used in mathematics education (Lemeris, 1990). Other widely known microworlds are Cabri-Gèométre (Laborde, 1986), and Voltaville (Glase et al., 1988).

Microworlds, view the computer as a medium for the student to solve problems and are based in the belief that the real impact of the computer can only materialize if the student has complete control over it without the teacher interaction.

The epistemology behind the learning support environments, is a constructivist one. Learning is seen as a autonomous process of construction of cognitive structures so a complete control for interacting with the computer is given to the student. The assumptions behind a microworld design are: 1) the learning of solution methods is produced as a colateral effect of the interaction of the student with the microworld tools which are specifically design for suggesting good solution strategies; 2) the student is motivated and interested in obtaining the knowledge that can be given through the microworld; 3) the student is eager to explore the microworld; and 4) the student is capable of extrapolating the knowledge acquired in the microworld to the real world.

Intelligent tutoring systems. In these systems, a flexible approach to teaching is implemented through the use of some kind of mecanism to evaluate the student's response. Depending on the student's answers to questions asked by the system, a decision is taken to branch to new material or to a remedial one. The author(s) of the system try to anticipate all possible student errors and specify remedial material for all the possibilities based on the idea of what might be the bad conceptions that cause the wrong answer. In order to guide properly the student interaction with the system a model of the student is internally implemented. There is a wide variation in the level of detail of the model of the student (learner) in different ITS. Some widely known examples of ITS are: SCHOLAR (Carbonell, 1970), GUIDON (Clamcey, 1979) and Anderson's Geometry and LISP tutors (Anderson, 1986). One of the main characteristics of ITS is the separation of the tutorial knowledge from the domain knowledge.

The underlying epistemology in most ITS is still the traditional epistemology

which sees knowledge as a kind of facility which can be delivered. Intensive research on ITS has given birth to a wide spectrum of ITS applications in many areas.

2. A NEW CLASS OF ITS

It is nowadays almost a consensus among educators that the student should construct their knowledge. If the computer is going to be used for improving education, this idea should be underlying the design of new educational computer environments. The constructivist epistemology developed by Jean Piaget, describes the knowledge acquisition as a continuos process of construction of cognitive structures launched by a situation that cannot be handled with the actual cognitive structures, that is by a desequilibrating situation. Cognitive structures are organized systems of mental representation of activities (operations) related by a form of performance which have an associated expected result. To construct his (her) knowledge the student has to be involved in some type of activity. A concept is constructed around its constitutive operations and it is the group organization of these operations which gives its flexibility for application in a variety of situations. The flexibility attained by the organization of operations in groups contrast to the rigidity of habits acquired in the traditional education. This focus on education, where the student is an active entity who builds his (her) own knowledge through interiorization and organization of operations is considered in learning support environments, but their openess does not guarantee that the student will explore the operations needed to acquired a particular concept and will explore their relations, nor guarantee that the student will extrapolate the concepts acquired in the microworld to the real world. The teacher guidance is convenient to organize the activities that might lead the student to interiorize the operations related to a concept or notion and build his (her) own knowledge. The teacher could also help the student to extend his (her) knowledge to other real world problems and to emphasize certain elements.

Extracting the best elements of both ITS and Learning Environments we propose a model of intelligent tutoring system which implements a constructivist approach of learning. Considering the actual limitations in our knowledge about the function of the mind, we support the idea of not trying to develop a computer system to substitute the teacher but to build a computer system that be his (her) allied in the teaching process, a kind of teacher's assistant; leaving the responsability of analizing the

student behaviour and taking specific decisions to guide the student activity with the system to the teacher. The main characteristics such an ITS should have are the following:

-The system should be capable of operating in a dual form: as an exploratory world where the student can investigate his ideas about a certain topic with the help of tools provided by the system, and a tutor environment where the activities which can help the student to build a concept are gradually provided in a similar form that a flesh teacher would do it.

-In the tutor mode of operation the order of presentation of the activities and the number of examples and exercicies provided to the student are guided by an underlying intelligent tutor, which is flexible and adaptable to the student's needs. More examples, exercises and explanations can be given at student request, for help him (her), adapting the tutoring to the student and never imposed.

-A student model may be present but it is not a vital part of the system because in this approach, the ITS is not a substitute of the teacher but a partner which share the responsability of teaching, and it is the teacher who has to make the tutoring decisions.

-Activities that might lead the student to develop interiorized actions (operations) related to a concept or notion of interest should be implemented for each concept to be teached. It is the responsability of the system designer to identify such operations following the Piaget's idea of their organization in groups, implement activities for the acquisition of the direct, asociative and inverse operations related to a concept, and provide facilities for their exploration.

-Facilities for the student to investigate around a concept should be provided by means of tools in the computer environment.

The computer environment with the above characteristics is an intelligent tutor because in one of its modes of operation, the tutor mode, the student activities with the system are dosified and organized in a similar way that it might be provided by a experienced teacher (tutor); this is attained by means of incorporating tutoring knowledge in the system. It is also intelligent because it is capable of solving problems and answer questions on its domain (Grandbastein, 1992; Balacheff, 1994), and it is constructivist, not only because of the facilities for student's action provided by the system in its exploratory mode, but also because its design is compatible with

the idea that the student builds his (her) knowledge through the interiorization of activities and the organization of related operations into flexible structures, that is into cognitive structures. A first step in this direction is given in the software Lirec (Cuevas, 1994).

3. SOME ACTIVITIES FOR AN ITS IN STATISTICS

In this section we present with two very simple examples the way in which our approach can be implemented for descriptive statistic teaching. The examples are referent to two of the most basic concepts, the mode and the mean. In spite of the apparent simplicity of the concepts, and therefore usually superficial treatment, it is frequently the case that the students do not understand the essence of those concepts and make a bad use of them (Torma, 1995). The activities presented can be used in an autonomous way by the student to construct by himself ideas about the central tendency measures and are sequenced only when the system is used in the tutor mode of operation.

With respect to the mode, we present simple and attractive problems for the student to solve, in which the need of obtaining a representative number of a group of data of the type of the mode might be recognized and is the solution to the problem. The student, hopefully, as a result of his (her) need to solve a problem, should figure out the need of a representative number of the type of the mode, for the group of data presented in the problem. The name "mode" could be given to the student later as well as its precise definition, as part of the tutoring activity of the system. Alternate problems could be given to the student as his request or as a result of his (her) performance with the first problem given. Activities of data manipulation will be suggested with the aim of inducing in the student the acquisition of the mental activity (operation) to get the measure called mode. In order to show the effects on the mode caused by data changes, slight data changes to the original data, like changing a number by a small and a large quantity, adding a new value, eliminating a value, will be presented and the solution of the problem with the new data would be asked. More complex data modification might also be given to the student in order to strengthen the operation. Activities to recognize the related inverse operation would be suggested; that is, given a value of the mode the student would be requested to exhibit a group

or groups of data which might have such a value for their mode. The combination of these two activities will induce the student to recognize the relation of the direct and inverse operation connected to the mode.

For the mean, the activities used will be such that they permit the acquisition of the concept of center of mass of a set of data. This can be attained with several activities, one of them is the following: using a scale, data from a set of data is drop one by one into the scale, pointing out how the supporting point of the scale, which is the mean, has to be moved to mantain the scale in a horizontal position. Other activities are the following: the data is dropped into the scale in the order in which it was taken or obtained; the data is dropped into the scale in ascending or descending order; the first data that is dropped into the scale is the central data and after that, the extreme data are dropped one by one. Aditional activities are those in which given the mean of a set of data, a relative big or small data is added to the set of data and the new mean is requested, with the purpose that the student make an association of the stability of the mean with respect to data which is relative far from it. It is important, that once the concept of mean is obtained and understood, to make the student to explore the effect that the same activities has on the mode. Once that the student has mentally constructed his (her) concept of the mean, the activities to perform are those that allow him to discover the relation of the center of mass concept of a data set with the algebraic expression of the mean. Additionally, a series of operatory exercices like the ones proposed for the mode have to be done in such a way that they permit the construction of the inverse operation; that is, given a mean how is the set of data from which it comes from.

4. CONCLUDING REMARKS

An ITS based on the constructivist epistemology has been proposed. This approach of computer educational systems might be welcomed by the teachers who will not see the system as a opponent to their labor but as a partner which facilitates their educational tasks. In general, the development of computer environments that help the student to construct congnitive structures about central concepts in specific domains could be a great help to both the student and the teacher. There is involved a lot of work and investigation in implementing a constructivist approach for teaching just a concept. Among the activities involved are the following: 1) to

recognize the operations involved in a particular concept; 2) to detect the relations among the recognized operations; 3) to define the activities that the student could perform to acquire the mentioned operations; 4) to define the activities that might lead him (her) to recognize the connections among the operations; 5) to implement in a computer medium the defined activities in an interesting form to the student. However, the benefits that might be obtained to both the student and the teacher could be worthwile the effort involved in the design and implementation of such systems.

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EARLY DEVELOPMENT OF ALGEBRAIC REPRESENTATION AMONG 6-13 YEAR-OLD CHILDREN: THE IMPORTANCE OF *DIDACTIC CONTRACT*

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This research is aimed to study the development of algebraic representation, contributing to the comprehension of its development through the analysis of problem-solving protocols produced by 72 children, with ages varying from 6 to 13 years. Two other interconnected independent variables were proposed: ***algebraic structures*** *(six) and* ***version*** *of problems (two: stressing transformation of a quantity into another, or stressing the equality between two quantities). Results show that, if certain written representations for algebraic problems (for both versions indistinctly) are only detected in age-levels corresponding to the moment of introduction to algebra at school (11-12 years of age, 6^{th} grade in Brazilian school system), many other and rich productions were detected among younger children (3^{rd} - 4^{th} grades),* ***in interaction with propositions from the observer/teacher****. These data suggest the non-spontaneous, school-rooted character of algebra, and also rich possibilities of pre-algebra schema development through* ***didactic contract.***

In Brazil, as in many other countries (Bodanskii, 1991), curricula in elementary mathematics proposes that algebra must "wait" to be formally presented until some arithmetic principles are well established. Because of this order of presentation due to *didactic transposition* of mathematics (Chevallard, 1985), the *passage* from arithmetic to algebra has become an important domain of interest and research in the field of psychology of mathematics education, and both epistemological and didactic *obstacles* (Bachelard, 1974) have been described in this context (Laborde, 1982; Filloy & Rojano, 1984; Vergnaud, Cortes & Favre-Artigue, 1987; Garançon, Kieran & Boileau, 1990; Cortes, Kavafian & Vergnaud, 1989). Among the above mentioned obstacles, one of the most important in the *conceptual field* (Vergnaud,1990) of algebra is the representational transposition from natural language (in which word problems are expressed) to algebraic-formal representation (Laborde, op.cit.; Da Rocha Falcão, 1992), in which context the equal sign has equally a new and complex meaning (Schliemann, Brito Lima e Santiago, 1992).

This study shares this interest in algebraic representation, and aimed to contribute to the comprehension of its development through the analysis of solving-problem protocols produced by 72 children, with ages varying from 6 to 13 years (six age-groups of 12 subjects in each, corresponding to six school-level groups from 1^{st} to 6^{th} grade of Brazilian elementary instruction). All children were invited to solve 12 algebraic problems, presented as short stories during an unique session conducted as a clinical interview. This set of problems covered six algebraic structures (see Table 1, next page), each structure being presented

under two problem versions: version A, stressing transformation of a quantity into another, and B, stressing equality between two quantities. A complete table of structures (Table 1) and an example of the two versions of problems mentioned above (Table 2) are presented below (for a complete reproduction of the 12 problems proposed, see Brito Lima, 1996):

Algebraic structures	1	2	3	4	5	6
	a + bx = c	x + ax = b	x + a = bx	$x+\frac{x}{a}=b$	$\frac{x}{a}+b=c$	a + y = b + x + y
Problems	1 and 7	2 and 8	3 and 9	4 and 10	5 and 11	6 and 12

Table 1: six algebraic structures explored

Problems versions	
A: stressing transformation	**B: stressing equality**
PROBLEM 1: Leonardo has started a collection of self-adhering stickers. He had 23 stickers to put in his album, and he was given two more equal packets of stickers. Now, he has a collection of 37 stickers. How many stickers were there in each of the packets he was given?	PROBLEM 7: Amanda and Tiane have collections of post-cards. Amanda's collection is composed by 68 post-cards, while Tiane had 22 post-cards, and received two more and equal sets of post-cards given by her father. The two girls have now the same number of post-cards in their collections. How many post-cards were there in each set of post-cards received by Tiane?
Problems 1, 2, 3, 4, 5, 6	**Problems 7, 8, 9, 10, 11, 12**

Table 2: examples of two different versions of problems (stressing transformation and equality) corresponding to algebraic structure 1.

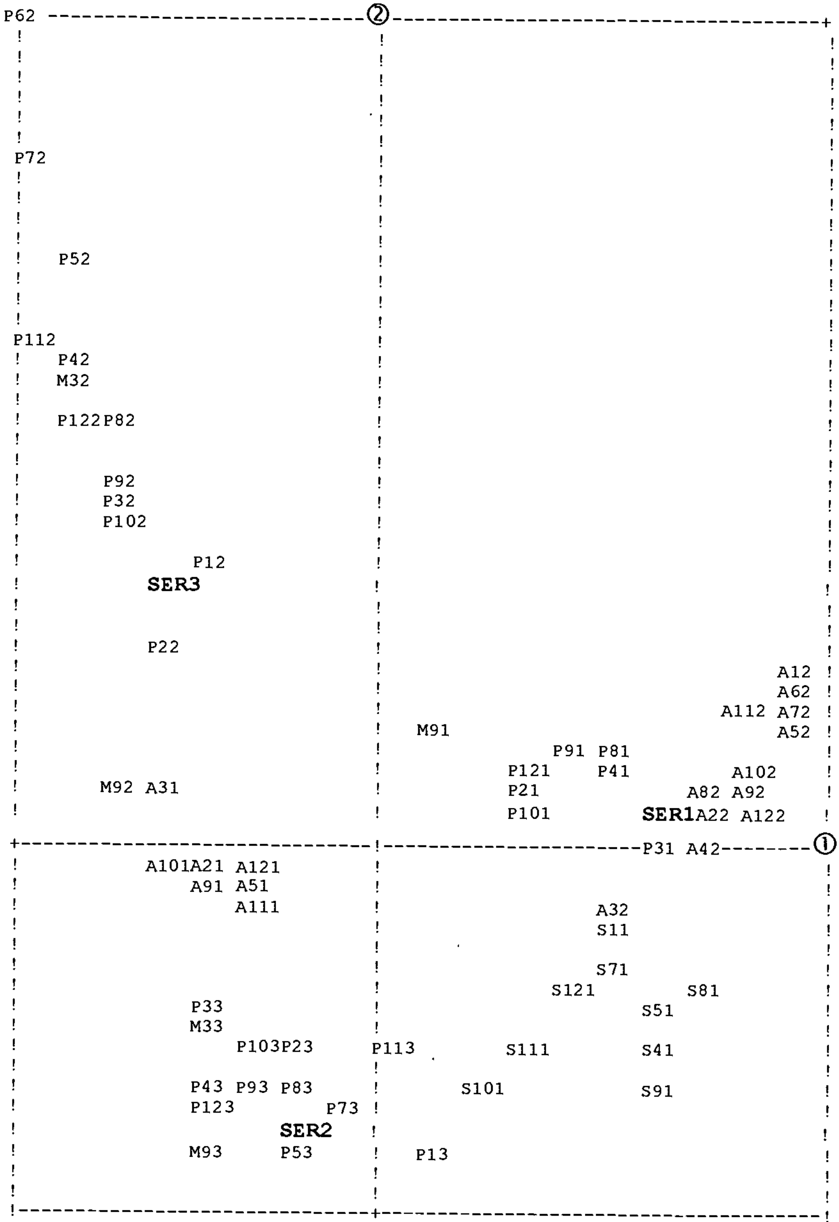
All children were given paper and pencil, and they were explicitly allowed to solve the problems in their own, without time-limits. Besides, subjects were encouraged to discuss their work with the observer, specially after their first representational proposition to one of the problems.

Subjects solving-problem protocols for the twelve problems were analyzed and classified according to a categorical scheme composed by four aspects, briefly described in Table 3 (next page). A multidimensional factor analysis (correspondence analysis of nominal data) was performed with these modalities of categories (completed by two additional qualitative categories - *sex* and *school level*, grouped in three composed levels: 1st and 2nd, 3rd and 4th, 5th and 6th - this grouping procedure being guided by a previous and specific factor analysis). The simplified factorial plan (Factorial plan 1, ahead) illustrates the more important aspects suggested by this study. First of all, neither *sex* nor *problem version* have had important contributions to the factorial plan (the modalities corresponding to these categories are absent of the plan). This result means that these categories didn't contribute to any significant partition of subjects in the sample.

Codes	A	P	M	S
Categories	**Simple performance**	**Basic Procedure**	**Problem Manipulation**	**Symbolic support**
Modalities	1. Right answer 2. Wrong / no answer	1. Arithmetic (immediate operations) 2. Previous representation (WS) 3. Previous representation (AS)	1. Numeric manipulation 2. Previous manipulation of numbers and / or other symbols (WS) 3. Previous manipulation of numbers and / or other symbols (AS)	1. Symbolic support for counting (fingers, matches, seeds, traits in the paper) and operating (school algorithms, schemes, drawings) 2. No explicit support

Table 3: items of categorical analysis of subjects productions with respective modalities (**WS** = without suggestion from the observer; **AS** = after having accepted suggestion).

On the other hand, the modalities of the category "*Simple Performance*" (code A) have an important contribution for the first (and more important) factor (Factor 1, represented by the horizontal axis in the orthogonal plan). As we can observe, there is a sub-cloud of modalities corresponding to "wrong answer" on the right extremity of the horizontal axis: code A, number of the problem and code of the modality "wrong", 2 (according to table 3 above) - A12, A62, A112, A72, A52, A102, A82, A92, A22, A122, A42, A32; opposed to this sub-cloud, there is another one on the left side of the axis, grouping modalities of the same category, but corresponding to the modality "right answer", code 1: A31, A41, A81, A101, A21, A121, A91, A51, A111 (absent modalities didn't have an *important* contribution - a percentile contribution over the percentile mean contribution of all modalities to the factor). Associated to this category, we can see the qualitative category "School level", with the modality SER1 (1^{st} and 2^{nd} level) on the right side, near the sub-cloud of wrong-answers, and SER3 (5^{th} and 6^{th} level), near the sub-cloud representing right-answers. The multidimensional frame is completed by another two sub-clouds, represented by the modalities of the category P, "Basic procedure": on the right side, a sub-cloud composed by the modality code 1 (Arithmetic: immediate operations) for problems 2, 3, 4, 8, 9 and 12: P91, P81, P121, P41, P21, P31; on the left side, we have a larger sub-cloud, composed by the modalities code 2 ("Previous representation without suggestion from the observer"), for all twelve problems: P62, P72, P52, P112, P42, P122, P82, P92, P32, P102, P12, P22. We can try to propose, now, a global interpretation for factorial axis 1: it splits the sample in two clearly opposite groups, corresponding to the lower (1^{st} - 2^{nd}) and higher (5^{th} - 6^{th})



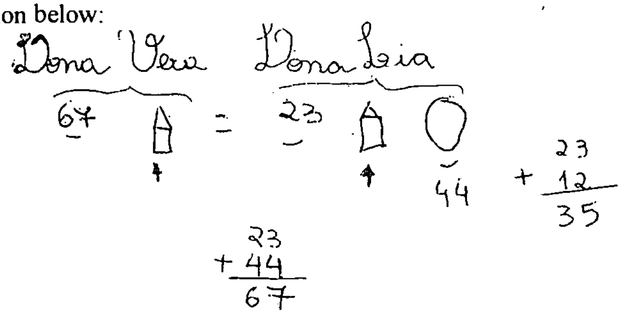
Factorial Plan 1: "cloud" of modalities distribution in the orthogonal plan produced by the two principal factors (factorial axes 1 and 2 explains 39,7% of total variance).

school-levels. Subjects from the first group had important difficulties with the problems, which they tried to solve arithmetically. Subjects from the second group, on the other hand, could answer easily the set of problems, proposing by their own initiative a previous representation for them. We have, in factorial axis 1, the classic arithmetic-algebra splitting, since older subjects from the left side of factor 1 (5^{th} - 6^{th} school levels) had just been initiated to algebra at school.

Factor 2, the second in importance, represented by the vertical axis in the orthogonal plan, shows an even more interesting aspect: the up-down opposition, here, is represented by modalities of Basic Procedure (code P): on the up-side, we have once more a sub-cloud of modalities related to the proposition of a "Previous representation without suggestion from the observer" (P62, P72, P52, P112, P42, P122, P82, P92, P32, P102, P12, P22) , also related to the qualitative category "School level" (SER3, 5^{th} and 6^{th} levels); on the down-side, we have a new opposition, represented by a sub-cloud of modalities equally related to the proposition of a "Previous representation", but this time *after specific suggestion from the observer* (P33, P103, P23, P113, P43, P93, P83, P73, P123, P53, P13), and associated to an intermediate school level, SER2 (3^{rd} and 4^{th} levels). Factor 2 is, in our opinion, more interesting than factor 1, because it shows a more tenuous opposition between an "algebraic" group (freshly introduced to elementary algebra, and able to represent problems in expressions and to manipulate them before operating arithmetically) and another group not yet introduced to algebra, but sensible to suggestions concerning previous representation, as illustrated below by an extract of protocol produced by K., 3^{rd} level, for problem 12, reproduced below:

Dona [*Mrs.*] Vera e Dona [*Mrs.*] Lia decided to go to the super-market in order to buy some fruits. Dona Vera bought 67 oranges and a packet of grapes, while Dona Lia bought 23 lemons, a similar packet of grapes and bag of apples. The two ladies came back home with the same quantity of fruits each. How many apples were there in the bag bought by Dona Lia?

K. accepted the suggestion of representing the problem beforehand, and produced the expression below:



Note that K. proposes an expression which includes the equal sign and different symbols for different fruits, the icon proposed for the packet of grapes being

repeated in both sides of the expression. She arrives to solve the problem implicitly manipulating the unknown represented by the packet of grapes, which is eliminated when she proposes that 23 lemons plus the apples (unknown represented by the circle) must be equal to 67 fruits. K. and many other subjects from SER3 (3rd and 4th level) show important pre-algebraic schemes involving specially the proposition of hybrid equations, where natural language, icons, numbers and mathematical formal operators coexist. These pre-equations are not manipulated in an algebraic way, serving only to guide the subject in choosing and executing arithmetic operations, as shown in another protocol, produced by T., 4th level, for problem 7 (reproduced in Table 2 above):

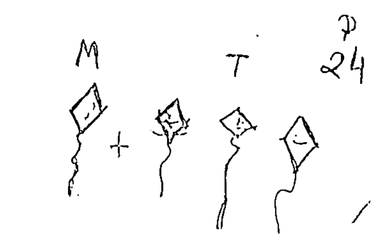
Sentenças	Cálculo	Resposta
$\square \times 2 + 22 = 68$	$\begin{array}{r} 22 \\ +23 \\ 23 \\ \hline 68 \end{array}$	cada bloco tinha 23 papéis de carta

First of all, T. proposes an expression ("Sentenças"), following not only the observer's suggestion, but also a very common didactic contract in the Brazilian elementary arithmetic classroom: before operating arithmetically (using operational algorithms), write down the corresponding mathematical expression. T. makes use of the small square (□) , the first symbolic representation of unknowns proposed in arithmetic activities like: $\square + 6 = 10$ $\square = ?$. Once the sentence proposed ($\square \times 2 + 22 = 68$), he by-passes explicit manipulations and operations to establish the numeric value for the unknown □, the only explicit operation ("Cálculo") being a verification of the solution (23), transposed to the third section ("Resposta", answer) of this elegant work.

It is also important to observe that, if intermediate school-level subjects appear more sensible than their younger colleagues towards invitations to "represent the problem first, try to solve it later" (Da Rocha Falcão, op. cit.), it doesn't mean that representational schemes are completely absent among younger sub-sample (1st - 2nd school level, 6-7 years of age). The extract of protocol on the next page, proposed by R., a 2nd grade subject, shows an interesting representational effort where important numeric relations of the problem are detected and correctly represented, after interaction with the observer/teacher, who wrote in the paper the letters M ("manhã", morning), T ("tarde", afternoon), and D ("dia", the whole day). After this initial help, R. could represent the first unknown quantity of kites by the drawing of a kite, and the second unknown by three similar drawings of kites, since the problem established that children had hand-crafted "three times the quantity of kites in the afternoon".

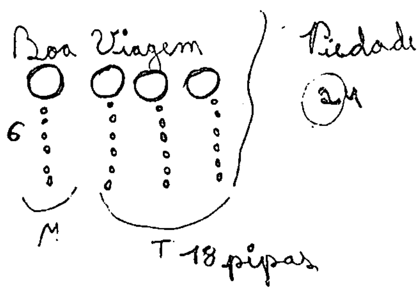
Problem 8: Last Sunday, children from P. and BV. [beaches in Recife - Brazil] participated in a context in order to chose the more beautiful kite. Working on Saturday, children from BV. hand-crafted a certain quantity of kites in the morning, and the triple of this quantity in the afternoon. Children from P. produced 24 kites in all. Knowing that both group of children prepared the same number of kites, how many kites did BV. children handicraft on Saturday morning?

R's representation of problem 8, after being aided by the observer:



Equally interesting and ingenious is the *spontaneous* representation for the same problem proposed by A, 2nd grade, representation which guides his solving-problem procedure: he starts a counting-on procedure, distributing small dots under one + three circles representing the day-production of BV. children, and stops when he reaches 24. Then, he counts the dots under the first circle (representing the morning-production, and arrives to the answer of the problem.

A's representation and solving-problem procedure (problem 8):



Many psychological schemes (Vergnaud, 1990) can be addressed and amplified by *contract*, as already shown in other research efforts involving the proposition of didactic sequences in elementary algebra (Da Rocha Falcão, 1995). Subjects from SER2 seem to be, as a matter of fact, in a *zone of proximal development* (Vygotsky, 1991) concerning representational algebra, where propositions and activities from teachers have an important role in algebra sense-making and solving-problem strategy. Data from the present study show that a very important work on this issue can be initiated long before the traditional curricular moment (5^{th} - 6^{th} level), involving a socio-cultural context in the mathematical classroom where the principal aspects of the conceptual field of algebra can be explored, understood and incorporated. If usually children can not perform this task spontaneously, specially when they are submitted to a long and previous arithmetic "immersion", they are disposable to accept certain *contracts* (Schubauer-Leoni, 1986) which are extremely important from a didactic standpoint, since algebra, like many other activities (Leontiev, 1994), is deeply embedded in school culture.

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THE AFFECTIVE DOMAIN IN MATHEMATICAL PROBLEM-SOLVING[1]

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We offer a research-based theoretical framework for describing the affective domain in mathematical problem solving, extending earlier perspectives of McLeod and of Goldin as well as our previous joint work. Key new ideas include the component of values/morals/ethics as it pertains to problem-solving affect, the notion of mathematical self-acknowledgment, and the concept of meta-affect. Our overall framework forms the basis for analyzing interactions between children's affect and cognition in a longitudinal series of task-based clinical interviews. Certain ideas are illustrated with examples from the interviews.

Research on mathematical problem solving has concentrated primarily on cognition, less on affect, and still less on interactions between them. In examining the strengths and weaknesses of research on affect in mathematics education, Leder (1993) argues convincingly for a "multi-layered approach" including students' "subtle responses and reactions." She adopts in her methodology "a deliberate strategy that questions the validity of the 'snapshot' approach to the measurement of attitudes." Our research is based on a similar strategy. In qualitative, exploratory investigations (DeBellis and Goldin, 1991, 1993) we examined in talented high school students and in elementary school children the influence of affect on strategic decision-making during non-routine mathematical problem solving. DeBellis (1996) studied children over two years through a series of five carefully-structured task-based interviews, designed to maximize non-directive mathematical problem solving and construction of external representations. Videotapes of four subjects were analyzed for interactions between affect and cognition, using fine-grained protocol analysis, inferences made by observers, and a validated facial movement coding system (Izard, 1983). Partly as a result of this work, we have come to the view that affect—and its detailed interplay with cognition—is the most fundamental and essential system of representation in powerful mathematical learning and problem solving.

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This article is not intended to summarize empirical results, but to offer a partial theoretical framework for describing the affective domain in mathematical problem solving. It extends some earlier perspectives of Goldin (1987, 1988) and McLeod (1989, 1992) as well as our previous joint work. New points include the component of values/morals/ethics in problem-solving affect, the notion of mathematical self-acknowledgment related to this component, and the concept of meta-affect.

Emotions, Attitudes, and Beliefs

McLeod (1989, 1992) usefully partitions the affective domain of responses to mathematics into emotions, attitudes, and beliefs. These differ from each other in stability, intensity, the degree to which cognition plays a role in the response, and the length of time that each takes to develop. Emotions—positive or negative feelings—are the most intense and least stable. They "may involve little cognitive appraisal and may appear and disappear rather quickly, as when the frustration of trying to solve a hard problem is followed by the joy of finding a solution" (McLeod, 1992, p. 579). Attitudes, where most of what Leder calls "snapshot" research has focused, are "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability" (1989, p. 249). They are seen as developing in two ways: from the automatizing of a repeated emotional reaction to mathematics, or from an assignment of a pre-existing attitude to a new but related task. Beliefs may be about mathematics as a discipline, or about oneself in relation to mathematics. McLeod sees beliefs as mainly cognitive, developing comparatively slowly. He sums up, "we can think of beliefs, attitudes, and emotions as representing increasing levels of affective involvement, decreasing levels of cognitive involvement, increasing levels of intensity of response, and decreasing levels of response stability" (McLeod, 1992, p. 579).

Affect as a System of Representation

A model developed by Goldin (1987,1988) describes five kinds of internal representational systems, constructed over time, that interact continually in symbolic relationships with each other as human beings engage in mathematical problem solving: (a) a verbal/syntactic system, (b) imagistic systems, (c) formal notational systems, (d) a system of planning and executive control, and (e) an affective system. The latter refers to changing states of feeling during mathematical problem solving (local affect), as well as more stable, longer-term affective constructs (global affect). Essential to the idea of affect as a *representational* system is that as states of feeling interact with other modes of representation, they encode important information (meaning) and influence problem-solving performance.

Emotional states are then to be considered as local affect. Attitudes and beliefs, as well as values, ethics and morals (see below), are to be considered aspects of global affect—relatively stable, self-regulating structures in the individual. Research on emotions in mathematics education has tended to focus on strong reactions—such as anxiety or phobia toward the subject, or elation with success—and not (as we think essential) on more subtle emotions, such as puzzlement, curiosity, frustration, or confidence, inherent in solving mathematical problems. Though we agree with most of McLeod's analysis, we differ with his assessment that the level of cognitive activity involved in emotions during problem solving is low (at least as compared to attitudes and beliefs). Our studies suggest it is very high, though the cognitions interacting with fleeting emotions may be difficult to identify.

Affective pathways (Goldin, 1988) are established sequences of (local) states of feeling, possibly quite complex, that interact with cognitive representational configurations. Such pathways serve important functions for experts as well as novices, providing useful information, facilitating monitoring, and suggest heuristic strategies during the problem solving. Two idealized examples of affective pathways interacting with heuristic configurations during problem solving are the following. (1) A positive pathway begins with curiosity and puzzlement at the outset of problem solving, which evoke exploratory and problem-defining heuristics and motivate the solver to better understand the problem. A state of bewilderment or impasse leads to feelings of frustration, which encode the information that to this point the strategies employed have led to insufficient progress. Heuristic processes are evoked to revise strategies and challenge previous assumptions. Feelings of pleasure, elation and satisfaction occur linked with insight (imagistic representation) as the problem yields to new approaches. Global structures are built that entail positive self-concepts and anticipation of positive affect in difficult mathematics problems. (2) A negative pathway also begins with curiosity and puzzlement, but these encode a search for "safe" procedures rather than an exploratory opportunity. When procedures fail the resulting frustration turns rapidly to anxiety and despair. These also evoke heuristic processes—reliance on authority, defense mechanisms, avoidance and denial. Global structures of mathematics- and self-hatred are built. Affect may thus empower or disempower students. Empowering affect serves as an impetus to persevere, take risks, engage with new external and internal representations, ask questions, construct new heuristic plans, etc. Disempowering affect hampers performance, blocks understanding or makes it unrecognizable when it occurs, and induces negative outcomes associated with "math anxiety" or phobia. In our view every individual constructs complex networks of affective pathways, contributing to or detracting from powerful mathematical problem-solving ability.

Values, Morals, and Ethics

Do the three components of emotions, attitudes, and beliefs adequately capture the spectrum of affective responses in mathematical problem solving? We think an important fourth component deserves attention, one that includes aspects of a solver's values, morals, and ethical judgments that interact with problem decision-making. A complex values/morals/ethics system (sometimes shared, sometimes highly individualistic) is one of the most powerful motivators of human beings. Developing in childhood (Kohlberg et al., 1983) such a system provides the psychological sense of what is good and bad—the feeling of being right, being justified, being wrong, or judging others to be in the right or in the wrong. It is a system powerful enough to energize profoundly creative, altruistic, or destructive behavior. But what has this to do with mathematical learning and problem solving? The importance of the values/morals/ethics component of affect pertains to the individual's feelings, tacit or overt, about learning, problem solving success, mathematical behavior, etc., as (morally desirable) virtues or values. Following the rules, or following directions (including mathematical rules), may be regarded by the child as "good", failing to do so as "bad". To us this is much more than a belief about what mathematics is, or what works to obtain solutions. Some students who do not follow established instructional procedures, as in addressing a non-routine problem, may actually be tacitly contravening their own moral values or self-expectations, while others (who value originality, rebellion, or self-assertiveness) may be acting consonant with them. Cheating in school may be considered evil or shameful, and doing mathematics with help may for the child be a form of cheating. The tacit commitments made by students to learn and to understand, their sense of goodness about themselves when they do as they "should" do, and wrongness when they fail to do as they "should", all fall within this component.

We have thus come to visualize the affective domain as a tetrahedron (see Fig. 1). The four components—emotions, attitudes, beliefs, and values/morals/ethics—pictured at the vertices, are inferred, internal, mutually interacting and mutually influencing facets of affective states. They interact with each other (as indicated by the line segments), with cognition, and with the external environment during mathematical problem solving. The emotions, attitudes, beliefs, and values of other individuals may influence problem solvers directly, as when the clinician communicates expectations to the child, or the child searches the clinician's face for a sign of encouragement, approval, or disapproval. Broader social and cultural conditions, situational and contextual factors, can also be understood in relation to the affective tetrahedron as they influence mathematical problem solving.

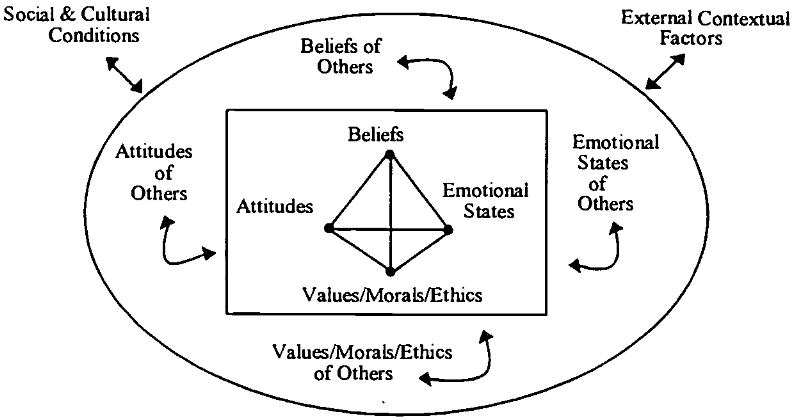

Figure 1. A Tetrahedron with Components of an Affective Representational System

Mathematical Self-Acknowledgment

We use the term mathematical self-acknowledgment to describe a learner or problem solver's ability (or willingness) to acknowledge an insufficiency of mathematical understanding. We place this construct, for many students, in the values/morals/ethics component of the affective domain, as it may relate directly to the student's value of self or sense of right and wrong in relation to mathematics. Important aspects of mathematical self-acknowledgment are: recognition of the insufficiency of understanding, the decision to take further action, and the nature of the action. Recognition that something does not make mathematical sense may be expressed to oneself (as the solver of the problem) or to someone else (e.g., a teacher). Either type of acknowledgment may carry specific value, moral, or ethical dilemmas for the solver, which can help or hinder the solver's admission of mathematical insufficiency. The solver may or may not decide to do something about an acknowledged insufficiency of mathematical understanding. If the decision is to act, mathematical performance may be hindered or helped, depending on the choice of action. Actions may include surface-level adjustments, explicit efforts at deeper understandings, or a combination of both. Examples of surface-level adjustments might include mathematical bluffing—pretending to know, hiding the fact of not knowing behind a plausible procedure, doing nothing, making up answers, forcing wrong answers to make sense, or guessing while creating the illusion of knowing. These we would tend to regard as impeding mathematical understanding. But not all surface-level adjustments impede understanding. Making rapid conjectures or an intuitive guess, or generalizing a rule to a new situation, can produce powerful

results. Deeper mathematical understandings may be sought through a variety of complex heuristic behaviors, which allow for better problem interpretation and sense-making. But deeper responses can also be unproductive, as in solving story problems by classifying them into types that require alternative algorithms. The strongest problem solvers seem to display straightforward recognition of insufficient understanding, and deep, productive responses.

When Stan (age 11) was asked in Interview #5 to explain what the fraction 3/1 means, he touched the side of his face. "... that fraction would equal three. Because, um, say if you had, well just say, you couldn't, well say if you had a pie and (5-second pause, presses lips together) and you could buy it (4 second pause) well say um (3 second pause) well (smiles while looking down at paper) ... well, I know that would be three because um (5 second pause, furrows brow, presses lips together) ..." After questions by the clinician, he explains that "three divided by one equals two (writes 3 - 1 = 2 in vertical form) and then you would take uh, two, div, um I mean subtracted by one equals one ..." We infer here that Stan recognizes an insufficiency of understanding, but is reluctant to acknowledge it. He bluffs at first by telling what he knows to be true (3/1 equals 3), and eventually finds a procedure (repeated subtraction) that gives him this result. The observer comments, "When Stan knows the answer, he is very confident and feels pretty good. However, when he is not sure, he tries his best but starts talking around the answer ... in circles."

There are various forms that denial of insufficiency in mathematical understanding may take. Denial when one does not know one's understanding is insufficient differs from denial when one knows it and is trying to cover it up. There is also an essential difference between "guessing" where the solver tries to create the illusion of understanding, and "guessing" as part of trial-and-error or testing a hypothesis.

In our view mathematical self-acknowledgment plays an important role in the construction of the individual's global affect. Feeling "stupid" with one's own mathematical insufficiency when an error is pointed out discourages behaviors that foster mathematical power. Disempowering affective pathways block mathematical self-acknowledging behavior. The alternative is powerful affect, feeling happy that one's hard work allowed another to point out an error that has become obvious. This type of affect allows mathematical progress and learning to continue.

Meta-Affect

Meta-affect may be characterized in relation to affect in a way similar to characterizations of metacognition (Lester et al., 1989). It includes (a) emotions

about emotional states, and emotions about or within cognitive states, and (b) the monitoring and regulation of emotion. When we say that educators need to help students analyze how their feelings interact with cognition during mathematical problem solving, and how they can better manage their own emotional responses, we are addressing meta-affective capabilities.

One example deals with the affect of "discomfort". At the outset of Interview #3, we inferred that Londa (age 10) felt discomfort or unease associated with nervousness or unfamiliarity in the interview situation. Later in the interview, she is asked "Which would be easier, to cut a birthday cake into three equal pieces or four equal pieces?" Three Styrofoam objects were located on the table in front of her; a circle, a rectangle and an equilateral triangle. She replied, "Like if you have a circle and cut it in three's that would be really hard because it would be something like that, like that (motions with finger) or if, if you did it like this (repeats same division) and you found out they weren't equal, you would have a hard time deciding. But if you had this (reaches for rectangle) all you'd have to do is cut it here and here." We inferred from gestures, facial expressions, and body movements that Londa was envisioning what cutting the circle in thirds would be like, an envisioning included letting herself feel "discomfort." The discomfort feeling was a way of encoding how difficult it would be to decide where to make the cut. Here it was an emotion she allowed herself to feel as part of her envisioning process of what it would be like to try to cut a circular cake in thirds. This discomfort—unlike the discomfort at the beginning of the interview—is an emotion she feels entirely comfortable about feeling. She is allowing herself to feel the discomfort in a hypothetical context, one more or less entirely within her control. She has shaped the context herself and the emotion is a contextualized, localized discomfort.

As this example illustrates, to say that a problem solver is in an affective state or feels an emotion may be a tremendous oversimplification. Meta-affective information is needed to understand how the solver feels about the affect in question, and how it relates to other affective and cognitive processes.

Conclusion

We have presented some suggestive ideas, a partial theoretical framework, and brief illustrative examples pertaining to affect in mathematical problem solving. We are not in any way saying here that "feeling good" is synonymous with powerful affect. The affect that makes for mathematical problem solving ability is complex, and includes negative as well as positive feelings. It entails structures of mathematical self-acknowledgment and meta-affect that serve to promote deeper understandings.

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Creating a Shared Context: The Use of a Multimedia Case in a Teacher Development Course

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Abstract

In this research study, we investigated the impact of a multimedia teaching case on the professional development of a group of pre-service mathematics teachers, most of whom were concurrently engaged in their student teaching experience. In particular, we examined the extent to which the case provided the pre-service teachers with a common context for analyzing teaching and learning, the degree to which it supported the reflection of pre-service teachers on their own developing practice, and the ways in which the materials fostered an examination of the complexity of the classroom environment. In this paper, we report on some of the findings from our study, which include the benefits as well as the limitations and weaknesses of the multimedia case in this particular context.

Introduction

The task of restructuring mathematics pedagogy continues to be a significant challenge for mathematics educators involved in teacher education. In this paper, we describe the results of one attempt to create a shared context for discussing mathematics pedagogy through the use of digitized video on CD-ROM. Our research goal was to investigate the impact of a multimedia case study of teaching on the professional development of a group of pre-service mathematics teachers. The multimedia case is based on a series of lessons focusing on the concept of volume. These lessons were conducted in a fifth-grade classroom in an urban school setting.

The multimedia case allowed us to go significantly beyond the use of text-based vignettes, such as are found in the publications of the National Council of Teachers of Mathematics ([NCTM], 1991). Unlike the vignettes, the case study included extensive video clips of a series of lessons developed over three days, interviews with the teacher on her planning and decision making, digitized copies of student work, video clips of student interactions in small groups, and links to text-based material on the NCTM curriculum standards (1989) and professional standards (1991). The first author of this paper, who was the instructor of the teacher development course, chose this multimedia case because the richness of these resources and the ability to selectively focus on particular aspects of the classroom environment appeared to provide an opportunity for pre-service teachers to carefully examine the complexity of the classroom and the teacher's role within that complex environment. Moreover, as pre-service teachers often have very different practicum experiences, the multimedia case study appeared to have the potential to provide a common context for understanding teaching and

learning while simultaneously supporting their reflection on their own individual student teaching experiences.

In this research study, we investigated three questions: (1) To what extent did the multimedia case study provide pre-service teachers with a common context for analyzing teaching and learning? (2) How did the case study support the pre-service teachers' reflection on their own developing practice? and (3) Did the materials foster an examination of the complexity of the classroom environment and an understanding of the role of the teacher in that environment? In addition to these questions, we examined some of the limitations and weaknesses of the multimedia case study that arose in this particular context.

Theoretical Framework

First and foremost, this research study is grounded in the view that classrooms are complex environments. This complexity presents particular problems for the novice teacher whose limited experience and knowledge make it difficult to effectively observe the complexity of interactions that occur, often with great rapidity, in a typical classroom. Beginning teachers are usually concerned with issues of classroom management and the planning and content of lessons as important priorities. But the pedagogical content knowledge that an experienced teacher brings to bear in effective classroom instruction is extensive and includes elements of epistemology, psychology, mathematics, philosophy, and pedagogy (Shulman, 1986). The number and intricacy of theories attempting to model mathematics teaching has increased substantially over the last two decades (Koehler & Grouws, 1992). For example, in elaborating his model of the professional development of teachers, Simon (1995) describes mathematics teaching as the cyclic interrelationships of teacher knowledge, thinking, decision making, and mathematical activities, all of which are influenced by the teacher's understanding and evolving hypotheses about students' learning. The pre-service teacher is thus faced with the challenging task of simultaneously understanding the complexity of classroom environments, while at the same time integrating a multiplicity of knowledge elements and beliefs into a coherent, emerging practice.

The second theoretical perspective that is brought to this study is the notion of the teacher as a reflective practitioner. In his recent work, Cooney (1996) advocates that reflection should be a central component of teacher education programs that aim to develop flexibility of thinking and adaptability to classroom constraints. In this research study, we explored the extent to which a multimedia case, in which the classroom teacher herself reflected on elements of her practice, would support and foster pre-service teachers' own critical reflection on their emerging practice. Pre-service teachers do not have a common student teaching experience. In contrast, their experiences vary tremendously in setting, host teachers, students, grade level, and so on. Thus when they come to their teacher

development class, disparate rather than shared experiences come to the fore. A central question was to determine whether a multimedia case could serve as a common experience for the pre-service teachers to support a more reflective analysis of teaching and learning.

Methodology and Data Sources

This qualitative research study was conducted with 13 students in a teacher development course at a mid-sized research university in the United States. The class met once a week for three hours. Ten of the students were taking the course concurrently with their second student teaching experience in a variety of middle school and secondary school settings. One student had already completed her student teaching, but had not yet had a full year of teaching experience; one would be completing her student teaching in the following semester; and one was a visiting student from abroad with several years of teaching experience. Five students were men, and eight were women; three were undergraduates, and ten were graduate students in a masters degree program.

Each student was given a copy of the CD-ROM "Investigations in Teaching Geometry" (Goldman, et al., 1994) developed at Vanderbilt University and a copy of the "Geometry Investigations" HyperCard stack for accessing video clips and descriptive text on the CD. The students used the materials on publicly available Macintosh machines on campus or on their own personal computers.

The primary purpose of the lessons on the CD was to introduce fifth-grade students to the concept of volume through informal investigations rather than through formal definitions. This was accomplished through three lessons that began by visualizing a three dimensional shape drawn on a two-dimensional plane. The students then worked with one inch cubes to fill a box and were encouraged to think in terms of layers in order to enumerate the total number of cubes. The final activity consisted of creating the largest possible box by cutting corners from a square sheet of paper and folding up the sides. The teacher of these lessons (the second author of this paper) was not the students' regular classroom teacher, but came to this classroom for the purpose of teaching these lessons.

The multimedia case materials included video of each lesson, copies of the teacher materials and student work, and the written lesson plans. The case also contained four "Investigations" to be explored by the pre-service teachers: (1) Planning and Teaching the Lessons, (2) The Teacher's Role, (3) Assessment of Student Learning, and (4) Key Mathematical Ideas. The investigation on the planning and teaching of the lesson included commentary and reflection by the teacher, in which she described her thoughts about what went into the lesson plans, anticipated student responses, and how she modified her plans. The assessment of student learning provided a database of students' written work that could be accessed to examine the strategies that students used in the lessons and to

assess the class's progress. (cf. Bowers & Cobb, 1995, and McClain & Barron, 1995 for further descriptions of these investigations).

Each of the three lessons in the case study was the subject of one class session in the teacher development course, beginning with the second class meeting. Each pre-service teacher was asked to complete a journal entry on the lesson, responding to a particular focus question. The focus questions provided the beginning point for class discussion. This process was repeated for each of the three days of the lessons. In their fourth journal assignment, the students were asked to select one of the "Investigations" to explore and discuss.

In addition to student journals, the class instructor (first author) took field notes during class and recorded her own reflections in a series of journal entries. The class discussion generated many questions and issues about the lessons. As the discussions unfolded, these questions were addressed via electronic mail to the teacher of the lessons (second author). Thus, the data sources for this study included student journal entries, journal entries and field notes by the class instructor, email exchanges with the teacher of the geometry lesson, course evaluations completed by each student, and a questionnaire on the use of the multimedia case study completed by each student. The results reported below are based on the analysis of this data.

Results

The use of the multimedia case clearly promoted a sense of a shared context for analyzing the teaching and learning environment. All but one pre-service teacher strongly agreed with the statement that the "multimedia case study was very effective in providing a context for our classroom discussion." This was articulated most clearly by the comment of one student who claimed: "We were all watching the same scenario and discussing teaching from a *common experience*, rather than just trying to express the same things from student teaching when no one else in the room saw what you are describing and can fully understand the context of what you describe" (emphasis added). Clearly this student saw the case study as providing a shared experience, rich in its own context, and something that is not possible to achieve with the separate experiences of student teaching.

These sentiments were echoed in the comments of other student teachers who suggested that the discussions in class were "very beneficial" and "interesting." One student commented that the class discussion was the most valuable aspect of the multimedia case. The field notes taken during class indicated that the discussions were extremely lively and animated, with the students having vivid and detailed recall of elements of the case study in support of their arguments. Finally, several students commented that the discussions in class brought out new perspectives on a common experience. One student explained that the most valuable aspect of the case study investigations was "the

discussions we had in class after we had watched part of it. I enjoyed it so much because people often picked up on things I hadn't even thought about." Thus, over several weeks, the video lessons became part of the shared experience of the class, with multiple perspectives being built on and developed within the conversation of this classroom. This observation is consistent with findings from research conducted in other pre-service courses (Barron, Bowers & McClain, 1996).

Three major themes about the teaching of the lessons emerged from these class discussions. First, the pre-service teachers felt that the teacher was extremely respectful of the students in the class. The pre-service teachers supported this claim with specific examples from the video in which the teacher used respectful language when addressing the students as a large group and in her interactions with them as pairs, over the full three days of the lessons. A second theme that emerged during the class discussion was that the pre-service teachers were very critical of the brevity of the wait time given to the fifth-grade students as they responded to the teacher's questions. This became a key question which was addressed via email with the teacher. The pre-service teachers felt supported in their observations as the teacher agreed that her wait time was poor, especially as the lessons progressed.

The third theme that emerged from the discussions was also related to the use of language in the classroom and centered around an incident where a student had given a response of "15 + 15" to describe how she had computed the volume by adding two layers of cubes, each containing 15 cubes. When restating the student's answer, the teacher changed that response and wrote "15*2" on the chalkboard. In the teacher interviews, the teacher explained that she intended to initiate shifts in the students' thinking by recasting additive solutions in terms of multiplicative ones when she felt it was appropriate. Some pre-service teachers argued that the shift to multiplication was the point of the lesson. But other pre-service teachers argued that the student's response should have been written as given by the student or the student should have been asked if the re-phrasing was acceptable. The teacher should have made that shift in representation visible to the class. Another problematic element with this instance for some of the pre-service teachers was the sense that it transformed the teacher's role into the controller and focal point of the discourse in the classroom and the central authority.

Each of these three themes became focal points for the pre-service teachers' reflections on their own practice. When asked "As a result of investigating this CD, what things did you do differently (or pay more attention to) in your student teaching?", two-thirds of the pre-service teachers responded that they attended to their own use of wait time. The issue of re-phrasing students' comments and answers became part of the classroom practice of the teacher development course, where the pre-service teachers would point out when

the instructor re-phrased their comments, and perhaps changed their meaning. Several pre-service teachers reported that they found themselves paying attention to using a student's own language and representations rather than immediately changing those expressions to some other form. One pre-service teacher also explicitly reported that he began to "write multiple solutions out on a chalkboard like Kay [the teacher in the video] does." Similarly, several other pre-service teachers commented that they paid more attention to having the students rather than themselves become the focal point of classroom communications. All but one pre-service teacher reported that the multimedia case helped them in reflecting on their own practice. Several suggested that critiquing the practice of a third teacher (in the case study) helped them in critiquing their own practice. As one pre-service teacher stated, "I found myself critiquing my own performance as a teacher as I critiqued Kay (am I doing the same thing I criticized Kay for?)." While many pre-service teachers mentioned wait time and becoming self-critical or evaluative, there were numerous other aspects of practice that were mentioned, such as how to adjust lessons, encouraging explanations, techniques for asking questions, and accepting multiple and/or incorrect solutions without judgment.

The breadth of the questionnaire responses to how the pre-service teachers reflected on their own practice suggests that the complexity of the classroom environment and their own roles within that environment began to emerge as the case was investigated. The pre-service teachers described the class discussions on the case study as "very involved," "heated," "thought provoking" and "enlightening," which is consistent with the instructor's field notes. Taken together, these data suggest that the multiple dimensions and aspects of teaching practice were brought to the fore through this shared experience. The pre-service teachers were keenly aware of the fact that the video tape of the lesson was edited. The fact that certain details were left out of the case video was problematic; in some cases, this resulted in the pre-service teachers making reserved judgments ("we didn't see, so we don't know"). For example, some pre-service teachers observed that Kay spent a great deal of time standing at the overhead; but others pointed out that it is difficult to tell, since other portions of the tape were edited out. Many of the pre-service teachers felt that these lessons were "staged" and not necessarily very realistic; many were suspicious of what parts had been edited out and why. It became clear that they were seeing a particular view of this particular classroom; but not having some of those omitted details made it difficult to fully come to grips with some of the aspects of the classroom environment.

Another indicator that the pre-service teachers experienced some of the complexity of the classroom environment was the emergence of "why" questions along with the "how-to's" that are often characteristic of beginning teachers' concerns. For example, in focusing on their email questions for Kay, the pre-

service teachers asked practical questions such as "How were the groups formed?" but also "Why was it done this way?". Other pre-service teachers asked complex questions such as "How did you accomplish creating an atmosphere where students were involved and eager to engage in discussion?" Finally, in commenting on the teacher interviews in the case, about half of the pre-service teachers argued that the value of the interviews is that they provided the "why" and the rationale for the teacher's decisions. Other pre-service teachers, however, felt that the teacher interviews were too brief to be of value.

Discussion and Conclusions

The richness of the resources in the multimedia case provided the pre-service teachers with a classroom environment that they could investigate, explore and critique. As their investigations of this environment were shared in the teacher development class, the three days of lessons on volume became a shared teaching and learning experience. As they discussed their own perspectives on these lessons, several common issues emerged: the respectful atmosphere of the classroom, the need for giving the students more wait time in responding to teacher questions, and the pedagogical dilemmas involves in re-phrasing the students' language in order to meet instructional objectives. In addition to the depth with which these issues were analyzed by the pre-service teachers and their instructor, the pre-service teachers identified a broad range of topics and issues that they used in reflecting on their own teaching practice. Nearly all the pre-service teachers reported multiple ways in which this case study supported their reflection on their own practice and influenced their student teaching.

This study confirms the earlier finding that the pre-service teachers found the classroom to be friendly and supportive (Bowers, 1996), but unlike the prospective elementary teachers in that study, these teachers were less convinced about the authenticity of the classroom. Their concerns stemmed from two perceived impacts: the presence of the video cameras and equipment in the classroom and the editing that was done to cut the video from three hours to 38 minutes. While some of the second set of limitations were mitigated by the ability to enter into electronic mail exchanges with the teacher of the lessons, this suggests that additional background material and more extensive teacher interviews should be included in the case materials. The pre-service teachers clearly recognized that the view of the video tape was inherently one perspective of the classroom and that this will always present limitations to what can be seen within the classroom. Nonetheless, the pre-service teachers were able to gain significant insight into the complexity of the classroom environment and the teacher's role within that environment.

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TRIPLE APPROACH: A THEORETICAL FRAME TO INTERPRET STUDENTS' ACTIVITY IN ALGEBRA

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We, in the GECO group, use a methodology called "triple approach" (psychology, mathematics and didactics).We present here three main notions of our theoretical frame: the local bits of knowledge, their dimensions and the three types of polarization of the subject's mathematical activity. The relationship between these notions, the very particular role the conformity polarization plays and the fact that the experimental actual data are the shifts of polarization rather than the polarizations themselves are discussed. Thus, we present here a synthesis of theoretical elements which have been dispersed in various papers until now, taking into account the most recent state of our research.

Depuis plusieurs années, nous développons au sein du GECO un cadre théorique pour l'interprétation de l'activité du sujet en mathématiques (en particulier en algèbre), basé sur une triple approche, psychologique, didactique et mathématique. Nous exposons ici trois notions centrales de ce cadre théorique: les connaissances locales, leurs dimensions et les trois orientations ('types of polarization') de l'activité mathématique du sujet. Nous explicitons les articulations entre ces notions, le rôle très particulier joué par l'orientation de conformité, et le fait que les observables sont en fait les changements d'orientation. Nous présentons ainsi une synthèse d'éléments théoriques jusque-là dispersés dans diverses publications, actualisée pour tenir compte de l'état actuel de notre réflexion sur le sujet.

Since many years the GECO group works on the learning of algebra at middle and high school level. To this end, we built up some theoretical tools: the local bits of knowledge, their dimensions and the three types of polarization. We use a methodology called "Triple Approach" (Psychology, Mathematics and Didactics), and we collect experimental data by interviewing students, during which they describe for themselves - and for us - their mathematical activity.

We built up progressively these notions, as they proved to be fruitful and helped us to understand some aspects of the learning of algebra. We presented them in various papers (Sackur & Léonard, 1985, Léonard & Sackur, 1991, Drouhard, Léonard, Maurel, Pécal & Sackur, 1994, Drouhard, 1995b, Sackur, 1995). These papers however focused mainly on the way we used these ideas (except Léonard & Sackur, 1991, published in French). The aim of this paper therefore is now to present some *recent* aspects of our theoretical framework (although not achieved). That is why this paper has an unusual mainly theoretical content.

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We shall describe here how our approach allows us to analyse the work of a student in algebra. What we want to observe, can be observed only if we "enter" into the way the subject thinks, if we focus on his/her private thought. We collect which we analyse with our theoretical tools. The interviews are led in such a way that the collected data is quite different (closer to the private thought of the student) from that collected by a teacher in the classroom, or even during a personal talk with a student (Sackur, 1995).

TRIPLE APPROACH

The "Triple Approach" has its origin from studies on the students' algebraic knowledge. We think that what we say here could be applied to various kind of knowledge, but we will not state this point here. The notion of local bits of knowledge is essential in the Triple Approach. Our aim, by emphasising this notion, is to stress the following idea: the errors the students make are not resulting from incoherence or misconceptions but rather from a particular kind of knowledge. To study this fruitfully, we were led to claim the following statement:

> *We claim that any knowledge results from interactions between a subject, a social group and the reality.*

This involves the existence of three areas: the psychological area (intra-cognitive relationship of the subject with himself), the social area (inter-cognitive relationship of the subject with a social group) and the area of 'reality' (relationship with a reality either material or conceptual in the case of mathematics.

We devised these three areas while trying to study the learning of mathematics within a Piagetian perspective. One may interpret Piaget's ideas (in a very sketchy way) of how knowledge is constructed as resulting from interactions between actions which take place in *two* areas, psychological and 'real' (Piaget, 1974). In order to take into account how (advanced) *mathematical* knowledge is built however, it seemed necessary to us to introduce a socially-related dimension, related to the society of past and present mathematicians and teachers amongst others. This dimension relies on the idea that mathematics is a social construction, and that if even some basic (e. g. logical) knowledge may be constructed just by the interaction of the child with his/her environment, it is highly improbable that s/he could build by him/herself advanced mathematical ideas (those which have been built through a long and uncertain historical path) just by interacting with his/her environment.

Obviously such ideas are related to those developed by Vygotsky (1987). One may take care however that we are not focusing here on the *social* construction of the mathematical knowledge, but rather on the construction *by an individual* of the *socially* - and historically - *already constructed* (advanced) mathematical knowledge, that is not the same thing (even if related).

LOCAL BITS OF KNOWLEDGE

Within this theoretical three-area framework we set out the notion of *local bits of knowledge*. Every knowledge is *local*, and may be said 'true' inside given limits. The subject ignores the existence of these limits, and of course their place in the whole knowledge field. These limits can be identified by an 'expert', i. e. anybody whose knowledge is more comprehensive than the subject's.

What do we mean by 'true'? The answer to this question takes place within the three areas. In the psychological area, the knowledge is *coherent* by itself for the subject; it does not contain any contradiction inside the domain where the subject may use it (it may be contradictory with other knowledge outside however). In the social area, the local bit of knowledge is *valid*, validated by a social group (or one representative) which recognises it as such. In the area of 'reality' at last, the bit of knowledge is true when it is *efficient.*

We claim that a local bit of knowledge is coherent, valid and efficient inside its limits, and loses simultaneously theses three qualities outside. We call these qualities the 'dimensions' of the bit of knowledge. It is viewed here as static, in a state of equilibrium.

According to this idea, we assume that the student constructs a local bit of knowledge in this way: his knowledge is, on the beginning, very local (like above for multiplying). Then it evolves towards more comprehensive bits of knowledge acceptable by the reference social group, the teachers and/or the mathematicians for instance.

A good example of local bits of knowledge we found when interviewing Leslie, a girl student, aged 15. She was working on quadratic equations, and found at the end of a computation the following expression:

$$24\,a^2 = -8$$

She then said that she could not find a because "a square number is always positive". The interviewer asked her *how* she knew that[2]. She said that it was clear, and to prove it "in front of a positive number one finds a "+" sign, and in $(a\text{-}b)^2 = a^2\text{-}2ab\text{+}b^2$, there is a "+" in front of the b^2".

The point here is that this "proof" is absurd for the expert, not for the subject, who coordinates in a coherent way (for her), three bit of knowledge, each of them being locally true:

LBK1: "a square number is always positive"

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2 note that, during our interviews, we try to avoid asking directly the subject a question like "why?". We know that, in this case indeed, s/he would (unconsciously) tend to re-construct her/his thought in order to give us an acceptable answer ('a posteriori rationalization') rather than to give us her/his original private thought (Vermersch, ibid.).

LBK2: "in front of a positive number one finds a "+" sign"

LBK3: "$(a\text{-}b)^2 = a^2\text{-}2ab+b^2$"

The domains of validity are easy to identify (for the expert). In LBK1, "number" must means real numbers, and this bit of knowledge turns false for complex numbers. In LBK2 in contrast, "number" must mean decimal writings of numbers (i. e. strings of digits, without letter or symbol, except a dot, e. g. "2.5" or "12"), and LBK2 is false if the "numbers" are denoted by letters. At last, LBK3 is valid whenever a and b denote elements of a commutative ring, otherwise false or undefined.

Separately, these local bits of knowledge are coherent for the student who refers to the convenient objects (numbers or writings), they are mathematically valid (in the related sets obviously) and are efficient, inasmuch as they yield exact results when applied to the corresponding mathematical objects.

We can make here two observations:

Firstly, both the mathematical domain of validity, the size of the reference social group (and the domain of efficiency of the knowledge) are increasing while the knowledge becomes less local, but of course it is not a term-to-term correspondence!

Then, we assume that, in general, all knowledge is local. There are universal mathematics statements indeed, which cannot be falsified since their mathematical domain of validity is explicitly stated, as:

$(\forall a \in \mathbb{R}\ \forall b \in \mathbb{R})\ (a\text{-}b)^2 = a^2\text{-}2ab+b^2$

Our focus however is not mathematical *statements* here, but rather psychological *knowledge* and, as seen above, the domain where the knowledge is 'psychologically true' is not just the domain where the corresponding formula is 'logically true'!

THREE TYPES OF POLARIZATION

Our aim now, as mathematics educators, is to identify the local bits of knowledge of a student in order to *act* on it. Our work then involves a *dynamic* point of view on bits of knowledge, their use and evolution.

When a local bit of knowledge is used inside its domain of validity, it is coherent, valid and efficient and therefore there is no need to modify it. When used outside its domain, it is not valid, not coherent and not effective. The system of Knowledge of the subject is then perturbed; s/he may not take it into account however, but if s/he does there is a possibility of evolution. So, let us focus now on the *dynamics* of the local bits of knowledge.

A knowledge is used in order to *act* indeed. Each action is *directed* towards an *intention*. Therefore, one knows whether intentions are reached or not according to 'reaching' *criteria*, which besides are 'stopping' criteria (allowing to know when to stop the action).

Let us now define more precisely the nature of both intentions and criteria. Related to the three areas, the *intention* of the subject's activity splits into three *types of polarization*: *understanding* in the Psychological area, *conformity* in the Social area and *performance* in the area of 'reality'.

Within the Area related to the type of polarization, the subject finds the feedback of his/her action and the clues to guide his/her action.

While working in 'performance', what the student knows about the (conceptual) mathematical 'reality' may lead him/her to reject a result like $|z| = -3$: this result cannot be said a "success" since it does not fit the mathematical "reality".

A student who is working in conformity on the other hand, takes his/her clues within the Social Area; his/her action tends to fit the rules (set by others, for instance the teacher, representative of the community of mathematicians, or other students in the case of a group work: students may come to an agreement on wrong Rules). When asked to explain his/her actions, a student working in conformity says formula-like sentences as: "When you have x above and below the line, you remove them", or "To solve an equation, you move the x to the other side".

We found necessary to give different names to intentions and criteria according to the type of polarization of the activity of the subject. This is summarised in the following table:

	Use and evolution of local bits of knowledge		local bits of knowledge
Areas	**Types of polarization**	**Standards**	**Dimensions**
Psychological	Understanding	Oneself	Coherent
Social	Conformity	Other(s)	Valid
'Real'	Performance	'Reality'	Efficient

Conformity

We shall focus now on conformity. This Type of polarization is nothing but harmful by itself: we mathematicians work in conformity whenever we need it. From the very beginning of algebra and Cartesian geometry, mathematicians have developed rules which permit them to make computations without referring to the meaning of the computations. It is a conceptual shortcut. The rules which permit to work on algebraic expressions are strict and there are plenty of them. An expert mathematician knows many things about them:

(a) s/he knows the rules.

(b) s/he knows where they come from: if s/he cannot, at one moment, demonstrate one rule, s/he knows that the demonstration exists, that s/he has known it (most of the time), and that s/he could find it again, either by her/himself or in a book.

(c) s/he knows how to control the result of the computations s/he makes by using the rules.

Algebra is such an important tool in mathematics because of the possibility of using conformity to work. What is harmful, is to work in conformity alone and never in understanding or in performance and, more generally, to work in one Dimension alone.

Identification

To identify indisputably the Type of polarization of a subject's activity, is far from easy. For instance, the rejection of $|z| = -3$ may follow (according to the context):

- from conformity, if s/he knows 'by rote' that "the length of a complex number is always greater than 0",
- from performance, if s/he knows that, in the mathematical 'reality', the lenght of a complex number is always greater than 0. The difference here is very subtle, and depends on *how* s/he knows it instead of *whether* s/he knows it. In mathematics indeed, rules of conformity describe mathematical 'reality' (facts which are necessary)
- from understanding at last, if s/he is aware that agreeing with $|z| = -3$ would be contradictory with the rest of his/her knowledge. Once again, it is a question of (subject's) point of view since the mathematical 'reality' is coherent (then the conform rules, too).

We may get plausible clues to the subject's Type of polarization however, by observing his/her non-verbal behaviour: tone of voice, pace of the discourse, gestures... That can not be easily described here in few words, since this observation relies on rather sophisticated interview techniques (Vermersch 1994). One may note however that a work in understanding is private, with few communication with the interviewer, often accompanied with murmurs, indistinct exclamations ("oh yes!", "how stupid!"...) and possibly an emphasis at the moment of the discovery (analogous to "Eureka!"). A subject who works in conformity, on the other hand, often recites rules with a more or less uncertain voice while looking for approval in the interviewers' (or teacher's) eyes.

TYPES OF POLARIZATION AND DIMENSIONS

While Dimensions are related to *static* aspects of the local bits of knowledge, Types of polarization are related to its *dynamic* aspects. Thus, we are led to claim the following statement:

a) *The **Dimensions** of a bit of knowledge are **indissociable**. A local bit of knowledge is Coherent, Valid and Efficient at the same time.*

b) *In contrast, the* ***Types of polarization*** *are* ***dissociated****. At any moment, the subject's action (involving his/her local bits of knowledge) is in one type of polarization alone.*

c) ***Shifts*** *of Type of polarization may be very quick; they may give us worthwhile clues about the Types of polarization.*

It must be emphasized that the shifts are often easier to observe during interviews than the subject's Types of polarization themselves. These shifts are often the actual (observable) data indeed, from which we assume the subject's Types of polarization.

At present we are working on the two following assumptions:

1) *A local bit of knowledge depends on the Types of polarization in which it is constructed.*

In other words, a subject's local bit of knowledge, if constructed (from the same field of bits of knowledge) in differing Types of polarization, differs. This hypothesis is rather strong, since it involves (among other consequences) that a local bit of knowledge, depending on the subject's psychological evolution and learning experience, is definitively more rich and complex than the related mathematical concepts (and obviously than the symbolic statement which express it).

2) *We can act on the Type of polarization in which the student works.*

This hypothesis is less trivial that it could appear at a first glance. Indeed, one cannot ask straightforwardly a student to act in performance or in understanding. Well known is the ineffectiveness of "Come on, please, consider it!" to lead the student in understanding. Our ongoing research address the question of how to guide the student by giving him/her tasks (like in the "Write False" interviews (Sackur, 1995)) which can have an influence upon his/her Type of polarization.

CONCLUSION

We remind that the aim of this paper is just to present the main points of a part of our theoretical framework: the local bits of knowledge, the three Dimensions and the three Types of polarization. Therefore, this paper contains no experimental evidence, since we refer to previous presentations of some aspects of our experiments. Hence, in this paper there is no discussion of experimental results, and the aim of the conclusion is just to prevent, one more time, a risk of misunderstanding.

We do think that there is no hierarchy between the Types of polarization. Understanding is in no ways a "good" way of doing mathematics in contrast to a "bad", low-level conformity, therefore learning mathematics is in no ways evolving

from the latter to the first[3]. Algebra for instance (and likely all mathematics), takes its power from the ability to allow the subject to work in conformity. Thus, s/he is not obliged to come back at any moment to the meaning (understanding) of both the algebraic expressions and their transformations. Every work in mathematics is done partly in conformity; on the other hand we observed that the students which have difficulties in algebra have been often working in conformity alone. That is why, according to our hypothesis on the possibility to act on the Types of polarization, our experimental work addresses preferably these students.

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3 we are fully aware that the words we use like "understanding", "conformity", or "efficience" carry a lot of parasite meanings; the problem remain the same however, for any other word we could choose.

WHEN DOES A WAY OF WORKING BECOME A METHODOLOGY?

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Abstract

We reflect on our way of working and try to position it in relation to that of others. We consider perceptions of methodology and discuss research questions and the way they affect and are affected by the methodology they are sited in. Through selected literature and the reactions of colleagues we aim to establish satisfactory criteria for an acceptable methodology and apply it to our way of working.

Having worked together for several years developing and using a theory of learning (Duffin and Simpson, 1995) we have come to a point when we feel the need to site our work in relation to that of others, and to reflect on our way of working in order to consider its validity as research.

Quite early in our working association we became aware of some specific characteristics of the way we work which we felt fed and were crucial to the conclusions we were reaching. More recently, as we became more aware of the importance of methodology to any piece of valid research, we have been trying to see where our work could fit into the range of existing methodologies. We sought this access both through discussions with others, a notable ingredient in our normal practice, and through selected literature.

One particular piece of literature stimulated a chain of thought which became a central focus in our attempt to clarify some unanswered questions about the nature of research itself and the place of our work within research.

Methodolatry and First Thoughts on Methodologies

It was the work of Daly (1973) that gave us our first signpost to what we were seeking. Her word 'methodolatry' captured our interest and began to take our thinking forward at this stage. The word appeared in Belenky et al. (1986) in the context of women's involvement in both learning and the research process and was strongly influenced by Daly's earlier work. Belenky discusses the idea that a set of acceptable methodologies can 'render invisible' those whose work does not fall

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within this category, work which wants to ask different research questions or use a different methodology.

Our attention was caught by the Belenky notion that a methodology, in part at least, can limit, even determine, the kinds of questions that can be asked in research. We began to see in this a duality: that a methodology can determine the questions that can be asked and that the questions a researcher is interested in may determine the methodology to be used. The two-way process that it indicates is a dominant feature, certainly of our work but perhaps, more generally, is part of any acceptable methodology. Indeed, even this may be too clear cut; it may be that questions and methodology are so bound up together that there is a constant ebb and flow between them in the research process as both become refined over a period of time.

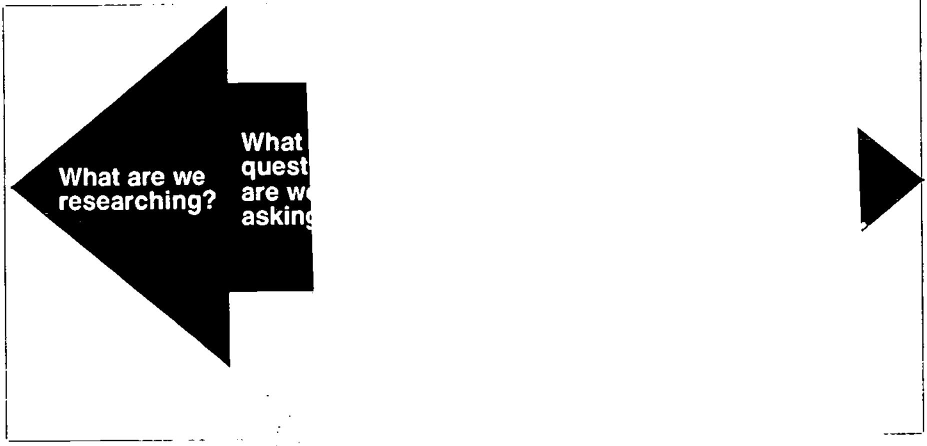
As we continued to clarify our thinking on this crucial issue, we began to realise that the idea of an 'ebb and flow' went beyond the relationship between just questions and method. We developed the idea of a *domain* and *range* of any research: by the domain of a piece of research we mean who or what is being researched, and by the range we mean those who can or want to use the results of the work. We became aware of the possible influences coming from this domain and range. For example, if the researcher wishes, as we do, to make their research available for teachers to influence their classroom practice, the form the research results take must be one that is accessible to teachers. This in turn will influence the kind of questions to be asked and thus also the focus of the research itself.

From this chain of thought we produced a list of five questions we felt would provide us with tools for our work and, if we could find answers to them, might enable us to site our work amongst that of others. These were

- who or what is being researched?
- what questions is the research asking?
- what form do the answers take?
- who can or does use the answers?
- what do they use them for?

Such a list suggests both a linear flow and a direction (from what is being researched to what use the answers are put to). In trying to remove these implications, we developed a diagram (figure 1) which illustrates our belief that there can be an ebb and flow of influence between the five questions and that there is no specified direction.

In order to consider some of these questions in relation to our work, and from our own viewpoint, we need to make explicit the details of our way of working.



Our Way of Working

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and, in the talking, enlarged and changed the perceptions of both of us.

It was only at a later date that our attention was drawn to the contrasts between the approaches of Piaget (seeing learning as rooted in the individual) and Vygotsky (seeing learning as rooted in interaction between individuals) as we listened to colleagues more experienced in research than ourselves. For us, both our individual ways of viewing what we experienced, and the important changes and developments arising naturally from our interaction, became cornerstones in our own development.

It was only recently, as our thoughts turned to ideas about connecting our work with that of others, that we began to feel the need to try to be explicit about these features of our way of working. We identified three essential characteristics:

- Introspection
- Co-spection
- 'As if from the inside'

By introspection we mean constantly seeking to discern our individual perceptions of experiences, both past and present, and our reactions to them. We suggest that looking at ourselves from the inside gives us an access to the mental processes of a learner that we cannot have in studying other people.

We use the term 'co-spection' to mean the sharing of our own personal reactions to experience, with the deliberate intention of using samenesses and differences to further both our individual and our shared perceptions. By 'as if from the inside' we mean that we try to approach the observation of the actions of others, usually in some kind of learning context, from a viewpoint which tries to take into account what individual learner's own perception of their experiences might be.

We recognise that some of the language we use in describing our way of working stems from that of Mason and from adapting our perception of his ideas to it. It is he who points out that the sole use of introspection as a research tool in psychology was strongly challenged by workers such as Watson (1913) and that the excesses of treating personal, inward looking accounts as unchallengeable partially led to the development of 'objective' behaviourism (Mason, 1994). Mason's work and his introduction of the terms extra-, intra- and interspection were influential for us both in concept and in the choice of vocabulary we made to describe one aspect of our way of working.

We accept that there are problems with introspection as a research method. Both Mason and ourselves are trying to solve these problems but we appear to be doing it in different ways. Mason does it by abandoning the term introspection in favour of extra-, intra- and interspection, while we do it through adding the notions of co-spection and 'as if from the inside' to validate, challenge and extend the findings of introspection.

For us it is the synthesis of all *three* characteristics - examining our individual responses to experiences, sharing them closely with the other who tends initially to respond differently to experiences, and using the similarities and differences of the internal processes we get from these to consider our observation of others 'as if from the inside' - which constitutes our way of working. The sharing through co-spection and the examination of any theory we develop through the 'as if from the inside' process provide us with a challenge to our individual, internal accounts which introspection alone cannot do.

The phrase 'as if from the inside' comes from reflecting on Mason (1987) when he classifies researchers as follows:

> We are all trying to model or describe the inner world of experience. Some of us proceed by contemplating and studying other people, or by studying ourselves as if from the outside; others proceed by contemplating and studying ourselves from inside.

We find a partial symmetry here: relating the two dimensions of studying ourselves/others and studying from the inside/outside. But while Mason relates the self to both inside and outside study, he only mentions the notion of studying others from outside. It seems to us that our attempts to look at others by considering what internal influences there might have been on their actions brings in a fourth concept of 'studying others as if from the inside', which appears to complete the symmetry.

This idea entered our work at an early stage before we had encountered the literature that gave rise to its vocabulary. It arose when we were developing our theory of learning, *natural, conflicting and alien* (Duffin and Simpson, 1993). Initially the theory was based only on the two concepts of natural and conflicting but, in examining an incident about a seven year old boy, we became conscious that a clear contradiction, which for us would have caused a conflict, did not perturb him in the slightest; he merely ignored it. This incident led us to consider what, in the mental processes of a learner, might lead to this kind of response, resulting in the third concept of our theory: alien. It also led us to realise that an essential feature of our work was that we were trying to observe learning incidents that came our way as if from the viewpoint of the learner involved: as if from the inside.

Connecting Our Way of Working

Our long term aim, then, is to try to connect our own way of working as described above with the five questions of figure 1. Later in this paper we aim to consider the ways in which interchange with colleagues has helped to crystallise our perception of research processes and methodologies generally. Before addressing this issue, however, it is important to note how our own work has shown the ebb and flow which we see as central to the research process itself.

In retrospect, and in the light of the processes through which we are now going, we can see that our earliest work, in which we were trying to describe and explain our different perceptions of the eight year old girl's work, was the result of a personally motivated question: how could the other see the incident in the way he/she did? This clearly influenced the form that any 'answers' we obtained took, since they were essentially merely answers for ourselves. The flow at that stage was predominantly from question to the form of answers. However, as we took our work out to others and sought their responses, we were encouraged to make those answers available in a form accessible to other researchers - and because of our own interests, to teachers as well. Thus we were led back from the issue of the

form of the answers we wanted to obtain, to the kinds of questions we were asking - reversing the flow.

This notion of ebb and flow makes the answering of the questions even harder. Whatever order we tackle them in will surely influence the way in which we answer them and is, in turn, influenced by our current view of our work. Thus it is important to recognise that we have only just begun to address the questions and whatever answers we give are extremely tentative.

If we start with the question 'who can or does use our research?' a first attempt at an answer might be:

> For a considerable amount of the time we have been working together, we have wanted to make our work available to teachers. Part of the reason for this comes from our own feeling that some research which might influence us, both as researchers and teachers, is inaccessible because we do not speak its language. We see it as only accessible to other researchers in the same field who do know the language. (Here we also see the influence of the question 'what are answers used for?' over 'who uses the answers?, which also influences the other questions we are postulating.) So we see our aim as the production of research for three main groups: ourselves (to take us forward in our own thinking), other researchers (to enable them to see our view of learning and compare it with their own and that of other people) and teachers (who may wish to use our theory to enable them to model the learning processes of their pupils differently or who may wish to use our way of working to develop their own models).

In producing that very tentative first answer, it is noticeable that we have had to bring in partial answers to other questions besides the one we started with: namely 'what is the research used for' and 'what form do the answers take'. Indeed, in coming merely to this initial answer to one question, we have become aware of how threatening all of these questions can be when they are separated: it appears that the research process entangles them so much that they cannot be dealt with easily on their own. Much work still awaits us before we can hope to arrive at satisfactory answers to all our questions.

Interchange with colleagues

As an extension of the co-spection element in our way of working, we make a practice of engaging colleagues in interchange about our ideas. At one of the most recent of these interchanges, in a session which included several research students as well as more experienced researchers, we raised some of the issues discussed above and asked them to relate the five questions to their own research.

What emerged was interesting on several counts. First, we found that the research students present seemed to have a somewhat rigidly defined perception of what constituted valid research, seeing it as being obliged to follow a laid down set of rules which resulted in a thesis format which told a linear story of their research, perhaps in one direction only in our figure 1. At the same time they were prepared to accept that, in doing the work, they were subject to the ebb and flow we postulate for the processes of research while being more sceptical about its relevance to the issue in question.

There was also an alternative view: that it was possible at later stages of a research career to, in the words of one participant, 'open up to alternative, perhaps more flexible ways of working' while still adhering to the view that, in the initial stages, it was important to have to conform to accepted norms and formats for working.

Yet a third view emerged: some contributors were prepared to suggest that it was possible to challenge the received view of what constitutes 'real research', that perhaps we can begin to take on board alternative forms of question and to use alternative methodologies. This was not said inconsequentially but was accompanied by references to Stenhouse (1984) from whom emerged the idea that research is "systematic enquiry made public".

Our discussion moved on to the idea that the central feature of work to be deemed research is not that of conformity to a *laid-down* rigour but instead requires that, to be valid, the procedures used in the research must be made explicit so that it is possible to measure those procedures against the reality of experiences. The validity then comes from that measuring against reality rather than coming from somebody else's perceptions of what valid research really is.

We return to the title of this paper: When does a way of working become a methodology? Perhaps a first tentative answer might be: A way of working becomes a methodology when it is made rigorous through being made explicit and can justify the intricate relationships between the questions it asks and the methods it employs.

We are starting on a long journey towards achieving this end. We have been explicit about the processes of our way of working and have started tentative attempts to answer the questions we have raised for our own work and its purposes.

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Development of Seventh-Grade Students' Problem Posing

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Three classes of seventh-grade students participated in a 3-month problem-posing program. Twenty-three children who displayed different profiles of achievement in routine and novel problem solving were selected for detailed observation. This paper reports on the children's development in one of several areas explored, namely, in the complexity and sophistication of the problems they were able to pose from open-ended situations. Consideration is given to the children's inclusion of critical information units, their use of semantic relations, the types of questions posed, and the complexity of problem solution. Links between students' problem-posing and problem-solving abilities are indicated.

Problem posing is recognized as a significant component of the mathematics curriculum and is considered to lie at the heart of mathematical activity (e.g., Brown & Walter, 1993; Moses, Bjork, & Goldenberg, 1990; Silver & Cai, 1996). The inclusion of activities in which students generate their own problems, in addition to solving pre-formulated examples, has been strongly recommended by several national bodies (e.g., Australian Association of Mathematics Teachers, 1996; National Council of Teachers of Mathematics, USA, 1989; Streefland, 1993). Despite its significance, problem posing has not received the attention it warrants from mathematics education researchers. We know comparatively little about children's abilities to create their own problems in different mathematical contexts, about the processes they use, and about the extent to which these abilities are linked to their competence in problem solving. There is also insufficient information on how children respond to programs designed to develop their problem posing (Silver, 1994).

The present study incorporated a 3-month problem-posing program that was implemented as part of the regular mathematics curriculum in seventh-grade classes. The study aimed to:

1. trace the development of students' problem posing across a range of mathematical situations;
2. trace the problem-posing developments of individual children as they participate in the classroom activities;
3. monitor changes in children's perceptions of, and attitudes towards, problem posing and problem solving;
4. identify links between students' problem-posing and problem-solving abilities.

Theoretical Perspectives

The study represents the final phase of a three-year investigation of primary school children's development of problem posing (English, in press a, b). Given the paucity of research on the topic, it was necessary to construct a theoretical framework that would guide the development and implementation of each phase.

The framework encompasses psychological and sociological components (cf. Cobb & Bauersfeld, 1995), as displayed in Fig 1.

KEY ELEMENTS OF PROBLEM POSING

Knowledge and Reasoning

- Understanding problem structures and recognising related structures;
- Knowing problem design, in particular, recognising critical information units;
- Being able to model and transform given structures, as well as build new ones;
- Knowing whether and how a posed structure constitutes a solvable problem;
- Being able to think in mathematically diverse ways;
- Knowing how and when to apply processes of analogical reasoning;
- Being able to reason critically in assessing problems and problem experiences.

Metacognitive Processes

- Communicating one's perceptions of, and preferences for, different problem types;
- Reviewing and enhancing self-efficacy expectations;
- Improving one's disposition towards problem posing and problem solving.

Sociological Factors

- Participating in classroom communities of philosophical and mathematical inquiry;
- Engaging in constructive dialogue and debate;
- Sharing and critiquing problem creations.

Figure 1 Key Elements of Problem Posing

Since this paper is concerned with children's problem creations from open-ended situations, those components pertaining to problem structure are reviewed. One of the fundamental elements of problem posing is understanding just what a problem is (Brown & Walter, 1993). This includes being able to recognise its underlying structure and to detect corresponding structures in related problems. Structure may be defined as "form abstracted from its linguistic expression" (Freudenthal, 1991, p. 20). While not denying the importance of problem context (Freudenthal, 1991), children need to recognise the mathematical structures of problem situations if they are to utilise these to generate new examples and questions; this requires them to place the contextual features in the background and bring the structural elements to the fore. That is, children need to construct meaningful mental models or representations that recognise the important mathematical ideas and how they are related (English & Halford, 1995; Nesher, 1992).

The complexity of problem structure is determined, in part, by its linguistic or syntactic properties (Mayer, Lewis, & Hegarty, 1992; Silver & Cai, 1996). Mayer et al. found that problem-solving difficulty seemed to be related to linguistic complexity, with problems containing assignment propositions easier than those with relational or conditional propositions (defined later). The nature and number of distinct semantic relations embodied in a problem also have a bearing on its complexity (Marshall, 1995; Silver & Cai, 1996). For example, a story problem that involves both multiplication and subtraction would be more complex

than a comparable case involving only one of these. Of interest in the present context is how the children's use of linguistic and semantic properties in generating their own problems developed over the course of the program.

Also of importance in children's facility with problem structure is their awareness of problem design. In generating their own problems, children must recognise the critical items of information that are required for problem solution (referred to here as "critical information units," addressed later). This awareness includes recognising the nature and role of the "known" and "unknown" information entailed in their posed problem, as well as any constraints placed on goal attainment (Moses et al., 1993). This knowledge is necessary for determining whether and how a posed problem structure constitutes a solvable problem, a basic element of problem posing (Brown & Walter, 1993). Children's inclusion of critical information units in constructing solvable problems was of interest in this study.

Methodology

Participants and Selection Procedures

Three classes of seventh-grade students from three state schools participated in this final study phase, conducted during 1996. Twenty-three students were chosen for in-depth observation and analysis (mean age of 11. 9 years in term 1). The 23 children (along with an additional six children serving as a small control group) were chosen on the basis of their responses to tests of number sense and novel problem solving; these were administered during the first term of the school year. The tests were modelled on examples that had been used successfully in the previous phases (English, in press a, b). The number sense test focused on facility with number and routine computational problem solving, while the novel problem-solving test included examples requiring a range of reasoning processes (e.g., deductive, combinatorial, spatial reasoning), as well as general strategies. The selected children displayed the following profiles of achievement:

1. strong in number sense but not so in novel problem solving ("SNS" profile; N=6)
2. not strong in number sense but strong in novel problem solving ("SNP;" N=5)
3. strong in both domains ("SB;" N=7)
4. average achievement in both domains ("AB;" N=5)

The intention was to include children from the first three profiles only, however difficulty in obtaining sufficient numbers necessitated adding the last category.

Procedures

The 29 children (including control group) were individually administered a comprehensive set of problem-posing activities during the second term and a parallel set towards the end of the fourth term. The problem-posing program was conducted during the third and fourth terms and comprised 12 weeks of classroom activities (approximately 1.5 hours per week). These incorporated a broad range of experiences that addressed the important elements of problem posing (Fig. 1). A variety of approaches was adopted, including small and large group discussions, class debates, sharing and critiquing of posed problems, and individual and whole

class reflections on students' progress and on the program itself. Throughout the program we tried to establish a community of inquiry involving meaningful dialogue or "connected talking" among the children and teacher (English, in press c; Yackel, 1995). All children maintained journals of their problem creations and we also video- and audio-taped all the activities of the selected children. The benefits of this form of research have been well documented (e.g., Cobb & Bauersfeld, 1995).

Analysis of Children's Responses to One of the Activities

This paper reports on the children's responses to one activity type that was included in the pre- and post-program activities as well as in the program itself (towards the end of the program). The children were required to construct three different problems from open-ended situations of the type shown in Fig. 2 (this example was a post-program activity; the program itself, included a number of open situations drawn from newspapers, travel brochures, and historical reports).

SPOOKY TRAVEL

A 5-day tour of the ghost castles on No Man's Island, departing from Munster Town, costs $1776 per person. A 4-day tour of the bat caves on No Man's Island costs $1400 per person. Departure from Cape Fear to No Man's Island costs $350 less per person. The cost of food for each of the trips is $450 per person if there is just one person travelling, and $400 per person if two or more people are travelling together.

An Figure 2 An Open-Ended Problem Situation

Children's responses to these situations were analysed using the following coding scheme (this scheme draws upon some of the ideas of Silver & Cai, 1996):

Problem creation and solvability. This was concerned with: (i) whether a mathematical problem was created, and (ii) whether the problem was solvable with a unique solution (although problems with more than one solution are important in the curriculum, such problems in the present context reflected a design weakness).

Problem complexity. This focused on: (i) the extent of critical information units included in the problem, (ii) the number of distinct semantic relations, (iii) the number of steps required for solution, and (iv) the type of question posed (assignment, relational, conditional). A critical information unit, as used here, refers to an item of information that is necessary for problem solution. For example, reference to the point of departure in the above example is a critical information unit, as is a statement on whether food is required. An assignment question addresses one variable, such as, "How much did the trip cost?" while a relational question compares two variables, such as, "How much more does it cost to go on the 5-day tour than the 4-day tour?" (Mayer et al., 1992). A conditional question imposes a constraint, such as, "How much would you have to pay if you wanted to depart from Munster Town and if you wanted to take a friend with you?"

Selected Findings

The children showed a distinct improvement in their problem generation between the pre- and post-program activities. On the pre-program activity, there were two

instances of a non-mathematical problem being generated (both from children in the AB profile) and 17 instances of an insolvable problem. Children from the SNP and AB profiles had the greatest difficulty in creating a solvable problem on the pre-program activities, while children from the SB profile were the most competent. On the post-program activity however, every child was able to create a solvable problem, with children in the SNS and SB profiles better able to create problems with unique solutions than children in the remaining profiles. In contrast, the six non-participants had difficulty in generating a solvable problem, with 45% of their creations being either non-solvable or a non-mathematical problem.

Developments in the complexity and sophistication of the children's problems between the pre- and post-program activities can be seen in Tables 1 and 2. Among the more noticeable developments was an increase in the number of critical information units the children included in their problems (reflecting an increase in solvable problems). Children from the SNP and AB profiles in particular, showed substantial growth, as was evident in Nathan's (SNP) case. He progressed from being unable to generate a solvable problem to creating the problem: *Which would cost more? Being a single person and leaving from Munster Town or having two people leave from Cape Fear to go to No Man's Island?* While children from the SNS profile also showed considerable improvement in their inclusion of critical information units, those from the highest profile (SB) showed little change between the pre- and post-program activity. These children had few difficulties in generating problems prior to the program and were able to create quite sophisticated examples during the program. For example, Adam posed this problem after examining a travel brochure: *I've taken a leap year off work and decided to go on as many holidays as possible. Each time I go on a holiday I have to take the time of the previous holiday to recover for the one coming up. If two holidays go for the same amount of time I'll go on the most expensive one, then the cheapest, then I'll go on another expensive one, then a cheaper one, and so on. What would be the average cost per day whether I'm at home recovering or on holiday? P. S. Money is not a concern..*

The program made little difference to the children's posing of relational questions. These were clearly not favored, reflecting the documented difficulties children experience with comparison problems (e.g., Stern, 1993). On the other hand, 59% of all the children's questions were of a conditional type and 35% were assignment questions. This is in contrast to Silver and Cai's (1996) findings where only 5% of their sixth- and seventh-grade students posed conditional questions. Interestingly, it was the SNP children who tended to favour conditional questions on both activities.

As indicated in Table 2, the children showed substantial shifts in their inclusion of semantic relations and in the complexity of their problem solutions. The SNP children demonstrated the greatest improvement, especially in their ability to incorporate several semantic relations in their problem; the AB children also showed marked gains. Children in the remaining two profiles displayed a noticeable

increase in the computational complexity of their problems, with 59% of their post-program problems involving 3 or more steps (in contrast to only 10% previously).

Although the sample is small and the data pertain to only one type of activity, there appear some links between competence in problem solving and problem posing. First, and not surprising, competence in both routine and novel problem solving appears associated with competence in posing problems from open-ended situations. Second, competence in number and routine problem solving appears associated with the construction of computationally complex problems. The third link, which was particularly evident in the classroom observations, is that children who are competent with novel problems but not so with routine numerical problems respond particularly favourably to problem-posing activities and demonstrate considerable divergence in their thinking and in their problem creations (these findings reflect those of the previous study phases).

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e 1 <u>Frequencies of Use of Critical Units and Question Types by Achievement Profile</u>

	First Problem Created							Second Problem Created							Third Problem Created						
	No. of Critical Units				Question Type			No. of Critical Units				Question Type			No. of Critical Units				Question Type		
Profile	0	1-2	3-4	5-7	A*	R	C	0	1-2	3-4	5-7	A	R	C	0	1-2	3-4	5-7	A	R	C
Pre-program Problems Generated																					
SNS**		4	2		3	1	2		4	2			1	5			6		1		5
SNP		5			2		2		4	1		1		4		4	1		1		4
SB		1	5	1	4		3		1	5	1	4	2	2		1	5	1	5	1	2
AB		5			2		3	1	4			2		2	1	3	1		1		3
Post-program Problems Generated																					
SNS			4	2	1		5			4	2	2	1	4			3	3	4		2
SNP		1	4			1	4		2	2	1	1	1	3			5				5
SB			3	4	2		5			5	2	4		3		1	4	2	3		4
AB		1	3	1	2		3		2	2	1	3		2			4	1			5

<u>Note</u>. * A: assignment R: relational C: conditional ** SNS: strong in number sense only (N=6); SNP: strong in novel problem solving only (N=5); SB: strong in both (N=7); AB: average in both (N=5). On the pre-program activity, there were 3 cases where a question type could not be assigned and 2 instances in which the question type was both relational and conditional. On the post-program activity, there was one of the latter instances.

: 2 Frequencies of Use of Semantic Relations and Solution Steps by Achievement Profile

	First Problem Created										Second Problem Created										Third Problem Created									
	No. of Semantic Relations				No. of Steps						No. of Semantic Relations				No. of Steps						No. of Semantic Relations				No. of Steps					
Profile	0	1	2	3	0	1	2	3	4	5>	0	1	2	3	0	1	2	3	4	5>	0	1	2	3	0	1	2	3	4	5>
Pre-program Problems Generated																														
SNS*	1	4	1		1	4	1				2	1	2	1	2	1	2	1				3	2	1		3	2	1		
SNP	3	1	1		3	1	1				2	2	1		2	2	1				3	2			3	2				
SB		4	3			4	3				1	4	1	1	1	4	1	1				3	4			2	4		1	
AB	2	3			2	3					1	4			1	4					3	1	1		3	1	1			
Post-program Problems Generated																														
SNS		2	3	1		1		1	2	2		4	2			1	1	4				1	4	1		1	1	1	2	1
SNP		2	3			2	2		1			1	4			1	4					1	4			1	2	2		
SB		1	4	2			3	2		2		2	5			2	3	1	1			2	3	2		1	2	2	2	
AB		4		**		4						2		3		2		2		1			5				4			1

Note. SNS: strong in number sense only (N=6); SNP: strong in novel problem solving only (N=5); SB: strong in both (N=7); AB: average in both (N=5). ** One child from the AB profile created a combinatorial problem for her first problem.

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A CLOSE LOOK AT THE USE OF MATHEMATICS-CLASSROOM-SITUATION CASES IN TEACHER EDUCATION

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Mathematics-Classroom-Situation Cases (MCS-Cases) are real or hypothetical classroom situations involving mathematics, in which the teacher has to respond to a student's question or idea. A previous study indicated that the use of MCS-Cases in teacher education has the potential to contribute to the development of pedagogical content knowledge. This article investigates the use of one such case, the Decimal Point Situation, in an in-service course for twenty elementary teachers. The Situation was used in four different settings: written questionnaire, individual interview, whole class discussion and student interview. The findings indicate that different settings highlight and lead to different outcomes regarding various aspects of pedagogical content knowledge.

INTRODUCTION

Past professional development activities for in-service mathematics teachers tended to focus on ways of implementing curricula developed by "experts". However, such sole approach cannot help teachers to fulfill a role as is envisioned by current reform movements. Consequently, there is a growing trend which aims to enhance the professionalism of teachers and to empower the teacher as a decision maker. In recent years, several innovative teacher education programs that belong to this trend were designed. Some of them center on and stem from how children learn and how children's mathematical ideas develop on particular topics (e.g., Fennema, Peterson, Chiang, & Loef, 1989).

As teacher education being rethought and experimented, and new understandings about teacher knowledge and its representation are developed (e.g., Shulman, 1986), teacher educators develop and examine innovative pedagogies for teacher education programs. One such pedagogy is the case-based pedagogy (e.g., Merseth, 1996) which fits naturally with current conceptions about teaching. Teaching is gradually recognized as a complex and ill-structured domain where theory cannot instantly determine action, i.e., the "right answers" cannot be derived through direct application of appropriate principles and theories. In many cases, teacher action derives from and builds on contextual and local situations and experiences. Therefore, teacher educators and staff developers began, in the last years, to develop and use a variety of case materials in their pre-service and in-service courses (e.g., Barnett, Goldenstein, & Jackson, 1994; Wilcox and Lanier, in press).

Our work in the last several years belongs to the same trend. We developed cases that describe real or hypothetical[1] classroom situations involving mathematics, in which the teacher has to respond to a student's question or idea. The situations designed for junior-high school teachers center on the function concept; those aiming at elementary school teachers focus on arithmetic and number sense--all deal with students' ways of thinking and with various characteristics of teachers' responses to students. We use these cases in teacher education courses (Even & Markovits, 1991, 1993; Markovits & Even, 1994), and we explore the potential of the cases to raise teachers' awareness and sensitivity to students' ways of thinking in mathematics; to expand teacher capacity to critically analyze teachers' responses to students' questions, remarks or hypotheses concerning subject matter; to develop teacher ability to reflect on their practice, etc.

Because the use of cases and case methods in teacher education is a rather new phenomenon, there is little research that examines their potentiality. Our research indicates that the use of the "function cases" with junior-high school teachers raised their awareness of students' thinking (Even & Markovits, 1993). Other developers and users of cases also report influence on teacher thinking, cognition and beliefs. However, as is emphasized in Merseth's (1996), very little is actually known about the nature of case practice in teacher education. What are promising ways of using them? What happen teachers work on them alone? in small groups? large groups? Should teachers discuss the cases among themselves? respond to them in writing? Our study focuses on the issue of case practice in teacher education. It investigates the use of one case, the Decimal Point Situation, in four different settings in the context of a teacher education course, and examines how the different settings highlight and lead to different outcomes regarding various aspects of pedagogical content knowledge.

DATA COLLECTION

Subjects

Twenty elementary school teachers participated in the study. These teachers participated in a two-year program for preparing mentors for elementary school teachers held in a teacher college. Almost all have participated in in-service courses on mathematics teaching; some completed a two-year course for elementary school mathematics coordinators. Most of the teachers had at least ten years of experience in teaching elementary school, usually in the upper grades (grades 4-6). Overall, their background in teaching mathematics was somewhat better than the average elementary school teacher.

[1]Note, that there are some differences between "our cases" and "cases" as they are commonly defined in the literature (for a comprehensive literature review of cases and case methods in teacher education, see Merseth, 1996). For example, in the literature, cases are usually real stories and include detailed background data. Our cases, on the other hand, may be hypothetical (although rely on research); and they focus on students' thinking in a rather short episode, leaving for the teachers' imagination to fill the background details according to their own experiences.

Mathematics-Classroom-Situation (MCS) Cases

The course "Mathematics Classroom Situations" was part of this two-year program. It was held in the second semester of the first year. Mathematics-Classroom-Situation (MCS) Cases formed a major part of the course. MCS-Cases center on real or hypothetical classroom situations involving mathematics, in which a student work or idea is described and the teacher is asked to respond to it. The situations are chosen so that they highlight students' ways of thinking and conceptions as known from research and personal experience. One such case centered on the Decimal Point Situation:

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A student was told that 15.24x4.5 = 6858, and was asked to locate the decimal point. The student said that the answer is 6.858 because there are two places after the decimal point in 15.24 and one place after the decimal point in 4.5. Together it makes three places after the point in the answer.

How would you respond?

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We chose this situation because it focuses on multiplication of decimals--a typical basic elementary school mathematical topic, and also because it opens for discussion the issue of the place of memorized rules and number sense in mathematics classes.

In addition to such situations, we develop and use an extended form of them which include responses that were given by other teachers to the situations. The course participants are asked to react to these responses. Following are the responses that accompany the Decimal Point Situation in its extended form:

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1. *I would tell the child: You located the decimal point correctly and also explained it correctly.*
2. *I would ask the child to find two whole numbers that are close to the given numbers and to multiply them. I would then ask him to look at his exercise and the given exercise and to check what is going on.*
3. *I would tell the child that the multiplication of the integer parts alone (15 x 4) is 60. So we get more than 60. That's why the answer should be 68.58. In addition, I would write down the exercise, and ask the child to multiply. The answer would be 68.580 and I will explain that 68.580 equals 68.58.*
4. *The child does not understand how to multiply decimal numbers. I would give him several exercises and ask him to solve them using the standard algorithm.*
5. *I would tell the child: You stated a correct rule but your answer is incorrect, because when you multiply 4 and 5 the answer has a 0 at the end. Zero is not shown in your answer, and that's why you made a mistake in locating the decimal point. The answer is 68.580.*

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Settings

The participants met the Decimal Point Situation in four different settings during the study: two occurred before the course, while answering a written questionnaire and during an individual interview; two during the course, at a whole class discussion and when interviewing students.

Written Questionnaire

Several weeks preceding the course, a questionnaire was administered to all participating teachers. Each of the eight tasks in the questionnaire, among them the Decimal Point Situation, described a MCS-Case.

Individual Interview

Individual interviews were conducted with six of the twenty teachers, several weeks after the questionnaires were handed in, and a couple of weeks before the beginning of the course. The interviews centered on the extended form of three situations, one of which was the Decimal Point Situation. The subjects were presented with responses given by other teachers to the same situations and were asked to react to these responses. After reacting to the other teachers' responses, the interviewees were asked to choose the response they liked the most.

Whole Class Discussion

At the first several course meetings the teachers discussed the ways they had responded to the questionnaire tasks, in an unstructured manner with no specific guidelines as to what to pay attention. Then, they were presented with other teacher responses and were asked to react to them. Finally, after reacting freely to other teachers' responses for several situations, the teachers were asked to analyze the responses according to the following criteria:

- Does the teacher understand what the student does not understand?
- Does the teacher's response concentrate on the student's misconception?
- Does the teacher's response emphasize rituals? Does it pertain to meaning?
- Is the response teacher-centered? Student-centered?
- Is there any problem regarding content knowledge?

The Decimal Point Situation was discussed during the first two course meetings. These meetings were videotaped and later transcribed.

Student Interview

As part of the final assignment for the course, the teachers were to explore students' ways of thinking about mathematical situations and teachers' explanations. To do that, each teacher interviewed a pair of sixth-grade students, either her own or from other classes. The teachers presented three situations to the students (the Decimal Point Situation was one of them) and asked the students to respond. (First they asked the children to solve the mathematical problems by themselves.) Then, they presented the students with teachers' responses to these situations and asked the

students to react. The teachers audio-recorded the interviews and later transcribed and analyzed them. These analyses were reported in writing to the instructors and were handed in together with the transcripts and the recorded cassettes.

THE DECIMAL POINT SITUATION IN DIFFERENT SETTINGS

Because of space limitations we describe very briefly the different aspects of the participants' mathematical and pedagogical content knowledge that characterize their responses, analyses and reflections in three of the four settings and elaborate more on the aspects that characterize the "whole class discussion" setting.

In the first setting of this study--the questionnaires--five of the teachers considered the answer given by the student as correct. The cause for this mistake seems to be an assumption that the given problem is a standard one and therefore one can use a mechanical way of solution. These teachers, of course, did not use any number sense. Eight of the teachers evaluated the student's answer as partly correct. The reason for this seems to be rooted in their noticing, on the one hand, that the answer is incorrect, while, on the other hand, the student was still using a well-known and commonly taught rule: When multiplying two decimal numbers, the location of the decimal point is determined by adding the number of digits after the decimal point in the two numbers. The other seven teachers evaluated the student's answer as completely wrong.

The teachers who thought that the student's answer was correct, focused their written response to him on a request for an explanation. Such a response is very interesting. These teachers consider as correct an answer that is wrong. Nonetheless, what bothers the teachers is whether the student remembers the explanation for the rule that produced the answer. Not only are the teachers not aware of the student's mistake, a mistake that was caused by a mechanical use of a memorized rule instead of an application of number sense, but superficially they claim to care about understanding, whereas their responses actually point again to preference of memorization--this time memorization of explanations.

The other 15 teachers used explanations in their written response to the student that were based on three different strategies. The first kind emphasizes estimation of the magnitude of the product when multiplying the whole parts in each number. The second kind of explanation is related to the "missing" zero, suggesting to multiply the last digits of the numbers. Another kind of explanation concentrates on actual performance of the standard algorithm for multiplication of decimal numbers. This, of course, leads to the appearance of the "missing" zero, and allows the use of the rule. The first two kinds of explanation are based on number sense while the third kind emphasizes a solution of the problem based on the well-known algorithmic way only.

The probing in the second setting--the individual interviews--caused the interviewees to re-examine their previous thoughts and conceptions of the student's answer and sometimes change them. A dilemma arose in relation to the feeling that

the rule was correct and still the final answer received by applying this rule was wrong. The interviewees' reactions to the other teachers' responses, as well as their preferences when asked to choose among the other teachers' responses, had similar characteristics to the responses they themselves gave when answering the questionnaire. This consistency was apparent in both the mathematical approach they used and in the way they chose to approach the child.

The unstructured discussion in the third setting--the whole-group discussion--took place at the first course meeting. The aim of this part was to allow the participants to hear each other's responses and discuss them collectively instead of individually as was the case when they answered the questionnaires or were interviewed. The Decimal Point Situation case was the third case they discussed at the meeting.

While participants' focus during the first two settings of using the Decimal Point Situation--questionnaires and interviews--centered on suggestions and evaluation of ways of response to the student, this was not the case during the unstructured discussion. The instructor launched the discussion by reminding the participants of the situation and asked them: "How would you respond?" Instead of suggesting ways of response as might be anticipated, one participant protested:

> He was misled from the beginning. They took off the zero... Because if the zero was there, the whole story would have ended differently... He knows the rule, but he was misled from the beginning, because we always say it is 68580, the zero is important in multiplication. He didn't use estimation, he was told that this is the answer and was asked to locate the decimal point, so he went ahead and used the rule.

The teacher who said this was one of the teachers who wrongly used the rule of counting places and therefore answered 6.858 on the questionnaire. However, she was not the only one to criticize the task. Several other participants (among them people who correctly answered 68.58) joined her in claiming that it was unfair to ask a student such a "trick" question. For example, "I think that it is unfair of the teacher to give an answer which, first of all, is not completely correct." Or, "this is a question that causes students to fail. It is an unfair question."

The raging debate about the "fairness" of the question continued for some time during the first course meeting until the meeting ended. We expected this to be the end of it, but at the beginning of the second meeting the participants initiated a renewal of the debate. Some of them described to the class how they were bothered by the task and therefore chose to present it to other people (students and teachers) during the week between the two course meetings. They then used the responses they received as support for their claim that the task was unfair.

The discussion of the fairness of the task led the group to examine the issue of the objectives of asking students questions. The people who enthusiastically argued against the fairness of the task seemed to implicitly assume that the only aim of asking students questions is to evaluate their performance of what they have learned

in class. For example, one teacher claimed: "It does not mean that he doesn't know. He knows how to do what he was asked to. He knows how to locate the point. This was the question." According to this approach, teachers are allowed to ask questions only about the material they have taught:

> If you want him to locate the decimal point, and you taught him to do it in the way the child did, then you have to give him a complete answer so that he can locate the decimal point.

Another suggestion was to "prepare" the child to answer such questions by gradually directing him through several tasks in which the teacher can show him that zero may disappear. For example,

> If we want to give such a question to a child to make him think, then there is a way to lead him to that. [We should give him] similar questions, several such exercises, showing him that zero is missing. Give him some exercises with zero. But rank the questions a little bit.

Some participants felt uncomfortable with this narrow approach to student questioning. They also referred to student questioning as a means to evaluate students' performance. However, they claimed that the teacher should evaluate not only the mastery of techniques but also evaluate problem solving performance and the availability of different tools such as estimation:

> Actually, the child needs to use all the tools available to him. I don't think that one can separate: 'I am teaching only techniques or I am teaching estimation.' When I give something, some topic, I want him to use all the tools. If I try to give a question that focuses only on techniques I think that I fail in my objectives. A child needs to use all the tools that I teach him.

One of the participants expanded the aim of using student questioning beyond evaluation of student performance. She claimed that student questioning is also a means to help them learn:

> I think that he [the student] must use some control. I think that it doesn't come to the student naturally but we have to build it. I think we should give such questions through which we educate him not to be a robot.

Analysis of the teachers' reflections on the interviews they conducted with students in the forth setting indicates that for most of the teachers talking to students with the aim of understanding their ways of thinking was very different from the way they usually teach. The non-mediation meeting with, and discovery of, their students' ways of thinking, caused the teachers to reflect on their style of talking to students, and to re-examine their own actual classroom teaching. This motivated them to decide to make changes in their teaching regarding content (e.g., teach estimation) and teaching style (less teacher talk).

CONCLUDING REMARKS

This study unpacks the potential of using MCS-Cases in teacher education. The findings indicate that different settings highlight and lead to different outcomes regarding various aspects of pedagogical content knowledge. Some of the aspects were explicitly raised by the teachers themselves, while others were implicit in their answers, responses and reactions, and became explicit in our analysis.

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ACTION-BASED STRATEGIES IN ADVANCED ALGEBRAIC PROBLEM SOLVING

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This paper analyzes the strategies of a group of undergraduate students to solve a set of problems involving divisibility. The focus is on action-based strategies, i.e. on strategies depending on physical manipulations which are performed with little semantical control. It is shown that problems requiring relational knowledge or impredicative reasoning may result difficult to a number of students even if only elementary concepts and methods are involved.

1. Introduction

The use of knowledge, including relational one and the semantical control on resolution procedures have proved crucial steps in advanced algebraic problem-solving. Unfortunately, instructional methods that may induce students to disregard meanings and to overestimate a little set of techniques (within well-fixed notation systems) as paradigms of doing mathematics are yet common in high school (and at university as well). These techniques are often applied (and assessed) with little care to context and conditions and it is usual to work within a little number of theoretical and to represent mathematical ideas in stereotyped ways. This may severely affect learning processes at university level. Let us see two examples.

Example 1

When dealing with problems like "For which m, $m \in \mathbb{Z}$, the equation $mx+3y=m$ has solutions in $\mathbb{Z}$?", some students write $y = \frac{m(1-x)}{3}$ and conclude that m must be a multiple of 3. This happens even after they have been taught the theory and algorithms appropriate to handle and solve Diophantine equations and may depend upon the custom of solving linear equations only within a field.

Example 2

Consider a problem like "Given the set $A:=\{0,1,2,3\}$, find out all the functions f: $A \to A$ such that $f(0)=1$ and $f(1)=0$". Some students answer "There is no function like this", meaning that no linear or quadratic polynomial function f: $A \to A$ satisfies the conditions. This happens after they have been taught the definition of function and have seen examples in different representation systems.

In the first example the students apply a technique utterly disregarding the conditions. Actually, the strategy they perform is little more than a physical manipulation, with very little semantical control. It is something very close to an action, in the sense of Dubinsky's action-process-object (APO) framework (see Dubinsky, 1991; Breidenbach et al., 1992; Dubinsky et al., 1994; Zazkis and Campbell, 1996). Through this paper I use the word 'action' to stress the lack or inadequacy of semantical control.

In the second example the students bound themselves to a narrow class of functions, which can be represented in a standard way, and show a poor understanding of the idea of function. In both cases the strength of behaviors and patterns acquired in their high-school experience overpowers any alternative teaching and are substantial obstacles to future learning.

Students' achievement of semantical control on their procedures is a central goal for many mathematics educators. At this regard, Ferrari (1996) has classified the performances of a class of freshman computer science students according to the ability to use algebraic knowledge and the semantical control on procedures. This classification has proved well correlated with students' general academic results.

In this paper I want to improve the analysis of students' strategies when solving a set of divisibility problems. In particular, I want to focus on action-based strategies and to see how the contents of problems can affect students attitudes and performances, in order to become aware that there are activities seemingly advanced but which encourage the acquisition of mathematical contents with no or little understanding.

Divisibility have been chosen because it involves some simple ideas that are usually taught since primary school and offers a wide range of problems that may allow students to use a variety of different strategies. Moreover, divisibility problems even simple may require the use of algebraic knowledge represented in relational form and involve a conceptual frame which is different from the ones generally overstressed in high school practices.

2. A set of problems

The following set of problems has been administered at the end of November, 1996, to a group of 39 freshman computer science students after approximately 30 hours of introductory algebra devoted to language of sets (about 4 hours), the idea of function (about 8 hours), introductory combinatorics (about 4 hours) and arithmetic (about 14 hours), with particular regard to divisibility, factorization, greatest common divisor and congruence. There has been no emphasis on divisibility criteria. The teaching was oriented to problem solving and was given by the author, with the help of some senior students (tutors). Tutoring was optional.

Students were allowed 1 hour to solve all the problems and to freely use books, papers and pocket calculator. The problems were the following.

1. Let $M = 3^4 \cdot 5^3 \cdot 7^6 \cdot 19^8$. Answer to the following questions and explain your answer.
   a. Is M divisible by 63?
   b. Is M divisible by 18?
   c. Is M+5 divisible by 10?
   d. Is there an integer x such that $M \leq x \leq M+10$ and 8 is a divisor of x?
2. Consider the function f, $f: \mathbb{N}\times\mathbb{N}\rightarrow\mathbb{N}$, defined by $f(m,n)=5^m 25^n$. Is f injective?
3. Find out the values of m and n ($m, n \in \mathbb{Z}$) satisfying the following relations:

a. $\gcd\{3m,5\} \mid m$ (the greatest common divisor of 3m and 5 divides m);
b. $\gcd\{10,2n\} \mid n$ (the greatest common divisor of 10 and 2n divides n).

3. A priori analysis

Questions 1.a, 1.b, 1.d are similar to some proposed by Zazkis & Campbell [1996].

Students dealing with questions 1.a, 1.b are expected to relate the notion of divisibility to factorization. They could be more or less aware of the meanings involved. In particular, they could have learnt some rule relating factorization to divisibility with little semantical control. In other words, these questions could be answered by the simple action of factoring 63 or 18 and materially searching for the corresponding factors in the given decomposition of M.

1.c cannot be solved by the simple inspection of the representation of M. The occurrence of the prime factors of 10 within the factorization of M+5 cannot be empirically checked but only inferred. Students could observe that M and 5 are both divisible by 5 and thus M+5 must be as well, that M+5 is even (as M and 5 are odd) and finally that an even number divisibile by 5 is divisible by 10.

Question 1.d involves the recognition of the existence of an object without seeing or constructing it. Students are asked to do something of the kind of a non-constructive proof of existence. In this case the construction is possible, but it is much simpler just proving the existence of such an x than actually computing it. The mathematical contents involved in this question are not much beyond the usual primary school curricula.

Problem 2 is similar to a problem already discussed by Ferrari (1996, p.346). It can be solved by procedures requiring little mathematical control but demands for some (at least procedural) understanding of the idea of function. It requires more mathematical knowledge than all the questions of problem 1, even if it could result easier (from a problem solving perspective) than questions 1.c, 1.d.

Problem 3 is an application of the definition of gcd. Some problems of the same kind have been previously proposed to all the group. The request for impredicative reasoning and the double occurrence of each parameter within the formulas could be an obstacle for action-based strategies. Nevertheless it could be solved by students even with a poor mathematical background who can accurately apply the definitions they have learnt.

4. Outcomes

Problem 1

For the sake of brevity, I introduce a table to summarize the results related to problem 1 (questions a, b, c, d). Each row represents a possible combination of answers (columns 1-4) and the number of students adopting such combination (column 5). 1 means correct answer to the corresponding question, 0 means wrong or missing.

1.a	1.b	1.c	1.d	n°of students
1	1	1	1	5
1	1	1	0	9
1	1	0	1	2
1	1	0	0	14
0	0	0	0	9

Questions 1.a, 1.b

30 students give a correct answer to both 1.a and 1.b. All the correct answers to 1.a explicitly use properties of factorization and someone actually find (by cancellation) that $\frac{M}{63} = \frac{3^4 \cdot 5^3 \cdot 7^6 \cdot 19^8}{3^2 \cdot 7} = 3^2 \cdot 5^3 \cdot 7^5 \cdot 19^8$. All the correct answers to 1.b point out that an even number cannot divide an odd one, or that there was no '2' within the factorization of M.

The following are some examples of wrong answers to 1.a and 1.b.

- *"M is divisible by 63 because they have some common factor".*
- *"63 does not divide M because the exponents of the prime factors of 63 are not multiple of the exponents of (the prime factors of) M".*
- *"M is not divisible by 63 because gcd(M) ≠ gcd(63)".* (Most likely, by gcd(M) he means "the largest prime factor of M").
- *"M is divisible only by its prime factors and their powers".*
- *"M is divisible only by the product of its factors".*

1 student tried to perform euclidean division in order to check divisibility and 2 others do not answer to any of the questions 1.a and 1.b.

Question 1.c

14 students give a correct answer to 1.c. All of them give a correct answer to 1.a and 1.b as well. The strategies (correctly) used are the following.

◊ 3 students remark that M+5 must be even (as both M and 5 are odd) and multiple of 5 (as both M and 5 are multiples of 5).

◊ 6 students provide a strategy similar to the previous one but, in addition, explicitly write down: $M+5 = 5 \cdot (3^4 \cdot 5^2 \cdot 7^6 \cdot 19^8+1)$, remarking that the number within the brackets (which is the successor of an odd number) must be even.

◊ 3 students remark that $M+5=10 \cdot \left(\frac{3^4 \cdot 5^2 \cdot 7^6 \cdot 19^8}{2} + \frac{1}{2}\right)$ and show that the sum of the fractions within the brackets is an integer.

◊ 2 students remark that the last decimal digit of M is a '5', and then M+5 must end with a '0', and so it is divisible by 10.

The answers to 1.c classified wrong have been 25, including missing answers; it is worthwhile to distinguish between students giving correct answers to 1.a, 1.b and the others.

Among the students providing correct answers to 1.a, 1.b:

C1 5 claim that M+5 is not divisible by 10 because there is no factor '2' within M+5;

C2 8 try to find out some factor '2' but perform some computations or manipulations uncorrectly;

C3 1 writes 'no' without any explanation;

C4 2 give no answer.

Among the students providing answers to 1.a, 1.b classified wrong:

C*1 2 claim that M+5 is not divisible by 10 because there is no factor '2' within M+5;

C*3 5 write 'no' without any explanation;

C*4 2 give no answer.

Question 1.d

Only 7 students provide a correct answer to 1.d. 5 of them correctly answer to all of the questions of problem 1, and all explain their answer by means of additive properties of of ℕ (*"at least one number of the form 8+8+... must lie between M and M+10"*). The remaining 2 students correctly answer to 1.a and 1.b but not to 1.c because they perform some computation uncorrectly. 21 students do not answer to question 1.d. Only 11 students provide wrong answers to this question. The wrong answers are the following.

D1 4 students provide answers like *"Of course it exists! Each of the numbers M, M+1, M+2, ..., M+10 fulfils the condition. But these numbers are not necessarily divisible by 8."* Most likely they interpret the question as two different questions to be fulfilled separately.

D2 1 student writes: *"M+x must be a multiple of 8. Thus it must be M+x = 8·M. But this equation has no solution, and there is no x like this."*

D3 1 student interprets '8 is a divisor of x' as '8 is divisible by x'.

D4 1 student interprets '8 is a divisor of x' as '8=x'.

D5 2 students claim that *"There is no factor 8 within any of these numbers."* All of them give correct answers to 1.a, 1.b, 1.c.

D6 2 students try to compute the number required. They take away M from both sides, search for an integer x, $0 \le x \le 10$, such that 8 is a divisor of x and find that it must be 8. Since M+8 is not divisible by 8, they conclude that the problem has no solution."

Problem 2

17 students give a correct answer to problem 2, and 5 students do not answer at all. All the correct answers consist in the presentation of one or more counterexamples. Some wrong answers are listed below.

E1 10 students base their answers on wrong interpretations of the ideas of function and of injective function; among them there are all the 4 students with a correct answer to 1.c and a wrong one to 2.

E2 5 students claim that $5^m 25^n = 5^x 25^y$ because of the uniqueness of decomposition into factors; all of them have given a wrong answer to 1.c.

E3 2 students claim that from $5^m 25^n = 5^x 25^y$ or even m+2n = x+2y follows m=x and n=y.

Problem 3

18 students give a correct answer to 3a, 20 to 3b and 13 to both questions. In other words 12 students correctly answer just to one of the questions, and 14 to none. It is remarkable that all students give some answer. The correct answers are generally based to the computation of gcd(3m,5) and gcd(10,2n) as functions of m or n and a careful analysis by cases. A large number of different wrong answers has been found, as listed below.

F1 2 students introduce the corresponding linear congruences (i.e. 3mx+5y=m and 10x+2ny=n) and try to solve them as they were within a field.

F2 2 students replace 'a divides b' with 'a≤b and b=ak for some integer k' and then use only the first clause ('a≤b').

F3 5 students claim that m must be a multiple of 5 (question a) or n must be a multiple of 10 (question b).

F4 6 students claim that m must not be a multiple of 5 (question a) or n must not be a multiple of 10 (question b).

F5 5 students compute gcd(3m,5) or gcd(10,2n) uncorrectly (in particular 2 disregard the occurrences of m, n).

F6 3 students consider only m = 1, 3, 5 for (a) and n = 1, 2, 5, 10 for (b).

F7 1 student, answering to question a, claims that *"If 5 divides m, then the gcd is 5 and there is no solution, because 5 is not a factor of m"*.

F8 3 students find only a finite set of values of m, n satisfying the relation (but not all of them).

5. Discussion

The sequence of questions a, b, c, d of problem 1 seem to provide a reasonable classification of students' skills, since students failing to answer to a question generally (with 2 exceptions) do not solve the subsequent ones. Only the weakest students give a wrong answer to 1.a and 1.b. Their mathematical competence is very poor and they seem unable to use words to express even elementary mathematical ideas an relationships. The answers to 1.c are remarkable. Students' need for actually recognizing the factors 2 and 5 within the representation of M+5 seem to affect their strategies very much. Students providing a correct answer try to make explicit these factor even when unnecessary. among the others, 7 explicitly claim that there is no factor 2 within M+5, and even those answering *"no"* with no explanation most likely have been dealing with the same obstacle. The answer 'no'

with no explanation is more popular among students providing a wrong answer to 1.a and 1.b, whereas a number of the students giving a correct answer to 1.a, 1.b (but not to 1.c) try to find out some factor '2' but cannot carry out the computations correctly. It is reasonable to think that their correct answers to 1.a, 1.b were based on actions, in turn based on some rule they have learnt but they cannot relate to other number properties (e.g., properties relating addition with divisibility).

Question 1.d has troubled students more than one could expect. The lack of an algorithm they have already learnt has induced a good number of them to provide no answer at all. Some of the errors are concerned with language (D1-D4). The position of 'x' within the second clause of the condition given may have deceived the students answering *"Of course it exists! ..."* (D1). Maybe the question would have been easier had I written "... and x is a multiple of 8", for students' interpretation of the condition is focused on x, and they expect to find it as the subject of the sentence. Answers like D2 and D4 might be related to the difficulty of translating relationships represented in words into algebraic formulas. Errors like these have been studied in detail by Bloedy-Vinner (1996). Other errors (D5, D6) are caused by action-based interpretations of divisibility, which regard divisibility as the material recognition of the factor 8 within the given presentation of the number x and thus the effective construction of a number like that. I do not yet know the final results, but I guess that students giving a correct answer to 1.d will prove the top group of the class.

A good number of wrong answers to 2 are caused by an inadequate understanding of the idea of function (E1). A detailed analysis of the errors of this kind, though interesting, is beyond the scope of this paper. The wrong answer E2 is remarkable because it implies the material recognition of the factor without semantical control (they apply uniqueness of decomposition as if the factors were prime); the fact that all of the students choosing E2 give a wrong answer to 1.c confirms this interpretation.

All of the students provide some sort of answer to problem 3. There is a wide range of wrong answers to this problem. Beyond 'operational' answers like F1 or answers seemingly depending on the search for a material occurrence of 5 within m (maybe F7), there is a number of errors related to algebraic language. Someone finds difficult to represent divisibility (F2); a discussion on this aspect can be found in Ferrari (1996, p.349). Others cannot handle the algebraic expressions involved. Answers F3, F4 may point at some trouble with impredicative relations with a double occurrence of the parameter. Maybe some students cannot coordinate the evaluation of 2 parameters at the same time. Someone focuses on the first expression (F4): if m is not a multiple of 5, the gcd is 1 and the relation is satisfied with no need for taking into account the expression on the right. In the same way others focus on the second expression (F3): if m is a multiple of 5 the relation is satisfied anyway. In both cases they find sufficient conditions that are by no means necessary. Even answers F5, F6 could be related to a poor command on algebraic language, mainly caused by impredicativity.

In general, a number of wrong answers to problem 1 seem to imply a sort of 'negation by failure': if the technique they have been taught and that usually gives an answer 'yes' cannot be applied, or does not work, then the answer is 'no'. This happens, for example, with almost all the negative answers to 1.c (C1, C*1, most likely C3 and C*3 and maybe C2) and with D1, D5, D6. Another example of a behavior like this is given in Ferrari (1996, p.348), with the distinction between subjective and objective interpretations of uniqueness. This seem to point out that students' knowledge is quite unstable: if the methods they have been taught or they are used to use do not work, they immediately are at a loss and cannot analyze the problem situation any more. This interpretation is strengthened by the fact that all students give some answer to problem 3, and 34 out of 39 give some answer to problem 2, whereas only 18 give an answer to 1.d. Problems 2 and 3 are typical algebra problems the student can recognize as the tasks they are required to deal with, and are related to contents and methods they have been taught, even if not elementary, or easy to understand. Question 1.d involves ideas that are quite elementary but it is not a typical school problem and is not related to methods they have been taught. There is a lot of problems which involve primary school concepts and require relatively simple methods but result difficult even to graduates in mathematics. Conversely, there are problems that involve seemingly advanced concepts and methods but require very little as regards modelization and problem solving, as they allow students to perform action-based strategies which do not imply any semantical control at all.

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Tacit Mechanisms of Combinatorial Intuitions

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Abstract

The problem which inspired the present research referred to the relationships between schemata and intuitions. Schemata are defined following the Piagetian line of thought, as programs of processing information and controlling adaptive reactions. Intuitions are defined as self-evident, global, immediate cognitions.

Our main hypothesis was that intuitions are generally based on certain structural schemata. In the present research this hypothesis was checked with regard to intuitive solutions of combinatorial problems.

Intuitions and Schemata

The main theoretical problem which inspired the present research refers to the relationships between intuitions and schemata.

An intuition is a cognition characterized by self-evidence, immediacy, globality, coerciveness and extrapolativeness (see Fischbein, 1987). In the present text, the term intuition has been used especially as a global, direct, relatively self-evident evaluation in contrast to a solution based on an explicit computation.

The concept of schema has, in the psychological literature, various meanings. In the present text it will be used as a program (roughly analogous to a computer program) aimed to interpret a certain amount of information and to prepare and control the corresponding reaction.

(For a larger analysis of the concept of schema and for more references, see the book of Howard, 1987; see also: Anderson, 1977, Attneave, 1957, Fischbein, 1978, Flavell, 1963, Hastie, 1981, Piaget, 1976, Rumelhart, 1980.) For instance, suppose one is asked to evaluate the number of groups which can be produced by changing the order of, let's say, five different objects (for instance, letters: a, b, c, d, e). One may try to evaluate intuitively (a global guess) or one may try to use a certain procedure, a *schema* for calculating (or producing) the number of groups.

The question addressed by us has been the following: An apparently spontaneous reaction (an intuitive cognition) is genuinely spontaneous or it is influenced, shaped by a kind of schema, expressed in a tacit computation? If such a tacit elaboration exists, what is its relationship with the correct, mathematical procedure (schema)?

It is necessary, at this point, to introduce a distinction which is, in our opinion, epistemologically important. There are general schemata which have a

basic, structuring role; and there are more content-bound, particular schemata with a more restricted impact.

For the first category, let us mention the schemata of classification, seriation, bijection, the concepts of measure and unit, number, the concepts of deterministic relationships and randomness, the concepts of proportion, probability, correlation and combinatorial operations, the concepts of formal vs. empirical proof, etc.

With regard to the second category - every concept with a specific meaning (triangle, chair, pencil, etc.), every reflex, every mathematical or scientific formula etc., represent preconditions for identifying an object or performing a certain operation. They are specific schemata.

As a matter of fact, one may consider, tentatively, that schemata are organized in hierarchies from very general ones to specific, content-bound ones.

The basic hypothesis of the present research was that, indeed, intuitions are, generally based on tacit, sequential structures.

Combinatorial Intuitions

In the research exposed in the present paper, our attention is focused on combinatorial problems.

Our interest in devising the present research was both theoretical and didactical. The theoretical interest is obvious. Both schemata and intuitive cognitions are of high theoretical importance. In both categories, one deals, generally, with stable, well structured, well integrated, highly influential mental-behavioral structures. What are the relationships between them?

On the other hand, combinatorics is, per se, an important chapter in mathematics and its relevance to various branches of mathematics is well known. Moreover, combinatorial capabilities constitute, according to Piaget and Inhelder, one of the basic schemata, reaching maturity during the formal operational stage. The propositional nature of formal reasoning is based on the combinatorial capability of the adolescent (see Inhelder & Piaget, 1958). (For an updated review of the literature concerning combinatorics, see the excellent work of Batanero, Godino & Navarro-Pelayo [1994]. It analyses the area of combinatorics from the mathematical, the psychological and the didactical points of view.) (See also: Deguire, 1991; English, 1994; Fischbein & Gazit, 1988; Fischbein et al., 1970; Inhelder & Piaget, 1958.)

Up to now, combinatorial intuitions were not analyzed specifically. The existing investigations refer either to the evolution with age of the combinatorial capacities of the child (expressed in the capacity to produce various subsets of elements from a given set of n elements) according to a certain definition; or to the techniques of teaching and learning combinatorics.

Methodology

Subjects:

a) Three groups of pupils enrolled in the following classes: grade 7 (N = 63), grade 9 (N = 62), and grade 11 (N = 62). b) Students enrolled in teachers colleges (N = 41). c) Adults with various mathematical backgrounds (N = 25). These adults were people with low mathematical education, enrolled in a course in mathematical literacy. None of the subjects had formerly attended any course in combinatorics.

Instruments:

A questionnaire was administered containing various combinatorial problems: Permutations, arrangements with and without replacements, combinations.

The subjects were asked to estimate, globally, the number of possible groups of elements which could be produced with a given set of elements according to a certain procedure.

After the subjects answered in writing, interviews were organized by which the same subjects were asked to explain their solutions. Twenty-five subjects were interviewed.

Procedure:

The session started with a general explanation with regard to combinatorial operations. After that introduction, the questionnaire was administered and the subjects were asked to estimate the numbers corresponding to the respective combinatorial problem. Orally, one has insisted that the subjects have only to estimate the answers (not to compute). The questionnaire was administered in usual classroom conditions. The subjects were allowed about 45 minutes to complete the questionnaire.

The interviews asking the subjects to justify their evaluations were organized some days after the questionnaire had been administered.

The following categories of problems were presented:

1) Permutations of 3, 4, 5 elements ($P_n = n!$)

2) Arrangements with replacement of 3,4,5 elements taken two by two ($A_k^n = n^k$)

3) Arrangements without replacement of 3, 4, 5 elements taken two by two.

$$A_k^n = n(n-1)\cdot(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

4) Combinations of 3, 4, 5 elements taken two by two

$$C_k^n = \frac{n(n-1)(n-2)\dots(n-k+1)}{1,2,3,\dots k} = \frac{n!}{k!(n-k)!}$$

Results

The averages of estimations for each type of problem are presented in Tables 1, 2, 3 and 4.

With regard to permutations, one may observe the following - for 3 elements the estimations are close to the correct answer. For 4 and 5 elements the global tendency is to underestimate the number of permutations.

With regard to estimations of the number of arrangements with replacement, the results remain close to the correct answers with a slight tendency to overestimate for 5 elements in older subjects.

With regard to arrangements without replacement, there is a general tendency to overestimate (about 20%) for every group of elements and of all age levels.

Table 1: Permutations. Averages of Estimations and Standard Deviations

	3 Elements		4 Elements		5 Elements	
	Mean	SD	Mean	SD	Mean	SD
Grade 7	5.82	0.77	14.79	5.80	35.47	31.01
Grade 9	6.92	2.65	16.80	11.80	44.60	29.50
Grade 11	8.09	6.02	26.41	15.40	37.10	35.60
College students	7.55	4.76	18.40	12.00	63.60	27.00
Adults	5.96	0.20	19.60	7.16	64.70	28.90
General Mean	6.97		19.27		48.45	
Correct Solutions	6		24		120	

Table 2: Arrangements with Replacement (k elements taken by 2): Means and Standard Deviations

	3 Elements		4 Elements		5 Elements	
	Mean	SD	Mean	SD	Mean	SD
Grade 7	6.55	1.77	12.80	2.80	20.00	4.80
Grade 9	8.90	4.54	16.50	7.09	30.40	19.80
Grade 11	12.04	7.30	18.60	12.50	29.40	20.50
College students	7.82	2.13	16.92	8.90	40.50	52.30
Adults	8.80	3.25	14.60	5.30	25.30	11.13
General Mean	8.95		16.05		28.79	
Correct Solutions	9		16		25	

Table 3: Arrangements without Replacement. Means and Standard Deviations

	3 Elements		4 Elements		5 Elements	
	Mean	SD	Mean	SD	Mean	SD
Grade 7	7.28	3.04	14.70	3.88	22.30	3.18
Grade 9	6.00	2.30	12.90	3.70	22.30	6.15
Grade 11	9.53	10.20	16.90	18.90	29.80	32.30
College students	7.97	2.80	16.48	7.17	30.36	32.60
Adults	5.84	2.81	10.28	4.30	18.72	6.96
General Mean	7.48		14.67		25.10	
Correct Solutions	6		12		20	

Table 4: Combinations. Means and Standard Deviations

	3 Elements		4 Elements		5 Elements	
	Mean	SD	Mean	SD	Mean	SD
Grade 7	3.46	11.24	6.22	3.09	8.13	3.56
Grade 9	3.69	1.70	6.92	3.98	9.71	5.75
Grade 11	8.01	7.30	15.38	12.80	28.30	30.01
College students	4.31	2.30	10.87	10.60	16.70	25.96
Adults	3.24	1.09	5.68	1.90	8.28	2.54
General Mean	4.74		9.33		14.91	
Correct Solutions	3		6		10	

- Finally, with regard to combinations, there is a still stronger overestimation (of about 50%) when referring to the global means though, in fact, the overestimations increase with age.

From these raw data we learn very little about the mechanisms of the respective intuitions. But let us turn to the interviews. The basic remark is that most of the subjects try to justify their estimations (that is a posteriori) by indicating one of the possible binary operations with the given numbers.

For permutations of 3 elements one has got, usually, the answer: 3x2 (which, by chance, yields the correct answer). For 4 elements one has got the following variants: 4x4, 4^2, 3x4 and 2^4. For 5 elements, similar multiplications have been obtained: that is 5x5, 2^5, 4.5, 5^2.

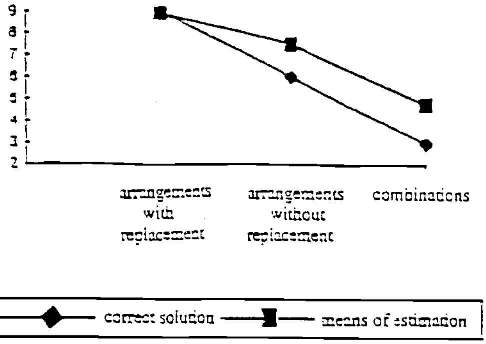
But the same binary multiplications -- in various proportions -- were obtained when asking the subjects to estimate the number of selections for arrangements with and without replications and for combinations! It is important to emphasize that the explanations given by the subjects to their guesses were usually, in accordance with their "spontaneous" reactions.

Nevertheless the means of the estimations are different for the different combinatorial problems. Moreover, when analyzing the averages of the estimations, one finds that the relationships of their magnitudes follow approximately the relationships between the correct solutions.

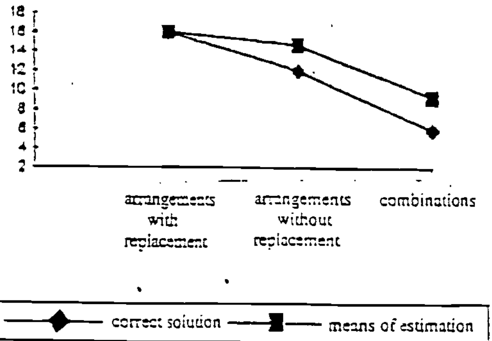
That is, considering the estimations (e): One has $eP_n > eA_2^n$ with replacement $> eA_2^n$ without replacement $> eC_2^n$ (see Figure 1).

In other terms, the intuitive guesses are, on one hand, based on some binary multiplicative operations (which by themselves, are not related to the correct formulae) and on the other hand, their magnitudes are influenced by what should be the correct answers. Let us try to summarize what has been said so far and may cast a light on the relationships between schemata and intuitions.

- The intuitive guesses are not wild guesses. In combinatorial problems, the global estimations express, generally, multiplicative operations which corresponds to



4 Elements



5 Elements

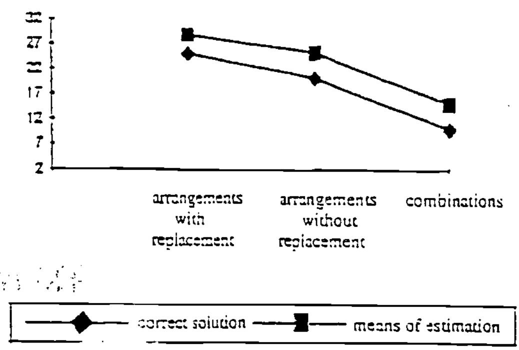


 Figure 1

the fact that combinatorial reasoning is really of a multiplicative type of reasoning.

- The multiplications invoked in subsequent interviews are always binary operations though in most cases, the correct answers consist in more complex operations (for instance, $P_n=1, 2, 3, ...n$). This finding leads us to the conclusion that intuitions: a) originate in some schemata (which in the present case are computational), and b) these schemata undergo a process of compression, yielding the appearance of a global, immediate guess (see Thurston, 1990). This process of compression seems to be essential for the transition from schemata to intuitions.

  A similar finding has been described by Tversky and Kahneman. Two groups of high school students were given 5 seconds to estimate the result of a multiplicative operation. The first group had to estimate the product 8 x 7 x 6 x 5 x 4 x3 x 2 x 1 and the second group had to estimate the product 1x 2 x 3 x 4 x 5 x 6 x 7 x 8. The correct answer is 40320, but for the first group the median estimate was 2250 while for the second group it was 512. The order of magnitude of the respective estimates has been determined by magnitudes of the first numbers of the products presented (the anchoring bias) (Tversky and Kahneman, 1982, p. 15). As a matter of fact we deal here with a similar type of compression process as described above with regard to combinatorial estimations.

- The computational schemata do not seem to be the only source of the respective intuition. A second adaptive-corrective process seems to take place originating in a further tacit consideration of the conditions of the problem involved. For instance, when passing from arrangements without replacement to combinations, a division intervenes:

$$\frac{n(n-1)(n-2)...(n-(k+1))}{k!}$$

The subject seems to interiorize, tacitly, the meaning of the instruction: "... in combinations, the selections do not depend on the order of the elements", (and, consequently, there are fewer possible combinations than arrangements without replacement).

It seems then that the impact of structural and specific schemata in shaping intuitions (which are sometimes not adequate) may be favorably counterbalanced by further information originating in the objective given conditions. We face here a tacit, complex process, the details of which we do not know yet.

Didactical Recommendations

We suggest to use the following steps in teaching the various combinatorial solving procedures:

- To start by asking the students to estimate without explicit computation the number of possible selections.
- To resort to explicit systematic procedures (for instance, tree diagrams) and to set up the corresponding formulae;
- To contrast the computational results with the initial estimation and to try to explain the process by which the initial guess has been obtained (under-and-over estimations)

We assume that this way, one may increase both the interest and the understanding of the students for combinatorial problems.

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SHADOWS ON PROOF

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ABSTRACT. In this paper we refer to an experiment in which students of the age range 14-17 have to proof a statement on natural numbers, writing all their thoughts while they are working on this task. We perform a kind of 'genetic decomposition' of the statement and single out some parameters, on which we base the analysis of the students' protocols. The main schemes found in students' proofs are the authoritarian, the empirical, the ritual and the symbolic. We study the relations of these proof schemes with the context chosen by the students to prove. Some students' behaviours allow to single out elements suggesting the influence of the algebraic or arithmetic contexts on proving this type of statement: we call it *algebraic or arithmetic shadow effect*.

INTRODUCTION

One of the issues that we try to develop in our research on proof is the idea that proof is not context-free, that is to say that the context has a strong conditioning role in the students' performances on proving. For example, in a recent study, (Furinghetti & Paola, to appear), we have presented to students questions having the same formal structure, but set in different contexts and treated with different languages (mathematical context with the mathematical language, situations related to usual life with the natural language). The result of this experiment has been that the students' performances differ according to the context, since the meaning of the field in which the tasks are set acts as an element of diversion in proving; we call this phenomenon «semantic shadow effect». In other occasions, see for example (Furinghetti & Paola, 1991), we have worked at the interior of mathematics, and we have found different performances according that the context was algebra or geometry, even if the statements presented to students had the same formal structure. In this paper we have considered the students' performances in proving a statement concerning natural numbers. The choice of this context was motivated by the fact that the students work in it from their early days in school and thus we thought that this context would have resulted particularly 'friendly' for them.

METHODOLOGY

As a first step we have carried out an in-service teacher training course on proof consisting of a part on theoretical topics (including elements of logic) and a part on educational issues. Participants teachers were asked to answer a questionnaire on their conception of proof. Their answers would have been the starting point for a discussion on the re-shaping of their style on teaching this topic. After this work we have invited the teachers to collaborate to our research putting their classes to our disposal. Three teachers have agreed; the fact that they were aware of the educational problems underlying proof and that they were motivated by the previous activities makes us confident that they would have observed the instructions we gave. We were lucky since the three teachers teach in four classes which differs for the ages of students (Teacher A: one class of students aged 14, Teacher B: one class of students aged 15, Teacher C: one class of students aged 16 and one 17) and for the types of curricula (with more or with less emphasis on mathematics).

The study consisted in analysing how the students solve the following exercise: «*Prove that the product of any three consecutive natural numbers is divisible by 6*». Students were asked to write all their attempts and thoughts. Our analysis has been performed on their protocols.

The instructions to the teachers were:
- to report the time emploied
- to not help or influence the students
- to push them to write all the things they were thinking in solving the exercise
- to make students aware of which project they were part and to encourage them to an active collaboration with the researchers; this awareness of students was promoted also to prevent them from being lazy or cheating the teacher by cribbing from a school-mate, since this would have polluted the experiment.

We succeeded quite completely in all these points. We have also asked to the teachers to make a prevision on the students performances. All the teachers agreed that the exercise was within the capacity of their students and no one considered that it would be difficult to deal with the technical issues of the exercise such as the interpretation of the terms involved in the statement (natural, divisible, consecutive). In the following we give some brief information.

Teacher A. His students (class A) are aged 14. The school where he teaches has a strong mathematical curriculum; in algebra, among other topics, he develops modular arithmetics with the remainders classes. He feels that his good students will be able to prove the statement through the remainders classes. He does not takes into consideration the exploration through less formal ways.

Teacher B. Her students (class B) are aged 15. The school where she teaches is oriented to give a good mastery of foreign languages; the mathematics program is in line with the recent curriculum changes in Italy, but mathematics is not an important subject. She thinks that her students will start with examples and afterwards will generalize. Many students will look for some formulas. The greatest difficulty will be to formalize the intuition in a logically correct sequence of statements. She feels that some students will make many examples with the aim of finding counterexamples which prove that the product of any three consecutive natural numbers is <u>not</u> divisible by 6.

Teacher C. Her students are aged 16 (class C I) and 17 (class C II). The school where she teaches is aimed at preparing the students in economical disciplines. The mathematics programs is rather innovative; the students learn also to program at the computer in the mathematics course. She thinks that few students will work on numerical examples; the majority will look for a formalization and will attempt to manipulate the expressions $(n - 1)n(n + 1)$ or $n(n + 1)(n + 2)$. Some students will argument in a quite descriptive way.

ANALYSIS OF THE RESULTS

To analyse the protocols we put us in a perspective similar to that of the 'genetic decomposition' presented in (Dubinsky, 1991), that is to say we analyse the exercise proposed to students in order to isolate its main conceptual or procedural components and the relations among them. As a result we have single out the following parameters which we shall use as a basis for studying the protocols.

- algebraic language (even if in a poor form) is used or is not used
- which kind of use of algebra is prevailing, in particular how the letters are used
- mastery of the concepts specific to the problem (divisibility, multiples)
- role of numerical examples
- algebraic or analgebraic thinking, with particular reference to the interpretation of algebraic expressions
- use of some kind of iconic language
- use of quantifiers
- proof schemes followed by students

We observe that some parameters mainly concerns algebra, others are more specific of the process of proof, even if we shall see that the distinction is not so clean. At the interior of algebra we distinguish between pure manipulative issues and issues linked to the mastery of critical concepts, such as variables and quantifiers.

According to this classification we have singled out general factors concerning all the classes A, B, C I and C II pointing out differences and analogies. Afterwards we have analysed more in details the behaviours of the class C II to have more precise elements.

The type of the present research does not imply a quantitative analysis of the numerical data; we only give some general figures which provide a first overview of the situation.

To make clear the figures in the table we note that:
- The sum of the numbers of the students who use algebra or not is in general greater that the number of the answering students since some student use both the languages adopting a sort of syncopated-like language.
- The data of the class C I have a meaning different from that of the others: the teacher has made a mistake in giving the text of the exercise, putting the number 2 instead of 3, so that the text has became «Prove that the product of any two consecutive natural numbers is divisible by 6», which is an impossible task. We shall see that also in this case interesting behaviours emerged.

Not in all the classes to solve the exercise was a compulsory task, nevertheless the third column of the table shows that students participate with good will; this fact is confirmed by the care employed in working on the exercise.

Classes	Students' age	Students answering	Algebraic language	Natural language	Iconic language	Right answers	Maths program
A	14	19 of 19	10	12	0	7	strong
B	15	11 of 16	11	1	0	0	weak
C I*	16	24 of 24	7	17	0	6	medium
C II	17	18 of 18	6	13	0	2	medium

One of the characteristics of this exercise is that it can be easily developed through the natural language by activating the frame of divisibility or of multiples. On the contrary, if the statement is translated into an algebraic expression, the attempts of manipulation may bring to a *cul de sac*. For this reason we have taken as a first parameter to consider the use or non use of the algebraic language. In the classes A and B the literal computation is a topic of the program; in particular in the class B the teacher focus on it (this is a quite common behaviour in the Italian tradition). The students of the classes C I and C II have left the study of algebra (literal computation and so on) since one or two years respectively. It is likely that the relatively high percentage of right answers in the class A is due to the fact that here the classes of remainders are part of the program developed: the protocols show that students understand the text and are able to activate a frame suitable to solve the problem.

In the class B all the students use the algebraic language. No one in the class B produces right answers. The protocols show that there is a lack of control in algebra which provokes a loss of control in proving. The students of class B are also victims of what in (Furinghetti & Paola, 1991) is called «irresistible impulse to calculate», that is to say they transform literal expressions and solve equations without any

precise purpose linked to the task. The proof scheme adopted, according to the classification in (Harel & Sowder, 1996a and b), is the ritual, since from the analysis of the protocols it emerges that students think that the justification has to be communicated via symbolic expressions or computations. In this case the ritual is combined with the symbolic proof scheme, since symbols are used «as if they possess a life of their own *without reference to their possible functional or quantitative relations to the situation*» (*ibidem*, p.61).

In the cases A and B it is not clear how much the students' behaviours are induced also by the authoritarian scheme. According to (*ibidem*, p.60) this scheme is present when « students are not concerned with the question of the burden of proof, and their main source of conviction is a statement given in a textbook, uttered by a teacher, or offered by a knowledgeable classmate». This scheme has been clearly evidenced in the class C I, thanks to the mistake made by the teacher in giving the text of the problem (see above). We are aware that the form of the text of the exercise «*Prove that* ...» was more commanding than the form «*Is the product of two consecutive natural numbers divisible by 6?*», thus it strongly pushes students towards the acceptation of the statement as surely true. Nevertheless the weight of the authoritarian scheme in conditioning the students behaviour is evident since among the 13 students who find counterexamples only 6 recognize that the statement proposed by the teacher is not true, while 7 try to forget it. In these last students the authoritarian proof scheme prevails on evidence. A confirmation of the fact that the authoritarian scheme conditions the students' performances is provided by Sara. She produces 6 examples which satisfies the statement and writes «The product of two numbers *must* [emphasis is our] give a multiple of 6». In this case the presence of the authoritarian scheme is unaware, in other cases is aware. For example, Alessandra writes: «- Natural numbers are the positive numbers. - Consecutive means one after the other. - The product is the result of a multiplication. Then I must prove that the result of the multiplication of two numbers, for example 3 and 4, is divisible by 6. ...I have understood the statement, but I'm not able to prove it». The analysis of the given statement performed by expressing the definition of the terms intervening in it with her own words is due to the doubt on the possibility to solve problem.

We can label this students' behaviour in class C I as the schizophrenia caused by the acceptation of the existence of two separate worlds - the world of the teacher and their own world - which have not necessarily points of contacts or at least analogies.

As we have observed in the case of the wrong text, also in the right text the form of the exercise («Prove ...» instead of «Is ...?») pushes students towards argumentation rather than conjecturing; this fact conditions the way they worked. Nevertheless we were expecting from the protocols to find some forms of iconic representation: in all the classes no one has used it. We are referring to the representation of the numerical rule, to the use of numbers patterns as in the primary school, to arrows for connecting formulas, to tables for connecting the various examples and so on ... Our findings are in accordance with some aspects emphasized in educational research

(Healy & Hoyles, 1996; Presmeg & Bergsten, 1995). This avoidance of the graphical language could be linked to the premature use of the algebraic language and of the formalization. This hypothesis comes from the results in (Dutto, 1996), where we find that students aged 11-13 use different iconic representations, when solving our problem and other similar.

ZOOM ON THE RESULTS OF THE CLASS C II

This class seems a good set for general considerations. While in the other classes the role of algebra (especially literal computation) could have been too much conditioning since it is the main part of the program, here students have left the study of algebra (literal computation and so on) since two years. Nevertheless they are working in topics (functions, programming with computer) which can add motivations to the algebra they have done before. For example, they have had the occasion to consider the concept of variable from different points of view. From the analysis of the protocols some facts emerge that we outline in the following.

• As observed in (Bloedy-Vinner, 1994; Furinghetti & Paola, 1994) one of the main problems in algebra concerns the use of quantifiers. For example, Erik writes the formula $6n = n(n + 1)(n + 2)$, ascribing the same status to the letter n on the left and on the right of the sign =. Here there is a lack of command in using the quantifiers: the student ignores that the right formulation would be «For any natural number n a natural number k exists such that $n(n + 1)(n + 2) = 6k$».

• We have found an empirical proof scheme, see (Harel & Sowder, 1996), based on the use of examples and confined to a level of pre-generalization. Myriam verifies the statement in a single case and writes «It works! ...But it could be by chance. Perhaps I have to try again with 5 or 6 numbers». In some cases the stage of pre-generalization is really naïve: for example, Maura checks the property expressed in the statement through 'little' numbers (3, 4, 5) and through 'big' numbers (1001, 1002, 1003), ascribing a property of generalization to these last ones.

The empirical proof scheme is present only when the natural language is used. The students who start writing the expression $n(n + 1)(n + 2)$ do not produce examples. This fact suggests that they do not interpret this expression as a function producing numerical values, as it was observed in (Bloedy-Vinner, 1995). This explains why we do not find the empirical proof scheme in protocols where the algebraic language is used.

• The use of letters is not necessarily evidence of an algebraic mode of thinking: in some cases we observe that letters are used as mere labels. For example, Silvia writes «$1 \cdot 2 \cdot 3 / 6 = 6/6 = 1$» and just after «$a(a + 1)(a + 2)/(a + 5)$» to indicate that the product of three consecutive natural numbers is divisible for 6: clearly here a is a label for the value 1. On the contrary Matteo (one of the two good solvers), after having proved the statement, writes the expression

$$\frac{\overset{3}{\cancel{9}}\cdot\overset{5}{\cancel{10}}\cdot 11}{1\cdot\cancel{2}\cdot\cancel{3}}$$

It seems that he uses the number in more general terms than the Silvia does with letters.

• There are only two students who solve rightly (Matteo and Mario). Matteo proves using the natural language, by activating the frame of divisibility evoked with the sentence «In the product of three consecutive natural numbers one is always divisible for 2, one for 3 and one for 1». He uses the formula $n(n + 1)(n + 2)$ only for synthesizing the thesis and afterwards he gives the example quoted above which seems to have a didactic function. Mario uses five examples for exploring the situation presented in the given statement and after it grasps that «Given three consecutive natural numbers one is even and one is divisible by three». At this point he gives his proof and after verifies the truth of the statement on an example which has a didactic purpose. The fact that in both the cases the examples are given after the proof with a didactic function suggests that the students consider examples as a privileged means for communicating.

• In spite of the teacher's expectation (see the chapter *Methodology*) not all the students showed a sufficient command on the terms appearing in the given statement. The *bête noire* was the word «divisible», some problems were given also by the zero (if it has to be considered belonging to natural numbers) and to the nature of natural numbers (are the negative integers natural numbers?). This makes our initial hypothesis on the property of natural numbers to be a friendly context too much optimistic.

The comparison of the expectations expressed by the three teachers and our findings would be an interesting starting point for discussing the didactic contract. For example, the students considered good by the teacher C used algebraic formalism and were not able to answer.

CONCLUSIONS

In all the classes the simple problem from which we started revealed itself a Pandora's vase of issues on students' behaviours both in proving and in doing algebra.

As for proof the most adopted is the empirical proof scheme. It is our opinion that this fact is strongly dependent on the arithmetic context. We have also observed the presence of ritual, symbolic and authoritarian proof schemes in students who used the algebraic language. The authoritarian proof scheme seems to be induced by the kind of didactic contract between the teacher and the students. The ritual and the symbolic seem more related to the specificity of the algebraic context that the students have chosen. We feel that algebra may hide the necessity to be convinced and students are strongly pushed towards the ritual scheme. The perception that they have of algebra

as a meaningless domain of symbols confirms their conviction that only a symbolic way is what the teachers is expecting from them.

The orientation towards formalism has not as a counterpart a good command in dealing with the tools of algebra. We have observed the poor use of quantifiers, but also of other basic tools such as variables (or parameters): the letters are often used only as stenographic signs.

Our exercise shows a *double shadow effect on proof*. From one hand we observe an *algebraic shadow effect* on the meaning that some students would have from arithmetic which prevents from using it in their attempts of proving. On the other hand there is also an *arithmetic shadow effect* which confine students to the empirical proof scheme and prevent them from generalizing.

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Perpendicular Lines - What Is The Problem?

Pre-Service Teachers' Lack of Knowledge on How to Cope With Students' Difficulties

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This paper deals with difficulties experienced by students in understanding the concept of perpendicular lines, and difficulties encountered by teachers when trying to explain it. We shall try to suggest a theoretical framework.
This is part of an ongoing research in which an attempt is made to identify students' and teachers' difficulties in geometry, to suggest an explanation based on cognitive theories and research, and subsequently to provide teachers with the relevant cognitive knowledge, recommended by the paradigm of Cognitively Guided Instruction (Carpenter & Fennema, 1992).

Background. Romberg & Carpenter (1986) pointed out two distinct disciplines of scientific inquiry: research on children's thinking and research on teaching. Carpenter & Fennema (1988, 1992) present a model for integrating cognitive and educational sciences (Cognitively Guided Instruction), in order to plan a more effective teaching program in mathematics. They suggest a program designed to help teachers understand children's thinking and use this knowledge to make instructional decisions.
Following these ideas, we are trying to detect and point out problems in geometry instruction; we then analyze them and attempt to provide an explanation for the difficulty which takes into account cognitive knowledge concerning perception, thinking and problem-solving strategies. In this paper we deal with the subject of perpendicular lines.

Methodology. Difficulties encountered by students and teachers were detected and identified by monitoring the work of pre-service teachers in their third year of studies. The monitoring process included reading their lesson outlines, videotaping their geometry lessons to 9th graders (slow learners), and interviewing the (pre-service) teacher after the lesson. The interview focused on an attempt to examine and comprehend difficulties which came up in the lesson.
In the next stage, an effort was made to understand the roots of these difficulties, analyzing them on the basis of cognitive theories and research. In light of the analysis, we examined the teacher's reaction to the difficulties and the degree of its effectiveness. This paper will specifically address the concept of perpendicular lines.

Analysis and Discussion. Let us first consider the following interview which was made by the researcher (R) after a lesson on the rhombus, with two teachers who taught two different groups of 9th graders:

Ora: As for perpendiculars, when I ask them a question such as what kind of angle is supposedly formed between two lines which run perpendicular to one another, I am not sure that they know this for real, but they say 90° straight out.
***R**: So what does this mean? That they know or that they don't?*
***Ora**: When I ask them what are the diagonals which are perpendicular to one another, what kind of angle is formed between two perpendicular lines, they tell me: aha 90°, but then they don't know what they have to check, i.e., they don't know where they need to check this angle that is indeed 90°.*
***R**: What do you mean, they don't know where? They don't know where the 90° is?*
***Ora**: They don't understand where the angle is drawn, yes, they don't understand where it is.*
***Keren**: They know that perpendicular lines represent 90° but do not understand what this means.*
***Ora**: They don't understand where it is.*
***R**: OK, what should be done to make them understand?*
***Keren**: I don't know, I really don't know.*

This Interview suggests two difficulties: first, students find it hard to identify the right angle between two perpendicular lines; second, the term 90° angle may not have a visual representation and may not be linked to the concept of right angle.

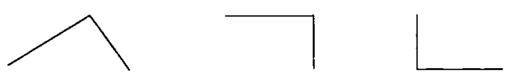
To analyze the difficulties, let us first examine the concept of perpendicularity and the various elements involved. The textbook or classroom definition is more or less as follows: "perpendicular lines" are two lines which intersect at a right angle.

This concept includes a number of elements, some of them are explicit and others are implicit:

1) Two lines (sometimes two segments or one line and one segment)
2) The lines intersect
3) A right angle is formed at the point of intersection
4) There are three more angles at the point of intersection
5) These three other angles are also right angles
6) A right angle is a 90° angle

A student facing this concept should already understand what lines and intersecting lines are (element 1 and 2).

The student should also identify a right angle presented in its basic form (i.e., two rays emanating from a common point). An expected difficulty at this stage is a limited prototype of a right angle, with the student being able to identify a right angle only if one of its rays is horizontal.



"a non- right angle" a right angle (?) a right angle

A difficulty of this kind requires some training with the concept of right angles.

Let us therefore assume that the student can identify some "basic" forms of right angle (element 3). At this point he or she still faces a number of difficulties:

A. **Identification**. In a configuration of perpendiculars, he or she must identify the right angle in its basic form. In other words, the student is required to identify a simple form within a complex figure (intersecting lines forming 4 angles).

This difficulty can be explained by the Gestalt principles pertaining to the organization of perception (Anderson, 1995, pp. 44-46). According to the "good continuity" principle there we tend to identify lines with better continuity than lines with sharp bends. For example, ask yourself what you see in the following drawing: (figure 1)

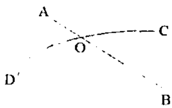


Figure 1.

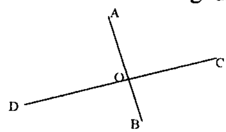


figure 2.

Most people's reply will be something like: a line AB and a curve CD. Rarely does one hear an answer such as "curve" AOC and another "curve" DOB. The "good continuity" principle explains this tendency of ours.

If we look at the other drawing (figure 2), we will find that the same principle leads us to see two segments, AB and CD, whereas the right angle can only be seen by perceiving, say, AOC—as a distinguished figure, which contradicts the "good continuity" principle.

Thus, the student who observes two intersecting lines does not necessarily perceive an angle between them, and in any case does not know where to look for a right angle. This difficulty came up in the interview.

Let us examine how visual information is processed in the student's mind, once it has been perceived and recorded in the cognitive system. After receiving visual information, it is organized unconsciously in units in such a way that each unit represents a part of the whole structure. Complex shapes are constituted of hierarchical units (Anderson, 1995, pp. 123-125). The figure + . for example, may be decomposed into four distinguished sub-figures, where each sub-figure is a segment: . In this case, identifying an angle pattern within the complex pattern is not trivial (the student has to compose sub-figures producing an angle, then compare it with the right-angle pattern in his mind e.g., he might identify an angle ┐ , while the pattern of the right angle in his mind is └). Of course, decomposing the above figure into ┤└ or ┘┌ , will make the right angle identification easier. But there is yet another barrier: what if the observed figure was: × ? It would most likely be decomposed into the following sub-figures: ><. Will the student be able to identify in this pattern the previous one of intersecting (perpendicular) lines, tilted by 45°? Devoting time to concrete examples of right angles and perpendicular lines, using paper cutouts and

puzzles, and asking the students to transform them in the plane improves flexibility in dealing with patterns by making them more familiar.

B. **Selection.** The student has to decide on which of the four angles in front of him to focus (elements 3, 5). The understanding that if one of them is a right angle then so are the rest, and that the selection is therefore arbitrary, cannot be taken for granted.

Such an understanding can be considered as "visual understanding" (level 1- **visualization** according to van Hiele's theory, e.g., Hoffer, 1983). As preparation, two perpendicular lines could be presented, where each of the four angles is emphasized in its turn. Changing the focus from one angle to another and then to the lines and vice versa may help perceiving the relations between the angles (and between them to the lines). Such an understanding can also be considered as a level 2- **analysis** (e.g., by taking apart and assembling the four right angles to form the perpendiculars with their four right angles; or by folding paper into four). This analysis involves mental activity and it is advisable to follow it up with a verbal description.

It may be assumed that when "quickly overviewing" the four angles before him, a student will check the one "closest" to the image he or she has in his mind of what a right angle is. "Closest" in what sense? Is the angle's size the criterion, or its orientation in the plane? How flexible are his mental transformations and do they enable a comparison between different positions in the plane?

Now, if we return to the problem of "selection" and the need to identify a certain angle as a right angle, one might assume that the angle selected will be the angle which bears the greatest resemblance ("resemblance" in one of the meanings mentioned) to the pattern in the student's mind.

C. **Inference.** The inference concerning the other three angles (in the configuration of the perpendicular lines) is not a trivial matter. There are two possibilities here. If the student knows that all four angles are right angles in case one of them is right angle, then the other three will be conceived as right angles. Otherwise, the student might fail to realize that all the other 3 angles are also right angles. (In such a case, if the teacher talks about an angle other than the one the student has chosen, it should come as no surprise if the student does not realize that it is a right angle). Moreover, concerning one angle, failing to recognize a right angle would cause a failure in recognizing perpendiculars though concerning another one could make the student succeed!

So far, we have dealt with right angles, without relating to its measure. The right angle is often defined in class as **a 90° angle**.

This time we have one more difficulty to boot: we do not know what concept image (Vinner, 1991) the student has of the right angle and whether it coincides with a 90° angle. The impression from the interview is that the students have heard the notions of 90° and right angle and are aware that they are synonymous. This does not mean that these terms necessarily have a meaning, and even if they do, it is not necessarily the same for both. We will use the term "conceptual behavior" (denoting the result of conceptual thinking processes, dealing with concepts, relations between them, ideas to which these concepts are related, logical relations, etc.) as opposed to

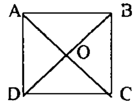
"pseudo-conceptual behavior", which might look like conceptual behavior, but which is brought about by mental processes which do not characterize conceptual behavior (Vinner, 1997). According to Vinner, in mental processes which lead to conceptual behavior, words are connected to ideas, whereas in mental processes which lead to pseudo-conceptual behavior, words are connected to words, without any ideas behind them.

The interview above demonstrates a pseudo-conceptual behavior. The students deal with the notions of angles, perpendiculars, 90°, but seem to be unclear about the relations between them (if any such relations exist in their minds), and ideas linking the concepts are not known (at least to the teacher). The words "90 degrees" are associated with "perpendicular lines", but there are probably no ideas behind them, and therefore the students do not know where to look for the "90 degrees".

Teachers frequently approach right angles "numerically": calculating angles, ascertaining perpendicularity according to the numerical size of the angle, etc. Weak students are also capable of solving such assignments. They rely on arithmetical knowledge, constructing their answers on verbal cues. The real situation will be exposed when moving on to non-computational problems.

Let us go back now to Ora and Keren. Ora has noticed that the students fail to "find" the angle (Ora: "They don't understand **where the angle is drawn, ... they don't know where they need to check this angle which is indeed 90°").** Nothing of what she says indicates that she understands the source of the problem. She does not refer to the difficulty of identifying a simple figure inside a complex one, nor to the tendency to see lines which form "good continuity". On the other hand, she does seem to define better the problem underlying the use of the notions 90° and right angle (Ora: "...**I am not sure that they know this for real, but they say 90° straight out**...When I ask them what kind of angle is formed between two perpendicular lines, they tell me: **aha 90°, but then they don't know what they have to check**"). Keren sensed this too (Keren: **"They know that perpendicular lines represent 90° but do not understand what this means**"). Despite this, they fail to make a further analysis and to characterize the problem accurately. In any event, they have no idea how to improve understanding in students! (Keren: "I really don't know").

Another dialogue took place between a (pre-service) teacher (T) and a girl student (S) during a geometry class for 9th graders (weak group), where students were asked to examine the properties of a square. The assignment was to check if the diagonals of a square were perpendicular to one another. In the Hebrew mathematical jargon this is expressed by the phrase: the diagonals "cut each other" at a right angle. "Cut" in Hebrew means: divide into two parts, intersect, split! The students were shown the following drawing of a square with its diagonals.



S: *(pointing at AC) They "cut" each other here, right? So here it's 90 (points at <ADC).*

(She probably means that the diagonal splits the square into two congruent parts. In the triangles obtained as a result, the right angle of the triangle, which is also an angle of the square, is a quite dominant figure).

T: *When we speak about perpendicular diagonals ... show me the diagonals.*

(The teacher tries to locate the source of the difficulty)

S: *(Points at AC and BD)*

T: *That's right. And where do they "cut" each other?*

S: *(points at diagonal AC and shows that it forms two triangles, ABC and ADC.)*

(She probably means that the diagonals split the square in two)

S (After a brief hesitation): *Oh, no, they "cut through" here (points at the four vertices)*

(Here, she probably thinks that the question is about the intersection points of the diagonals with the square)

T: *Where do these two diagonals "cut each other"?*

S: *Here (points to O)*

T: *Well done. Now, when I ask if they are perpendicular to one another, what I mean is that as they "cut each other", are 90° angles formed there (points to O)?*

S: *Yes! (points to the right angle in each of the two triangles ABC and ADC, i.e., angles <ACB and <ADC.)*

(After a while):

Oh, no, these (points to the two other triangles and their angles, i.e., <BAD and <BCD)

T: *When I speak about diagonals which are perpendicular to one another, their point of intersection has 90°. Where is that point?*

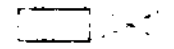
S: *(points to O)*

T: *Now tell me if an angle of 90° is formed between this diagonal (points to AC) and this diagonal (points to BD).*

S: *(points to <ABC!).*

The dialogue shows that the student does not master the concept of "intersecting lines". She therefore interprets intersection in different ways: the diagonal dissects the square into triangles, the diagonals intersect the circumference of the square (at the vertices). Even after the teacher explains to the student where the intersection point is and the student identifies it clearly, she constantly turns to another angle when asked to check whether the angle at the intersection point is right. A number of explanations are possible:

1. The right angles in triangles ABC and ADC fit the right angle pattern in the student's mind, and therefore when looking for a right angle, she first "focusing" on angles which fit the pattern.

2. As a second thought, she realizing that her answer is rejected, she identifies two other right angles in the drawing and then points to triangles BAD and BCD.
3. Following the thought process, we might discover that during the visual information processing stage, the student decomposes the figure into two sub-figures, the two triangles constituting it:  . Such decomposition makes it hard to identify the required angle for two reasons: firstly, the required angle is not included in the sub-figures! Secondly, the right angle within the sub-figures (right-angle triangle) provides her with an answer and with no motivation to "keep on searching". A different decomposition of the figure into the two sub-figures (e.g., ) is not at all simple, especially because an inappropriate decomposition has already been done previously.

One may wonder what the student's answer would have been had the square in the drawing been rotated by 45°. The diagonals would then run parallel to the paper margins and the right angle would appear in its prototypical shape and would therefore be easier to identify. Hershkowitz's research findings concerning right-angle triangles confirm this hypothesis (Hershkowitz, 1989). On the other hand, dealing with squares which were rotated by 45° ("diamonds") is harder for most students (who fail to identify them as squares).

Going back to the teacher, we shall try to point out communication failure between her and the student.

When the teacher is given the "surprising" answer concerning the location of the right angle, she checks whether the student knows what diagonals are, suspecting that the difficulty lies in identifying them. Her suspicion proves wrong. She therefore moves on to verify that the student identifies the point of intersection. Since the teacher has no clue as to what makes the student say the things she says (points to the diagonal, points to the four vertices), she guides her ("Where do these two diagonals cut each other"?) and puts words in her mouth ("What I mean is that as they "cut each other"... When I speak about diagonals which are perpendicular to one another, their point of intersection has 90°)".

We have analyzed and probed the student mind as well as the teacher's reactions. Teachers who understand thought processes will almost certainly change their instruction method and, more importantly, their response to situations of noncomprehension by students. For example, this topic of perpendicular lines will call for extensive treatment of all the visual links between right angles, perpendicular lines, and the complex figures in which they appear. Training can be carried out by means of concrete models, drawings, and mental verifications.

Conclusion. In this paper, we have presented some difficulties observed in students coping with the concept of "perpendicular lines". We have noted a difficulty to understand the combination of various elements which make up the concept (lines which are also angle rays; not one angle but four of them; a special angle—a right

the relation is between other right angles "nearby" and the right angle which appears in the definition).

We suggested various explanations for the origin of these mistakes: hypotheses concerning ways by which visual information is processed, thought processes (pseudo-conceptual) and students' concept images.

Finally, we examined the teachers' reaction to the difficulties which cropped up and found that they were helpless on one hand, and doing their best to "lead" the students to the solution on the other hand. Our clear conclusion is that a teachers' lack of tools which would help them understand students' difficulties makes them incapable of coping efficiently and providing proper instruction.

It can be assumed that training teachers in order to provide them with the relevant cognitive knowledge and with experience in analyzing such situations as shown in this paper could contribute significantly to improve instruction and learning (first attempts of this kind already carried out by the researcher have yielded promising results).

Moreover, the authors believe that the crucial issue in teacher training is not what the most appropriate explanation is, but the question how to understand the student's thought processes.

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Actions and invariant schemata in linear generalising problems

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In this paper we outline some results obtained from an ongoing research on the students' process of generalization. A written test and a task based interview was administered to eleven secondary students (aged 15-16 years). The teorethical model for generalization developed by Dörfler (1991) has probed to be usefull in analyzing the students' processes. We report briefly actions performed and invariant schemata established by students related to two different setting: numerical and spatial.

In its usual presentation, a linear generalising problem is a word problem that includes the first three terms of a numerical sequence and some pictures to illustrate the situation described. From a mathematical point of view we have an affin function $f(n)=dn+b$, $(b\neq 0)$, and there is an essential relationship namely $f(n+1)-f(n)=d$, to say that the difference between consecutive terms in the sequence is a constant.

Current research on student perception and generalization of numerical pattern has identified and classified the strategies used to solve linear generalising problems (Stacey, 1989; Orton & Orton, 1994 and 1996; García-Cruz & Martinón, 1996a). The SOLO taxonomy was used by Redden (1994) to classify the students' responses to a written test and to set a developing model for generalization. Also Taplin (1995) found an observable progression in the children's ability to recognise generalisations from their representations of spatial patterns, which fits the SOLO model.

However, little attention has been paid to the process through which one the students construct and develop a generalization in these types of problems. The role played by the drawing in the generalization process has been partially sketched by García-Cruz & Martinón (1996b).

In this paper we report briefly some results of our ongoing research focused on the students process of generalization when solving linear generalising problems. The research questions were:

a) Do students use a visual or a numerical strategy?
b) How do students check their patterns?

Generally, a *visual strategy* is defined as the method of solution that "involves visual imagery, with or without a diagram, as an essential part of the method of solution" (Presmeg, 1986, p.298). In this paper, a *visual strategy* is defined as one in which the drawing plays an essential role in the process of abstraction. A *numerical strategy* is defined, accordingly, as one in which the numerical sequence plays an essential role in the process of abstraction.

1 Theoretical Background

There is a broad agreement that the essential characteristic of mathematical knowledge is its generality and abstractness. Abstraction and generalization are important as a product but

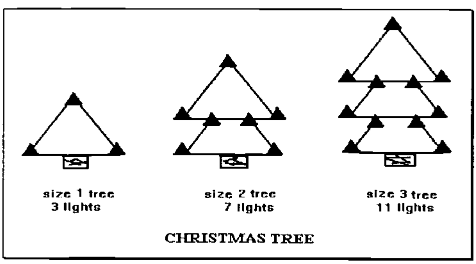
from a didactic point of view the associated processes of abstraction and generalization are much more important. W. Dörfler (1991) has modelled in detail the process of what Piaget called reflective abstraction, within this model the abstraction is the mean to construct a generalization. In our study we have adopted this theoretical model.

The essential features of this model are the emphasis on *actions* as the genetic source for abstraction and generalization. The *actions* that are material, imagined or symbolic are the starting point for the process of abstraction, even mathematical operations must be regarded as *actions*. Thus, the starting point is an *action* introduced by the student that concerns the elements given in the problem (either the drawing or the numerical sequence) as a response to questions which state an objective (calculate the numbers of components $f(n)$ for an object of a given size n). This *action* or *system of actions* directs the student's attention to some relations and connections between the elements of the *action*, size and components of the given object, and as a result to establish an *invariant* for the action.

This establishing of an *invariant* and its symbolic description has the character of a process of abstraction because some certain properties and relationships are pointed out and attention is focused upon them. Thus, the *action* or *system of actions* determines to some extent the directions and the content of the generalizations, i.e., the *invariants*, which operative character (the rules stated) results from the genesis out of the *actions*. To develop to a certain degree a generalization the student has to establish the *schema of the action* (*invariant*) as a general structure, i.e., to construct an *intensional generalization*. At this point the generality thereby constructed does not represent the qualities of things but relations between things, n and $f(n)$, which have been established and constructed by *actions*. The result of this process is a variable cognitive model that has two complementary aspects, first an expression of a cognitive activity of the subject and second as part of the objective knowledge, the mathematical content.

2 Methodology

The research was carried out in two stages. In the first stage a written test was administered to all students (N=168) in the last year of compulsory secondary education (aged 15-16 years) at a suburban high school and at the begining of the school year. For 133 out of 168 students the written test was the problem-1.



CHRISTMAS TREE

Problem-1

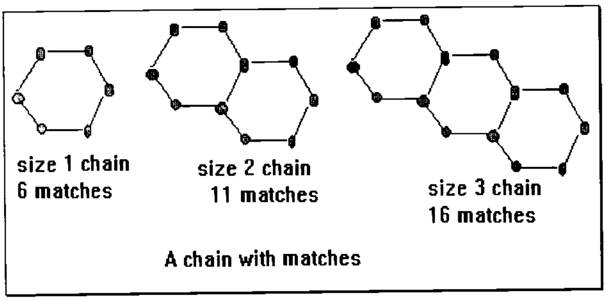
a)How many lights are there on a size 4 tree?
b)How many lights are there on a size 5 tree?
c)How many lights are there on a size 10 tree?
d) How many lights are there on a size 20 tree?

Explain how you found your answer.

In order to get a better analysis of the numerical strategies the following version of problem-1(with no drawing) was administered to a small group of students (N=35).

Problem -1a . *Ana and Juan are building up the Christmas Tree. In the instructions' booklet they found the following: A size 1 tree would need 3 lights, a size 2 tree would need 7 lights and a size 3 tree would need 11 lights. Try to help Ana and Juan answering the following questions:*
a) How many lights would need a size 4 tree?
b) How many lights would need a size 5 tree?
c) How many lights would need a size 10 tree?
d) How many lights would need a size 20 tree? Explain how you found your answer.

After the analysis of the written responses eight students were selected from the group were the problem-1 was administered (students *S1* to *S8*). Two of them shown in their responses that the drawing was used, two shown that only the numerical data were used and four gave no explanation or from their explanation not a clear conclusion could be obtained about the use of the drawing or the numerical data. From the second group, problem-1a, three students were chosen (student *S9*, *S10* and *S11*). The whole group of eleven students was chosen for the variety and quality of their responses to the written test. In the second stage these students were given individual interviews and asked first about some questions on problems 1 and 1a that may have not been clearly state from the written responses and second they were confronted with some questions about the situation stated in problem-2.



Problem-2

How many matches would you need to make the same sort of chain with size 4 ?

How many matches would you need to make the same sort of chain with size 23 ?

The objective of this second task was to verify *in situ* how students develop the process of abstraction and generalization and to what extent they recognize the second problem as similar to the first one. A consequence of this methodology was that the researchers did not ask necessarily the same questions to each student. Also they did not know if the students had received a specific instruction on arithmetic sequences, a topic related before, but they were aware that no student had had any training in sequences from the begining of the school year. Only two students belonging to the group interviewed had received specific instruction on arithmetic sequences the year before, but this fact was discovered during the interviews.

3 Results

The process of abstraction and generalization has actions introduced by students concerning the elements of the situations as its genetic source within the theoretical framework adopted in this study. The objective of these actions is to find out the number of elements *f(n)* corresponding to an object of size *n*.. Acting upon the numerical sequence or upon the drawing the elements of the actions are conceived as variables while certain relationship is maintained,

i.e., the invariant. Now we will describe briefly the actions and the invariant schemata developed by these eleven students, so the list should not be considered exhaustive.

Actions

Concernig the drawing.

Drawing a picture of the whole object required and counting all the elements is an action used in the introductory questions, *f(4)* or *f(5)*, and do not lead to a generalized strategy obviously. Students usually did it as a mean to check the validity of their calculations, as we will see below. Actions which lead to a generalized strategy are:

A1: Imaging or sketching to some extent a picture of the object required and adding similar parts while each new part has a number of elements equal to the constant difference *d*. The special feature here is that not direct counting at all is performed by students in the sketch done.

A2: Imaging an object of a certain size as constituted by aggregation of other objects of lesser size, i.e., a ten-size object as built up from two five-size objects.

Concerning the numerical data.

A3: Counting from a given term (i.e., *f(4)* but not *f(1)*) the number of *d* (the constant difference) which must be added to get a specific term (i.e., *f(10)*).

A4: Similar to action *A2* but performed upon the numerical sequence.

A5: Find a functional relationship between the object size *n* and the number of components *f(n)*.

A6: Applying the algorithm *rule-of-three*, which consists in giving three numbers to calculate a fourth number using the following schema:

$$\begin{array}{l} 5 \text{ ——— } 19 \\ 10 \text{ ——— } x \end{array} \qquad x = \frac{10\times 19}{5} = 38$$

Obviously the result of this calculation does not correspond with any term in the sequence but after doing that calculation the student *S3* checked it and made some arrangements leading her to get the correct answer. Below we will analyze in more detail the performance of this student. Here we have a system of actions instead of only one action.

A7: Applying the symbolic expresion for arithmetic sequences learned before. During the interview the student *S4* recognized the numerical pattern of problem-1 as arithmetic and after some calculations and checking he reconstructed the corresponding general symbolic expresion $f(n)=f(1)+(n-1)d$; later while he was confronted with problem-2 he applied automatically this formula showing an explicit knowledge of the similar structure of both problems.

A8: Successive addition of the constant difference to extend the numerical sequence.

Invariant schemata as result of actions

As result of the actions described above students established the following invariant schemata:

I1: f(n)=d(n-1)+f(1). Developed from actions upon the drawing, also from the numerical sequence.

I2: f(n)=6n-(n-1). This invariant was developed by *S2* within the problem-2. The important feature of this invariant is that neither the constant difference nor the first term in the sequence is an essential part of it.

Both invariants *I1* and *I2* were derived from action *A1*. Thus, the same action performed upon the drawing can lead to two different invariants. In *I1* the constant difference *d* and the first term *f(1)* play a prominent role while in *I2* both elements are not essential parts.

I3: f(n)=d(n-m)+f(m), m>1. This invariant was developed by student *S6* acting upon the drawing and by student *S9* acting upon the numerical sequence.

I4: f(2n)=2f(n).

I5: f(n)=dn+b. This invariant states the functional relationship between *n* and *f(n)*, corresponding to *4n-1* and *5n+1* in problems 1 and 2 respectively.

I6: Derived from action *A6*. The student *S3* developed this invariant for problem-1 and here its symbolic expression corresponds with *f(2n)=2f(n)+1*.

I7: f(n)=dn. Derived from action *A8*, assuming that repeated addition of *d* implies *f(n)=dn*.

Table-I summarizes the correspondence between actions and invariant schemata established by the eleven students in our study:

Table-I	Problem-1 and 1a				Problem-2			
	Drawing		Number Sequence		Drawing		Number Sequence	
students	Act.	Inv.	Act.	Inv.	Act.	Inv.	Act.	Inv.
S1	*A1*	*I1*			*A1*	*I1*		
S2	*A1*	*I1*			*A1*	*I2*		
S3			*A6*	*I6*			*A6*	*I6*
S4			*A6*	*I1*			*A6*	*I1*
S5			*A4*	*I4*	*A1*	*I1*		
S6	*A1*	*I3*			*A2*	*I4*		
S7			*A5*	*I5*			*A5*	*I5*
S8	*A1*	*I1*					*several*	*no inv.*
S9	-----	-----	*A3*	*I3*			*A6*	*I1*
S10	-----	-----	*A4*	*I4*			*A8*	*I7*
S11	-----	-----	*A5*	*I5*			*A5*	*no inv.*

4 Discussion

Although students use more than one action we have placed in Table-I only the last action with which they have completed the process of establishing an invariant. To establish an invariant the student has to apply the same rule abstracted from a specific calculation, i.e., *f(4)*, at least to another calculation, i.e., *f(10)*, showing that he or she has made an *intensional generalization* (establishing the schema of the action as a general structure) and an *extensional generalization* (extending the range of *n*).

From Table-I we gather the following indications: Different invariants can be established from only one action, thus action *A1* performed upon the drawing leads to three different

invariants, this is so because students' attention is focused in some aspects of the drawing highlighting these from other aspects. The actions performed upon the numerical sequence leads to the stating of only one invariant, due to the specific feature of the mathematical operation involved. Only two students out of eleven did not establish an invariant within the problem-2. A special case, student *S8*, will be discussed later.

Checking the rule abstracted should be considered an action as well. This action turns to be absolutely essential for students performing their calculation in the numerical setting as it is shown in Table-II.

Table II	where do students check	
	upon drawing	upon num. sequence
problem-1	*S3, S4, S6, S8*	*S5, S7*
problem-1a	------------	*S9, S11*
problem-2	*S1, S3, S5, S6, S9*	*S7, S8, S11*

If we compare Table-I and Table-II we can see that students performing their actions upon the drawing do not check their rules using the given numerical data. These students showed during the interviews more confidence that students performing their actions upon the numerical sequence. The actions upon the drawing fits the general structure of the rule in the students' cognition more precisely (intensional generalization), and the subsequent application of this rule to any other calculation was done with no doubt and confidently. However, some students whose actions were performed upon the numerical sequence (*S3* and *S4* within problem-1; *S3* and *S9* within problem-2) check the validity or their rules in the spatial setting, the drawing. Only two students, *S2* and *S10* did not check their rules during the process of solution in both problems. Student *S4* did not check his rule for problem-2 because he recognized the problem structure as similar to problem-1 and then he automatically applied the same invariant. He did not remember the symbolic expression for the general term of an arithmetic sequence, but he was able to reconstruct it while solving the problem. This case should be considered as an outstanding performance of developing a generalization.

The usual way of checking the validity of an invariant is counting on a drawing or extending the numerical sequence till the term needed. The use of routine activities for checking reinforces the students' confidence on the rule abstracted. Only two checks (student *S7* and *S11*) were done, using a known pair of values *(n, f(n))* and substituting this pair on the corresponding abstracted rule in both problems. This proved to be essential in establishing the invariant *I5* through the action *A5*.

An outstanding check was performed by student *S3*. She assimilates the calculation of *f(10)* in problem-1 to the existing schema, *rule-of-three*, using the pair *(5,19)* obtained adding the constant difference to *f(4)*. After doing the corresponding calculations she obtained the value 38 for *f(10)* and comparing this value to the available sequence 3,7,11,15,19 she notes that the number 38 (even number) does not fit in that sequence (odd numbers). Thus, she turns to draw a sketch of a size-ten-tree and performs a direct counting of the lights needed obtaining 39. Then she says that she must add one to the resulting calculations obtained applying the *rule-of-three*. For the next calculation, *f(20)*, she applied the rule derived without

checking on the drawing. She accommodated the particular schema of *rule-of-three* to fit the situation, establishing an invariant which symbolic expression is equivalent to $f(2n)=2f(n)+1$.

To develop a rule for a specific calculation it should no be considered as the establishing of an invariant. During the interview the performance of student *S8* should be considered as paradigmatic of the employ of many actions leading to the establishing of no invariant. He starts calculation of $f(4)$ sketching a picture of a size-four-chain to end counting on the sketch to get $f(4)=f(3)+5=16+5=21$. When promted to calculate $f(15)$ he applied the *rule-of-three* using data $(4, f(4))$ but the outcome calculation was not a whole number, so after some numerical explorations he used action *A3* to get $f(15)=5\times 11+f(4)=76$. Then he was promted to calculate $f(32)$ and, instead of applying the rule developed before, he said that the solution should be $f(32)=2f(15)$ plus something else. At this point he was encouraged to reflect on previous calculations. Suddenly he starts action *A5* sketching roughly a table using the available data and after some trying he concluded the task with the expression $f(32)=32\times 5+1$. It seems that he has established an invariant after all, but when asked about the validity of that rule for further calculations he said that he had not enough confidence in that rule unless he had tried it before, because that rule could fail in some case and then he would have to develop another one. The whole activity of student *S8* shows that neither intensional nor extensional generalization has been achieved. He has developed a specific rule for every calculation. The most perplexing thing for the interviewer was that he had successfully developed and established an invariant for problem-1, but this was done acting on the drawing instead on the numerical sequence. For this student the numerical setting probed to be harder than the spatial setting.

Otherwise, for student *S1* and *S2* the drawing probed to be the best setting for developing a generalization. Though an example, the calculation of $f(4)$, they were able to establish an invariant. They felt very confidences with the rule abstracted and that rule was successfully applied to any further calculations. For these students, the variable elements in the rule abstracted are detached from their original values that they are associated and gain meaning by themselves.

The students' performance discussed shows that the role played by the drawing is twofold. First is the setting for developing (abstracting) a rule and second is the setting for checking the validity of a rule developed upon the numerical sequence.

5 Conclusions

The theoretical model (Dörfler, 1991) for generalization has proved useful for describing the students' developing of generalization in this type of problem. To establish an invariant could need many actions, or system of actions, for some students while other only need of one action.

We have distinguished within the process two key aspects: first to abstract a rule for a specific calculation and second to establish the general structure of this rule and to extend the range of the variable elements (intensional and extensional generalization). The same action could lead to different invariants and also the invariants could include (or not) essential elements (*d* and *b*) of the underlying mathematical object: the affin function $f(n)=dn+b$.

The students' use of learned knowledge, although not appropriate for these problems, is an important feature of students' performance. The case of the assimilation-accommodation of

the *rule-of-three* leading to the establishment of an invariant is an outstanding student's behaviour. The consistency of students' choice of the numerical or spatial setting is a relevant conclusion derived from this study (see Table-I and Table-II).

Space limit does not allow us to extend in the analysis of the visual and numerical strategies, but we hope the above discussion can serve as an outline of our still not finished research in this area. It should be emphasized here the important role played be the drawing in the visual and numerical strategies, being the setting were students develop their rules during the process of abstraction in the former and being the setting for checking that rules in the later. We also think that more research is needed here to clarify in more detail the students' behaviour when establishing an invariant and the special features of the drawing that could lead a student to establish an invariant like *I2*.

Finally our ongoing research is now mainly directed to the study of the particular symbolizations students use in these type of problems, also to the meaning they give to the usual standard mathematical symbolization. But there is another research question: By what means do students recognize a similar mathematical structure among these type of problems?

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A CLASSROOM DISCUSSION AND AN HISTORICAL DIALOGUE: A CASE STUDY

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This report deals with a comparison between a mathematical discussion in the classroom and an historical dialogue. Both regard the mathematical modeling of the phenomenon of the fall of bodies and in particular the possible dependence of the fall speed on the traversed space. The protagonists of the classroom discussion are 8th grade students, while the protagonists of the historical dialogue are Simplicio, Sagredo and Salviati (Galilei, 1638). Analysis and comparison of the two 'discussions' raises issues concerning: interpretation of the analogies between them; and the conditions that allowed the classroom discussion rapidly to cover some important steps in the development of scientific thinking represented in the historical dialogue (this was read after the discussion!).

1. Introduction

From a Vygotskian and Bachtinian perspective, one of the main features of a discussion is the presence of *voices*. The term *voice* is used after Bachtin to mean '*a form of speaking and thinking which represents the perspective of an individual, his/her conceptual horizon, his/her intention and his/her view of the world*' (Clark & Holquist, 1981; see also Wertsch, 1991). The *voice*, described and defined by Bachtin in the literary field, is an innovative expression that communicates meanings recognisable at the social level, and assumes a universal character.

Drawing on the metaphor of mathematical discussion as a '*polyphony of articulated voices on a mathematical object that is one of the motives of teaching-learning activity*' (Bartolini Bussi, 1996), it is interesting to analyse what happens when, under the teacher's guidance, the objects under discussion are the *voices* themselves. *Voices* may be emitted by the students, the teacher, or even by history when the teacher introduces in the discussion the *voices* of scientists from the past on the topic being examined.

Henceforth, we shall use the expression '*voices and echoes game*' to express the idea that, when a suitable task is assigned, the '*source voice*' (of a student, of the teacher, of history) triggers an '*echo*', i. e. a link (expressed as a discourse) with the object of the *voice*. In this way, the student strives to link this *voice* to his/her conceptions, experiences and *personal senses* (Leont'ev,1978). The *echo* idea was originally elaborated during the design and implementation of the teaching experiment reported in this paper. In the *voices and echoes game* the classroom discussion may happen *to 'anticipate' a voice from history.* This report concerns one such episode.

2. The teaching experiment.

The case study presented in this report concerns a discussion in an 8th-grade classroom of 21 students; the topic is the possible dependence of the speed of falling bodies on the traversed space. The discussion is part of a teaching experiment designed to study the functioning of the *voices and echoes game*, making extensive use of *voices from history* (see 2.1 and 2.2). For the past three years the students have been taught by the same teacher and, under her guidance, have learnt to engage productively in mathematical discussion, acquiring argumentative skills and payin~

full attention to argumentation consistency. These were necessary conditions for the implementation of the teaching experiment, as ' *long acquaintance with the active discussion of schoolfellows' or teacher's utterances put the students in an active attitude, that allows to discuss and criticise both the voice of history and that of the teacher* ' (Bartolini Bussi, 1996).

2.1. Rationale of the teaching experiment.

From previous experiences carried out by the Genoa group, we knew that students' spontaneous knowledge about the fall of bodies was limited to perceptual data that had not been developed much from a cultural perspective. Indeed, students' relationship with eveyday culture and their personal experiences afford them little opportunity to go beyond obvious facts. Our hypothesis was the following: some *voices of history* (Aristotle) can represent fully and precisely the perceptual universe of students, while other *voices* (Galilei) can lead them to challenge the Aristotle's theory. We hypothesised that pursuing suitable classroom tasks could produce *echoes* to such *voices*; in this way the teacher could mediate some crucial steps in the scientific revolution of the 17th century.

2.2. Phases of the teaching experiment.

a) The first individual task is designed to introduce 'the fall of bodies' phenomenon: *'What do you think is the reason why a feather or a leaf fall to the ground more slowly than a stone?'* . Working individually, many students hypothesise that the relevant variable is weight. A 'balance' discussion follows.

b) Aristotle's *voice* is introduced with a classroom reading of excerpts from *De Coelo,* where Aristotle claims that fall speed is proportional to the weight, which in turn represents the tendency of bodies to reach their natural place. Later on, the 'voices and echoes game' begins with suitable tasks (*'If you were Aristotle, how would you explain the fact that a feather falls more slowly than a stone?* 'and so on)

c) When the students have appropriated Aristotle's *voice*, the focus shift to mathematical formalisation by means of present-day sign systems (algebraic formulas and cartesian graphs).

d) At this point, Galilei's *voice* is introduced by reading and interpreting the famous excerpt (Galilei, 1638) where he refutes the two hypotheses of Aristotle, according to which fall speed is directly proportional to body weight and inversely proportional to density. This passage from Galilei challenges previous student acquisitions. In the discussion that follows and in individual texts the students admire Galilei's arguments but at the same time are troubled by them. Aristotle's theory and the conceptions of many students are in crisis: yet it is difficult to think that *'a lead drop can go as fast as a cannon-ball'.*

e) The next day the teacher starts a new discussion concerning the possible dependence of fall speed on the height from which the fall starts, an idea mentioned in passing the previous day. This new discussion anticipates the theme and structure of an excerpt from Galilei (see 3).

f) This excerpt from Galilei is subsequently presented to the students. The *voice of history* is perceived (and used didactically) as an *echo* of classroom discussion.

3. The object of the case study: a classroom discussion and a dialogue by Galilei. We shall compare the classroom discussion (phase e) and the dialogue by Galilei successively read by students (phase f), by looking for analogies and differences in the following areas:

- linguistic utterances: unlike other languages, Italian has not changed very much from Galilei's times. In the English translation of the discussion we have tried to maintain original consistency between Galilei's text and the discussion;
- the logic of argumentation (alternation between specific examples and generalisation; use of mental experiments);
- argument content (particular conceptions and examples, etc.).

To make comparison easier, we have divided the classroom discussion into four parts and paralleled it with suitable excerpts from the dialogue by Galilei. The points of greatest consistency between the two have been underlined.

3.1. Classroom discussion.

[...]

[3] T. Let us go move on another issue: height. In yesterday's discussion the matter of height was raised: what do you think?

[4] Eleonora: If we make an object fall from the desk and drop a similar one from a higher place, they should arrive together, since, as Daniele C. said, speed increases with height.

[5] Daniele C.: No, I' said that <u>speed increases with height</u>, not that the two objects arrive together.

(Nobody appears to understand the meaning of Eleonora's words)

[6] T: What Eleonora is saying makes sense, as we shall see later. Let's go back to speed in relationship to height. Yesterday what example did you make?

[7] Daniele C.: <u>If an object falls from 10 meters it makes a hole this big *(gestures)*; if it falls from 200 meters it makes a much bigger hole.</u>

[8] T.: Galilei also introduces height: in your view, what is the relationship between speed and height?

[9] Sebastiano: The higher the object, the more speed it acquires.

[10] T.: Right! Galilei thinks that speed increases with height, too. Give me some examples.

[11] Enzo: If somebody dives from a 2-meter board, he makes a certain splash, but if somebody dives like a bomb from 20 *(meters)*, the water goes out of the swimming pool.

[12] Daniela M.: If you throw yourself from the first floor, you don't hurt yourself; if you throw yourself from the third *(floor)*, you kill yourself.

[13] Vincenzo: If I want to increase speed I need space, I go to the 80- meter track!

(Other examples follow).

From *Dialogues Concerning Two New Sciences*:

Sagr. So far as I see at present, the definition might have been put a little more clearly perhaps without changing the foundamentale idea, namely, uniformly accelerated motion is such that its <u>speed increases in proportion to the space traversed</u>; so that for example, the speed acquired by a body in falling four cubits would be double that acquired in falling two cubits and this latter speed would be double that acquired in the first cubit.

Because there is no doubt that <u>a heavy body falling from the height of six cubits has, and strikes with, a momentum [*impeto*] double that it had at the end of three cubits</u>, triple that which it had at the end of two.

Daniele C, in his interventions, expresses first the law [5] and then an example, exactly as Sagredo does. Furthermore both Daniele's and Sagredo's examples concern the effect produced by the falling body. Eleonora's *voice* is not grasped: even if correct it is too far removed from the student's way of thinking; hence it dies out. The following comments echo Daniele's *voice*.

3.2. Classroom discussion

[20] T.: Right, but this is a description; from a mathematical point of view, what form of regularity could you draw? Be careful though, we're not yet dealing with a mathematic law.
[21]: Enzo: When somebody falls from a height, for each meter his speed increases by a certain amount.
[22] T. Explain that better.
[23] Enzo: Say the speed he gets per meter is 6 km/h , if he falls from 2 meters, the speed is 12; if he falls from 3 meters, the speed is 18.
[24] T.: What's the name of this in mathematics?
[25] Enzo: Proportionality.
[26]: T.: Proportionality.

From *Dialogues Concerning Two New Sciences:*

Salv. It is very comforting to me to have had such a companion in error; [...] but what most surprised me was to see two propositions so inherently probable that they comanded the assent of everyone to whom they were presented, proven in a few simple words to be not only false, but impossible.
Simpl. I am one of those who accept the proposition, and believe that a falling body acquires force [*vires*] in its descent, its velocity increasing in proportion to the space, and that the momentum [*momento*] of the falling body is double when it falls from a double height; these propositions, it appears to me, ought to be conceded without hesitation or controversy.

In the *voices and echoes game*, Enzo represents the collective *voice* of the classroom; he is the spokesman for the 'theory' implicit in previous comments. In fact, from now on the students will always refer to 'Enzo's law'. We notice consistency between Enzo and Simplicio concerning both words and logical structure: first the law [21] is expressed in a general way, then mathematisation takes place [23]. We may observe that while proportionality is explicit ('the double of..') in Galilei's example, in the Enzo's example it is made explicit only after the teachers' request. In the comparison between the two excerpts we note that in the classroom discussion the role of Salviati (who is first to state that the conclusion is false) is still missing.

3.3. Classroom discussion.

[27] Fabio P.: If it travels at 6 km/h in one meter, at 2 meters, when it has to traverse the second meter, it goes faster, so it takes less time. That regularity cannot exist.
[28] Andrea: It speeds up little by little.
[29] Vincenzo: But then Enzo says that even with a greater height, speed does not increase.
[30] T.: Be careful, Enzo says that the fall speed is proportional to height; Vincenzo says that this means... what? I don't understand well.
[31] Vincenzo: ... having a greater height does not affect speed. I agree with Fabio, who says that it goes faster. But Enzo says that it goes the same speed.

[32] T.: Attention, Enzo does not say that speed is constant, but that it is proportional to height.
[33] Stefania: I don't agree with Enzo: if you throw a stone from, say, 1000 meters, the fall, if what you say is true, would be instantaneous, and this is impossible.
(Buzz)

From *Dialogues Concerning Two New Sciences:*
Salv. And yet they are false and impossible as that the motion should be completed instantaneously; and here is a very clear demonstration of it.

In the discussion Enzo's *voice* triggers some dissonant *echoes*: Fabio introduces the 'time' variable [27], while Stefania [33] performs a mental experiment, using practically the same words as Salviati.

3.4. Classroom discussion.

[34] T.: Good girl! A good objection! How do you respond?
(troubled silence)
[35] T.: Repeat your objection.
[36] Stefania: If what Enzo says is true, when you throw a stone from 1000 meters, the fall is instantaneous.
[37] Elisa: It is not instantaneous: it only goes much faster. Enzo says that speed always increases by the same amount.
[38] T.: What you're saying is not right. Be more precise.
[39] Elisa: If it falls from 1 meter it has a certain speed; if it falls from 2 meters, the speed is double; if it falls from 10 meters it is ten times faster.
[40] T: Stefania doesn't agree with you: if it falls from 1000 meters it is 1000 times faster and falls instantaneously.
[41] Elisa: No, even if it falls 1000 times faster ...
[...]
[44] Cristina: I don't understand what instantaneous means.
[45] Stefania: That if you let it go it has already fallen to the ground.
[46] Cristina: But Enzo says that if it falls from 2 meters it falls with double speed: I don't understand why you say instantaneous
(Overlapping voices)
[47] Cristina: Well, it falls proportionally to its height.
[48] Daniele: But isn't the 1000 speed the final one? Then it means that it has reached it, that at the beginning it was 1, then 2 and so on.
[...]
[60] T. Let's go back to the objections: Stefania says that if what Enzo says is true, when I drop it from 1000 meters it falls instantaneously. Fabio's objection: the relationship here is not proportionality, if in the first meter it goes at a certain speed in the second meter it goes faster.
[61] Fabio S.: What Fabio says is wrong. If if you drop something from 1000 does it arrive 1000 times faster?
(doubts)
[62] Daniele C.: It arrives faster...
[63] If he said proportional, that means 1000 times faster, so it arrives 1000 times earlier.
(Disagreement)
[64] Daniela M.: If you put something 1 meter high and something else 2 meters high, if it doubles speed at two meters, then they arrive together.
[65] Fabio S.: That's what I said.
[66] T.: I don't think so. You said that they arrived 1000 times earlier.
[67] Fabio S.: Well, that's what I meant

[...]
[71] Daniele C.: As soon as you drop it, it's already on the ground.
[...]
[78] Daniela M.: I wanted to ask Enzo something.
[79] T.: Enzo has become the reference point, but remember that at the beginning nearly all of you agreed with him.
[80] Daniela S.: In the longest route, in the first part, does the object go faster than in the other route?
[81] Enzo: If the first has twice as far to travel as the second, when this has gone halfway, the other has gone halfway too, and they fall to the ground together.
[82] Daniela M.: But the shorter one cannot take the same time to cover half the distance *(To the teacher)* May I go to the blackboard?
(Daniela draws two vertical lines, one twice the length of the other, and a horizontal line in the middle of the longer one)
Supposing, I stop the body from the highest point in the middle, where is the other body?
[83] Daniele: The other has gone halfway too.
[84] T. According to Enzo's theory, when the first has gone halfway, the second has gone halfway too. Hence they arrive at the same time.
[85] Daniele C.: But that means if you throw one from 1 meter and the other from 5000 meters they tie
[86] T.: Exactly. Galilei realised this mistake after 24 years. If the idea of proportionality had been true, it would have happened like Stefania, Daniela M. and the others said.
[87] Daniele C.:If you ask me, they cover the first meter in the same way, then in the second meter it picks up speed, bit by bit.
[88] T. Do you have any idea of how the body would fall, according to what you say. Try to draw how it would fall.
[89] Daniele C.: If I ldrop one body from 1 meter and another from 10 meters, they have to tie the first meter at least, so the first reaches the ground and the second does not.
[90] T. If you had to photograph a body every second it falls, how could you draw it?
(Daniele draws a representation of the stroboscopic graph)
[91] Daniele C.: the more space it has to cover, the more speed it picks up, but a little at a time.
[92] Fabio S.: Speed increases as the body falls.

From *Dialogues Concerning Two New Sciences:*

Salv.If the velocities are in proportion to the spaces traversed, or to be traversed, then these spaces are traversed in equal intervals of time; if, therefore, the velocity with which the falling body traverses a space of eight feet where double that with which it covered the first four feet (just as the one distance is double the other) then the time-intervals required for these passages would be equal. But for one and the same body to fall eight feet and four feet in the same time is possible only in the case of instantaneous [*discontinuous*] motion; but the observations shows us that the motion of a falling body occupies time, and less of it covering a distance of four feet than of eight feet; therefore it is not true that its velocity increases in proportion to the space.

We note that Stefania's [33] comment puzzles her classmates, as they have to equate it with 'Enzo's theory'. Stefania proceeds in a Galilean way, but, as opposed to Galilei, she produces a limit case to refute Enzo's idea and to show the impossibility of proportionality. The word *'instantaneous'* creates unease among the schoolmates, who react by returning to the proportionality between speed and space. Daniela M.

[64] unblocks the situation: if two falling bodies start and arrive together, it means that the spaces are covered in equal times. In comments [82] and [89], which are connected to Daniela's utterance, the speaker tries hard to elaborate a mental experiment which is the same as Galilei's. In this excerpt congruences with the Galilei's dialogue concerning both the expressions and the arguments, may be noted.

As a concluding remark, we note that the roles played by the students in the discussion are consistent. At the very beginning, Daniele C. is Sagredo: he postulates the dependence of speed on the traversed space. Enzo is the spokesman for the Aristotelian perspective; he puts into practice the idea that perception leads to the law and thus plays Simplicio's role. Finally, acting collectively, Stefania, Fabio P., Daniele C. and Daniela M. play the most difficult role, that of Salviati, by disproving the hypothesis that speed is proportional to traversed space.

4. Discussion.

It should be noted that the comparison is between a real discussion and an imaginary dialogue (which was read after the discussion). This implies that in the dialogue the aim is absolutely clear in the author's mind: the protagonists (Simplicio, the Aristotelian philosopher; Salviati, the clever thinker; and Sagredo, the cultured man of his age) act according to an established script, which, on the one hand, traces the steps of Galilei's twenty-year search and on the other points out the contrasts between Aristotle's and Galilei's theories. It is evident that this does not happen in the classroom: the teacher's aim is unknown to the participants in the discussion, who are not following any established script. This makes the discussion less linear in comparison with Galilei's dialogue, but more lively and varied, from the viewpoint of reference to experience.

In spite of these differences, the underlinings and comments in Section 3 point out strong analogies between the classroom discussion and Galilei's dialogue. The verbal expressions used are often similar. Moreover, both debates have a similar sequence of phases: general law, example, mathematisation, mental experiment. The contents too are analogous: the most surprising example is the argument that disproves the hypothesis of proportionality (a hypothesis that, as we know, Galilei long supported). In effect, in one hour a class of 13-14-year-olds managed to cover all the steps Galilei took in over twenty years of research. How was it possible? Precise interpretation of this episode (observed in a particular teaching - learning situation) seems to require adjustment and specification of the general hypotheses of Piaget & Garcia (1985) regarding the relationships between ontogenesis and phylogenesis of scientific hypotheses. In our view the following elements may have brought about the above analogies:

- some of the teacher's comments ([6], [30], [34], [40]), which single out particular student comments, explain their content and orient the discussion in the direction of Galilei's dialogue;
- cultural aspects arising from the *voices and echoes game* played earlier in the same teaching experiment. Previous reading of excerpts from *Dialogues* (Phase d) may have provided students with the material needed to take further steps. For instance, Stefania commented on an excerpt from Galilei (Phase d) as follows: *'I was struck by the reductio ad absurdum made by Simplicio, when he says that speed depends on density, hence the lower the density, the faster the object. But if we made the density zero, the object would fall in an instant, and this is impossible as in a vacuum an*

object does not move'. Her utterance is an echo of an excerpt from Galilei; she uses it to take on Simplico's argument and in doing so interiorises the 'limit' method, which allows her to produce the utterance [33] in the following discussion. Another example is given by Daniele C., who contributes towards the construction of Galilei's mental experiment . In a preceding discussion he said:*'I agree with Galilei, because at the very beginning I thought that there was regularity and proportionality between weight and speed ... Moving to the limit case, as Galilei does, I understand that it is impossible that if a body has covered 100 meters the other has covered none'.* In this case the interiorization concerns both the method and the content;

- aspects from the students' own culture (experience gained either inside or outside school): as opposed to Galilei, today's students seem to find it easy to grasp concepts such as 'speed in a given instant' or 'speed variation instant by instant' (see Daniele C. [48], [87], [91]), possibly because of their experience with car and motorbike speedometers. Indeed, Galilei initially conceived speed as the ratio between the whole traversed space and the time spent. Only after many years did he elaborate the idea of 'speed at a given instant' as the ratio between space and time at that instant. In addition students possess powerful sign systems (algebraic formulas, cartesian graphs) that were not used by Galilei. Finally, some aspects of Galilei's way of reasoning, the same adopted by modern science (relationship between hypotheses and quantitative data obtained from experiments, use of mathematics in natural sciences and so on), may have influenced the students' thinking during their previous school experience, through cultural experiences and by means of information collected in and out of school settings;

- elements inherent in the structure of debates (hence shared by an imaginary dialogue and a real discussion) such as the search for possible logical or factual contradictions to an idea that is not agreed with, or the alternance of general statements and examples. These may derive from three years' experience in mathematical discussion, besides the standard development of argumentative skills in the present out- of- school setting.

We believe that more detailed specification of the above aspects would make the reproduction of this teaching experiment easier and, more generally, would improve teacher' management of teaching experiments that focus on historical sources for the mediation of important steps in scientific thinking.

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The Importance of Social Structure in Developing a Critical Social Psychology of Mathematics Education[1]

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Abstract

In this paper I develop the argument (started at PME 20) that we need to consider ways of re-evaluating a psychology of mathematics education, by looking towards critical social psychology as an alternative. In this we would need to consider both the notion of ideology as well as social theory of Anthony Giddens and Pierre Bourdieu for an understanding of how humans operate and in particular how the mathematics education classroom practices develop.

A view of the current situation

As mathematics educators, we are perhaps more aware than most of the importance and significance placed upon mathematics in our society. However this importance is usually seen as *functional*, rather than *constituting*. By *functional* I mean that learning mathematics is often seen as enabling the learner to be able to carry out some function, that it has some purpose to which we should be ascribing. In this I include basic skills, vocational preparation, preparation for entry to the next stage of education and so on. By *constituting*, I mean that learning mathematics plays a role in shaping the society in which we find ourselves. I make the distinction deliberately, because by constituting I do not want to imply that it plays a positive enabling role. On the contrary mathematics is not (just) a 'gateway to', it is also a 'filter out'. Mathematics, along with other school subjects plays its part in justifying the present social reality. (I guess I ought here to use the phrase 'school mathematics' to show that what is taught is a particular subset of the discipline. I will not do so as it will become cumbersome. This needs to be how this paper is read.)

In Great Britain, you will inevitably see those pupils who find themselves placed in lower attainment groups[3] and become labelled as 'special needs', 'in need of remedial help' and so on, usually come from lower social classes. There is a dominant rationale for this - that social conditions make it difficult for them to get on in mathematics. (There are other rationales of course, some based on genetic disposition)

There is an alternative viewpoint - a radical viewpoint - that school mathematics has the effect of alienating certain social classes. Valerie Walkerdine (Walkerdine 1988) and others (Gilligan 1982), have written about the process by which school mathematics alienates women and racial groups for example. Barry Cooper has shown how the national Standard Assessment Tasks in the UK can result in discrimination between pupils of different social classes (Cooper 1996). There is a crucial debate about the degree of intentionality of such a process. I will not develop this here.

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[3] It is common practice in Great Britain for pupils to be placed in groups for mathematics lessons. It is claimed that these are based upon 'ability' or 'attainment'. It is also claimed by those who support this practice that such discrimination is a positive force in tea hing pupils.

It is sometimes difficult to see just what alternative there could be to the current dominant practices within mathematical education. This can in part be due to the way in which rationalising disciplines, such as psychology, are not merely descriptive, but are also produced and sustained by and within the dominant social structures. They thereby define our way of seeing the world. Alternatives which seem to challenge this hegemony can be ostracised and marginalised. Consider Henry Giroux's view of knowledge:

> [A radical view of knowledge] would be knowledge that would instruct the oppressed about their situation as a group situated within specific relations of domination and subordination. It would be a knowledge that would illuminate how the oppressed could develop a discourse free from the distortions of their own partly mangled cultural inheritance. On the other hand it would be a form of knowledge that instructed the oppressed in how to appropriate the most progressive dimensions of their own cultural histories, as well as how to restructure and appropriate the most radical aspects of bourgeois culture.
>
> (Giroux 1983, p 35)

I know of no school or government which encourages such a radical view of knowledge. In addition it is likely that many who are reading this paper see this viewpoint as - at best - marginal and idealistic; possibly (most likely!) unrealistic. Such a view of knowledge could hardly be further away from the experiences of most pupils in schools today. It might be claimed that Henry Giroux is a hopeless idealist.

However all is not well with the current model of mathematics education and many children leave school as failures. Why might this be? Some call for 'better psychology' to better understand the child's learning process. However it is important to ask - though a challenge to many 'sacred cows' what part might child-centred pedagogy play in this failure? Child-centred pedagogy, and with this I place forms of constructivism, derive from the tradition of developmental psychology. However developmental psychology is based upon a set of claims which are historically situated and make claims to truth about the capacities of the child. The possibility of success of a project intended to make developmental psychology more adaptable has been critiqued by Valerie Walkerdine who sought:

> to demonstrate the problem in assuming that the way out of dilemmas about the possibility of both a liberatory pedagogy and a 'social' developmental psychology is in the limit-conditions of the project of a developmental psychology itself. Because of the way that the object of a developmental psychology is formulated, it is impossible to produce the radical theory which would fulfil the hopes of many within the discipline.
>
> (Walkerdine 1984, p 154)

Current pedagogic practices are "saturated with the notion of normalised sequences of child development so that those practices help produce children as the object of their gaze" (Walkerdine 1984, p 155) and in turn, these practices actually produce the child and the child's development in their own image. Schools are so structured that the mechanisms for this saturation permeates the architecture, the curriculum materials, the organisation of the classroom and the assessment arrangements.

> Developmental psychology is productive: its positive effects lie in its production of practices and pedagogy. It is not a distortion of a real object 'the developing child' which could be better understood in terms of a

radical developmental psychology, for the very reason that it is developmental psychology itself which produces the particular form of naturalised development of capacities.

(Walkerdine 1984, p 163-4)

Developmental psychology then is a product/construct which in turn structures how we view the world. With this in mind, when I look at mathematics classrooms a number of questions are raised for me. Why is mathematics still a socially unpopular subject? Why does it alienate learners from lower social classes? Why is it unrelated to children's real experiences? Why is there so much resistance in mathematics classrooms. (Note my use of the word resistance. Others may use 'disruption', 'indiscipline'. The choice of words we use is politically loaded). Looking at papers presented to PME conferences leads me to ask further questions. Why do colleagues feel the questions they work on are important? What interests are served by some of the research which is carried out? Why is there not a Sociology of Mathematics Education conference? (The answer to this is likely to be both politically and historically situated.)

Teachers and ideology

Over the years there have been numerous disputes about various aspects of mathematics education: What is the nature of mathematics? Is mathematics absolutist or fallibilist? What is an appropriate epistemology? However all this goes on in the face of increasing disadvantage in our societies. In many ways it may be seen that schools and the models of psychology on which they are based, actually legitimate social disadvantage. This is not because schools are failing in their duty, it is rather because it is the purpose of institutionalised schooling to maintain social disadvantage.[4]

In all of this though we have to admit that teaching is carried out by teachers who all have views on what they are doing. A lot of research has investigated the nature and structure of teachers' knowledge, and there is an extensive literature. However a dimension which often does not get attention is the influence a teacher's ideology has upon the nature of the practices which go on in the classroom.

I want to argue that classroom decisions made by teachers (either interactive or prior planning decisions) are not rational choices made by looking objectively at the situation. There are influences and structures of thought which impose themselves on teachers and these influences and structures of thought are just what is included in the notion of ideology. In some ways ideology is not a fashionable notion to write about - it is also not an easy one. Current interest in post-structuralism has suggested that ideology as a notion has no further significance and indeed is no longer credible. This comes through the deconstruction of the 'subject' and rejection of essences. It is not my intention to give a thorough critique of ideology. This discussion will therefore be a limited exposure to those parts of the debate which I feel are essential. A fuller critique of the notion of ideology can be found in (amongst others) (Althusser 1971; Eagleton 1991; Hall 1996; Laclau 1977; Larrain 1979; Larrain 1983; Larrain 1996; Marx and Engels 1846)

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[4] I recognise this may be considered a controversial claim to make. However it is no more controversial that to claim that it is the duty of schools to help individual pupils to reach their full potential. What is crucial here is that this is not considered controversial as it does not conflict with dominant hegemonic perspectives.

In claiming that schools are mechanisms of reproducing domination, I am not claiming that it is teachers themselves who are singularly guilty of that oppression. This is for (at least) two reasons. First, individuals do not only make society, they are also agents for it. As Jorge Larrain tells us:

> Material conditions and social institutions have been produced in human practice, but they have acquired an independence over and above individuals, constituting an objective power which dominates men and women
>
> (Larrain 1983, p 20)

Secondly there is the whole area of the "unintended consequences of intentional conduct" (Giddens 1979, p 59). In a paper presented to PME 20, Tony Cotton and I suggested that we needed to draw together the social and psychological if we want to change mathematics education for a more just society (Cotton and Gates 1996) and in responses to that paper some colleagues said that this was non-contentious. However there are contentious issues when we begin to explore the unintended consequences.

A well used quote from Karl Marx is pertinent here :

> The ideas of the ruling class are in every epoch the ruling ideas, i.e. the class which is the ruling *material* force of a society, is at the same time its ruling *intellectual* force. The class which has the means of material production at its disposal, has control at the same time over the means of mental production, so that thereby, generally speaking, the ideas of those who lack the means of mental production are subject to it. The ruling ideas are nothing more that the ideal expression of the dominant material relationship. . . hence among other things [they] rule also as thinkers, as producers of ideas, and regulate the production and distribution of the ideas of their age.
>
> (Marx, 1846 #85)

Interestingly the word 'ideology' is noticeable by its absence and there is considerable disagreement about what 'ideology' is. I will therefore give some indication of my understanding and use of the term. First Alex Callinicos gives a helpful clarification:

> More precisely, ideologies are practices which function symbolically, usually through the generation of utterances, subject to definite norms and constraints. Very often these norms and constraints derive from the prevailing structure of class power.
>
> (Callinicos 1983, p 135)

The two important ideas here are the symbolic nature of practices generated through utterances, and secondly the derivation of the norms from the class nature of society. On the first claim, I want to cite such practices as ability discrimination (or 'setting'). There is a discourse in the UK which gives legitimacy to the practice and it is my claim that these discourses are generated by social norms. On the second, I claim that the class nature of society and in particular the economic basis lie at the root of such discourses. I am currently developing this argument. It is I know very unfashionable - but this very unfashionability is of itself a discourse derived by the social structure.

Why am I so interested in ideology? I will quote at length form Stuart Hall, who articulates my position better than I could:

> The problems of ideology is to give an account, within a materialist theory, of how social ideas arise. We need to understand what their role is in a particular social formation, so as to inform the struggle to change society and open the road towards a socialist transformation of society. By ideology I mean the mental frameworks - the languages, the concepts, categories, imagery of thought, and the systems of representation - which different classes and social groups deploy in order to make sense of, define, figure out and render intelligible the way society works. The problems of ideology therefore concerns the ways in which ideas of different kinds grip the minds of masses and thereby become a 'material force' In this, more politicised perspective, the theory of ideology helps us to analyse how a particular set of ideas comes to dominate the social thinking of a historical block, in Gramsci's sense; and thus helps us to unite such a bloc from the inside and maintain its dominance and leadership over society as a whole. It has especially to do with the concepts and the languages of practical thought which stabilize to a particular form of power and domination; or which reconcile and accommodate the mass of the people to their subordinate place in the social formation. It has also to do with the processes by which new forms of consciousness, new conceptions of the world arise, which move the masses of the people into historical action against the prevailing system.
>
> (Hall 1996, p 26 - 27)

My interest in ideology then has to do with structuring frameworks, but in a predictive way by incorporating some element of being able to predict how individuals might act/think. This may involve: the nature of idea, where these derive, how they are mediated, the relation between belief and activity, the construction of 'common sense' interpretation and its justification. Ideology calling on these characteristics would seem to be one source of power and discourse (in a Foucauldian sense). This is not to suggest that ideological forms exhibit any particular form of coherence or consistency. The lack of consistency in ideology is well known to us all. We accept everyday the acceptance of contradictory views as well as actions which seemingly contradict our views. *Rationalisation* is a significant feature of ideology too.

Terry Eagleton further suggests that enduring ideologies depend less on blatant falsehoods than on accurate but partial representations - ideologies are therefore notable for what they do not discuss (Eagleton 1976, p 35). In addition Clifford Geertz suggests that ideologies represent the world in a way which people find reassuring (Geertz 1973). It should therefore come as no surprise that some of the arguments I may develop come to be seen as uncomfortable

Social discrimination and mathematics pedagogy

There is a debate within the mathematics education community regarding the nature of a 'critical mathematics education'. Such a critical mathematics education would see itself as preparing individuals to assume critical stances in society, to recognise and oppose oppressions. Mathematics can play a part by empowering learners through discussion, conflict of opinion, challenging the teacher, by demonstrating the injustice in society. It would require learners to pose real problems rather than the fantasy world currently on offer.

However such debates take place, while little changes in schools. No government to my knowledge fosters such an approach - certainly not the UK; it is not difficult to see

why. However it is important to look into schools, and work with teachers to illuminate the processes by which mathematics is a force for dis-empowerment.

> By acknowledging the need for a critical social psychology, educators can begin to identify how ideologies get constituted, and they can then identify and reconstruct social practices and processes that break rather than continue existing forms of social and psychological domination. Bourdieu argues that the school and other social institutions legitimate and reinforce through specific sets of practices and discourses class-based systems of behaviour and dispositions that reproduce the existing dominant society.
>
> (Giroux 1983, p 39)

There is a sense in which the everyday reality of schools forces teacher to operate within a discourse which they may feel unempowered to challenge. By adopting a critical stance and opening up a dialogue on practices and the relation to the unintended consequences, we would be developing a more democratic mathematics education challenging the hegemony of dominant ideologies and how these organise the practices in schools. We see this socially articulated in the following way:

> [Schools] set such a store on the seemingly most insignificant details of dress, bearing, physical and verbal manners. . . The principles embodied in this way are placed beyond the grasp of consciousness, and hence cannot be touched by voluntary, deliberate transformation. The whole trick of pedagogic reason lies precisely in the way it exhorts the essential while seemingly to demand the insignificant: in obtaining respect for forms and forms of respect which constitute the most visible and at the same time the best hidden manifestations to the established order.
>
> (Bourdieu and Passeron 1977: 2nd Edition 1990)

Bourdieu's insight here exposes the duality of social norms: they are both imposed upon individual's thinking, as well as articulated by individuals. Bourdieu's description has considerable relevance to mathematics education. Not only do teachers of mathematics play their part in demanding the insignificant, the dominant practices demand a respect for forms of authority and respect whose unintended consequence is the continuance of the established order. In particular I am referring to common practices such as ability grouping, 'investigations', authoritarian pedagogy, assessment strategies.

Robyn Zevenbergen places constructivism in a social and political context and draws the conclusion that it is a 'liberal bourgeois discourse' which serves to legitimise the dominance of powerful social groups (Zevenberger 1996). Michael Apple sees a similar problem:

> Most discussions of the content and organisation of curricula and teaching in areas such as mathematics have been strikingly internalistic. Or, where they do turn to "external" sources other than the discipline of mathematics itself, they travel but a short distance - to psychology. . . though it has brought some gains . . . it has, profoundly, evacuated critical, social, political, and economic considerations from the purview of curriculum deliberations. In the process of individualising its view of students, it has lost any serious sense of the social structures and the race, gender and class relations that form those individuals.
>
> (Apple 1995, p. 331)

Agency and structure: the psychological and the social

This leads us into perhaps the central problem of modern social theory: the relationship between human agency and social structure. (Archer 1988, p ix)
The essence of the issue here is given by Marx:

> Men make their own history, but they do not make it just as they please; they do not make it under circumstances chosen by themselves, but under circumstances directly encountered, given and transmitted from the past.
>
> (Marx 1852)

In this comment, Marx sums up the tension in acting in a social world - pre-empting social constructionism. Social life is a product of active subjects, working within constraints which they may chose to ignore. Some are self imposed, others imposed from without. I see a form of recursion here. Society is not merely individuals working within an imposed and constraining social structure. Rather it is a dynamic system in which structure is both formulated by and imposed upon actors. More recently Anthony Giddens puts it this way:

> All human action is carried out by knowledgeable agents who both construct the social world through their action, but whose action is also conditioned and constrained by the very world of their creation.
>
> (Giddens 1981, pps 53 - 54)

Embedded here is the argument for a reorientation in the psychology of mathematics education - to see the frameworks and theories we use as politically located, and legitimising particular social norms. Arguments that place psychology outside of or above social structure are no longer tenable.

> Structure is not 'external' to individuals: as memory trace, and as instantiated in social practices, it is in a certain sense more 'internal' than exterior to their activities in a Durkheimian sense. Structure is not to be equated with constraints but is always both enabling and constraining.
>
> (Giddens 1984, p 25)

This is a helpful development, and for me brings together Marxian perspectives of consciousness, Bourdieu's notion of habitus and Foucault's notion of power. In order to develop the nature of mathematics pedagogy then we need to adopt a perspective which explores the social structure of society and the roles played by teachers, learners and theories of learning. In addition it requires us to recognise the existence and nature of oppression, and how this comes about both through social stratification and human practices. What is important is to look at then is:

> How it comes about that structures are constituted through action and reciprocally, how action is constituted structurally.
>
> (Giddens 1976, p 11)

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MEANING OF PROOFS IN MATHEMATICS EDUCATION [1]

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Abstract

The main characteristics of the meaning of proof in different institutional contexts -logic and foundations of mathematics, professional mathematics, empirical sciences, daily life, and school mathematics- are analyzed. Consequently, the necessity of inserting the study of the epistemological and didactic problems posed by the teaching of proof in mathematics classrooms within the more general framework of human argumentative practices is deduced.The superposition is also observed at different teaching levels for the different institutional and mathematical meanings of proof, which might explain some students' difficulties and cognitive conflicts.

1. Introduction

Growing interest in the problems of the teaching and learning of proof is presently to be found within Mathematics Education (Hanna, 1995; 1996). Research publications on this subject have increased over the last five years (see the special issue in *Educational Studies in Mathematics*, 1993), though we also find earlier relevant contributions (Lester, 1975; Bell, 1976; Fischbein, 1982; Balacheff, 1987; etc.)

This interest is justified by the essential role of validation situations within mathematics, and by the students' poor level in understanding and building mathematical proofs (Senk, 1985; Recio and Godino, 1996; Harel and Sowder, in press).

In spite of the aforementioned research, there is still room for research into clarifying the meaning of mathematical proof, its different types and mutual relationships. In particular, the idea of demonstration ,which is understood in a rigid and absolute way by the mathematical community, seems to be the sole, valid conception. We consider it necessary to carry out a systematic study of the various meanings of proof, not just from the subjective point of view, but also in the different institutional contexts. This study would allow for a comparison of the different research contributions, posing new investigation problems, alternative interpretations of students' difficulties and elaborating new didactic proposals.

[1] This report has been founded by DGES, MEC. Project PS93-0196.

In this research report, we analyze the different meanings that the idea of proof takes in different institutional contexts, using the theoretical framework by Godino and Batanero (1994) and Godino (1996), concerning mathematical objects and their meanings. Here, this implies taking the validation situations and the corresponding argumentative practices as primitive notions. Proof notions emerge from argumentative practice systems. We, furthermore, distinguish between personal and institutional dimensions thereof.

2. Situations of validation and argumentative practices

The word 'proof' is used with various senses in different contexts. Sometimes these various senses are recognized through terms such as 'explanation', 'argumentation', 'demostration', etc. Though in all theses cases there is a common idea, - that of justifying or validating a statement (thesis) by providing reasons or arguments -, in fact, the differences in the types of situations in which they are used, their characteristic features and the expressive resources used in each case can be different. These changes in situations and argumentative practices suggest different senses of the concept of proof, e.g., various "object proof" according to our ontosemantic model.

In this paper we shall use the term 'proof' to refer to the objects emerging from *argumentative practices* (or *arguments*) *systems* accepted at the heart of a community, or by a person, in validation and decision situations. That is to say, situations that require justifying the truth of a statement, or the efficacy of an action.

An important distinction between *analytical and substantial arguments* is made by Krummheuer (1995), following Toulmin. Analytical arguments, characteristic of correct logic deductions are tautological. That is, a latent aspect of the premises is visibly elaborated, but they add to the conclusion nothing more than what is already a potential part of the premises. Substantial arguments, on the contrary, expand the meaning of the propositions to the extent to which they adequately relate a specific case by means of updating, modification, and/or application.

From a cognitive viewpoint, we consider that the relationships between *reasoning* and argumentative practices are those established between a construct and its empirical indicators. Balacheff (1987, p. 148) defines the reasoning as the "*very often not explicit intellectual activity of manipulating information to produce new information from data*". From our perspective, this intellectual activity gives rise to personal or institutional argumentative practices, which constitutes its

ostensive dimension. Simultaneously, reasoning is developed by means of such practices, so that the study of reasoning is intrinsically linked to the study of argumentation.

In the next section we shall show that a proof is a contextual and pragmatic attribute of a discursive practice.

3. Meanings of proof in different institutional contexts

From a cultural viewpoint, Wilder (1981) wote that, "*we must not forget that what constitutes 'proof' varies from culture to culture, as well as from age to age*" (p. 346). We are trying to show that this relativity must be widened to different institutional contexts, when we are interested in psychological and didactic problems involved in the teaching of proof.

We consider that a context or institutional framework is a local viewpoint or perspective concerning a given *'problématique'*, characterized by using expressive resources and specific tools, as well as habits and specific behavior procedures. In this section, we shall study the diversity of proofs according to the following institutional contexts: logic and foundations of mathematics, professional mathematics, daily life, empirical sciences and the teaching of elementary mathematics (including primary, secondary and university levels). We have to recognize that in each of these contexts it is also possible to identify more local viewpoints in which the problem of truth and proof takes on specific connotations. However, we consider that the level of analysis adopted in this paper is sufficient to show the diversity of identificable 'object proof', and in particular that there is no uniform theory and practice firmly established about mathematical proof.

3.1. Logic and foundations of mathematics

In these contexts, the veracity of a theorem rests on the validity of the logic rules used in the proof; the theorem appears as a logical and necessary consequence of the premises, through the corresponding deductive inference. A statement (or theorem) accepted as true has a universal and intemporal validity.

It is also important to emphasize the nature of the problematic situations that are faced in these contexts. The aim of the validation process is to justify, with the maximum guarantees, the truth of the system of mathematical propositions, or at least part thereof. This implies looking for the minimal independent, non contradictory and complete system of axioms (self-evident truths), such that the other mathematical propositions may be derived by applying the inference logic rules. Hence, it deals with the theoretical problem of organizing and structuring the

system of mathematical knowledge. The use of formal languages is required to achieve the greatest guaranties and rigor in this work.

The 'object proof' in these institutional contexts may be synthetically described as emerging from the system of *analytical argumentative formal practices*, and its meaning is given by the intensional, extensional and representational characteristics described.

Nevertheless, we recognize that substantial argumentations are also used to justify some statements even in these institutional contexts. In any mathematical system, the acceptance of axioms or postulates is necessarily reached through intrinsically inductive arguments. Let's remember what Poincaré (1902)wrote :

> *"What is the nature of mathematical reasoning? Is it actually deductive as it is ordinarily believed? A deep analysis shows us that it is not so; it participates to some extent in the nature of inductive reasoning, and that is why it is productive"* (p. 15)

3.2. Professional mathematics

As regards to the real practice of mathematics, the notion of proof clearly differs from formal logic and foundation studies in mathematics .

Formal proofs become extraordinarily complex, which in practice makes the complete formalization of proofs in many mathematic investigations impossible, even when it would be feasible, in principle.

> *"They may require time, patience, and interest beyond the capacity of any human mathematician. Indeed, they can exceed the capacity of any available or foreseeable computing system"* (Hersh, 1993, p. 390).

As asserted by Resnick (1992), this makes contemporary mathematics full of "working proofs", i.e., informal and non axiomatized proofs.

In the field of professional mathematic, proofs are deductive but not formal. They are expressed through ordinary language completed with symbolic expressions. There is no generally accepted standard of rigor for systemizing mathematical proof.

In this way, mathematical theorems in fact lose their character of absolute and necessary truths. Real mathematics acquires a falibilist, social, conventional, and temporary character. This situation induces us, in real mathematical practice, to describe proof as a *'convincing argument, as judged by qualified judges'* Hersh (1993, p. 389).

The problem faced by professional mathematicians is to solve new problems, to increase the knowledge body, and, secondarily, to organize and found the whole

system of mathematics. The highest degree of safety of the work carried out by people interested in the foundations of mathematics is not required.

3.3. Experimental sciences and daily life

Proof, in these contexts, is mainly based on substantial arguments (empirical inductive, analogical, etc.) from which we conclude that what is true for some individual in one class is true for all the members of that class, or that what is sometimes true, will be true in similar circuntances, or with a given probability. The simultaneous use of deductive arguments, in particular statistical inferences, is not discarded:

- the validity of the statements does not have a universal and absolute character;
- their validity is increased when more facts supporting the statement are shown or produced;
- an example that is not fulfilled does not thoroughly invalidate the sentence.

Proof uses the expressive resources of ordinary language, symbols and any type of concrete devices.

In the experimental sciences, experiments or observations are made with maximum care, controlling all possible factors that might influence the results. They also use symbolic resources.

Reasoning by analogy plays an important role in natural reasoning showed in our daily inferences. All analogical inferences start from the similarity of two or more things, concerning one or more aspects, concluding with the similarity of those things in another aspect.

3.4. Proof in the mathematics classroom

As a rule, mathematical theorems are necessarily true for secondary and university level curricula, textbooks and mathematics teachers. But arguments establishing their truth are frequently informal-deductive, not deductive, or they are even based on external authority criteria.

Elementary mathematics -including mathematics at university courses - is a knowledge whose truth is considered to be completely certain. There are some proofs for theorems accepted by the generality of the professional mathematicians. Therefore, this knowledge has not the falibilist character attributed to advanced mathematics, or at least, is presented in this way in textbooks and in mathematics classrooms.

In these institutional contexts, particularly at the higher levels, students are expected to acquire the capacity of understanding and carrying out mathematical

proofs, to establish the truth of theorems with absolute safety, and to convince themselves and any person of such unquestionable truth.

This is an idiosyncratic use of proof, different from what is done by professional mathematicians. Mathematicians must develop proofs to convince referees for journals; mathematics students must convince themselves, and convince the teacher of the necessary and universal truth of theorems.

4. Personal meanings of proof

The process employed by a person to suppress doubts about the truth of a conjecture is called *proof scheme* by Harel and Sowder (in press): "*A person's proof scheme consists of what constitutes ascertaining and persuading for that person*" (p. 12). The different categories of proof schemes they identify represent a cognitive stage, an intellectual ability in students'mathematical development, and are derived from the actions taken by the students in their process of proving.

In the ontosemantic model developed by Godino and Batanero (1994), these proof schemes could be *personal* or *mental objects,* and their meanings are the systems of practices carried out by the person involved in decision and validation situations.

Harel and Sowder distinguish three main proof scheme categories: based on external convictions (ritual, authoritarian and symbolic), empirical (inductive and perceptual) and analytical (transformational and axiomatic).

For these authors, the high incidence of the three subtypes based on "external convictions" and of empirical-inductive proof schemes in the students could be explained by the influence of school habits, which reinforce such types of argumentative practices.

The analysis presented in the previous sections suggests, indeed, that within elementary mathematics classes argumentative, not analytical practices, may prevail, mainly at primary and secondary school teaching levels. These arguments - unconsciously implemented by mathematics teachers - might be extrapolated from other institutional contexts, such as daily life or empirical sciences.

Furthermore, the role played by substantial argumentation in the phases of searching and formulating conjectures in problem-solving should not be forgotten. Analytical arguments, characteristic of mathematical proof, are not the only argumentative practices of professional mathematicians to convince themself about the truth of their conjectures. This form of reasoning is frequently sterile, even an obstacle, in the phases of creation and discovery in problem-solving, where forms of substantial argumentation, in particular empirical induction and analogy, are

allowed and even necessary. We may recall the words of Polya (1944, p. 116): *"Mathematics presented with rigor is a systematical, deductive science, but mathematics at the embryo stage is an experimental, inductive science".*

5. Conclusions and implications for research and teaching

Certainly, we may appreciate some common features in the uses of the word 'proof' in the different institutional contexts described. This allows us to think about proof in a general sense. But this generic, abstract, metaphysical way of thinking, should not conceal the rich and complex variety of meanings acquired by the concept of proof, or, better, by the diversity of 'object proof' each one of them exists with a local meaning for the members of such institutions. We believe it is interesting to consider that there is not just a single concept of proof but several, depending on the subjective and epistemological viewpoint, when we are interested in the psychological and didactic problems involved in the processes of validating mathematical propositions (Godino and Batanero, 1994).

By recognizing this diversity of objects and meanings, we shall be in a better position to study the components of meaning, the circumstances of their development, the roles performed in the different contexts. In fact, we would better understand the *ecological relationships* established between objects and the systemic nature of their meaning. This ontosemantic modelization can help to take into account the cognitive conflicts posed to each person forced to participate as a subject in different institutional contexts.

Since students are simultaneously subjects of different institutions, at the heart of which different argumentative schemes are carried out, it seems reasonable that students may have difficulties in discriminating the respective use of each type of argumentation. Consequently, we consider that such institutional proof schemes might be explanatory factors for subjective schemes, and therefore they should be taken into account and investigated in depth.

It is necessary to somehow articulate the different meanings of proof, at different teaching levels, thereby developing progressively among the students the knowledge, discriminative capacity and rationality required to apply them in each case. Informal proof schemes cannot just be considered to be incorrect, mistakes or deficiencies, but rather as stages in achieving and mastering argumentative mathematic practices.

Understanding and mastering deductive argumentation by students require the development of a rationality and a specific state of knowledge. It demands "*the adhesion to a problem that it is not that of the efficiency (exigency of practice) but*

rather that of rigor (theoretical exigency) (Balacheff, 1987, p. 170). But the construction of this rationality is a progressive process that requires time, as well as ecological adaptations of the 'object proof' (didactic transpositions) at different teaching levels.

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A VISUAL PRESENTATION OF A LONGITUDINAL STUDY: DESIGN AND ANALYSIS

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This paper discusses the presentation and analysis of students' perceptions of a concept as they interact with different dynamic representations made available through computer environments. The diagrams used as tools in this process represent a historical sequence and aim to show the limitations in students' perceptions, their generality and the way they are interconnected.

One of the problems analysing a longitudinal study of students interacting with different representations of a concept is how to capture students' progress through the environments. A methodology was developed (Gomes Ferreira, 1997)[1], while investigating students' perceptions of function as they interacted with the different dynamic representations made available through computer environments. This culminated in a visual presentation of the evolution of perceptions of a concept — the blob diagram which is the subject of this paper.

Brief description of the research

In this research project, a selection of properties of function (range, periodicity, variation, turning point and symmetry) was distinguished. The study sought to analyse how students come to discriminate, generalise, and synthesise these properties while working with software in activities designed to encourage exploration of the dynamic features of the programs.

Two software programs which exploit the possibilities of computers to explore representations of functions by continuous movement were selected: DynaGraph (DG) (Goldenberg et al, 1992) and Function Probe (FP) (Confrey et al, 1991a). DynaGraph allows students to vary-the-variable of a function in a visual representation and to observe the variation of its image. Function Probe allows continuous and direct transformations of graphs, which change the status of the Cartesian system into an action representation (Kaput, 1992). Thus, the research was designed specifically to investigate how the dynamic tools of DynaGraph and Function Probe might structure students' perceptions of function properties. Both programs were used in the creation of microworlds consisting of the software tools and a set of activities.

The design of the microworlds involved: the selection of four families of functions (constant, linear, quadratic and sine functions) from which twelve functions were

[1] This research was sponsored by CAPES and developed at the Institute of Education.

chosen which highlighted the properties through exploring with the software; elaboration of activities of description/guessing and classification of the functions which led to developing a language for discussion; adaptations of DynaGraph, DG Parallel (with x-axis and y-axis disposed in parallel) and DG Cartesian (with Cartesian system) to enable exploration of the selected functions without the students having access to the corresponding equations. In both adaptations, the activities involved a game where the variable y was represented by a striker. For a complete description of the microworlds, see Gomes Ferreira (1997).

In order to investigate the use of these microworlds against a background of the Brazilian curricula, the study was undertaken with four pairs of Brazilian students who had already studied functions at school. A pre-test and an analysis of the school approach to functions served as starting points. Both focused on the chosen properties, revealed students' previous perceptions and pointed some obvious potential over-generalisations and barriers. These were compared to the range of epistemological obstacles revealed during the research activities.

The pairs of students followed the microworld activities in two different sequences: two pairs did the activities in both DG Parallel and DG Cartesian followed by the activities in FP, and the other two pairs followed the activities in the opposite order.

By working with multiple representations of function, the study investigated how the students came within each of the microworlds to discriminate and generalise each of the function properties. It also investigated the syntheses made between perceptions derived from activities in different microworlds and those constructed in school. A final interview was undertaken to investigate how students made link during the activities as well as to motivate new links if possible.

A longitudinal analysis was undertaken tracing the evolution of students' perceptions of the function properties while interacting with the microworlds, giving consideration to the origins of these perceptions and the set of functions to which these perceptions could be applied from a mathematical viewpoint. This analysis attempted to identify the main aspects of each of the microworlds which appeared to contribute to the students' progress.

Representations and concepts

Schwarz & Bruckheimer (1988) argue that "Although the concept of function and its subconcepts are not theoretically linked to a particular representation ... the properties of a function are often understood in their representational context only and no abstraction of these properties is made by the beginning students" (p.552). This argument shows the unfeasibility of disconnecting concepts from their representations. An alternative notion of concept offered by Confrey et al (1991b) takes the position that: "representations and ideas are inseparably intertwined. Ideas are always represented, and it is through the interweaving of our actions and representations that we construct mathematical meaning" (p.17). In an approach

which considers that conceptual understanding arises from making connections across different representations (see Noss & Hoyles, 1996), the main interests were: to investigate the characteristics of each perception as articulated within different representations and to investigate whether the use of multiple representations leads to some convergence across representations

The research used the ways students described functions as evidence of their understanding of the function properties and the different perceptions as revealed in the interactions with the software can be interpreted as a map of students' understanding of the concept.

A model to analyse students' perceptions

Researchers have been working with a model to analyse students' understanding which classifies the acts of understanding into four categories (Hoyles & Noss, 1987; and Sierpinska, 1992): Using, Discriminating, Generalising and Synthesising. 'Using' is the act of using a concept as a tool for the functional purpose of achieving particular goals. 'Discriminating' is the act of explicating different parts of the structure of a concept. 'Generalising' is the act of extending the range of applicability of these parts. In the process of generalising, new aspects of the structure of a concept are discovered. Finally, 'Synthesising' is the act of integrating different representations of the same knowledge in different symbolic forms derived from different domains into a whole. Thus, conceptual understanding arises from making connections across different domains.

The research adopted three of the phases — DGS. It investigated these perceptions through different representations embodied in different microworlds, and the analysis needed a model which could categorise acts of perceiving within and between representations. As the study examined different properties of functions, the model could not be linear model. DGS is not linear and the categories are not necessarily followed in ascending order. Rather, it is spiral considering that students can be working simultaneously in different categories depending on the property as well as the representation. The research tried to trace the path of students' perceptions of each function property as revealed in the interactions with the three different microworlds.

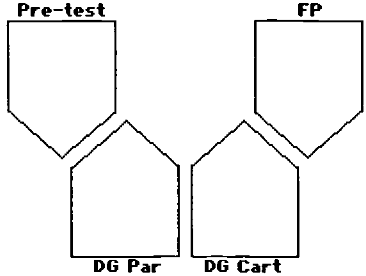
Construction of a blob diagram

The longitudinal analysis was divided into three phases. First, a summary of students' previous knowledge was made from the analysis of their pre-tests. Second, the students' perceptions of the property constructed during their interactions in the research environment was examined, in particular how and when they came to discriminate, generalise, associate, and spontaneously synthesise their different perceptions of the properties. Finally, the connections motivated in the final interview were identified. From this analysis a report of students' perceptions was created for

each property, together with a table of the students' perceptions of the property for each function in each microworld. All these analyses were brought together in a blob diagram.

The blob diagram is an adaptation of the one used by Hoyles & Healy (1996) which presented information about a longitudinal approach. Here, an improved version will be presented to illustrate the perceptions of monotonicity developed by Jane & Anne, one of the pairs of student subjects in the research.

Diagram 1 — Disposition of the pentagons in Jane & Anne's blob diagram



The interactions in each microworld (and pre-test) are summarised in one pentagon. The pentagons are displayed to allow two microworlds to be linked without passing through a third microworld and to keep the sequence of the microworlds. Diagram 1 shows the disposition of the pentagons for Jane & Anne, who followed the activities from DG to FP.

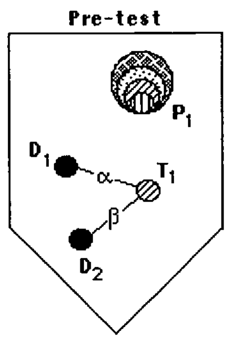
Each perception evidenced in the report was represented by one blob which is shaded. As a topological diagram the position of each blob inside a pentagon has no meaning. The shade of each blob indicates the families of functions to which the students discriminated and generalised the perception:

- Constant functions
- Linear functions
- Quadratic functions
- Sine functions
- The set of function was not clearly identified — a general perception.

The blobs are labelled to differentiate perceptions as follows:

T - referring to a term;
D - defining a term (in the pre-test only)
P - characterising functions.

Diagram 2
Pentagon of the pre-test

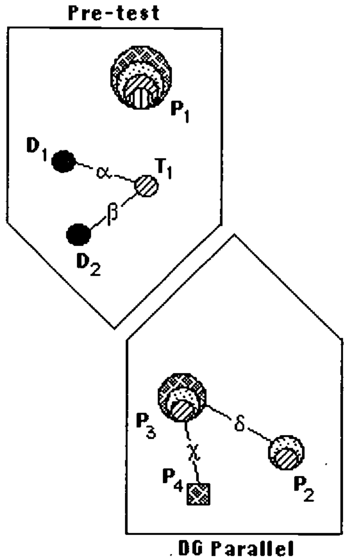


Anne defined the term 'increasing' by "when a>0" referring to linear coefficient. Thus, two blobs are placed inside the pre-test pentagon. One for the terms 'increasing' and 'decreasing' labelled T_1, and one for "when a>0" labelled D_1. Both blobs were linked by a line labelled α which represents this connection (see diagram 2). Connections between different perceptions are shown by lines linking the blobs. Connections were only represented if there was clear evidence from the transcripts that a link had been made by the students. Each link is denoted by a Greek letter to facilitate reference in the text.

Jane's definition emphasised polarisation when analysing graphs: "increasing is a function that reaches positive value at the system (y>0)" (D_2) (see link β). Despite being given as general perceptions, both definitions are in fact valid only for linear functions, thus, diagram 2 presents the shade of blob T_1 for linear functions.

Despite the fact that these definitions were linked only for linear functions, Jane & Anne were also able to interpret any graph in a variational way (P_1) when they were asked about the behaviour of y when x increased. The generality of this perception is represented in diagram 2 by inscribing blobs into blobs. The separation between P_1 and the other blobs highlights the evidence that the students used T_1 only for linear functions.

Diagram 3
Pentagon of DG Parallel



In DG Parallel, Jane & Anne developed a variational perception of monotonicity articulated within the microworld — 'when x is positive, y follows x or y doesn't follow x' (P_2) — to linear functions and then to generalise to parabolas (see diagram 3). Generalisations of a perception are represented by inscribing one blob inside another. The rule 'when x is positive, y is positive' (P_3 in diagram 3) was used by the students to recognise monotonicity for linear functions. P_3 was also generalised among quadratic functions and sine functions. Link δ was included because P_3 was a result of the students' attempt to generalise P_2 while analysing the function of y=x-6. Jane replaced P_2 for P_3 which involves a polarisation of knowledge.

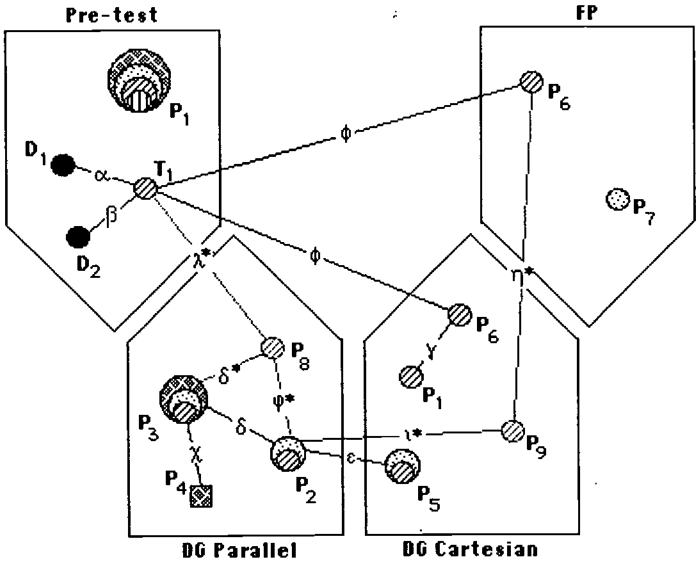
The analysis of the sine functions in DG Parallel also led the students to revise P_3. However, they only classified the striker of a sine as "the striker changes many times" (P_4) (see link χ). Note that P_4 has no direct correspondence to monotonicity from a mathematical viewpoint. Blobs are divided into two types: circles and squares. Perceptions without correspondence with the property from a mathematical viewpoint are represented with squares, instead of circles. They also present shades.

Diagram 3 clearly shows the separation between knowledge from the pre-test and those built in DG Parallel. It also shows two kinds of perceptions the students had of monotonicity. The first is connected with the term 'increasing' which reflects their previous knowledge about monotonicity. The second group of perceptions is variational.

Reflecting this separation, in DG Cartesian Jane & Anne presented two distinct sets of perceptions (see diagram 4). The perception P_2, which was articulated within DG

Parallel, was brought to DG Cartesian. On analysing the behaviour of x and y only, they changed 'y follows x' into "when x is going to positive, y is going to the positive" (P_5) (see link ε). The second set of perceptions (P_6 and P_1) was linked with their previous perception of monotonicity by the direction of the straight line traced by (x,y) (P_6) (see link ϕ). Although the perception of the term 'increasing' was confined to linear functions, the students gave a variational interpretation (P_1) for the property in DG Cartesian (see link γ). This perception was also presented in the pre-test, but there, P_1 was not linked to the term 'increasing'. That is the reason to repeat the label P_1. Same perceptions in different microworlds are given the same label. Note that link γ reduced the sample in which the students generalised P_1 to linear functions which is very clear in diagram 4 by the shades of P_1. This confirms the existence of an obstacle while using the term 'increasing' — a terminology used in school mathematics.

Diagram 4[2] — Addition of the findings from DG Cartesian to the final interview



2

T_1 - Terms 'increasing' and 'decreasing'

P_1 - Read through a graph the behaviour of y while x increases

P_2 - When x is positive, 'y follows x' or 'y doesn't follow x'

P_3 - Rules like: when x is positive, y is positive

P_4 - When x is positive, y changes orientation

P_5 - Rules like: when x is going to the positive side, y is going to the negative side

P_7 - Graph stops growing and starts decreasing

P_8 - 'y follows x' or 'y doesn't follow x'

P_9 - Straight lines is recognised by y moves in only one orientation

D_1 - $a>0$

D_2 - Graph reaches the positive ($y>0$)

In FP the students used 'direction of straight lines' (P_6) to recognise whether a linear function was 'increasing' or 'decreasing' . Note that it is a similar link to the one already discussed between perceptions in DG Cartesian and the pre-test, thus, it is also labelled by ϕ. As in the case of the perceptions, similar connections are represented by the same letter. As the link was evidenced only with the term not with P_6 in DG Cartesian, they were not linked in the diagram despite being the same label and having an indirect connection.

Once more a variational perception of monotonicity (P_7) when applied to non-linear functions were not connected to the terms 'increasing' and 'decreasing' from school knowledge. While exploring extreme values in FP, Jane & Anne interpreted the graph of $y=-0.25x^2$ as increasing or decreasing (P_7).

Diagram 4 shows two compartmentalised sets of perceptions: one linked to the terms but restricted to linear functions, other more generalisable but disconnected from the school term.

As the final interview investigates the connection between different perceptions in different microworlds, their blobs and links had to be added in the pentagons. The lines and the blobs have two colours to distinguish the perceptions and connections built spontaneously while working with the microworlds and pre-test and motivately in the final interview: black and grey. The black ones were used for spontaneous perceptions and links while the grey ones are for the motivated ones. Motivated links are distinguished by an asterisk as a visual aid in the text. For example, Jane & Anne connected 'direction of a straight line' (P_6) to 'y follows x' or 'y does not follow x' (P_2 and P_8) (see links ι^*, φ^* and η^*). They also connected this perception to the term 'increasing' or 'decreasing' restricted to straight lines (see link λ^*). These two connections passed through an association (P_9). In order to achieve the above-mentioned syntheses, the students identified 'strikers that move in only one orientation' as being 'straight lines'.

On trying to generalise the connection to the striker of $y=-0.25x^2$, they used the rule (P_3) 'when x is positive, y is positive' meaning 'y follows x' (see link δ^*). This is similar to link δ but elaborated in the final interview, thus, it received an asterisk.

Conclusion

The main assertion of this paper is that the construction of the blob diagram served two purposes: it was a tool for analysis and helped to identify points of development as well as a means of presentation of the longitudinal analysis of students' interactions with different microworlds. Studying the diagrams for different students showed visualisation of:

- isolated perceptions;
- continuity in the process of constructing an idea;
- revision and generalisation of perceptions;
- connections between perceptions expressed in different microworlds;

- how one microworld (DG Cartesian in the case) served as a bridge between perceptions in other microworlds;
- difficulty of perceiving a property in a microworld;
- dominant perceptions;
- the path traced through the sequences of microworlds;
- how some perceptions were blocked by others.

The diagram also presents a historical analysis which includes perceptions from the pre-test to the final interview for which a post-test could be substituted. Moreover, its design is easily adaptable to more microworlds or settings of further studies. The use of this diagram allowed Gomes Ferreira (1997) to extract the main points of students' perceptions from the detailed analysis of transcripts derived from her study.

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