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ABSTRACT

Research reports from the annual conference of the International Group for the Psychology of Mathematics Education include: "A Comparison of Children's Learning in Two Interactive Computer Environments" (Edwards); "On Building a Self-Confidence in Mathematics" (Eisenberg); "Classroom Discourse and Mathematics Learning" (Ellerton); "Constructivism, the Psychology of Learning, and the Nature of Mathematics" (Ernest); "Cognition, Affect, Context in Numerical Activity among Adults" (Evans); "Teachers' Pedagogical Knowledge: The Case of Functions" (Even; Markovits); "Cognitive Tendencies and Abstraction Processes in Algebra Learning" (Filloy-Yague); "On Some Obstacles in Understanding Mathematical Texts" (Furinghetti; Paola); "Toward a Conceptual-Representational Analysis of the Exponential Function" (Goldin; Herscovics); "Duality, Ambiguity and Flexibility in Successful Mathematical Thinking" (Gray; Tall); "Children's Word Problems Matching Multiplication and Division Calculations" (Greer; Mc Cann); "Children's Verbal Communication in Problem Solving Activities" (Grevsmuhl); "The 'Power' of Additive Structure and Difficulties in Ratio Concept" (Grugnetti; Mureddu Torres); "Why Modeling? Pupils Interpretation of the Activity of Modeling in Mathematical Education" (Gortner; Vitale); "A Comparative Analysis of Two Ways of Assessing the van Hiele Levels of Thinking" (Gutierrez; Jaime; Shaughnessy; Burger); "A Procedural Analogy Theory: The Role of Concrete Embodiments in Teaching Mathematics" (Hell); "Variables Affecting Proportionality: Understanding of Physical Principles, Formation of Quantitative Relations, and Multiplicative Invariance" (Harel; Behr; Post; Hesh); "The Development of the Concept of Function by Preservice Secondary Teachers" (Harel; Dubinsky); "Monitoring Change in Metacognition" (Hartl); "The Use of Concept Maps to Explore Pupils' Learning Processes in Primary School Mathematics" (Hasemann); "Adjusting Computer-Presented Problem-Solving Tasks in Arithmetic to Students' Aptitudes" (Hativa; Pomeranz; Hershkovitz; Mechmandarov); "Computer-Based Groups as Vehicles for Learning Mathematics" (Heal; Pozzia; Hoyles); "Pre-algebraic Thinking: Range

of Equations & Informal Solution Processes Used by Seventh Graders Prior to Any Instruction" (Herscovics; Linchevski); "LOCI and Visual Thinking" (Hershkowitz; Friedlander; Dreyfus); "Two-step Problems" (Hershkowitz; Nesher); "Evaluating Computer-Based Microworld: What Do Pupils Learn and Why?" (Hoyles; Sutherland; Noss); "Inner Form in the Expansion of Mathematical Knowledge of Multiplication" (Ito); "Some Implications of a Constructivist Philosophy for the Teacher of Mathematics" (Jaworski); "Teachers' Conceptions of Math Education and the Foundations of Mathematics" (Jurdak); "Games and Language-Games: Towards a Socially Interactive Model for Learning Mathematics" (Kanes); "Translating Cognitively well-organized Information into a Formal Data Structure" (Kaput; Hancock); "A Procedural-Structural Perspective on Algebra Research" (Kieran); "Consequences of a Low Level of Acting and Reflecting in Geometry Learning--Findings of Interviews on the Concept of Angle" (Krainer); "The Analysis of Social Interaction in an 'Interactive' Computer Environment" (Krummheuer); "Can Children Use the Turtle Metaphor to Extend Their Learning to Include Non-intrinsic Geometry" (Kynigos); "Pre-schoolers, Problem Solving, and Mathematics" (Leder); "La fusee fraction. Une exploration inusitee des notions d'equivalence et d'ordre? (Lemerise; Cote); "Critical Incidents' in Classroom Learning--Their Role in Developing Reflective Practice" (Lerman; Scott-Hodgetts); "Human Simulation of Computer Tutors: Lessons Learned in a Ten-Week Study of 20 Human Mathematics Teachers" (Lesh; Kelly); "Advanced Proportional Reasoning" (Lin); "Rules without Reasons as Processes without Objects--the Case of Equations and Inequalities" (Linchevski; Sfard); "Everyday Knowledge in Studies of Teaching and Learning Mathematics in School" (Lindenskov); "The Knowledge about Unity in Fractions Tasks of Prospective Elementary Teachers" (Llinares; Sanchez); "Describing Geometric Diagrams as a Stimulus for Group Discussions" (Lopez-Real); "Pupils' Perceptions of Assessment Criteria in an Innovative Mathematics Project" (Love; Shiu); "Developing a Map of Children's Conceptions of Angle" (Magina; Hoyles); "The Construction of Mathematical Knowledge by Individual Children Working in Groups" (Maher; Martino); "The Table as a Working Tool for Pupils and as a Means for Monitoring Their Thought Processes in Problems Involving the Transfer of Algorithms to the Computer" (Malara; Garuti); "Interrelations Between Different Levels of Didactic Analysis about Elementary Algebra" (Margolinas); and "Age Variant and Invariant Elements in the Solution of Unfolding Problems" (Mariotti). (MKR)

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CONTENTS OF VOLUME II

A COMPARISON OF CHILDREN'S LEARNING IN TWO INTERACTIVE COMPUTER ENVIRONMENTS

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The nature of middle- and high school students' learning in two interactive computer environments for mathematics was investigated. Each environment was concerned with a different mathematical topic (in one case, graphing equations and in the other, transformation geometry), yet they shared a similar underlying pedagogical structure. It is proposed that this structure helped to support the same kind of learning in children working in the different environments; specifically, the students used visual feedback from a game-like situation to build and "debug" their understanding of graphs and transformations, respectively.

Introduction

Computer-based instruction in mathematics and science has taken many forms, and widely-differing instructional philosophies have been used, explicitly or implicitly, in designing educational software (Edwards, 1986; Newman, 1985; White, 1981). In many cases, the differences in underlying assumptions about the nature of learning are so great that the only thing educational software packages have in common is the very fact that they are implemented on a computer. Early examples of computer-based instruction were in most cases based on the "transmission" metaphor for learning, or even on behaviorist principles. The information to be learned was programmed into the computer via a sequence of scripted interactions (a "tutorial"), or was presented as a series of drill-and-practice problems. Feedback on the students' attempts to solve problems was generally given immediately. However, as in traditional classroom teaching, these corrections were handed down by an external authority. There was no way that learners could use their own abilities to correct their own work. Von Glasersfeld has noted that this kind of correction is not completely satisfactory:

> From the constructivist point of view, it makes no sense to assume that any powerful cognitive satisfaction springs from simply being told that one has done something right, as long as "rightness" is assessed by someone else. To become a source of real satisfaction, "rightness" must be seen as the fit with an order one has established *oneself*. (Von Glasersfeld, 1987, p.15)

Another feature of early computer-based instruction, and one that is still found in many popular packages, was the use of "rewards" or reinforcements for good performance by the students. Such rewards were often completely unrelated to the material being presented, and instead consisted of colorful displays or "exciting" sound effects. Dugdale (1981) has pointed

out the deficiencies of this kind of feedback, and has stated a number of principles which should be followed in designing learning environments for mathematics. A paraphrase of these principles would include the following recommendations:

- The environment should consist of a "working model" of the concepts to be learned, in which the mathematics is intrinsic. Students should be able to explore and manipulate this working model.
- Direct, meaningful feedback should be provided which the learners themselves can interpret in order to diagnose and correct their own errors.
- The environment should include a set of inherently-interesting problems which can be explored by students of varying abilities and inclinations.

The research reported here focuses on two pieces of educational software for mathematics, one created by Dugdale and one by the author, whose design followed the principles listed above. Thus, the two environments share a common pedagogical structure and, it is argued, therefore support a similar kind of learning, even though each addresses a different mathematical topic. These two computer environments can be classified as members of an emerging class of interactive learning environments, sometimes called "microworlds," which have the potential to support a constructive kind of learning (Papert, 1987; Thompson, 1987; Edwards, 1990).

The goal of both of the studies was to investigate how children interacted with the microworlds, and how the game-like situations which were central to each environment supported their mathematical learning. Specific questions included:

- What kinds of information do the students attend to when playing the games?
- What strategies do they use in trying to achieve their goals?
- Are there changes in the students' goals over the course of interacting with with the microworlds?
- Are there common patterns of interaction and learning, even though the microworlds address different mathematical content?

It was hypothesized that, indeed, there are common patterns of learning and interaction, and that the students would make use of visual feedback from the environments to refine and correct their mathematical understandings while playing the games. This report will briefly describe each environment, summarize the results of two qualitative studies of children's mathematical learning while engaged with the software, and discuss the relationship between the common pedagogical structure of the microworlds and similarities in the nature of the children's learning.

The Computer Environments

The first study used the Graphing Equations software (created by Sharon Dugdale and David Kibbey; see Dugdale, 1981), which is commonly called "Green Globs." The software links the symbolic representation of an algebraic equation with its graph. Students may type in any equation, and the graph is drawn for them on a coordinate grid. The game, Green Globs, presents a set of large dots scattered over the grid; the purpose is to enter equations which will pass through as many dots as possible. An illustration of a Green Globs screen, in the middle of a game, is found in Figure 1, in the Results section of this paper.

The second study used a Logo-based microworld for transformation geometry, called TGEO (for full details, see Edwards, 1989). Again, in this environment, the students may enter a symbolic representation, this time specifying any of a set of geometric transformations such as rotations, reflections and translations, and see the effect of the transformation displayed visually. The initial set of euclidean transformations had the following syntax: SLIDE 10 30 (translate 10 turtle steps horizontally and 30 vertically); ROTATE 25 25 90 (rotate the entire plane around a center point at (25,25) through an angle of 90° clockwise); and REFLECT 10 0 45 (reflect the plane over a mirror line passing through the point (10, 0) with a heading of 45°). The microworld included a game, called the Match Game, in which the goal is to superimpose two congruent shapes by applying a sequence of transformations to one of the shapes. This game is also illustrated in the Results section, in Figure 2.

Overview of the Two Studies: Method

The first study investigated the learning of 6 high school students (all girls; average age: 16) who spent a single session of approximately 1 1/2 hours with the Green Globs environment. In the second study, the subjects were 12 middle-school students (9 boys, 3 girls; average age: 11 years, 9 months) who worked with the TGEO microworld for transformation geometry. The second study, as a whole, extended over a period of 5 weeks; however, the portion reported here consists of the first two sessions only. In these sessions, the students also had about 1 1/2 hours of experience with the computer environment. In the first study, there were two pairs of students working together and two students working alone; in the second study, all of the students worked in pairs.

In both studies, the students were shown the computer environment, given a brief explanation of how it worked, and allowed time (10-20 minutes) for free exploration before playing the game. The students in the Green Globs study had time to play one or two games; in the TGEO study, the number of games completed ranged from 3 to 7. All the interactions in each study were videotaped. In addition, any written notes created by the students were collected, and the sequence of moves for each game was recorded, either on paper or by computer.

Results

A first point to make is that the students in both studies were engaged by the games, said that they enjoyed them and learned from them, and would have liked to continue playing them, if there had been more time. This is not a trivial point; an interpretation is that the students truly felt an "ownership" of the problems posed by the game; they wanted to succeed, not in order to gain any extrinsic reward, but, as Von Glasersfeld has stated, in order to increase the "fit" between their own understanding and the new experiences they encountered in playing the games. The analysis which follows will describe the strategies and patterns of interaction which emerged in each study, first separately and finally attempt a synthesis.

The Green Globs Study

The high school students who used the Green Globs software had all had previous instruction in algebra, yet there was a wide range in the amount that each student remembered about graphing. Correspondingly, a range of strategies were applied in trying to "hit" the dots on the screen. The more able students called upon their existing mathematical knowledge to generate likely solutions on paper before typing them in; the less able students tended to immediately type in a guess at the keyboard and then use successive approximation to come closer to the targets. When interviewed after the session, one student, Phe, alluded to the importance of existing mathematical knowledge. When asked how well she did, she answered, "As well as I thought I did. You have to use your knowledge; if you don't have much knowledge, you can't do very well." One pair of students seemed to want to use only the knowledge of which they were most certain, graphing straight lines. Virtually all of their solutions were straight lines, with the slope and intercept calculated in advance on paper. Other students used circles, ellipses and parabolas, although straight lines were the most popular type of graph for all but one student.

After typing in their solutions, the students were very attentive to visual information on the screen, and made use of this feedback to improve their "shots." Phe carried out what turned out to be quite a sophisticated exploration of parabolas while playing the game. The first 9 shots of Phe's game are illustrated in Figure 1 (the graphs are labeled alphabetically in the order entered; gray dots indicated "globs" which were hit). Phe began by immediately typing in what may have been a prototypical equation for her, Y=X (labeled "a"). She then switched to a parabola, again, typed in quickly without much attention to the screen: $Y = 1/2\,X^2 + 1$ (graph b). This parabola did not hit any dots, and Phe then began to vary the parameters of the parabola in a somewhat systematic fashion. Her next three shots were: $Y = 3\,X^2$ (graph c); $Y = -(1/2\,X^2) - 2$ (graph d); and $Y = -(1/4\,X^2) - 2$ (graph e). In this sequence, she seemed to be working to gain understanding and control over the first parameter, which determines the curvature of the parabola. Her later inputs varied the placement of the parabola, by changing the

value of the constant. In the end, Phe entered a total of 17 different parabolas in this game, a also used 6 lines to hit isolated globs.

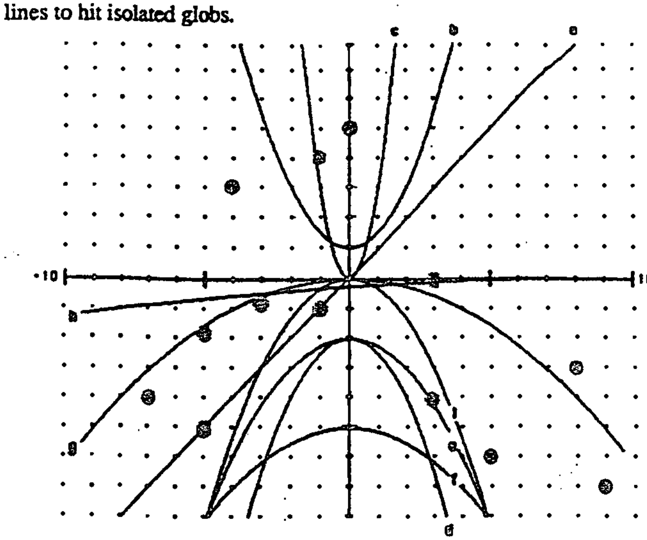


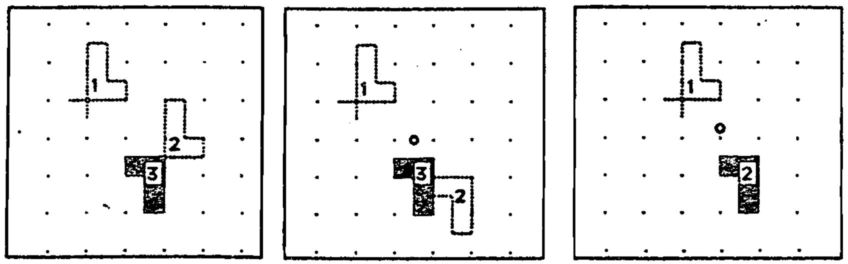
Figure 1: Phe's first 9 moves in <u>Green Globs</u>.

Phe quickly moved from a strategy of apparent guesswork, in her first two inputs, to one of systematic variation. In committing herself to building a family of parabolas, she did choose an easier strategy, that of using only straight lines. In fact, she only switched to strai lines when the interviewer reminded her that they were available, and soon returned to her parabolas. If Phe's goal had been only to hit globs, she could have fulfilled it more easily by using straight lines, but she chose to continue her exploration of parabolas. I would interpre this behavior as indicative of a modification in this student's goals: it was apparently as important to her to understand how parabolas worked as it was to win the game. Phe had bee the student who remarked upon the importance of knowledge in playing this game, yet she al made good use of the feedback from the microworld in order to refine and increase her knowledge. She stated at the conclusion that she wished her school had the software, and th if she played it long enough, she would be ready for a test. Phe, and the other students in the Green Globs study, seemed aware of the effectiveness of the game in helping them to better understand the graphing of equations.

The TGEO Study

In the second study, there were also a variety of strategies applied in playing the Matc Game, involving both analytic methods and visual estimation. A typical game is illustrated ir Figure 2. This figure shows a sequence of replays of the same game; the object is to move th L-shape labeled "1" onto the final black L-shape. In this sequence, the students Jos and Tar

first employed the most common strategy for winning the game: simply slide the L-shape until corresponding vertices are superimposed, and then pivot (a simple version of ROTATE) to bring the headings into alignment. This "Slide-Pivot" strategy was discovered by all of the students who played the game. However, when asked whether they could improve their score, by finding a single move solution, all of the students played additional games to seek this optimal solution. As can be seen in Figure 2, it took Jos and Tar two tries before they discovered the solution. They used visual feedback from screen, as well as an analysis of the vertical and horizontal distances between the L-shapes, to determine the final solution.



SLIDE 20 -15
PIVOT 180

First Replay:
ROTATE 15 -10 180
SLIDE -10 5

Second Replay:
ROTATE 10 -7.5 180

Figure 2: Jos and Tar's Solutions to the Match Game

As stated above, the students quickly discovered a simple, fool-proof strategy for solving any problem posed in the Match Game. However, once they were asked to look for a more efficient solution, the students no longer used this simple strategy as a first attempt. Instead, they would try, over and over, to use the more powerful yet more complex commands, ROTATE and REFLECT, in spite of the fact that they ended up taking longer to complete a game. They would only fall back on using SLIDE or PIVOT when hopelessly lost in a game.

This again, I believe, is an example of the change in students' goals as they played the game. Initially, they were focused on the purported goal of the game, superimposing the two shapes. However, they soon adopted a new goal, that of understanding the more complex transformations, and were willing to spend a great deal of time and mental effort into gaining control over these operations. They moved from playing the computer game, to playing the "game" of mathematics, which involves generalization and abstraction of the properties of these mathematical entities.

Discussion

Both Green Globs and the TGEO microworld are designed in such a way as to support children's "debugging" of their own solutions, and of their own conceptual models of the mathematical entities instantiated in the environment. This debugging is possible because the microworld makes available a set of linked representations for the entities; as Hoyles and Noss put it when describing what could be called the original microworld, the language Logo, "the learner is both engaged in the construction of executable symbolic representations and is provided with informative feedback" (Hoyles and Noss 1987, p. 133). The basic model for learning with an interactive computer environment of this kind is as follows:

- Instead of directly teaching the properties of the mathematical entities, these entities are incorporated into a game-like situation, in which the learner must <u>use</u> them to solve a problem.
- In order to use the entities effectively, the student must understand how they work.
- This understanding is build through an iterative process of conceptual debugging: Students generate solution attempts based on their current model of how the entities work; if these attempts fail, they compare their internal model of the entities with what they see on the screen. In their next attempt, they refine their solution based on this visual feedback. The process continues until the learners have gained a sufficient understanding of the entities to succeed in the game.

If the game is well-designed, the students will also abstract the essential properties of the mathematical entities.

This analysis of learning in a microworld is supported by explicit comments made by students who participated in the studies. For example, in the Green Globs study, one participant described how the software would be useful to other students: "It gives you a chance to see your mistakes. You type in the equation and it gives you what that is, and you can go back from there and compare." Another student pinpointed the strength of computer tools for mathematics: "You may have an equation and graph it wrong, but what this does is, you plug in whatever you have and it will give you back a specific graph. Some people are confused on parabola, hyperbola, the pluses and minuses. They can tell on this computer which it is."

In conclusion, the Green Globs and the TGEO environments are examples of computer software packages which support a similar kind of mathematics learning. This type of learning is constructivist, in that the learner must build upon his or her existing knowledge, and the microworld provides the tools needed to correct and refine this knowledge. These environments also have the potential to allow students more independent and self-directed

exploration of mathematical patterns, in which learners can go beyond the goals of the game and continue to satisfy their own desire to find meaning and order in their educational experiences.

References

Dugdale, S. (1981). Green globs: a microcomputer application for graphing of equations. CERL Report E-21. Urbana, Illinois: University of Illinois.

Edwards, L. (1986). Tools, toys, and tutors: Computers and mathematics learning. Unpublished manuscript, University of California at Berkeley, Group in Science and Mathematics Education, Berkeley, CA.

Edwards, L. (1989). Children's learning in a computer microworld for transformation geometry. Unpublished doctoral dissertation, University of California at Berkeley.

Edwards, L. (1990). The role of microworlds in the construction of conceptual entities. In G. Booker, P. Cobb & T. de Mendicuti (Eds.), Proceedings of the Fourteenth Annual Conference of the International Group for the Psychology of Mathematics Education. Mexico City, Mexico.

Hoyles C. & Noss R. (1987). Children working in a structured Logo environment: from doing to understanding. Recherches en Didactique des Mathématiques, 2 (12), 131-174.

Newman, D. (1985). Functional environments for microcomputers in education. The Quarterly Newsletter of the Laboratory of Comparative Human Cognition, 7 (2), 51-57.

Papert, S. (1987). *Microworlds: Transforming education.*. In R. Lawler & M. Yazdani (Eds.) Artificial Intelligence and Education: Volume One. Norwood, NJ: Ablex.

Thompson, P. (1987). Mathematical microworlds and intelligent computer-assisted instruction. In G. Kearsley (Ed.), Artificial Intelligence and Instruction: Applications and Methods. Reading, MA: Addison-Wesley.

Von Glasersfeld, E. (1987). *Learning as a constructive activity*. In C. Janvier (Ed.) Problems of representation in the teaching and learning of mathematics. Hillsdale, NJ: Erlbaum.

White, B. (1981). Designing computer games to facilitate learning. AI-TR-619. Artificial Intelligence Laboratory, Massachusetts Institute of Technology.

On Building Self-Confidence in Mathematics

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The pursuit of good feeling in education is a dead end. The way to true self-esteem is through real achievement and real learning. (Krauthammer, 1990).

A Dangerous Myth: Either you have it in math or you don't, and the job of the teacher (or school) is to find out who has it. (Usiskin, 1990).

The attribution theory of "intrinsic/extrinsic causality factors" to describe one's perceived competence in mathematics has been used in an effort to explain observable gender differences in mathematics achievement. In short, this theory says that when boys do well in mathematics, they attribute their success to internal factors: "I'm smart; I'm creative, I have talent, etc." When boys do not do well in mathematics, they attribute their failure to external factors: "The questions were tricky; I studied the wrong material, the teacher was bad", and to other similar statements. But for girls this is not the case. When girls do well in mathematics they often attribute their success to external factors: "The test was easy, I was lucky, I studied the right material, etc." and when they fail, they attribute their failure to internal factors: "I'm stupid, I've never done well on this sort of test, and I got what I deserved."

Over the years there have been many surveys assessing the general validity of this theory. These studies have been conducted in an effort to ascertain if there really are gender differences in the perception of one's mathematical accomplishments, and if so, why they exist (Fennema & Sherman (1978), Fennema (1980), Wolleat (1980) and Amit (1988)). Although the results are inconclusive with respect to gender differences, the data do show that many individuals hold these intrinsic/extrinsic perceptions of themselves -- and that these perceptions are highly correlated with mathematics achievement. In other words, if one doesn't have a high self-concept with respect to his ability in mathematics, then hard problems aren't even attempted , and this "defeatist attitude" seems to go way beyond school mathematics (Fennema, 1980).

Research mathematicians work on hard problems which often take years to get a handle on. But they work on them because they believe that they have a chance to crack them. Their perception of their own ability is an extremely important factor in the mathematics they do. A low self-opinion of one's ability in school mathematics can be devastating. So the major question is, how can we, as teachers, build a healthy self-concept in our students of their own ability to do mathematics?

The thesis of this paper is that one's perceived ability in mathematics is formed mainly as the serendipitous outcome of success in problem solving. Although psychological support from teachers enters into the growth of one's self-confidence, e.g., when should the classroom teacher encourage, compliment or chastise a student in the process of solving a problem--*the end result is that real self-confidence is the result of success*--and it is something that is far too important to be left to develop by chance. But how

does self-confidence develop? Let us consider two extreme positions on this matter; the individualistic approach of R.L. Moore and the cooperative learning approach of N. Davidson.

The Moore Method

There are many ways of teaching mathematics. The "chalk and talk" method is probably best known, but it forces students into a receptive and passive role. Students take notes from a learned professor and digest the content in them. They absorb knowledge and often apply it through well designed problem sets, but they seldom create knowledge themselves. Essentially, students become receptive validators of information created by others. Students certainly do internalize the concepts, but it is a digestive process, rather than a creative one. The R.L. Moore method of teaching mathematics is built from a different premise.

R.L. Moore (1882-1980) was a professor of mathematics at the University of Texas. His style of teaching was unique and very, very successful as evidenced by the cadre of individuals he developed who surged to the forefront of 20th century mathematics[1,2]. His technique was to pit one student against another in a competitive situation. He would present to students the definitions and the hypotheses of theorems, and they themselves would construct appropriate conclusions and proofs of the statements. In other words, the students under his tutelage, built mathematical structures, day by day, year after year. Moore had his students promise that they would never look anything up in textbooks nor in journals, and that they would address all of their questions on the material to him, and to him alone. His thesis was that progress and creative abilities are nurtured through competition--not friendly competition--but cut-throat competition. The easiest way to get a person to solve a problem, is to tell him that an (unnamed) dunce (of another class) solved it!

During class Moore would call on student X to present a particular problem. If others in the room had not yet solved the problem themselves, and they did not wish to see the solution, they could leave the class and return when student X had finished his presentation. There is a movie produced by the American Mathematical Association which shows Moore in action. In this movie it shows him asking the class if someone had solved a particular problem. Several students had solved the problem and as he invited one of them to the board to present the solution, three-fourths of the rest of the class walked out. They had not yet solved the problem and they were not ready to have it explained to them. There are stories, probably apocryphal, that Moore and the student presenting the problem were oftentimes the only ones in the room. Moore's premise was that Mathematical progress is a result of intense competition, and that confidence in one's ability is a result of mathematical progress. In other words, there is a symbiotic relationship between progress and confidence. Strange as this might sound, the system really worked, at least for Moore.

The details of the Moore method of teaching have been documented, studied and analyzed within the mathematics community (Jones, 1977). The results he obtained from his teaching method are far-famed. Whether or not one agrees with his teaching technique is not the question. What is

important, is that this method imparted to students an incredible amount of self confidence--a confidence that they did not have before they started studying with him. When one studied mathematics with Moore, one studied all (or nearly all) of their mathematics with Moore, and only with Moore--he became their guru, and one cannot argue with the results. Mary Ellen Rudin, Professor of Mathematics at the University of Wisconsin, is one of the leading topologists in the United States. She studied under Moore every semester that she was at the University of Texas, from the time she took her first course in her freshman year until the time she was awarded her Ph.D. degree. Although her comments on her mentor's classroom method are sometimes critical, the confidence he built in her comes through as seen in this interview:

> "He built up your ego and your competitiveness. We were against each other, but at the same time we were a team for each other".
>
> It builds your ego to be able to do a problem when someone else can't, but it destroys that person's ego. I never liked that feature of Moore's classes. Yet I participated in it.
>
> [He never referred in class to other people's work?] Never, never, never.
>
> I felt cheated...I didn't know any algebra, literally none. I didn't know topology. I didn't know any analysis--I didn't even know what an analytic function was. I had had my confidence built, and my confidence was plenty strong.
>
> Your give me the definitions, and I'll solve the problem. I 'm a problem solver, primarily a counter example discoverer. I still have this feeling that if a problem can be stated in a simple form that I can understand, then I will be able to solve it even if doing so involves building some complicated structure. Of course, I have had some failures.... [But you've never failed in confidence?]... No,having failed 5000 times doesn't seem to make me any less confident. At least I don't feel bound by any serious constraints or doubts about my ability.
>
> He built your confidence that you could do anything. No matter what mathematical problem you were faced with you could do it. I have that total confidence to this day." (Albers &Reid (1988)).

The Moore method of teaching is Socratic teaching with a large dosage of competition thrown into the formula. Halmos (1985, p. 260) commenting on this method stated: "I am convinced that the Moore method is the best way to teach (mathematics) that there is....".

But for all the good things there are to say about the Moore method there are also bad things, and Halmos was aware of these too: Eg. he stated:

"Moore felt the excitement of mathematical discovery and he understood the relation between that and the precision of mathematical expression. He could communicate his feeling and his understanding to his students, but he seemed not to know or care about the beauty, the architecture, and the elegance of mathematics and of mathematical writing. Most of his students inherited his failings as well as his virtues (diluted, of course);..." (p.255).

The Moore method is built upon the Chinese proverb:
"I hear, I forget; I see, I remember; I do, I understand." Moore simply had his students construct mathematics--something that most teacher's don't even try to get their students to do. For most of us, we are content if students can understand the mathematics, but many of us believe that to have them construct mathematics is out of the realm of their ability. This, as we will see, is nonsense.

One reason we hold these views is that few of us have created mathematics ourselves, another reason is technical; Moore dealt with highly motivated graduate students--we see unmotivated undergraduates or high school students, with deeply ingrained inferiority complexes with respect to their mathematics ability. As we will see, both of these can be overcome.

It is well documented that Moore helped shape 20th century mathematics, and that his main method for doing this was to build one's confidence in their mathematical ability. But it is also rumored that for every mathematician he created, he also sent five to insane asylums. While this is probably also apocryphal, the adage "I do, I understand" the "constructivist approach" (Cobb, 1987, Confrey, 1987) was the heart of his program.

Humanizing Moore: The Davidson Approach

Moore was successful in producing research mathematicians, but his method of pitting one student against another is rather repugnant, at least to many in the education field. So several intense efforts were made to humanize Moore. One of the more successful of these approaches was developed by Neil Davidson of the University of Maryland--and this turned into a new international educational movement called "cooperative learning". Davidson (1986) developed over a twenty five year period a teaching method which is built upon Moore's principles but which factors out the competitive element of the Moore method. Davidson divides classes into groups of four, poses definitions and theorems to the groups and then stands back to let the students develop the solutions themselves. The role of the instructor is drastically changed, from being the presenter of information to that of acting solely as a resource individual. Davidson has a long list of guidelines to aid the instructor in deciding when he can join in with discussions of the students and when he should remain quiet, but the idea is that students construct their own mathematics. Davidson is interested in the mathematics as well as the social dynamics of the working sessions. But a serendipitous outcome of his teaching style is that students have their self-confidence bolstered. Indeed,they often create novel proofs of theorems (Davidson, 1986, p 350). I personally observed a group of

average students in Davidson's class construct the proof that the limit of product of two functions is equal to the product of the limit of the functions (assuming that the individual limits exist). This is a very difficult theorem and to think that these students did this with only the barest of mathematical machinery attests to the power of his methods.

But just as the Moore Method has draw backs, so does Davidson's. The most serious one is that it takes a tremendous amount of time to get students to construct the mathematics; and often times only one individual in the group does the actual work--and explains it to the others. True, the students may benefit from the method, but it isn't a very efficient way of imparting knowledge. University life is built around timetables and lists of courses with specific content which must be covered; the Moore method and Davidson's modification of it does not seem to be applicable in the large, but both seem to be very successful in developing confidence in students, and this confidence seems to be a direct outcome of their success in problem solving activities.

This idea of cooperative learning is relatively new in mathematics, but it seems to work. Uri Treisman of the University of California at Berkeley has made an interesting observation with respect to the study habits of Asian versus Black American students. He noticed that many Asian students would do their mathematics together in small groups, whereas Black American students would attempt to do all of their work alone. For the Asian students, the mathematics was an extension of their social life; for the Black Americans it seemed to be separate from their social life. In other words, Asians would be observed talking about particular problems and solutions in the same way one talks about a sporting event, or other events in the social milieu. Mathematics for them seemed to be socially acceptable--even to those who weren't classroom stars. But the Black American students were seldom observed talking about mathematics with one another. Mathematics was a private thing. As one would guess, the Asian students on the whole, succeeded; the Black students on the whole, failed. So Treisman developed a program getting students to talk mathematics, to bring mathematics into their social milieu. Like Davidson, this dealt with getting the students to work in small groups. His method has been so successful that it is being adopted throughout the United States (Jackson, 1988).

It seems then that a prerequisite for being successful in mathematics, is to have a strong sense of self-confidence in your mathematical ability. The Moore and Davidson programs have used interesting classroom techniques to fete out mathematical creativity and unexpectedly, mathematical self-confidence. Although their programs have dealt largely with advanced students, elements of their approaches can be be applied to everyone studying mathematics. In other word, there is a middle ground, one need not completely adopt a Moore or Davidson teaching style to build self-confidence in students. Confidence in mathematics, which is simply how one perceives of oneself, can be built through problem solving in "recreational" mathematics .

Recreational Mathematics

A major goal of school and university mathematics is to impart to students the self-confidence to be able to think. One tactic that I have used is to give students problems from the smorgasbord of recreational mathematics. The problems generally have nothing to do with what the students are discussing at the moment, but they are of a nature that they should be within in the ability of every student to solve. Students love them--particulary the novel problems. There is noting new in any of this--but one should make a conscious effort to present two or three such problems per week in his lessons. It is wonderful to see what happens. Students often get completely absorbed in the problems--and their attention must be torn away from them and directed back to the lessons--and confidence develops.

Student confidence in problem solving is built in a very systematic way. The new thing here is to consciously bring recreational problems into the classroom. There are thousands of problems which can be used for this purpose, it depends on the course and on the level of the students. Several of my favorites ones are listed in Table1, along with sources of where to find others. All I am asking is that we encourage teachers to use two or three such problems each week for one semester--and then to sit back and reflect on what they have done. I firmly believe that we can nurture self-confidence through problem solving in recreational mathematics. Usiskin (1990) has asked: *If everybody counts, why do so few survive?* The answer, I think, is that many students do not have the self-confidence to try. Having problems from recreational mathematics as part of the curriculum can help change that.

Table 1

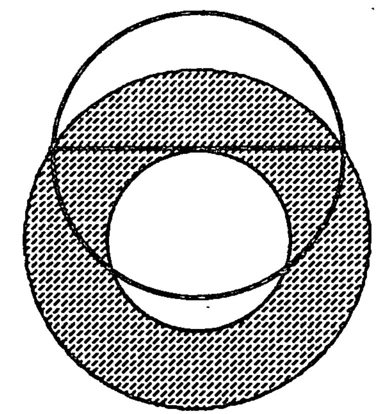
1. A man wants to sell a woman some insurance. He is just about to wrap up the deal but needs to know the ages of her three children. She tells him that the product of the ages of her three children is 36. He makes note of this but says that he needs to know more. So she tells him that the sum of their ages is equal to the number of the house next door. The man runs outside and returns. He is exasperated--and says that he needs more information. The conversation goes on with the woman saying that she thought he was a smart insurance man and that she would cancel her order with him for he is not as smart as she thought etc. She also tells him that she is in a hurry for she must take her oldest to her violin lesson. The salesman jumps up and claims that he now knows the ages of her children. What were they?

2. A fly and spider are in a 12 X12 X 30 box. The fly is one the middle of one of the 12 X12 walls, one foot from the floor. He is so scared that he does not move. The spider is in the middle of the other 12 X12 wall one foot from the ceiling. What is the minimal distance the spider must crawl to get to the fly? (The answer is not 42).

3. A billiard ball shot from the lower left hand corner of a rectangular billiard table at a 45°. If the table is 3X3, the ball will end up in the upper right hand corner. If the table is 3X5 it will end up in the upper left hand corner and it is 4X3

it will end up in the lower right hand corner. Where will the ball end up if you have a 302X512 table?

4. Take any circular track made of two concentric circles. Can you show that the area of the track is equal to the area of a circle whose diameter is a cord of the larger circle, but tangent to the smaller circle?



Sources:

1. Dunn.A. Mathematical Bafflers Vol. 1 &2. Dover Publications,
2. Gardner M. Mathematical Puzzles of Sam Loyd. Vol 1.&2. Dover Publications.
3. Dudeney, H.E. Amusements in Mathematics, Dover , 1970.
4. Schaaf, W.L. A Bibliography of Recreational Mathematics, NCTM, 1978.
5. The Monthly Calendar in each issue of the Mathematics Teacher.
6. Pappas, T. The Joy of Mathematics, Wide World Publishing/Tetra.1990.

Notes:

1. The top 15% of the most productive mathematicians in the United States and Canada during the period of of 1915-1954 were listed. 25% of them had earned their Ph.D. at the University of Texas, 5% from the U. of California at Berkeley, 8% from the University of Chicago, 16% from Harvard and 20% from Princeton. (Jones, 1977).

2. Among his students were: R. Wilder, G.T. Whyburn, J.H. Roberts, R.H. Bing, E. Moise, R.D. Anderson, and Mary Ellen Rudin.

Bibliography

Albers,D. and Reid, C. (1988). "An interview with Mary Ellen Rudin". The College Mathematics Journal. 19(2), 115-137.

Amit, M. (1988). " Gender Differences in achievement, level of study and causes attributed to success and failure in high school mathematics." M.Sc. Thesis. Department of Science Teaching, The Technion, Haifa, Israel.

Cobb, P. (1987). "Information processing, psychology and mathematics education--a constructivist perspective. Journal of Mathematical Behavior 6(3) 3,40.

Confrey, J. (1987). "The constructivist". Proceedings of PME11. Montreal,Canada, Vol. 3, 307-317.

Davidson, N. (1986). Cooperative Learning in Mathematics. London. Addison Wesley.

Dubinsky E. (1990). See the UME Trends, (News and Reports on Undergraduate Mathematics Education, edited by E. Dubinsky, Purdue University).

Fennema, E. (1980). "Sex-related differences in mathematics achievement: where and why". In Mathematics: People, Problems and Results Vol. 3. Edited by D. Campbell and J. Higgins, Belmont, Calif. (Wadsworth Pub.) 1984., pp. 209-221.

Fennema,E. & Sherman, J. (1978). "Sex-related differences in mathematics achievement and related factors: a further study". Journal for Research in Mathematics Education, 9 (2), 189-203.

Halmos, P. R., (1985). I Want to be a Mathematician. Washington, D.C: The Mathematical Association of America.

Jackson, A. (1988). Research mathematicians in mathematics education: The National Scene. Notices of the American Mathematical Society 35(8),1123-1131.
Jones, F.B., (1977). "The Moore Method." The American Mathematical Monthly, 84(4), 273-278.

Krauthammer, C. (1990). "Education: Doing Bad and Feeling Good." Time Magazine, Feb. 5., p.78.

Usiskin, Z. (1990). "If Everybody Counts, Why do So Few Survive?" UCSMP Newsletter, No. 6, (Winter), 5-11.

Wolleat, P,. Becker, A. Pedro J., and Fennema, E. (1980). "Sex differences in high school students' causal attributions of performance in mathematics". Journal for Research in Mathematics Education, 11(5), 356-366.

CLASSROOM DISCOURSE AND MATHEMATICS LEARNING

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The tension between mathematics content acquisition and the creation of quality mathematics teaching/learning environments is illustrated schematically, and case studies based on five mathematics classrooms are summarised. While a wide range of mathematical discourse is covered by the case studies, each of the teachers concerned was attempting to establish a learning environment in which individual students would be most likely to construct mathematical concepts and relationships in response to problems associated with their own personal worlds. That is to say, the teachers wanted their students to "own" the mathematics they were learning. It is concluded that the data point to the overriding importance of establishing mathematical learning environments that are both student-centred and teacher-centred, and that there is no single best way of teaching mathematics. Even so, extensive, interactive communication seems to be a common feature of quality mathematics learning environments.

The Tension Between Mathematics Content Acquisition and Ownership

Research findings have consistently shown that the reality of everyday mathematics classroom situations has made it difficult for teachers to maintain more open forms of interaction (e.g. Bourke, 1984; Clarke, 1984; Cobb, 1988; Desforges, 1989; Mousley & Clements, 1990). Moreover, there appears to be an inherent tension between the idea of creating teaching/learning environments in which children own the mathematics they learn, and the expectation that school mathematics will assist children to acquire an understanding of a more or less fixed set of mathematical concepts, skills and principles. This is illustrated schematically in Figure 1.

Who Owns the Mathematics Being Studied?

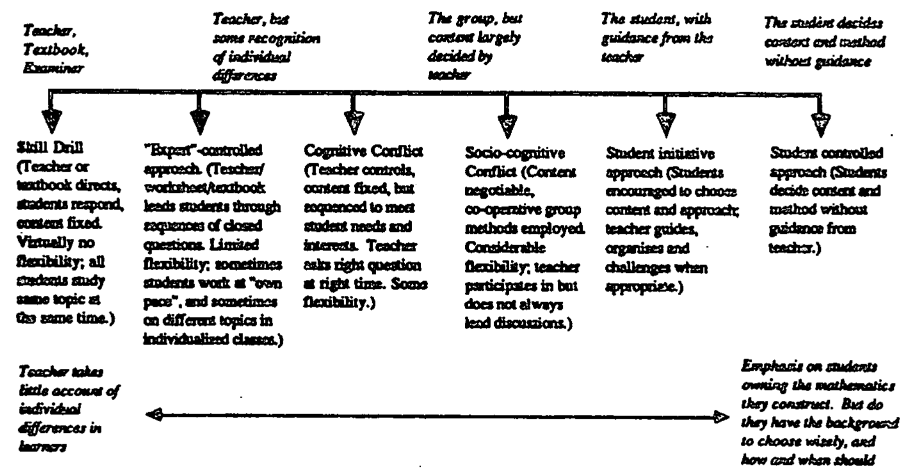


Figure 1. The ownership/content tension.

Direct Explanation and the Quality of Mathematics Learning

The assumption among many leading mathematics educators that direct explanation is a poor way to teach mathematics to young children, is not supported by classroom studies carried out in Australia, the United Kingdom, and the United States of America (e.g. Bourke, 1984; Bennett, 1976; Stallings, 1976). These studies report that children who are taught mathematics

traditional methods, in which the teacher regularly uses whole-class teaching, often outperform children from more individualised settings on standard tests emphasising computation, and on a range of other tests including tests of mathematical problem solving.

In summarising their research comparing mathematics instruction in Japan, Taiwan, and the United States, Stigler and Baranes (1988) commented that their findings should cause educators to question several assumptions that underlie American mathematics education:

> First is the assumption that direct explanation is not a useful way to teach mathematics to young children. There is a widespread belief in our society that concrete experiences are the best way to teach young children, and that language will either go over their young heads, or lead to learning of a superficial kind. There are two important findings from the Japanese observations that we need to ponder: (a) Young children are capable of responding to, and apparently understanding, complex verbal explanations, and (b) it is possible to stress *both* concrete experiences and verbal explanations at the same time. It is possible, in fact, that both are necessary to promote high levels of learning. (p. 299)

Given such findings, further research into the effects of various forms of discourse in the mathematics classroom would appear to be in order.

Some Reflections on the Roles of Mathematics Teachers

Before commenting further on how language, communication, and mathematics learning might be interconnected, it will useful to see what classroom teachers do when they attempt to establish learning environments in which their students would be most likely to link mathematical ideas with situations that arise in their own personal worlds. We shall now provide five examples of what teachers have done when they were faced with the challenge of creating learning environments of this type. Through these examples it will become clear that many teachers are not clear about what they need to do to create such environments.

Examples 1 and 2: Making Cardboard Containers

Early in 1991 I arranged for two teachers to organise a problem-solving activity with their classes. Each teacher was given 20 kg of rice, and enough clear plastic "glasses" (each 250 ml capacity), thin cardboard squares (each 21 cm x 21 cm), scissors and masking tape to meet the needs of up to ten groups of four students each. I described the task to the teachers as follows:

> What I would like you to do is to divide the class into groups of four, and ask each group to "make" a container out of the cardboard just big enough to hold exactly one "glass" of rice; when groups have completed this task, ask them to make another container, just big enough to hold two glasses of rice; when this has been completed, then you can encourage the groups to think about what other kinds of questions related to the cups of rice and the cardboard might be worth exploring, and the groups could then be asked to develop solutions to their own problems.

Both of the teachers wanted me to provide more details about the activity; however, I told them that I preferred them to organise the activity in whatever way they felt was most appropriate. I did make it clear, though, that I was hoping students would become actively involved in the learning process, and that each group would develop a pride of "ownership" in whatever strategies they devised in responding to the task.

As it turned out, the two classes I observed were "streamed"; one was a "bottom" stream class of Grade 8 boys, and the other a Grade 7 "middle" stream class, containing girls and boys.

The "bottom" stream of Grade 8 boys. The teacher of this "bottom" stream of boys (Teacher A) decided to devote a "single period" of 40 minutes to the task. He said that the boys often worked in small groups for their mathematics classes.

Teacher A divided the class into groups of four, and gave explicit directions to the class as a whole. After pointing out the available equipment (cardboard, rice, etc.), he told the class that each group would have a maximum of ten minutes to construct an open box that held exactly one glass of rice; no trimming with scissors was to take place. The groups responded positively to these instructions, and after quickly collecting the necessary equipment, they proceeded to talk enthusiastically about the task.

After ten minutes, much talk, and much trial and error, Teacher A invited each group to show the whole class its open box full of rice. Then Teacher A said: "OK. Now I want you to make another open box, only this one should hold exactly two glasses of rice. Do you think this will be possible?" This question led to some discussion (because some of the boys believed that since the cardboard squares were all the same size, the boxes should all hold the same amount of rice). However, it was generally agreed that Teacher A's request for an open box holding exactly two glasses of rice was an interesting one calling for immediate investigation. Without further ado each group began working on the task and, after much animated discussion, quickly succeeded in making an appropriate box.

As soon as a group finished the task of making a box that held two glasses of rice Teacher A challenged the group members to make an open box that held exactly three glasses of rice. The lesson closed with Teacher A asking the boys to decide whether it would be possible to construct an open box (from the same-sized cardboard) that would hold exactly four glasses of rice.

The "middle" stream class of Grade 7 boys and girls. The teacher of the "middle" stream class of boys and girls (Teacher B) also decided to devote a "single period" of 40 minutes to the task. He said that the class had never worked in small groups before.

Teacher B divided the class into single-sex groups of four, the groups being largely pre-determined by the seating arrangements. He told the class that each group had to use the cardboard, scissors and masking tape to make a container that would hold exactly one glass of the rice. Then he left the groups to themselves to work out what they had to do. He did not move around talking to the groups, but stayed at the front of the room, filling in a form concerned with some aspect of the administration of the school. The groups were unsure of what was expected of them, and talked about what they should do. After about ten minutes each

...oup had folded a piece of cardboard to form a conical container, and had trimmed this with ...ssors so that the rice neatly filled it.

Teacher B then told the class that they should now make a container that held exactly two glasses of rice. He then left the room for a period of ten minutes. In his absence each of the groups made another cone, appropriately trimmed, that held exactly two glasses of rice. The students did not know what to do next, and tended to talk to each other, informally, about everyday matters not related to the task in which they had been involved.

About fifteen minutes before the end of the lesson Teacher B returned to the classroom; he proceeded to point out to the class that there were many shapes they could have made, and that it was interesting that the same-sized piece of cardboard could be folded into containers that held different amounts of rice. He then asked each group to make an open box that would hold exactly one glass of rice. The groups experienced difficulty in attempting this task, however, and Teacher B moved around showing each group how such a box might be made. Finally, he raised the possibility of making a box that would hold two glasses of rice, and some groups were able to do this before the lesson ended.

Example 3: Nurturing a Community of Mathematicians

In 1989 I observed, on two occasions, 28 children (aged 5 to 12) engaged in a range of mathematical activities. The children were working with their two teachers in adjacent classrooms. The teachers, each with eight years' teaching experience, were committed to an approach that was primarily concerned to help the children realise that they were responsible for constructing their own mathematical meanings. The children themselves decided what aspect of mathematics they would investigate on any particular day, whether they would work individually or in a group, whether they needed equipment, whether they needed to talk to their teacher, what books they would use, and how they would record their findings.

In an article that they jointly authored, the two teachers (Waters & Montgomery, 1989) pointed out that although some children worked in pairs or in groups, most worked individually. Even so, there was constant discussion, often between an older and a younger child (p. 82).

At the conclusion of their article on their mathematics program, the two teachers said they believed that teachers should see themselves as practising mathematicians. Thus, in the classroom they themselves should work on their own mathematical investigations while their students worked on theirs. By doing this, they would not only show the students the kind of mathematics that was worth studying, but they would also demonstrate what it was to be a healthy learner of mathematics - self-motivated, self-directed and self-regulated. They said that they were attempting to create a "risk-free" learning environment in which learners' attempts were valued.

The two teachers were adamant that it was not their responsibity to define precisely what mathematics their students should study on a day to day basis, and maintained that it was more important for their students to see them continually doing mathematics than for them to be moving around the classroom asking questions of their students. In short, they wanted their

students to be "young mathematicians," fully responsible for their own learning (Waters & Montgomery, 1989, p. 85).

Example 4: Towards Creating Cognitive Links

Clements and Del Campo (1989) reported a teaching study they carried out with a composite class of 21 children in Grades 3 through 6. After interviewing the children, and taking note of the imagery and strategies employed by each of the children when asked to share circular and triangular pizzas among two, three and four people, they divided the 21 children into 5 groups. The groups were given 15 minutes to talk about the "pizza-sharing" tasks.

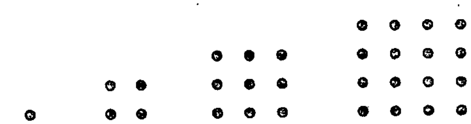
The 21 children then came together again, seated in a semi-circle on the floor, and each group reported its findings in its own words words. After this "reporting back" session, the teacher took the opportunity to link the sharing notion with the formal mathematical language "one-third of" and "one-quarter of." Then the groups were reformed, and given scissors and outline drawings of circles and equilateral triangles. Each group was required to cut the circles and equilateral triangles into thirds and quarters.

Following this activity session, each child was once again interviewed, and five weeks later, without any prior warning, each child was interviewed a further time. Data reported by Clements and Del Campo (1989, pp. 30-32) demonstrate clearly that the intervention program was successful. Children who previously had not been able to divide circles and equilateral triangles into quarters and thirds, learnt to do so, and in almost all cases this learning was retained over a period of five weeks.

Example 5: A Whole-Class Approach

Readers who have taught mathematics will not have too much difficulty identifying with the frustrations of the Grade 7 mathematics teacher in the following transcript of a class discussion which is reproduced from Clements and Ellerton (1991).

Teacher: Here are some numbers that have a special name: 'one, three, five, seven, ...'
Grade 7 pupil:: They're all odd numbers?
Teacher: Good girl. Now here are some other numbers that have a *different* name: 'one, four, nine, sixteen, twenty-five, ...'
Another Grade 7 pupil: One is one times one; four's two times two; nine's three times three, ...
Teacher: What's the name of numbers like that?
Pupil: Even numbers?
Teacher: No. The even numbers are two, four, six, eight, etc. Have a look at this. [Teacher draws the sequence of dots shown in Figure 2 on the blackboard, and labels the appropriate groups of dots 1, 4, 9, and 16.] What's the next number?
Pupil: Twenty-five, they're *square* numbers.



1=1x1 1+3=2x2 1+3+5=3x3 1+3+5+7=4x4

Figure 2. Odd numbers and square numbers

Teacher: Good. Now have a look at this. [Teacher writes the following on the blackboard.]

$1 + 3 = 4$
$1 + 3 + 5 = 9$
$1 + 3 + 5 + 7 = 16$

Teacher: What do you think I'm going to write next?
Pupil: One plus three plus five plus seven plus nine equals twenty-five.
Teacher: Great. You see, if you add the first four odd numbers together, you finish up with the fourth odd number multiplied by itself. In other words, you get four by four. What do you think you'd get if you added the first six odd numbers together? What do you think Eve?
Eve: The first six odd numbers?
Teacher: Yes. What's one plus three plus five plus seven plus nine plus eleven? [Teacher writes $1 + 3 + 5 + 7 + 9 + 11 = ?$ on the blackboard.]
Eve: Ahm, one, four, nine, sixteen, twenty-five, ... ahm, thirty-six?
Teacher: Yeah, but what's a quick way of getting it?
Another pupil: Six by six is thirty-six.
Teacher: Yeah, that's right. You'll always get a square number. That's obvious from the dots I drew before on the board. [The teacher proceeds to extend the 4 x 4 array of dots she had previously drawn, adding nine, then 11 dots, as in *Figure 3*.

Teacher: You could keep on with the same pattern. What would be the next square number?
Pupil: Forty-nine - seven by seven.
Teacher: Yeah, you add 13 to 36. The picture proves that the sum of the first 100 odd numbers is 100 times 100. What do you think this equals? [Teacher writes $1 + 3 + 5 + 7 + ... + 19 = ?$ on the blackboard.] I'll give you a couple of minutes to work that out, and to explain your reasoning in your workbook. You can draw a picture if you like. Away you go.

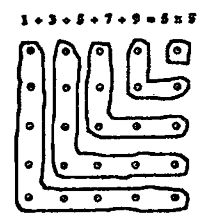


Figure 3. Towards a proof.

[Following this instruction, the pupils worked individually in their workbooks. Inspection of what they wrote revealed that many simply added each successive term of the series $1 + 3 + 5 + ... + 19$, to get 100 (or something fairly close to that). Others thought that the answer should be $19 \times 19 = 361$, and they often drew a 19 by 19 array of dots to illustrate their answer. Only a handful of pupils, in a class of 25, stated that the answer had to be 10 times 10 and illustrated their answer with a suitably partitioned 10 by 10 array of dots.]

As Bishop (1988, p. 118) has stated, although reasoning based on a diagram like Figure 3 does not fully qualify as proof in the strictest mathematical sense, it is *proving*. It does make the appropriate connection overt and explicit.

Notice that at no point did the teacher or her pupils make use of deductive reasoning. Even so, the pupils were left with the impression that the relationship (whatever it was) had been well established by the teacher. For them, all of their teacher's authority was to be associated with what had transpired. She had proved *something*, and it was *their* business to *try* to understand it.

Mathematics Learning Environments that Nurture Interest and Understanding

In all of the five classrooms summarised, the teachers were striving to create environments in which, ultimately, the students "owned" the mathematics they learned - that is say, they wanted their students to construct the concepts and relationships in an autonomous way. In each of the cases there were tensions arising from the need to cover content, the desire to use school time responsibly, and the challenge of establishing a successful mathematics learning environment.

In four of the five cases the content for the mathematics to be covered was more or less decided beforehand by the teacher(s) - the exception being Example 3, where the two teachers encouraged their students to generate their own mathematics tasks. The learning environments established by the teachers differed markedly, however, even with Examples 1 to 2 when the two teachers were given identical instructions, and were provided with the same equipment. In Figure 4 each of the five cases has been located on an abbreviated form of Figure 1, according to the extent to which the learners' thinking was teacher-directed in the different learning environments. Interestingly, the extent to which an example appears towards the right of Figure 4 does *not*, in my view, reflect the quality of the corresponding learning environment.

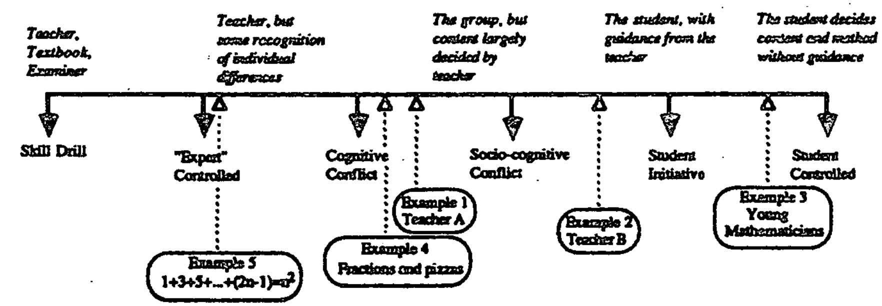


Figure 4. Locating the five examples of mathematics learning environments.

Some Concluding Comments

According to Cobb (1990, pp. 209-210) there is research support for moving to establish mathematics classroom environments that incorporate the following qualities:

1. Learning should be an interactive as well as a constructive activity - that is to say, there should always be ample opportunity for creative discussion, in which each learner has a genuine voice;
2. Presentation and discussion of conflicting points of view should be encouraged;
3. Reconstructions and verbalisation of mathematical ideas and solutions should be commonplace;
4. Students and teachers should learn to distance themselves from ongoing activities in order to understand alternative interpretations or solutions;
5. The need to work towards consensus in which various mathematical ideas are co-ordinated is recognised.

From my perspective the learning environments that were established by Teacher A, with his eighth-grade, bottom stream class of boys (Example 1), and the two teachers of Example 3 ("young mathematicians"), came closest to incorporating these five characteristics. Yet, these two cases are not close to each other on the continuum in Figure 2. The reason is simple: Teacher A's approach, so far as choice of content and method, was much more directive than the

approach of the two teachers in Example 3. Interestingly, while Teacher A and the two teachers in Example 3 both encouraged creative student-student dialogue, teacher-student and student-teacher dialogue both occurred more often in Teacher A's class than in the composite classes in Example 3. It is noteworthy, too, that the approach used in Example 4 ("fractions and pizza") was also directive, so far as content and method were concerned, yet it too generated enthusiasm, "ownership," and impressive understanding of mathematical content.

My conclusions must necessary be tentative. I believe that the classroom data reported in this chapter point to the overriding importance of establishing mathematical learning environments that are *both* student-centred and teacher-centred. The data summarised in this paper suggest that while there is no single "best" way of teaching mathematics, extensive interactive communication seems to be a common feature of quality mathematics learning environments.

References

Bennett, N. (1976). *Teaching style and pupil progress.* Cambridge, MA: Harvard University Press.

Bishop, A.J. (1988). *Mathematical enculturation.* Dordrecht: Kluwer.

Bourke, S. (1984). Insights into the teaching of mathematics in primary schools. *Victorian Institute of Educational Research Bulletin, 52*, 6-18.

Clarke, D. (1984). Secondary mathematics teaching: Towards a critical appraisal of current practice. *Vinculum, 21* (4), 16-21.

Clements, M.A., & Del Campo, G. (1989). Linking verbal knowledge, visual images, and episodes for mathematical learning. *Focus on Learning Problems in Mathematics, 11* (1), 25-33.

Clements, M.A., & Ellerton, N.F. (1991). *Polya, Krutetskii and the restaurant problem.* Geelong: Deakin University.

Cobb, P. (1990). Multiple perspectives. In L.P. Steffe & T. Wood (Eds), *Transforming children's mathematics education: International Perspectives* (pp. 200-215). Hillsdale, NJ: Lawrence Erlbaum.

Cobb, P. (1988). The tension between theories of learning and theories of instruction in mathematics education. *Educational Psychologist, 23*, 87-104.

Desforges, C. (1989). Classroom processes and mathematical discussions: A cautionary note. In P. Ernest (Ed.) *Mathematics teaching: The state of the art* (pp. 143-150). Barcombe, East Sussex: Falmer Press.

Mousley, J., & Clements, K. (1990). The culture of mathematics classrooms. In K. Clements (Ed.), *Whither mathematics?* (pp. 397-406). Melbourne: Mathematical Association of Victoria.

Stallings, J.A. (1976). How instructional processes relate to child outcomes in a national study of follow through. *Journal of Teacher Education, 27*(1), 43-47.

Stigler, J.W., & Baranes, R. (1988). Culture and mathematics learning. In E.Z. Rothkopf (Ed.), *Review of Research in Education 15, 1988-89* (pp. 253-306). Washington, DC: American Educational Research Association.

Waters, M., & Montgomery, P. (1989). Simulating the early learning environment in a maths classroom. In B. Doig (Ed.), *Every one counts* (pp. 78-85). Melbourne: Mathematical Association of Victoria.

CONSTRUCTIVISM, THE PSYCHOLOGY OF LEARNING, AND THE NATURE OF MATHEMATICS: SOME CRITICAL ISSUES

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Constructivism is one of the central philosophies of research in the psychology of mathematics education. However, there is a danger in the ambiguous and at times uncritical references to it. This paper critically reviews the constructivism of Piaget and Glasersfeld, and attempts to distinguish some of the the psychological, educational and epistemological consequences of their theories, including their implications for the philosophy of mathematics. Finally, the notion of 'cognizing subject' and its relation to the social context is examined critically.

Constructivism has become one of the main philosophies of mathematics education research, as well as in science education and cognitive psychology. I use the term 'philosophy' deliberately, for I do not believe that constructivism is well enough defined to be termed a theory. Neither its key terms, nor the relationships between them are sufficiently well or uniformly defined for the term 'theory' to be strictly applicable. The multiple ways in which 'constructivism' is understood are shown by Malone and Taylor (1989) whose brief survey distinguishes between 'personal', 'radical', 'socio', 'pragmatic' and 'C1 and C2' varieties of constructivism.

The growing importance of this philosophy means that a critical discussion of its theoretical basis and practical implications is necessary. A further reason is the risk of teachers and researchers both dividing into camps of non-believers and believers, and the latter identifying constructivism with the child-centred educational ideology of 'progressivism'. My aim is to ask a number of questions from a critical but non-partisan standpoint. What is constructivism? What are its implications for the psychology of mathematics education, and for epistemology and the philosophy of mathematics? To avoid ambiguity I shall first consider Piaget's constructivism, and then Glasersfeld's radical constructivism.

Piaget's constructivism

It is primarily the influence of Jean Piaget which has established

constructivism as a central philosophy in the psychology of mathematics education. His constructivism has a number of components including an epistemology, a structuralist view, and a research methodology, although I do not claim that these are either exhaustive or independent.

Piaget's epistemology has its roots in a biological metaphor, according to which the evolving organism must adapt to its environment in order to survive. Likewise, the developing human intelligence also undergoes a process of adaptation in order to fit with its circumstances and remain viable. Personal theories are constructed as constellations of concepts, and are adapted by the twin processes of assimilation and accommodation in order to fit with the human organism's world of experience. Indeed Piaget claims that the human intelligence is ordering the very world it experiences in organising its own cognitive structures. "L'intelligence organise le monde en s'organisant elle-meme" (Piaget, 1937, cited in Glasersfeld, 1989a: 162)

Piaget's structuralism involves a belief that in organising itself, the human intelligence necessarily constructs a characteristic set of logico-mathematical structures. Ultimately, these are the three mathematical mother structures distinguished by Bourbaki. Piaget posits an invariant sequence of stages through which an individual's cognition develops, in constructing these structures (Piagetian Stage Theory).

Piaget's methodology centres on the use of the clinical interview. In this procedure an individual subject is required to perform certain carefully designed tasks in front of, and with prompting and probing from an interviewer. A series of sessions are likely to be needed for the researcher to develop and test her/his model of the subject's understanding concerning even the narrowest of mathematical topics.

Each of these three components of Piaget's constructivism has strengths and weaknesses and important educational implications (although I shall leave epistemology until the next section). Piaget's structuralism is the weakest of these three areas, and can be regarded as inessential to constructivism (but not to Piaget) so that some recent accounts of constructivism discuss only the epistemological and methodological aspects (Noddings, 1989). The two main features are (1) the assumption that the normal development of the human intelligence necessitates the construction of the three logico-mathematical Bourbakian mother structures, and (2) Piagetian Stage Theory, which I shall not discuss here. The Bourbakian

account of mathematics in terms of three mother structures (algebraic ordering and topological structures) represented an advance in th formulation of modern structural mathematics mid-20th century. But it i neither complete nor timeless. It represents mathematical knowledge as single, fixed hierarchical structure founded in set theory and logic an developing level by level, not to greater abstraction but to greate complexity. This structure is neither necessary nor sufficient to represen the bulk of modern mathematics. The account leaves out combinatoria mathematics which has grown dramatically in import with the development o the electronic computer, including, for example Chaology. In short, th Bourbakian approach, for all its undoubted strengths, is only one way o conceptualising mathematics, and does not have the essential an foundational character it was taken to have by Piaget. Thus thi structuralist feature of Piaget's theory can be rejected. Elsewhere, provide a more complete critique of Piaget's structuralism (Ernest, 1991).

Piaget's clinical interview method is undoubtedly an importan contribution to research methodology in the psychology of mathematic education. With its accompanying methodological assumptions it is among th most widely used approaches of today. Quite rightly so, when in-depth information about an individual's thinking and cognitive processing i required. However, as even its title suggests, the clinical interview is not the invention of Piaget. Depth psychology, sociological and anthropological interview approaches all offer parallels from outside Piagetian psychology. Indeed the very title 'ethnomethodological approaches', widely used in educational research, explicitly refers to another disciplinary tradition. Piaget deserves credit for introducing the clinical interview methodology to the psychology of mathematics education. But it cannot be claimed to result uniquely from his constructivism, given that broadly similar approaches also result from widely different theoretical bases.

Glasersfeld's constructivism

Ernst von Glasersfeld has extended the foundational work of Piaget significantly, developing a well founded and elaborated constructivist epistemology. He bases this on the following two principles.

Principle A: The 'Trivial' Constructivism Principle

"knowledge is not passively received but actively built up by the cognizing

subject" Glasersfeld (1989, page 182).

Principle B: The 'Radical' Constructivism Principle

"the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality." op. cit.

Principle A has important psychological and educational implications. It means that knowledge is not transferred directly from the environment or other persons into the mind of the learner. Instead, any new knowledge has to be actively constructed from pre-existing mental objects within the mind of the learner, possibly in response to stimuli or triggers in the experiential world, to satisfy the needs and wants of the learner her/himself. An immediate consequence is that the transmission model of learning, as assumed in the crudest forms of the 'lecture method', is seen to be a grossly inadequate model. Although lectures may succeed in communicating to learners, the underlying mechanism is far more complex than that implied by the 'transmission model'. One major difference is that individual learners construct unique and idiosyncratic personal knowledge, even when exposed to identical stimuli. This model of learning has had a profound impact on research on the psychology of mathematics education in the past decade, and also underpins many recent developments in teaching.

Despite this great influence on the twin practices of research and teaching, two important caveats concerning Principle A should be stated. First, although it may help to reconceptualize the teaching of mathematics, the Principle does not strictly imply or disqualify any teaching approach. Rote learning, drill and practice, and passive listening to lectures can, as they always have, give rise to learning. The activity that is necessitated by the Principle takes place cognitively, and so visible inactivity on the part of the learner is irrelevant. Some teaching techniques may be more or less efficient than others (although I suspect that this is only the case given also certain learner and contextual factors). However, this is not the issue here. The 'trivial' constructivist view of learning does not rule out any teaching techniques in principle. Nor does it equate to the 'discovery method' or problem-solving teaching approaches, as Goldin (1989) also argues, although it can be used to support them.

Secondly, the model of learning and the concomittent psychological and educational implications of Principle A are not the unique consequences of it. The unique and idiosyncratic nature of learners' mental constructions

also follow, to a very large extent, from other psychological approaches such as those of Ausubel, Kelly, and others, as well as from Repair Theory, information processing and other cognitive science approaches; from sociological theories such as those of Mead, Schutz, Berger and Luckmann; and from the hermeneutic and interpretivist research paradigms. Once again, although constructivism (and in particular, Principle A) may have been a great stimulus in mathematics education research on idiosyncratic learner meaning-construction, it is only one of many theories in the social sciences giving rise to comparable theoretical insights.

Principle B adds a further important epistemological dimension to constructivism (transforming it into 'radical constructivism'). It claims that all knowledge is constructed, and that none of it tells us anything certain about the world, nor presumably, any other domain. This is not entailed by Principle A, for it is consistent to assume that objective truth exists, but that the cognizing subject constructs idiosyncratic personal representations of it, following Principle A. This is trivial constructivism's claim, which accepts A but rejects B. Principle B has been criticised for leading to the denial of the existence of the physical world (Goldin, 1989; Kilpatrick, 1987). However, this is an incorrect conclusion (see Ernest, 1990). Radical constructivism is consistent with the existence of the world. All that it denies is the possibility of any certain knowledge about it. Glasersfeld (1989b) has explicitly made the point that radical constructivism is ontologically neutral.

The implication of Principle B that there is no certain knowledge is very important, both philosophically and educationally. Applied to mathematics, important consequences can be shown to follow, provided that additional premises are assumed, notably a set of values (Ernest, 1990, 1991). However, the claim I wish to make is that Principle B, whether alone or in combination with Principle A, does not in my view lead to any new practical implications for education and psychology, beyond those of trivial constructivism. The one possible exception to this is the adoption of a sceptical epistemology, although what this adoption signifies in practical terms is not clear. Any good modern scientific practice must be ready to admit that its theories are refutable, following Popper, whether in the domain of subjective or objective knowledge. Hence even trivial constructivists should accept the refutability of scientific knowledge.

Even if it is allowed that there are stronger implications for practice than I have indicated, it can be said that such implications are by no means unique to Principle B, or indeed to any formulation of radical constructivism. There is a long sceptical tradition in philosophy which denies the existence of certain knowledge, which in the modern era finds expression in the work of Dewey and the pragmatists; Wittgenstein, Putnam, Rorty; and which constitutes a central tradition in the philosophy of science, in the work of Kuhn, Feyerabend, Toulmin and Lakatos. In the sociology of science and knowledge the dominant paradigm takes all knowledge to be a fallible social construction. Likewise in post-structuralist and post-modernist thought, such as that of Derrida, Foucault and Lyotard, no true knowledge of the world is assumed. Thus the denial of certain knowledge is far from unique to radical constructivism.

The Nature of Mathematics

Brouwer's Intuitionism is a view of the nature of mathematics sometimes associated with radical constructivism. Steffe (1988) argues that these views are consistent, and implies that Intuitionism is the appropriate philosophy of mathematics to consider. There are indeed parallels, for Brouwer argues for the subjective construction of mathematics, and maintains that mental constructions have philosophical priority over linguistic forms. However the weakness of Brouwer's position is the assumption that all the many subjective constructions of mathematics by different persons lead to the same body of knowledge. This is an essentialist view which assumes that a given structure of knowledge, notably mathematics, must emerge from any individual's consciousness. It also assumes that there is a unique body of mathematical knowledge, and so that any individual's construction, if correct, must produce a part of it. These conclusions contradict both principles. However, my intention is not to demonstrate that radical constructivism is inconsistent. As Lerman (1989) shows, radical constructivism and intuitionism are distinct and independent.

Glasersfeld does not commit this error, for his treatment is closer to the 'full-blooded' conventionalism of Wittgenstein (1978). Glasersfeld (1989c) draws the analogy between mathematics and chess, arguing that the seeming necessity of mathematical theorems follows from the social acceptance of its rules and definitions. However, the radical constructivist account of other persons is that they are hypothetical entities constructed by the cognizing subject to account for certain apparent regularities in its experiential world (Glasersfeld, 1989b). Since the warrant for accepting a mathematical theorem, is therefore that it conforms to socially accepted rules of justification, like those of chess, its certainty can be no stronger than any link in the overall chain of hypotheses and reasoning. The existence of other persons is part of the constellation of assumptions. Therefore mathematical knowledge rests on foundations at most as firm as the existence of certain hypothetical entities, notably persons. Given Principle B, this is a less than certain basis for mathematical knowledge.

I see this outcome as a strength of radical constructivism. A growing tradition in the philosophy of mathematics (in the works of Bloor, Davis, Hersh, Kitcher, Lakatos, Putnam, Tymoczko and Wittgenstein) regards mathematical knowledge as uncertain and fallible, contrary to traditional absolutism. Radical constructivism is consistent with this new tradition (see Ernest, 1990, 1991). But although these parallels offer support, they also mean that this conclusion does not follow uniquely from Principle B.

The Primacy of the Individual over the Social

Following the rational tradition in philosophy since Descartes the radical constructivist account begins with the assumption of a cognizing subject. The genesis of all knowledge is described in a narrative based on a epistemological/biological metaphor. This posits an homunculus, a miniature being at the core of the cognizing subject, which is necessitated by both principles. This subject is the active creator of knowledge, and the adaptive organiser of the experiential world. However, a weakness is that radical constructivism assumes the subject to be unproblematic. There is no question as to its origin or constitution: is it unitary and indivisible?

Post-structuralist psychology, founded on the work of Foucault, Lacan and others, elaborated in Henriques *et al*. (1984), and applied to mathematics by Valerie Walkerdine, questions the unity and unproblematic nature of the cognizing subject. It regards the individual-social dualism that it necessitates as a false dichotomy, viewing the individual as constituted by social and discursive practices, including positioning in power relations. Other psychological traditions also deny the inviolable primacy of the cognizing subject, including the symbolic interactionism of

Mead, the Soviet psychology of Vygotsky, and the Activity Theory of Leont'ev and Davydov. Thus a problem for radical constructivism is to account for the cognizing subject itself, which cannot be taken as unproblematic. A range of theoretical perspectives suggests that the subject is at least partly constituted by the social context, and that language plays a more central part in the formation of mind than radical constructivism allows.

The secondary role accorded to the realm of the social is also a problem. Other persons are regarded as subjective constructs, and so the social domain, the source of parental and human love, of language, culture and mathematics, has no intrinsic primacy. To regard the social as secondary to the pre-constituted cognizing subject is again problematic, and this difficulty is not adequately resolved by radical constructivism.

Conclusion

Radical constructivism leads to significant consequences, but in each case these also follow from other theories. So what is unique about it? Although a number of theories lead to similar consequences in methodology, learning theory, sceptical philosophy and fallible philosophy of mathematics, none offers an account so inclusive and wide ranging. Hence a unique strength of radical constructivism is its breadth. This is important, for a single broad theory is more powerful than a set of narrow ones. Perhaps for this reason, despite the problems raised here and elsewhere, constructivism remains one of the most fruitful philosophies of mathematics education research today.

References

Ernest, P. (1990) Social Constructivism as a Philosophy of Mathematics: Radical Constructivism Rehabilitated? Poster, PME-14, Mexico.
Ernest, P. (1991) The Philosophy of Mathematics Education, London: Falmer.
Glasersfeld, E. von (1989a) in Husen, T. ed. International Encyclopedia of Education (Supplementary Volume), Oxford: Pergamon, 162-163.
Glasersfeld, E. von (1989b) Poetics, 18, 435-448.
Glasersfeld, E. von (1989c) A Constructivist Approach to Teaching. Unpubl.
Goldin, G. (1989) Proceedings of PME 13, 2, Paris, 15-22.
Henriques, J. et al. (1984) Changing the Subject, London: Methuen.
Kilpatrick, J. (1987) Proceedings of PME 11, 1, Montreal, 3-27.
Lerman, S. (1989) Educational Studies in Mathematics, 20, 211-223.
Malone, J. and Taylor, P. (1989) ICME-7 Topic Group Proposal. Unpubl.
Noddings, N. (1989) Constructivism in Mathematics Education. To appear.
Steffe, L. et al. (1988) A1 ICME-6 Program Statements, Budapest, 1988: 9-17.
Wittgenstein, L. (1978) Remarks on the Foundations of Mathematics, Cambridge: MIT Press.

COGNITION, AFFECT, CONTEXT IN NUMERICAL ACTIVITY AMONG ADULTS

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Describing the specific discourses within which a subject is positioned when confronting a "mathematical" problem can help us to understand its emotional charge for that person. In addition it can be argued that the subject's responses, and indeed those of the researcher, can be understood only when the effects of anxiety and unconscious defences are acknowledged. These points are illustrated from an analysis of interviews with adult students about a range of numerical problems.

THEORETICAL FRAMEWORK

Many researchers have pointed to the inseparability of affect and cognition. For example, D'Andrade (1981) suggests that the distinction "affective" vs. "ideational" is analytic only; thus, saying "the stove is hot", or "Joe is a cheat" convey both types of meaning. The strength of the affective component of such sayings is due to its being communicated "through face and voice by the important people in one's life" (p.193).

However, much of the research on adult cognition in maths or numeracy seems to make little reference to affect. For example, Carraher, Carraher and Schliemann (e.g. T. Carraher, 1988) and Lave (1988) have challenged traditional cognitivist assumptions in stressing the context of problem solving. Yet the former emphasise the "strategy" used, esp. whether it is oral or written, and Lave's emphasis on goals, expectations, "value", rather than feelings and desire, means that emotions are seen as being experienced in a rather cognitive way.

Some, but by no means all, quantitative research finds a relationship between performance and affect that is seen as general across a set of individuals; see e.g. Llabre and Suarez (1985). However, Legault (1987) argues the need to study particular cases, since success or failure (in maths) is inscribed in a whole dynamic particular to each subject: it is not possible to explain these outcomes in a general fashion, nor to tie them to a specific affective factor (p.123).

Valerie Walkerdine's work focusses on social difference in child development and cognition, esp. gender and social class differences, and on the pain, anger and anxiety that form part of the "lived experience" of these differences (e.g. 1990).
(i) Cognitive and affective are viewed as complementary aspects of the same ensemble; mathematical meanings are learned within specific practices in which learners are "regulated", e.g. by parents or teachers, and develop their subjectivity; hence, these practices are emotionally charged:
"So 'mathematical meanings' are not simply intellectual, nor are they comprehensible outside the practices of their production." (Walkerdine et al., 1989, pp. 52-53).
(ii) Thus, different contexts, such as home and school, are characterised by different practices and related languages and meanings: "discursive practices".
(iii) These discursive practices "position" subjects: for example, in "eating out", being the one(s) who pay(s) is determined in many contexts in a complex interplay of gender and age positioning, certain ploys, etc.
(iv) In many practices, it can be argued that the emotional charge takes the form of deep-seated desires, anxieties and fantasies, e.g. "Reason's Dream" (Walkerdine, 1988); see also Taylor (1988). Anxiety is taken as a signal of unconscious conflict that leads to the mobilisation of defences. This points to the importance of the unconscious, and of psychoanalytic ideas.
These ideas led to a number of research questions, including:
(1) how to determine which practices are called up by the setting, and by the problems posed in the interview;
(2) how to describe the performance of the subject; and
(3) the relationship between cognition and affect, esp. anxiety. Here I focus on (3); on (1) and (2), see Evans (1988).

METHODOLOGY

The setting was a U.K. Polytechnic with a relatively high proportion of "mature" students (over 21 years of age, returning to study after some years of work or child-care).
Over three years, new entrants to two degree courses were asked to complete questionnaires including items about previous numerate experiences, and a "performance" scale, followed immediately by a version of the Maths Anxiety Rating Scale. At the end of their first year, a subsample (n=25) were given an interview based on a number of "practical" problems - e.g. reading graphs, deciding how much (if at all) they would tip after a meal out, deciding which bottle of sauce to buy.

Question (3) above may be addressed in my research in a number of ways: here I focus on (A) between-subject analyses of the questionnaires, and (B) within-subject analyses of the

interviews. Between-subject analysis of questionnaires used correlations of indicators of performance - separated into "school maths" and "practical" performance, with those of anxiety, separated into "maths test anxiety" (relating to school maths), and "numerical" anxiety (relating to everyday contexts). Within-subject analysis of interviews included description of illustrative incidents, plus in-depth analysis of the flow of the interview.

RESULTS

For (A), I found an inverted U relationship between school maths performance measure and maths test anxiety score, over the whole sample (n>900), when students' qualification in maths, gender, age, parents' occupation, and own previous occupation (if any) were held constant: using this non-linear model, the maximum performance was estimated to correspond to a neutral anxiety scale score ("neither anxious nor relaxed"), with performance levels estimated to decrease for students reporting either more, or less, anxiety. However, in order to explore what such a general relationship between scales might mean, it was decided to explore the meaning of performance and anxiety in a range of episodes in a sample of interviews.

For (B) some results of the within-subject analysis of one interview will be presented. The subject was female, aged 19, middle class (by parent's occupation), with A-level in maths.
A student of Town Planning, she worked part-time, currently as an electronics assembler, and previously in a shop.
1. Her performance on the questionnaire had been strong: 22 of 23 Qu. correct, except for one on the practical maths scale - "Suppose... (in) a restaurant... the bill comes to a total of 3.72; if you wanted to leave a 10% tip, how much would the tip be?"; her response: "37.2 p". Scored "wrong" (as 1p was then the smallest money unit). Her maths anxiety responses on the Q're were in the 3rd decile of the whole sample, on both numerical anxiety and maths test anxiety - but in the top decile on a measure of general anxiety. Thus, she certainly would be well below average for women on both dimensions of maths anxiety - viz. numerical (practical) and maths test anxiety (School Maths), but above average on general anxiety. Thus she seems to fit well with the "inverted U" relationship between performance and anxiety shown in (A).

2. "Performance" in interview: Four problems, incl. Qu.2 [10% of 6.65] were done in her head, and "correctly". Qu.4 [15% tip on a meal of £3.53] and Qu.5 [9% increase on wages of £66.56] were done with pencil and paper, ultimately correctly, though Qu. 4 begins with a slip - see below. She expresses overwhelming confidence after almost every question in the interview, e.g. for Qu. 1 [reading a pie-chart] : "very familiar, know exactly what it means... don't have to think about it..." (interview transcript, p.3) - except for Qu. 4!

3. For most questions, this S seems to be positioned in school maths (SM), e.g. for Qu.2. Qu.4, where she was first asked to "choose from a menu", seems to call up the practice of "eating out at restaurants". However, she reverts to using pencil and paper - indicative of SM - to calculate a 15% tip. That more than one discourse may have been called up for Qu. 4 is an illustration of being "positioned inter-discursively" (e.g. Walkerdine) or of the "proportional articulation of structuring resources" (Lave, 1988).

4. Qu. 4 deserves more detailed consideration. When I ask if she ever goes to a restaurant with a menu like that shown, she seems to reply very quietly and hesitantly. After she "chooses" the seafood platter [£3.53], I ask how much she would tip for a restaurant meal: she replies, somewhat hesitantly, "...well, 15%, I suppose..." (p.7).

5. When I ask what a 15% service charge would be, she says "Well, I'd have to use pencil and paper". Then -

S : [7 sec. / inaudible / coughs / 6 sec.] Well, 23 1/2 pou- no, that's wrong... [12 sec.]...what I've done wrong, oh (JE: Is it wrong?) Yeah, umm [laughs nervously]... I don't know what I'm doing... [She realises she has divided 15% into £3.53, instead of multiplying]
(JE [2 lines]) S: [15 sec.]... 52.95, 53 pence.

She explains that she rejected the answer produced by dividing because "I just saw that it was obviously not right... it was far too small" (p.7).

6. Further on (p. 8), she indicates that in restaurants, "I don't usually pay". But she "looks at prices and things,... add them up in my head...". Even if she's not paying?: "I don't want to be an expense".

DISCUSSION

I. The slip in 5., and what she says in 6., are suggestive of anxiety. Possible interpretations of this:
(a) anxiety to do with the interview itelf, which she is experiencing as an evaluative situation (supported by an item by item analysis of her responses to the anxiety scales);
(b) anxiety experienced about the question itelf, since she is not confident about doing a slightly more complex calculation for Qu. 4 (a 15% tip) than she has so far had to do (NB:

Almost all students were asked to calculate a 10% tip at this point in the interview, though of course she would not know this.);
(c) anxiety recalled about doing the right thing in a restaurant - comes up before the 15% calculation, and interferes with her doing it.
Support for (c): her hesitation, etc. at the presentation of the restaurant menu: and "I don't want to be an expense".
Counter-indication for (a) and (b): In the next Qu., she had to calculate a 9% pay increase, also on paper, also a "complex" calculation, but she got it right.

II. The anxiety evinced might appear to be "mathematics anxiety", since it appears while doing something that an observer could choose to label as "mathematical" - though it could also be labelled as "being interviewed" or even as "talking about eating out", etc. However, calling the anxiety "mathematical" would seem to be accurate only if we were to assume that she was positioned solely in a school maths discourse, as for (b). For (a), she is positioned inter-discursively as an interviewee, as well as in SM, and for (c) positioned inter-discursively in "eating out" as well as SM.

III. Her performance appears very competent in both Q're and interview (though many of the questions should not be difficult for someone with A-level maths) - except for the slip on Qu. 4. In terms of affect, she gives a picture of overwhelming confidence about maths and the use of numbers, but seems to show anxiety when she applies the incorrect operation for tipping. Is it possible to interpret her expression of confidence as a defence against anxiety?

RE-ANALYSIS

This last point suggests a need for insights from psycho-analysis in analysing the interviews, here done within an ethnographic perspective. Hunt (1989) suggests some guiding principles:
(i) everyday activity is mediated by unconscious meanings linked to complex webs of signification ultimately traceable to childhood experiences; thus, (ii) transference is a routine feature of many fieldwork relationships; (iii) intrapsychic conflict routinely mobilised vis-a-vis social events, esp. if anxiety is aroused or links made to unresolved issues from childhood (pp.25ff.). These insights may be used to add a deeper level to the analysis of the tipping incident:

5'. my asking her a more difficult question than other Ss: She was one of the very few students to suggest a 15% tip as normal. In the interview, I was struck by the fact that she had A level maths, attained by only 7% of that year's entry, and impressed by her expressions of confidence. I recall feeling some anxiety myself: unlike most of the other interviewees, I had

not met this student before; I was concerned about how the interview was going, and esp. that she might be bored with such easy questions. Then came her responses to Qu. 4. Here, given the "opening" by her mentioning 15% as a tip, and with the above "reasons" in play, I asked her what a 15% tip would be, likely a more difficult problem than that [10%] asked to other Ss.

6'. her "slip": First, we can note that she seems anxious from when the "menu" is presented: she hesitates, and speaks very quietly. She begins the calculation by asking for a pencil and paper, then divides 15% into 3.53, instead of multiplying - not a mistake that one would expect from someone with A-level maths (!) Recall that her only question marked "wrong" on the questionnaire was the calculation of a 10% tip on a restaurant bill of £3.72 as "37.2 p". We can conjecture that anxiety does seem to have been triggered by this question - and that it is anxiety about the context of eating out, perhaps about the relationship(s) in this context. We might next conjecture that these anxieties are related to the fact that she made a slip, and further related to the content of the slip: the latter, involving division rather than multiplying, led to a result that was smaller than it should have been; when we remember that she later admits to not wanting to "be an expense", we might say that her slip was "motivated" by the anxiety.

7'. her anxiety about being an "expense": Here the term "expense" signifies in different ways - as an amount which could be arithmetically calculated, or as being a burden within a relationship with other(s) more powerful in resource terms and on whom the subject is dependent (a parent or partner). The play across these different senses, related to different positionings, is likely to provoke anxiety, associated with the activity of eating out, with the related social relationship(s), and with any operations involved in the activity, e.g. choosing a dish, calculating the total cost of her meal. In what she says, the relationship, and her position within it, is referred to by a quantitative term.

CONCLUSIONS

(1) This analysis shows the role that unconscious anxiety may have played in the responses made by a student solving "maths" problems. It questions whether anxiety which might seem at first seem so "mathematical" does indeed have these origins. And it argues further that anxiety is not a general attribute, but rather is specific to the practice(s) concerned.
(2) I also show that the relationship between anxiety and performance, rather than being general, can only be fully grasped through analysis of particular cases, as here.
(3) This points to the conclusions that cognition and affect are aspects of a whole; that the relationship relates to social differences and specific discursive practices; that affect can interfere with (or support) cognition.

(4) Analyses such as these contribute both to the develoment of an adequate theory of mathematical learning and its difficulties, and to programmes aiming to acknowledge classrooms as "settings where teachers and students mutually produce mathematical meanings" (Noss et al., 1990) - by illustrating once again the sorts of meanings that students may bring with them into the classroom.

REFERENCES

CARRAHER T. (1988). Plenary Address; in A. BORBAS ed. PME-12, Veszprem, vol I, 1-23.
D'ANDRADE (1981). Cognitive Science, 5, 3, 179-195.
EVANS J. (1988). In A. BORBAS ed. PME-12, Veszprem, vol. I, 296-303.
HUNT J. (1989). Psychoanalytic Aspects of Fieldwork. Sage.
LAVE J. (1988). Cognition in Practice. Cambridge U.P.
LEGAULT L. (1987). In J. BERGERON ET AL. eds. PME-11, Montreal, vol. I, 120-125.
LLABRE M. AND SUAREZ E. (1985). J. Counselling Psych., 32, 2, 283-287.
NOSS. R., HOYLES C. AND SUTHERLAND R. (1990). In G. BOOKER ET AL. eds., PME-14, Mexico, Vol. I, 175-182.
TAYLOR N. (1988). In C. KEITEL ET AL. eds., Mathematics, Education and Society. UNESCO Sci. & Technol. Ed. Doc. no. 35.
WALKERDINE V. (1988). The Mastery of Reason. Routledge.
WALKERDINE V. (1990). Plenary Address, PME-14.
WALKERDINE V. AND GIRLS & MATHS UNIT (1989). Counting Girls Out. Virago.

TEACHERS' PEDAGOGICAL KNOWLEDGE: THE CASE OF FUNCTIONS

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This study concentrates on two interrelated aspects of teachers' pedagogical content knowledge: One being teachers' knowledge and understanding of students' conceptions, preconceptions and misconceptions. The other, teachers' responses to students' questions, remarks or hypotheses concerning subject matter. The goals of the study are: (1) to study teachers' pedagogical knowledge according to those two aspects, and (2) to investigate the potential use of the research instrument and results in teacher education. The contextual content in which we conduct our research is mathematical functions. A questionnaire that included several tasks was administered to junior-high teachers. Each task described a situation in which the teachers had to react to a student's question or idea. The teachers' answers were analyzed according to the following dimensions: (i) content knowledge, (ii) awareness of the student's difficulties, and (iii) kinds of teacher responses. Activities for raising teachers' awareness with regard to the above aspects were designed and conducted.

INTRODUCTION

Research and theoretical work on teachers' knowledge has now shifted its focus from general knowledge to subject matter specific knowledge. One category of teachers' knowledge of the latter kind is pedagogical content knowledge which has gained greater attention in recent years. Shulman (1986) describes this kind of knowledge as knowing the ways of representing and formulating the subject matter that make it comprehensible to others; understanding what makes the learning of specific topics easy or difficult; knowing the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning.

The quality of teachers' pedagogical content knowledge is crucial. Teachers need to focus on what students are doing and why, and use this information to make knowledgeable decisions about appropriate actions in order to help the student. This kind of attitude to teaching is a natural consequence of accepting a (even a partial) constructivist point of view to learning. Since, as von Glasersfeld (1984) says, "the teacher's role will no longer be to dispense 'truth' but rather to help and guide the student in the conceptual organization of certain areas of experience...[one of the things teachers need in order to do this is] an

adequate idea of where the student is". Only when the teachers know where the students are, can they, as Confrey (1987) states, "build from these limited conceptions towards more sophisticated conceptions".

Being knowledgeable about students' ways of thinking about the subject matter is not all that there is to pedagogical content knowledge. Pedagogical content knowledge is the process of integrating knowledge (Ball, 1988). When teachers represent mathematics they are influenced by what they know and believe across different domains of knowledge: mathematics, learning, learners, and context.

Our research concentrates on two interrelated aspects of teachers' pedagogical content knowledge. One aspect has to do with teachers' knowledge and understanding of students' conceptions, preconceptions and misconceptions. The other aspect deals with teachers' actions: how they respond to students' questions, remarks or hypotheses about the subject matter. The goals of the study are: (1) to study teachers' pedagogical knowledge according to those two aspects, and (2) to investigate the potential use of the research instrument and results in teacher education. The contextual content in which we conduct our research is mathematical functions.

METHOD

The participant subjects in this study are junior-high teachers (the concept of function is usually introduced in those grades). A questionnaire that included several tasks was administered to the subjects. Each task described a situation in which the teachers had to react to a student's question or idea. For most of the tasks the subjects first had to explain what the student might have had in mind when asking the question or stating the idea. Then, the teachers were asked to describe how they would respond to the student in that situation. A written reaction to the questionnaire was also collected from the teachers. Then, a whole-group discussion was held on some of the tasks. After a non-structured open discussion, the teachers were presented with examples of other teachers' answers (from a pilot study) and analyzed them according to the following dimensions: (i) content knowledge, (ii) awareness of the student's difficulties, and (iii) kinds of teacher responses.

Some of the tasks in the questionnaire were based on students' misconceptions as reported in the literature (e.g., Markovits, Eylon, & Bruckheimer, 1986; Vinner & Dreyfus, 1989) as illustrated by Tasks 1&2 below. Other tasks dealt with the development of the concept of function and its definition as illustrated by Task 3.

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Task 1
A student was asked to find the equation of a line that goes through A and the origin O (see Figure 1).

Fig. 1 Fig. 2

The student said: "Well, I can use the line y=x as a reference line. The slope of line AO should be about twice the slope of the line y=x, which is 1 (see Figure 2).
So the slope of line AO is about 2, and the equation is about y=2x, let's say y=1.9x."
a) What do you think the student had in mind?
b) How would you respond?

Task 2
A student is asked to give an example of a graph of a function that passes through the points A and B (See Fig.1).

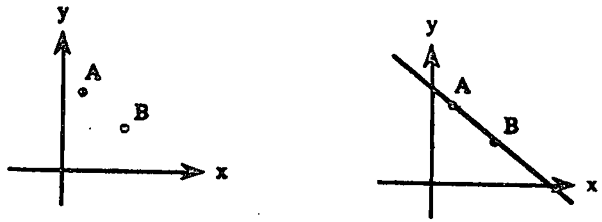

Fig. 1 Fig. 2

The student gives the answer as shown in Fig.2.
When asked if there is another answer the student says that there is only one such function since through two points passes only one straight line.
a) What do you think the student had in mind?
b) How would you respond?

Task 3
A student asks you why there is a need for the requirement of a unique image for every element in the domain in the definition of function. How would you respond?

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WHAT DOES IT TELL US?

Several dimensions were used for the analysis of the questionnaire data. They are illustrated by examples from the subjects' responses.

Content Knowledge

Teachers' content knowledge plays an important role in their understanding and evaluating of students' questions and answers, as was found, for example, in a study conducted in the USA of prospective secondary mathematics teachers' knowledge about teaching mathematical functions (Even, 1989). The questionnaire was not designed to directly explore content knowledge. However, the way the subjects answered revealed, in many cases, their own subject matter knowledge. For example, as a response to Task 2 a teacher said: "Through two points passes one line, but this line can be represented by a number of functions". Having this (wrong) conception about the relationship between lines and functions, it is questionable whether this teacher can help his/her students understand the subject matter.

Understanding What the Student Does Not Understand

Knowledge of the subject matter is not enough. A teacher may know the content and still have difficulties understanding what is it that the student does not understand. "Getting into a student's head" is not easy. But it is necessary for a teacher to do in order to be able to help the student construct his/her own knowledge. Some teachers ignore students' ways of thinking and their sources. Instead, they evaluate students' work only as either right or wrong .

For example, one teacher "explained" what the student had in mind in Task 1 as: "The student wanted to find the slope and to use the slope for finding the equation. But she didn't do it right". This "explanation" is very different in its nature from an explanation of another teacher: "The student meant that the angle between line AO and the x-axis is almost twice the angle between the line y=x and the x-axis, and by mistake mixed up angles and slopes". The first teacher completely ignored the (wrong) connection between angles and slopes which the

student had made and just remarked: "She did not do it right." This teacher did not relate to the source of the student's mistake which the second teacher had done. Ignoring the specific process the student used in Task 2 was also found in Even (1989) where the results show that about one-third of the prospective teachers did not pay attention to the student's way of thinking. A teacher who merely says to a student that his/her answer was wrong cannot help the student realize that his/her assumption about the linear relationship between a slope and an angle was wrong.

Teacher Response

Teacher response to a student's question or idea depends on several factors. For example, the teacher needs to choose an appropriate representation of the subject matter (Wilson, Shulman & Richert, 1987), take into account the place and time where the situation takes place, the learner's knowledge, etc. The subjects in this study could not take all these factors into consideration when they answered the questionnaires. In our analysis of the data we chose to concentrate on several important aspects that the subjects could have considered. Some of them are interrelated with the two dimensions described above.

Concentration on the student's misconception. A teacher's response may be directed towards the student's misconception. For example, "I'll tell the student that there is a difference between a slope and an angle..." (Task 1). On the other hand, the response may be general without any specific reference to the student's difficulty, as shown in the following response: "There is a need to explain again the topic of slopes, and how to calculate them". This teacher did not respond to the wrong connection of slopes and angles the student made but rather decided to explain the whole topic again.

Ritual versus meaning orientation. A teacher response may emphasize rituals or attend to meanings. As a response to Task 3, for example, one subject chose to repeat the rule in different words: "Because every function is a relation, but if one represents functions algebraically then for each x there must be only one y. But the other way around is not needed." In contrast, another subject's response attended the "why?": "The request of a one

to one correspondence will limit the applied area of functions. For example the quadratic functions or functions such as $f(x)=|x|$ will not be functions any more". Although being incomplete, in this answer the teacher tried to attach meaning to the explanation.

Teacher versus student centered. Some responses were teacher-centered, i.e., the teacher tried to teach by <u>telling</u> the student how to do things--to "<u>transfer</u> knowledge" whereas the student was expected to stay relatively passive. A response to Task 1 illustrates this point: "<u>I'll explain</u> the slope as $\frac{\Delta y}{\Delta x}$ and <u>I'll show</u> the student, by drawing a line, what the slope of a line that goes through the origin and has $\frac{\Delta y}{\Delta x}$ is." Other responses were student-oriented and emphasized the <u>construction</u> of knowledge by an active student. For example, "I'll <u>ask the student</u> to draw the graph of the function s/he suggested, y=2x, so that s/he can see the mistake".

Richness of responses. In most of the responses the subjects suggested only one explanation for the student's way of thinking and chose one way to react. Still some suggested more than one way to explain and to respond. For example, one subject said that there may be two interpretations of the situation in Task 2: Either the student studied linear functions only--in which case she would not do much, or the student had a broader knowledge about functions--in that case she would get into a deeper explanation. Suggestions of only one explanation to complex situations were also found in Even (1989). The reason for this is not clear. It might be that the subjects were not aware of the complexity of the situation. It also may be attributed to not wanting to spend too much time answering the questions. This issue will be further studied during interviews.

It is clear that the responses depended on the tasks. There were tasks, for example, in which understanding what the student might have had in mind was not difficult (e.g., Task 2), while in others it was more difficult (e.g., Task 1).

But it seems that the responses depended also on the subjects' teaching style. Whereas some of the subjects responded qualitatively differently to the various tasks, there were also

subjects whose responses to all tasks were of the same kind. This is illustrated by the following selection of one subject's responses that were all teacher centered: "I'll explain to the student that a slope is not an angle but the tangent of the angle, or another way if the student does not know tangent (Task 1)", "I'll give the student another example of a function that passes through A and B (Task 2)".

WHAT DOES IT TELL THE TEACHERS?

After handing in the questionnaire, the subjects were asked to write down their reaction (no specific instructions were given). Most of them emphasized the importance of paying attention to students' ways of thinking (the first part of the tasks). But almost none related to the second part of the tasks--the teachers' responses. For example,

> During the learning process students make mistakes, and we, as teachers, correct them, but pay no attention to the source of the mistakes. The questionnaire made me aware of the need to think about the source of the mistake and not only to correct it.

It seems that after answering the questionnaire, the subjects recognized the central role that teacher's understanding of students' thinking should play in teaching. But most of them did not recognize the importance of teacher's reaction.

After a non-structured open discussion on some of the tasks, the teachers were presented with the dimensions that we used for the analysis of the questionnaire data. They were also given examples of other teachers' answers (from a pilot study) and were asked to analyze them according to the given dimensions. It seems that this activity helped to create teachers' awareness of the importance of teachers' reaction. For example, one of the participants said:

> This is the first time that I debate about how to respond to a student while taking into consideration his/her mistake. First, I have to understand what the source of the mistake is, and try and think that may be my explanation was not good enough or perhaps I shouldn't rush and explain, but ask the student for an explanation of his/her way of thinking, and how he/she came to that conclusion.

CONCLUDING REMARKS

Pedagogical content knowledge is crucial in teaching and therefore should become an integral part of preservice teacher education. Even though a teacher education program cannot cover "everything" that is related to pedagogical content knowledge, it can and should provide a solid basis for teachers to build their own pedagogical content knowledge. Later on, inservice teachers should be guided to construct different aspects of pedagogical content knowledge. The tasks in our questionnaire involve both subject matter and teaching style. Therefore, preservice and inservice teachers can benefit from working on it while the emphasis can be on different aspects according to the background and needs of the teachers.

REFERENCES

Ball, D.L. (1988). Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education. Unpublished doctoral dissertation. Michigan State University, East Lansing, MI.

Confrey, J. (1987). The constructivist. In J.C. Bergeron, N. Herscovics, & C. Kieran (Eds.). Proceedings of the 11th International Conference for the PME. Montreal.

Even, R. (1989). Prospective secondary teachers' knowledge and understanding about mathematical functions. Unpublished doctoral dissertation, Michigan State University, East Lansing, MI.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. For the Learning of Mathematics, 6 (2), 18-24, 28.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15 (2), 4-14.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. Journal for Research in Mathematics Education, 20, 356-366.

Von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), The invented reality. New York: Norton.

Wilson, S.M., Shulman, L.S., & Richert, A. (1987). "150 ways of knowing": Representations of knowledge in teaching. In J. Calderhead (Ed.), Exploring teacher thinking. Holt, Rinehart, and Winston, Sussex, pp. 104-124.

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Cognitive Tendencies and Abtraction Processes in Algebra Learning

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In the present paper we discuss eleven kinds of phenomena which make themselves manifest at the time students are learning new mathematical concepts and operations. These phenomena can be observed both in the classroom and during clinical interviews. When learning mathematics new rules are constantly being formed, due to the fact that new ways are found to extend conceptual networks previously developed. A central aspect in this point of view is the idea of sense, as opposed to meaning, when speaking about a MSS. In all the comments, we have tried to distinguish between these two notions: meaning and sense.

INTRODUCTION

The results of more than fifty clinical interviews performed at the Centro Escolar Hermanos Revueltas, in Mexico City during the last five years are discussed. In order to further clarify some of the terms introduced, and to show their interrelationships and significance, we shall use examples from studies such as Filloy (1986 and 1990); Figueras, Filloy and Valdemoros (1986a and 1986b); Filloy and Lema (1984); Filloy and Rojano (1984, 1985a and 1985b). Basically, the experimental development of the present study is depicted in those works; and the theoretical framework where the notion of Mathematical Signs Systems (MSS) and its use are discussed can be found in Filloy (1990).

We shall begin by presenting, in episodic form, the description of a typical clinical interview where, as a teaching sequence, a strategy is being used in which the starting point is a concrete model. A more detailed description of the study for which these interviews were effected can be found in Filloy and Rojano (1989), where a discussion and analysis of this kind of model are also offered.

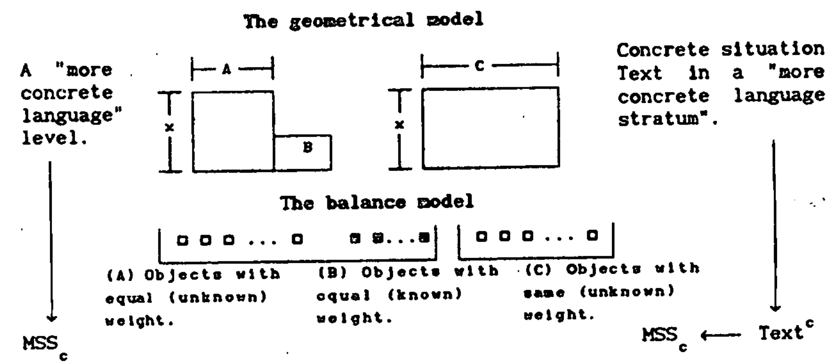
Episodic Analysis of a Typical Interview

The didactic sequence is designed to provide the student with a number of problem solving situations which are described in a concrete (balances, piles of stones, plot exchanges, etc.) language (both iconic and written). The aim is that, at the end of the sequence, students can solve linear equations, syntactically. We are going to designate the Mathematical Signs System that we wish to teach as MSS_a (the subscript a denoting that this system is more abstract than the MSS_c, which is more concrete; signs, in the latter, bear a more direct relationship with certain meanings derived from more concrete situations: land plots and geometrical properties, balances and the equilibration properties, etc.).

Thus, the first type of situations in MSS_a are text such as:

$Ax + B = Cx$, where A, B and C are given positive integers and in this case, $C>A$. (The children were presented with particular numerical instances),

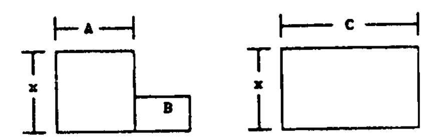
which, at a more concrete level, within the MSS_c mentioned above, assume the form:



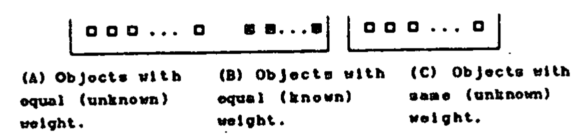
FIRST EPISODE

STEP A Interpreting Text^a as Text^c

Step A: translating the equation into the model

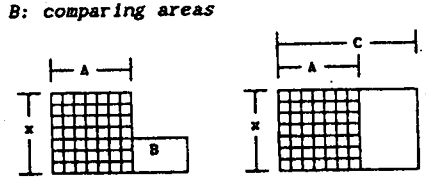


Step A: translating the equation into the model

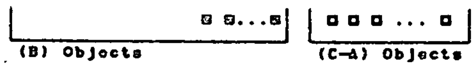


STEP B Unchaining of known actions in the more concrete language stratum, in order to decode the problem situation.

Step B: comparing areas



Step B: repeated removal in pairs of objects of unknown weight, maintaining equilibrium, until none remain in the left hand pan.



STEP C Performing concrete actions.

STEP D Decoding the problem situation until a solution is reached: a solution which is described in another Text^c.

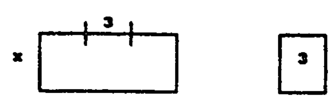
Comment 1. The assumption presupposes that these actions are well developed, as are also their properties, and that this permits solving the problem situation posed by Text^c in the "more concrete" language stratum MSS_c.

SECOND EPISODE

STEP E Translating into a Text^a, which is the solution to the problem situation described in Text^c.

Step E: writing the new equation, $(C-A)x = B$

STEP F Solving the equation $(C-A)x = B$. Previous teaching permits solving this "arithmetical" equation.
Of course, this is not quite that simple, for we can envisage an intermediate problem situation (in a "more concrete" stratum), either in an Text^a, or in a Text^c.
Here is an example of an intermediate problem situation:



"reduced equation $3x = 3$"

Sometimes, steps B, C and D must be performed anew.

THIRD EPISODE

At this point in the teaching sequence, the teaching strategy of REPETITION AND PRACTICE comes to the fore, with more examples to be solved, although with, for instance, larger and larger numbers (a, b, c) in the equation $ax + b = cx$, $c>a$.

STEP G. C, D and F are once more performed, giving rise to steps H and I.

STEP H Abbreviation process tending to leave aside many of the meanings which appeared in Text^c, and pointing to the arrival to a procedure at a syntactic level.

STEP I Production of intermediate personal codes to represent the actions performed and the intermediate results (see [7]).

Comment 2. An intermediate language stratum has been created, where meanings arise from the personal syntactic rules just created. Up the levels H-I, all actions performed to Text^c are DEPENDENT ON THE SENSE OF THE (CONCRETE) CONTEXT. Thus far, we have merely created a DIDACTIC DEVICE to solve equations of the type $ax + b = cx$, $c>a$. But what happens when a different Text^a is presented to the student; for instance, the new Text^a: $8x + 5 = 3x + 15$?

FOURTH EPISODE

STEP J This is the recognition that the problem situation is a new one, and that it cannot be reduced to a reading effected by means of the intermediate language strata just created, which are "more abstract" than the original concrete language stratum.

STEP K A new learning process is achieved (through discovery) by virtue of the teaching strategy itself, and thereby it becomes possible to unchain A, B, C, ... up to step J.

Comment 3. However, new "facts" come into play:

STEP L It takes less time to effect step H.
In step I, a refinement is made in the direction of having available meanings which are now freer from dependence on the sense of the concrete context. Step L is a denial of a part of the meanings arising from the language stratum in which the text in question is described. DIFFERENT SITUATIONS, NON-REDUCIBLE TO EACH OTHER, can NOW be interpreted in the same way; and the syntactic rules constructed in step I are, now, applicable to both Texts^c. It is only NOW, that both texts are recognized as belonging to the same type of problem situation and, therefore, the same solving process is unchained.

FIFTH EPISODE

STEP M A return is made to step K, with new types of problem situations. An example of a new Text^a would be: $8x - 3 = 5x + 6$.

STEP N Once again, steps A, B, ..., L are unchained.

STEP O Operations are created in a new language stratum in which senses are no longer dependent upon the concrete context, and new meanings are given to the new abstract objects.

Comment 4. By means of the strategy of repetition and practice, a new, more abstract language stratum is created, through which more abstract situations can be modelled (or into which such situations can be translated).

By this whole process (the five episodes), a collection of stratified MSSs has been created, with interrelated codes that allow the production of texts whose decoding will have to refer to several of those strata; the working out of the text will use actions, procedures, and concepts whose

...operties are described in some of the strata.

Two texts T and T', both produced with a set L of stratified MSSs will be called transversal if the user cannot work out T as in the decoding of T' --i.e., if T is not reducible to T' with the use of L (remember Comment 3). What happens usually is that the learner can work T and T', but cannot recognize the two codifications as a product of the different strata of L.

If we now have another stratified MSS, M, in which T and T' can be decoded and the working out of both can be described through the same actions, procedures, and concepts in M, the meaning of which has as referents the actions, procedures, and concepts used in the decoding of T and T' in L, then we will say that M is more abstract stratified MSS that L for T and T' (remember Step O).

To accomplish this, the actions, procedures and concepts used in M have lost part of their semantic-pragmatic "meaning", they are more abstract (see [2]).

The Learning of Elementary Mathematics: A Scenario where Concepts are Modified. We are paraphrasing here some of Wittgenstein's positions (see [10]) on what he thinks Mathematics is; here, however, we are only asserting such standpoints with respect to the cognitive processes within which the learning and, therefore, the teaching of mathematics develops. Oservations such as we have previously described, and such as follow, lead us to consider teaching situations, including those where mathematical demostrations are performed, as a source of new concepts through the establishment of, or by changes in the meanings of the MSS in which what has already been taught is described. This idea stands in marked contrast against what mathematicians (who are mostly Platonists) argue when they uphold the point of view that proofs achieve an exploration among preexisting notions.

Thus, it is our suggestion that when learning mathematics new rules are constantly being formed, due to the fact that new ways are found to extend conceptual networks previously developed. A central aspect in this point of view is the idea of sense, as opposed to meaning, when speaking about an MSS. In all the comments to the discussion above, we have tried to distinguish between these two notions: meaning and sense. Let us now proceed to the analysis of a number of "facts" which always make themselves present when, in a teaching situation, the learner is trying to pass from a more concrete language stratum MSS_c, to a more abstract one by following an approach such as the one described when analyzing the episodes in the typical interview.

Eleven Cognitive Processes Present when Learning more Abstract Concepts.

ONE The presence of an abbreviation process of concrete texts, which permits producing new syntactic rules. Consider the first episode; but, above all, the fourth episode and Comment 3 (see also, [7]), and step H.

TWO CONFERRING INTERMEDIATE SENSES. See: Comment 2; the analysis of step L in Comment 3; the progress described in step O.

THREE Returning to more concrete situations upon the occurrence of an analysis situation. This fact is always present in most actions of mathematical thinking and has been reported in many other investigations ([3], [4], [5], [6], [7], [8] and [9]). In the discussion above, it can be observed in Step F, and particulary in Step J, where a return is made to the use of parts of the concrete model which had already been discarded in the previous steps. The return to a more concrete situation is also observed in step M.

FOUR The impossibility of unchaining operations which a few moments ago could be performed. See Filloy [2], where such a behaviour is described when an attempt is made at solving the equation $Ax = B$. In [3], situations arise where operativity with fractions is inhibited by the presence of wrong spontaneous readings of a geometrical type concerning the notions of ratios and proportions of magnitudes, in a teaching sequence where the proof of Thales' Theorem is the leading string to provide new senses to the uses of concepts included in the arithmetic of fractions. See also step F, regarding ($Ax = B$) arithmetical equations whose operativity had been completely mastered by the whole population; yet, when submerged in the chain of steps described above, this causes that a majority of students loose their great operational ability for solving such equations.

FIVE Focusing on readings made in language strata that will not allow solving the problem situation. See again, in Filloy [2], the remark on 12-13 year-olds' performance when trying to solve problem situations based on the solving of equation $Ax = B$. Also, in [3], an example can be found of focusing on the wrong geometrical reading as regards the order of magnitude between ratios of magnitudes. In the above discussion, this behaviour can also be found in step F or step I, specially due to what is asserted in Comment 2 about the dependence of sense on the concrete context upon which the aforesaid "focusing" occurs.

SIX The articulation of erroneous generalizations. Literature on the mistakes made by students is brimful with such behaviour. The subject tends to free himself from the behaviour described under point FIVE by raising a rule to other contexts where its application makes no sense; we are faced here with a wrong use of such concepts and operations.

SEVEN The presence of appellative mechanisms which focus attention on the unchainment of wrong solving processes. In many cases, certain subjects can not adequately solve what we discuss in step F, due to this behaviour. Many of the phenomena which are mentioned under point NINE are also attributable to this behaviour.

EIGHT The presence of inhibitory mechanisms. The examples given above are typical of this behaviour; but also, within the framework of our discussion, the presence of negative solutions makes way for the obstruction of syntactical rules which had already been mastered. Insisting on not to begin to analyze a problem, on refusing to solve simple equations where radicals appear, the inability to use --in the intermediate steps of an analytical chain for solving a problem-- little mastered syntactical facts, etc., are further examples of this kind of behaviour. In our previous discussion, this phenomenon can occur in step O, when working on equations in which the coefficients of the unknown are negative.

NINE The presence of obstructions arising from the influence of semantics on syntax, and vice versa. Solving problems and conferring meanings to algebraic signs is a process which predisposes the individual to the use of syntax. Most of the phenomena described in point FOUR can be interpreted in such a fashion. The case is even observed that the subject writes a simple arithmetical equation (in the middle of the solving process for a problem) and does not recognize it as such, notwithstanding the fact that he or she has been very aptly solving it for years. In the case of syntax, the tendency to focus attention on more concrete use strata inhibits adequate readings of more abstract texts.

TEN The generation of syntactical errors due to the production of intermediate personal codes in order to confer senses to intermediate concrete actions. Consider step I, and Comment 3. Also, in [7] a description is given of this cognitive tendency. It can be observed, in that same paper, how this can generate syntactical errors.

ELEVEN The need to confer senses to the networks of ever more abstract actions, until converting them into operations. Gathered together, all the steps in the clinical interview stand as an example for this assertion.

REFERENCES

[1] FILLOY, E. Teaching Strategies for Elementary Algebra and the Interrelationship between the Development of Syntactic and Semantic Abilities. Proceedings of the Eighth Annual Meeting for the Psychology of Mathematics, North American Chapter. East Lansing, Michigan, 1986, pp. 108-113.

[2] FILLOY, E. PME algebra research. A working perspective. Plenary Address. Proceeding of the XIV PME Conference. Mexico 1990, pp. PII.1-PII.33.

[3] FILLOY, E. and LEMA, S. "Obstructions to Learn Elementary Geometrical Variation Concepts. Thales' Theorem". Proceedings of the Sixth Annual Conference of the Internacional Group for the Psychology of mathematics Education. North American Chapter. Madison, Wisconsin. 1984a.

[4] FILLOY, E. and ROJANO, T. From an Arithmetical to an Algebraic Thought (A clinical study with 12-13 year olds). Proceedings of the Sixth Annual Conference of the Internacional Group for the Psychology of mathematics Education. North American Chapter. Madison, Wisconsin. 1984b, pp. 51-56.

[5] FILLOY, E. and ROJANO, T. Obstructions to the Acquisition of Elemental Algebraic Concepts and Teaching Strategies. Proceeding of the Ninth Annual Meeting of the PME. International Group, 1985a.

[6] FILLOY, E. and ROJANO, T. Operating the Unknown and Models of teaching (A clinical study with 12-13 year olds with a high proficiency in Pre-Algebra). Proceedings of the Group for the Psychology of Mathematics Education North-American Chapter, 1985b.

[7] FILLOY, E. and ROJANO, T. Solving equations: the transition from Arithmetic to Algebra. For the Learning of Mathematics 9.2 (June 1989). FML Publishing Association Montreal. Quebec, Canada.

[8] FIGUERAS, O.; FILLOY, E. and VALDEMOROS, M. "Some considerations on the graphic representations of fractions and their interpretation". Proceedings of the Eighth Annual Conference of the Internacional Group for the Psychology of mathematics Education. North American Chapter, Michigan, EE.UU. 1986a.

[9] FIGUERAS, O.; FILLOY, E. and VALDEMOROS, M. "Some difficulties which obscure the appropiation of the fraction concept". Proceedings of the Eleventh Annual Conference of the Internacional Group for the Psychology of mathematics Education. North American Chapter, Michigan, EE.UU. 1986b.

[10] WITTGENSTEIN, L. "Remarks on the Foundations of Mathematics", 2nd edition, eds. von Wright, Rhees, Anscombe, Trans. Anscombe. Blackwell 1964.

ON SOME OBSTACLES IN UNDERSTANDING MATHEMATICAL TEXTS

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ABSTRACT. *The research reported in this paper is aimed at singling out and discussing some causes of student failures in mathematical proof. For this purpose, we shall consider a rough schematization of the various aspects of mathematical proof and focus our attention on the basic activity of "understanding mathematical texts".*

The study is carried out by means of a questionnaire on the subject addressed to students aged 14 and 17 years. From the analysis of their answers we find that the causes of failure are both "technical" obstacles (mathematical language, quantifiers, implication, generalization,...) intervening in proof and other obstacles due to elements (that we shall term "of diversion") linked with student' behaviour towards the learning of mathematics.

INTRODUCTION

Although it is widely recognized that the role of proof is central in the study of mathematics (Hanna, 1983), research such as that by Hanna (1989a, 1989b) and others, carried out in different contexts and from different points of view, indicates a lot of elements that render the activity of proving difficult for students and leads to conclude that in teaching the role of proof is in crisis. This conclusion concurs with the opinion of upper secondary school teachers (age range 14 - 19) we have repeatedly interviewed on the subject: they think that proof is out of their students' reach. The same teachers, when asked to explain which are the causes of failure in proving on the part of their students, are often unable to single out and analyze them critically; they rather give generic explanations such as «students are not well prepared at previous school levels» (this is an "evergreen" explanation for any kind of student failure), «students are not sufficiently clever», «students do not concentrate when working».

In order to offer teachers materials for elaborating more precise ideas on the subject, we carried out an investigation by means of a questionnaire on *the understanding of statements and the recognizing when an explanation is a proof* which was administered to 289 students aged 14 (first year of upper secondary school) and 299 students aged 17 (fourth year of upper secondary school). In the present research report we outline some results of this investigation.

We point out that, in programming the research, our aim was not to ascertain once again students' poor performance in activities linked with proving, but to single out some of the causes of such performance. The causes we have found belong to various categories; in particular, among them, we may focus our attention on what we term "*elements of diversion*", i. e. elements not specifically due to lack of knowledge (of mathematical notions and concepts and of "technical" issues related to proving), but due to other factors affecting student behaviour. We will see in the following that the check at two different ages, i.e. at different stages in maths education, offers further features useful to investigate in this direction.

DESIGN OF THE RESEARCH

As we know that the problem of mathematical proof in teaching presents various aspects, we need to give a framework to our work. For this purpose we shall consider the following rough schematization of activities *related to mathematical proof* (left-hand side) and outline some *issues playing a role in these activities* (right hand side):

the "understanding" of a mathematical text
- the logical-linguistic abilities of interpreting the meaning of words and concepts in relation to the mathematical context
- to follow the steps of a reasoning made by others (teachers, textbooks,...)

the reproduction of already learned proof
- the unambiguous and efficient use of colloquial language
- the correct use of mathematical language
- the autonomous use of semantic connections
- the transmission and explanation to other persons

the production of proof
- the production of conjectures
- the choice among different conjectures
- the autonomous production of procedures to deduce the truth of one statement from another

The previous activities concern intellectual performances oriented at different directions. In the first two, the difficulties encountered by students are mainly *logical* and *linguistic*: in the third activity there are also difficulties of *heuristic* type, which are encountered both in advanced mathematics thinking and at more elementary levels, when the informal use of ideas precedes their logical analysis and students try to produce convincing arguments in practical situations.

In the research reported in the present paper we confine ourselves to the *first activity*, which we assume to be an important premise to any kind of activity regarding mathematics proof.

At this level a questionnaire may be sufficient for collecting information. The questionnaire, see Chiarugi, Fracassina and Paola (1990), we have used consists of 10 questions which require to decide whether a given statement is true or to assess given explanations with respect to consistency. For every question we propose 4 options *[a]*, *[b]*, *[c]*, *[d]* only one of the first three options is the right answer, the fourth option *[d]* consists on the possibility of writing a personal answer if the student does not agree with our proposed answers. In collecting the data we classified as *[d]* also the few cases in which two options were chosen simultaneously. We have added the option *[d]* in order to enrich the information given by the questionnaire. The allowed time was 1 hour.

In designing the questionnaire we have considered a core of basic contents known to all students involved and have pursued a certain balance among questions concerning algebraic environment (5) and those concerning geometric environment (4): the results show that, except in a very few cases, the ignorance of mathematical topics was not the cause of failure.

Virtually all students involved have had no previous significant experience in answering questionnaires structured in this way. Moreover they were not used to analyze mathematical texts with the care required by us; thus, in certain cases, the questions may have appeared to them a bit fastidious and overdemanding, since, in their opinion, they were required to analyze in detail all the "nuances" of mathematical language; nevertheless they strove to answer with good will. No preliminary presentation of the topic had been given to students. After the performance the questionnaire has been used in the classes involved in the research as a starting point for a discussion on some aspects of the mathematical language and logical deduction.

The students involved in the investigation belong to schools with different bias and different maths programs. This difference in maths programs is not relevant at the age of 14, while it is significant at the age of 17. In both cases more than 50% of the students involved in the investigation follows the curricula of a school (scientific lyceum) in which mathematics plays a very important role: in particular, the contents at the age of 17 years include exacting topics such as the formal introduction of real numbers and calculus.

RESULTS OF THE QUESTIONNAIRE AND COMMENTS

In the following we set out the questions of the questionnaire and give some comments on students' performance. Obviously we are conscious that alternative explanations could be given to students' responses: here we focus on the interpretations which emerge as consistent from the analysis of the answers to the different questions and that are also suggested by the optional explanations offered in *[d]*.

In parentheses near the option there are the percentages of answers: the first percentage refers to 14 years old students, the second refers to 17 years old students. The underlined percentages are those referring to the right answers.

• Question 1. FOR ALL REALS X, $2X - 1 = 3$

[a] the statement is true only for $x = 2$ (53%, 76%)

[b] the statement is false: $2x - 1 = 3$ is not true for $x = 0$ (<u>9%, 6%</u>)

[c] the statement is true or false, depending on the values given to x (35%, 14%)

[d]...(3%, 4%)

We are not surprised that students fail to choose the right answer *[b]*, since it is well known that the use of counterexamples is one of the difficulties in proving. What may be surprising is the low percentage of correct answers and the share of percentage of the two wrong answers. *A priori*, we considered that the main obstacle would consist in recognizing a proposition as being true *or* false, fostered by the ambiguity of colloquial language in which the possibility of considering a statement true or false at the same time, according to the different situations, is contemplated. Nevertheless the experiment shows that the main obstacle is not this point: the great majority of students disregards of the word *any* and chooses option *[a]* since they do not resist the *impulse of finding a solution*, when presented with a first degree equation. This irresistible impulse to manipulate formulas constitutes a strong *element of diversion*. This conclusion is confirmed by the fact that virtually all students who add their personal comments

correctly write that the statement is false, but they are not satisfied with the justification we provide in the text by means of the counterexample and write that the statement is false because «the solution of the equation $2x - 1 = 3$ is $x = 2$»: the rule-bound approach to algebraic problems, already observed in Lee and Wheeler (1989) prevails on the ascertaining that a simple counterexample works in proving the statement not true.

The percentages show that this behaviour is growing along *with the raising of school level* (in this regard see also the comments to the questions 5 and 7), since in classroom practice students are pressed by the learning of formulas and algebraic manipulations. Thus they perceive the "didactic contract" as based on the particular aspect of doing mathematics linked with the manipulation of formulas and with the ritual idea of applying rules.

The following results of question 2, which presents some analogous difficulties, seems to confirm our hypothesis about the *element of diversion*, since, even if the difficulties on the use of counterexample persist (less than 40% guess the correct solution), the percentages show different student behaviour.

• Question 2. ISOSCELES TRIANGLES ARE EQUILATERAL.

[a] the statement is true or false, depending on the considered triangle (25%, 41%)

[b] the statement is false: if a triangle has two sides of equal length it cannot have three sides of equal length (22%, 11%)

[c] the statement is false, for example the triangle with the sides of length 1, 3, 3 is isosceles, but not equilateral (<u>40%, 36%</u>)

[d]...(12%, 11%)

Firstly we observe that, unlike question 1, here a higher percentage of students (the majority of younger students) guesses the correct solution through the counterexample. In no other case the answers requiring counterexample collected such a high percentage of preferences. This fact may lead to think that the absence of a strong *element of diversion* such as the impulse to calculate addresses differently students' answers, on the other hand that the attitude to the counterexample is conditioned by the context. If we accept this hypothesis, we may think that in certain situations the geometrical environment may offer good stimuli to students' reasoning, as hinted in Mesquita (1989).

The answers *[d]* shed light on the *element of diversion* which addresses the answers to option *[b]*, that *a priori* seems to us so clearly wrong. The usual comment given in *[d]* is «a triangle isosceles has two sides of equal length, but the third side has different length» and is related to the habit, already observed by other authors, for example Becker (1982), of adding further arbitrary information during the process of deduction. Here students *add the hypothesis* that the third side of the isosceles triangle cannot have the same length of the other two.

• Question 3. THE ASSOCIATIVE PROPERTY OF THE DIVISION DOES NOT HOLD FOR RATIONALS:

$(2 : 3/4) : 7/5 = (2 \cdot 4/3) : 7/5 = 8/3 \cdot 5/7 = 40/21$ AND

$2 : (3/4 : 7/5) = 2 : (3/4 \cdot 5/7) = 2 : 15/28 = 2 \cdot 28/15 = 56/15$.

[a] the explanation proves the statement: in our example the numbers are chosen at random, it is possible to repeat the reasoning for three other rationals (34%, 26%)

[b] the explanation is not a proof, it is only an example (29%, 47%)

[c] the explanation proves that the statement is true: we have given a counterexample to the statement "the associative property holds for rational numbers" (<u>26%, 19%</u>)

[d]..........(11%, 7%)

In this case we find again that counterexamples have no appeal, specially for the older students. The fact that they address the majority of their answers to option *[b]* seems to hint that the increased attention paid to generalization in the maths programs at the age of 17 leads to a kind of suspicion towards the single cases. We consider this conjecture confirmed by the small percentage of the same students who in the following question 4 get into the pitfall of considering the presentation of a few examples sufficient to prove a statement (even if the percentages of option *[c]* in question 4 show that all students have not clear ideas on this point). On the other hand in Lee and Wheeler (1989) it is observed that, in general, the attempts to "check whit numbers" are not a widespread students' strategy in solving algebraic problems. This observation may constitute a further explanation for the lack of production of *numerical* counterexamples.

The younger students address the majority of their answers to *[a]*, because they get into the pitfall of thinking that if a statement is true in some cases *chosen at random* then it will be true in *any case*. A *naive use of probability* is applied to proving.

In this question both the categories of students obtain the lowest percentage of correct answers. The sentences given by students in *[d]* provide an explanation for this failure: the usual comment offered is «property associative is not valid for rational numbers and thus is not necessary to prove anything». In this case the lack of motivation to prove is due to an *element of diversion* constituted by the *confusion between properties (here of rational numbers) that have to be proved and axioms given when defining concepts.*

- Question 4. FOR ALL REALS X, $2X + 4X = 3X + 3X$

[a] the statement is true: if $x = 2, 2x + 4x = 4 + 8 = 12, 3x + 3x = 6 + 6 = 12$

if $x = 3, 2x + 4x = 6 + 12 = 18, 3x + 3x = 9 + 9 = 18$

if $x = 4, 2x + 4x = 8 + 16 = 24, 3x + 3x = 12 + 12 = 24$

and so on (24%, 9%)

[b] the statement is true for the properties of equality: for all real numbers $2x + 4x = 6x, 3x + 3x = 6x$ (<u>27%, 40%</u>)

[c] the statement is true, both [a] and [b] are a correct proof of it (44%, 47%)

[d]..........(6%, 3%)

The main issue of this question is that about a half of the students simultaneously accept an inductive and a correct deductive argument as being mathematically valid proof. The high percentage of acceptance of both kinds of proof show that an empirical view of mathematics as an experimental science still persists also at the school level of the older students, where the mathematics teaching is aimed at generality and abstraction. The persistence of the empirical view is confirmed also by certain justification written in *[d]* «*[b]* is *always* correct, while *[a]* is correct only in *single cases*». Our conclusions on this point confirm the observations in Martin

and Harel (1989) and in Hershkowitz and Arcavi (1990), as for the behaviour of preservice and inservice teachers towards inductive arguments and the checking of a proof on examples.

- Question 5. A REAL NUMBER X SUCH THAT $X^2 + 1 = 0$ DOES NOT EXIST: $(-1)^2 + 1 = 2$.

[a] the explanation is a correct proof: the giving of a counterexample is sufficient to prove the statement (31%, 22%)

[b] the explanation could be a correct proof, if more numerical examples were considered (13%, 6%)

[c] the explanation is not a proof: it is necessary to prove that for any rational number x it is $x^2 + 1 \neq 0$ (<u>47%, 50%</u>)

[d]..........(9%, 21%)

This question is one of the two (the other is question 6) in which both the categories of students obtain the highest percentage of correct answers. Moreover it is the one with the most answers to option *[d]* (15%) and thus we have a lot of hints on students behaviour. The strategy in answering can be illustrated by some *[d]*, were in defence of the truth we were offered (specially by older students) some very accurate explanations on the impossibility of finding real solutions of the given equation. In this question the command in the algebraic manipulation constitutes a help, even if the algebraic *obviousness of the statement* (which is linked to the impulse to calculate seen in question 1) may constitute an *element of diversion*: some students write «for any x it is $x^2 + 1 \neq 0$ and thus the justification given in *[a]* is sufficient»: according to Fishbein and Kedem (1982), in this case since the statement to prove is "obvious" the proof appears to be useless. Another effect of the "obviousness" emerging from answers *[d]* is that it hinders students to understand that a statement may be true, even if the given justification is not correct: students concentrate their efforts in proving that $x^2 + 1 \neq 0$ and disregard to discuss the explanation we offered.

- Question 6. THE BISECTING LINES OF TWO ADJACENT ANGLES ARE PERPENDICULAR: FOR EXAMPLE, IF WE CONSIDER TWO RIGHT ADJACENT ANGLES BÂC AND CÂD WE HAVE 1/2 BÂC = HALF RIGHT ANGLE, 1/2 CÂD = HALF RIGHT ANGLE AND SO THEIR SUM IS A RIGHT ANGLE.

[a] the explanation is not a proof: it proves the statement in the particular case in which the two adjacent angles are right. We do not know if the statement is true in other situations (<u>50%, 64%</u>)

[b] the explanation is a proof: it is allowed to choose at pleasure the two adjacent angles right or not (17%, 13%)

[c] the explanation is a proof: it is allowed to repeat the same reasoning even in the case in which the two angles have not the same width (25%, 14%)

[d]..........(8%, 8%)

The main information offered by the percentages is the confirmation of the conjecture we have made in question 2 that the geometrical environment may allow better achievements in performance at the level we are considering. In particular, we observe that the younger students show a better command over the difficulties intervening in options *[b]* and *[c]*, in respect to the analogous difficulties appearing in question 3.

• Question 7. A RATIONAL NUMBER x SUCH THAT $x^2 = 1$ EXISTS

[a] the statement is false: two numbers (1 and -1) satisfy the equality (32%, 41%)

[b] the statement is true: $1^2 = 1$ (40%, 24%)

[c] the statement is false: both 1 and -1 are integer and not rational (20%, 23%)

[d]..(8%, 12%)

The younger students were much better in answering this question. We point out two *elements of diversions* provoking the failures of older students. The first is the students' habit already observed by Freudenthal, of telling not only *the truth*, but *all the truth* : here *all the truth* is represented by telling (even if not required) that the equation $x^2 = 1$ has two solutions: for this reason the majority of older students choose *[a]*. This *element of diversion* was expected, since we have often observed that in the students' mind the "didactic contract" requires to provide as much information as possible. The other *element of diversion* has emerged from the comments given in *[d]* and from the percentages of answers to *[c]*: the students' attention was attracted by the adjective *rational* and the explanations they give are centred on proving that the number 1 is rational, more than on proving that $1^2 = 1$. No problems were observed in the questions where *real* numbers were mentioned: students have worked without considering the adjective *real*, even those (the younger) who only have an informal and intuitive idea of real numbers.

• Question 8. SQUARES ARE RECTANGLES.

[a] the statement is true (28%, 44%)

[b] the statement is false: squares are not rectangles, since have not only the angles, but also the sides of the same length (52%, 35%)

[c] the statement may be true or false: for example a rectangle with sides measuring 2 and 5 is not a square (10%, 9%)

[d]..(10%, 11%)

The complex explanations given by students in *[d]* help to understand why such a high number of answers refers to option *[b]*, behaviour observed also in Burger (1985): students add the arbitrary information that the sides of a rectangle necessarily cannot have the same length, on the other hand they are again victims of the syndrome of telling *all the truth* about the properties of rectangles and squares. This muddle of information hinders the organization of reasoning and the right implication.

• Questions 9 and 10. Taking into account that after our investigation the questionnaire would be used also for discussing in classroom some aspect of the mathematical language and of proof, we have also inserted two questions (one on algebra, one on geometry) about the inversion of theorems. The results in these questions confirm the difficulties observed at different school levels in this regard, see, for example, Tall (1989).

CONCLUDING REMARKS

We have outlined some elements of reflexion as to students behaviour in the kind of performance involved in the questionnaire. We may add a final remark on a common feature emerging from the answers to option *[d]*, i. e. the students' habit of rewriting in their own

words the statements we proposed, in some cases using practically the same words. This habit reveals a difficulty in conforming themselves to the requests of the teachers and shows a substantial insecurity. Students personalize their statements with words of reinforcements such as *always, never, completely*. Very often when writing their own explanation students modify the sentence «the statement is *true*» in «the statement is *correct*». An interpretation of this change could be that they identify the mathematical truth on what their maths teacher defines correct. Is this a further evidence (in addition to that emerging from the percentages) of the weight of the "didactic contract" on students' behaviour?

Finally we may add that when we designed the questionnaire the teachers interviewed forecasted that there would be few chances for 14 years old students, while expressing more optimism about 17 years old students. The analysis of our data indicates a balance in the performances of the two categories. This gap between teachers' expectation and the results of the experience we carried out may strengthen the doubt that sometimes teachers are not quite aware of their students learning process.

REFERENCES

Becker, G.: 1982, 'Difficulties and errors in geometric proofs by grade 7 pupils', in Vermandel, A. (editor) *Proceedings of P.M.E. 6*, Antwerp, 123-127.

Burger, W. F.: 1985, 'Geometry', *Arithmetic teacher*, 32 (6), 52-56.

Chiarugi, I.; Fracassina, G. and Paola, D.: 1990, *Questionario sulla lettura di testi matematici*, Dipartimento di Matematica, Università di Genova.

Fishbein, E. and Kedem, I.: 1982, 'Proof and certitude in the development of mathematical thinking', in Vermandel, A. (editor) *Proceedings of P.M.E. 6*, Antwerp, 128-131.

Hanna, G.: 1983, *Rigorous Proof in Mathematics Education*, OISE Press, Toronto.

Hanna, G.: 1989a, 'Proofs that prove and proofs that explain', in Vergnaud, G.; Rogalski J. and Artigue M. (editors) *Proceedings of P.M.E. 13*, Paris, 2, 45-51.

Hanna, G.: 1989b, 'More than formal proof', *For the Learning of Mathematics*, 9, 1, 20-23.

Hershkowitz, R. and Arcavi, A.: 1990, 'The interplay between student behaviors and the mathematical structure of problem situations - Issues and examples', in Booker, G.; Cobb, P. and De Mendicuti, T. N. (editors) *Proceedings of P.M.E. 14*, Oaxtepec, 2, 193-200.

Lee, L. and Wheeler, D.: 1989, 'The arithmetic connection', *Educational Studies in Mathematics*, 20, 41-54.

Martin, W. G. and Harel, G.: 1990, 'Proof frame of preservice elementary teachers', *Journal for Research in Mathematics Education*, 20 (1), 41-51.

Mesquita, A. L.: 1989, "Sur une situation d'éveil à la déduction mathématique en géométrie", *Educational Studies in Mathematics*, 20, 55-77.

Tall, D.: 1989, 'The nature of mathematical proof', *Mathematics Teaching*, 127, 28-32.

TOWARD A CONCEPTUAL-REPRESENTATIONAL ANALYSIS OF THE EXPONENTIAL FUNCTION

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A partial theoretical analysis is provided for the construction of more advanced conceptual schemes in mathematics, as illustrated by the exponential function. The analysis is based on a synthesis of ideas from two relevant models: a model of understanding proposed by Herscovics and Bergeron, and the model of problem-solving competence based on cognitive representational systems proposed by Goldin. The resulting framework allows a broader interpretation of "intuitive understanding", and can help us chart the complexity of the constructions expected of mathematics students (and characterize their cognitive obstacles) in learning about the exponential function meaningfully.

Models of Understanding and Problem Solving

During the past fifteen years, the psychology of mathematics education has witnessed the development of a body of research on the construction of conceptual knowledge, and has seen important advances in the study of problem solving. These two domains of investigation seem to have followed somewhat parallel but separate paths, with little cross-pollination occurring between them. Out of them have emerged, on the one hand, several *models of understanding* (Herscovics and Bergeron, 1983, 1984, 1988) which have proven useful in the epistemological analysis of fundamental mathematical ideas, and, on the other hand, *models of competence* (Goldin, 1983, 1987) that provide detailed descriptions of mathematical problem-solving processes. In

this paper, we draw on ideas from both domains to describe a model of understanding suitable for the analysis of more advanced mathematical concepts; i.e., concepts which are constructed on *prior mathematical knowledge.* We illustrate the model's applicability by considering the exponential function.

In attempting to characterize the different aspects of understanding achieved when fundamental mathematical ideas such as "number", "addition", etc. are constructed, Bergeron and Herscovics found these could not be described by classical concept-formation theory in terms of exemplars and non-exemplars. Instead the notion of a *conceptual scheme* was introduced, defined as a *network of related knowledge together with the problem situations in which the knowledge can be used* (Bergeron and Herscovics, 1990). They then designed various models of understanding to identify the different components involved in the construction of specific conceptual schemes. In 1984, four modes of understanding were described as follows:

Intuitive understanding refers to an informal mathematical knowledge characterized as the case may be by pre-concepts (e.g. surface is a pre-concept of area), or a type of thinking based on visual perception (e.g. non-conservation of number), or unquantified actions (e.g. adding to and joining are two actions associated with arithmetic addition), or estimation based on rough approximations (e.g. more, less, etc.).

Procedural understanding refers to the acquisition of mathematical procedures which the learners can relate to their intuitive knowledge and use appropriately (e.g. counting-all and counting-on are two arithmetic procedures which quantify the actions of adding to or joining).

Mathematical abstraction refers to both abstraction taken in the usual sense as a detachment from any concrete representation and procedure (e.g. when the number 7 exists in the child's mind without requiring the presence of objects or the need for counting), and abstraction taken in the mathematical sense as the construction of invariants (e.g. conservation of number), or as the reversibility and composition of mathematical transformations and operations (e.g. subtraction viewed as the inverse of addition; strings of addition seen as equivalent to fewer operations), or as generalization (e.g. perceiving commutativity of multiplication as a property applying to all pairs of natural numbers).

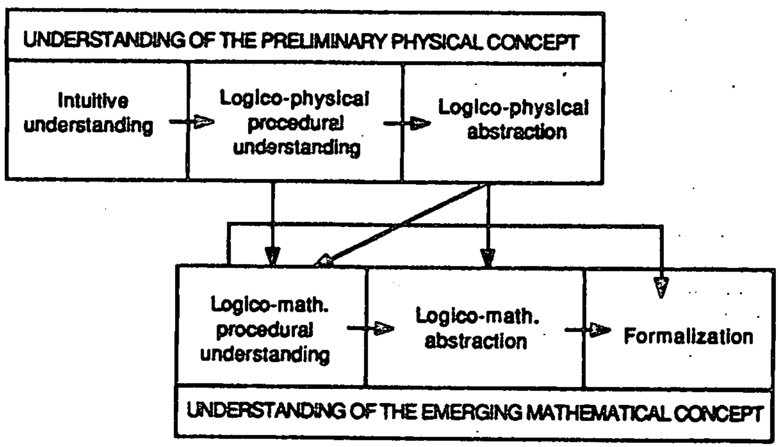
Formalization refers to its usual interpretations, that of axiomatization and formal mathematical proof. But we assign to formalization two additional meanings, that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior abstraction or procedural understanding has occurred to some degree.

(Herscovics and Bergeron, 1984, pp. 190-196)

Applying this model to describe the understanding of the major conceptual themes in children's early mathematics learning led to the differentiation between two tiers: the *understanding of preliminary physical concepts*, and the *emerging mathematical concept.* The

result was a two-tiered "extended model of understanding", for which Figure 1 provides a schematic overview (Herscovics and Bergeron, 1988; Herscovics, 1989). In the extended model, the first tier involves three levels: *intuitive understanding* refers to a global perception of the notion at hand, based essentially on visual perception, and provides rough non-numerical approximations; *logico-physical procedural understanding* refers to the acquisition of logico-physical procedures (dealing with physical objects) which the learners can relate to their intuitive knowledge and use appropriately; and *logico-physical abstraction* refers to the construction of logico-physical invariants, the reversibility and composition of logico-physical transformations, and generalizations about them. The second tier also involves three components, corresponding to the latter three of the four modes of understanding described in 1984, but now identified as *logico-mathematical procedural understanding, logico-mathematical abstraction*, and *formalization*.

Figure 1.



This model distinguishes between *logico-physical* understanding, which results from thinking about procedures applied to physical objects and about spatio-physical transformations of these objects, and *logico-mathematical* understanding, which results from thinking about procedures and transformations in connection with mathematical objects. In this framework, one can discuss reflective abstraction in relation to actions operating in the physical realm, without necessarily considering it as having to be somehow mathematical.

The model has been used successfully in describing construction of the number concept (Bergeron and Herscovics, 1989), early addition (Herscovics and Bergeron, in press), early multiplication (Nantais and Herscovics, 1989), and physical length and measure (Héraud, 1989). However, it proves inadequate to describe the understandings involved in more advanced mathematical concepts, such as trigonometric, exponential, or logarithmic functions. Its inadequacy is due to the fact that *more advanced mathematical concepts are based on prior mathematical knowledge to a greater extent than they are based on physical pre-concepts.*

One might conclude that for such concepts it would be sufficient to retain only the second tier in Figure 1 to describe the conceptual schemes. One could then try to *iterate* this tier, to take account of the *mathematical* pre-concepts that are prerequisite to the advanced concepts under discussion. However, such a reduction to only three aspects of understanding would still not do justice to the complexity and richness of the cognitions involved in advanced mathematical ideas. An alternative is to reconsider the 1984 model in relation to advanced mathematical concepts. All the criteria used to characterize the four modes of understanding seem to be acceptable, *except for some of those introduced in the definition of intuitive understanding.* While the notions of pre-concept, of visual apprehension, and of non-numerical estimation remain valid, it is no longer sufficient to refer only to *physical* actions such as adding or joining physical objects. Instead, we propose to fill the gap by drawing on the *imagistic representational system* introduced in the unified model proposed by Goldin to describe problem-solving competence.

In this model five kinds of internal cognitive representational systems are proposed: (a) a system of *verbal/syntactic* representation, which refers to the problem-solver's use of words, grammar, and syntax; (b) *imagistic* representation, which refers to internal visualization, spatial representation, or auditory/rhythmic or tactile/kinesthetic representation; (c) *formal notational* representation, referring to mathematical symbols such as numerals, algebraic expressions, etc. (d) a system of *heuristic planning and executive control*, referring to choices of problem-solving strategies, and (e) *affective* representation, referring to the changing states of feeling that occur during problem solving, which can influence strategic and other decisions.

Each of these systems of cognitive representation consists in a certain sense of primitive characters or signs (cognitive entities), together with rules for forming permitted configurations of these, and for moving between configurations. They also include higher-level structures of various kinds. For example, to describe a verbal system of representation we might take the characters to be words, and the higher-level structures to be grammatical configurations of words. Most importantly, configurations or structures in one system can stand for or *symbolize* those in another--words can stand for visualized objects, mathematical symbols can

stand for kinesthetically-encoded action sequences, and so forth.

Representational systems are themselves viewed as *constructed* over time, and three stages of construction are posited (Goldin, 1988): (1) an *inventive-semiotic* stage, in which the entities in a new system are given meaning in relation to a previously existing system; (2) a *structural developmental* stage, during which the new system evolves, with its structure largely driven by the previously existing system, and (3) an autonomous stage, during which the new system has a "life of its own," independent of the system from which it was constructed.

The model was motivated by the desire to characterize in detail the complexity of problem-solving competencies in mathematics, and how these develop; for more detail, see Goldin (1983, 1987, 1988). But it does not fully explain why some problem solving *results in the construction of important new knowledge*, while other problem solving does not. One of our longer-range objectives is to achieve a full synthesis between the models for understanding and the model for problem-solving competence, which would enable us to understand the constructive learning process in greater detail.

Here, before moving to our discussion of the exponential function, we make the following observation which is a step toward such a synthesis. We have seen that there is a lack in the criteria for "intuitive understanding" (as in the 1984 model, or as in the first tier of the extended model) when applied to the construction of advanced conceptual schemes, because we no longer have the necessity of interaction with external physical objects. However, the gap can be filled by drawing on *imagistic representational systems* as they occur within the model for problem-solving competence. For advanced mathematical concepts, imagistic representation *can* be, but is not *necessarily*, a precursor to formalization. This theoretical observation leads to a model of understanding that is powerful enough to provide a framework for analysis of more advanced conceptual schemes.

Some Aspects of Understanding the Exponential Function

Let us illustrate how this set of ideas permits us to gather many of the ideas relevant to the construction of the conceptual scheme for the exponential function into a cognitive matrix.

Intuitive understanding. Making use of comprehension characterized by means of imagistic representation, we include here understandings associated with spatial visualization and kinesthetically encoded actions, even when these are not tied to external physical objects. For example, there is the relation between *geometric dimension* and exponents, in which 1, 2, and 3-dimensional objects (e.g. the length of a line segment, the area of a square, and the volume of a

cube) provide an interpretation for exponents of 1, 2, and 3 (for an arbitrary positive base). There are "tree" diagrams, in which the number of branches from a given branch is a fixed factor at each of several discrete stages. Later on, there are graphs of exponential functions, and the attendant understandings obtained through visualization of such graphs. We stress again that unlike the situation for earlier arithmetical concepts, intuitive understanding in this sense is not a necessary *prerequisite* for meaningful understanding in other modes. Some procedural understandings, (e.g. the interpretation of exponentiation as repeated multiplication) are likely to precede some intuitive understandings (e.g. those associated with graphical representations).

Procedural understanding. This type of understanding involves those explicit mathematical procedures used in the evaluation of exponential functions. The initial (procedural) definition of the exponential function as repeated multiplication can be related, for example, to the more tedious procedure of counting the branches on a tree diagram. Very quickly, the student will find procedural shortcuts when larger exponents are involved (e.g. $2^6 = 2^3 \times 2^3 = 8 \times 8$). The use of the calculator can greatly accelerate these discoveries, particularly when the calculator does not have an exponential key.

Mathematical abstraction. This aspect of understanding, defined to include mathematical generalizations and the construction of mathematical invariants, is involved whenever the initial definition of exponentiation (a natural number raised to the power of a natural number) is expanded to a wider domain. Such an expansion occurs in distinct, rationally sequenced paths. Using the original definition, the domain of the base can be expanded to include integers (positive and negative) and rational numbers. Through the construction of numerical patterns, the domain of the exponent can be expanded to the integers. To arrive at rational exponents requires a new conceptual scheme, that of the root function $a^{1/m}$. Indeed, if all of these generalizations are to stem from the initial idea of repeated multiplication of the base, there will be a reversal in the original problem: to evaluate $5^{1/2}$, for example, one must find the number a such that $a^2 = 5$; i.e. the exponent is fixed, and the base has to be determined. The procedure is no longer one of direct computation, but a system of gradual approximation that may involve complex heuristic planning (systematic trials, monitoring, etc.). It is only when the power function and the root function are combined that one can define the exponential function for rational exponents.

Since mathematical abstraction is also defined to include the construction of invariants, one must include here the various laws that can be derived from the initial definition--those that involve addition and subtraction of exponents, as well as the composition of power functions leading to the multiplication of exponents. That these laws continue to be valid for each generalization of the original definition is a fact that is far from obvious, and needs to be studied

carefully by the learner for powerful understanding to be present.

Formalization. It is difficult to see how the meaningful construction of the exponential function would be possible without this fourth aspect of understanding. Formalization refers to the process of providing mathematical form to a body of mathematical content--and of course there is constant interplay between the development of form and content. Initially, 3^4 is nothing but an alternative representation for 3 x 3 x 3 x 3; but very quickly, the new formal symbolic representation acquires a life of its own, capable of expressing sophisticated ideas such as $(ab)^{1/m} = a^{1/m} b^{1/m}$. Cognitive obstacles can occur when form and content do not develop in a synchronized fashion--the content may be difficult to comprehend if it lacks an adequate formal representation; or, more often, the mathematical form becomes devoid of meaning for the learner.

Conclusion

As can be seen from this brief analysis, the construction of the exponential function is far from a trivial task. This explains why so many students find it difficult, and have even greater problems when learning about its inverse, the logarithmic function.

Constructivist researchers argue that knowledge, rather than being "transmitted" or "communicated", is constructed by each individual. The three stages described above for construction of new representational systems find direct interpretation in the context of the exponential function: (1) At first, the new notation (a raised exponent) is given meaning in relation to the previously existing multiplicative notation. (2) Next, the new system undergoes structural development, as patterns (e.g. laws of exponents--the power of a product, of a quotient, or of a power) are obtained by reasoning from the earlier, multiplicative system. (3) Finally, in the autonomous stage, the new exponential representation "separates" from the older, multiplicative notation, enabling the transfer of conceptual understanding to new domains; the exponential function is perceived *as a function* (an understanding enhanced by graphical representation). If the root function is initially constructed as a separate entity, it too goes through similar stages; and, as we have noted, its construction will be somewhat more difficult for the student: evaluating $5^{1/3}$ is a complex problem, in which the iterated multiplication occurs within an overall process of successive approximation. The laws of exponents previously discovered by the student will also need to be rediscovered in the context of radicals.

In short, we find that ideas drawn from the theory of cognitive representation, joined with a model of understanding, are useful in characterizing the meaningful learning of more advanced mathematical ideas, such as the exponential function discussed here.

References

Bergeron, J. C. and Herscovics, N. (1989), "A Model to Describe the Construction of Mathematical Concepts from an Epistemological Perspective," in L. Pereira-Mendoza & M. Quigley, eds., *Procs. of the Thirteenth Meeting of the Canadian Mathematics Education Study Group*, Brock University, St. Catherine, Ontario, 27-31 mai 1989, 99-114.

Bergeron, J. C. and Herscovics, N. (1990), "Psychological Aspects of Learning Early Arithmetic," in P. Nesher & J. Kilpatrick, eds., *Mathematics and Cognition*, Cambridge, U.K.: Cambridge University Press, 31-52.

Goldin, G. A. (1983), "Levels of Language in Mathematical Problem Solving," in J. C. Bergeron & N. Herscovics, eds., *Procs. of the Fifth Annual Meeting of PME-NA*, Vol. 2. Montreal: Concordia Univ. Dept. of Mathematics, 112-120.

Goldin, G. A. (1987), "Cognitive Representational Systems for Mathematical Problem Solving," in C. Janvier, ed., *Problems of Representation in the Teaching and Learning of Mathematics*, Hillsdale, N.J.: Erlbaum, 125-145.

Goldin, G. A. (1988), "The Development of a Model for Competence in Mathematical Problem Solving Based on Systems of Cognitive Representation," in A. Borbas, ed., *Procs. of the Twelfth International Conference of PME*, Veszprem, Hungary: OOK Printing House, Vol. 2, 358-365.

Héraud, B. (1989), "A Conceptual Analysis of the Notion of Length and its Measure," in G. Vergnaud, J. Rogalski, & M. Artigue, eds., *Proceedings of the Thirteenth International Conference of PME*, Paris: Université de Paris, La Sorbonne, Vol. 2, 83-90.

Herscovics, N. (1989), "The Description and Analysis of Mathematical Processes," in C. A. Maher, G. A. Goldin, & R. B. Davis, eds., *Proceedings of the Eleventh Annual Meeting of PME-NA*, New Brunswick, N. J.: Rutgers University Center for Mathematics, Science, and Computer Education, Vol. 2, 1-28.

Herscovics, N. and Bergeron, J. C. (1983), "Models of Understanding," in *Zentralblatt für Didaktik der Mathematik*, Karlsruhe, Germany, ZDM 8312, 75-83.

Herscovics, N. and Bergeron, J. C. (1984), "A Constructivist vs. a Formalist Approach in the Teaching of Mathematics," in B. Southwell *et al.*, eds., *Proceedings of the Seventh International Conference of PME*, Australia: University of Sydney, 190-196.

Herscovics, N. and Bergeron, J. C. (1988), "An Extended Model of Understanding," in M. J. Behr, C. B. Lacampagne, & M. M. Wheeler, eds., *Proceedings of the Tenth Annual Meeting of PME-NA*, DeKalb, Illinois: Northern Illinois University Dept. of Mathematical Sciences, 15-22.

Herscovics, N. and Bergeron, J. C. (in press), "Analyse Épistémologique des Débuts de l'Addition," in A. Warbeq, ed., *Actes de la 41e Rencontre de la CIEAEM*, Université Libre de Bruxelles.

Nantais, N. and Herscovics, N. (1989), "Epistemological Analysis of Early Multiplication," in G. Vergnaud, J. Rogalski, & M. Artigue, eds., *Proceedings of the Thirteenth International Conference of PME*, Paris: Université de Paris, La Sorbonne, Vol. 3, 18-24.

Duality, Ambiguity and Flexibility in Successful Mathematical Thinking

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In this paper we consider the duality between process and concept in mathematics, in particular using the same symbolism to represent both a process (such as the addition of two numbers 3+2) and the product of that process (the sum 3+2). The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema. We hypothesize that the successful mathematical thinker uses a mental structure which is an amalgam of process and concept which we call a procept. We give empirical evidence to show that this leads to a qualitatively different kind of mathematical thought between the more able and the less able, causing a divergence in performance which is eventually manifested as continuing success or failure.

The *duality* of process and concept

The passage from process to concept has long been a focus of research in mathematical education. Piaget speaks of the *encapsulation* of a process as a mental object when

> ... a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower. (Beth & Piaget 1966, p. 247).

Dienes uses a grammatical metaphor to describe how a predicate (or action) becomes the subject of a further predicate, which may in turn becomes the subject of another. He claims that

> People who are good at taming predicates and reducing them to a state of subjection are good mathematicians. (Dienes, 1960, p.21)

In an analogous way, Greeno (1983) defines a "conceptual entity" as a cognitive object which can be manipulated as the input to a mental procedure. The cognitive process of forming a (static) conceptual entity from a (dynamic) process has variously been called "encapsulation" (after Piaget), "entification" (Kaput, 1982), and "reification" (Sfard, 1989). It is seen as operating on successively higher levels so that:

> ... the whole of mathematics may therefore be thought of in terms of the construction of structures,... mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures. (Piaget 1972, p. 70).

The *ambiguity* of symbolism for process and concept

The encapsulation of process as object is seen as a difficult mental activity for "How can anything be a process and an object at the same time?" Sfard (1989). Our observation is that this is achieved by the simple device of *using the same notation to represent both a process and the product of that process*. Examples pervade the whole of mathematics.

- The process of adding through *counting all* or *counting on* and the concept of *sum* (5+4 evokes both the counting on process and its result, 9),
- The process of *division* of whole numbers and the concept of *fraction* (e.g. $\frac{3}{4}$),
- The process of "add two" and the concept of signed number +2,
- The expression 3*x*+2 as both the process of adding 2 to 3*x* and the resulting sum,
- The trigonometric ratio $\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$ as both process and concept,
- The process of *tending to a limit* and the concept of the *value of the limit* both represented by the same notation such as $\lim_{x\to a} f(x)$.

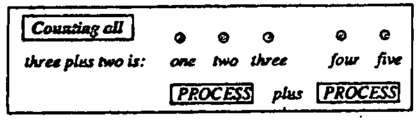
It is through using the notation to represent either process or product, whichever is convenient at the time, that the mathematician manages to encompass both – neatly sidestepping the problem. We believe that this ambiguity is at the root of successful mathematical thinking. It enables the processes of mathematics to be tamed into a state of subjection.

The *flexible* notion of procept

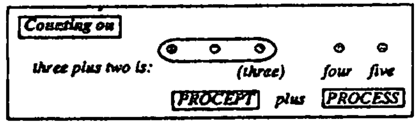
We will refer to the combination of process and concept represented by the same symbol by the portmanteau name "procept". A procept is of course a special kind of concept. It is one which is first met as a process, then a symbolism is introduced for the product of that process, and this symbolism takes on the dual meaning of the process and the object created by the process. As a child learns mathematics, the introduced symbol takes on a life of its own. It can be *written* (say, 3+2), it can be *read*, it can be *spoken* ("three plus two"), it can be *heard*. It is an external object that different people can share, so it has, or seems to have, its own external reality. It is the *construction of meaning* for such symbols, the processes required to compute them, and the higher mental processes required to manipulate them, that constitute the abstraction of mathematics. Indeed the *ambiguity* of notation to describe either process or product, whichever is more convenient at the time, proves to be a valuable thinking device for the professional mathematician.

As an example of the use of the notion of procept to produce a new theoretical synthesis of the development of mathematical concepts, we consider the development of concept of addition, (see, for example, Carpenter *et al*, 1981, 1982, Fuson, 1982, Secada *et al*, 1983). For considerations of space we must compress the discussion which follows somewhat.

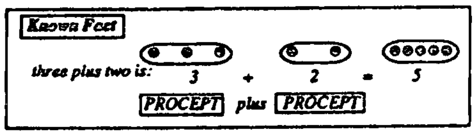
...m of two numbers, say 3+2, is a procept, first conceived as the process of "counting all" or "counting on". COUNTING ALL therefore may be viewed as PROCESS & PROCESS.



"Counting on" is a more subtle procedure in which the first number (three) is already seen as a whole, and the counting-on process counts on two more numbers ("four", "five").
COUNTING ON therefore consists of PROCEPT & PROCESS.



Finally PROCEPT & PROCEPT are embodied in a KNOWN FACT.



A proceptual known fact should be distinguished from a rote learned fact by virtue of its rich inner structure which may be decomposed and recomposed to produce *derived facts*. For instance, faced with 3+2, a child might see 3 as "one more than 2" and might know the double 2+2=4 to derive the fact that 3+2 is "one more", namely 5. For the proceptual thinker this gives a powerful feedback loop using known facts to derive new known facts.

Such proceptual facts may develop great flexibility, where 3+2=5 may be seen equivalently as 2+3=5, 5–3=2, 5–2=3, allowing subtraction to be seen as directly related to addition at the proceptual level, giving a fluent and easy way to develop subtraction facts.

The weakness of unencapsulated process

Meanwhile, the less able child who sees addition only as a process, is faced with a far more difficult task. We have observed that those children who perceive addition as count all or count on, often do so in time, so that although they may produce the right answer (8+4 is 9, 10, 11, *12*), by the time they reach the end of the process they may have forgotten the beginning, and so the sum 8+4=12 is not available as a new fact.

For a child whose concept of addition is mainly "count on", or "count all", the strategies of subtraction can only be in terms of a reversal of these processes. "Take away" involves counting the total set, removing the number to be subtracted and counting the remainder. "Count up" relies on counting the subset to be taken away, then counting up to the total. "Count back" relies on counting back the number to be taken away from the total. All of these involve sophisticated double-counting procedures which invariably need concrete or imagined props, such as a number line or ruler, or assigned parts of the body to support the counting process. The corresponding abstract processes prove difficult to carry out if such props are absent.

The less able child thus not only has a weak grasp of known facts as a foundation for knowledge, but also uses more complex procedures with more possibility of error. When the concept of place value is introduced later and the child meets two and three digit addition and subtraction problems, the difficulties are compounded. What might be a simple combination of proceptual ideas for the more able becomes the coordination of several complex processes for the less able.

Empirical Evidence

Seventy two children were selected by their teachers in two "typical" schools to represent the chronological ages 7+ to 12+, with each school providing three pairs of children in each year to represent the below average, average, and above average attainers. These children were interviewed individually for half an hour on at least two separate occasions a week apart, and in each session were asked to solve between eighteen and twenty arithmetic problems at various levels of

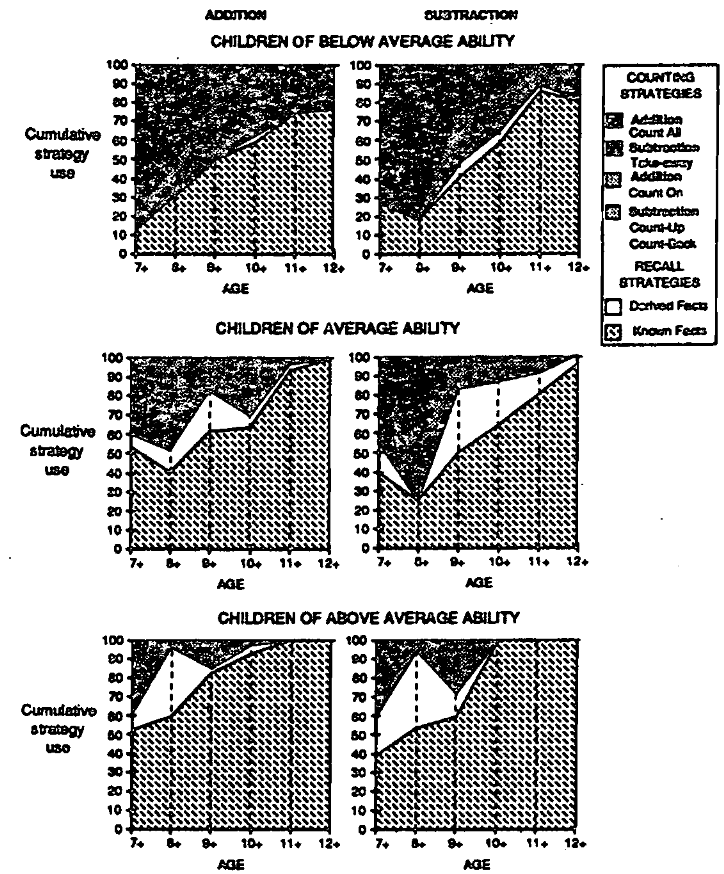


figure 1 : Strategies for solving addition and subtraction involving numbers up to ten

difficulty. Figure 1 (taken from Gray, to appear) illustrates the different strategies used by children of differing abilities in solving single-digit addition and subtraction problems.

Note the almost complete absence of derived facts in the less able (particularly in addition), whereas the average and above average start with a high proportion of known facts and use derived facts to generate other facts. As the ages of the children increase, the proportion of known facts increase, but to a lesser extent in the less able.

Figure 2 concentrates on the performances of the below and above average ability groups on three different levels of subtraction problem:

- **A** single digit subtraction (e.g. 8–2),
- **B** subtraction of a single digit number from one between 10 & 20 (e.g. 16–3, 15–9),
- **C** subtraction of one number between 10 and 20 from another (e.g. 16–10, 19–17).

Note the absence of derived facts for any of these levels in the 8+ below average children and the high incidence of derived facts in the same age above average children. See how the above average have 100% known facts in category A by the age of 10+ whilst the below average even tail off in performance at 12+ as the effects of attempting to cope with more complicated arithmetic begins to affect their competence in performance at this level.

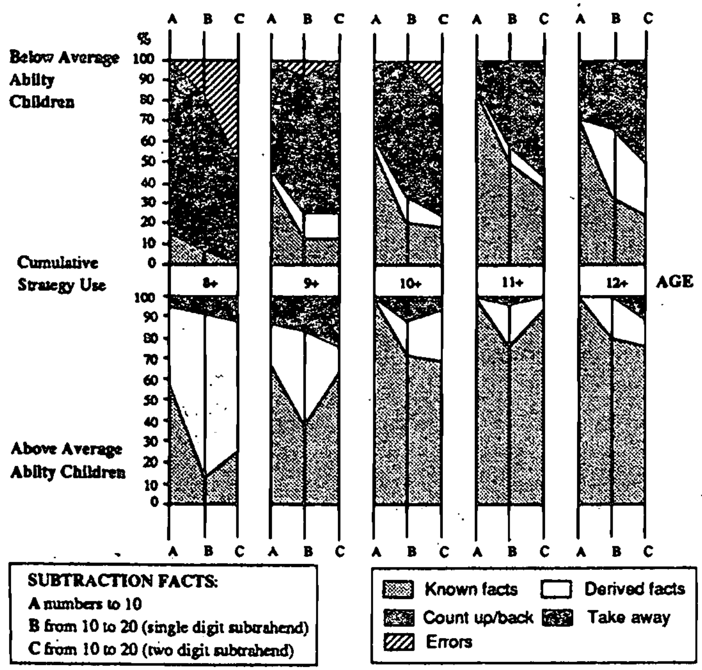


Figure 2 : Strategies for subtraction by below and above average children

The proceptual divide

We have seen that the more able have a proceptual structure available to them with built-in feedback loop. Gray (in press) has observed that more able children initially build up an increasing array of known facts to support their arithmetic, but then realise that their new ability to derive facts removes the burden of needing to remember them all. *For the more able, arithmetic eventually becomes increasingly simple*. Meanwhile, the less able become trapped in long sequential processes which increase the burden upon an already stressed cognitive structure so that, *for the less able, arithmetic becomes increasingly more difficult*.

This lack of a proceptual structure provokes a major tragedy for the less able which we call the *proceptual divide*. We believe it to be a major contributory factor to widespread failure in mathematics. It is as though the less able are deceived by a conjuring trick that the more able have learned to use. They are all initially given processes to carry out mathematical tasks but success eventually only comes not through being good at those processes, but by encapsulating them as part of a procept which solves the tasks in a more flexible way.

Figure 3 shows the total range of strategies used by more able and less able children in the ages 8+ to 12+ for specific problems whose answer was not obtained as a known fact.

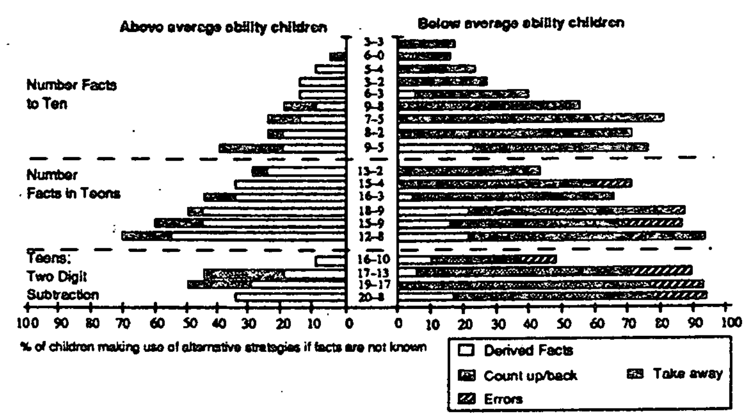


Figure 3 : Strategies for solving problems whose answer is not immediately known

The left hand side shows the above average children using almost all derived facts and a few examples of counting, whilst the right hand side shows few derived facts and a large percentage of counting, take away and errors. The proceptual divide is clearly shown.

The cumulative effect of the proceptual divide

Proceptual encapsulation occurs at various stages throughout mathematics: repeated counting becoming addition, repeated addition becoming multiplication and so on, giving what are usually considered by mathematics educators as a complex hierarchy of relationships (figure 4).

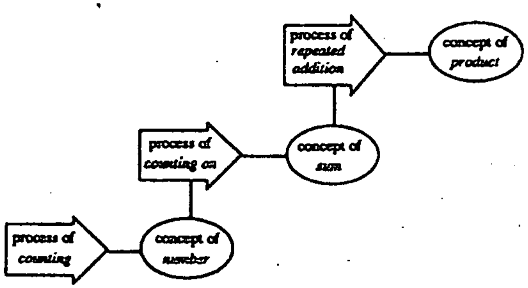


Figure 4 : Higher order encapsulations

The less able child who is fixed in process can only solve problems at the next level up by coordinating sequential processes. This is, for them, an extremely difficult process. If they are faced with a problem two levels up, then the structure will almost certainly be too burdensome for them to support.

The more able, proceptual thinker is faced with a different problem. The symbols for sum and product again represent *numbers*. Thus counting, addition and multiplication are operating on the same procept which can be decomposed into process for calculation purposes whenever desired. A proceptual view which amalgamates process and concept through the use of the same notation therefore *collapses the hierarchy* into a single level in which arithmetic operations (processes) act on numbers (procepts).

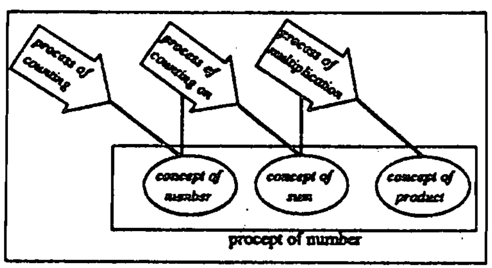


Figure 5 : Collapse of hierarchy into operations on numbers

We hypothesise that this is the development by which the more able thinker develops a flexible relational understanding in mathematics, which is seen as a meaningful relationship between notions at the same level, whilst the less able are faced with a hierarchical ladder which is more difficult to climb. It also provides an insight into why the practicing expert sees mathematics as such a simple subject and may find it difficult to appreciate the difficulties faced by the novice.

Examples from other areas of mathematics

The examples given in simple arithmetic by no means exhaust the possibilities in the mathematics curriculum. We have evidence that the lack of formation of the procept for an algebraic expression causes difficulties for pupils who see the symbolism representing only a process: indeed a process

such as $2+3x$ which they are not able to carry out because they do not know the value of x (Tall and Thomas, to appear). We have evidence that the conception of a trigonometric ratio only as a process of calculation (opposite over hypotenuse) and not a flexible procept causes difficulties in trigonometry (Blackett 1990).

The case of the function concept, where $f(x)$ in traditional mathematics represents both the process of calculating the value for a specific value of x and the concept of function for general x, is another example where the modern method of conceiving a function as an encapsulated object causes great difficulty (Sfard, 1989).

We therefore are confident that the notion of procept allows a more insightful analysis of the process of learning mathematics, in which the precision of definition of modern mathematics ("a function is a set of ordered pairs such that ...") causes seemingly inexplicable difficulties to the student. The ambiguity of process and product represented by the notion of a procept provides a more natural cognitive development which gives enormous power to the more able. It exhibits the proceptual chasm faced by the less able in attempting to grasp what is – for them – the spiralling complexity of the subject.

References

Beth E.W. & Piaget J., 1966: *Mathematical Epistemology and Psychology* (W. Mays, trans.), Dordrecht: Reidel.

Blackett N., 1990: *Developing Understanding of Trigonometry in Boys and Girls using a Computer to Link Numerical and Visual Representations*, unpublished Ph.D. Thesis, University of Warwick.

Carpenter T.P., Hiebert J. & Moser. J.M., 1981: Problem Structure and First Grade Children's Initial Solution Processes for Simple Addition and Subtraction Problems, *Journal for Research in Mathematics Education* 12 (1), 27-39.

Carpenter T.P., Hiebert J. & Moser J.M., 1982: Cognitive Development and Children's solutions to Verbal Arithmetic Problems, *Journal for Research in Mathematics Education*, 13 (2), 83-98.

Dienes Z. P., 1960: *Building up Mathematics*, Hutchinson Educational, London.

Fuson K.C., 1982: An Analysis of the Counting-on Solution Procedure in Addition. In T. Carpenter *et al.* (Eds.) *Addition and Subtraction: A Cognitive Perspective*, New Jersey: Laurence Earlbaum.

Gray E. M. (to appear): An analysis of diverging approaches to simple arithmetic: preferences and its consequences, *Educational Studies in Mathematics*.

Greeno G. J. ,1983: Conceptual Entities. In Genter D. & Stevens A. (Eds.), *Mental Models*, 227-252.

Kaput J., 1982: Differential effects of the symbol systems of arithmetic and geometry on the interpretation of algebraic symbols. Paper presented at the meeting of the American Educational Research Association, New York.

Piaget J., 1972: *The Principles of Genetic Epistemology* (W. Mays trans.) London: Routledge & Kegan Paul.

Sfard A., 1989: Transition from Operational to Structural Conception: The notion of function revisited, *Proceedings of PME XIII, Paris*.

Secada W.G., Fuson K.C. & Hall J.W., 1983: The transition from counting all to counting on in addition, *Journal for Research in Mathematics Education*, 14 (1), 47–57.

Tall & Thomas (to appear), Encouraging Versatile Thinking in Algebra using the Computer, *Educational Studies in Mathematics*.

CHILDREN'S WORD PROBLEMS MATCHING MULTIPLICATION AND DIVISION CALCULATIONS

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A systematic investigation of the ability of children aged between 9 and 15 to write word problems corresponding to given multiplication and division calculations is reported. Separate groups of children were tested with calculations involving integers, fractions and decimals. Rates of correct responses exceeding 50% occurred only for cases where an integer could be used as the operator. Patterns in the data reflecting the distinctions between multiplicand and multiplier and between partitive and quotitive division were found. It is argued that children should not be taught to do calculations without knowing what they are for; further, it is suggested that the applications of arithmetical calculations should be taught within a modeling framework.

According to Hersh and Davis (1981, p. 71): "There are many children who know how to add, but do not know when to add". This statement generalizes to the other basic arithmetic operations, and has been demonstrated in studies requiring children to nominate the appropriate operations to model situations described in word problems (for a summary, see Greer, in press). A different perspective can be obtained by using the inverse task, in which calculations are presented and children asked to generate word problems corresponding to them.

Both methodologies were used in the Concepts in Secondary Mathematics and Science project (Hart, 1981). The stories made up by children for addition calculations were fairly evenly distributed between union (combine), add-on (change) and compare situations, whilst those for subtraction were almost alway: "take-aways" (change). For multiplication, common choices were repeated addition and rate models (equal groups in the terminology of Greer (in press)), with some multiplicative comparisons; Cartesian product responses were very

rare. For division, stories reflecting partition (sharing) dominated.

Using the calculation 5.3 + 4.6 = 9.9, Shiu (1988) reported several cases of stories indicating a lack of understanding of decimal numbers, including some in which the numbers were used in contexts which demand integers, such as:

> Robert had a jigsaw. He had fitted together 4.6 pieces, then Stella fitted together 5.3 pieces so altogether 9.9 pieces were fitted together.

Pimm (1987, p. 13) commented that many of the children's problems "exhibit the apparent irrelevance . . . of the surrounding story in mathematics classes. The stories do not have to be plausible or even make sense provided they contain the requisite numbers and a guide to the operation".

A more subtle point is illustrated by another of Shiu's examples, raising the question of the appropriateness of the operation to model the situation:

> Sally had a piece of string which was 4.6 cm long, she found a bit 5.3 cm long and tied both bits together so her string was 9.9 cm long.

This is reminiscent of Kilpatrick's (1987) discussion of a problem about a clother line, whereby an initial oversimplified model is refined by taking into account the amount of line realistically required for knots.

Af Ekenstam and Greger (1983) and Bell, Fischbein and Greer (1984) also presented children with multiplication and division calculations with decimals. Both studies report children "forcing" calculations into inappropriate frameworks as in this example (Bell et al., 1984, p. 139):

> Emma had 0.74 sweets. She wanted to share them so each doll got the same. There was 0.21 dolls. How many did each doll have?

Considerable ingenuity was shown by some children in accommodating calculations within familiar schemata, as in these examples from Bell et al:

> Mary has 4 lb apples. She has got to share them between her and her 23 friends. How many apples do they each get? (12 apples to a lb) [4 + 24]
>
> Mr Woolworth is a shopkeeper who sends packages of material to local women or tailors. One woman rings up and says she only wants half a box of material leaving 10.5 boxes instead of 11.

If each woman/man wants 0.71m of material and there is 10.5 boxes how much material will he send to customers? [10.5 x 0.71]

Experiment

The current study was a more systematic investigation of children's ability to write appropriate word problems for multiplication and division problems. Separate tests for integers, fractions and decimals were administered to different groups of Northern Irish children aged between 9 and 15. Details of the items and groups can be found in the two tables below.

Table 1. Results for integers (percentages of appropriate word problems)

Ages	9	11	13	15
Number tested	40	38	49	54
5 x 3 =15	49	60	69	80
4 x 46 = 184	30	50	58	74
16 x 18 = 288	30	44	56	75
Means	36	51	61	76
12 ÷ 3 = 4	37	41	50	78
468 ÷ 3 = 156	43	42	61	76
352 ÷ 88 = 4	34	35	51	76
945 ÷ 35 = 27	49	37	51	79
Means	41	39	53	77

Results for integer calculations are shown in Table 1. Some of the major patterns in the data were:

1. Size of integers had relatively little effect. In particular, the lack of apparent effect for division is consistent with the suggestion in Hart (1981, p. 36) that the division sign is a direct cue for "sharing".

2. Most of the appropriate stories for the multiplication calculations were in terms of a number of equal groups of discrete objects. Identifying the

number per group as the multiplicand and the number of groups as the multiplier, for 3 x 5 = 15 and 16 x 18 = 288 the overall percentages of appropriate stories in which the first number was used as the multiplier were 65% and 49% respectively, implying that 3 x 5 may be interpreted in either way (e.g. as "3 fives" or "3 times 5", or as "3 multiplied by 5"). For 4 x 46 = 184, the 4 was used in 77% of the cases as the multiplier, which intuitively seems simpler and suggests flexibility on the part of the children as to which number to use as multiplier.

3. For divisions, virtually all the stories were partitive (sharing of a collection of discrete objects into the given number of equal groups), as has been found consistently in previous studies.

Table 2. Results for fractions and decimals (percentages of appropriate word problems)

Ages	11	13	15		13	15
Number tested	110	98	104		100	86
18 x 2/3 =12	38	18	41	5 x 0.80 = 4	37	59
8 x 2/3 = 16/3	37	15	38	6 x 0.14 = 0.84	33	68
4 x 3/14 = 12/14	26	28	36	8 x 0.57 = 4.56	39	60
Means	34	20	38		36	62
14/3 ÷ 8 = 7/12	21	18	20	3.25 ÷ 13 = 0.25	43	62
4/5 ÷ 6 = 2/15	19	15	25	0.54 ÷ 3 = 0.18	30	52
6 ÷ 8 = 3/4	19	14	40	9 ÷ 25 = 0.36	23	26
Means	20	16	28		32	47
10 ÷ 2/3 = 15	30	11	25	12 ÷ 0.75 = 16	3	29
10/3 ÷ 5/6 = 4	10	14	11	4.50 ÷ 0.25 = 18	19	32
3/4 ÷ 1/8 = 6	23	15	23	0.78 ÷ 0.26 = 3	18	32
Means	21	13	20		13	31

Results for fractions and decimals are shown in Table 2. The classes of 13-year-olds were of lower ability than the others (according to their schools), so no particular significance should be attached to their comparatively poor performance. Patterns in the data include:

1. For the multiplication items involving fractions, the fraction was predominantly used as the multiplier, and part/whole stories produced e.g.

> A man measures a piece of wood 8 ft long, but he only needs $2/3$ of that, how much does he need?

The percentages with the fraction as multiplier were 67% and 77% for the 13- and 15-year-olds respectively. However, for the corresponding decimal calculations, the integer was predominantly used as the multiplier.

2. In the division problems, partitive stories were predominant for division by an integer (97% for fractions, 100% for decimals, overall), whereas quotitive stories were predominant for division by a number less than 1 (87% fc fractions, 96% for decimals, overall).

Across all the tests, the contexts of the appropriate word problems generated were consistent with previous findings: equal groups and equal measures predominated when they were appropriate -- for more difficult numbers, the most common contexts used were price, speed, length, weight and capacity (but not area).

Likewise, the distribution and nature of errors was largely consistent with previous findings (with the exception that the inclusion of the answer in calculations such as $9 \div 25 = 0.36$ virtually eliminated reversals -- as was intended). However, a quite common error not previously reported was asking the question "How much was left?" instead of "How much was used?" as in this example:

> 8 boxes of crisps were at a party, $2/3$ were took home, how many were left?

Another common error is exemplified by this story:

> A petrol tank has 8 gallons in it. If it now increases its load by $2/3$ times, how much will there be in the tank now?

Despite the inclusion of the answer in the calculation presented, children writing this sort of story assumed that multiplying 8 by $2/3$ would give 12.

Discussion

Even allowing for the difficulty and unfamiliarity of the task, the most general point to be made is that many children are being taught to do calculation without being able to describe situations in which those calculations are applicable. This reflects the disproportionate emphasis on computational skill typical of many schools in many educational systems; word problems are often thinly disguised practice of computation. An informal analysis of the textbooks used by the children studied in this experiment suggested that they contribute substantially in various way; systematic analysis of such textbooks would be an appropriate focus for further investigations, as would the analysis of examples generated by teachers for their pupils (again, informal analysis in the present study suggested that the teachers involved may use a restricted range of situation for word problems).

The term "modeling" is sometimes used in the sense that physical and graphical representations provide models for the arithmetic operations (e.g. a rectangular array for multiplication). This use is often associated with the aim o legitimizing calculation procedures and promoting understanding of the formal properties of number and operations (e.g. commutativity of multiplication). From this perspective, a limited number of concrete representations is adequate.

An alternative approach uses "modeling" in the opposite sense, and treats arithmetical operations within a general framework of mathematical modeling (Greer, in press). Hilton (1984, p. 8) declared that "the separation of division from its context is an appalling feature of traditional drill arithmetic . . . the solution to the division problem $1000 \div 12$ should depend on the context of the

problem and not the grade of the student". Greer (in press) argues that "instead of the sanitized view of the world generally presented in 'the word problem game' . . . pupils should experience a wide range of examples in which the natur of the relationship between reality and calculation and the appropriate interpretation of the calculation vary, and in which implicit modeling assumptions . . . are made explicit" (cf the discussion by Kilpatrick (1987) referred to in the introduction).

From this point of view, describing a variety of situations modeled by a given calculation and, moreover, variation in the nature of the modeling process itself, would be an appropriate teaching exercise. This approach has been implemented by Streefland (1988, p. 81):

> Again and again the opportunity is given to invent problems, among which [are] illustrations of bare numerical division. An example: 6394 + 12, invent stories belonging to this sum such that the result is respectively

532	532.84 remainder 4
533	532.833333
532 remainder 10	about 530

Further research is under way to look at intra-cultural differences in the types of word problem generated by children within different educational systems. Variation is to be expected because of linguistic differences, such as the availability in Hebrew of a simple way of expressing multiplicative comparison (Peled & Nesher, 1988) and because of curricular differences, such as the greater emphasis in France on Cartesian product.

References

Af Ekenstam, A., & Greger, K. (1983). Some aspects of children's ability to solve mathematical problems. *Educational Studies in Mathematics* , *14* , 369-384.

Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and context. *Educational Studies in Mathematics* , *15* , 129-147.

Cockcroft, W. H. (1982). *Mathematics counts.* (Report of the Committee of Inquiry into the Teaching of Mathematics in Schools). London: HMSO.

Davis, P. J. & Hersh, R. (1981). *The mathematical experience.* Boston: Birkhauser.

Greer, B. (in press). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on learning and teaching mathematics.* Reston, VA: NCTM and New York: Macmillan.

Hart, K. (Ed.) (1981). *Children's understanding of mathematics: 11-16.* London: Murray.

Hilton, P. (1984). Current trends in mathematics and future trends in mathematics education. *For the Learning of Mathematics*, *4* (1), 2-8.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* , pp. 123-147. Hillsdale, NJ: Erlbaum.

Peled, I., & Nesher, P. (1988). What children tell us about multiplication word problems. *Journal of Mathematical Behavior*, *7* , 239-262.

Pimm, D. (1987). *Speaking mathematically.* London: Routledge & Kegan Paul.

Shiu, C. (1988). A problem-solving approach to diagnostic assessment. In D. Pimm (Ed.), *Mathematics, teachers, and children* , pp. 43-51. London: Hodder & Stoughton.

Streefland, L. (1988). Reconstructive learning. In A. Borbas (Ed.), *Proceedings of the Twelfth International Conference for the Psychology of Mathematics Education, (Vol.1)* , (pp. 75-91). Veszprem, Hungary: OOK Printing House.

[IL]DREN'S VERBAL COMMUNICATION IN PROBLEM SOLVING ACTIVITIES

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A model is proposed as a tool to investigate the use of children's verbal communication as an instrument of learning while working in pairs at arithmetic word problems. In an empirical investigation with a class of third year primary school children more than 2000 speech acts made by the students were classified according to the various domains of the model: cognitive, communicative, affective-emotional and social. In addition for each of the students an evaluation was made by the teachers which provided details of student's achievements and behaviour in the learning situation. The statistical analysis of the data provides further insight into the dynamics of mathematical conversations between children.

1. INTRODUCTION

In the past several models have been developed to describe and study the structures of human communication (eg Watzlawick et al 1969, Spanhel 1973, Bellack et al 1974, Bales (Piontkowski 1976), Diegritz/Rosenbusch 1977, Maier/Bauer 1978). Depending on their aims and intentions these models vary a great deal in their approach and consequently also in the dimensions which they include. Although some of the models have been applied to the special situations of teaching and learning, none of them fully account for the processes of problem solving and of communicative behaviour.

In order to investigate children's verbal communication while working in pairs at a word problem, it is therefore necessary to set up a model which comprises cognitive aspects of the problem solving process and of information processing, communicative and affective-emotional components as well as components of social interaction. In a preliminary study children's verbal communication in problem solving activities was analysed in a qualitative way where various components of communication were classified (Grevsmühl/Storbeck 1989). The purpose of the present investigation is to complete these findings by a more quantative analysis of the dynamics of children's mathematical conversation.

In the investigation 22 students from a mixed-ability third year class of a local primary school (age 9 years) were tested in pairs outside of their regular class-setting. Five arithmetic word problems were used to promote mathematical conversations between the children. One of them is the well-known 'snail problem' which has also been used in educational advisory tests (eg BBT 3-4, Beltz):

> *'A snail is sitting at the bottom of a well and wants to crawl out. The well is 16 meters deep. Each day the snail crawls four meters up the side of the well but during the night it slides back one meter. On which day does the snail reach the top of the well?'*

At the beginning of the test the supervisor asked one of the students to read out loud the problem. Paper and pencil was provided and the students were asked to solve the problem together. All conversations were recorded and transcribed but only 33 of them were evaluated further where there was no communication with the supervisor and which contained a substantial amount of mathematical conversation between the students. Monologues or dialogues with predominantly private or merely organisational conversations were not included.

2. MODEL OF CHILDREN'S VERBAL COMMUNICATION IN PROBLEM SOLVING ACTIVITIES

For the investigation of children's verbal communication speech acts are evaluated on the basis of the cognitive, communicative, affective-emtional and social domains. A speech act is defined as the smallest unit of human communication carrying a content, a communicative function (relational aspect) and an intentional aspect. It may consist of only one or several words. A verbal expression may therefore consist of several speech acts. The 33 conversations contained more than 2000 speech acts which were classified according to the ten components of the model (1.1 to 4.2 of Table 1) and evaluated statistically. It was not the intention to include details of the various strategies used by the children in tackling the word problems but to investigate their mathematical conversations in a problem solving environment.

In the cognitive domain the problem solving process has been analysed by the stage model put forward by Newman and described by Watson (1980). Each of the speech

Table 1: MODEL TO DESCRIBE CHILDREN'S VERBAL COMMUNICATION WHILE WORKING IN PAIRS AT WORD PROBLEMS (relative frequency in %)

1. COGNITIVE COMPONENTS				
1.1 Level in the Problem Solving Process	c:	f:	ind:	su:
reading ability	4.3	0.3	0.0	4.7
comprehension	4.3	1.3	7.6	13.2
transformation	11.5	17.3	21.8	50.6
process skills	7.8	0.7	15.9	24.3
encoding/ interpretation	4.1	1.3	2.1	7.5
1.2 Information Processing (see Table 2)				
of the speaker's own though				50.9
of a thought adopted from the partner				38.6
of information gained from the word problem				10.4
1.3 Approach Taken by the Speaker				
searching/ clarifying				29.0
arguing				40.6
neutral				30.5
2. COMMUNICATIVE COMPONENTS				
2.1 Type of Conversation				
mathematical conversation				91.3
organisational conversation				8.2
private conversation				0.4
2.2 Type of Language				
mathematical language				7.4
colloquial language				79.3
situative language				13.2
2.3 Type of Verbal Expression (see Table 3)				
constativa				40.0
positional				28.7
evaluativa				31.2
3. AFFECTIVE-EMOTIONAL COMPONENTS				
3.1 Attitude towards the Task				
like				1.7
dislike				2.3
neutral				95.9
3.2 Attitude towards the Partner				
friendly				1.0
hostile				6.0
neutral				93.0
4. COMPONENTS OF SOCIAL INTERACTION				
4.1 Type of Joint Activity				
cooperative				69.4
competitive				7.9
neutral				22.6
4.2 Way in which Power is Executed				
dominating				15.7
subservient				7.6
neutral				76.8

acts was not only allocated to one of the five stages but it was also recorded if the content of the speech act was correct (c), false (f) or indefinite (ind). The latter notation was also used if a correct conclusion was drawn from an incorrect thought made previously. The analysis shows that half of the speech acts occur in the transformation stage and nearly a quarter while processing skills. The majority of false or indefinite speech acts also occur in these two stages.

The component information processing was used to record and analyse the flow of information between speaker, partner and problem (Table 1). In more than half of the speech acts the speaker expresses his own thoughts, about a third of which contain a new thought, nearly a third the continuation of a thought and nearly a quarter the repetition or confirmation of a thought (Table 2). In 39% of the speech acts a thought of the partner was taken up by the speaker where in nearly half of the cases the thought was repeated by the speaker, in 17% it was carried futher and in nearly a third the partner's thought was either probed, corrected or questioned. Only in 10% of the speech acts information was taken from the word problem where in nearly half of the cases the information was provided and in a third it was repeated or confirmed.

The approaches taken by the speakers in the conversations are for 29% of the speech acts searching-clarifying, for 41% of an arguing nature while the rest is neutral (Table 1).

For the communicative domain a linguistic analysis of the speech acts was undertaken according to the type of conversation, of language and of verbal expression. Due to the choice of conversations it was expected that more than 90% of the speech acts would be mathematical conversations and contain also a substantial amount (7.4%) of mathematical language (Table 1). A speech act was said to use mathematical language if it contains a mathematical expression or a phrase used in the teaching of mathematics. Situative language occurs in about 13% of the speech acts. These acts do not make sense on their own but only in relation to the activity and are often accompanied by non-verbal forms of communication.

Verbal expressions were analysed according to a system of classification suggested by Diegritz/Rosenbusch (1977) in which each speech act is allocated to one of the classes constativa, positional or evaluativa. In constativa the speech act is task-orientated with an emphasis made on the content rather than on the relational aspect. In the class of positional the position of the speaker becomes visible, for instance by his effort to influence the position of others. There the relational aspect is all important. The person and attitude of the speaker are predominant and the speech act reflects more the speaker's own ideas. In the class of evaluativa the speaker persues a subject or person orientated evaluation. Again the relational aspect plays a decisive role but in order to prevent overlapping with the class of positional an evaluativa is defined as being the reaction to a speech act made by others (eg a contradiction can only be evoked by a provocative act proceeding it). The analysis shows that 40% of all speech acts are constativa, 40% of which are statements. Furthermore, about 30% of the speech acts are positional and about 30% are evaluativa with more than half of them being confirmations (Table 3).

For the affective-emotional and social domains the dimensions dominance, cooperation and friendliness were taken into account as suggested by Bales (Piontkowski 1976) for the analysis of interaction processes and for modelling interpersonal ratings. Whereas in nearly all of the speech acts the attitude towards the task is neutral, 6% of the speech acts show a hostile attitude towards the partner compared to a mere 1% which is friendly. In the social domain 70% of the speech acts indicate a cooperation with the partner, 8% a competition. As regards to the way in which power is executed 16% of the speech acts reflect a dominating, 8% a subservient role of the speaker.

3. TEACHERS' EVALUATION OF STUDENTS' BEHAVIOUR

For each of the 22 students an evaluation was made by the teacher of Mathematics and independently by the teacher of German. A modified test form developed by the Institut für Bildungsplanung und Studieninformation (1975) was used which contains

achievements in Mathematics, in the writing of essays and in dictation, behaviour in the learning situation, emotional and social behaviour.

The fairly high Pearson correlation coefficients between independence in work, ability to convers fluently and clear and intelligible presentation of tasks indicate a common parameter within the behaviour in the learning situation (0.71-0.95). Moderate to strong correlations were also found between these parameters, the achievements in Mathematics (0.63-0.93) and in essay (0.74-0.81) and the dominance while working in pairs in Mathematics (0.74-0.83) and in German (0.63-0.77). Furthermore, the dominance while working in pairs also correlates strongly with the achievements in Mathematcs (0.83) and in essay (0.71).

4. CORRELATIONS WITH COMPONENTS OF THE MODEL

Pearson correlation coefficients were calculated between the various components of the model and also between the components and parameters of students' behaviour. As a report and discussion of all the correlations would be beyond the scope of this paper, only some of the results can be presented here.

In the information processing (Table 1: 1.2) moderate to strong correlations were found between the total number of speech acts with the speaker's own thought and his independence in work and dominance while working in pairs (0.6). Correlations of similar strength but negative exist between the total number of speech acts with a thought adopted from the partner and these parameters (0.6).

Positive evaluativa (Table 3) correlate strongly (0.7-0.9) and positively with the number of speech acts with repetition/ confirmation of a thought adopted from the partner (Table 2) and negatively with the achievement in essay, with self-confidence and with finding contacts to others.

The constativa reasoning (Table 3) correlates strongly (0.9) with correct interpretation in

Table 2: INFORMATION PROCESSING (relative frequency in %)

	speaker's own thought	thought adopted from partner	information from problem
new thought/ providing information			
correct	19.8	-	41.2
false	14.1	-	7.7
continuing	31.0	16.9	7.3
reasoning	3.1	3.1	0.0
repeating/ confirming	22.8	48.7	33.8
probing	3.4	8.6	4.8
correcting	2.6	9.0	1.2
questioning	3.1	13.7	3.9

Table 3: TYPE OF VERBAL EXPRESSION (relative frequency in %)

Constativa			40.0
stating	40.3		
supposing	13.8		
reasoning	17.0		
answering	3.0		
reading out	8.3		
repeating	17.5		
informing	0.0		
Positional			28.7
Optativa		36.6	
questioning	22.3		
requesting	0.7		
reaffirming	13.6		
Imperativa		20.2	
inviting	7.0		
instructing	8.0		
ordering	5.2		
Constructiva		14.6	
suggesting	14.6		
Specific Positional		28.6	
maintaining	2.8		
considering	5.9		
undecided	11.1		
persisting	8.0		
commenting	0.7		
Evaluativa			31.2
positive		62.2	
confirming	52.6		
conceeding	9.6		
praising	0.0		
negative		37.8	
rejecting	16.0		
reproaching	0.6		
contradicting	6.4		
objecting	10.6		
correcting	4.2		

the problem solving process, the continuation of one's own thought and the questioning of information gained from the problem.

The number of speech acts with an arguing approach (Table 1: 1.3) correlates strongly with the achievements in Mathematics, in essay, with independent and careful work, with self-confidence and also with finding contacts to others (0.7-0.8). On the other hand a neutral approach correlates strongly but negatively with these parameters.

5. CONCLUDING COMMENTS

There is no doubt that the results of this investigation depend on the choice of sample made (children, choice of problem and of presentations, choice of conversations, etc). Still the model itself provides important information of the dynamics of children's verbal communication in a mathematical problem solving environment. In order to gain further insight into the dynamical aspects, it is necessary to investigate the iterative nature in which the stages of the problem solving process are passed through and also the sequence of verbal expressions within the conversations.

REFERENCES

Bellack A / Kliebard H M / Hyman R T / Smith F L, Die Sprache im Klassenzimmer, Düsseldorf 1974

Diegritz Th / Rosenbusch H, Kommunikation zwischen Schülern, München 1977

Grevsmühl U / Storbeck C, Lehrersprache, Schülersprache - Aspekte sprachlicher Kommunikation im Mathematikunterricht der Grundschule, Beiträge zum Mathematikunterricht 1989, Bad Salzdetfurth, 163-166

Institut für Bildungsplanung und Studieninformation (IBS), Merkmalsbogen zur Verhaltensbeschreibung durch den Lehrer, Stuttgart 1975

Maier H / Bauer L, Zum Problem der Sprache im Mathematikunterricht, unpublished, Regensburg 1978

Piontkowski U, Psychologie der Interaktion, München 1976

Spanhel D (ed), Schülersprache und Lernprozesse, Düsseldorf 1973

Watson I, Investigating Errors of Beginning Mathematicians, Educational Studies in Mathematics 11(3), 319-329, 1980

Wygotski L S, Denken und Sprechen, Stuttgart 1964

THE "POWER" OF ADDITIVE STRUCTURE AND DIFFICULTIES IN THE RATIO CONCEPT

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This report deals with the aim of observing the relationship between the "power" of additive structure and difficulties to recognize ratio questions. It is well known that often errors are a result of the student's conceptual system: of his or her conceptions, intuitions, methods of dealing with problems and so on. This system, of which the student need not be conscious, may be different from the system the teacher wants to communicate. Often the student does not use the rules or methods the teacher has taught when he is asked to solve problems out of the immediate context of practising that rule. In our research an incorrect strategy on a ratio question was to apply the additive structure. As K.Hart "This is not surprising, but what is astonishing is how many children continue to use additive methods long after they have outgrown their usefulness and are very cumbersome". May we have some help from epistemology ?

1. Background to the research

In a context not explicitly addressed to testing the acquisition of the ratio concept, some facts have emerged that seem significant in relation to the use of proportional reasoning.
During the development of a teaching activity (cfr.[4]), the main aim of which was to compare the strategies used by pupils to solve problems of the Liber Abaci (1) of Fibonacci and the strategies of that ancient author, we set the pupils, in particular, the following problem:
<<Two towers A and B of 30 and 40 steps in height are 50 steps apart; between the 2 towers there is a fountain. Two birds fly off from the top of the two towers, at the same speed and they arrive at the fountain at the same moment; what are the two horizontal distances from the center of the fountain to the two towers?>>.(2)

(1) Leonardo Pisano (Fibonacci) wrote his Liber Abaci in 1202. We are referring to the Edizione critica di Baldassarre Boncompagni, Roma, Tipografia delle Scienze Matematiche e Fisiche, 1857.

(2) The original problem is :<<In quodam plano sunt due turres, quarum una est alta passibus 30, altera 40, et distant in solo passibus 50; infra qua est fons, ad cuius centrum volitant due aves pari volatu, descendentes pariter ex altitudine ipsarum; queritur distantia centri ab utraque turri.>>.

2. The research problem

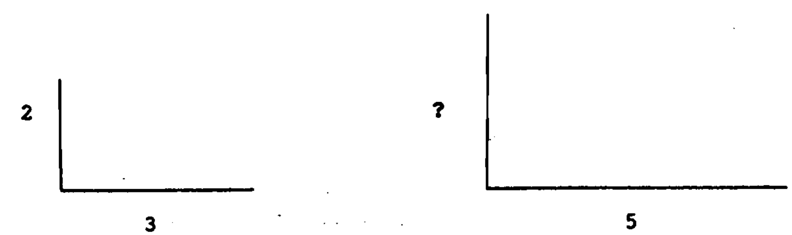
That Fibonacci's problem was proposed to 12 year old pupils and to 14-15 year olds.
The 12 year olds, who did not know Pythagorean theoreme, generally solved the problem using an additive model that, in this case, is obviously incorrect.
On the contrary, the 14-15 year old pupils used Pythagorean theoreme. However, we were alarmed by the "power" of the additive model and we thought it was important to organize a research in order to verify if the additive model prevails also in 14-15 year old pupils when they are in a situation different from that of Fibonacci's problem, in particular in a ratio context, and when we remove the "safety" of resorting to Pythagorean theoreme.

The aim of our research, that is still in progress, is also to investigate by ratio questions the interpretations of the dominance of the additive model in order to obtain useful elements for developing a series of teaching strategies able to place, constructively, that dominance in a critical position.

3. Context of the research

We are working with 288 12 year old pupils of 12 classes of the second year of junior high school and with 140 14-15 year olds of 5 classes of the first two years of non-scientific specialization secondary high school.(3)

After Fibonacci's problem, we proposed to fifty per cent of the sample the following question (cfr. K. Hart [5] p. 544):

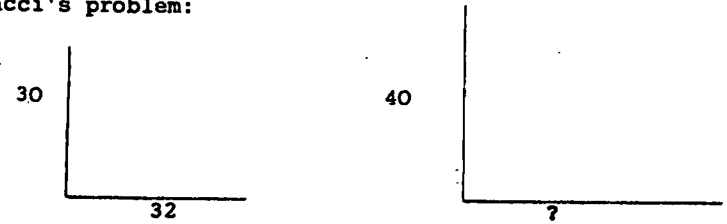


Which number do you replace to ? to obtain an enlargement of the first figure?

Explain your answer.

(3) In the area where these classes are, the percentage of pupils who begin the senior high school after the junior high school is ~ 90 %.

proposed to the other fifty per cent of the sample a similar question where the numbers remind, in some a way, those of Fibonacci's problem:



Which number do you replace to ? to obtain an enlargement of the first figure?

Explain your answer.

4. Results

The 12 year olds who solved Fibonacci's problem by an additive strategy continued to do so.
Under the same circumstances, also some 14-15 year olds used it.
In particular, the results were as follows:

- first problem (Fibonacci's one):
  76.4% (220) aged 12 (288) used the additive model;
  90% (126) aged 14-15 (140) solved the problem by Pythagorean theoreme.

- second problem (no differences have arisen between the two versions):
  85% (245) aged 12 (288) used the additive model;
  27% (38) aged 14-15 (140) used the additive model;(some of those who correctly answered using the ratio method, were perplexed to obtain a recurring number as the result!).

5. Interpretation of results and Research to come

We are faced with an obstacle that is not easily surmontable.
The difficulty in surmounting it may derive from the nature of the obstacle itself and the understanding of its nature is not only a didactical problem, but also an epistemological one, as is well known.

We are concerned with a stucture, the additive one, which put in a critical position the ratio concept also with respect to some 14-15 year old pupils. Certainly this is not a novelty; in particular, as K.Hart [5] p.544 <<What is astonishing is how many children continue to use additive methods long after they have outgrown their uselfulness and are very cumbersome>>.

As epistemology teachs us that a given structure proceeds to a different level of complexity only when it is destabilized by its own limits and then restructured, we decided to develop the analysis and the interpretations of pupils'answers by considering:

1) an epistemological/cultural/anthropological level;

2) a didactical/cognitive level.

1) In common parlance generally no distintion is made between the modalities whereby <<when one quantity increases, so too another increases>>.
The analysis of protocols (ratio question) indicated that some of the recurrent ways of explaining the result are the following:
<<As in the first figure 30 was augmented by 2, in the second figure too we must augment by 2.>>; <<I wrote 42 obtained by adding 10 to 32, because I saw that 30 has become 40.>>; <<Because if a measure augments by 2, the other one too must augment by the same quantity.>>; <<As the difference between 40 and 30 must be equal to the difference between the two other numbers, I have put 42.>>.

In such a situation we believe that a teaching strategy, in order to evidence the limits of the additive model, could provide:
a) an activity by merely numerical exercices in which it is possible to expressely point out the non-applicability of that model;
b) a discussion activity with pupils about:
   I) the deficiency of the modelling accomplished with the common parlance;
   II) the comparison of the additive model with the multiplicative one, both in those cases where the first model agrees with the second one and in those cases where the contradiction is evident.

This approach was designed to enable pupils a consciously overcome the error, seen as evolutive step of the process of the pupil's building up of knowledge.

2) The predominance (during the first years of school) of additive problems-in the words of G. Vergnaud [11] p.35 <<Several important concepts are involved in additive structures: cardinal, measure, state, transformation, comparison, difference, inversion and directed number are all essential in the conceptualizing process undertaken by students.>>- and the fact that, also later, in many situations, teaching always begins from the "linear" cases, can create an "additive imprinting". This imprinting represents the first resource to which the pupil clings when he must solve a new problem.
if this interpretation is valid, then a subsequent strategy could provide:

a) "difficult" problems that pupils will probably solve by an additive model;

b) moments of reflection on this "imprinting" of "additivity", to render explicit to pupils, once again, not only the error, but also the possible reasons for interpreting it.

References

[1] Blair E., 1981, Philosophies of Mathematics and perspectives of Mathematics Teaching, Int.I.Math.Educ.Sci.Technol., Voll 2 (2).

[2] Boero P., 1989, Campi semantici nell'insegnamento-apprendimento della matematica..., rapporto di ricerca, Seminario Nazionale di Ricerca in Didattica della Matematica, Pisa.

[3] Booker G., 1987, The role of errors in the construction of Mathematical knowledge, Proceedings of the 39th CIEAEM Sherbrook (Canada), pp.63-69.

[4] Grugnetti L., 1989, The role of the History of mathematics in an interdisciplinary approach to mathematics teaching, Zentralblatt fur Didaktik der Mathematik, Karlsruhe, n.4, pp.23-32.

[5] Hart K., 1980, Methods children use to solve Mathematics problems, Proceedings of ICME 4, Berkley, pp.543-545.

[6] Noelthing G., 1980, The development of proportional reasoning and the ratio concept, Part I, Differentiation of Stager, Educational Studies in Mathematics 11,pp.210-253.

[7] Piaget J., Inhelder B., 1967, The child's Conception of Space Routledge and Kegan Paul, London.

[8] Piaget J., Rolando G., 1980, Psicogenesis e Historia de la Ciencia, Mexico, Siglo XXI.

[9] Serpinska A., 1985, La notion d'obstacle epistemologique dans l'enseignement des mathématiques, Proceedings of the 37th CIEAEM Meeting in Leiden.

[10] Stricklan Jr G.F., De Nitto J.F., 1989, The power of proportions in Problem Solving, Mathematics Teacher, vol. 82 n.1, Genuary, pp. 11-13.

[11] Vergnaud G., 1988, Theoretical frameworks and empirical facts in the psycology of Mathematics Education,proceedings of ICME 6, Budapest, pp. 29-47.

WHY MODELING?

PUPILS INTERPRETATION OF THE ACTIVITY OF MODELING IN MATHEMATICAL EDUCATION

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Most research on modeling in mathematics education has examined how pupils perform the activity and what cognitive obstacles they have to overcome. Through repeated classroom discussions conducted during a one year interdisciplinary course on mathematical model building we have tried to capture 8th graders' understanding of why to build models. Some cognitive operations and obstructions involved in pupils' interpretation of the purposes and interest of building models in mathematics are described and illustrated. The epistemological status they confer to a model and its autonomy with respect to facts are analysed. Possible relations between pupils' conception and standard school presentation of models are drawn.

1. Introduction

Modeling activities have received, in recent years, a large amount of attention in pedagogical research and curriculum studies related to mathematical education, both with and without the use of adhoc computer softwares. Pragmatic definitions of these activities have been proposed in a didactical context, as well as typologies of the different kinds of mathematical skills required in the process (see, for instance, Blum, Niss, and Huntley, 1989).

Several cognitive components intervene in the description of a modeling activity: in particular the recognition of the need of a model for the solution of the problem; the capability of verbal; gestual and graphic representations of the underlying problem; the definition of the significant variables, parameters and functional dependences needed for its mathematical description.

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Most of the published research on modeling in mathematics has been centered on how to build models, and on the various difficulties associated with operations like the transformation of tables of datas into graphic representations and the construction of an appropriate algorithm or function (see Leinhardt, Zaslavsky, & Stein, 1990 for a review). We believe that, while this study is very important and probably decisive in the actual classroom practice of problem solving through mathematical modeling, the question of pupils' understanding of the why to build models still deserve a more careful and in depth exploration from the psycho-cognitive point of view (see Vitale,1988; 1988a; 1989).

In this paper, we present some elements of the theoretical framework underlying our research program on the psycho-cognitive aspects of modeling, as experienced in the classroom context by pupils between 13 and 15 years, as well as some preliminary data gathered from an experimental, long-range interdisciplinary course in modeling. Pupils' understanding of the purposes of model building will be infered from extensive class discussions conducted around the activity of modeling. More specifically we have tried to capture 8th graders' implicit answers to the three following questions :

a. What is the need for modeling?
b. What is the epistemological status of a "model", as compared to that of an "empirical fact"?
c. What is the autonomy of a model, as compared to that of an "empirical fact"?

2. Role of models in mathematical education

Common school curricula tend to present facts and models in a way that can easily lead to misconceptions about their relationships.

a. Facts are generally presented as "neutral", independently from any underlying model or theory. Teachers and examiners can usually be satisfied by simply (re)phrazing such statements as "the orbits of the planets are ellipses with the sun in one of the foci", "parallels are lines which never cross", etc. The consideration that "facts" are not neutral, and cannot in general be even stated without a representational frame (which implies a number of models), is generally ignored.

b. Models are commonly presented as approximate (or simplified, idealized, ...) representations of a set of factual data, of the actual evolution of a dynamical process. This leads to a subtle psychological handicap for models: being considered - more or less - approximate, they are seldom taken seriously and assessed at their face value. What is needed is a dialectical approach that sees and explores the interrelation between facts and algorithms. Facts can be fitted by an algorithm that, in turn, clarifies factual data. Declarative and procedural thinking are not opposed, but complementary.

c. The way models are used is also very likely to make them look as problem-dependent, and therefore allowed no autonomy. Reality is associated to the factual information, mathematical modeling to its formal description; this leads to a neglect of the internal, autonomous life of a model and of its deep reality. What we need is again a dialectical approach that sees and explores the interrelation between formal thinking (the realm of logico-mathematical structures) and causal thinking (the realm of physical structures). Physical phenomena can suggest models that, in turn, uncover deep and new mathematical realms.

3. Psychocognitive components and obstructions in the modeling activity

In accordance with the theoretical frame described very briefly above, we have concentrated our investigation on some psycho-cognitive components of the

activity of modeling and possible corresponding obstructions (columns 1 and 2 of Table 1).

cognitive operation	possible obstruction	main findings
Going beyond data	Perceived characteristics of facts and of their representation	Diagrams as explanations of changing processes
Slicing time (for dynamical models)	Continuity of time changing processes	Continuous change as more immediate than state transitions
Selecting variables and parameters	Conception of an explanation's characteristics	Best fits as models
	Factors' role extraction from graphics	Difficulties with the role of time in evolution processes
	Limited understanding of functional dependance	Applying functional dependance more difficult than its intuitive understanding
Constructing an algorithm	Role of variables and parameters in functional dependance	Programming can help understanding the separation of algorithm from variables and parameters

Table 1: Some cognitive operations involved in the activity of modeling, possible obstructions and relative findings.

4. Experimental set-up

Our research program takes place at present in a secondary school, in two 8th grade classes in 1989-1990 (13-14 years old pupils) and in two 9th grade classes in 1990-1991. The research goes on during the whole academic year, followed by the teachers and monitored by the researchers. The two teachers collaborating in the program teach to their class both informatics and science (respectively, physics and biology).

Factual data taken out of various contexts have been used during the classroom activity on modeling. Some of them come from private or public data banks (such as the changes in height and weight of the pupils, starting, when possible, in infancy or the evolution of AIDS epidemics in Switzerland); some other data are drawn from direct science laboratory experimentation (such as the temperature decrease of previously heated liquids or the growth of microbes on Petri dishes).

Once the data are obtained, they are presented as numerical tables, put into graphical form, analysed from the point of view of qualitative description, and used where possible as the starting point for the construction of a local model. Once a plausible local model has been proposed, the programming of its integration (in LOGO), the running of the program and the exploration of the best fit (as a function of the parameters) are discussed and realized by the whole class.

Note that while the construction of local models is within reach of the pupils for some of the processes studied here, it is definitely beyond the pupils' mathematical capacities in other cases. This is done on purpose to avoid giving to the pupils the false impression that every process can easily be explained by simply following its evolution in time. Questions such as the spread of the AIDS epidemic, already difficult to model for scientific reasons, are made even more complex because of sociological reasons such as the changes which have occur during the last few

years in test policy and in peoples attitudes towards testing. Such apparent unsuitability lead however to highly interesting discussions in the classroom about the problem at hand but also, at a more general level, about the connections relating science, and mathematics in particular, to sociological issues, such as the evolution of peoples attitudes and values.

5. First results and conclusions

Cognitive difficulties were found on all issues discussed in point 3 as can be seen in the last column of table 1. First of all, the need for a model was often obscure; request by the teachers that one should try and go beyond factual data seemed often, and for a long time, unreasonable to the pupils. Why bother, for instance, to "find a model for the temperature decrease of a body, when we already have the whole diagram in front of us?"

Time slicing (the freezing of evolution into static states and jumps between them) often looked arbitrary; change seemed to many to be more primitive and immediate than transitions.

Going from the set of factual data (first descriptive level) to a local model (first explicative level) and then integrating the local model into a global one (second descriptive level) was often hard to motivate. Having found a curve which fits the data was for the pupils often and firmly synonymous with having explained the process. Understanding out of a graphic which factors are constant and which are relative is a difficult activity (Chevallard, 1989), as already noticed in other contexts (Hillel, Gurtner, & Kieran, 1989). In particular, the actual effect of time, whether an active or just a passive factor in the process remained unclear in most cases and will deserve closer observation. On a more technical level, a lack of understanding of what the notion of functional dependence (although a rather intuitive idea, Gurtner 1989; Janvier, 1987) implies and allows when applied to the construction of

global models, constituted a major obstacle towards effective modeling (Karplus, 1979). For instance, even when the temperature of the environment was recognized as the limit value of the temperature decrease of a heated liquid, the need to incorporate it into the local as well as into the global formula was far from being evident.

The separation of algorithm from variables and parameters created often difficulties but seemed however acceptable to the pupils, at least in the abstract domain of the school problems to which they had been exposed.

As a very provisional and tentative conclusion, we would like to insist on the fact that research on the most fundamental and deep grounds for modeling (motivation, formalisation, graphical representations, integration, and so on) is essential for an efficient use of modeling activities in school practice. The imposition of problem solving and modeling activities to pupils whose cognitive obstructions to modeling are still active will only result in yet another abstract, and arbitrary, school drill, with no influence whatsoever on the psycho-cognitive development of the pupils.

References

Blum, W., Niss, M., & Huntley, I. (Eds.). (1989). Modeling, applications and applied problem solving. Teaching mathematics in a real context. Chichester: Ellis.

Gurtner, J. (1989). Undersanding and discussing linear functions in situations. A developmental study. *Actes de la treizième conférence internationale Psychology of Mathematics Education* (vol. 2, pp. 31-38). Paris.

Chevallard, Y. (1989). Le passage de l'arithmétique à l'algèbre dans l'enseignement des mathématiques au collège. *"Petit x", 19,* 43-72.

Hillel, J., Gurtner, J., & Kieran, C. (1989). Structuring and destructuring a solution: An example of problem solving work with the computer. *Proceedings of the twelth international conference for the Psychology of Mathematics Education* (Vol. 2, pp. 402-409). Montreal.

anvier, C. (1987). Representation and Understanding: The notion of Function as an Example. In C. Janvier (Ed.). *Problems of representation in the teaching and learning of mathematics* (pp. 67-71). Hillsdale, NJ: LEA.

Karplus, R. (1979). Continuous functions: students' viewpoints. *European Journal of Science Education, 1*, 397-415.

Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research, 60* (1), 1-64.

Vitale, B. (1988). The unbearable lightness of models, in N. Fleury, S. Joffily, J.A. Martins Simoes and A. Troper (Eds.). *Leite Lopes Festschrift; a pioneer physicist in the third world.* Singapore: World Scientific, pp.47-63.

Vitale, B. (1988a). Psycho-cognitive aspects of dynamical model building in LOGO; A simple population evolution and predator-prey model. *Journal of Educational Computing Research, 4*, 229-253.

Vitale, B. (1989). The exploration of the space of informatics and the realm of open mathematics, in W. Blum, M. Niss, & I. Huntley, (Eds.). *Modeling, applications and applied problem solving. Teaching mathematics in a real context.* Chichester: Ellis, pp.109-115.

A COMPARATIVE ANALYSIS OF TWO WAYS OF ASSESSING THE VAN HIELE LEVELS OF THINKING

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Abstract

This study provides the results of a comparison between two different ways of determining the Van Hiele Levels of reasoning. On one hand, we have compared a clinical interview versus a paper and pencil test. On the other hand, we have compared the assignation of students to a single level of reasoning versus their assignation to a degree of acquisition of each Van Hiele level. Subjects from both test environments were assigned to the Van Hiele levels by using the two assessment procedures. Comparisons of the two assessment procedures yield close agreement on Van Hiele levels acquisition of many of the subjects, particularly the American students.

Introduction and Rationale

In the late Seventies, a growing interest for the Van Hiele Model of Reasoning started in the Western Countries. Since then, there has been a continuous research activity analyzing and applying the Van Hiele Levels; the assessment of the students' level of thinking has played a relevant role in this research. Several types of tests have been used for this assessment:

a) Paper and pencil tests with multi-choice questions (Usiskin, 1982).

b) Paper and pencil tests with open-ended questions (De Villiers, 1987; Jaime, Gutiérrez, 1990).

c) Clinical interviews with open-ended questions (Burger, Shaughnessy, 1986; Mayberry, 1981).

d) Learning sequences (Fuys, Geddes, Tischler, 1988).

Each one of these kinds of tests has been used in several works, and has proved to have both advantages and inconveniences, although the usefulness of the first one has been questioned because it offers serious doubts about its reliability to reflect the student's thinking (Crowley, 1989; Wilson, 1990).

On the other hand, independent of the kind of test used, several ways of assignment of the students to the Van Hiele Levels have been used by researchers:

a) Each <u>question</u> in the test is assigned to a specific Van Hiele level, and the answers are marked as good or bad. The student is assigned to an overall level on the basis to the number of good/bad answers for each level (Mayberry, 1981; Usiskin, 1982).

b) Each student's <u>answer</u> is assigned to the specific Van Hiele level it reflects. The student is assigned to an overall level on the basis of the number of answers in each level (Burger, Shaughnessy, 1986; De Villiers, 1987; Fuys, Geddes, Tischler, 1988).

c) The Van Hiele levels are considered in a continuous way, and each student's <u>answer</u> is assigned to a point in that continuum. The degrees of the student's acquisition of each level are determined on the basis of that assignments (Gutiérrez, Fortuny, Jaime, (w.d.); Jaime, Gutiérrez, 1990).

A current problem in the research related to the Van Hiele Levels is the absence of standardized tests and criteria for determining the students' level of thinking, valid and reliable enough as to be used as reference when designing a new test or way of assessment. The aim of the research we are reporting is to compare and analyze two kinds of tests and two ways of determining the Van Hiele levels of thinking, by means of a crossed application: We have selected subsamples of the students who participated in the researches reported in Burger, Shaughnessy (1986) and Jaime, Gutiérrez (1990), and we have made a twin assignment of each student in those subsamples to the Van Hiele levels by using our two assessment criteria independently. In this report we present the results of the study.

There are several complete descriptions of the Van Hiele Levels in the literature; in particular, in Burger, Shaughnessy (1986) and Jaime, Gutiérrez (1990) appear the descriptors we have used in this research. Here we just make short statements of the characteristics of the levels to help the reader:

<u>Level 1 (Recognition)</u>: Students judge a geometrical object by its appearance and consider it as a whole.

<u>Level 2 (Analysis)</u>: Students identify the components of geometrical figures, and they describe them by means of their properties. Deductions are based on observation and measurement.

<u>Level 3 (Informal deduction)</u>: Students are able to logically classify families of figures. Definitions are meaningful for students, and they can give informal arguments for their deductions.

<u>Level 4 (Formal deduction)</u>: Students understand the role of the elements of an axiomatic system, and they can perform formal proofs.

Method

<u>A) The Tests.</u> The basis for this work were two tests built and administered as part of previous research projects carried out by the members of the team.

• One was the test used in the Project "Assessing Children's Intellectual Growth in Geometry" (Burger, Shaughnessy, 1986 and 1990). It had 8 open-ended tasks dealing with geometric shapes, and it was administered by means of clinical interviews to a range of North-American students from grades K to 12 and university mathematics majors. We will refer to them as the American test and American students. In short, the items were: 1) Drawing of several different triangles. 2) Identifying and defining several sorts of triangles. 3) Sorting triangles according to attributes chosen by the student and by the interviewer. 4, 5, and 6) Similar to the previous ones, but for quadrilaterals. 7) "What's my shape?" that is, identifying a shape from a list of clues. 8) Equivalence of definitions of parallelogram, and knowledge about axioms, theorems and postulates.

• The other test was the one used in the Project "Design of a Curricular Proposal for the Learning of Secondary Geometry According to the Van Hiele Levels" (Jaime, Gutiérrez, 1990). This test had 6 open-ended items (some of them divided into two parts), and it was administered in a written form to several classes of first grade students in Spanish Secondary Schools (students aged from 14 to 16). We will refer to them as the Spanish test and Spanish students. In short, the items were: 1) Identifying regular, irregular, concave, and convex polygons. 2) Identifying several sorts of quadrilaterals, according with the student's definitions. 3) Identifying squares and rectangles according with given definitions. 4) Identifying appearances of "anla" (a non-standard polygon) in a set of polygons by using its definition. 5) Classifying "anlas" and regular polygons or quadrilaterals. 6) Building a definition for obtuse triangle from a given relationship of properties.

The main difference among the two tests was the way of administration. There is no doubt that a clinical interview is the best way for assessing the

student's thinking level, since the interaction between student and interviewer may give an in depth knowledge and the interviewer may ask for more explanations when necessary. But, on the other hand, clinical interviews are very time-consuming and are not appropriate for big samples; in this case, researchers are forced to use a written test. The weakness of a written test is that usually students tend to write short explanations, and then it is difficult to know their real level of thinking; the result may be the assignment to students of a Van Hiele level lower than the real one. When designing the Spanish test, we were aware of this problem, and we stated the items asking the students to give complete explanations and stating concrete questions about the reasons for their choices.

B) The Marking Schemes. The two ways of assigning students to the Van Hiele levels used in our researches represented two different interpretations of the process of acquisition of a level: The American Project assumed the hypothesis of the discreteness of the Van Hiele levels, while the Spanish Project assumed the hypothesis of the continuity of the levels.

- The American researchers made assignations (each researcher independently) of each student's answer to the predominant level of thinking exhibited saying that "it was in level n", although in some cases they realized that the answer had clear indications of two consecutive levels, and then they assigned it to a rating like 1-2, indicating that the answer showed a transition from level 1 to level 2. Finally, each researcher made an overall assignation of students to the Van Hiele levels, based on the ratio between the levels of the different student's answers, which was either a level or a transition between two consecutive levels (Burger, Shaugnnessy, 1990).

- The Spanish researchers tried to reflect in their assignations how strongly each student's answer was rooted in a given level (they made independent assignations and afterward they put them together, looking for a consensus in the discordant assignations). Every answer was assigned both to the Van Hiele level it better reflected, and to a certain "type of answer", depending on how clearly the level was reflected and on its mathematical accuracy (from the point of view of the reflected Van Hiele level). The types of answers were:

Type 0. No reply or answers which cannot be codified.
Type 1. Answers indicating that the learner has not attained a given Van Hiele level but which give no information about any lower level.

Types 2/3. Wrong/correct answers which contain very few explanations but giving some indication of a given level of thinking.
Type 4. Answers with clear and sufficient justifications which clearly reflect characteristic features of two consecutive levels of thinking.
Type 5. Wrong answers which clearly reflect a given level of thinking.
Types 6/7. Incomplete/complete, correct, and sufficiently justified answers which clearly reflect a given level of thinking.

After marking the answers to a test, the types of answer were quantified and the student's "degree of acquisition" of a given Van Hiele level was determined by calculating the arithmetic average of the values of the student's answers to those items that could have been answered at that level. The overall assignation for a student was a vector with four values, from 0 to 100, reflecting the student's degree of acquisition of the Van Hiele levels 1 to 4. For a detailed description, see Gutiérrez, Fortuny, Jaime (w.d.).

C) The Sample. The American researchers selected 6 students from the sample in the American project, aiming to represent the different ages and thinking abilities present in the whole American sample. In the same way, the Spanish researchers selected 6 students from the sample in the Spanish project, aiming to represent the different thinking abilities present in the whole Spanish sample.

D) The Process. Our purpose was to obtain a twin assignation of both American and Spanish students to the Van Hiele levels, in order to compare the results and to have conclusions referring to the kind of test (oral/written) and the marking criteria (discrete/continuous Van Hiele levels).

Then, the Spanish researchers marked, according to their own marking schemes explained above, the students in the American subsample. The working material were the audiotapes and students' drawings from the clinical interviews carried out by the American researchers. Similarly, the American researchers marked, according to their own marking schemes explained above, the students in the Spanish subsample. The working material were the texts from the written tests administered by the Spanish researchers.

Results

Tables 1 and 2 present the twin assignation of students to the Van Hiele levels of thinking. The first rows contains vectors with the student's numeric

degrees of acquisition of the Van Hiele levels 1, 2, 3, and 4. The second and third rows contain the independent assignations of students to the levels made by the American member of the team.

Table 1. The results for the Spanish students (written tests).

	Susana	Mayte	Juan
Spanish	(100, 39, 13, 0)	(50, 25, 3, 0)	(90, 27, 3, 0)
American (S)	1	1	1-2
American (B)	1	1-2	1*

	Maria	Salud	Yolanda
Spanish	(38, 2, 0, 0)	(100, 58, 32, 0)	(100, 17, 0, 0)
American (S)	1-2	2	1
American (B)	1-2	2*	1

* With indication of the upper level.

Table 2. The results for the American students (oral tests).

	Amy	Tyrone	Don
Spanish	(100, 81, 0, 0)	(95, 27, 0, 0)	(100, 98, 64, 33)
American (S)	2	1	3
American (B)	2	1	3

	Karen	David	Tom
Spanish	(100, 84, 26, 0)	(98, 55, 0, 0)	(100, 100, 74, 52)
American (S)	2	1	4
American (B)	2	1	4

The first thing that catches the attention is the difference between the two kinds of results. While the assignation to a single level (or a range of two levels) just tells us which was the dominant level (or the two dominant levels), the degrees of acquisition tell us how confidently and satisfactorily the student used each of the four levels in the tasks.

It is very difficult a perfect agreement between such different ways of assignation, but an analysis of Tables 1 and 2 looking for similarities and discrepancies shows that there is a rather good agreement between the three assessments. The numeric degrees of acquisition of the levels may be transformed in qualitative ones (as seen in Table 3): From 0 to 15, there is no

acquisition of the level; from 15 to 40, there is a low acquisition; from 40 to 60, an intermediate acquisition; from 60 to 85, a high acquisition; and from 85 to 100, there is a complete acquisition of the level. For example, the vector (100, 58, 32, 0) means complete acquisition of level 1, intermediate acquisition of level 2, low acquisition of level 3, and no acquisition of level 4. In this way, we can easierly compare the results from the different assignations.

Table 3. Qualitative results for the Spanish assignations.

Susana	Mayte	Juan	Maria	Salud	Yolanda
(C, L, N, N)	(I, L, N, N)	(C, L, N, N)	(L, N, N, N)	(C, I, L, N)	(C, L, N, N)

Amy	Tyrone	Don	Karen	David	Tom
(C, H, N, N)	(C, L, N, N)	(C, C, H, L)	(C, H, L, N)	(C, I, N, N)	(C, C, H, I)

C = Complete; H = High; I = Intermediate; L = Low; N = No acquisition

The better agreement between the three assignations was in the American Students; only the cases of David and, mainly, Tom presented a slight disagreement between the American and the Spanish assignations. Tom was one of the university majors, and may be the American researchers were so expectant of his high Van Hiele level that his answers were unconsciously over-valued. This may be a danger of any method of assessment based on subjective estimations without clear objective directions. Similarly, in a group of good students, we may have an unconscious tendency to under-value a poor student.

The bigger disagreements appeared in the Spanish subsample (Mayte and Maria); it has relation with the fact that these were the poorest students, since a poor student usually gives few answers and short explanations, making more difficult the task of analyzing the answers and giving an accurate assessment; in fact, Maria answered only 4 out of the 9 questions.

There is another reason for a better agreement in the American Students: in an interview there is the possibility of asking the student for an explanation of a confusing answer, so the interviewer may modify the script if necessary. When marking the written tests, there was a general claim between the researchers that we would like to have the opportunity of interviewing them for asking for some explanations.

Conclusions

In this paper we have stated some point that should receive further

attention. There is still a lot of work to do in relation to the Van Hiele Levels, mainly in curricular development, and, previously, it is necessary a detailed work for improving the current ways of evaluating the levels of thinking. With respect to the tests, there is not a clear direction; we think that the most promising possibility is a kind of test based on paper and pencil open-ended questions followed by short interviews devoted to ask students for explanations on the dark parts of their written answers. On the other hand, it is quite clear that the traditional assignation of students to a single level is a simplistic view which lost part of the richness of the student's answers, so research should be done aiming to develop new methods of evaluation based on the observation of the ability of students in using the four Van Hiele levels, as a way for obtaining a more complete picture of the student's thinking.

References

Burger, W.F.; Shaughnessy, J. (1986): Characterizing the van Hiele levels of development in geometry, *Journal for Research in Mathematics Education* 17.1, pp. 31-48.

Burger, W.F.; Shaugnnessy, J.M. (1990): *Assessing children's intellectual growth in geometry* (final report). (Oregon State University: Corvallis, USA).

Crowley, M.L. (1989): *The design and evaluation of an instrument for assessing mastery Van Hiele levels of thinking about quadrilaterals* (Ph.D. dissertation). (University of Maryland: Maryland).

De Villiers, M.D. (1987): *Research evidence on hierarchical thinking, teaching strategies and the van Hiele theory: Some critical comments.* (Research Unit for Math. Educ.; Univ. of Stellenbosch: Stellenbosch, R. South Africa).

Fuys, D.; Geddes, D.; Tischler, R. (1988): *The van Hiele model of thinking in geometry among adolescents,* "Journal for Research in Mathematics Education Monograph" n° 3. (N.C.T.M.: Reston, USA).

Gutiérrez, A.; Fortuny, J.M.; Jaime, A. (w.d.): An alternative paradigm to evaluate the acquisition of the van Hiele levels, *Journal for Research in Mathematics Education,* to appear.

Jaime, A.; Gutiérrez, A. (1990): Study of the degree of acquisition of the van Hiele levels by secondary school students, in *Booker; Cobb; Mendicuti (1990): Proceedings of the 14th International Conference of the P.M.E.* (Authors: México) vol. 2, pp. 251-258.

Mayberry, J. (1981): *An investigation of the van Hiele levels of geometric thought in undergraduate preservice teachers.* (University Microfilms Internat.: USA).

Usiskin, Z. (1982): *Van Hiele levels and achievement in secondary school geometry.* (ERIC: Columbus, USA).

Wilson, M. (1990): Measuring a van Hiele geometry sequence: A reanalysis, *Journal for Research in Mathematics Education* 21.3, pp. 230-237.

A PROCEDURAL ANALOGY THEORY: THE ROLE OF CONCRETE EMBODIMENTS IN TEACHING MATHEMATICS

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Abstract

In mathematics education there seems to be a significant lack of success in explaining the relationship between the use of concrete embodiments in teaching mathematics and learning mathematical concepts and procedures. The Procedural Analogy Theory reported here draws on research in mathematics education and cognitive science to outline a teaching approach involving ***declarative encoding, proceduralization, procedural analogy****, and* ***automatization****. This theory predicts that the pedagogical usefulness of an embodiment is a function of the degree of similarity of the procedure for the embodiment and the procedure for the initial symbolic representation: and that this degree of isomorphism can be quantified.*

Introduction

For quite some years now, and especially since the 1960s, mathematics educators have emphasised the need to teach and learn mathematics through the use of concrete materials. There's no doubt in many teachers' and teacher educators' minds that the value in this kind of approach is more-or-less self evident. Certainly the numerous articles appearing in a range of journals over the years, especially the kind describing "how to...." or "how I....", and the anecdotal accounts one hears both at conferences and more casually from classroom practitioners, suggest that the principle is both well accepted and widely practised.

But if it is well accepted where in the research literature is the body of empirical data supporting the use of embodiments? What research indicates that concrete representations of mathematical concepts are actually valuable, that they do represent the concepts we intend, and that they do have a real and measurable impact on students' learning? And where is the data indicating exactly how concrete representations allow the learner to better arrange his or her cognitive structure so that learning is more effective?

Research Literature

There are numerous publications suggesting the use of *multi-based arithmetic blocks* to clarify the principles of place value (Booker, et al, 1982; Dienes, 1973; Trewin, 1970; Williams, 1972), and the usefulness of diagrams and pictures for teaching fractions is virtually universal (Hall, 1981b; Post, 1988; Trewin, 1970).

However, in surveying the literature in mathematics education there seems to be a significant lack of success in explaining the relationship between the use of concrete embodiments such as these in teaching mathematics and in the learning of mathematical concepts and procedures. To some extent this may reflect the nature of reporting procedures where, for the sake of brevity, full details of the research are not able to be included. At the same time the research literature has many examples of comparisons between an experimental teaching method and a traditional approach. The characteristics of the experimental and traditional teaching methods are generally not specified in sufficient detail to allow replication of the research, or application to classrooms. (Refer to Behr, et al, 1984; Bell, et al, 1983; Fennema, 1972; Heibert, 1986; Lesh and Landau, 1983; Raphael and Wahlstrom, 1989; Sowell, 1989; and others.)

Empirical support for the belief in the pedagogical usefulness of embodiments is lacking. Few studies address the question whether embodiments function as intended, and those that do are not encouraging. For example, Resnick and Omansson (1987) carried out a teaching experiment in which students were taught subtraction with regrouping by mapping back and forth between the MAB blocks and the symbolic algorithm. And Ohlsson, Bee, and Zeller (1990) evaluated two computer-based microworlds for fractions in which the computer reacted to changes in graphical objects by displaying isomorphic changes in the corresponding arithmetic symbols, and vice versa. Neither study found that the students learned to solve the relevant class of problems. It seems that the belief in the pedagogical effectiveness of embodiments is based on a combination of ideological preference, common sense and classroom experience.

The teaching sequence described in Figure 1 is unlikely to be unique to this author, but has been described in a number of forms (Hall, 1980, 1981a, 1981b, 1982), and has been the basis of much of the author's work in teaching mathematics education.

Figure 1: *A Pragmatic Teaching Approach*

Unstructured play
Using materials in any manner to achieve correct answer
Using materials in a prescribed manner to achieve correct answer
Using materials, writing corresponding expanded algorithm
Using materials, writing corresponding contracted algorithm
(Algorithm only, check with materials)
(Algorithm only, place value language)
Algorithm only, face value language

The sequence in Figure 1 describes one approach when using embodiments to teach mathematical skills. Learners need first to have time to familiarise themselves with the materials, through unstructured play. There is then a period where materials are used as an aid to the solution to a problem, this solution leads to an algorithm that corresponds to the structure and movement of the materials. Finally, this algorithm is reduced to its simplest form. As indicated in Figure 1, each of these steps may be supplemented by additional steps. What is the rationale behind such an approach, and how can its effectiveness be explained?

A Procedural Analogy Theory

The Procedural Analogy Theory described here was developed by Ohlsson and Hall (1990), and is based on four different concepts from learning theory: *declarative encoding, proceduralisation, procedural analogy* and *automatisation*. In this theory the function of the embodiment is to enable the declarative encoding of the embodiment procedure; this encoding is proceduralised to become executable. The embodiment procedure is moved to the realm of symbols through the development of a corresponding algorithm, by analogy. The analogy requires a high level of isomorphism between the embodiment and the symbolic representation. That is, the theory predicts that the pedagogical usefulness of an embodiment is a function of

the degree of similarity of the procedure for the embodiment and the procedure for the initial symbolic representation. An intermediate expanded notation may be needed to improve the isomorphism. Once a symbolic representation has been established, it is simplified until it becomes the final procedure.

Proceduralisation requires the teacher to explain the embodiment procedure, the student constructs a declarative encoding of that procedure, and then practices the embodiment procedure in order to convert the declarative encoding into an executable procedure. This complexity of proceduralisation explains why people are unlikely to be able to execute a procedure that has been described to them, where there has been no opportunity for practice. Correct proceduralisation requires correct declarative encoding, and this depends on the effectiveness of the teacher's description and explanation. This leads Ohlsson and Hall (1990) to propose that embodiments are beneficial *because embodiment procedures are easier to describe than symbolic procedures.* That is, it is easier to describe concrete objects and actions on these objects than it is to describe conceptual operations on abstract objects such as numbers.

Central to this Procedural Analogy Theory is the notion of quantifying comparisons between *traces* of pairs of procedures. The trace of a procedure is the sequence of actions that the learner generates when executing the procedure. For example Figure 2 shows one trace for addition of whole numbers with expanded notation.

Figure 2: *Trace for Addition with Expanded Notation*

0. Add (50000 + 30 + 2) and (100 + 40 + 9).

1. Process column 1.
    1.1 Recall (2 + 9) => 11.
    1.2 Regroup 11.
        1.2.1 Decompose 11.
            1.2.1.1 Recall 11 = 10 + 1.
            1.2.1.2 Write 1 in column 1.
        1.2.2 Carry 10.
            1.2.2.1 Write 10 in column 2.

2. Process column 2.
    2.1 Recall (30 + 40) => 70.
    2.2 Recall (70 + 10) => 80.
    2.3 Write 80 in column 2.

3. Process column 3.
    3.1 Recall (0 + 100) => 100.
    3.2 Write 100 in column 3.

4. Process column 4.
    4.1 Write 0 in column 4.

5. Process column 5.
    5.1 Write 50000 in column 5.

Total number of goals is 20.

There are clearly a number of alternatives in writing traces for any given procedure. The selection of the appropriate trace is a reflection of those procedures the teacher wants students to acquire, and requires consistency in terms of identifying goals and subgoals, and consistency in identifying corresponding and non-corresponding subgoals.

The degree of isomorphism between two traces may be quantified by mapping the traces of the two procedures onto each other, beginning with the top goal. If two goals fulfill similar functions in the two procedures, then these goals are corresponding. When a goal cannot be mapped onto the goal in the other procedure, then that goal is not corresponding.

The degree of isomorphism between two traces may be expressed by the index

$$I_{(1,2)} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2},$$

where N_1 and N_2 are the total number of entries (goals and actions) in the two traces, and D_1 and D_2 are the number of entries in each trace that does not have a match in the other trace. If there is to be any degree of isomorphism the two top goals must correspond to each other, so 2 is subtracted from the total number of entries. The index varies between 0 and 1. By taking the

number of corresponding entries in proportion to the entire set of entries, the index is independent of the absolute size of the traces.

The Procedural Analogy Theory predicts that the degree of isomorphism between the embodiment procedure and the symbolic procedure is a major determinant of pedagogical effectiveness. To measure this variable for a particular use of an embodiment involves six steps:

Design a teaching sequence for the embodiment procedure,
Generate a trace by running the embodiment program on a problem,
Write a program for the symbolic procedure,
Generate a trace,
Map the traces onto each other, and
Calculate the isomorphism index.

This sequence allows the teacher or researcher to experiment with various teaching approaches using embodiments so as to maximise the isomorphism index between the embodiment and symbolic procedures.

Applying Theory to Completed Research

In Resnick and Omansson (1987) a teaching approach for subtraction used a vertical algorithm with MAB representation of the minuend only. Data available to Ohlsson and Hall (1990) from this research allowed traces to be developed, and a calculation of the isomorphism index for Resnick and Omason's teaching sequence. It was found

$$I = \frac{(33 + 26 - 2) - 31}{(33 + 26 - 2)} = \frac{57 - 31}{57} = \frac{26}{57} = 0.46$$

This value of the ismorphism index is low, indicating that the similarity between the MAB procedure and the symbolic procedure in this case is not very high. This result is one possible explanation for the negative result in the Resnick and Omansson (1987) study. The mapping instruction did not work, because the target for the mapping was too dissimilar to the source.

Ohlsson, Bee and Zeller (1990) obtained lower than expected results from elementary school children in activities associated with size, equivalence and addition of fractions. In a further analysis of these results, through the development of traces and calculation of the isomorphism index, the index was found to be 0.27. This index is low, indicating that the two procedures have different structures. A revision of the manner in which the embodiment was used, led to a new trace being developed, so that the isomorphism index was increased to 0.92.

Discussion

A Procedural Analogy Theory has been described which appears to have the potential to explain how embodiments impact on learners, and the potential to provide a powerful pedagogical tool in planning mathematics teaching through the use of embodiments.

The theory argues that the pedagogical value of an embodiment depends on the ease with which learners can follow the embodiment procedure as described by the teacher, the ease with which this description can be proceduralised, the degree of isomorphism between the embodiment and the symbolic procedures, and the extent to which this initial symbolic procedure can be altered to the final target procedure through simplification.

The work reported here has concentrated on the isomorphism aspect of the theory. There has been no attempt to establish measures of effectiveness of teacher description, learning proceduralisation or simplification of algorithms. The theory is relatively simple to apply, it explains the importance of teacher talk, and it has provided an explanation for results obtained in two major pieces of research. Clearly the theory now needs further investigation, but if Ohlsson and Hall are correct, teaching approaches where the isomorphism index is low suggest the embodiment will be unsuccessful in helping learners.

References

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal of Research in Mathematical Education*, 15(5), 323-341.

Bell, A. W., Costello, J., & Kuchemann, D. E. (1983). *A Review of Research in Mathematical Education. Part A. Research on Learning and Teaching.* Berks, UK: NFER-Nelson.

Booker, G., Irons, C., & Jones, G. (1982). *Numeration.* Canberra: Schools Commission.

Dienes, Z. P. (1973). *The Six Stages in the Process of Learning Mathematics* (P. L. Seaborne, Trans.). Paris: OCDL. (Original work published 1970.)

Fennema, E. H. (1972). Models and mathematics. *The Arithmetic Teacher.* December

Hall, N. (1980). *Strategies for Teaching Remedial Mathematics.* Albury, Australia: RCAE. (Available as ERIC fiche ED 226 992)

Hall, N. (1981a) Teaching Decimals - Diagnosis and Remediation. In A. Rogerson (Ed.) *Mathematics: Myths and Realities.* Melbourne: MAV.

Hall, N. (1981b) Teaching Fractions - Diagnosis and Remediation. In A. Rogerson (Ed.) *Mathematics: Myths and Realities.* Melbourne: MAV.

Hall, N. (1982). Towards a theory of teaching mathematics. In J. Veness (Ed.), *Mathematics: A Universal Language.* Sydney: AAMT.

Hierbert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Erlbaum.

Lesh, R., & Landau, M. (Eds.) (1983). *Acquisition of Mathematics Concepts and Processes.* NY: Academic Press.

Ohlsson, S., Bee, N., & Zeller, (1990) Tech. Rep. No. KUL-90-01. Pittsburgh, PA: University of Pittsburgh, Learning Research & Development Center.

Ohlsson, S., & Hall, N. (1990) *The Cognitive Function of Embodiments in Mathematics Instruction.* (Tech. Rep. No. KUL-90-02). Pittsburgh, PA: University of Pittsburgh, Learning Research & Development Center.

Post, T. R. (1988). *Teaching Mathematics in Grades K-8.* Boston, MA: Allyn and Bacon.

Raphael, D., & Wahlstrom, M. (1989). The influence of instructional aids on mathematics achievement. *Journal for Research in Mathematics Education*, 20(2) 173-190.

Resnick, L. B., & Omanson, S. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in Instructional Psychology. Vol. 3.* Hillsdale, NJ: Erlbaum.

Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics*, 20(5), 498-505.

Trewin, A. H. (1970). *Project mathematics, Vols. 1-4.* Sydney: Macmillan.

Williams, J. D. (1972). *Teaching Technique in Primary School Mathematics.* London: NFER.

VARIABLES AFFECTING PROPORTIONALITY: UNDERSTANDING OF PHYSICAL PRINCIPLES, FORMATION OF QUANTITATIVE RELATIONS, AND MULTIPLICATIVE INVARIANCE[1]

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The analysis presented in this paper is part of several extensive analyses we have been conducting in the last few years dealing with multiplicative concepts in the global context of the multiplicative conceptual field. Here, we propose three global variables believed to affect multiplicative and proportional reasoning. The first variable deals with the physical principles underlying the problem situation. It intends to explain the conceptual basis for the variance in difficulty of proportion problems due to differences in context. The second variable ties theories on multiplication and division concepts (Fischbein, Deri, Nello, & Marino, 1985; Vergnaud, 1983, 1988; Nesher, 1988) with research on the concept of proportionality, by analyzing the type of multiplicative relations which exist between the problem quantities (e.g., whether this relation is of the multiplier-multiplicand type or product of measure type). The third variable deals with mathematical invariance. It is suggested that interpreting multiplicative and proportion situations in terms of variability and invariability and change and compensation can be powerful in problem solutions and should be a target for mathematics curricula.

In their literature review on proportional reasoning, Tourniaire and Pulos (1985) described the type of tasks used in research to investigate the concept of proportionality and the variables found in this research to affect children's performance on proportion problems. They classified these tasks into *physical* (e.g., the balance scale task), *rate* (where ratios of dissimilar objects are compared; e.g., speed problems), *mixture*, and *probability* tasks, and distinguished between *missing-value proportion problems* (e.g., "Find x in 3/x=2/7") and *comparison proportion problems* (e.g., "Which is greater, 3/7 or 4/9?). The variables affecting children's performance they divided into *task-centered* variables--presence of integer ratio, order of the missing value, and numerical complexity--and *context variables*--presence of mixture, referential content (descrete versus continuous), familiarity, and availablity of manipulatives.

Of the many traditional proportion tasks, *three* types of tasks have been widely used: *rate and mixture* tasks (e.g., Noelting, 1980a, 1980b), the *balance scale* task (Inhelder and Piaget, 1958; Siegler

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...J Vago, 1978), and the *fullness task* (Bruner and Kenney, 1966; Siegler and Vago, 1980). In the discussion below we refer to these three types of tasks as representatives of traditional proportion tasks. Recently we have developed a class of proportion tasks which we called "multilevel" tasks to distinguish them from the traditional tasks which we viewe as "single-level" tasks. They were so labeled because it can be shown (see Harel, Behr, Post, & Lesh, in press) that the "multilevel" tasks consist of three subtasks, two are isomorphic to two variations of a traditional task, and the third is an additive task which involves a coordination between two order relations. An example of a "multilevel" task is[1]:

> **2 blocks of kind ALFA weigh the same as 3 blocks of type BETA. If possible find whether 5 blocks of type ALFA weigh the same as, more than, or less than 7 blocks of type BETA?**

In a previous work (Harel and Behr, 1988) we extended the task centered variables as defined by Tourniaire and Pulos (1985) and analyzed their impact on children's problem representations and solution strategies of missing value proportion problems. In this paper we present more global accounts for the reasoning and difficulties children encounter in dealing with multiplicative problems. These accounts will be discussed in the context of the two categories of tasks mentioned earlier: the three representatives from the traditional proportion tasks and the new class of "multilevel" tasks.

VARIABLES AFFECTING PROPORTIONALITY

Tourniaire and Pulos reported that mixture problems have been found to be more difficult that other proportion problems. They gave the following explanation to account for this finding in the case of mixture and rate problems:

> *First, the elements of the ratio in a mixture problem constitute a new object, e.g., red and yellow paint mixed makes orange, or a modified object, e.g., orange juice mixed with water makes a weaker orange juice. By contrast, no new object emerges in rate problems. Second, mixture problems require that the subject understands what happens when the two elements are mixed. Third, in most mixture problems, the quantities are expressed in the same unit, e.g., ounces, whereas in most rate problems the quantities involved are in different units, e.g., ounces and dollars. Dealing with quantities expressed in the same unit may be more confusing.* (pp. 183-184).

These are peculiar explanations; it is not clear how they differentiate between mixture problems and rate problems or how they account for the relative difficulty between these two types of problems. First, the ratio elements in rate problems, as in mixture problems, *do* constitute a new object; the quantity of speed, for example, is a new quantity which emerges from the quantities of time and distance. Second, both rate problems and mixture problems require that the subject understands the physical situation when two

[1] These "multilevel" tasks were generated from another task called the Blocks task which was investigated with seventh graders (Harel et al., in press) and with experts (Harel & Behr, in press).

quantities are combined. This is true whether this is a mixture of water and orange concentrate or a comparison between time and distance. Third, why should problems in which the quantities are expressed in the same unit be more difficult than those in which the quantities are expressed in different units?

In the discussion below we propose three variables which affect proportionality. The first two can account for the relative difficulty of proportion problems; these are the physical principles underlying the problem situation and the type of multiplicative relations which exist between the problem quantities. The third variable deals with advanced multiplicative reasoning in which ratios and products are compared in terms of changes and compensations. This type of reasoning is viewed as a culminating point in multiplicative reasoning and should be a goal for school mathematics. To reach this goal children must experience in early age situations that deal with multiplicative change and compensation and grasp an intuitive understanding of the mathematical principles (see Harel et al., in press; Harel, Behr, Post, & Lesh, 1990) that constitute these situations.

Physical Principles Underlying the Problem Situation

In this section we hypothesize a conceptual basis for the variance in difficulty of proportion problems due to differences in context. A fundamental difference among proportion tasks lies in the principles underlying the physical interactions between or among the problem quantities. We believe that these principles are the basis for taking account of the proportionality constraints in solving the tasks. For example, in the problem above, it is assumed that the weight of each set of blocks is equally distributed among the individual blocks in the set; and if some of the numerical data in this problem were fractions, one must assume that the distribution of the weight within each block is homogeneous. The physical principle of homogeneous distribution of weight, or homogeneous density of matter, is intuitive and spontaneous, in the sense that it is acquired in everyday activities, such as lifting objects. Mixture and rate tasks require an intuitive understanding of other physical principles which are less spontaneous than the homogeneous density principle. In the mixture task the principle involved is about uniform diffusion between liquids; in the rate tasks the principle is about uniform rate, such as, speed or work. The fullness task involves the principle about liquid, which states that the pressure is the same at all points at the same level within a liquid at rest; this principle guarantees the uniform level of the liquid and the absolute separation of the water space from the empty space. The balance scale involves the principle of conservation of angular moment, which states the conditions of equilibrium that must be satisfied if a

balance scale is to remain balanced or fall toward one of the two sides of the fulcrum. These conditions are non intuitive and less spontaneous when dealing with the summation of the products of weight and distance on each side of the fulcrum.

It should be emphasized that we are *not* claiming that in order for children to solve one of these tasks successfully they must *explicitly* know the physical principles underlying the task. Our argument is that an intuitive understanding of the physical principles underlying the situation of the task is necessary for solving the task correctly.

Multiplicative Relations Between Problem Quantities

The semantic relationship between the problem quantities in the "multilevel" problem presented above is conceptually different from those in the mixture task, the balance scale task, or the the fullness task. In that problem the weight of each set of blocks can be thought of as a product of the number of blocks in the set and the weight of each block. Accordingly, the role played in the product by the first quantity (number of blocks) is conceived of as an (integral) *multiplier* and the role of the other (the weight of each blocks) as the *multiplicand.* This multiplier-multiplicand relationship--simple proportion relationship in Vergnaud's (1983, 1988) terms, or mapping-rule relationship in Nesher's (1988) terms--involves two measure spaces, number of building-blocks and weight. The use of this relationship is ordinary in everyday activities and in school word problems, and usually is expressed as repeated addition. It is based on a set-subset relationship, the operation union of sets, and the concepts of cardinality and measure. All of these are acquired informally through everyday activities.

The balance scale task, in contrast, involves three measure spaces: the quantities multiplied are derived from two independent measure spaces--weight and distance--and their product creates the measure space of moment. The semantic relationship between these three quantities--product of measure in Vergnaud's (1983) terms--is formal, in the sense that it is acquired through instruction: moment is a vector quantity (not a scalar quantity) which is defined as the *cross product* of two vectors, weight (the net gravitational force acting on an object hung on one side of the fulcrum) and (directional) distance from a fixed point (the fulcrum) to the point on which the object is hung. Indeed for many balance scale task variations this definition is not necessary because they can be solved based on an intuitive knowledge acquired through inactive experience. These include many non-numeric variations, such as, "If two boys, Tom and John, sit on opposite ends of a seesaw, in an equal distance from the center, and Tom is heavier than John, which side would go down, Tom's side or John's side?" The definition of moment, however, is the foundation for the physical principle that $w_1 \cdot d_1 = w_2 \cdot d_2$ (where w_1 and w_2 represent the weight of two objects each is hung on another side of the fulcrum and d_1 and d_2 their distances from the fulcrum, respectively) if and only if the two sides of the fulcrum are balanced, which children do not acquire spontaneously from everyday experience, and it must be used in solving numeric variations of the balance scale task.

The other two types of tasks, rate and mixture tasks and the fullness task, are of ratio type. The semantic relationships between the problem quantities within a ratio can have different meanings, among which the *partitive division* meaning, the *quotitive division* meaning, and the *functional* meaning. Consider, for example, the ratio 12 cups of water to 4 cups of orange concentrate. One can change 12:4 to the unit-rate 3:1 and think of 12:4 as the number of cups of water per one cup of orange concentrate, or change 12:4 to 1:1/3 and think of the number of cups of orange concentrate per one cup of water, both indicate a partitive division interpretation of the relationship between the ratio's quantities. The ratio 12:4 can also be thought of as a rate, namely, that the amount of water (orange concentrate) is some number of times as much as the amount of orange concentrate (water); this relationship, in contrast to the former one, is a quotitive division interpretation. Finally, the ratio 12:4 can have a functional meaning; namely, one of the ratio quantities, 12 cups of water or 4 cups of orange concentrate, is dependent on the other. In the following section we will elaborate more on this viewpoint.

Multiplicative Invariance: Functional and Relational

Ratio and proportion problems can be interpreted in terms of functional dependencies, variability and invariance. Before we define this idea, we indicate that the term 'invariance' is used in advanced mathematics in the context of whether a set of objects preserves its mathematical structure when it undergoes a certain change. For example, if U is a subspace of a vector-space W and T is a linear operator on W, it is important--for reasons related to transformation representations--to determine whether T(U) is still a subspace of W, or, using different words, whether U as a vector-space is invariant under T.

Questions of variability and invariability are important not only in advanced mathematics but also in the context of the mathematics taught in school, and in particular in the context of the multiplicative conceptual field. Missing-value and comparison proportion problems and missing-value and comparison product problem (e.g.,"Find x in $3 \cdot x = 2 \cdot 7$" and "If possible, determine which greater, $a \cdot b$ or $a' \cdot b'$ if $a > a'$ and $b < b'$?") can be analyzed by solvers in terms of changes and compensations. Consider, for example, the missing-value proportion problem, "Find x in $3/7 = 9/x$." One *can* view this problem as asking: if 3

...es a numerical change which results in the number 9, what change must 7 undergo so that the value of 3/7 is left unchanged? What constitutes this thinking is the understaning that the value of 3/7 is not invariant under the change 3 -> 9, and that a certain compensation, 7 -> x, for this change is needed to leave 3/7 unchanged (i.e., 3/7=9/x). We called this *functional invariance* because this change and its corresponding compensation each have only *one* outcome value, which is the "uniqueness to the right" property (i.e., for one input of a function there is exactly one output) that constitutes the idea of function.

Functional invariance reasoning can be applied to comparison proportion problems as well. Consider the problem, "Which is greater, 2/3 or 8/15?" In a manner similar to the above analysis one might view 8/15 as a result of a change made to 2/3. For example, 2 -> 8 and 3 -> 15 can be thought of as instantiation of the changes "times by 4" and "times by 5," respectively. Since the change 3 -> 15 does not compensate exactly for the change 2 -> 8, the value of 2/3 is *not* viewed as equal to the value of 8/15. One can go a step further and think that since the latter change (3 -> 15) EXCEEDS the compensational change which would leave 2/3 unchanged (i.e., 3 -> 12), the result, 8/15, must be SMALLER than 2/3.

Relational invariance is when changes are interpreted in terms of the directionality of the order relation between corresponding quantities. Consider, for example, the problem: "Which is greater 6/7 or 3/8?" Similar to the above analysis, we can view 3/8 as a result of a change applied to 6/7, where 6 is changed to 3 (6 -> 3) and seven to 8 (7 -> 8). But, unlike the functional invariance in which 6 -> 3 and 7 -> 8 are specified as instantiations of *functional changes*, in relational invariance they are interpreted in terms of "increase" and "decrease" relations.

Functional invariance reasoning is conceptually very powerful because it is an abstraction of other interpretations such as those discussed earlier and can be applied not only in ratio problems but in a wide range of multiplicative problems. It emerges from the idea of variability and invariability which is central to mathematical reasoning and thus should be an important target for mathematics curricula.

In previous reports (Harel et al., in press; Harel, Behr, Post, & Lesh, 1990) we presented two classes of principles which are believed to constitute *relational invariance reasoning*: that is, they consitute the intuitive basis for deciding whether the order relation between two ratios or two products is determinable (the order determinability principles class), and if yes whether it is the =, >, or < relation (the order determination principles class). Our current experimental work examines teachers and children's understanding of these principles and their ability to apply them in solving multiplicative and proportion problems.

CONCLUDING REMARK

To a large extent and until recently, research of the multiplicative domain has looked at the development of individual multiplicative concepts or subdomains, without much effort to deal with the interrelations and dependencies within, between, and among the mathematical/cognitive/instructional aspects of these concepts. For example, research on the learning of the rational number concept did not take into account children's conceptions of multiplication and division (e.g., Fischbein intuitive models) or the role of unit in children's development of different rational number interpretations; research on the learning of the decimal system is somehow separated from research on fractions and proportionality. It may well be that in the course of a scientific exploration, this path of research development was inevitable; namely, in the process of exploring models for the conceptual development of multiplicativity, research had to deal with separate multiplicative concepts before it could reflect on what was learned and raise questions about the interrelations between and among these concepts so that the accumulated knowledge can be described in general terms, in terms of a conceptual field. We believe however, that we have reached a point where accumulated research knowledge in this domain is such that a reflection is necessary and possible. It is time to focus on questions concerning the acquisition of the *multiplicative structures* by taking into account what is known from research on the learning of (seemingly) isolated mulitplicative concepts.

In the last few years, we have made an effort in this direction. We have focused our research on theoretical analyses of the multiplicative conceptual field (MCF), based on our own and others' previous work on the acquisition of multiplicative concepts--such as, multiplication, division, fraction, ratio, and proportion--and relationships among them. Our goal has been to better understand the mathematical, cognitive, and instructional aspects of multiplicative concepts in terms of a conceptual field. For example, in Harel and Behr (1989) we theorized models for problem representations and solution strategies of missing value proportion problems that take into account a wide range of multiplicative variables, such as the intuitive rules of multiplication and division and knowledge of the numerical component and the measure component of quantities; in Behr, Harel, Post, and Lesh (in press a, in press b) we introduced an analysis and notational system that focuses on the notion of "unit types" and "composite units" (Steffe, 1989) whereby we established links between research on rational numbers and research on other conceptual areas such as whole number arithmetic (e.g., Steffe, 1989), intuitive knowledge of multiplication and division (e.g., Fischbein, Deri, Nello, & Marino, 1985), exponentiation (e.g., Confrey,

1988), and beginning algebra (e.g., Thompson, 1989); and in Harel, Behr, Post, & Lesh (in press) we have presented an analysis, partially addressed in this paper, in which multiplicative situations were analyzed with respect to a wide range of task variables and basic principles upon which intuitive knowledge about fraction order and equivalence and proportional and multiplicative reasoning can be based were introduced.

REFERENCES

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3-17

Behr, M., Harel, G., Post, T., & Lesh, R. (in press). Rational numbers: toward a semantic analysis, emphas on the operator construct. In T. Carpenter & L. Fennema (Eds), *Learning, Teaching, and Assessing Rational Number Concepts: Multiple Research Perspectives.*

Behr, M., Harel, G., Post, T., & Lesh, R. (In press). Rational number, ratio, and proportion. In D. Grouws (Ed) *Handbook for Research on Mathematics Teaching and Learning.*

Brunner, J. S., & Kenney, H. (1966). On relational concepts. In J. S. Bruner, R. R. Olver, & P. M. Creenfiel (Eds.), *Studies in cognitive growth.* NY: Wiley.

Harel, G., & Behr, M. (1989). Structure and hierarchy of missing value proportion problems and their representations. *The Journal of Mathematical Behavior*, *8*, 77-119.

Harel, G., & Behr, M. (In press). The blocks task on proportionality: Expert solution models, *Journal of Structural Learning.*

Harel, G., Behr, M., Post, T., & Lesh, R. (in press). The blocks task; comparative analysis of the task with other proportion task, and qualitative reasoning skills of 7-th grade children in solving the task. *Cognition and Instruction*.

Harel, G., Behr, M., Post, T., & Lesh, R. (1990). Understanding the multiplicative conceptual field, The *Proceedings of 14th Annual Conference of the PME.* Mexico: Oaxtapec, Vol. 3, pp. 27-34.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence.* New York: Basic Books.

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades* (pp. 19-40). Reston: National Council of Teachers of Mathematics.

Noelting, G. (1980a). The development of proportional reasoning and the ratio concept: Part I, Differentiation of stages. *Educational Studies in Mathematics, II*, 217-253.

Noelting, G. (1980b). The development of proportional reasoning and the ratio concept: Part II, Problem structure at successive stages: problem solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics, II*, 331-363.

Siegler, R. S. & Vago, S. (1978). The development of a proportionality concept: Judging relative fullness. *Journal of Experimental Child Psychology, 25*, 371-395.

Thompson, P. (1989). Cognitive model of quantity base reasoning in algebra, paper read at the AERA Annual meeting, San Francisco, California.

Tournaire, F. (1986). Proportions in elementary school. *Educational Studies in Mathematics, 17*, 401-412.

Tournaire, F., Pulos, S. (1985). Proportional reasoning: a review of the literature. *Educational Studies in Mathematics, 16*, 181-204.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh &M. Landau (Eds.), Acquisition of Mathematics Concepts and Processes (pp. 127-124). New York: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades* (pp. 141-161). Reston: National Council of Teachers of Mathematics.

THE DEVELOPMENT OF THE CONCEPT OF FUNCTION BY PRESERVICE SECONDARY TEACHERS: FROM ACTION CONCEPTION TO PROCESS CONCEPTION[1]

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As a result of the instructional treatment essentially all students participating in this research progressed towards a process conception of function from starting points varying from very primitive conceptions to action conceptions of function. The main question this paper deals with is how far beyond an action conception and how much into process conception was each student at the end of the instructional treatment? In this connection, there are four important factors: ***1. Restrictions students possess about what a function is.*** *The main restrictions are: the manipulation restriction, the quantity restriction, and the continuity of a graph restriction.* ***2. Severity of the restriction.*** *In regard to the manipulation restrictions, for example, some students feel that before they are willing to refer to a situation as a function, they personally have to know how to manipulate an explicit expression to get the output for a given input. Other students are satisfied with the presence of an expression even though they admit that they don't know how to deal with it.* ***3. Ability to construct a process when none is explicit in the situation. 4. Uniqueness to the right condition.***

Breidenbach, Dubinsky, Hawk, and Nichols (in press) have distinguished four Levels of function concept. A *prefunction* conception does not display very much understanding of function and is not very useful in performing the tasks that are called for in mathematical activities related to functions. An *action* is a repeatable mental or physical manipulation of objects involving, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). A student whose function conception is limited to actions might be able to form the composition of two functions given via algebraic expressions by replacing each occurrence of the variable in one expression by the other expression and then simplifying, but he or she would probably be unable to compose two functions in more general situations, e.g., when they were given by different expressions on different parts of their domains, or if they were not given by expressions at all, but by algorithms. A *process* conception of function involves a dynamic transformation of quantities according to some repeatable means that, given

[1] This paper is a short version of the paper, The Nature of the Process Conception of Function, by Ed Dubinsky and Guershon Harel, which will appear in a book on the concept of function edited by these authors.

same original quantity, will always produce the same transformed quantity. The subject is able to imagine the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. Notions such as 1-1 or onto become more accessible as the subject's process conception strengthens. A function is conceived of as an object if it is possible to perform actions on it, in general actions that transform it. These distinctions are in agreement with other researchers viewpoints and findings (see, for example, Harel and Kaput, in press; Sfard, 1989; Dreyfus & Eisenberg, 1987; Dreyfus & Vinner, 1982)

This paper concerns the development of the concept of function along the action-process path. It is important to indicate that as with all cognitive transitions, the progress along this path is never clear cut and it can be quite difficult to determine with certainty that a function concept of an individual is limited to action or that he or she has a process conception. This may be the case even with individuals who have made significant progress in the transition from an action to a process conception of function. One purpose of the work described here is to contribute towards a clarification of the distinction between action and process conceptions of functions. Although we feel that we have made progress, this problem is far from solved and more research will be necessary.

The individuals whose concepts of function were studied were undergraduate students. In Breidenbach et al (in press), a teaching experiment is reported in which the students made significant progress in moving from prefunction and action conception of function towards a process conception. The present report deals with the question: *How far beyond an action conception and how much into process conception were those students at the end of the instructional treatment?* We address this question, not in terms of a "progress scale", but rather in terms of the following four factors which we consider to be important components of a process conception:

1. Restrictions students possess about what a function is:
   (a) the manipulation restriction (must be able to perform explicit manipulations or not a function),
   (b) the quantity restriction (inputs and outputs must be numbers),
   (c) the continuity restriction (i.e., a graph representing a function must be continuous.)
2. Severity of the restriction (to refer to a situation as a function, you personally have to know how to manipulate an explicit expression to get the output for a given input.)

3. Ability to construct a process when none is explicit in the situation, and the student's autonomy in such a construction.
4. Understanding the distinction between the uniqueness to the right condition and 1-1.

These factors were not determined prior to our analyses of the interviews, but rather they emerged from our interpretations of the observations of students' apparent ways of thinking about functions in a wide range of mathematical situations. In the next section we will discuss some protocols of student responses which will illustrate some of these restrictions.

THE STUDY

Procedure

A group of 22 students participated in a course in Discrete Mathematics, in which the instructional treatment used was based on a constructivist theory of learning and involved computer activities designed to foster development of the students' conceptions of function. Several observations of the students concerning their apparent understanding of the concept of function were taken before, during, and after this instructional treatment.

The pre-observation consisted of a written instrument in which students were asked to describe a given situation (see Table 1) by one or more functions; they were presented with 24 such situations. The post-observations consisted of interviews of 13 of the 22 students about their reactions to 6-8 of the situations. They were asked to give their definition of function and to explain their thinking in giving the original response, if they still felt the same way, or wanted to change their answer.

Table 1

Examples of Situations Presented to the Students to Be Described by One or More Functions

Situation	Characterization	Description
S1	Set of ordered pairs	{[x, 2x+1]: x in the set of all integers}
S2	Finite sequence	[2n+n^3: n in [1..100]]
S4	Equation	$y^4 = x^2$
S5	String of characters	"Purdue Women's Basketball Team Wins NCAA"
S9	Equation	$x^2+3x+2=0$
S10	Sequence with Boolean values	[2n>n^2+3n: n in [1..100]]
S13	Equation	$2xy-\sqrt{x}\log y=2$
S15	Equation	$y^4=x^3$
S17	Set of ordered pairs	{[x, y]: x, y in the set of all rational numbers}
S20	Table	Name: Sue, John, Terri, Mike / Owed: \$17, \$8, \$45, \$51

Results and Discussion

The first question introduced to all subjects was: What is a function, what does it mean to you?

Subj1's response to this question was:

Subj1: Um, a function to me is where you have a set of numbers, your domain, that are numbers that you are going to put into some type of process, and then the process may or may not manipulate the numbers that you are inputting, and then give you some numbers out. And then those numbers that you, I don't know, you just get some numbers out.

It is not reasonable, from such a general answer to such a general question, to conclude anything definitive about Subj1's function concept. We can only use such a statement as a first approximation, from which we can make guesses to be tested against her responses to more detailed questions. Subj1's reference to numbers going in resulting in some numbers coming out suggests a process-conception. Subj1's emphasis on "numbers" (repeated six times) raises questions about the "quality" of her process-conception. Is this emphasis an indication that Subj1's function-conception is still at the action level? Does her function scheme includes the quantity restriction and/or the manipulation restriction? Her response to another situation (S17; Table 1) is a little more helpful:

Subj1: No because you're not really doing anything to the x and y that you're ... I mean you're just putting them into a set of tuples. You're not really manipulating the numbers in any sort of way. You're just taking two numbers and just setting them aside in a tuple......You're putting something in and I just, I just don't see it doing anything.

In this response, when trying to justify her position that this situation is not a function, the main reason that Subj1 appears to be giving is that there is no manipulation. This appears to contradict her earlier statement that "the process may or may not manipulate the numbers..." The important question for us is whether the phrase "doing anything" which she uses in her repetition a moment later refers to manipulation or something more general. When she was asked to compare S17 to S1 which she correctly described by a function, she said:

Subj1: Your x value in the first part of the tuple [in S1], but then you're manipulating that x value to put it, to get a number for the second number in your tuple.

And when the interviewer asked:

I: Okay, so what, the difference, what distinguishes this [S1] yes from the previous [S2] no, was that this one did what?

Subj1: This one [S1] manipulated the x.

The severity of this restriction can seen in the following response in discussing S13:

Subj1: Like for the other ones, [Situations] 4 and 15, whatever value you put in for x you could manipulate the rest of the equation to find out what y was.

An important fact is that in this example, it is not really possible to actually perform the manipulations. This is a typical situation for an "implicitly defined" function. When the interviewer points this out by asking:

I: What if you couldn't manipulate to find out for y? What if I give you an x value and there was no way you could manipulate this?

Subj1: If, if you couldn't manipulate ... Maybe, maybe you could get two, have two input values, one for x and one also for y and then manipulate them. Then your output would be true or false that the equation is ...

I: And the manipulation would be what?

Subj1 switches to a different kind of manipulation:

Subj1: What this equation says, once you plug those numbers in and you solve for the left hand side of this equation and then you come up with your output of being true or false if that equation holds true.

This last manipulation is not so clearly suggested in the given formula, at least according to how such an expression is dealt with normally in school and college mathematics, but must be constructed by Subj1 herself. This particular construction, if not invented on the spot is certainly very new to her and was mentioned only briefly in class. Her final comment is a fairly decent general description that might be given by someone with a process conception of function. Certainly, Subj1's function schema includes a restriction on the nature of the "process that relates the input to the output". However, the fact that she was able to suggest almost instantly a new interpretation that would satisfy her function conception is an indication of flexibility, and it is evidence that her repertoire of possible interpretations for a given situation is not limited to a single one.

The ability to construct a process that is not apparent in the situation was quite common with these students. Consider the following response by Subj2 to S1.

Subj2: Yes, this is a function...Because it is a set of tuples. Your first ... This is a set of tuples. Your first element of your tuple is going to be an integer, and if it's the set of all integers you're not going to have any repeats in there.

That last comment is an explicit statement of the requirement that no value appears more than once as a first component of a tuple or pair.

I: Okay. So go back to your definition for me. So how does that satisfy your thought of what it needs to be?

Subj2: Okay, when I look at these I think of putting them into the computer. Okay, I'm going to put this set into the computer...Then like if I were going to call up like t(1), okay, I'm going to get out 3, is going to be my answer. I input the first element of the tuple, output comes the second, the second element of the tuple.

Here she has referred to the process, and, on request, she explains in the next few statements the nature of that process.

I: What about that manipulating, that process, you were talking about?

Subj2: The process is you input the first element of tuple. It looks at its tuples and finds one that had this first element in it, and outputs the second element.

The point here is that the process which Subj2 describes is not at all explicit in the information she is given in the presentation of the situation. It is something that she has to construct herself, a structure that she imposes on the situation in order to make it a function. We are not suggesting here that Subj2 has invented this idea herself. It is, no doubt, something she learned in the course. What she has learned is to deal with certain kinds of situations by constructing a function, in this case by constructing the process of going from the first component to the second component of an ordered pair.

Subj3 is a student who is not restricted by the manipulation restriction, but seems to be bothered by the quantity restriction.

Subj3: The tables's like tuple. You have the name as the first component and the amount he owes as the second component. So that's what it seems to me. You, you're, you put in the first name and you get out what's owed. I mean it's a tuple of values.

Despite the fact that Subj3's schema of function seems to be free of the "manipulation restriction," there are indications that he is bound by the "quantity restriction" which means that in his view the inputs and outputs must be quantities (as opposed to propositions, for example). Subj3's responses to S9 and S13 are those which indicate the "quantity restriction" in his process conception of function.

Subj3: $x^2+3x+2=0$. Um, I, well, ..., you can solve for x but you can't, ... I don't think it's a function because you're, you can't ... it equals zero. You can't, it doesn't ask to put in anything to receive an answer to put an integer through that equation to get an answer. I mean it's going, unless it's one that has to be zero. But I don't think it's a function; it's just an equation.

I: So you'll put in an x and it either satisfies that equation or not. Right?

Subj3: Well, yea. Well, it could return ... Your equation could be true or, I mean you could return true or false.

I: If you thought about it that way what would it be?

Subj3: If, uh, ... Yea, for x, if you put in x with this equation and it does equal zero then you could do a function that returns true. And if it doesn't equal zero it returns false.

Again, we see here that two ideas are present and it takes only a slight suggestion from the interviewer to bring out the more powerful, but less familiar notion. A few minutes later, when Subj3 was asked what he thinks about S13, he responded:

Subj3: Well, it looks similar to number 9 in that it's an equation with an answer. You would either need an x and y value to plug in to see if that equation equals two and you could return true or false. But I would answer no now because it's an equation. You don't have any specifics on what it's going to return. What you're, what the process ... If this is process, what do you want it to return? Do you want it to return true or false or do you want it to return just values it is equal to?

We see in this last exchange, a possible emergence of an issue of authority. Who has the right to specify the domain and range, to construct the function?

SUMMARY

We can summarize a number of observations from these interviews. There is a certain amount of ambiguity regarding both the manipulation and quantity restrictions. They are often present, but not always with absolute certainty. A subject can have the restriction in one situation, dispense with it in another. Moreover, the two seem to be independent.

The question of whether or not the manipulation can actually be performed is a crucial factor for some students in determining whether they see a situation in terms of a function. For some, it requires only a prompt for them to drop the restriction. We see this as an example of a student's Zone of Proximal Knowledge (Vygotsky, 1978). For other students, this restriction, and perhaps others, seem to be a matter of authority. It is not always clear to the student that he or she has the "right" to construct a function, to specify its domain, range and other features. It may be that the student's autonomy in constructing a process depends on whether some form of algorithm is present.

We have been able to make some additional observations which, for reasons of space, are not illustrated here. A student can offer an intriguing analysis in which he or she invents a process to describe a given situation. A strong process conception appears to help students avoid the confusion between "uniqueness to the right" and 1-1. Most students in this study overcame a continuity restriction (the graph of the function must be an unbroken line) and some showed an awareness of the fact that they had previously been bound by this restriction.

REFERENCES

Breidenbach, Dubinsky, Hawk, and Nichols. (in press). The development of the process conception of function, *Educational Studies in Mathematics.*

Dreyfus T. & Eisenberg T. (1987). On the deep structure of functions, *Proceedings of PME XI,* Montreal, Vol I, pp. 190-196.

Dreyfus T. & Vinner S. (1982). Some aspects of the function concept in college students and junior high school teachers, *Proceedings of PME VI,* Antwerp.

Harel & Kaput. (in press). Harel, G., & Kaput, J. (in press). The role of conceptual entities in building advanced mathematical concepts and their symbols. In D. Tall (Ed), *Advanced Mathematical Thinking.*

Sfard A. (1989). Transition from operational to structural conception: the notion of function revisited, *Proceedings of PME XIII,* Paris, Vol III, pp. 151-158.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

MONITORING CHANGE IN METACOGNITION

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This research project studied the development of or change in metacognition in 9 middle-grade mathematics teachers who were provided an experience designed to motivate increased metacognitive activity and construction of knowledge that was productive for problem solving.

The mathematical problem-solving performance of school children in the United States remains a national concern. While the complex environment of the mathematics classroom provides a tangled web of factors that interact to impede or enhance the mathematical performance of the learner, from the field of psychology, metacognition theory (Flavell, 1979) offers a viable perspective for investigation on this problem.

Metacognition theory consists of (at least) two components, metacognitive activity and metacognitive knowledge. The former consists of the monitoring and subsequent regulation of what you know, and of what you do with what you know. In contrast to this active component of metacognition, metacognitive knowledge--often referred to as beliefs (Flavell, 1979)--exists as information about the nature of mathematics, the teaching of mathematics, ones self as a mathematical person, appropriate strategies in mathematics, etc. Metacognitive knowledge is taken to be a subset of all the knowledge an individual may have, differing primarily in the substance of what is known. These two dimensions of the theory of metacognition will provide the framework for the discussion of this research project.

Focus of the Inquiry

It has been suggested that metacognition, then, is an important link in understanding mathematical problem-solving performance (Schoenfeld, 1987). Metacognitive activity or the ability to monitor and regulate ones knowledge and thinking appears to give the problem solver dominion over his or her resources and ultimately makes him or her more successful in a task (Schoenfeld, 1987).

Metacognitive knowledge or beliefs have been suggested to be the driving force of mathematical problem-solving performance (Silver, 1982). And while much effort has been made to identify the nature of beliefs (Hirabayashi & Shigematsu, 1987) and the consistency of beliefs (Cooney, 1985; Hart, 1987), little has been done to study change in metacognitive knowledge.

The focus of the inquiry being reported was to extend the body of knowledge by studying development of or change in both metacognitive activity and knowledge in a set of teachers who were provided an experience designed to motivate this change. We also studied the interaction of metacognitive activity and metacognitive knowledge.

Methods

Fifteen middle-school teachers in the metro-Atlanta area were participants in the Problem Solving and Thinking Project (PSTP), a four-year National Science Foundation sponsored research project (MDR 8650008).

The Experience

The experience designed to elicit change was the Institute on Problem Solving and Thinking, an 11 week off-campus course offered through Georgia State University, and taught by the

project directors, Schultz and Hart. It was designed to (a) facilitate increased metacognitive activity, (b) provide experiences to motivate the construction of metacognitive knowledge that was productive for problem solving, and (c) integrate (a) and (b) to enhance problem solving performance. The following will describe each.

Facilitate Increased Metacognitive Activity. A set of experiences were designed to facilitate increased metacognitive activity. These included (a) engaging the teachers in Whimbey & Lochhead-type (1982) paired problem solving where one teacher acted as an external monitor for a second teacher during a problem-solving experience; (b) responding to structured questions about a problem about to be solved (See Table 1); (3) engaging in oral group reflections after a problem solving session; and (4) viewing video tapes of themselves and others to monitor problem solving performance, and observable metacognitive activity and metacognitive knowledge.

Table 1: Structured Questions Asked Before and After Problem Solving

Belief	Initial Questions	Final Questions
Task	Do you think this problem is hard or easy for you?	Did you think the problem was difficult or easy?
Strategy	What strategies do you think you will use?	What strategies were useful or not useful for this problem?
People	How do you think you will do on this problem?	What do you think of your performance on this problem?

Each activity to facilitate metacognitive activity followed a model/experience/reflect recursive sequence. For example, the project directors would model paired problem solving in front of the class. Teachers would have the experience of observing the struggle of solving an unfamiliar problem. Finally the group

ould reflect on the experience. Then the recursion--the roject directors would model how a teacher could facilitate paired problem solving in the classroom. The teachers, as students in the course, engaged in a problem-solving experience and the group reflected on the activity.

Motivate Productive Beliefs. In order to change metacognitive knowledge (beliefs) teachers needed to have experiences that were contrary to what they believed about themselves as problem solvers or the nature of mathematics. For example, if teachers believe there is usually one right solution to a problem and only one way to arrive at that solution, then they need experience with problems with more than one right answer and numerous solution paths. The belief literature suggested various domains to frame the experiences (Table 2). However, the Institute was not limited to these domains. In each

Table 2: Belief Domains

Belief	Institute focus
1 Mathematics is something to be memorized (formulas, algorithms, etc.).	Mathematical thinking and reasoning were emphasized.
2 Mathematics should be done quickly.	Mathematical problems were used that took varied amounts of time.
3 Mathematics is computation resulting in a right answer.	The focus of problem solving was the process with emphasis on strategies.
4 The teacher is the authority whose role it is to transmit knowledge.	Students (in this case Institute teachers) discussed and resolved most questions.
5 There is one way to solve a problem.	Several solutions processes were explored for every problem.
6 There is one correct answer.	Problems with no answer, one answer, and more than one answer were explored.
7 Mathematics should be done alone.	Mathematical problems were frequently solved in groups.
8 Mathematics is always elegant and parsimonious.	Problems and solutions were sometimes messy and unclear.

case, effort was made to monitor all beliefs and provide experiences that supported beliefs that were productive for mathematical problem solving.

The Interaction. In order for learners to coordinate new knowledge, whether it be cognitive or metacognitive, they must be conscious of and get control over their understandings and procedures. The process of becoming aware was a critical component of every experience. We not only wanted to facilitate change in metacognitive activity and knowledge but we also wanted to make teachers aware of their metacognition and how it might be impacting their problem-solving performance. We hypothesized that much of metacognition was unconscious, that is, teachers were not aware of the beliefs they held that were motivating their problem-solving behavior and they were not aware of when or if they were monitoring their mathematical behavior. In order to facilitate awareness of these components of their mental processes, videotaping was frequently used. The project directors modeled analyzing videotapes for evidence of beliefs. Teachers experienced through observation and then reflected with the directors on the experience. They were then given viedotapes of other teachers in the institute to take home and analyze. The analysis was shared with the other teacher and the class. In addition the role of the external monitor in paired problem-solving sessions was reflected on in the Institute. The external monitor modeled the kind of questions individuals should ask themselves during problem solving to monitor their own problem-solving performance.

Data Collection

Pre, interim, and post institute data were collected from

15 teachers in the following forms: videotaped pre and post (a) individual problem solving using structured and unstructured interviewing and think aloud techniques, (b) small group problem solving, (c) teaching of self-defined good problem solving lessons to a class of their students. In addition, weekly written reflection logs and pre and post non-routine paper-and-pencil problem-solving tests were analyzed. This variety of data sources allowed up to triangulate for reliability.

Analysis and Observations

Nine of the fifteen teachers were studied. Analysis followed a constant comparison method as described by Lincoln & Guba (1985). Some general observations from the group were as follows. From the analysis on metacognitive activity--monitoring and regulation--we were able to develop a model to describe the domains of activity. The model has two primary dimensions: (a) when montoring occurs and (b) what was being monitored (Table 3).

Table 3: Domains of Metacognitive Activity (Monitoring)

		Point in time at which monitoring ocurred		
		Before	During	After
Substance of Monitoring	Cognitive knowledge		*	
	Cognitive activity		*	
	Metacognitive knowledge			
	Metacognitive activity			

Metacognitive activity. Prior to the Institute most metacognitive activity (monitoring) occured in the two cells with astericks, i.e., during a problem solving experience and about specific content they knew--or did not know--and strategies they were engaged in. Teachers seldom made any comments about the problem before they began working it and seldom reflected back over the experience except to ask the correct answer. Over the 11 weeks the teachers began to spontaneously monitor and reflect

on their problem solving before they began a problem and after they were finished. In addition they began to demonstrate awareness of beliefs that might be impacting their behavior.

Metacognitive knowledge. Analysis of the data for evidence of beliefs held by the teachers prior to the Institute showed that they demonstrated more beliefs that were unproductive for mathematical problem solving. Commonly held unproductive beliefs were found in all the domains listed in the left column of Table 2 except in category 5 (there is one way to solve a problem) and category 7 (problem solving should be done alone). Analysis of post-institute data continued to show evidence of unproductive beliefs, but also showed evidence of an increase in commonly held productive beliefs, particularly in category 3 (in mathematical problem solving the process is more important than the product) (Lee, 1990).

The interaction. Some of the most interesting data came from the post-data interviews. It was in that format that teachers were most able to describe their awarenesss of the impact metacognition was having on their problem-solving performance. For example, one teacher shared, "I cannot see a change in my monitoring and regulating behaviors on either the [post] videotapes or the written transcriptions, however, I have a distinct memory of pleasure/success in improving what I perceived to be my monitoring behaviors. I was surprised to find no evidence of this on tape."

Discussion

In summary, we found little change in problem-solving performance over the 11-week period. We also found only moderate evidence of change in metacognition. What we did find was an

increased awareness of the role metacognition may have on problem-solving performance, and an increased willingness to engage in the problem-solving process and an attempt to implement many of the ideas from the Institute into the mathematics classroom. Our belief is that 11 weeks is not long enough to make a real impact and that sustained and substanative change will take a much longer period of time.

References

Cooney, T.J. (1985). A beginning teacher's view of problem solving. Journal for Research in Mathematics Education, 16(5), 324-336.

Flavell, J.H. (1979). Metacognition and cognitive monitoring: A new era of cognitive-developmental inquiry. American Psychologist, 34, 906-911.

Hart, L.C. (1987). Interactive domains, metacognition and problem solving. In J. Bergeron, N. Herscovics & C. Kieran (Eds.), Proceedings of the International Group for the Psychology of Mathematics Education (Vol. II, pp. 236-242) Montreal, Canada: University of Montreal.

Hiravayashi, I. & Shigematsu, K. (1987). Metacognition: The role of the "inner teacher". In Bergeron, J.C., Herscovics, N., & Kieran, C. (Eds.), Proceedings of the Eleventh International Conference on the Psychology of Mathematics Education (pp. 243-249). Montreal, Canada.

Hwe-Young Lee. (1990, July). A Study of the Relationship between Magnitude of Beliefs and Problem-Solving Performance of Middle School Mathematics Teachers. Unpublished doctoral dissertation. Georgia State University, Atlanta.

Lincoln, S.Y., & Guba, E.G. (1985). Naturalistic inquiry. Beverly Hills, CA: Sage.

Schoenfeld, A.H. (1987). What's all the fuss about metacognition. In A.H. Schoenfeld (Ed.), Cognative Science and Mathematics Education. Hillsdale, NJ: Lawrence Erlbaum Associates.

Silver, E.A. (1982). Knowledge organization and mathematical problem solving. In F.K. Lester & J. Garofalo (Eds.), Mathematical Problem Solving: Issues in Research. Philadelphia, PA: Franklin Institute Press.

Whimbey, A., & Lochhead, J. (1982). Problem Solving and Comprehension (3rd ed.). Philadelphia: The Franklin Institute Press.

THE USE OF CONCEPT MAPS TO EXPLORE PUPILS' LEARNING PROCESSES IN PRIMARY SCHOOL MATHEMATICS

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In a project on "Individual Mathematical Learning Processes"*) concept maps were used to find out and to describe pupils' individual interpretations of concepts related to several topics of primary mathematics. It turned out that - regarding all pupils in a class - the concept maps differed widely whereas the maps which had been constructed by a single pupil before and after a sequence of lessons, resp., very often were similiar in structure. An interpretation of these facts leads to the assumption that a pupil's achievement in mathematics depends very much from her/his prior knowledge and what (s)he focuses her/his attention on when confronted with the examples or the learning environment created by the teacher to teach some mathematical topic.

Among mathematics educators who are interested in the cognitive aspects of learning and understanding there seems to be a consensus that "to learn mathematics means to construct mathematics" (Fischbein, 1990, p. 7). This means in terms of Tall and Vinner (1981), for example, that these researchers are more interested in (students') "concept images" than in "concept definitions". In cognitive science to describe these individual interpretations of (mathematical) concepts several models have been suggested as, e.g., "frames" (Davis, 1984), or "overlays" (Ohlsson, 1986). In addition, methods how to elucidate such personal constructions are discussed: written tests, "out loud" protocols, "clinical interviews", etc. (cf., e.g., Ginsburg et al., 1983; Schoenfeld, 1985; or Hasemann, 1988). Nevertheless, all efforts should be made to find and to prove new methods for the exploration of individuals' mathematical learning processes.

At the PME 13 conference in 1989, at least two reports were given on research on learning processes in mathematics where a method called "concept mapping" was used (cf. Mansfield, 1989; Hasemann, 1989). This method was put up for discussion by Novak (cf. Novak et al., 1982); in the research just mentioned,

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wever, this method was used in a quite different way than it is described by Novak et al. (but Mansfield and Hasemann seem to have used it in the same way which will be explained below). From discussions on findings presented by Hasemann (1989) the question arose whether "concept mapping" is a reliable method at all to explore an individual's mathematical learning processes. So it was decided to establish a project with the objective to check the value of this method for research in mathematics education. First results will be presented here:

If it is intended to check the effects of a sequence of lessons usually students' prior knowledge is explored before the lessons take place, and later on students' knowledge after the lessons (by pre- and post- or delay-tests, resp., interviews, etc.). Especially with young pupils, one crutial point in this design is the evaluatuion of children's prior knowledge as, on the one hand, by written tests alone we can hardly find out an individual's prior knowledge about some mathematical topic, and on the other hand it's rather difficult to make sense of interviews with the young children.

In our experiments we used concept maps
a) before the sequence of lessons was carried out,
b) after the sequence of lessons was carried out,
with the aim to check whether we could elucidate learning effects from pupils' concepts maps, i.e. we looked at the maps to find signs for changes in the pupils' conceptual framework (see below). We took different mathematical topics (arithmetic and geometry) in classes of different grades (2, 4, and 6), and we carried out the experiments in different areas (Hamburg, Hannover, and Osnabrück). In this paper, I'll restrict myself to the presentation and the discussion of concept maps which were constructed by second graders in relation with a sequence of lessons on "folding and reflecting figures".

In these lessons, the pupils saw that there are "symmetrical" figures which can be folded (once or more often) in such a way that there are "two equal halves" which "look alike". In addition, the symmetry of figures can be checked with the help of a mirror (i.e., by reflection); the line, therefore, was called

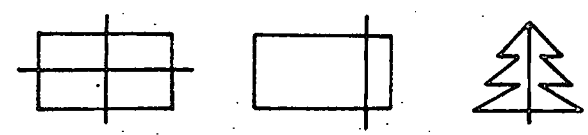
"folding axis" or "reflecting axis". Using lattices, the pupils learnt how to construct symmetrical figures.

Before these lessons a pre-test was carried out with 14 (of a class of 26) children in the following way: One child at a time firstly solved three problems taken from the mathematical problem in question (i.e. using only informal knowlegde about this topic), and then this child carried out the concept mapping experiment. The discussions between the child and a (student) teacher were audio-taped and transcribed. The post-test (about two weeks after the lessons) was carried out in the same way with 9 of the 14 children in the pre-test (but the problems were a bit more complicated as those used in the pre-test). In the pre-test, the pupils worked out these problems:

1. The pupils were asked to fold a sheet of paper and to cut out a figure: Look at the figure and describe what you see.
2. Given were some figures, drawn on a sheet of paper (rectangles, triangles, and a star): Check whether these figures can be folded or not. In case, mark the folding line(s).
3. Given were a lattice, a (vertical) line and (half of) a figure which was formed by some coloured pieces of wood and which adjoined to the line: Put more pieces on the lattice such that both sides of the new figure look equal.

(Remark: The items are, of course, not unambiguous. The pupils' solutions, however, will not be discussed here as the main reason to present them was to get the pupils in the right mood for the following concept mapping experiment; it turned out, however, that, on average, these problems were not very hard).

To construct the concept maps, in the pre-test the pupils got a sheet of paper and twelve small card with either terms that describe concepts ("to fold", "folding line", "look alike", "two equal halves", "star", "right", "left", "symmetrically"), or figures:



(In the post-test, some terms/concepts were changed correspondingly to the content of the lessons, see fig. 1 and 2). The pupils were asked to distibute the cards on the sheet of paper such that concepts or figures that are related are put together closely whereas those concepts or figures that are not related are seperated on the sheet. The pupils was said with insisten-

ce that there is no "correct" or "wrong" answer and they should deal with the cards as they felt it is best. In a second step the pupils were asked to find names for groups of concepts and figures (if possible) and to mark and to name relationships between such groups or between single concepts or figures.

It is expected that by this procedure at least parts of a pupil's actual conceptual framework (related to the context of the problems which had been worked out just before) are "mapped" from the pupil's mind to the outside (for critical remarks about this assumption see the discussion at the end of this paper). The concepts and figures on the small cards were chosen to support this process: "right" and "left", for instance, are concepts as well known from every day life as useful for the description of the symmetry of figures; in addition, at least this pair of cards might be grouped together easily. Most of the other concepts or figures were taken from the problems, or they were expected to be useful to solve the problems (except "symmetrical" - this term was unknown for the pupils when the pre-test took place). The terms "star" (and "beetle" in the post-test) were chosen as in the problems figures of a star or a beetle, resp., occured; both were symmetrical.

Most interesting is the comparison of pupils' concept maps in the pre- and the post-test, resp. (see fig. 1 and 2). It should also be remarked that Maike's concept maps stand for a group of four pupils' maps, just as Saline's for another group of four; Johanne's seem to be a special case.

In Maike's concept map in the pre-test (see fig. 1) there is a structure, but this seems to be related to every day life experience: For example, she grouped together star and pine-tree because "both things belong to christmas" (the experiments took place in the summer, by the way), and stars are "on the right and on the left" of the sky. Mathematical aspects were hardly considered by the girl, but she was right when she excluded the unknown term "symmetrical". From her map in the post-test we learn that she was now able to deal with this term correctly: on the right and on the left there are "two equal halves"

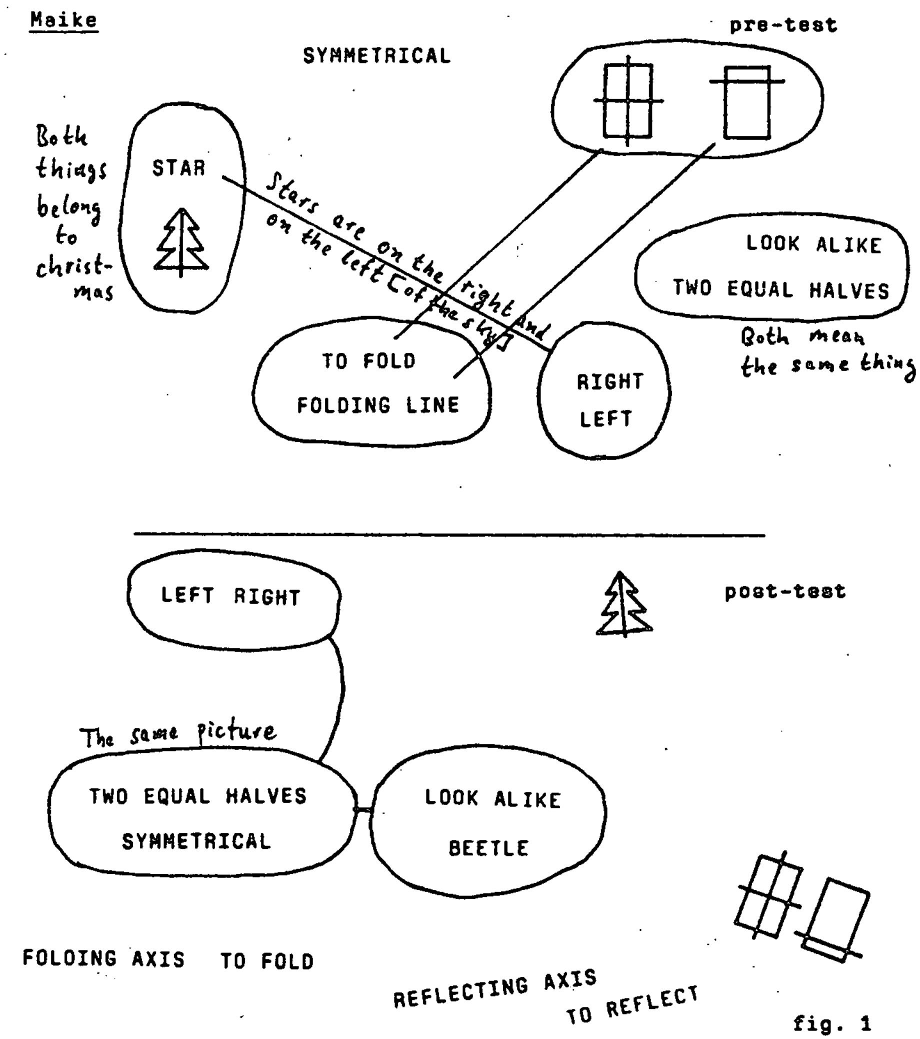


fig. 1

This group is connected with "look alike" and "beetle", but not with two other examples for symmetrical figures. Seen as a whole, the structures in Maike's maps look rather vague.

Things are quite different in Saline's maps (fig. 2). From both maps we recognize a conceptual structure related to symmetry, in the pre-test more intuitively generated (look at the ex(?)-cluded unknown term "symmetrical"!), in the post-test well constructed and explicitly excluding the "asymmetrical" rectangle.

Selina

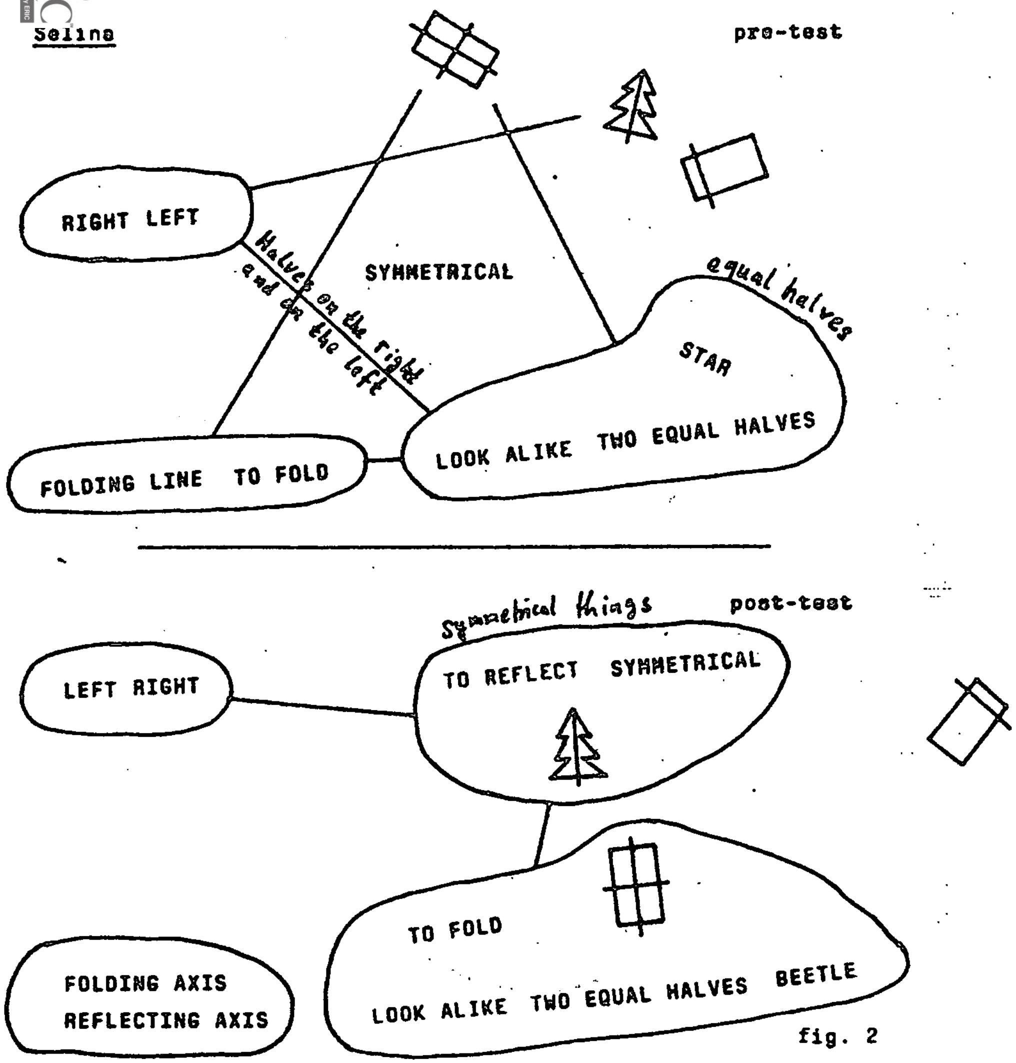


fig. 2

Johanna's concept maps (which couldn't be copied here) show that there are also pupils who have changed their conceptual structures in mind during the the lessons: In the pre-test, Johanna's central idea was that there were three figures given, but in the post-test she realized the symmetry of the figures.

In an experiment with 4th-graders the maps looked quite similiar, and even with an arithmetical topic precisely these three types of concept maps occured: (1) maps with hardly any mathe-

matical structure neither before nor after the sequence of lessons (instead, the pupils related the concepts to every day life experience, and they seem to have learnt in the lessons just the meaning of words or how to work out algorithms); (2) the maps are quite well structured with regard to the mathematical topic in question even before the instructions took place (and still better afterwards); (3) there is, from the mathematical point of view, an improvement in the maps (this group, however, is the smallest with wide margin). Regarding also pupils' behaviour in the lessons and with the problems we were led to the conclusion that there is a correlation between the kind of a pupil's concept maps and his/her achievement in mathematics: maps of type (1) very often come from pupils of lower, type (2) of higher, and type (3) of average abilities.

How to explain these results? Do we have to conclude that success in learning mathematics depends above all from the pupil's prior knowledge, and in the lessons they hardy learn anything (except some formalism)? In fact, Weinert et al. found out that a student's prior knowledge is a significant - perhaps the most significant - factor for learning behaviour as well as for achievement (cf., Weinert et al., 1987, p. 24). But, in case they are right: Were does this prior knowledge come from?

I think that concept maps in the first place do not refer to (prior) knowledge. Instead, from the maps we can see how a child relates the concepts or figures on the cards to some conceptual framework in mind: either to symmetry (i.e. to the topic of the problems which were worked out just before) or, for instance, to christmas or the stars in the sky (i.e. to non-mathematical experience). From these different aspects the children focus their attention on we can deduce another explanation for achievement in mathematics: When the pupils are confronted with examples, (concrete) objects (or whatever learning environment created by the teacher) there are children who focus their attention on the mathematical aspects of the examples or objects whereas some others prefere to think about, for instance, colour, materials, situations in which these things have already been used, etc. Obviously, the former group of pupils is more likely to understand the teacher's intentions than the latter

one. (It can be assumed, however, that there is a strong relationship between the pupil's focus of attention and his/her prior knowledge - according to Piaget's idea of learning as an equilibration between assimilation and accommodation).

Final remark: Of course, the results just mentioned do not definitely prove the reliability of the "concept mapping" method. It has to be tested further whether structures visible in an individual's concept maps are stable, at least for a considerable period of time. In the ongoing second part of the project, therefore, we carry out a long-term investigation using the same concepts repeatedly with the same samples of pupils, and we compare these results with those from psychological tests on cognitive styles, especially on field-dependence/independence.

References

Davis, R.B. (1984): Learning mathematics. Beckenham, Kent.

Fischbein, E. (1990): Introduction. In: Nesher, P. & Kilpatrick, J. (ed.): Mathematics and cognition. Cambridge, 1-13.

Ginsburg, H.D. et al. (1982): Protocol methods in research on mathematical thinking. In: Ginsburg, H.D. (ed.): The development of mathematical thinking. New York, 7-47.

Hasemann, K. (1988): Kognitionstheoretische Modelle und mathematische Lernprozesse. In: J. Math.-Didaktik, 9, 95-161.

Hasemann, K. (1989): Children's individuality in solving fraction problems. In: Proc. PME 13, vol. 2, 67-74.

Mansfield, H. & Happs, J. (1989): Using concept maps to explore students' understanding in geometry. In: Proc. PME 13, vol. 2, 250-257.

Novak, J.D. et al. (1983): The use of concept mapping and knowledge Vee mapping with Junior High School science students. In: Science Education, 67(5), S. 625-645.

Ohlsson, S. (1986): Some principles of intelligent tutoring. In: Instruc. Science, 14, 293-326.

Schoenfeld, A. (1985): Making sense of "out loud" problem-solving protocols. In: J. Math. Behav., 4, 171-191.

Tall, D. & Vinner, S. (1981): Concept image and concept definition in mathematics. In: Educ. Stud. Math., 12, 151-169.

Weinert, F.E. et al. (1987): The development of individual differences in learning performance and in school achievement. In: Second European Conf. for Research on Learning and Instruction, Abstract Vol. Tübingen, Sept 19-22, 1987, 24/25.

Adjusting computer-presented problem-solving tasks in arithmetic to students' aptitudes

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Summary

The current goals of mathematics education are based on the idea that all real learning involves thinking, and that thinking ability can be and must be cultivated in every student. This study offers a method that stimulates students' thinking while doing arithmetic, through adjusting mental challenges in arithmetic to all students in a class. These challenges are arranged in an hierarchical sequence of increasing difficulty and their adaptation to students' aptitudes is achieved through five strands that differ in the amount of steps skipped while going through this hierarchy. All 36 students of a third-grade class worked individually in a microcomputer lab for ten 40-minutes weekly session. Each student was assigned, to start with, to one of the five strands on the basis of a combined score on the researchers' designed pretest and on teacher-administered class tests. Continuous adjustment of level of challenge to each student's aptitude was made throughout the period of experiment on the basis of student's performance. On completion of the assigned sequence of tasks, a posttest and an attitude questionnaire were administered. These tests and the questionnaire were also administered to students of two other third-grade classes in the same school who did not work with the computer assignments, serving thus as a control group. On the basis of only partial data analysis we conclude that all students learned to operate with numbers beyond the number domains that they were familiar with and greatly enjoyed working with the challenging tasks. The innovative method for adaptation of challenges to students' aptitudes worked very well, and differences in patterns of performance and in the development of problem-solving strategies were identified between students of high and low aptitude.

Although mathematics offers special opportunities for children to discover the power of thought, many students never learn the value of mental effort (Everybody Counts, 1989). In support of this observation, the analysis of the fourth NAEP (National Assessment of Educational Performance) in mathematics (Silver, 1988) reveals that many students face difficulties with items that do not involve routine, familiar tasks even when these items test basic concepts. This is a very unsatisfactory state of affairs because the new goals in mathematics education are based on the ideas that all real learning

involves thinking, and that thinking ability can be and must be nurtured and cultivated in every student (Resnick and Klopfer, 1989). One of the major current aims of mathematics education is teaching children to be good problem solvers. The essence of problem solving is being able to deal with novel situations or with problems one has not been specifically trained to solve. (Lesgold, 1988, p. 205).

To be good problem solvers in arithmetic, students should use a variety of problem-solving strategies; invest mental activity (ibid), and feel confidence in their ability to overcome mental challenges in mathematics (Kloosterman, 1988). To achieve this confidence, students should experience and practice challenges in arithmetic. However, the mathematics curriculum in elementary schools is largely uninspired, boring, and superficial (Porter, 1988). The major part of teachers' mathematics instruction time is spent teaching computational skills of the four basic operations (Ibid), activities which usually exclude problem-solving challenges.

The learning material in our study is designed to provide arithmetic practice on a different level than that of the regular curriculum. Rather than boring technical drill, we propose challenging tasks that require the investment of substantial mental effort. These tasks are not word problems, as are most of school-arithmetic solving-problems tasks, but numerically-based activities. Our activities emphasize the student's identifying the appropriate operation and estimating the size of number to add or subtract, rather than on performing the particular computation (which is done by the computer). We expect that through the process of problem solution, students should develop problem-solving strategies, estimation abilities, a sense of generalization across number domains, and an understanding of place-value concepts as well as other aspects of number sense.

The tasks are arranged in an hierarchical sequence of increasing level of difficulty. One objective of our study has been to explore the adaptation of the tasks to the students' aptitudes so that each task presented to a student poses substantial mental challenge to him or her. There is a variety of methods and techniques for adapting learning material to students. Wang & Resnick (1978) suggested the use of diagnosis tests to place individual children in the hierarchical sequence of tasks, thus adjusting hierarchically-arranged learning tasks through assigning different starting points in the hierarchy to students with different initial knowledge. Resnick & Ford (1981) suggested to adapt instruction to each learner through skipping certain steps in the learning tasks hierarchy. In our study, we adopt this suggestion to hierarchies of problem-solving challenges through adjusting the amount of steps to be skipped in the hierarchy of the tasks.

Method

The computerized tool, named here in short the "Broken Calculator" (Schwartz, 1989), presents on a microcomputer screen a basic calculator with 10 digits, the 4 basic arithmetic operations, and one memory. The user can disable any of the Calculator's keys; set a goal to be reached; print on the screen any two numbers and an arithmetic operation to be applied to these numbers; get the result of the arithmetic operation as computed by the computer; and get a screen printout of the list of the operations used in solving a particular problem. When the user reaches the prespecified goal after performing one or several operations, the number of steps for reaching the goal is presented on the screen.

The hierarchy of our tasks is based on the following principles (Table 1):

Table 1

INSTRUCTIONS: Use only the given key(s) and the least number of addition and subtraction, up to 4 operations.

No.	Keys	Goal	Solution (w/min. # of steps)	Students' aptitude VL	Lo	Med	Hi	VH
6	1	14	11+1+1+1					
11	3	21	33-3-3-3-3					
17	2	38	22+22-2-2-2					
20	1	134	111+11+11+1					
72	1	2,100	1,111+1,111-111-11	V				
73	2	4,002	2,222+2,222-222-222+2	V	V			
74	1	2,099	1,111+1,111-111-11-1	V				
75	1	1,989	1,111+1,111-111-111-11	V	V	V		
76	2	4,398	2,222+2,222-22-22-2	V				
77	2	4,178	2,222+2,222-222-22-22	V	V			
78	3	6,327	3,333+3,333-333-3-3	V				
79	3	5,967	3,333+3,333-333-333-33	V	V	V	V	
87	3	30,399	33,333-3,333+333+33+33	V	V	V	V	V
102	2	220,442	2,222,222-2,222+222+222-2					
113	1	1,010,111	1,111,111-111,111+11,111-1,111+111					
203	1	-109	1+1-111					
220	1	2.221	1.111+1.11					

- All tasks are of the same type: One should use only a given single number key, only the + or - operations, and achieve the goal (a given number) using the minimal number of arithmetic operations up to a maximum of 4 operations.

- The tasks require investing mental effort, using problem-solving strategies;
- The increase in level of difficulty throughout the sequence of tasks is obtained due to the fact that each task requires at least one more step of generalization than the previous task. The generalization is obtained through either one, or a combination of the following methods:

(a) Increasing the number of operations, that is, if a certain task requires two operations, the next task requires three or four operations of the same type. To illustrate: if one task is to get the goal of 244 using only 2's (solution: 222+22) then the next task may be--getting the goal of 468 using only 2's (solution: 222+222+22+2).

(b) Replacing one operation (e.g., addition) with the other (subtraction). To illustrate, if one task is to get the goal of 1,245 using only 1's (solution: 1,111+111+11+11+1), then the next task may be-- getting the goal of 1,021 using only 1's (solution: 1,111-111+11+11-1)

(c) Crossing the tens position, the hundreds position, etc., through changing the particular digit used. To illustrate, if one task is getting the goal of 48 using only 2's (solution: 22+22+2+2), then the next task is getting the goal of 72 using only 3's (solution: 33+33+3+3). This second task crosses the 60 and goes over 70. This "crossing" adds difficulty to the solution.

(d) Crossing the tens position, the hundreds position, etc., through replacing one operation with the other. To illustrate, if the first task is getting the goal of 1,023 by using solely 1's (solution: 1,111-111+11+11+1), the following task may be-- getting the goal of 979 also by using solely 1's (solution: 1111-111-11-11+1). The solution of the second task demands crossing a thousand, that is, subtracting below a thousand, which causes substantial difficulties to children.

(e) Changing a number domain. To illustrate, if one task is using only 2's to get 402 (solution: 222+222-22-22+2), then the next task may be: using only 2's to get 4,402 (solution: 2,222+2,222-22-22+2) or even: using only 2's to get 4,022 (solution: 2,222+2,222-222-222+22).

The adjustment of level of difficulty of the tasks sequence to students' aptitudes was done in the following manner:

Five strands for going through the hierarchical tasks sequencewere designed to provide an optimal amount of "jumps" (number of steps to be skipped in the hierarchy) for students of different aptitudes (see Table 1), as explained next.

-The first strand, for students with the lowest aptitude, goes through each step of the original sequence without skipping any task. However, it does limit the number domains involved: For example, it involves whole numbers only up to 9,999.

- The second strand skips approximately every second task, and goes, again, up to 9,999.

- The third strand skips approximately every second task of the second strand, that is, approximately every fourth task on the original list of tasks, and goes up to 99,999.

- The fourth strand skips approximately every second task of the third strand, that is, approximately every eighth task on the original list, and goes up to millions.

- The fifth strand skips approximately every second task of the fourth strand, that is, approximately every 16'th task of the original list of tasks, and goes up to millions.

Procedure

A preliminary two-stage study examined the use of the tasks by students working in pairs. The first stage consisted of observation of one pair of students working with the software and the tasks. The second stage involved a full third-grade class working in pairs on these assignments for two months. The two stages served to develop the ideas underlying the tasks, including designing the tasks and structuring the hierarchy; designing the five strands, and planning the procedure. The pair-work for a full class proved to be unsuccessful because the task solutions were largely based on students' aptitudes, and even assigning students to homogeneous groups did not prevent comparisons and competitions between the two members of the pair. On the basis of this experience we decided to assign students to work individually in our main study.

The study took place during September-November 1990 in a third-grade class in a Tel Aviv suburban school for mixed population of middle and low SES population. All 36 students of the class worked individually in a microcomputer lab for ten 40-minutes, weekly sessions. The class was divided into two groups that attended the computer lab separately because the lab included only 20 computer stands. Each student was assigned, to start with, to one of the first four strands of tasks on the basis of a combined score on a pretest administered by us and of grades in three teacher-administered class tests. The pretest examined knowledge of place-value concepts and of other elements of number sense that we expected to be promoted through solving our tasks. The students got the assignments printed on worksheets and typed

them into the computer. In the lab, the students were supervised by their class teacher and by one of the researchers. Students who faced difficulties with three consecutive tasks were assigned to a "lower" sequence. When the tasks showed to be too easy for a student, he or she was moved to a "higher" sequence. This was our method for adjusting the level of challenge to individual aptitude. On completion of the assigned sequence of tasks, each student took a posttest-- identical to the pretest--and responded to a questionnaire that examined attitudes towards arithmetic in class and with the "Broken Calculator". The pre- and posttests and the attitude questionnaire were also administered to students of two other third-grade classes in the same school, who thus served as a control group.

To sum up, our data sources were students' individual diskettes, with saved solutions; paper-and-pencil solutions on the worksheets, and performance on the pre- and post tests.

Results and Conclusions

The process of data analysis has not yet been fully completed. Students' solutions on diskettes and worksheets are being analyzed to identify strategies for overcoming problems and patterns for solutions. Results of the pre- and post tests will be compared and the questionnaire on students' attitudes will be analyzed. Presently we can report intermediate results based on partial analysis of our data sources and on our impressions from supervising the students' work in the lab:

- Almost all students greatly enjoyed to work with the assigned tasks and seemed to be highly motivated by the challenges.
- All students learned to operate with numbers beyond the number domains that they had been familiar with, that is, with large whole numbers, negatives and decimals and they developed their problem-solving strategies.
- The adaptation of challenges to students' aptitudes through skipping steps in the hierarchy of tasks seemed to work very well.
- Substantial differences were identified among students of different abilities. The students with very low aptitude needed 20 tasks on average to master the tens' range, and 70 tasks to master the hundreds' range, and they were not able to complete the thousands' range within the 10 sessions. Their mastery in the two completed number domains was never perfect. That is, assigning them they would err again in tasks that they had solved several times in previous sessions. In comparison, the highest-ability students (of the 4th strand) needed (on average) 5 tasks to prove mastery in the tens' range, 3 in the hundreds' range, 9 for the thousands, 5 for the ten-thousands, 4 for the hundred-thousands, and 10 for the millions. All these took them 6-7 sessions

to complete. Their mastery was almost perfect. They never erred when assigned a previous task. In the three last sessions, having completed the whole-numbers tasks, they solved tasks with negatives and decimals. All other students were observed to perform between these two extremes. In addition, the problem-solving strategies that were developed throughout the study were more effective and were identified faster by the higher-ability students than by the lower-ability students.
- The management of students' work in the computer lab initially required substantial adult involvement for the evaluation of students' on-line solutions, for providing help, and for moving students up or down in the five strands. However, after the 4th session, when all students mastered the method of work with the software and with the assigned tasks, the class teacher managed to work comfortably with both halves of the class at the computer lab, without additional adult help.

To summarize, this study reveals the potential of the use of the microcomputer in implementing arithmetic instruction based on current thinking on students' learning of arithmetic. It is also clear that beyond its beneficiality to their understanding of arithmetic, the method is challenging and exciting to the students.

References

Everybody Counts: A report to the nation on the future of mathematics education. (1989). Board of Mathematical Sciences; Mathematical Sciences Education Board; National Research Council.

Hativa N. (1988). CAI versus paper and pencil--discrepancies in students' performance. Instructional Science, 17(1), 77-96.

Kloosterman P. (1988). Self-confidence and motivation in mathematics. (1988). Journal of Educational Psychology, 80(3), 345-351.

Lesgold, A. (1988). Problem solving. In R. Sternberg & E.E. Smith (Eds.) Psychology of human thought. Cambridge, MA: Harward University Press.

Porter, A. (1988). A curriculum out of balance: The case of elementary school mathematics. Institute for Research on Teaching Research Series, No. 191. East Lansing, MI: Michigan State University.

Resnick L.B. & Ford W.W. (1981). The psychology of mathematics instruction. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Resnick, L.B. & Klopfer, L.E. (1989). Toward the thinking curriculum: An overview. In L.B. Resnick & L.E. Klopfer (Eds.), Toward the thinking curriculum: Current cognitive research. Alexandria, VA: Association for Supervision and Curriculum Development.

Schwartz, J. (1989). What do you do with a broken calculator? Boston: Sunburst Publications.

Silver, E.A. (1988). The fourth NAEP mathematics assessment. Learning Research and Development Center, University of Pittsburgh.

Wang, M.C. & Resnick, L.B. (1978). The primary education program. Johnstown, PA: Mafex.

COMPUTER-BASED GROUPS AS VEHICLES FOR LEARNING MATHEMATICS

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This paper presents some research being carried out as part of the "Groupwork with Computers" project[1], together with a description and analysis of two groups working on a Logo-based mathematics task. In particular, we examine the nature of collaboration and the reciprocal influence this has both on group outcome and individual progress.

Background and Aims

Proponents of computer-based groupwork suggest potential benefits include: the externalisation of ideas, through interaction, the consideration of alternative perspectives, a greater diversity of skills and knowledge enabling exchange of information and ideas and increased attentiveness and on-task behaviour. Our own research has indicated higher levels of discussion in computer-based mathematical environments as compared to paper and pencil environments (Healy et al 1990). Research studies into learning resulting from computer-based groupwork have however produced conflicting evidence. Higher levels of achievement resulting from group as compared to individual situations are reported by, for example, Johnson et al (1986) and Mevarech et al (1987), but not found by Trowbridge (1987) and Salomon and Globerson (1989). These differences are not surprising given that computer-based groupwork is not a single phenomenon; the nature of the group, the curriculum content and the type of software are all likely to engender different styles of working and pupil response.

The work reported here forms part of the "Groupwork with Computers" research project (Eraut and Hoyles, 1988). Our aim is to identify factors influencing effective computer-based groupwork in the context of mathematics learning. We examine the relationship between level of collaboration and both group and individual learning outcomes and explore the reciprocal role of the computer with respect to outcomes and group processes.

Research Design

The research takes place in seven classes in six schools. Eight experimental groups of six students (aged 9-12 years) have been selected, each consisting of three girls and three boys, a girl and boy from each of the achievement levels high(H), middle(M) and low(L), as assessed by their class teacher. Each group undertakes three research tasks, two involving Logo programming and one database work. During the research session each group is given one copy of the task and three computers are available for use. Each session lasts for approximately 2 hours 30 minutes.

The research tasks vary in terms of mathematical content, but each consists of a *construction activity* and an *application activity*, each with an

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...ciated *group outcome*. The intention is that these group outcomes are appropriated by all group members. The tasks have been designed so that to be successful in terms of group outcome, the following have to be negotiated: *Task Management* — the organisation of people, task components and resources (including computers); *Global Targets* — the mathematical and programming ideas which relate to the group outcome; *Local Targets* — subcomponents of the task which can be legitimately allocated to sub-groups.

Data Collection and Analysis

Process data is collected by researcher notes, video recording of the whole group and the screen output of one computer. During the administration of the research tasks the researchers do not intervene (except in exceptional circumstances). Group outcomes (GO) are scored by reference to computer products and completed task sheets. Individual learning is measured by a series of pre, post and delayed post tests involving the mathematical ideas embedded in each task. The group is given a semi-structured interview after each task and a further interview is conducted informally with the class teacher to obtain background information to help contextualise the research.

The process data is analysed by classifying the discussion and interactions of the group into a number of *episodes* — group interactions concerning task management, global and local targets. We intend to examine both group collaboration and individual involvement. We measure the collaboration across the whole group on task management (CS_m) by means of a collaboration score based on the proportion of the group party to discussions and interactions during all the task management episodes:

$$CS_m = \frac{\text{No. of pupils party to Task Management episodes}}{\text{No. of Task Management episodes x group size}}$$

Individual management involvement scores in task management (IS_m) are also calculated as follows:

$$IS_m = \frac{\text{No. of Task Management episodes to which a member is party}}{\text{No. of Task Management episodes}}$$

Collaboration and involvement scores relating to global target episodes are similarly calculated (CS_g, IS_g).

The intention is that the whole group will work together but when this does not occur and the group splits into subgroups, interactions about global targets are distinguished for each subgroup and separate collaboration scores calculated. In these cases the group outcome is scored for each subgroup.

Some Findings

We are currently involved in the data collection phase of the project. We present an analysis of two groups working on the *Bicycle Spokes Task*. Tests of individual learning measure understanding of the number of degrees in a complete turn and the relationship between the number of divisions in a radially divided circle and the turning angle for each division. Figures 1 and 2 show the task and associated global and local targets. Further data will be presented at the conference.

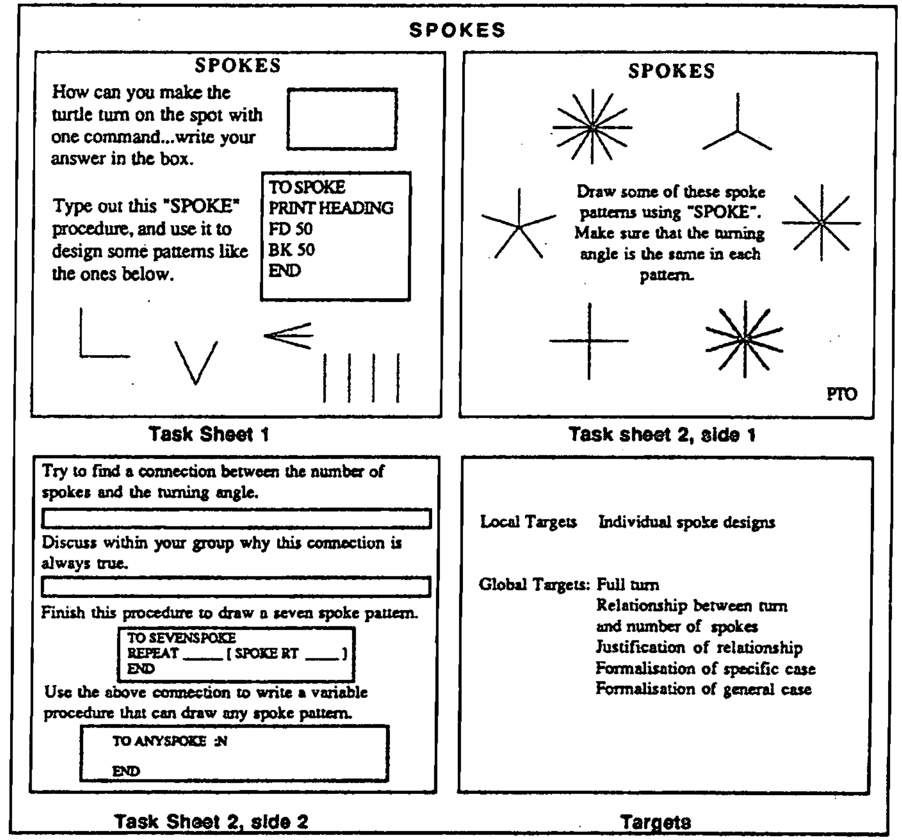

Figure 1: The Bicycle Spokes Task: Construction Activity

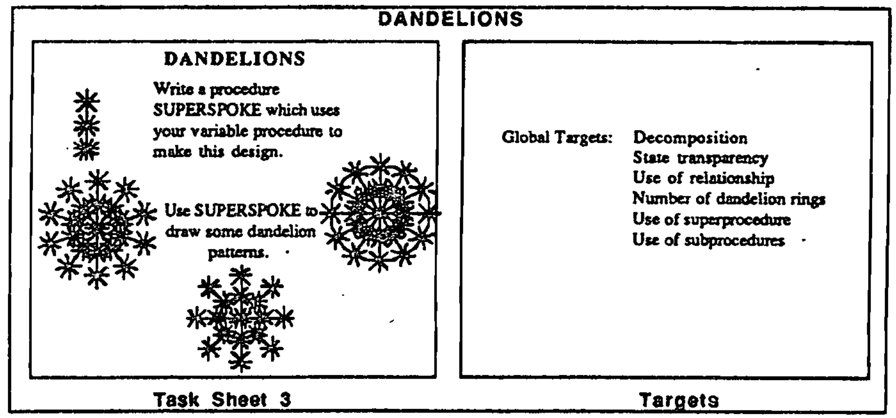

Figure 2: The Bicycle Spokes Task: Application Activity

Group A

Elly(L), Ann(M), Carla(H), Steve(L), Guy(M) and Chris(H) make up Group A. The start of the group interaction involved a degree of "task sheet snatching" — individuals taking sheets at random with no group planning or task management. All group members moved quickly to the computers where they split into three sub-groups. Steve, Guy and Chris made up SG1, while Carla was involved in both the other subgroups, with Ann at one computer (SG2) and with Elly at the third (SG3). Since they did not have the construction activity task sheets, both SG2 and SG3 moved quickly onto the application activity. The screen effects they produced provoked SG1 to abandon the construction activity in preference for the application. Thus none of the global targets of the construction activity were addressed and task sheet 2, side 2 was not completed.

Throughout the session, Steve and Guy tended to flit on and off task, both having some difficulty in concentrating throughout complete episodes. When Guy was on task he played an active role, making appropriate suggestions and comments. Steve's main role was that of task sheet snatcher. He made few active contributions during global target episodes. Chris was responsible for encoding (expressing in the Logo language) and typing, although he tended not to articulate his actions. With limited help, Chris wrote a superprocedure which produced a two ring dandelion, making use of the underlying mathematical relationship.

	Collaboration Score		Involvement Score												GO Score
	CSm	CSg	ISm						ISg						
			Girls			Boys			Girls			Boys			
			E	A	C	S	G	C	E	A	C	S	G	C	
Whole Group Construction	0.59 $(\frac{3}{5}, 21)$*		0.4	0.5	0.6	0.7	0.6	0.5							
Application	0.43 $(\frac{2}{5}, 10)$		0	0.7	0.4	0.6	0.4	0.5							
SG1(S,G,Ch) Construction		0 $(\frac{0}{5}, 0)$							0	0	0	0	0	0	0/23
Application		0.79 $(\frac{6}{6}, 16)$							0	0	0	0.5	0.9	1	13/21
SG2 (A,C) Construction		0 $(\frac{0}{5}, 10)$							0	0	0	0	0	0	0/23
Application		1 $(\frac{3}{6}, 9)$							0.3	1	1	0.3	0.1	0.1	11/21
SG3 (E,C) Construction		0 $(\frac{0}{5}, 0)$							0	0	0	0	0	0	0/23
Application		1 $(\frac{3}{6}, 6)$							1	0.5	1	0	0	0	11/21

Table 1: Collaboration, Involvement and Group Outcome Scores (Group A)

Carla sat in a position which enabled her to move freely between two computers. In both her sub-groups she took control of all the encoding but shared the typing with the other group member. Her style involved instruction

* The first number is the proportion of management or global targets addressed; the second is the number of episodes about these targets.

rather than explanation when giving help to either Ann or Elly. Ann tended to try to make sense of these instructions and made a number of suggestions of her own, in contrast to Elly who simply typed in what Carla said. Both groups produced a three ring dandelion in direct drive which made use of one spoke procedure, although the third ring was not rotationally symmetric.

Table 1 presents the collaboration, involvement and group outcome scores associated with Group A. The CS_m for the group during the construction activity was 0.59, dropping to 0.43 during the application activity as the task-sheet snatching strategy proliferated. All the girls were party to all global target episodes within their subgroups (CS_g's = 1). SG1 had a lower CS_g score (0.79) mainly because Steve was involved in only half of the episodes.

All three sub-groups obtained a group outcome of 0 for the construction activity. Two factors seem to have contributed to this; the mismanagement of the task sheets and an element of competition due to the public nature of the computer screen. Despite this, all three groups were reasonably successful on the application activity. Individuals who brought with them knowledge of the mathematical relationship always took control of encoding which explains the groups' relative success.

	Pre (Max Score = 7)	Post (Max Score = 7)
Elly (L)	0	0
Ann (M)	2	2
Carla (H)	5	6
Steve (L)	0	0
Guy (M)	7	7
Chris (H)	7	7

Table 2: Pre and Post Tests Scores (Group A)

Only one student, Carla, improved her test score. The other five members obtained the same score before and after the group task, despite the fact that all were involved in some episodes in which the mathematical idea being tested was used. Three group members brought to the session an understanding of this idea and two of them (Carla and Chris) took responsibility for calculating the appropriate angles for any specific case and encoding in Logo without communicating this understanding to the other group members. We suggest that this is because Chris encoded and typed in his ideas without explaining them, and Carla acted as "instructor" rather than "teacher" — telling her contemporaries what to do but not why.

Group B

Group B was made up of Erica (L), Haley (M), Jenny (H), Jim (L), David (M), and Tom (H). Following the task introduction, the group split into two single-sex sub-groups. The boys (SG1) took the first and last of the three task sheets and the girls (SG2) took the second.

Jim answered the full turn question and SG1 went to the computer to check this. They experimented with the SPOKE procedure, but chose not to use it in their subsequent attempts to produce spoke designs. They had difficulties when entering and editing their procedures, resulting in syntax errors which they did not understand. After a short break, David and Tom negotiated a

strategy to produce a dandelion, involving the production of one large spoke design embellished with smaller ones by rotation. Tom made constructive suggestions but had difficulty in encoding his ideas in Logo. Thus, although they shared the calculation of angles appropriate for specific cases, David took care of both encoding and typing — and clearly felt that a lot of the work had been left to him. Jim seemed unsure as to what David was doing and made few active contributions. By the end of the session, SG1 had an almost complete three-ring dandelion.

Jenny, Haley and Erica (SG2) planned spoke design procedures on paper, Jenny taking control of encoding. At the computer Erica, or sometimes Haley, typed in the commands dictated by Jenny. The group had difficulties when it came to describing the relationship between number of spokes and turning angle in natural or formal language. Jenny used the algorithm on specific cases, but the group were not able to articulate the general case and the construction activity was abandoned. Their approach to the application activity was to place spoke designs in various positions on the screen, without the use of the rotational symmetry of the dandelion. They wanted to complete the task as quickly as possible, displaying a competitive attitude towards the other subgroup. As screen effects from SG1 looked more and more promising, this group lost motivation.

	Collaboration Score		Involvement Score												GO Score
			ISm						ISg						
			Girls			Boys			Girls			Boys			
	CSm	CSg	E	H	J	Ji	D	T	E	H	J	Ji	D	T	
WholeGroup Construction	0.63 $(\frac{1}{3}.4)$		0.8	0.8	0.8	0.5	0.5	0.5							
Application	0.6 $(\frac{1}{3}.5)$		0.8	0.8	0.8	0.4	0.4	0.4							
SG1 (Ji,D,T) Construction		1 $(\frac{1}{5}.2)$							0	0	0	1	1	1	1/23
Application		1 $(\frac{6}{6}.6)$							0	0	0	1	1	1	18/21
SG2 (J,E,H) Construction		1 $(\frac{3}{5}.4)$							1	1	1	0	0	0	3/23
Application		1 $(\frac{3}{6}.6)$							1	1	1	0	0	0	8/21

Table 3: Collaboration, Involvement and Group Outcome Scores (Group B)

Table 3 presents the collaboration, involvement and group outcome scores associated with Group B. The girls tended to be more involved in management decisions, reflecting a greater pre-occupation with the task sheets. Collaboration over the sharing of the task sheets was not high (CS_m-construction = 0.63, CS_m application = 0.6).

Both sub-groups were maximally collaborative during global target episodes. The involvement scores show no inter sub-group communication on global targets. This ruled out the possibility of SG2 receiving further ideas as to the expression of the general relationship in either natural or formal language. In SG1 the application activity was carefully planned with none of the global targets avoided while SG2 focussed more on getting a design on the

screen than on analysing the structure of the dandelion, resulting in avoidance of global targets and a less sophisticated product.

	Pre (Max Score = 7)	Post (Max Score = 7)
Erica (L)	4	3
Haley (M)	5	3
Jenny (H)	7	7
Jim (L)	1	2
David (M)	7	6
Tom (H)	7	5

Table 4: Pre and Post Test Scores (Group B)

In the pretest, Erica and Jim were the only two in the group who seemed unaware of the 360° relationship, and their post test results showed no improvement. Jenny's score remained the same, while the scores of David, Tom and Haley went down. Although both Erica and Jim were party to global target episodes, the nature of their involvement appears not to have led to individual gain. Erica had the role of typist, but played no active role in encoding. Jim's role within SG1 is less clear, but given this group did not confront the mathematical relationship in the construction activity, and Jim became inattentive during the application activity, the same lack of active participation in the constructions may be a significant factor.

Concluding Remarks

The most striking results about the two groups are: the low group outcome scores for the construction activity; the single-gender groupwork; the apparent absence of any individual learning regarding the mathematical relationship; the absence of any articulation of the mathematical relationship in general terms in either natural or Logo language (although both groups contained pupils who could confidently predict angles for specific cases); the apparent absence of any relationship between collaboration score and group outcome or individual learning. Our tentative explanations of the above are:

Absence of Intervention: There were points during the session when a simple intervention may well have considerably changed the group dynamics. We intend to identify when and where intervention would seem to be necessary.

Pupils' Constructing their Own Goals: Despite its careful design, the pupils (not surprisingly) tended to restructure the task to fit their own goals. These goals emerged implicitly through computer interaction rather than through discussion. In particular, we noticed a centration on the final screen product at the expense of other group targets. Both groups chose not to work together on a single outcome but to split into subgroups without managing the task as a whole or satisfactorily sharing the resource sheets. Perhaps the lure of the computers contributed to this early splitting into subgroups — curtailing group discussion but also fostering a change in atmosphere from insecurity to "busyness".

Competition: Within both groups there was at least one competitive subgroup, aiming to produce an impressive screen design more quickly than the others. It may be that gender was an influencing factor, with single sex subgroups setting "girls against boys". A degree of competition appeared to be caused by or at least fuelled by the public nature of Logo work.

Group Dynamics : All five subgroups contained at least one pupil who already had some knowledge of the relationship in question. In four of the subgroups, each characterised by a desire to complete the task quickly, one of these pupils took exclusive responsibility for employing and encoding the appropriate algorithm and dominated the solution process. In contrast, the fifth subgroup (Group B, SG1) were more interested in making sense of the mathematical structure of the task than in completing it as quickly as possible. This subgroup was characterised by a high level of discussion and negotiation, with two pupils making use of the relationship (although for both this involved consolidation of knowledge). Negotiation of a joint solution strategy may have contributed to the relatively high GO score obtained by this subgroup.

We suggest that when groups become competitive or when their main focus is to finish as quickly as possible, then they are content to let one pupil take control of the task solution process. If these pupils are already able to confidently use the ideas embedded in the task, then this may indeed be the quickest root to completion. However, in these cases the pupils in control seem unlikely to spontaneously engage in peer tutoring. Discussion and negotiation leading to the externalisation of ideas and the exchange of knowledge could be seen as diversory and a waste of time. Within these scenarios it is not surprising that individual progress is limited or non-existent.

References

Eraut, M., and Hoyles, C. (1988) *Groupwork with Computers*, Journal of Computer Assisted Learning, 5(1), 12-24.

Healy, L., Hoyles, C., and Sutherland, R. (1990) *The Role of Peer Group Discussion in Mathematical Environments*, Institute of Education, University of London.

Saloman, G and Globerson, T. (1989) *When teams do not function the way they ought to*, International Journal of Educational Research, 13(6), 89 - 99.

Johnson R., Johnson, D. and Stanne, M. (1986) *Comparison of Computer-Assisted Cooperative, Competitive and Individualistic Learning*, American Educational Research Journal 23(3), 382 - 392.

Mevarech, Z., Stern, D. and Levita, I. (1987) *To cooperate or not to cooperate in CAI: that is the question*, Journal of Educational Research 80, 164 - 167.

Trowbridge, D. (1987) *An investigation of groups working at the computer*, Berger, K. Pezdek and Banks, W. (eds) Applications of Cognitive Psychology: Problem Solving, Education and Computing, Hillsdale, N.J.: Erlbaum

Pre-algebraic thinking: range of equations and informal solution processes used by seventh graders prior to any instruction

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Filloy and Rojano (1984) have suggested that a demarcation point exists between algebra and pre-algebra, a demarcation point they have called the **didactic cut** *which occurs in first degree equations when the unknown appears on both sides of the equation. Our study based on individual interviews of a whole class of 22 seventh graders shows that the notion of didactic cut is a valid one but that it must be defined in terms of a cognitive obstacle, not a mathematical form. We suggest that it be defined as the* **inability to operate with or on the unknown.** *Our data shows that the didactic cut exists even when the unknown occurs twice on the same side of the equal sign, and even when the unknown occurs only once but as a subtrahend or as a divisor.*

One of the most interesting ideas in this field has been presented by our Mexican colleagues who have suggested that one can find a sharp demarcation between arithmetic and algebra as evidenced by what they called the **didactic cut**, that is, the the occurrence of the unknown on both sides of the equality symbol in a first degree equation in one unknown as in $ax + b = cx + d$ (Filloy & Rojano, 1984). In their research on the didactic cut, they have investigated the impact of some instruction sequences involving models that would provide meaning for operating with or on the unknown. They have used the classical balanced scale model and another original model based on the equivalence of distinct areas (Filloy,1987, Filloy & Rojano, 1989, 1985a,b, 1984, Gallardo & Rojano, 1987).

The students that were studied in the above research were subjected to some instruction regarding the solution of equations involving the single occurrence of the unknown: they were taught to use inverse operation(s). It is after such instruction that they were examined on the solution of more complex equations involving the double occurrence of the unknown ($ax + b = cx$) and given further instruction using the geometric or the balance models.

It is difficult to judge from this research how seventh graders might respond to the task of solving equations **prior to any** instruction whatsoever. Although partial answers to this question have been provided by Carolyn Kieran (1984,1981), both her samples of subjects (6 and 10 respectively) and the number of equations tested (14 and 9 respectively) were too small to bring out any general trends. In prior assessments dealing with this topic, little attention was paid to equations in which the unknown appeared as the subtrahend or the divisor (e.g. $273 - n = 164$; $525 : n = 15$) or to equations in which the unknown appeared twice but on the same side of the equality symbol (e.g. $11n + 14n = 175$). Although Kieran used one of each type in her work, her results did not indicate whether or not the students operated on **or with** the unknown. More generally, the relationship between the **form** in which an equation appears (using concatenation between letter and numeral, using the multiplication sign, or using a placeholder), its acceptance by the students, and the effect it might have on their success in solving equations has not been clearly established. To find answers to some of these questions was the objective of our investigation.

Methodology

We decided to interview a whole class of seventh graders in order to observe a wider range of mathematical abilities. Of the 27 students in the class, 23 were given parental authorization and were interviewed individually in two 45-minute sessions. Nine of these students (mostly the weaker ones) needed a third session lasting from 15 to 30 minutes. One of our subjects had to be dropped since her responses in the second interview could not be viewed as spontaneous for she had sought instruction from her father after the first session. The average age of the remaining 22 students was 12 years and 9 months. The class was chosen in a parochial school on the basis of easy access, a very cooperative staff that allowed individual students to be interviewed whenever needed, and especially, a classroom teacher who collaborated with us in avoiding any introduction of algebra prior to our investigation (except for the meaning of concatenation and numerical substitutions for literal symbols in the context of formulas) and also making all the

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essary arrangements. The classroom teacher provided us with the results of the students' formance in school mathematics in their regular tests and exams: 3 students had results better ... 90%, 8 students were in the 80% range, 5 in the 70% range, 5 in the 60% range, and 1 in the 50% range. Thus the class could be divided into two groups, 11 achieving 80% or better and 11 between 50 and 80%.

In order to focus the student's attention on the equation solving processes, we wished to eliminate most of the arithmetic drudgery and wanted also to avoid the pupil's problem of keeping track of his/her various attempts. Hence, students were strongly encouraged to use a calculator provided during the interview, a very simple calculator that did not take into account the order of operations. Each equation was presented on half a page with ample space for the students to write whatever they wanted. Students were asked to "think aloud" and the interviewer, sitting right next to the subject, wrote down each attempted solution on the side of the student's worksheet. The student was free to consult the interviewer's notes at all times. An observer was present during each interview in order to take notes and also to provide a second opinion in the evaluation of the student's responses.

Preliminary assessment

The function of the preliminary assessment was to gather some relevant information about each student. Questions were raised about their knowledge of the word "equation", the different uses of the equality symbol, the order of operations, their ability to view a string of operations globally, their familiarity with concatenation, and their preferred form in which an equation was presented (placeholder or letter, concatenated or not.)

Most students interpreted the word "equation" as an arithmetic operation and the examples they gave were written vertically. Two of the weaker subjects did not accept the use of the equal sign to indicate the decomposition of a number into a sum (e.g.34 = 19 + 15), whereas all students accepted its use to denote arithmetic equivalence (e.g. 15 + 7 = 10 + 12) as found before in Herscovics & Kieran (1980). Although the order of operations had been introduced in class, 17 of our 22 students (77%) gave us 110 as the answer to the first string of operations (5 + 6 x 10 = ?) thereby indicating that they first performed the addition. However, when working on the second string (17 – 3 x 5 = ?) 19 subjects found immediately the correct answer 2. Students were then challenged by the interviewer about their answer to the first string. All of them corrected themselves and mentionned that they had learned the mnemonic BOMDAS (Brackets first, Multiplication and Division, then Addition and Subtraction).

Another element we wanted to assess was the student's ability to perceive a string of operations more globally, by somehow distancing themselves from the string and taking so to speak an **overview** of it. We felt that this could be related to a more deliberate approach to equation solving. We asked students to evaluate 17 + 59 - 59 + 18 - 18 =? Only 5 students (three of them among the weak ones) saw that no operation had to be performed because of the cancellations, 4 more students perceived only one of the two cancellations, but 13 (59%) did not realize at all that cancellations were possible and proceeded to perform all the operations sequentially. The interviewer pointed out the cancellations and presented another string: 237 + 89 - 89 + 67 - 92 + 92 = ? Results were somewhat better with 8 students perceiving both cancellations, 8 more perceiving the first one (+89 - 89), but not the second one (-92 + 92), and still 5 subjects not being aware of any cancellation.

The final part of the preliminary assessment ascertained the students' familiarity with **concatenation.** All our students knew that 3n represented 3 x n and when asked what they would get if they replaced n by 2 or 5, all of them answered 6 and 15 respectively. This is in sharp contrast with the results obtained with students who had previously been exposed to concatenation but in a post test reverted back to an arithmetical frame of reference and therefore thought that the substitutions would yield 32 and 35 respectively (Herscovics & Chalouh, 1985). Finally, since we thought that the form in which an equation was presented might affect the student's ability to solve it, we gave each subject the choice of selecting the format he/she preferred. Among our 22 students, 12 chose the concatenated form (3n), 8 chose the form using the multiplication sign (3 x n), and 2 chose to work with placeholders (3×□).

Selection of equations

We wanted to submit to our students equations involving all **four operations**, as well as variations in the **position of the unknown**. It was conceivable that when faced with equations in which the unknown was the subtrahend or the divisor, the students might solve them by operating with or on the unknown. The **size of numbers** was also taken into account. Many equations where repeated in our list, the difference being the size of the numbers. Even for equations with smaller numbers their size was usually beyond the pupil's range of memorized number facts. However, some equations were repeated with three digit numbers in order to determine if we would witness a shift in solution strategies. This was a reasonable assumption since in many cases, students will use primitive processes on equations they find very easy and use their more sophisticated procedures when they feel they are warranted. Whenever the same type of equation was repeated, the one with the smaller numbers was presented first in order not to discourage the subject. For afterwards, when tackling the same equation with larger numbers, the availability of one solution procedure might induce in the pupil the freedom to search for a more efficient one.

Another variable that was taken into account was the **direction of the equation.** For each of the four operations, some equations representing the decomposition of a number into a sum or difference, a product or a quotient, were included in our selection. Prior to our preliminary assessment, we had no idea how well the students accepted the equality symbol to represent either an arithmetical decomposition into an operation or an arithmetical equivalence. We also varied the **number of operations.** We introduced 7 equations with an increased number of **arithmetical operations** to verify one of Kieran's early results (Kieran,1984) showing that some students were having difficulties in solving an equation involving several arithmetic operations (e.g 4 + n - 2 + 5 = 11 + 3 - 5). We also included in our list 8 equations in which the **order of operations** played an important role.

The last and most critical area of our investigation was the testing of equations in which the same **unknown occurred twice.** Since none of our students had ever seen such equations, we had to informed them that "solving" here meant "that the same number had to be used in each of the two occurrences of the letter n". We included 10 equations in which the unknown appeared **on the same side** of the equality symbol in order to assess if students would group the terms involving the unknown and thereby exhibit some ability to operate with or on the unknown. We also included 3 equations in which the unknown appeared on **both sides** of the equality symbol since 5 or all 6 of Kieran's pupils had not been able to solve such equations (Kieran, 1984).

Results

Since nearly all equations were solved by **all** our students, we will denote those that were not. We will focus our attention on the solution procedures that proved to be the most frequently used as well as on the specific difficulties encountered by the students with some of the equations. Our very high success rate might be due to several reasons. The first one is the availability of a **calculator,** the second one is the availability of the interviewer's note eliminating the student's need of keeping track. But we feel that the main reason is the fact that students were required to go through a **verification** procedure: once they thought they had found a solution, they had to write it on top of the unknown in the equation and verify its validity on the calculator. Of course, an incorrect solution induced the search for a correct one.

Addition and subtraction For the five equations involving addition the most common procedure was the expected use of the inverse operation. The first equation involving subtraction (eq.6) did not lend itself to a solution using an inverse operation. Out of 22 students, 16 **subtracted the difference from the minuend**: 37 - 18 = 19. Most of them tried to continue this procedure with the next equation. A quick verification led them to use the inverse operation which they also applied to the following equation. In the solution of equation 9, one student could not solve it for he was reading the equation from right to left and the solution he had found (23 + 37 = 60) when substituted for n made perfect sense to him since he read it as "60 minus 37 is 23" The same reading prevented this student to solve the following equation. Three other students read their solution from right to left but corrected themselves when asked by the interviewer to read the equation a second time. It is interesting to note that two students felt the need to re-write this equation as 37 - n = 23 and did so with equations 12 and 13. Reading backwards was also

the reason for one student failing to solve equation 11 and two students failing to solve equation. 13.

The following table indicates the equations, the most frequent procedure used and its frequency.

Equation	Most freq proc	%	Equations	Most freq.proc	%
1) 14 + n = 43	Inverse operat	82	26) 23+n+18=44+16	Grp & Invers	64
2) 35 = n + 16	Inverse operat.	77	27) n + 15 – 9 = 61	Group-Inverse	77
3) n + 596 = 1282	Inverse operat.	86	28) 39 + n – 12 = 74	Subst.-Approx	41
4) 437 + n = 984	Inverse operat.	86	29) 4+n–2+5=11+3–5	Grp-Subst.	41
5) 1269 = 693 + n	Inverse operat.	86	30) 4n + 17 = 65	Inverse-Inver.	41
6) 37 – n = 18	Subtr.differen.	73	31) 13n + 196 = 391	Inverse-Inver.	77
7) n – 13 = 24	Inverse operat.	77	32) 3n – 12 = 33	Inverse-Inver.	50
8) 17 = n – 15	Inverse operat.	77	33) 16n – 215 = 265	Inverse-Inver.	73
9) 23 = 37 – n	Subtr.differen	73	34) 420=13n+147	Inverse-Inver.	73
10) 273 – n = 164	Subtr.differen	73	35) 188 = 15n – 67	Inverse-Inver.	68
11) n – 872 = 167	Inverse operat	77	36) 6 + 9n = 60	Inverse-Inver.	41
12) 235 = n – 163	Inverse operat	86	37) 63 – 5n = 28	Sub.dif+no.fact	41
13) 364 = 796 – n	Subtr.differen	73	38) n + n = 76	Div.by 2	68
14) 16n = 64	Inverse operat	68	39) n + 5 + n = 55	Inver-Div.by 2	32
15) 32n = 928	Inverse operat	82	40) 3n + 4n = 35	System.Subst.	55
16) 2088 = 174n	Inverse operat	82	41) 9n – 4n = 20	Solv.first subst	59
17) 84 : n = 4	Divis.by quot.	77	42) 11n + 14n = 175	System.Subst.	82
18) 525 : n = 15	Divis.by quot	77	43) 17n – 13n = 32	System.Subst.	86
19) n : 6 = 13	Inverse operat	86	44) 5n + n = 78	System.Subst.	86
20) n : 8 = 57	Inverse operat	86	45) 7n – n = 108	System.Subst.	91
21) 15 = n : 7	Inverse operat	86	46) 7n+5n+7=55	System.Subst.	55
22) 23 = 115 : n	Divis.by quot	73	47) 3n+5+4n = 19	System.Subst.	36
23) n + 15 + 27 = 61	Group- Invers.	64	48) n + 15 = 4n	Solv.1st subst	59
24) n + 34 = 29 + 38	Group - Invers	73	49) 4n + 9 = 7n	System.Subst.	55
25) 14 + n + 17 = 50	Group - Invers	68	50) 5n+12 = 3n + 24	System.Subst.	86

Multiplication and division. The equations involving multiplication were solved by all. If only 16 students used the inverse for handling equation 14, it is because 4 other students simply used number facts However, in the next equation involving larger numbers, three of them used the inverse operation, while one of them performed systematic substitutions. Regarding the equations indicating division, students found these quite easy. Equation 17 and 18 were solved by dividing the dividend by the given quotient: 84 : 4 = 21 . The next three equations were handled using the inverse operation. Equation 22 proved to be more difficult with one student failing to solve it (again reading from right to left). Of the 16 students who used division by the quotient, 9 of them did so in their second attempt at solving the equation. Perhaps this is due to the fact that the previous three equations had been solved using the inverse operation. The two students who had problems accepting the use of the equal symbol for decomposition rewrote equations 16, 21, 22.

The results indicate that essentially two types of procedures are used. For equations in which only addition or multiplication are present, students will overwhelmingly revert to solve these by using the inverse operation, but the numbers in the equation must be large enough to go beyond known number facts. These results are very similar to those obtained by Kieran (1981) but differ from those of Gallardo & Rojano (1987) who found "some confusion between the various operations, interpreting addition as subtraction, subtraction as division".

For equations involving subtraction and division, one must distinguish between cases where the unknown is a minuend or a dividend from those cases where it is a subtrahend or a divisor. In the former cases, over 77% of the students use the respective inverse operations. In the latter cases, students respectively subtract the given difference from the given minuend or divide the given

dividend by the given quotient. At no time did we see any evidence of students directly performing operations on or with the unknown. There was never any observable attempt to transform 37 – n = 18 into 37 = 18 + n or to transform 84 : n = 4 into 84 = 4 x n . Thus we can conclude that students solve these equations by **working around the unknown** at a purely numerical level.

Grouping of numbers Equations 23 to 27 were solved by all students. Grouping the numbers on either side of the equal sign was the predominant procedure. Equation 23 was also solved by 4 subjects using inverse operations consecutively (61-15-27=19) and 2 students grouping the numbers on the left and then using known number facts. A similar procedure was also used by 3 students to solve equation 24. Using consecutive inverse operations was also used by 5 students on equation 25. While the predominant procedure used to solve equation 26 was to group the numbers on the right (60), then the numbers on the left (41), followed by an inverse operation (60 - 41 = 19), 3 students did not bother with this last step since they used a known number fact; 4 other students grouped the numbers on the right and then used consecutive inverse operations.

Procedures were markedly different for equations 28 and 29 Quite obviously the appearance of a negative sign was having an impact. Although only one students failed to solve equation 28, half the students (11) were not able to do so with equation 29. The most common procedure used to solve equation 28 was substitution and consecutive approximation (9 students); 7 others used consecutive inverse operations (74+12-39), while 4 subjects grouped the numbers on the left and then subtracted the result from the right. The presence of a negative sign seems to induce a major change in procedure if we compare those used here with those used to solve equation 26.

It is equation 29 which gave us the most startling results. However they compare with those of Kieran (1984) who found that 3 of her 6 subjects provided incorrect solutions. In our study, the 11 students who failed (5 in the upper half and 6 in the lower half of the class) did so as a result of grouping the numbers on the left incorrectly. They ignored the negative sign preceding 2 and simply focused on the following addition sign thus **adding 2 + 5** ! Having thus obtained 7 they ended up solving 4+ n -7 = 9 and found that n = 12 as their solution. Even after inserting the solution in the equation in order to verify, they maintained their erroneous grouping and hence were not in a position to detect their mistake. This unexpected detachment of a number from the subtraction symbol preceding it also occurred in our preliminary assessment and we will deal with it later on.

Additive and multiplicative operations . All equations were solved by all students excepting equation 35 (one student failed) and equation 37 (3 students failed).If the frequency of repeated use of inverse operations seems low for the solution of the first equation in the above table, it is because 8 students used a known number fact as soon as they compared 65 – 17 = 48 with 4n. Equation 31 in which the numbers are large enough shows that most students (17 out of 22) solved the equation by using consecutive inverse operations. To solve equation 32, 9 students used consecutive inverse operation, 3 started with an inverse operation and then used a known number fact, but 6 used a systematic substitution procedure. Equation 33 was similar to the previous one but the larger numbers resulted in 16 subjects using consecutive inverse operations and only 4 using systematic substitution. Nearly identical results were obtained for equations 34 and 35 although one student had to re-write them and change the direction of the equation.

Equations 36 and 37 proved to be more difficult. For equation 36, 9 students used consecutive inverse operations, 4 first used an inverse operation followed by a known number fact, and 3 used only known number facts. Interestingly, it is here that 5 subjects felt the need to insert brackets around the product, 4 of these were using a form of the equation showing a multiplication sign between the number and the literal symbol, the remaining student was using a form involving placeholder notation. For equation 37, 9 students subtracted the difference (63-28=35) and then used a known number fact in comparing the result with 5n; 6 students used the inverse operation to complete this last step. The three students who did not solve this equation were all reading the difference from right to left (as "5n – 63").

Our results differ from prior investigations. For equations with large enough numbers, Kieran (1981) found that only 2 of her 10 subjects were "undoing twice". Filloy & Rojano (1984) found that for two-step equations their high-ability subjects used a mixed strategy, "undoing the operation on the first step and applying 'plugging-in' (non-systematic trial and error strategy) or specific fact procedures on the second one". Our results show that with large enough numbers

[Eq]uations 31, 33, 34, and 35) the percentage of students using inverse operations consecutively [ra]nged from 68% to 77%.

Double occurrence of unknown on the same side All the equations (from 38 to 47) were solved by all students.Of course, since none of them had ever seen these before, the interviewer had to explain: **When we have the letter (or box) twice in the equation, it means that you have to replace each letter (or put in each box) with the same number.** The first two equations were found easy and did not require any sophisticated procedures: for equation 38 most students (15) simply divided by 2, 5 others used a systematic substitution process, and 1 student immediately used a known number fact. For the next equation, 7 students used an inverse operation (55 – 5) and then divided the result by 2; 6 students did not indicate any operation but simply used known number facts; 3 students performed the subtraction operation and then a known number facts; 5 students used a substitution procedure, 3 of them succeeded on their first try.

Equations 40 and 41 used only small numbers and thus we found that on the former one 12 students used a systematic substitution procedure and 8 others succeeded on their first try; for the latter equation, 13 subjects succeeded in their first substitution attempt whereas 8 others used a repeated systematic substitution. Three students felt the need to bracket the operations involving multiplication. However, it is with these equations that we witnessed the first appearance of an algebraic procedure: two students solved equation 40 and one of them solved equation 41 by grouping the terms in the unknown.

Equations 42 and 43 involved larger numbers. The most frequent solution procedure was that of systematic substitution (18 and 19 respectively). For equation 42, 2 students succeeded on their first substitution and 1 used a combined substitution and approximation technique. This last procedure was also used by 1 student to solve equation 43. The two students who had spontaneously discovered the possibility of grouping the two terms involving the unknown used them once again with these two equations.

Equations 44 and 45 were somewhat similar in that they involved a double occurrence of the unknown but one of the terms did not exhibit explicitly any coefficient. The most frequent procedure used was that of systematic substitution (19 and 20 respectively) while some other students succeeded on their first substitution attempt (3 and 2 respectively). It should be noted that the two subjects who had used an algebraic procedure on the previous equations did no longer apply it here or later in the remaining equations.

Equations 46 and 47 were slightly more difficult since a numerical term appears with the two terms involving the unknown. The most frequent procedure used to solve equation 46 was that of systematic substitution (13 students); 5 others first used an inverse operation (55 – 7) and then succeeded with their first substitution, so did two other students who did not bother to use the inverse operation; one student (other than the previous two subjects) used an algebraic procedure: she performed the first inverse operation, grouped the terms with the unknown and then divided by 12. Regarding equation 47, 8 students used a systematic substitution procedure and 8 others succeeded on their first attempt; 4 subjects used first an inverse operation and then a known number fact; the student who had solved the prior equation algebraically was able to do so again.

The set of equations in which the unknown appears twice on the same side (equations 38 to 47) gave us the opportunity to explore the way students would handle this new type of problem. Kieran (1984) had found that 5 of her 6 subjects solved incorrectly the equation $3a + 5 + 4a = 19$. This is in sharp contrast with the fact that all our 22 subjects solved the same equation (our equation 47). Equations 38 and 39 ($n + n = 76$ and $n + 5 + n = 55$) must be considered as trivial cases that were meant to familiarize the students with a double occurrence of the unknown and thus cannot be seen as representative. However, where the numbers used were large enough (equations 42, 43, 44 and 45), the preferred mode of solution was that of systematic substitution, the frequency ranging from 82% to 91%.

It is in this set of equations that we found isolated cases of spontaneous algebraic methods. One student was able to solve equations 40 to 41 by regrouping the two terms in the unknown, a second student did so only for equations 40 and 43, and a third student solved equations 46 and 47 by first performing an inverse operation and then grouping the unknown. It is noteworthy that none of these students extended the grouping procedure to equations 44 and 45 where one of the terms in the unknown did not indicate a coefficient. The sporadic incidence of this algebraic behavior prevents us from viewing these students having achieved any stable

algebraic knowledge. Instead, the overwhelming evidence points to the fact that grouping of the unknown is not a procedure that is acquired spontaneously. The possible explanation here is that it would involve operating with or on the unknown, a process that constitutes a natural cognitive obstacle in the algebraic development of the students.

Unknown on both sides Only 2 students failed to solve equation 48. One of these started random substitutions and hence could not develop any approximation; the other student very unexpectedly started assigning to the term 4n a place value interpretation thinking that upon substituting 5 4n became 45. In previous studies, we had seen this interpretation among novices who had not been exposed to concatenation (Chalouh & Herscovics, 1983). Due to the small size of the numbers in this equation, 13 students found the solution upon their first substitution; 7 others substituted more than once but in a systematic way.

For equation 49, 12 students used a systematic substitution procedure, whereas 7 others succeeded on their first attempt. The three students who failed used a systematic substitution approach but were unable to perceive an appropriate numerical pattern in their approximations. Our results differ from those of Kieran (1984) whose 6 subjects came up with incorrect solutions for the equation $4x + 9 = 7x$. Equation 50 was solved by all but 2 students. Among those who solved it, 19 used a systematic substitution procedure and one subject succeeded on his first attempt.

These three equations were more difficult to solve by substitution. Whereas in previous cases the systematic substitution yielded a sequence of approximations to a given number, this was no longer the case when the unknown appeared on both sides. The process of systematic substitution yielded a comparison of the two functions on either side of the equal sign and at some point the difference between the two was inverted and the respective numerical values became increasingly divergent.

Detachment from the minus sign As mentioned earlier in the analysis of the procedures used to solve equation 29, we found among some students a tendency to ignore the minus sign preceding the number 2 in $4 + n - 2 + 5 = 11 + 3 - 5$. In fact, **16 of the 22 students** did at one point add 2 and 5. However, 4 of them corrected their initial mistake during the verification of their solution. The detachment of the minus sign was not restricted to instances where a number was subtracted from an unknown for we found evidence of this in the preliminary assessment on the questions dealing with global perception of a string of operations. For the string $17 + 59 - 59 + 18 - 18 = ?$ we found that 6 students were ignoring the minus sign in front of -59 and adding 59 to 18, then re-inserting the minus sign to get –77. In the second string $237 + 89 - 89 + 67 - 92 + 92 = ?$ we found that 5 students were ignoring the minus sign in front of – 89 and instead were adding 89 to 67 ; but even more students (10) ignored the minus sign in front of – 92 and were adding 92 to 92 and then considering the sum as –184. Perhaps it is such a detachment of the minus sign that explains the great difference in the results on the two strings assessing the order of operations (17 students evaluated $5 + 6 \times 10$ as 110 but only 3 evaluated $17 - 3 \times 5 = ?$ erroneously).

This detachment of the minus sign was somewhat of a surprise to us. The high incidence of this mistake indicates that the problem is not idiosyncratic but may well reflect unsuspected cognitive obstacles (Herscovics, 1989). Perhaps this problem is somewhat induced by the introduction of the order of operations. By learning that multiplication and division take precedence over addition and subtraction, some students may conclude that addition takes precedence over subtraction or simply that they can start in the middle of a string.

Another plausible conjecture reaches more fundamental issues of a structural nature. Both Kieran (1989) and Booth (1989) have pointed out that students construct their algebraic notions on their previously acquired experience in arithmetic. As such, their algebraic system inherits the structural properties associated with the number systems known to them. This was often reflected in our investigation. Whenever students used procedures yielding decimal numbers they were convinced that these could not be the right answer. They explained that all the numbers used in our equations were natural numbers and thus they expected that the result had to be one too. Booth (1989) has suggested that "the students' difficulties in algebra are in part due to their lack of understanding of various structural notions in arithmetic". Of course, as seen in the procedures used to solve equation 29, these problems are compounded when the surface structure indicates that a number 2 has to be subtracted from an unknown. Unless the students know that they can use commutativity to obtain $(x + 4) - 2 + 5$ and then use associativity to perform $((4 - 2) + 5)$, they must resort to other means. Since our students had not seen negative numbers, the first term

in the expression $-2+5$ could not make any sense to them. The addition of 2 and 5 can then be seen as an attempt to transform the problem into a meaningful one.

Conclusion

Our study does bring new light to the upper limits of informal solution processes. The existence of a didactic cut as suggested by Filloy and Rojano (1984, 1987) has definitely been confirmed. But its definition needs to be refined. Filloy and Rojano define the didactic cut as "the moment when the child faces for the first time linear equations with occurrences of the unknown on both sides of the equal sign". The weakness of this definition is that it narrows down a cognitive obstacle to a mathematical form. Our results show that most of our students were able to solve such equations without any prior instruction. But more importantly, the solution methods they used when the unknown was on both sides of the equal sign were the same as those used for equations where the unknown occurred twice on the same side. This leads us to view the didactic cut not in terms of a mathematical form but in terms of a cognitive obstacle. We define **the didactic cut as the student's inability to operate with or on the unknown.**

That the terminology is appropriate follows from the fact that to overcome this obstacle some didactical intervention is necessary. In fact, Filloy and Rojano have succeeded remarkably well with their innovative geometric model. But this model, as well as the classical balance in equilibrium, have the specific purpose of endowing with some meaning some of the transformations involving operations with the unknown. By interpreting the didactic cut in terms of operations with or on the unknown, one can include equations where the unknown appears only once but involving more sophisticated number systems. It would be hard to imagine an untrained student solving equations such as $3 : (x + 2) = 1/2$ without operating on the unknown.

References

Booth, L. (1989), Grade 8 students' understanding of structural properties in mathematics, **Proceedings of PME-XIII**, Vergnaud, G., Rogalski, J. & Artigue, M. (Eds.), Paris, France, 141-148

Chalouh, L. & Herscovics, N. (1983), The problem of concatenation in early algebra, **Proceedings of PME-NA-V**, Bergeron, J.C. & Herscovics, N., (Eds.), Montreal, Canada, 153-160

Filloy, E. (1987), Modelling and the teaching of algebra, **Proceedings of PME XI**, Bergeron, J.C., Herscovics,N. & Kieran,C. (Eds.), Montreal, Canada, vol.1, 295-300

Filloy,E. & Rojano,T. (1989), Solving equations: the transition from arithmetic to algebra, **For the Learning of Mathematics**, 9, 2, 19-25

Filloy,E. & Rojano,T. (1985a), Obstructions to the acquisition of elemental algebraic concepts and teaching strategies, **Proceedings of PME IX**, Streefland,L. (Ed.), OW & OC, State University of Utrecht, the Netherlands, 154-158

Filloy,E. & Rojano,T. (1985b), Operating the unknown and models of teaching, in **Proceedings of PME-NA VII**, Damarin,S. & Shelton,M. (Eds.), Columbus, Ohio, 75-79

Filloy,E. & Rojano,T. (1984), From an arithmetical thought to an algebraic thought, **Proceedings of PME-NA VI**, Moser,J. (Ed.), Madison, Wisconsin, 51-56

Gallardo,A. & Rojano,T. (1987), Common difficulties in the learning of algebra among children displaying low and medium pre-algebraic profiency levels, **Proceedings of PME XI**, Bergeron, J.C., Herscovics,N. & Kieran,C. (Eds.), Montreal, Canada, vol.1, 301-307

Herscovics, N. (1989), Cognitive obstacles encountered in the learning of algebra, **Research Issues in the Learning and Teaching of Algebra**, Wagner, S. & Kieran, C. (Eds.), Reston, Virginia: NCTM, and Hillsdale, N.J. :Erlbaum, 60-86

Herscovics, N. & Chalouh, (1985), Conflicting frames of reference in the learning of algebra, **Proceedings of PME-NA VII**, Damarin,S. & Shelton, M. (Eds.), Ohio University, Columbus, Ohio, 123-131

Herscovics, N. & Kieran, C. (1980), Constructing meaning for the concept of equation, **The Mathematics Teacher**, vol.73, no.8, 572-580

Kieran, C. (1989), The early learning of algebra: a structural perspective, **Research Issues in the Learning and Teaching of Algebra**, Wagner, S. & Kieran, C. (Eds.), Reston, Virginia: NCTM, and Hillsdale, N.J. :Erlbaum, 33-56

Kieran, C. (1984), Cognitive mechanisms underlying the equation-solving errors of algebra novices, **Proceedings of PME-VIII**, Southwell et al.(Eds.), Sydney, Australia, 70-77

Kieran,C. (1981), Pre-algebraic thinking among 12 and 13-year olds, **Proceedings of PME-V**, Equipe de Recherche Pédagogique (Eds.) Grenoble, France, 158-164

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LOCI AND VISUAL THINKING

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LOCI is a software developed to give students the opportunity to acquire a qualitative, geometric, visual concept image of the notion of geometric locus. Classroom activities were designed to lead students from an initial acquaintance with the features of the environment to the stage where they were able to define and investigate their own loci.

One of ten items from a questionnaire administered after learning with LOCI was analyzed. The item required to identify drawings of loci that satisfy a given condition. Student responses were given on different levels of visual thinking. The most frequently employed ways of reasoning were categorized as local and global. Some students reasoned systematically in all, or most cases, but others changed their reasoning from one case to the other.

Background

In typical high school texts the concept of geometric loci is formally defined, explained and exemplified in a few lines. Then computational methods for finding the algebraic equations of a rather limited class of geometric loci are discussed at length. Clearly, textbook authors expect students to become proficient in these computational algebraic methods, and so do teachers. No attention seems to be paid to intuitive, qualitative and geometric aspects, which are at the center of the idea of a locus; the very essence of the concept of a locus remains hidden behind the computations. The process required from the student consists of analyzing a situation which is usually presented verbally, translating it into algebraic language and then performing computations according to formalized rules. Processes of intuiting, visualizing, exploring, conjecturing, defining, constructing and dynamically transforming, which are so important in mathematics, have no place in this activity.

The Software

With the aim of giving students the opportunity to develop a qualitative, geometric, visual concept image of the notion of geometric locus, we developed a computerized learning environment, named LOCI. When working with this environment, the student is fully in

control of his own activity. He may choose four kinds of action: Defining a locus, constructing points of the defined locus, conjecturing its shape, or transforming the locus by changing the data in the definition. These actions will now be briefly described.

Defining a locus: The user is asked to fix two geometric elements on the screen. These can be two points or two straight lines or a point and a straight line. Then, the user is asked to decide whether the locus should be determined by the sum, the difference or the ratio between distances from these two elements. To give a concrete example, one could decide on the locus of all points such that the sum of their distances from a given point and a given line is constant. The user is also asked to choose the value to be given to this constant sum. The two elements, the value of the distance between them, as well as a verbal formulation of the definition appear on the screen (Figure 1). To our surprise, we found that this rather basic setup leads to 64 different cases, depending not only on the choice of the elements and the relationship between the distances (sum, difference or ratio) but also on the relative position of the elements and on whether the chosen value is smaller, equal or greater than the distance between the elements. In other words, a well delimited but rich range of situations to be investigated has been created.

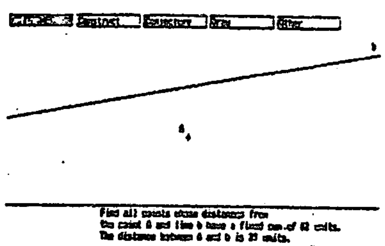


Figure 1

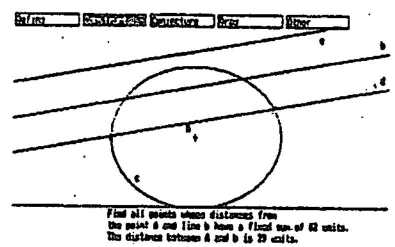


Figure 2

Constructing: Once a locus has been defined by the geometric conditions of its points, the student may attempt to construct points which belong to it. For this purpose, he/she may draw circles (of a chosen radius around a chosen point) and parallels (at a chosen distance to a chosen line), as well as mark intersection points between such circles and parallels. For instance, in Figure 1, one could intersect the circle around A with radius 50 with the parallels to b at distance 32 (Figure 2.). Finally, it is possible to erase the auxiliary constructions, but leave the

intersection points on the screen. This enables the student to check at a later stage whether these points are indeed points on the locus.

Conjecturing: On the basis of the constructions carried out and of qualitative geometric reasoning, the student may soon be able to venture an educated guess about the shape of the entire locus. The environment proposes a list of 22 choices for the locus, such as "a point", "an ellipse", "empty", "parabola segments" and "other". If the correct conjecture is made, the computer will draw the locus on the screen as well as describe it verbally. In our example, the locus consists of two parabola segments (Figure 3.). At any time, the user may choose to construct some more points or make further conjectures. If sufficient number of constructions have been carried out, it is also possible to obtain the locus on the screen without first correctly conjecturing its shape.

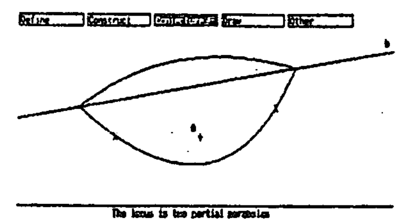


Figure 3

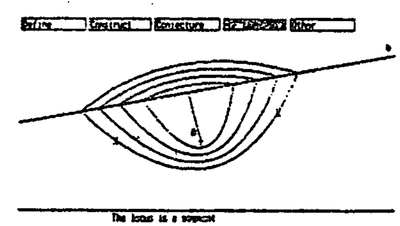


Figure 4

Transforming: After the locus has been obtained on the screen, it is possible to transform it dynamically by changing either the position of the two basic elements or the chosen value of the sum, difference or ratio of distances. In our example, one could, for instance, decrease the value of the sum of the distances from the point A and the line b until it equals the distance between the two given elements; at this time, the locus degenerates into a straight line segment (Figure 4). If the sum is further decreased, the locus becomes empty, and the computer will confirm this.

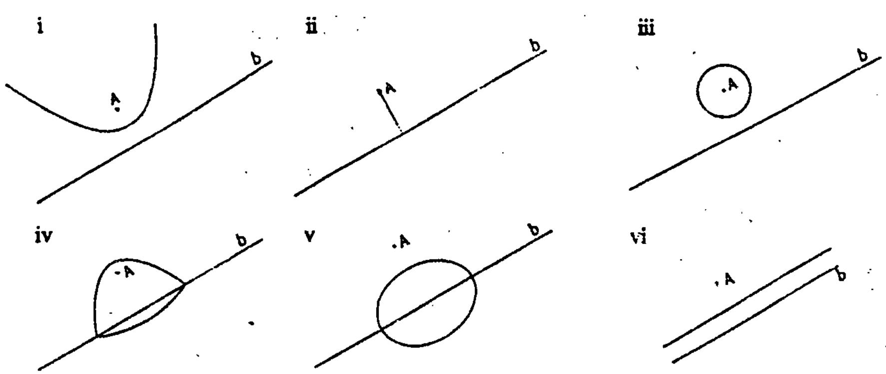
The Experiment

The LOCI learning environment has been used in two experimental settings: with an entire class (grade 10) and with a small group of students (grade 9). In both settings, students were guided by given worksheets and worked for about four class periods. During these activities, the students were led from an initial acquaintance with the features of the environment to the stage where they were able to define and investigate their own loci. The students built discrete points according to given definitions of loci, used their intuition to visualize their corresponding shapes and explored the transformations caused by changing the constants involved in the original definition. During the activities, the teacher acted as a consultant and at the summary stage, he led a class discussion.

Finally, a questionnaire was administered to the experimental class to check the students' understanding concepts related to loci. In order to illustrate the obtained results, we will analyze student responses on the following questionnaire item:

Given a line b and a point A.

Decide for each of the following drawings whether it is the locus of points with a fixed sum of distances from the given point and line. Explain your answer.



Drawing iv is the defined locus with the sum greater than the distance between A and b, and drawing ii is the locus when the sum equal to that distance. All the other drawings do not satisfy the given condition.

Table 1. presents some of the obtained data on student responses. As we can see, most of the students were able to match the correct drawing to the given definition.

Table 1. Students' responses to the item.

		Correctness of reasoning*				Type of reasoning**			
drawing	correct answer*	correct	partially correct	in-correct	no reasoning	"looks like"	local	global	else
i	91	65	22	9	4	-	32	46	21
ii	87	4	70	22	4	-	-	71	29
iii	100	65	9	13	13	-	18	68	14
iv	65	4	22	35	39	13	30	20	27
v	91	35	22	22	22	4	42	31	23
vi	100	61	13	13	13	4	17	71	16

*Percent from total number of students

** Percent from total number of reasonings

Ways of Reasoning

In most cases some kind of reasoning was given, but not necessarily a correct one. The percentages of the correct arguments were above 60 whenever the drawing did not correspond to the given definition (except in the case of the ellipse).

The number of correct arguments dropped when the drawing was the required locus. Here, even those who gave correct answers failed to consider both aspects of geometrical locus:

1. All the points on the drawing satisfy the defined conditions, and
2. Only these points satisfy the conditions.

Most of the students related only to the first aspect. For example; Amir said in case ii that the segment is the defined locus and explained:

" The sum of the distances of each point [on the segment] from the line [b] and the point [A] is equal to the distance between the line [b] and the point [A]."

Only Yiftah considered the second aspect as well. He wrote in case ii:

" There are no other points [besides those on the segment] whose sum of distances [from line b and a point A], equal to the segment's length."

These responses may be the result of the fact that the LOCI software emphasizes mostly the first aspect of the locus concept.

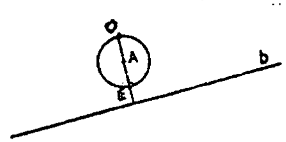
Most of the arguments were based on different levels of visual thinking:

a. Very few students used arguments like: *"Because this is how it looks to me"*. This may be considered a reasoning on the first van-Hiele level (judging the drawing by its appearance as a whole and not by analyzing its properties).

b. More than 70 percent of the reasoning (except in case iv), employed visual-analytical thinking. This sort of reasonings can be divided into local and global.

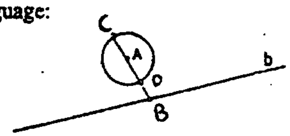
Local reasoning. This kind of reasoning is based on the analysis of one or a limited number of points on the drawing.

This reasoning is appropriate whenever the drawing is NOT an example of the defined locus, since then points can serve as counterexamples. For example, Tal explained in case ii:



[Circle A is not the defined locus, since] the distances of E and O from A are r, but the distance of O from b is greater than the distance of E from b.

A similar local reasoning was given by Yiftah as well, but he expressed it in a symbolic language:



$$AD + DB = AC + CB$$

Both students chose two points on the drawing and made an analysis based on visual considerations, to show that these points do not fulfill the required conditions of the locus.

Global reasoning. Many students used global considerations in the following three ways:

* One or several points on the locus were used as representatives of all the locus points. We may consider as examples Amir's explanation given above (where the drawing represents the defined locus) or Dudu's following explanation in case v (where the drawing does not represent the defined locus):

"Each pair of symmetrical points is at equal distances from b, but at different distances from A. Therefore, the sum will not be fixed".

This kind of reasoning reflects students' understanding of locus concept as the set of all points with a given property.

* Other students employed global reasoning by considering the drawing as a continuum of points. Amir, for example, wrote in case i:

"No! [the parabola is not the locus] because as we go up on the parabola, the distances from both A and b are increased."

Hilah explained in case iii (the circle):

"...because the distances from the point A are always the same and the distances from the line b are changing".

* There were students who related globally to different parts of the drawing. Tal gave the following reasoning in case v (the ellipse):

"The distances of points on the lower arc from b = to the distances of points on the upper arc from b. But the distances of the points on the lower arc from A are greater then the distances of points on the upper arc from A."

All the reasonings presented above show some level of visual thinking and except the first level all, except the first, were analytical as well.

Additional Results

Finally, we would like to present some additional findings.

* Almost all global reasonings that were given were correct or at least partially correct.
* Few students employed both local and global reasonings together in the same case.
* Other reasonings (included in Table 1. under "else"), were mostly some verbal repetition of the defined conditions. A few students based their reasoning on *"seeing similar cases"* during their work with the software.
* Some students reasoned systematically in all, or most cases, but others changed their reasoning from one case to the other. For example, Yiftah used global reasoning in case i, ii v and vi, local reasoning in case ii, and first van-Hiele level reasoning in case iv.

It seems that the level of the problem situation affects the level of reasoning -- the latter drops as situations become more complex.

TWO-STEP PROBLEMS - THE SCHEME APPROACH

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Introduction

Teaching two-step problems raises the question of what is analogous in the more complicated cases to the semantic categories found in the one-step problems (i.e., Change, Combine and Compare in the additive case, and Mapping Rule problems, Compare problems and Cartesian-Product problems in the multiplicative case).

Clarifying Terms Used in this Work

The main terms used are: "component", "underlying structure" and "scheme" (elaborated by Nesher and Hershkovitz in this volume). Let us start with an example of a one-step problem:

Problem 1: There are 12 tulips and 8 roses in a vase. How many flowers are there in the vase?

In this problem there are three components in the text, two of which convey the numerical information and one which is the question component.

1. There are 12 tulips in the vase.
2. There are 8 roses in the vase.
3. How many flowers are there in the vase?

Each component has two basic parts: the number and the description of the quantified objects. In the first component, the number is 12 and the description is "tulips in the vase". In the second component the number is 8 and the description is "Roses in the vase". In the third component we are missing the number (which is the target of the problem) but we know its description - "flowers in the vase". We will call the components that include both the number and the description, a "complete component" and the question component (description only without a number) the "incomplete component".

The incomplete component is crucial for understanding that Problem 1 calls for addition. If the third component would be "How many more tulips are there in the vase than roses?", the problem would be completely different (comparison rather than joining groups). Thus, we regard a simple word problem as a 3-place relation R(a,b,c), where a,b and c are the three components (propositions) in the text of the problem. We call this 3-place relation contained in each (one-step) word problem, the "underlying structure" of the text. Finally, the term scheme is used by us as a combination of two or more structures.

Categorization of Two-Step Problems

We would like to propose that the basic units of analysis should be a complete structure (a 3-argument relation) and not an operation or a single component. We propose only two building blocks, the additive structure and the multiplicative structure. Only three schemes can be constructed out of these two structures.

The three schemes are:

A. Hierarchical scheme.
B. Shared whole scheme.
C. Shared part scheme.

(according to Shalin's categorization, Shalin & Bee, 1985)

To illustrate, the generality of the schemes, and the two underlying structures let us examine Scheme B with all problems which can be derived from it. An example of a Scheme B problem is the following:

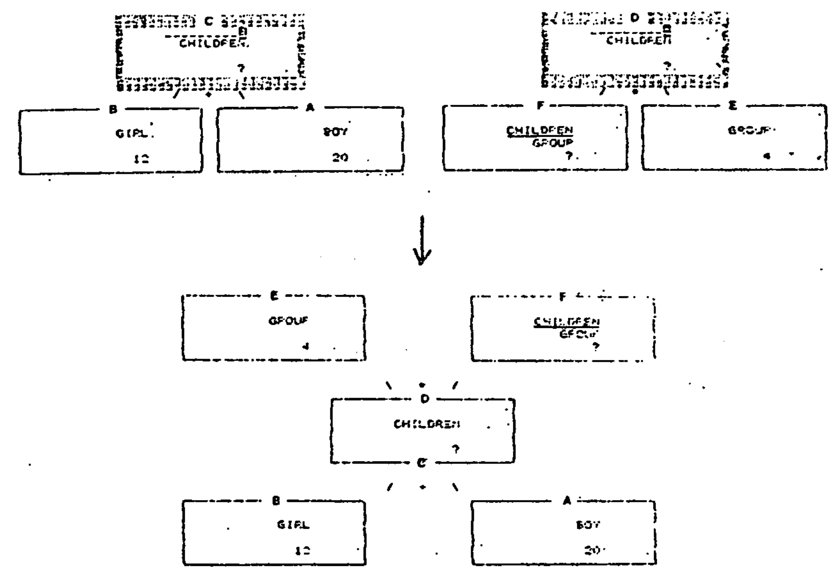
<u>Problem 2</u>: There are 20 boys and 12 girls in the camp. They are divided into 4 equal groups. How many children are in each group?

The components of Problem 3 are:

Additive structure
- [A] 20 boys in the camp (complete component).
- [B] 12 girls in the camp (complete component).
- [C] How many children are there in the camp? (latent component).

Multiplicative structure
- [D] There are {X} children in the camp (latent component).
- [E] 4 equal groups (complete component).
- [F] How many children are there in each group? (incomplete component).

The graphic representation of the structures and the scheme for this problem would be:

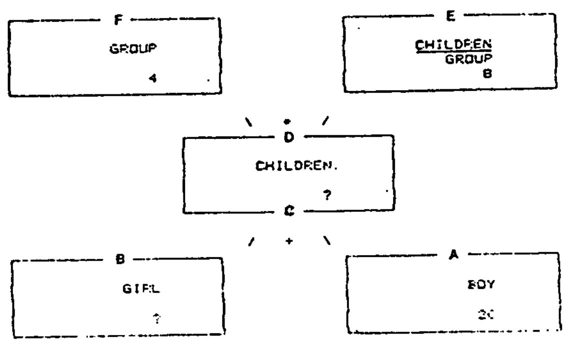


Note: Explicitly in the problem we have indentified only four components (three complete components and one incomplete component). We claim that each two step problem contains an additional latent component which should be deduced from the given text as can be seen from our analysis. The latent component in Scheme B, which is shared by both structures, is the sum (whole) of the additive structure and the product of the multiplicative structure. This is also the connection between the two structures. The situation described by Problem 2 can serve as a source for other three problems, all of which would be connected by the sum (or product) but would vary according to the location of the

incomplete component. We could ask for the number of girls, the number of boys or the number of groups, and each would form a distinct problem. For example, a problem asking for the number of girls:

Problem 3: There are 4 groups in the camp, 8 children in each group. There are 20 boys in the camp. How many girls are there in the camp?

The scheme for this problem would be the same - note the changed location of the question mark.



Implementation

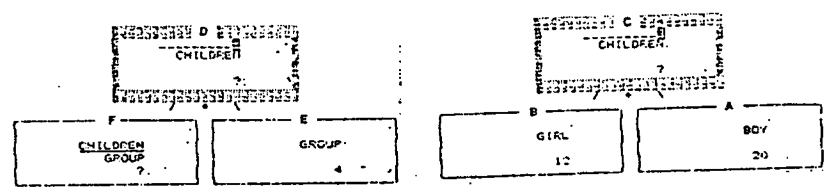
The above theoretical analysis led us to develop a computerized tool, named SPS (Schemes for Problem Solving), which applies the above analysis of two-step problems in the solution process performed by the student. The basic blocks for the student's activity are the structures, and the student makes the connection between structures to form one of the three available schemes

The system also provides immediate feedback concerning the relationship of the components in each structure and the connection between the structures.

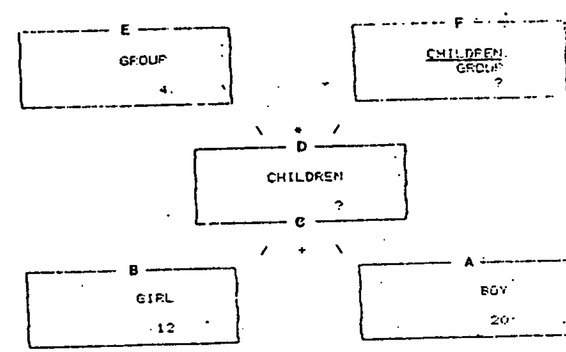
We would like to contrast our approach with three other computerized implementation one by Schwartz (1986); one by Thompson (1988) and the third by Reusser (1990). One basic difference exists between our program and the other three. Our building blocks, conceptually and practically, are only two

structures: the additive structure and the multiplicative structure. Our conception envisages each structure not as a mere collection of components, but rather as an expression of a 3-argument relation from which the different operations can be deduced. Actually, if 2 of the 3 arguments are complete, finding the third, the incomplete component, is determined by a specific operation. Thus when substituting a problem in a structure, one has to decide what is the role of each component before completing the structure. For example, let us see how the student will solve Problem 2 by using SPS.

First he has to decide that the problem involves a multiplicative structure and an additive structure. His next step will be to identify the role of the different components within each structure and to fill them in the correct places. Note that he will have to insert the latent component as well.



After completing the graphic representation of the structure he will choose the two identical components that are the latent. By overlaping the identical components, one above the other, he produces the graphic representation of the scheme.



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At this stage, the computer will ask about the correctness of the dimensions.

> You get from your structure
> that the addition of
> BOY & GIRL
> is equal to CHILDREN
> is that what you thought Y/N

> You get from your structure
> that the multiplication of
> $\text{GROUP} \cdot \dfrac{\text{CHILDREN}}{\text{GROUP}}$
> is equal to CHILDREN
> is that what you thought Y/N

Then the student will write down the mathematical sentence for the solution in component F [(12+20)/4].

If the student doesn't know how to produce the string in component F, he can move to component C and fill in the first operation (12+20). He - then moves to component F and fills in the second operation (32/4).

In the other three implementations, the building blocks are separate components that appear in the text. Given the components, the choice of the operation is the major task that needs to be accomplished by the problem solver.

For example, Problem 2 in Schwartz's program (The Algebraic Proposer, 1986, 1987) would be solved as follows:

First the student writes down the given explicit information, i.e., the three complete components. He does not need to analyze the role of each component, or the given information of the incomplete component.

	HOW MANY	WHAT		NOTES
A	20	boys	A	in the camp
B	12	girls	B	in the camp
C	4	groups	C	in the camp
D	◄		D	
E			E	
F			F	
G			G	
H			H	
I			I	
J			J	
K			K	

From here on the student will decide which is the appropriate operation, and the feedback from the program will be the queries

about the combination of units (dimensions) that will result from the proposed operation. The student then decides himself whether to continue or not.

> Quantity 20 boys
> Operation +
> Quantity 12 girls
> OK (Y/N) or ! to RETURN to pad

> boys and girls
> are both what
> ? children
> enter ! to start over

The next step will be:

	HOW MANY	WHAT			NOTES
A	20	boys		A	in the camp
B	12	girls		B	in the camp
C	4	groups		C	in the camp
D	32	children	A+B	D	
E	◄			E	
F				F	
G				G	
H				H	
I				I	
J				J	
K				K	
L				L	
M				M	
N				N	

Similarly to the above solution the student have to find out the second operation (division):

> Quantity 32 children
> Operation /
> Quantity 4 groups
> OK (Y/N) or ! to RETURN to pad

> the units of the answer are
> children /groups
> press ! to start over
> any other key to continue

	HOW MANY	WHAT			NOTES
A	20	boys		A	in the camp
B	12	girls		B	in the camp
C	4	groups		C	in the camp
D	32	children	A+B	D	
E	8	children /groups	D/C	E	
F	◄			F	
G				G	
H				H	
I				I	
J				J	
K				K	
L				L	
M				M	

In contrast, SPS does not monitor a single operation but rather a triple-relation, and therefore the feedback is deterministic

whereby the student cannot complete the computation before resolving the fit of the dimensions among the three components.

Although Thompson (1988) and Reusser (1990) use frames similar to Shalin's and the SPS for describing the components, they employ these components in a manner similar to Schwartz's mode, i.e., as two components combined through a binary operation to produce a third quantity, as they wish. As a result, they all deal with four basic operations (+,-,x,:), unlike the SPS which deals with two basic structures as undivided wholes.

At P.M.E, some observations of an experiment with the two approaches will be presented.

References:

Nesher P. & Hershkovitz S. (1991) "Two-step problems - research findings" this volume.

Reusser K. (1990): "From text to situation to equation Cognitive simulation of understanding and solving mathematical word problems". Learning and Instruction Vol 2.2 p.477-498

Schwrtz J.L. (1986): The Algebraic Proposer a mathematical enviroment for analysis, modeling & problem solving.

Schwartz J.L. (1987): "The Algebraic Proposer - a mathematical enviroment for analysis, modeling & problem solving". Published by True Basic, Inc.

Shallin V.L. & Bee N.V. (1985): "Structural Differences between Two-step Word Problems". Presented at the Meeting of the American Educational Research Association.

Thompson P.W. (1998): "Quantitative Reasoning and Algebra" Reasearch report funded by the National Science Foundation, Grant No. MDR 87-51381. Illinois State University.

EVALUATING A COMPUTER-BASED MICROWORLD: WHAT DO PUPILS LEARN AND WHY?

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As part of the Microworlds Project[1] we developed a computer-based microworld (the "Ratio and Proportion Microworld") and evaluated it formatively and summatively with an experimental class of 28 thirteen-year-olds. The summative evaluation indicated for the experimental class: significant improvement in correct responses; a shift in pupil strategies away from addition; and a more conscious and explicit awareness of strategy. Formative evaluation indicated the crucial role of off-computer activities in making computer-based strategies explicit.

Microworlds can be viewed as computational environments which 'embody' mathematical ideas. Conventional wisdom asserts that by exploration and use of the computer tools and reflection upon computer feedback learners will come to understand the mathematical structures and relationships which have been planted according to a priori learning objectives. Our experience suggests however, that mathematical learning tends not to be unproblematic. Key issues of debate centre on the degree of explicitness and timing of pedagogical intervention whilst maintaining a climate of pupil decision-making and exploration. For us, a microworld consists of software together with carefully sequenced sets of activities on and off the computer, organised in pairs, groups and whole classes designed with both the mathematics in mind and the pupils' developing conceptions of that mathematics.

We designed a ratio and proportion Logo-based microworld — the design features and the results of the pilot study were reported at the PME conference Paris 1988 (Hoyles, Noss and Sutherland, 1989(a); see also Hoyles, Noss and Sutherland (1989b). Modifications to the pilot study were also reported in these papers. Here we describe briefly the main study and report its results (a full report is given in Hoyles, Noss and Sutherland 1990).

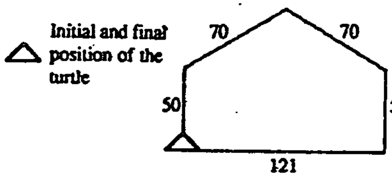
Two computational objects formed the basis of the technical component of the microworld. The first was HOUSE, a *fixed*[2] procedure drawing a closed shape (see Figure 1). The lengths of the sides of HOUSE were not related simply — there was no obvious multiplying factor connecting them. The second was a *general* procedure, LESLI, which drew a pinfigure (see Figure 2). LESLI was made up of variable subprocedures, pre-written according to proportional function relationships.

Using both HOUSE and LESLI, we intended that pupils would be encouraged to predict the outcome of running the procedures (with various inputs where appropriate) and to make patterns. We recognised that turtle-orientation and turtle-turn are sometimes sources of confusion for children with limited Logo experience (for elaboration of research into this and other aspects relating to Logo and mathematics, see Hoyles, Noss and Sutherland, 1986; Hillel and Kieren, 1987). These potential obstacles were avoided and pattern-

1 Funded by the Economic & Social Research Council, 1986-1989, Grant No: C00232364

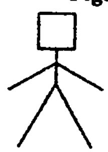
2 A procedure without inputs, in contrast with a *general* procedure.

generation facilitated by the provision of procedures JUMP and STEP which respectively moved the turtle (without drawing) up and across the screen if used with positive inputs. Activities were designed so that pupils would come up against visual conflict if they did not use proportional strategies for either enlarging and diminishing HOUSE (focussing on scalar operators), or whilst editing the procedure for LESLI to change the relationship between the parts of the body (focussing on function operators).




```
TO HOUSE
HT
FD 50
RT 60
FD 70
RT 60
FD 50
RT 60
FD 50
RT 90
FD 121
RT 90
END
```

Figure 1: HOUSE, a fixed closed shape



```
TO LESLI :SIZE
HEAD :SIZE * 2
NECK :SIZE
ARMS :SIZE * 2
BODY :SIZE * 2
LEGS :SIZE * 3
JUMP :SIZE * 3
END
```

Figure 2: Pinfigure LESLI

Methodology

For the main study, the microworld was taught by the researchers over a period of 6 weeks, $1\frac{1}{2}$ hours per week during the spring term of 1989. The school in which the research took place was a Secondary Comprehensive school near London. The experimental class consisted of 28, 13-year-old pupils in the third set of six (organised into 3 ability bands with two classes in each band). Set 1 was classified as the 'top' set according to school tests: thus the experimental group could be regarded as of 'average' mathematical attainment. The mathematics teacher had been a participant on the Microworlds Course (see Sutherland, Noss & Hoyles, 1990). Calculators formed a part of the normal classroom activities. Prior to the experimental work the children had experienced about 25 lessons in Logo of 35 minutes each and had undertaken a Logo project on tesselations. Thus, most of the children were relatively familiar with using the computers and with simple Logo syntax. Prior to the microworld experience, none of the children in the experimental class had been formally taught ratio and proportion in their mathematics classes.

The researchers were responsible for the teaching of the experimental class, for setting and marking homework assignments and giving feedback to the class teacher. Reciprocally, the class teacher was present at each session and gave us ongoing feedback. The rationale for this methodology was that the researchers wished to gain a 'feel' for the pedagogical constraints of teaching a whole class in school, as well as allow us to respond spontaneously to questions from the pupils and make immediate assessments of their reactions to the work.

We felt that gaining the vantage point of a teacher would enable us to give an interpretative gloss to the more formal microworld evaluation.

The microworld was evaluated summatively by:

- pre-, post- and delayed post- written tests for the experimental group, the same tests to be administered on each occasion[1]. The questions involved ratio and proportion but this was not made explicit.
- the same pre- post- and delayed-post-tests administered at the same three times to other classes in the same year group. These classes together are termed the *contrast group*. Because of the arrangements for setting by attainment in the school, the classes in the contrast group had different mathematics attainment levels than the experimental class. They too had no formal teaching of ratio and proportion during the period of research.
- audio-recorded interviews subsequent to the pre and post-tests with a group of pupils, called *case study pupils*, from the experimental class; case study pupils were chosen so as to obtain a spread of mathematics attainment (as judged by the mathematics teacher), a distribution of girls and boys which reflected the distribution with the year group, and a range of responses to the pre-test questions.

The progress of the case study children during the microworld was monitored by the researchers through observations and clinical probes during the activities (e.g. asking *why* they had adopted a certain strategy). This process data assisted in the interpretation of the pupil response profiles on the written tests and in the evaluation of he microworld as a teaching/learning sequence. In addition, dribble[2] files of these pupils' Logo work was collected.

The microworld was thus evaluated formatively by means of process data consisting of:

- observational notes on the case study pupils;
- marked homework assignments for the whole class administered after each session;
- computer printouts of the procedures written by the pupils within the class.

It is important to emphasise that we wanted to find the effect of the microworld by listening to the children, watching them work and trying to tease out their intuitions rather than by imposing a way of working. Thus at no time during the microworld experience was any attempt made to:

- transfer the meanings and understandings developed in the microworld to a non-computer context (or indeed to another computer context apart from that made available in the microworld);
- teach *rules* for working out proportion questions in any explicit or formal sense (such as the 'rule of three');
- teach pupils how to answer the questions in the written tests either by pointing out that there *was* an underlying proportional structure to these questions or by teaching a technique.

Overview of Microworld Sessions

The microworld consisted of computer-based activities (CB), class activities (C) (off the computer) and small-group activities (G) (on and off the computer). Both C and G activities began with discussion of a range of pupil responses (both correct and incorrect) to the activities. A summary is given below.

- *Computer-based activities (CB)*

i) Encouraging a sense of multiplication by decimals — the Target Game (involving iteratively reaching a set target number by multiplying by decimal numbers).
ii) Playing with HOUSE and building proportional HOUSEs (see Figure 1)
iii) Playing with LESLI and constructing new proportional 'families' (see Figure 2).
iv) Project with HOUSE and LESLI.

1 In the delayed post- test the values of quantities were altered slightly.
2 A dribble file records all keyboard interactions automatically on disc.

- *Class Activities (off-computer) (C)*

i) Exploring ideas of similarity and in proportion – the Classification Task (involving sorting figures according to the criteria of similarity).
ii) The meaning of in proportion — the "I" Task (confronting pupils with a range of responses to proportional questions).
iii) Enlarging HOUSE in proportion — the "Hole in the HOUSE" Task (involving discussion of the implications of using an additive strategy).
iv) Flow of Control in Logo (using the "Little People" metaphor)
v) Making proportional families — the construction of the PINHEAD Task (confronting the use of an additive strategy).

- *Small Group Activities (on and off-computer) (G).*

i) Enlarging HOUSE in proportion — Find the Method (where pupils are forced to discuss and defend their strategies used to construct proportional HOUSEs).

Summative Evaluation

The Tests: The pre- and post-tests were divided into two parts. The first part consisted of a *recognition* question involving the identification of a range of given rectangles which could be different-sized plans of a swimming pool of given dimensions. The second part involved the pupils in the *construction* of an unknown value given two measure spaces and three initial values where the situations 'demanded' a proportional relationship. Questions were set in two different contexts designed so as to be understandable to the pupils and aimed at provoking intuitions of proportionality: one context involved mixing paint (*paint context*) and the second enlarging photographs of rugs (*rugs context*).

Analysis of Correct Responses: The test papers from the pre-, post- and delayed post-test papers of the experimental group were marked, and correct and incorrect responses were recorded. Additionally we recorded instances when the response coincided with an answer which would have been obtained if the *addition strategy* had been used (i.e. when the relationship was seen as additive rather than multiplicative).

The first type of analysis involved looking only at correct responses. Totals and percentage totals of correct responses were calculated — for the tests overall and within the subcategories defined by the context (paint and rugs) and within the subcategories defined by whether the scale factor required could be an integer or had to be a fraction. The integer questions were divided into function or scalar according to whether the between-measure operator or the within-measure operator respectively was an integer.[1] Table 1 presents the raw data for the total number of correct responses per item. Table 2 presents the data divided into the four subcategories, paint integer, paint fraction, rugs integer, rugs fraction.

We begin by observing a noticeable overall effect: the data in Tables 1 and 2 suggest that the microworld experience led to an improvement in response but this was influenced by both context and scale factor. A repeated measures analysis of variance was undertaken for all the scores with respect to gender, time, context, scale factor and the interactions between these variables. The variable of time did not significantly interact with any of the other variables.

[1] If an item is designated, 'function integer', it means that if a between measures operator is chosen it would be integral.

However there was a highly significant improvement in response rate overall ($F = 13.90$, $p < 0.00001$, no. of degrees of freedom 2.52).

CONTEXT	SCALE FACTOR	NUMBER OF CORRECT REPONSES PER ITEM (Maximum = 28) PRE	POST	DELAYED
	Function Integer	12	20	21
	Scalar Integer	12	15	17
	Scalar Integer /	13	20	15
	Function Integer /	17	20	18
PAINT	Fraction > 1	4	11	9
	Fraction < 1	5	10	7
	Fraction < 1	2	6	7
	Fraction > 1	0	3	5
	Scalar Integer	18	24	25
	Function Integer	9	17	16
	Scalar Integer	21	22	24
	Function Integer	14	19	24
	Scalar Integer/Ans. Non Integer*	8	17	17
RUGS	Function Integer/Ans. Non Integer*	7	17	11
	Fraction > 1	3	10	5
	Fraction < 1	4	8	5
	Fraction > 1	3	11	9
	Fraction < 1	3	9	4

Table 1: Total Number of Correct Responses Per Item (max. possible = 28)
* These items are included in the sub-category rugs fraction in future analysis.

SUBCATEGORY		NUMBER OF CORRECT RESPONSES PRE	POST	DELAYED
Paint Integer	TOTAL	54.00	75.00	71.00
(no.of questions = 4)	%	48.21	67.00	63.39
Paint Fraction	TOTAL	11.00	30.00	28.00
(no. of questions = 4)	%	9.82	26.80	25.00
Paint	TOTAL	65.00	105.00	99.00
(no. of questions = 8)	%	29.02	46.90	44.20
Rugs Integer	TOTAL	62.00	82.00	89.00
(no. of questions = 4)	%	55.36	73.20	79.46
Rugs Fraction	TOTAL	28.00	72.00	51.00
(no. of questions = 6)	%	16.67	42.90	30.36
Rugs	TOTAL	90.00	154.00	140.00
(no. of questions = 10)	%	32.14	55.00	50.00
	OVERALL TOTAL	155.00	259.00	239.00
	%	30.75	51.39	47.42

Table 2: Total and Percentage Correct Responses (n = 28) in Subcategories Paint Integer, Paint Fraction, Rugs Integer, Rugs Fraction

Thus crudely, it is possible to ignore all other influences and eliminate all interactions and point to an overall programme effect comparable for boys/girls, across scale factors and contexts. There were also significant differences in terms of *context* [$F = 8.20$, $p < 0.01$, no. of degrees of freedom 1.26], *scale factor* [$F = 108.7$, $p < 0.00001$, no. of degrees of freedom 1.26], and one significant interaction between variables, *context* x *scale factor* x *gender*.

s last finding indicates context effects are to some extent dependent on gender or scale factor or both.

Building a Log Linear Model of Correct Pupil Responses: Our next step was to build a model using GLIM[1] which best fitted the data. This analysis indicated that the best model to describe pupil responses can be summarised as follows:

context + time + scale factor + context x *scale factor + context* x *operator*.

Thus, *context, time* and *scale factor* are significant influences on pupil responses, *but* the influence of context *cannot* be separated from the scale factor and whether the questions give preference to function or scalar operators.

Pupil Response Profiles: Having established that the microworld had an effect on the overall distribution of pupil correct responses to the written tests, we sought to focus on individual pupils and on any changes in individual pupil response profiles (pooling post and delayed-post tests). First, at the pre-test stage, we found that pupils with no idea as to the meaning of "in proportion" or who did not pick up context clues as to the structures underlying the question, adopted a "pattern-spotting" approach (that is, looked for patterns in the numerical values irrespective of any of their referents or underlying meanings). In the numerical paint context, the pattern-spotting strategy within integer questions tended to be adding; while in a rugs context, it tended to be non-adding as pupils were more influenced by visual clues gleaned from the context. In fraction questions, adding was the predominant response for both contexts. We interpret this latter finding as suggesting that pupils are fearful of fractions and decimals — even with a calculator available. In the face of this sort of obstacle, pupils will tend to revert to the easiest strategy, namely addition, particularly when they are not convinced that there is only one right method.

	INTEGER	FRACTION
PAINT	Majority of changes from addition to a correct strategy	Likelihood of change from incorrect (both add and other) to correct strategy approximately equal to change between incorrect strategies (from add to other and vice versa)
RUGS	Likelihood of change from incorrect to correct strategy (addition *not* dominant in pre-test)	Significant move away from addition, change equally likely from add to correct as add to other

Table 3: Pupil strategy changes between pre- and post-tests

An analysis of individual pupil profiles of responses within the four subcategories *paint integer, paint fraction, rugs integer* and *rugs fraction* revealed significant differences in profiles of response which is summarised in Table 3. We then endeavoured to look more generally at strategy changes using our interview data to assist in interpretation. We discerned a shift between pre- and post-tests from a perceptual to an analytic approach; from inconsistent or pattern-spotting approaches towards an awareness of the underlying mathematical structures and the consequent 'need' for a consistent approach; and from implicit strategies towards a conscious, explicit formulation of the

1 The software used was GLIM (General Linear Interactive Modelling Program) system, Release 3.77, obtainable from: Numerical Algorithms Group Ltd, NAG Central Office, Mayfield House, 256, Banbury Road, Oxford.

operation which had to be performed. Although these latter two shifts were influenced by context, we suggest that pupils had developed a 'feel' for proportional situations and were more convinced that a multiplicative strategy was 'better' than an additive strategy. These findings are illustrated in Table 4.

		MICROWORLD EXPERIENCE
RECOGNITION	Perceptual	→ Analytic
	Dominant use of Pattern Spotting	→ Mixture of strategies
PAINT	Dominant addition strategy	→ Consistent but implicit use of multiplicative strategies
	Little use of context clues	→ Explicit proportional strategies
	Some pattern spotting	→ Explicit proportional strategies
RUGS	Mixture of strategies	→ Consistent but implicit use of multiplicative strategies
	Some use of context clues	

Table 4: Changes in Pupil Approach

Formative Evaluation

In terms of development in understanding of ratio and proportion, our evaluation indicated that activities CB (ii) and CB (iii) were important in building intuitions of these mathematical ideas and alerting pupils to contextual clues[1]. However in the computer work with HOUSE and LESLI, the underlying proportionality of the activities is embedded and implicit: it stands at a second level of abstraction and feedback needs to be *interpreted* in contrast to the feedback in the Target Game (CB (i)). Thus pupils are not *necessarily* in a position to appreciate the feedback, if, for example, they do not have articulated intentions. They can be seduced by the 'picture' and can edit this to make it 'look alright' rather than reflect upon the way the picture was constructed. This points to the crucial significance of the off-computer class activities (activities C) and the Small Group Activities (activities G).

The microworld experience created a significant perturbation of pupil strategies evidenced by a move away from addition, towards *consistent* attempts, not always successful, to apply multiplicative operations. Pupils were less happy about changing strategies either across contexts or scale factors. There was also a shift towards *a more conscious and explicit awareness of strategy*, away from perceptive to more analytic approaches. This finding was particularly encouraging — even surprising — since there was *no* teaching for transfer of proportional ideas from computer to non-computer work. In particular, we observed a similarly surprising move (given that the

1 We believe that project work such as C (iv) is important but cannot infer this from our research since it was very curtailed due to time constraints.

computational work was largely in the graphical domain) away from incorrect to correct strategies in the non-visual paint context. We interpret this as a shift away from addition as the *default strategy*.

Our formative evaluation highlighted the importance of playful but carefully constructed computer-based activities to reveal and build intuitions and develop a language for representing these intuitions. We noticed that when pupils approached ratio items using Logo as a medium for their expression they developed strategies which *differed* from those observed in a paper and pencil setting — strategies which helped them to solve the problem by taking advantage of the computer. This *scaffolding* role of the computer needs more investigation. We also emphasise the crucial significance of off-computer activities, either in groups or in teacher-organised classwork in order to make the different computer-based pupil strategies explicit, to discuss any misconceptions which might not otherwise have been confronted and to validate those responses that are correct from a mathematical perspective. The role of discussion seems to have been particularly important in helping children come to terms with school mathematics discourse. We should point out here that we are not merely following the current fashion for discussion irrespective of what that discussion is *about*. On the contrary, a key facet of the discussions was that they used pupil responses as a starting point. Thus the discussions had a two-fold purpose: first to help children to *explicitly* confront mathematical notions and distinguish them from everyday ones; and second, to act as a window into pupils' thinking to enable us to develop sensitive activities within the framework of the microworld we had designed. Finally we point to what we see as the pivotal importance of the small group work — to serve as a bridge between pupil meaning and mathematical meaning.

References

HOYLES, C., NOSS, R., & SUTHERLAND, R., (1986), *'An Inset Programme Developing Microworlds for Secondary Mathematics'*, ESRC, Occasional Paper ITE/10/86, Research in Progress.

HOYLES, C., NOSS, R. and SUTHERLAND, R. (1989a) A Logo-based Microworld for Ratio and Proportion, *Proceedings of the Thirteenth Psychology of Mathematics Education Conference*, Paris.

HOYLES, C., NOSS, R. and SUTHERLAND, R. (1989b) Designing a Logo-based Microworld for Ratio and Proportion, *Journal of Computer Assisted Learning*, Vol. 5, No. 4, December, p.208-222.

HOYLES C., NOSS, R. & SUTHERLAND, R. (1990) *The Ratio and Proportion Microworld*. Report of the *Microworlds* Project, Volume 3. London: Institute of Education.

HILLEL J. AND KIERAN, C. 1987: 'Schemas used by 12-year-olds in solving selected turtle geometry tasks'. *Recherches en Didactique des Mathematique*, Vol. 8, 1.2 pp. 61-102.

SUTHERLAND, R., NOSS, R. & HOYLES, C. (1990) The Microworlds Course: An Approach to Computer-Based Inservice Teacher Education for Mathematics. In SCHUTYEN, G. *E.C. Logo Project: Teacher Training in Logo-Based Environments*

INNER FORM IN THE EXPANSION OF MATHEMATICAL KNOWLEDGE OF MULTIPLICATION

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ABSTRACT

Inner form is the mental rule which underlies pupil's reasoning. It is assumed to be a basis on which pupils expand their own mathematical knowledge. Focusing on the contrast with the basis of growth in mathematics, this paper aims to elucidate the notion of inner form, using multiplicative word problems. Interviews with four pupils show that each pupil has his/her own inner form. Some are related to physical systems in nature, others are related to abstract numbers. Furthermore, by examining the difference of performances between two pupils, the nature and role of inner form playing in the expansion is discussed.

1. Introduction -Form and Inner form-

We would like to explain what we mean by "form" first. On the process of growth, mathematics has recoursed to symbol representation as a driving force of its expansion. It seems that some properties represented symbolically are held while the others are more or less changed. Form is the represented properties held. For instance, "$a \div b = a \times (1/b)$, $b \times (1/b) = 1$" enables us to extend the divisor from whole number to fraction and to define division by fractions. In this case, represented property "$a \div b = a \times (1/b)$, $b \times (1/b) = 1$" is a form.

In mathematics teaching and learning, teachers often hope that pupils also expand their own mathematical knowledge, based on form. It is, however, difficult for pupils to do that. When considering the difficulty, in the situation where pupils are hoped to expand their mathematical knowledge based on form, we cannot ignore a phenomenon that they are apt to reason and to proceed by their own. "Inner form" is the mental rule which underlies pupil's such reasoning and which he/she continues to obey when being confronted with the novel situation. By "inner" here we mean "internal" or "of the mind but not overtly expressed". Inner form is expected to clarify how the pupil's own basis has relevance to form, for the purpose of expanding mathematical knowledge richly in the pupil (see Ito, 1990a, 1990b. In 1990a, capital letter FORM is used in place of inner form).

For instance, before recieving instruction from math teachers, many Japanese sixth graders compute $8 \div (3/5)$ as follows: $8 \div (3/5) = 3/(8 \times 5)$. One of those pupils says, 'I did so because I had learned to compute $(a/b) \div c = a/(b \times c)$'. His explanation indicates that he did not compute it haphazardly but invented the computation by applying his rule, "in division of fraction by whole

...er, put the whole number on the denominator". He carefully identified the ... as the property to be held and accepted it as applicable. As this example illustrates, it is plausible to suppose inner form in pupils' reasoning. This pupil expanded his mathematical knowledge of division based on not form but inner form.

The purpose of this paper is to identify inner form in pupils and to examine how inner form produces the expansion of mathematical knowledge. Our sample consists of five pupils. Since inner form depends on individual pupil, we need to shed light on each pupil. This is the reason for the number restriction.

2. Method

In this paper, we will approach inner form by searching pupils' behaviors, taking following two points into account:

Ⅰ. To set a developmental and problematic situation and to make pupils encounter to this situation;

Ⅱ. To notice expressions which pupils write on paper, especially to notice what kind of symbols and connections between symbols are seen.

Items The six items below were used in the situation of buying a stick.

> Imagine that your teacher asked you to buy a stick in a store.
> She also asked you to tell him the weight of the stick you bought.
>
> ① A stick weighs 5kg per meter. When you buy 3 meters of it, how much does it weigh?
>
> ② A stick weighs 0.2kg per meter. When you buy 4 meters, how much does it weigh?
>
> ③ A stick weighs 0.2kg per meter. When you buy 6 meters, how much does it weigh?
>
> ④ A stick weighs 4kg per meter. When you buy 0.5 meter, how much does it weigh?
>
> ⑤ A stick weighs 4kg per meter. When you buy 0.3 meter, how much does it weigh?
>
> ⑥ A stick weighs 8.2kg per meter. When you buy 0.4 meter, how much does it weigh?

These are classified into two types according to their structures(Table 1). TYPE Ⅰ can be solved using multiplication as "repeated addition". It is predicted that TYPE Ⅰ can be easy for the pupils. TYPE Ⅱ, on the other hand, cannot be solved by "repeated addition". It is impossible to imagine how to add four 0.3 times repeatedly instead of add four 4 times. The discrepancy might be called a kind of "didactical cut", as Filloy & Rojano(1984) refered in algebra learning. It is, therefore, predicted that pupils get confused terribly when they are confronted with TYPE Ⅱ items.

Table 1

Classification of items

TYPE Ⅰ		TYPE Ⅱ	
(I)×(I)	(D)×(I)	(I)×(D)	(D)×(D)
①(5×3)	②(0.2×4) ③(0.2×6)	④(4×0.5) ⑤(4×0.3)	⑥(8.2×0.4)

I:Integer D:Decimal

Sample and procedure Our sample consisted of five 4th graders at a public elementary school near Tokyo. Two of them were males and the others were females. Among them, Yuko is removed from the discussion, except for Table 2, because she learned how to mupliply decimals in advance.

For the remaining all pupils, they studied both multiplication and division of integers. They have enough ability to use their knowlege when handling word problems, as we will see below. They studied the concept of decimals and addition/subtraction of decimals as well. However, they have neither studied multiplication nor division involving decimals yet.

Each pupil was asked to solve each item in front of the author. Every time after he/she finished solving one item, an audio-taped interview was held. In the interview, the author asked some questions: e.g. "In what way did you find an answer?" "(pointing out specific handwritings)Why did you write this way?" "Would you explain your solution by pictures of a stick?" "How about making expressions? Explain why.". To complete the whole items each took 50-70 minutes.

3. Results

First, we would like to bring the response patterns for the six items of all five pupils in our sample. This is shown in Table 2. Table 3 shows the solutions each four pupil gave when he/she answered correctly.

The first point to be made is that the TYPE Ⅰ differ markedly in difficulty from the TYPE Ⅱ. ①-③ were solved correctly by all pupils, where ②, ③ will appear in their textbook later on. This shows that repeated addition is extremely easy, as is pointed out by many other researches (e.g. Bell et al,1984). With respect to④-⑥, however, pupils felt difficulty. For these problems, differences among pupils were remarkable and we saw their ways of reasoning and solutions by choices.

Table 2

Response Patterns for ①- ⑥

		①	②	③	④	⑤	⑥
Aki	(M)	○	○	○	×	×	×
Yuh	(F)	○	○	○	△	/	/
Nao	(F)	○	○	○	○	×	×
Masa	(M)	○	○	○	○	○	×
Yuko	(F)	○	○	○	○	○	×

○ :Correct answer ×:Incorrect answer /:No answer

△ :Correct with help of the author M:Male F:Female

Table 3

Solutions for correct answers

	①	②	③	④	⑤	⑥
Aki	A	A	A	×	×	×
Yuh	A	A	A	△	/	/
Nao	A	A	A	R	×	×
Masa	A	I	I	I	I	×

A :Repeated addition R :Ratio I :Change decimals into integers

When paying attention to basis which underlies their reasoning, we can see marked coherence within each pupil. In this regard, we can notice inner form each pupil has. Now we would like to identify inner form for each pupil.

4. Inner form underlying pupils' reasoning

Inner form identified were shown below. They are differet from one another in nature:

(Aki)Inner form relevant to the relationship among three physical systems

(Yuh)Inner form relevant to the conceptions of " ×"(multiplication) and "÷"(division)

(Nao)Inner form relevant to the relationship between two physical systems

(Masa)Inner form relevant to "integralization"

Explanations of these are given below. No special order observed.

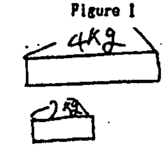
Inner form relevant to the conceptions of "×"(multiplication) and "÷"(division)

Yuh, in solving ④, first wrote 0.5×4 but put it out at once・・got stuck! She said, "it is strange because the result comes to less than 4." When told to use measure, she reasoned, "0.5 meter corresponds perhaps to this part・・we have to divide because it is a half ・・(silence) ・・ I wonder it might be 2 kilograms." After the interview, she said, "the problem asks for the weight of 0.5 meter・・So I was puzzled whether to multiply or to divide". We can catch a glimpse of something underlying her reasoning. Namely, <u>if a is the weight per meter, the result of a×b must be greater than a</u>. Not only in ①- ③ but also in ④, she continues to obey the rule above. This is the main cause of her puzzlement. We can predict that she will be puzzled again if she tries to solve ⑤.This rule is the inner form she has.

Inner form relevant to the relationship between two physical systems

Nao solved ④ by drawing a picture(see Figure 1). After drawing the picture, he said, "since 0.5m is a half of 1m, the answer must be a half of 4kg". Both his picture and what he said indicate a following rule underlying his reasoning: <u>Among four quantities (1m, xm, akg, bkg), suppose that 1m corresponds to akg and xm corresponds to bkg. Under the condition that 1:x equals to 1:n or n:1 (n:natural number), a:b equals to 1:n or n:1 respectively</u>. As we will see in the next section, in solving ⑤ and ⑥ for which he was puzzled, we can identify above rule behind his performance. The rule is always present as a basis for supporting his reasoning whenever he is confronted with a problematic situation. Thus, it may be called inner form.(In the discussion below, we will call it "IF(P2)".)

Figure 1



Inner form relevant to "integralization"

Masa solved ④ by 4×5 ÷10 and said, "make 0.5 five by multiplying by ten, compute 4 ×5 and get the result 20, and turn it back by dividing by ten and get the answer 2". In both his handwritings and what he said, we find the following rule underlying his reasoning: <u>To get the answer, we have to treat the expression involving decimals. In order to avoid inconvenience, change decimals into integers and compute in a more familiar way. Be careful not to forget to arrange the result according to the operation treated.</u> In solving ⑤ and even solving ⑥, as we will see in the next section, he continues to obey above rule. Thus, the rule always exists as the basis for his reasoning in a problematic situation. It is reasonable to call it inner form. (In the disccusion, we will call it "IF(I)")

Inner form relevant to the relationship among three physical systems

In solving ④, Aki thought that 0.5 meter equaled to 5 centimeters (this is wrong!) and that this was a problem of division. She then carried out computation by integralization: 0.5m ←→ 5m 5×4=20(kg) 20÷10=2(kg). However, when asked to explain it by picture, she became confused whether 0.5m equaled to 5cm or 50cm. Finally she changed her solution from 20 ÷10=2 to 20 ÷2=10. For ⑤, she said that it would be okey to take a half of 12 in the same way as above and computed 3m×4=12 12-6=6 6kg. For ⑥, she repeated the above

reasoning and computed 8kg2g×4=32.8kg 32.8+2=16.4 A 16.4kg.

Although Nao solved correctly at first, she finally choose dividing by 2 instead of 10. Why "2"? Her reasoning is difficult to understand. Therefore we need much discussion about her. However, for want of space, we omit it here and describe only the underlying rule that can be smelled out. Namely, Consider three systems: weight; length(the unit is the meter); length(the unit is the centimeter). Consider quantities in each system respectively: { s_i } ; { t_i } ; { u_i } . Suppose that s_i ,t_i ,u_i are corresponding one another; Under the condition that ratio of two quantities in one system equals to 1:n of n:1(n: natural number), ratio of corresponding two quantities in another system equals to 1:n or n:1 respectively.

5. Nature and role of inner form in the expansion of mathematical knowledge of multiplication

All four pupils doesn't know multiplicaion of decimals at all. In order to know both the reason for and the way of multiplication of decimals, they need to change the meaning of multiplication from "repeated addition" to "multiplication based on proportionality". This is a hard work and in fact, pupils were puzzled when solving TYPE II. Nevertheless, they strived for resolving the confusion and overcoming the difficulty. Of course, they cannot solve all six items correctly. Even though they are halfway in overcoming the difficulty, they begin to know the multiplication involving decimals. This is the reason that we use "expansion" here.

In this section, we consider how inner form identified above enable the expansion of mathematical knowledge of multiplication. Therefore, we examine how inner form causes the difference of performance among pupils. Because of paper restriction, we would like to examine only the difference between Nao and Masa in detail.

The main difference between the performances of Nao and Masa is that Nao didn't give the correct answer for⑤ while Masa did. This difference can attributed to the difference of nature between IF(P2) and IF(I).

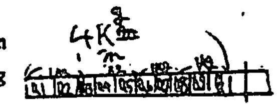
We identified IF(P2) for Nao. IF(P2) is related to physical quantities in two systems, especially to ratios formed within a system ($p_1:p_2=q_1:q_2$). Freudenthal(1983) designates this kind of ratios as internal. According to Freudenthal, IF(P2) is of an internal-ratio nature. IF(P2), however, is too restricted in the sense that $p_1:p_2(q_1:q_2)$ must be 1:n or n:1 and it doesn't care for ratios between two systems. Depending on this inner form, Nao tried to solve ⑤ as follows.

He first used a measure and moved his finger as indicated in Figure 2. He

said, "1 divided by 3 equals 3 with remainder 1・・". He then said that he had to measure the length that corresponds to 1kg. All these suggested that he did make efforts to find "n" in the IF(P2). Since 0.3 doesn't divide 1, Nao was at a loss and attempted to obtain the intermediate result "0.25m corresponds to 1kg". The reason why he thought of this correspondence is clearly explained by his effort to find "n"(in this case n=4) in IF(P2).

Figure 2

Figure 3



He then tried to compensate the difference between 0.25m and 0.3m. In order to do that, he derived 1.05kg as the answer by using addition(1+0.05). As was shown by previous researches (e.g. Vergraud,1983), on this kind of complex problem, pupils often confuse whether to multiply or to add. Since IF(P2) is of the ratio nature, his reasoning for 1.05 seems to have nothing to do with IF(P2). Nevertheless, in his performance, we can find IF(P2) as a basis of his choice of addition instead of subtraction. In thinking out 1.05, Naò was gazing at his picture(see Figure 3). His reasoning seems to be that "since 0.3m is greater than 0.25m comparatively, we must add the difference(0.05) to 1kg".

In this way, IF(P2) underlies Nao's reasoning and contribute toward emerging and resolving (even failed) his problematic situation.

Masa, who was identified IF(I) on the other hand, tried to solve ⑤ as follows. When solving ⑤, he first tried to multiply both weight and length by ten and to divide the weight by the length: 4×10=40 0.3×10=3 40÷3=13...1. After confusing for a while, finally he multiplied only the length by ten and gave the correct answer: 4×3=12 12÷10=1.2. About the final expression he offered, he said as follows. "First change only 0.3 into three・・・and 4×3 equals to 12・・・then turn 12 back by dividing by ten and get the answer."

In the reasoning above, we can easily find IF(I) or integralization. IF(I) is related to numbers that appeared in each problem rather than to physical quantities in two systems. As a proof of this, Masa tries to change decimals into integers regardless of the weight or length the decimals express. Since IF(I) is related to abstract numbers and number structures, it transcends specific physical meanings. This indicates some difference between IF(I) and IF(P2). We would like to notice that, because of this difference, Masa could cope with ⑤ rather successfully.

Furthermore, it is interesting to see that IF(I) is rather independent of choice of operations. In fact, Masa first believed that this was a problem of division, which led him to reason that he must multiply both two numbers by ten simultaneously. He didn't become aware for a while that it was a problem of

multiplication.

As we discussed above, the difference of nature of inner form subtlely causes the difference how mathematical knowledge expands. As a result, some pupils make errors while others do well without errors. Thus, the nature of inner form plays a crucial role in causing and overcoming difficulties.

6. Concluding remarks

The purpose of this paper is to elucidate the notion of inner form as a model of expanding process of pupils' mentality. Although the number of subjects examined are small and the analysis is not sufficient, we find inner form for each pupil that serves as a basis and supports his/her reasoning. In addition, interwiews with four pupils show that assuming that inner form is present has a possibility of seizing pupils' learning process. Every pupil expands their mathematical knowledge by his/her own inner form. Whatever the expansion is, it accompanies some restrictions. What is matter is that some inner form causes expansion that accompanies too much restriction. Inner form, therefore, affect mathematics learning deeply.

Investigating the nature and role of inner form has important implications for teaching. In Japan, teachers treat TYPE I and TYPE II quite differently. TYPE I problems are usually taught based on "repeated addition", while TYPE II are taught based on the methods of integralization. On the other hand, each pupil tries to expand by his/her own way. We should notice the crucial discrepancy underlying between teacher and pupils. We need to investigate each pupil's inclination more thoroughly and to understand what he/she really want to know. In this regard, inner form gives us a perspective.

References

Bell, A., Fischbein, E., and Greer, B (1984) 'Choise of operation in verbal arithmetic problems: the effects of number size, problem structure and context" in Educational Studies in Mathematics, 15 pp.129-147

Freudenthal, H. (1983) 'Ratio and proportionality' in Didactical Phenomenology of Mathematical Structures p.183

Ito, K. (1990) 'A characterization of the process of aquiring a mathematical concept by FORM-The case of YUKI' in Tsukuba Journal of Educational Study in Mahtematics, Vol 9-B pp.31-40

Ito, K. (1990) 'The conception of inner form: Nature and role in learning mathematics' in Proccedings of PME 14 p.225

Vergnaud, G. (1983) 'Multiplicative structures' in Acquisition of Mathematics concepts and Processes, Academic Press 127-174

SOME IMPLICATIONS OF A CONSTRUCTIVIST PHILOSOPHY FOR THE TEACHER OF MATHEMATICS

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The questions central to this paper are:

> When a mathematics teacher espouses a (radical) constructivist philosophy, what implications for the classroom arise, and what issues have to be faced?

The paper draws on an ethnographic study of the teaching of secondary mathematics teachers whose philosophy of teaching and learning mathematics is compatible with constructivism[1]. It focuses on issues which have arisen in this study which seem germane to teaching more generally, in particular the role of the reflective practitioner.

Background

Implications of a constructivist philosophy for mathematics education have been discussed extensively (eg von Glasersfeld, 1987a,b, Cobb, 1988, Lerman, 1989). Explorations of teaching, mostly at primary level, have sought to explore these implications in more practical terms (eg. Steffe, 1977, Cobb, Wood and Yackel, 1990). However, little seems to have been done at secondary level, and this is where my own study fits.

The study involved six secondary teachers, two in each of three phases, who, in my judgement, employed an *investigative approach* to teaching mathematics (Jaworski, 1985). Very broadly, this carried connotations of exploratory tasks and a child-centred environment. The field work for each phase took place over half a school year during which I studied the work of the teachers with one class of their choice. Study methods were chiefly those of participant observation of mathematics lessons and informal interviewing of teachers and pupils. Phase 1 was a pilot study in which theory and methodology evolved, loosely-defined notions of an investigative approach became embedded in a constructivist philosophy of knowledge and learning, and appropriate styles of observation and interviewing developed. In Phase 2, analysis of data from one teacher (see Jaworski, 1988) led to the emergence of a theoretical model, the *teaching triad*, through which to view mathematics teaching. The model was shown to have generalisable potential by its use in analysing data from the second teacher in this phase. Phase 3 was used both to validate the teaching triad, and to explore more overtly issues arising for a teacher working from a constructivist philosophy.

The purpose of an ethnographic study might be seen to be to provide a 'thick' description (Lutz, 1981) of the culture observed, in this case that of the mathematics classroom. This may be regarded as an inappropriate source for generalisation beyond that particular culture. However, in determining salience and focusing on certain issues, a researcher brings a particular perspective to the study which is reflected in descriptions given or conclusions drawn. (See, for example, Furlong and Edwards, 1977) The rigour of such research lies in the production of a reflexive account (Ball, 1990) in which a researcher is explicit about the research process on which claims are based. From this researcher-as-instrument position, it is possible to suggest characteristics or relationships 'which may upon further investigation be found to be germane to a wider variety of settings' (Delamont and Hamilton, 1984). In my study it has been important both to justify characterisations of mathematics teaching with explicit accounting of situations from which they have arisen, and to suggest ways in which such characterisations might be more generally applicable. I have also sought to distil issues

[1]*Constructivism* throughout this paper should be read as *radical constructivism*. I follow the definition of this provided in von Glasersfeld 1987b

might be seen to arise from a constructivist approach to mathematics teaching more generally.

From theory to practice

There is a fundamental paradox in the linking of theory and practice which is implicit in my study. I set out to identify aspects of teaching and learning mathematics in classroom situations from which it would be possible to characterise an *investigative* approach, or an approach based in a constructivist philosophy, to the teaching of mathematics. From a constructivist theoretical standpoint, practical situations have been observed and data from them analysed and this has resulted in further theoretical positions. The purpose of this research was to speak about the *practice* of teaching, but ultimately what is presented is a theoretical perspective of the practice of teaching. Although one could argue that the resulting theory is closer to the practice which it seeks to inform than the theories on which it is based, a translation to classroom acts is still necessary. The study suggests that the teacher working from a constructivist philosophy will be a *reflective practitioner* (Schön, 1985), and indeed I would claim that the translation of theoretical principles to classroom practice can only be done by the reflective practitioner. I believe that one cannot speak of a constructivist *practice*. Practice is something which is done, and this requires the practitioner to make appropriate translations from theory. What this paper goes on to present are some of the conclusions of my research, together with practical manifestations of them, which might then be related to practice more generally.

Tensions of teaching

The interpretation of a constructivist philosophy in the classroom raises many issues for a teacher of mathematics, and my research had pointed to two over-riding tensions with which the teacher has to contend. One is the *didactic* tension of which John Mason has written (e.g. Mason, 1988) based on Brousseau's (1984) 'Topaze Effect', and 'Didactic Contract'. The second, which I have called the *didactic/constructivist* tension incorporates the 'teacher's dilemma' (Edwards and Mercer, 1987). My interest in these tensions goes beyond their identification, to their implications for mathematics teaching.

The didactic/constructivist tension

This lies in the dichotomy for teachers between recognising that pupils will make their own constructions of whatever they are offered in the classroom while mathematics curricula require certain formal constructions to be made by pupils and tested by the establishment. The former recognition results in a desire to create an ethos in which such constructing is broad, rich and open, whereas the responsibility of serving the latter makes apparently contradictory demands. All of the teachers with whom I have worked have identified this tension in some form. It is particularly overt in the situations below where the tension is not only manifested, but seen to be tackled in the classroom.

The context of the teaching triad

The teaching triad is a device, through which to view or to characterise mathematics teaching[2]. It involves close linking of three domains of teaching - *management of learning* (ML), *sensitivity to students* (SS) and *mathematical challenge* (MC). During my Phase 3 research, after some weeks of viewing the practice of one teacher, Ben, in terms of the teaching triad, I sought his view of the triad, asking, quite straighforwardly, what he would make of its three 'headings'. I did this at the end of our conversation following one lesson, and I left him to think about what I had asked. When we met again before the next lesson, he had jotted down some notes which he handed to me, with a global remark, "I feel that management of learning is my job as a teacher. I think that those (referring to SS and MC) are a part of management of

[2] For origins of the teaching triad see Jaworski, 1988. A paper providing detail and example of the triad and its relation to a constructivist philosophy has been submitted recently for publication. (see references) A copy can be obtained from the author.

learning. As a teacher, that's my role in the classroom - as opposed to managing knowledge." I have reproduced his notes, as faithfully as possible below.

Management of learning –	*as opposed to management of knowledge?*	
I like to be a manager of learning –		
my role : –	*organiser of activity or questions*	
	chairperson	*devil's advocate*
	challenger	*listener*
	learner	*making pupils aware of other pupils*
I am not a judge		
Sensitivity – feelings – threat (need for success)		
	(everyone should be able to start the activity)	
	success breeds success	
choosing the activity -	*level of difficulty chosen by the pupil - not today*[3]*!*	
to the needs of 30 pupils -	*what a challenge*	
to pupils by pupils -	*my role*	
Mathematical Challenge – everywhere	*- from the teacher good*	
	- from the pupils – and it takes off!	
But how do we get there?		

We spent considerable time discussing the detail of what he had written here, and this led into a discussion of the lesson which would follow our discussion - the *vectors* lesson.

A more didactic lesson than usual

The teacher said that the lesson would be 'more *didactic* than usual'. Situation 1, below, is the conversation which followed. T represents 'teacher', R 'researcher', myself.

Situation 1 - "Very didactic"

1	T	*Very didactic, I've got to say, compared to my normal style. But we'll see what comes out. There's still a way of working though, isn't there?*
	R	That's something that I would like to follow up because you say it almost apologetically.
	T	*Yeah, cos I ...*
		Yeah, I do. Erm
		We're back to this management of learning, aren't we?
	R	Are we?
5	T	*Can I read what I put here? (Refering to his written words on ML above) I put here, "I like to be a manager of learning as opposed to a manager of knowledge", and I suppose that's what I mean by didactic - giving the knowledge out.*
	R	Mm. What does, 'giving the knowledge' mean, or imply?
	T	*Sharing my knowledge with people. I'm not sure you can share knowledge. Mathematical knowledge is something you have to fit into your own mathematical model. I've told you about what I feel mathematics is?*
	R	Go on.
9	T	*I feel in my head I have a system of mathematics. I don't know what it looks like but it's there, and whenever I learn a new bit of mathematics I have to find somewhere*

[3] I believe this referred to the imminent *vectors* lesson, suggesting that one consequence of a more didactic approach was that pupils would have less opportunity to choose their own level of difficulty.

that that fits in. It might not just fit in one place, it might actually connect up a lot of places as well. When I share things it's very difficult because I can't actually share my mathematical model or whatever you want to call it, because that's special to me. It's special to me because of my experiences. So, I suppose I'm not a giver of knowledge because I like to let people fit their knowledge into their model because only then does it make sense to them. Maybe that's why if you actually say, 'Well probability[4] is easy. It's just this over this.', it doesn't make sense because it's got nowhere to fit. That's what I feel didactic teaching is a lot about, isn't it? Giving this knowledge, sharing your knowledge with people, which is not possible?

Ben seemed to be saying that if we offer probability to pupils as simply a formula, "this over this" it is likely to have little meaning for pupils because they have no means of 'fitting' it into their experience. Ben's statement uses 'fit' in the von Glasersfeld sense (eg von Glasersfeld, 1987a), and statement 9 seems as clear an articulation of a constructivist philosophy as I would be likely to find 'off the cuff' in a discussion of what a lesson was going to be about. Two aspects of this conversation stand out:

1) the result of my probing at statements 4 and 6. This caused Ben to define what he meant by *didactic* teaching - 'giving knowledge out' - perhaps a *transmission* view of teaching'. His words (statements 7 & 9) seemed to eschew both a transmission view, and also some absolute view of knowledge, indicating a relationship between the building of knowledge and a person's past experience.

2. the third sentence in statement 1, "There's still a way of working though, isn't there?" I understood by this that, despite using a didactic approach, he believed there might nevertheless be a way of working which would fit with constructivist views.

In response to Ben's words above, I pushed harder towards what I saw as being a fundamental tension, the *didactic/constructivist tension*, of didactic approach versus constructivist philosophy.

Situation 2 - "A conjecture which I agree with"

1	R	I'm going to push you by choosing an example. *Pythagoras* keeps popping up, and Pythagoras is something that you want all the kids in your group to know about. Now, in a sense there's some knowledge there that's referred to by the term 'Pythagoras'. And, I could pin you down even further to say what it is, you know, what *is* this thing called Pythagoras that you want them to know about?
	T	*My kids have made a conjecture about Pythagoras which I agree with. So, it's not my knowledge. It's their knowledge.*
	R	How did they come to that?
	T	*Because I set up a set of activities leading in that direction.*
5	R	Right, now what if they'd never got to what you class as being Pythagoras? Is it important enough to pursue it in some other way if they never actually get there?
	T	*Yeah.*
	R	What other ways are there of doing that?

He laughed and, after a brief pause, continued.

	T	*You're talking in the abstract which then becomes difficult, aren't you now? Because we're not talking about particular classes or particular groups of pupils etc. Because I've always found in a group of pupils if I've given them an activity to lead somewhere there are some pupils who got there. It sounds horrible that. Came up with a conjecture which is going to be useful for the future if I got there, yes? And then you can start sharing it because pupils can then relate it to their experiences.*
	R	So, it's alright for them to share with each other, but not alright for you to share with them?
10	T	*If I share with them I've got to be careful because I've got to share what I know within those experiences.*
	R	OK. So, if we come back to didactic teaching then, if you feel they're at a stage that you can fit -whatever it is that you want them to know about - into their experience, isn't it then alright? You know, take the probability example this morning. If you felt ...
12	T	*That is nearly a definition, isn't it? That is, I suppose that's one area I'm still sorting out in my own mind. Because things like $\overrightarrow{AB}$ and vector are definitions. What work do you do up to that definition?*

[4]I had watched him take a lesson, on probability, earlier that day, for another teacher who was absent. This had involved helping pupils work with an exercise involving rules of probability which the other teacher had set the class. He was unhappy with pupils' use of the rules, believing that they were being used instrumentally rather than conceptually.

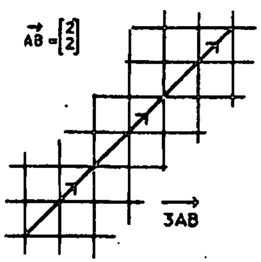
The tension seemed to be between having some particular knowledge which he wanted pupils to gain (like rules of probability, or Pythagoras' theorem, or vector notation) and the belief that he could not give them the knowledge. The above conversation seemed to summarise his pedagogical approach - the presentation of activities through which the pupils could construct knowledge, and his monitoring of this construction, "My kids have made a conjecture about Pythagoras which I agree with. So, it's not my knowledge. It's their knowledge." Implicit in this is his need to know about their construction, to gain access to their construal, "a conjecture about Pythagoras which I agree with". Pupils have to be able to express their thoughts in a coherent way for the teacher to make this assessment, so he has to manage the learning situation to encourage such expression. In Situation 1 he distinguished between being a *manager of learning* and a *manager of knowledge* (statement 5). In Situation 2, statement 12, he referred to a *definition*. The probability example involved a definition, as did the notion of vector and its representation as $\overrightarrow{AB}$. His, "I'm still sorting out in my own mind" seemed to refer to the status of a definition in terms of knowledge conveyance or construction. There seemed to be some sense in which you could only *give* a definition. If this is the case, what preparation needs to be done so that pupils are able to fit that definition meaningfully into their own experience? This seems to be an example of the *teacher's dilemma* (Edwards & Mercer 1987) - pupils construct their own meanings; the teacher offers something from which they can construct. There is some concept which the teacher needs to elicit or to inculcate. However, inculcation is likely to result in lack of meaning, and eliciting of what the teacher wants may never occur. (For further discussion, with further examples from this teacher, see Jaworski 1989)

Manifestations from the vectors lesson

The *vectors* lesson followed the above conversation. It involved teacher and pupils working together on meaning of vector, vector notation and the length of a vector. I interpreted the teacher's words about the didactic nature of the lesson to mean that he would be offering definitions and rules of which the pupils would need to make sense. However, in the beginning this was not the case. Some aspects of vectors had already been introduced to the pupils and the teacher began by asking them to express what they understood by certain forms, such as $\overrightarrow{AB}$ or $3\overrightarrow{AB}$. Pupils came out to the board to explain and draw. Others commented. There was a lot of talk between pupils themselves. There were many questions for the teacher.

The vector, $\overrightarrow{AB}$, had been described, by pupils, as 'a journey from A to B', and the teacher had accepted this description. When $3\overrightarrow{AB}$ was being discussed, someone asked, "But, where is *B* in $3\overrightarrow{AB}$?" After some discussion in which there were many suggestions from pupils, it became

clear that the teacher needed to provide some answer himself to this question. As he spoke he drew a diagram on the board:



"There's the vector AB. There's another vector AB, and there's another vector AB. And so I've got three vectors; three lots of AB.

I think at some point we have to go away from this idea that the vector is a journey from A to B, to a point where a vector is - this line, this quantity; and to think of that as the vector AB, as that line, not necessarily a journey."

This was followed by further discussion in the class regarding forms of notation and their meaning. Then again the teacher interjected, referring to notation on *vector* and *vector length*, written on the board:

> "We're just talking about the ways a mathematician writes things down, yes? We're not learning anything really new. Those two are different. ($\vec{AB}$ and $\vec{BA}$). Those two are the same (AB and BA). You tend to write the first one down because they're in alphabetical order, and we rarely write down BA, yes? We're just talking about what mathematicians write."

The notation was not negotiable. Pupils were required to understand and use these forms of notation. I saw the teacher providing opportunity for pupils to consider their meanings through expression and discussion of their own images and perceptions. This allowed the teacher to monitor their construal of the vector notation, and where necessary try to correct mis-interpretations by offering his own account. This seemed to me totally in keeping with a constructivist philosophy since he was overtly encouraging pupils to make sense of what was on offer. Although he was in some sense offering them his own knowledge - 'giving this knowledge, sharing your knowledge with people, which is not possible' (Situation 1, statement 9) he was nevertheless trying to fit it to the pupils' own experiences - 'to share what I know within those experiences'(Situation 2, statement 10).

The Didactic Tension

This may be expressed as follows: a teacher needs to make clear to pupils what is required of them, but, 'the more the teacher is explicit about what behaviour is wanted, the less opportunity the pupils have to come to it themselves' (Mason, 1988) As with the didactic/constructivist tension, the didactic tension is bound up in encouraging pupil construal of what the teacher values. Manifestations of this tension occurred throughout my study and one example, from the vectors lesson, follows.

A task for pupils to undertake

After the discussion described above, the class were set a task concerning vectors and their lengths. The teacher said, "I would like you to make your own questions up and write your own answers out and then share your questions with a neighbour. Could you be inventive please. Don't put up a whole series of boring questions"

There were some pupils who found it hard to make a start, wanting the teacher to be explicit about what he wanted them to do, what sort of questions he wanted. He mostly resisted spelling out particular questions, although he helped pupils to get started in some cases. He said later of one pupil,

> "Jessica was floundering. She wanted questions given to her and she wasn't getting them. She asked me three times for a question and I said the same thing three times. And then she says, 'What's Becky doing?' I said, 'Ask her'. 'What's Nicky doing?', 'Ask him!' She was trying everything to get a question out of me."

We had discussed the implications of this task before the lesson. Ben felt that most of his pupils would be able to tackle the task, but in doing so might not tackle all the different situations which he might cover by giving them a particular set of questions. I commented, after the lesson,

> "I think there are questions about management of learning in here, in the way this was done. I mean, in deciding that you would go along with the idea of letting them make up their own questions you were allowing for the possibility that the questions they made up wouldn't include all the different cases that you might have included if you'd set them an exercise. And yet if you set them the exercise they don't get the chance to think it through and investigate for themselves."

In the event, most pupils engaged with the task, and many came up with interesting lines of enquiry. There were many questions on vectors which had not been tackled, but which Ben, possibly because of his syllabus, *wanted* to tackle. However, the setting of an exercise with all such questions would not guarantee that pupils would gain what the teacher wanted.

The reflective teacher

Striving to encourage, provoke or enhance meaning-making seems a fundamental objective for the teacher. Creating opportunities through exploration and questioning, and encouraging of negotiation and expression of ideas are approaches which can serve this objective. The interpretation of these terms in the mathematics classroom cannot be prescribed, but the experiences of teachers such as Ben suggest that the teacher has simultaneously to explore and question the practice of teaching mathematics alongside their pupils' exploration of mathematics itself.

The phrase, "I'm still sorting out in my own mind" (Situation 2, statement 12) is one of many examples which this teacher gave indicating his continued working on his own practice. On another occasion we had been talking about planning and making judgements. Ben had asked, "How can a teacher make a right judgement then on all those things?" I asked, "Do you think it helps having made them explicit?" By 'them' I meant certain issues or concerns which affect planning or judgements. Ben said

> "I think it might change my judgements. That's me developing as a teacher, isn't it, changing judgements, changing directions."

On yet another occasion, when we talked about 'making explicit', in terms of what he felt about making particular aspects of his philosophy of learning explicit for pupils, he responded:

> "Well that's why I'm not very good at it. I'm not very good at it because I'm not sure in my own mind. I'm still developing what needs to be made explicit, and what doesn't."

As I probed his beliefs and motivations I was often struck by the contrast between his views of teaching confidently expressed, like those on management of learning above, and his recognition of areas of question, where perhaps he was still exploring his approach.

The paradox in linking theory and practice might be resolved by a symbiotic relationship between the teacher as theory maker and the teacher as developer of practice. Traditionally, teachers have not seen themselves as theory-makers, and have often spurned theory offered by researchers as irrelevant to their practical task. However, all the teachers I worked with had their own theories which varied in degree of explicitness. Often the conversations which we had resulted in these theories becoming more explicit. Sometimes the teachers found my probing disturbing. Ben said that sometimes it threatened his professionalism.

Conclusion

The tensions elaborated above are a theoretical synthesis from my analysis of classroom data. They have arisen from much discussion with teachers of situations in which we have participated and the teacher's own motivations and beliefs. For each of the teachers concerned

they take different forms and raise different questions. Each teacher works on these questions in their own way. More work needs to be done in studying this development of practice. Where the teachers in my study are concerned I have evidence that our work together contributed to their own thinking and development. However, this was more to do with repeated discussion and application of ideas than with overt theory building. Yet I firmly believe that the teachers were building theories, even if these were not expressed as formally as the ones in this paper.

Perhaps the most significant implication of a constructivist philosophy for the teaching of mathematics is the recognition that such tensions will be there and must be tackled, rather than any particular issue or tension which the teacher must face. Teaching itself needs to be an exploratory task. What activities, what classroom approaches, what classroom ethos will support both ideals and responsibilities? What is gained and what is lost in particular cirumstances? What can be learned from particular experiences?

In my study, classroom manifestations cannot be separated from the theory which has developed from them. Theory is sterile without the manifestations to provide substance. Manifestations are *not* examples of theory - they complement theory by indicating the richness and diversity of teaching situations, whereas examples suggest a completeness which denies metamorphosis.

References

Ball S.J. (1990) 'Self doubt and soft data : social and technical trajectories in ethnographic fieldwork'. in Qualitative Studies in Education, Vol 3 no 2, 157-171

Brousseau, G. 1984, 'The crucial role of the didactic contract in the analysis and construction of situations in teaching and learning mathematics.' In H.G. Steiner (ed.) *Theory of mathematics education*. Occasional paper 54, Institut fur Didactik der Mathematik, Universitat Bieleveld.

Cobb P. 1988 'The tension between Theories of Learning and Instruction'. In Educational Psychologist, 23(2) 8/-103. Lawrence Erlbaum Associates Inc.

Cobb, P. Wood T. and Yackel E (1990) 'A constructivist approach to second grade mathematics', in E. von Glasersfeld (ed.) *Constructivism in mathematics education*. Reidel, Dordrecht, Netherlands

Delamont S. and Hamilton D.(1984) 'Revisiting classroom research: a continuing cautionary tale' in Delamont S. (Ed) Readings on interaction in the classroom, Methuen, London pp 3-24

Edwards D. and Mercer N. (1987) *Common Knoweldge*, Methuen, London

Furlong V.J. & Edwards A.D. (1977) 'Language in classroom interaction: theory and data", in Educational Research, Vol 19, No 2, pp 122-8

Jaworski B. (1985) *An investigative approach to teaching and learning mathematics*, The Open University, Milton Keynes

Jaworski, B. 1988, 'One mathematics teacher' in *PME XII proceedings*, Vesprem, Hungary

Jaworski, B. 1989, 'To inculcate versus to elicit knowledge' in *PME XIII proceedings*, Paris

Jaworski, B. *Mathematics teaching: what is it?* Unpublished manuscript available from the author.

Lerman, S. 1989, 'Constructivism, Mathematics and Mathematics Education', *Educational Studies in Mathematics 20*, 211-233 Kluwer, The Netherlands

Lutz, F.W.. (1981) 'Ethnography: the holistic approach to understanding schooling", in Green J. & Wallat C. (Eds) Ethnography and Language in Educational Settings, Ablex, (Vol V in the series Advances in Discourse Process, ed. Freedle R.)

Mason, J.H. 1988, 'Tensions' in D. Pimm (ed.) *Mathematics, Teachers and Children*.. Hodder & Stoughton, London

Schön D.A. (1987) *Educating the reflective practitioner*, Jossey-Bass, Oxford

Steffe, L.P. 1977 *Constructivist models for childrens' learning in arithmentic*. Research workshop on learning models, Durham N.H.

von Glasersfeld E. 1987a 'Learning as a Constructive Activity'. In C. Janvier (Ed) *Problems of representation in the teaching and learning of mathematics*. Lawrence Erlbaum, Hillslade, NJ.

von Glasersfeld E. 1987b 'Constructivism'. In T Husen & N. Postlethwaite (eds) *International encyclopedia of education*: Supplement Vol 1. Oxford: Pergammon.

TEACHERS' CONCEPTIONS OF MATH EDUCATION AND THE FOUNDATIONS OF MATHEMATICS

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The profiles of teachers across three conceptions of the foundations of mathematics (logicism, intuitionism, formalism) and across three conceptions of mathematics education (conveyance, development, negotiation) were examined. A questionnaire consisting of items requiring forced-choice rank-ordering into one of the three conceptions was administered to 82 mathematics teachers. Results of cluster analysis show that four distinct profiles can be identified and characterized across the three conceptions of the foundations of mathematics. However, these four profiles did not exhibit clear characterization across the three conceptions of mathematics education.

Teachers of mathematics seldom have the need or the opportunity to articulate or demonstrate explicitly their conceptions of the foundation of mathematics in their professional life as teachers of mathematics. However, questions such as the following linger in the back of the minds of teachers and students: What is the relationship of mathematics to logic, language, thought or reality? What is the basis of axioms, proofs, consructions in mathematics? What is infinity? What is the subject matter of mathemtics? This is not to imply that such questions linger idly in the mind, but rather to suggest that the answers teachers give to such conceptions may affect their conceptions of mathematics education and their teaching behaviors in the classrooms.

It is hypothesized that teachers' conceptions of the foundations of mathematics form coherent "perspectives" which can be described in terms of aspects of the foundations of mathematics as derived from three known major schools of philosophy of mathematics i.e. logicism,

itionism, and formalism. The rationale for this hypothesis is that t, the three schools have existed historically as theories advanced by prominent mathematicians in this century; second, these philosophies present highly contrasting perspectives from a metamathematical point of view and different programs in mathematics itself; and, third, the contrasting perspectives of the three schools rest upon few simple and elemental ideas. Hence, we expect that teachers differ metamathematically i.e. have different conceptions of the foundations of mathematics without assuming that they are aware of the technical differences in the different mathematical programs. A second hypothesis is that teachers' with different conceptions of the foundations of mathematics have different conceptions of mathematics education. Recently three contrasting perspectives of mathematics education have been suggested (Voigt, 1989): Conveyance perspective (teaching of mathematics is the transmission of meanings of an external body of knowledge by an expert to a novice), developmental perspective (teaching of math is helping the internal cognitive structures to unfold), and, negotiation (teaching is the negotiation of meanings through social interaction).

It is not possible in this paper to give any meaningful account of the foundations of mathematics from the points of view of logicism, intuitionism, and formalism. Rather we mention a basic reference (Benacerraf and Putnam, 1983) which includes a selection of the original writings (or their translation to English) of the proponents of logicism (Russell and Carnap); intuitionism (Brouwer and Heyting); and formalism (Hilbert and Neumann). The relationship of these three schools to mathematics education has been only discussed theoretically (Ernest, 1985; Lerman, 1983). However, a number of empirical studies on teachers' conceptions of the nature of mathematics and its teaching have been reported (Thompson, 1984) and Middleton, Webb, Romberg, and Pittelman (personal communication, 1990). However, no study has been reported on teachers' conceptions of

the foundations of mathematics specifically as it is based on existing philosophies of mathematics.

Procedure

Sample

The sample consisted of 80 teachers selected from prospective math teachers at the American University of Beirut and the math teachers in private schools in Lebanon. The private schools normally serve middle and high socioeconomic communities and form more than 50% of all education in Lebanon. The qualifications of the teachers in the sample vary from completion of three years of university study to Ph.D. in math education. However, all subjects were involved in math teaching either at the pre-university level or at the level of teacher education. Teaching experience varied between 0 to 40 years.

Instrument

The development of the questionnaire passed in three stages. In the first stage statements which represent each of logicism, intuitionism, and formalism with regard to the foundations of mathematics were selected from the literature, put in a Lickert-type format, and given to a sample of teachers. The second stage was to revise the instrument. The challenge was to be able to have statements which represent each of the schools, have face validity to the teachers, and present clear contrasting options. Based on the experience of the second stage, it was decided to change the format to a forced-choice rank-ordering format. The third stage was to write the instrument whose final version consisted of eight items on the conceptions of the foundations of mathematics and eight items on the conceptions of mathematics education. Each item was made up of a stem and three choices. The stem represented an issue and the choices the three perspectives of the three schools of thought regarding that

issue. For each item, subjects were instructed to rank order the three choices according to their degree of agreement with their beliefs regarding that issue (1 for best agreement and 3 for least) and no two choices were to be ranked the same.

For the conceptions of the foundations of mathematics, the issues were: 1) Relationship of logic to mathematics; 2) basis of selection of axioms in mathematics; 3) nature of mathematical concepts; 4) establishing existence of mathematical objects; 5) subject matter of mathematics; 6) infinity; 7) nature of mathematical methods; 8) metaphorical description of mathematics.

For the conceptions of mathematics education, the issues were: Role of the teacher in the teaching of "meanings" of math concepts; 2) relationship of the teacher to the social and physical environment in classroom teaching; 3) dependence of the assessment of students' responses on the context and teacher; 4) explanation of gap between intended teaching and actual learning; 5) type of teaching behaviors while teaching solving problems in the classroom; 6) nature of conceptual framework employed in preparing for teaching a mathematical task; 7) universality of math concepts; 8) metaphorical description of the relationship between student and teacher in classroom teaching.

<u>Scoring</u>

Six scores subject was defined, each as the sum of ranks on the items belonging to each of the six conceptions. The maximum score for each conception is 24 and the minimum is 8.

An attempt was made to develop criteria for classifying a teacher as logicist, intuitionist, or formalist as far as the foundations of mathematics is concerned and as conveyer, developer, or negotiator as far as mathematics education is concerned. From a study of the distribution of responses, it can be demonstrated that if a score is at most 13 then the difference between the number of 1's (best agrees) and 3's (least agrees) is at least 3 out of 8 (i.e. 37.5%). Similarly, if a score is at least 19 then the difference between the number of 3's (least agrees) and 1's (best agrees) is at least 3 out of 8 (i.e. 37.5%). Accepting 37.5% as an adequately discriminating difference between agreement and disagreement with a certain perspective, the scores of 13 and 19 were set as cut-off scores. Consequently, for each of the six conceptions, a subject was classified as: a) belonging to the category represented by the conception if the score is at most 13; b) not belonging to that category if the score is at least 19; and, c) not classifiable if the score is between 14 and 18 inclusive.

<u>Statisitical Analysis</u>

Teachers were partitioned according to the pattern of their responses across the three conceptions of the foundations of mathematics using Ward's method of hierarchical cluster analysis (SPSS, 1988) with the scores on these three conceptions as the clustering variables. The profiles of the clusters thus extracted were examined in two ways. First, to characterize the clusters in terms of the perspectives of logicism, intuitionism, and formalism, and second, to examine whether the clusters differ in the conceptions of mathematics education.

Results

The cluster analysis produced four distinct clusters. The profiles of the four clusters across the three conceptions of the foundations of mathematics and three conceptions of mathematics education appear in Table 1. Teachers in Cluster 1 ($n = 31$) appear to disagree strongly with the formalistic conception of the foundation of mathematics (their mean 18.8 is not significantly different, $p < .01$, from the cutt-off point

Mean Scores and Standard Deviations on the Conceptions of the Foundations of Mathematics and Mathematics Education for the Four Clusters

	Foundations of Math						Mathematics Education					
	Logicism		Intuition		Formalism		Conveyance		Develop.		Negotiat.	
Cluster	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
1	14.0	1.6	15.1	1.0	18.8+	1.1	16.3	2.6	14.8	2.0	16.8	2.4
2	16.9	1.2	15.1	1.6	15.7	1.1	17.3	2.7	14.5	2.0	16.1	2.3
3	12.7*	1.4	18.8+	1.4	16.5	1.4	17.8	1.9	14.3	2.1	15.9	2.0
4	18.2	0.6	16.8	1.3	11.8*	1.3	19.2+	2.1	12.6*	1.7	16.4	1.3
All	14.5	2.5	16.2	2.2	17.0	2.2	17.2	2.6	14.4	2.2	16.3	2.2

* not significantly different from 13 ($p < .01$)

\+ not significantly different from 19 ($p < .01$)

Table 2
Cross-Tabulation of Cluster Membership by Different Coneptions of the Foundations of Mathematics

		Logicism		Intuitionism		Formalism		Unclassifiable
Cluster	n	+	−	+	−	+	−	
1	31	10(32%)	0(0%)	1(3%)	0(0%)	0(0%)	20(60%)	11(35%)
2	21	0(0%)	2(10%)	0(0%)	3(14%)	0(0%)	0(0%)	17(81%)
3	25	19(76%)	0(0%)	0(0%)	12(48%)	0(0%)	1(4%)	5(25%)
4	5	0(0%)	2(40%)	0(0%)	1(20%)	5(100%)	0(0%)	0(0%)

Note: +, − denote categories (according to the criterion) of subjects having and not having that conception respectively.

of 19), but do not appear to be classifiable in any category of the conceptions of matheamtics education (none of the means was significantly different from 13 to 19). Teachers in Cluster 2 (n = 21) do not reveal any commitment to any of the three conceptions of the foundations of mathematics or of mathematics education. Teachers in Cluster 3 (n = 25) agree strongly with the logicist conception of the foundations of mathematics ($\bar{x}$ = 12.7) and disagree strongly with the intuitionist conception ($\bar{x}$ = 18.8). With regard to the conceptions of mathematics education, teachers in Cluster 3 do not appear to be classifiable in any of the three categories. Teachers in Cluster 4 (n = 5) appear to identify strongly with the formalistic conception ($\bar{x}$ = 11.8), disagree strongly with the conveyance conception ($\bar{x}$ = 19.2) and agree strongly with the developmental conception of mathematical education ($\bar{x}$ = 12.6). Teachers in the total sample seem to subscribe more to the logicist conception of the foundations of mathematics than the intuitionistic and formalistic conceptions ($\bar{x}$ = 14.5 compared to 16.2 and 17.0 respectively). As for mathematics education, teachers in the total sample ranked themselves closer to the developmental conception than the negotiator and conveyer conceptions ($\bar{x}$ = 14.4 compared to 16.3 and 17.2 respectively).

Table 2 presents the results of the cross-tabulation of cluster membership by the positive and negative classifications (according to the criterion defined before) of the three conceptions of the foundations of mathematics. The results corroborate the profiles of the clusters in Table 1. Cluster 1 is predominantly in strong disagreement with formalism (65%), Cluster 2 is unclassifiable (81%), Cluster 3 is in strong agreement with logicism (76%) and in disagreement with intuitionism (48%), and Cluster 4 is in very strong agreement with formalism (100%).

Results of this study support a number of hypotheses. First, the responses of teachers on an instrument which requires self-reported rank-ordering on different aspects of the foundations exhibit distinctive profiles

which reveal coherent yet different conceptions of the foundations of mathematics. Second, the profiles thus generated can be characterized clearly and consistently in terms of the epistemological positions of logicism, intuitionism, and formalism. Profile analysis indicate that Clusters 1 and 4 are almost opposite with regard to agreement or disagreement with the formalistic and logicisticistic positions. Third, the profiles as represented by the four clusters do not seem to differ in their conceptions of mathematics education except for Cluster 4 which was identified as developmental non-conveyer. It is conjectured that the conceptions of the foundations of mathematics are more related to teaching behaviors than to self-reported conceptions of mathematics education which normally reflect the expectations of significant others of what constitutes "good" teaching. This conjecture calls for further research.

References

Bennacerraf, P., & Putnam, H. (Eds.) (1983). Philosophy of mathematics-selected readings. (2nd ed.). New York: Cambridge University Press.

Ernest, P. (1985). The philosophy of mathematics and mathematics education. International Journal of Mathematical Education in Science and Technology, 16, 603-612.

Lerman, S. (1983). Problem-solving or knowledge-centered: the influence of philosophy on mathematics teaching. Internatinal Journal of Mathematical Education in Science and Technology, 14, 59-66.

SPSS Inc. (1988). SPSS-X User's Guide (3rd ed.). Chicago: Author

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. Educational Studies in Mathematics, 15, 105-127.

Voigt, J. (1989). Social functions and routines and consequences for subject matter learning. International Journal of Educational Research, 6, 647-655.

GAMES AND LANGUAGE-GAMES: TOWARDS A SOCIALLY INTERACTIVE MODEL FOR LEARNING MATHEMATICS

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Mathematical games offer an attractive alternative to standard teaching methodologies. Unfortuneately, however, the development of games for these purposes has run into both practical and theoretical difficulties. Concentrating on the latter, this paper will explicitly highlight language rather than thought as the mediating factor in mathematical experience. Following Wittgenstein, mathematical concepts are interpreted as language-games; the linguistic character of mathematical games is also considered and the new concept of metagame introduced. A speculative typology of metagames, suggesting a possible structure for 'play' will also be offered. Reference is made to theoretical constructs within cognitive psychology, the philosophy of Wittgenstein, Freud, Lacan, and allusions are made to the writings of poststructuralist authors such as Foucault, Derrida and J-F Lyotard.

§ 1 Introduction

Within primary schools especially there is a great deal of interest in the utility of games as instruments for learning (see Bright, Harvey, Wheeler [1985]). This movement - renegotiating the playground boundaries, in a sense - contrasts with the traditional relationship of exclusion between games and the teaching and learning of mathematics. In a review of research on mathematical games, Larouche, Bergeron and Herscovics [1984] have noted that these can be associated with three main teaching objectives: 1) maintain and consolidate previously acquired skills; 2) improve problem solving ability; 3) develop and construct mathematical concepts. Their analysis has indicated, however, that a problem in analysising and implementing games of type 3) has been the lack of an adequate epistemological framework '... providing a detailed description of how a child may construct the notion involved ...' (p209).

This paper explores the relationships between the concepts of 'language', 'game' and 'mathematics': mathematical concepts are seen as language-games (§2) and the 'games of mathematics' are shown to be linguistic in character (§3). It is hoped that these results will shed light on one possible account of the epistemological framework sought by Larouche *et al.*. Later they may serve as a basis for constructing an integrated model for teaching and learning mathematics.

This paper is part of a research project (conducted within the Institute for Learning in Mathematics and Language, Griffith University) on mathematical games within primary and secondary schools. An essential aim of this project is to generate materials suitable for sustained application in classrooms.

§ 2 Language games in the construction of mathematical concepts

According to common sense notions a *game* is an activity involving one or more players in which, typically, actions are constrained by *rules* and result in the transformation of the game from one state of play to another. It is conventional to refer to such admissable actions as

's in the game, and any given state of play as an interim or provisional outcome of the game. A game is *played* by moving through a sequence of admissable moves. For instance the North African game of *mancala* operates with two players. Possible outcomes of the game coincide with the set of possible distributions of seeds; a minimal set of rules govern exactly how one distribution of seeds may be transformed into another. A final outcome of the game is when one player is divested of all seeds - this represents the game's objective. Card games, chess, computer games, other games involving chance and/or strategy - a wide diversity of games, infact - exhibit this commonly understood structure.

For some time reseachers have studied the significance for mathematics education of games involving mathemtical principles. However, relatively little consideration has been given to the consequences for mathematics education of a remark by Wittgenstein that mathematical concepts are themselves games [Wittgenstein, p183]. Before exploring this remark, it will be necessary to first indicate how Wittgenstein broadens the concept of game in his treatment of language-games. For Wittgenstein [§23] "... the term 'language-*game*' is meant to bring into prominence the fact that the *speaking* of langauge is part of an activity, or of a form of life." (italics in the original). J-F Lyotard, in his exposition of this concept, emphasises that the rules of the language-game "... do not carry within themselves their own legitimation, but are objects of a contract, explicit or not, between players..." [Lyotard, 1984, p10]. If there are no rules, there is no game - even infinitesimal modifications alter the game. Every utterance should be thought of as a move. Examples given by Wittgenstein himself [§23] indicates the scope intended for this concept, a selection of these include

Giving orders, and obeying them-
Describing the appearance of an object, or giving its measurements-
Constructing an object from a description (a drawing)-
Reporting an event-
Forming and testing an hypothesis-
Presenting the results of an experiment in tables and diagrams-
Making up a story; and reading it-
Play acting-
Solving a problem in practical arithmetic-

Wittgenstein's concept includes the common sense notion of game (*eg* mancala), for which the rules are explicit and known, at least in principle. More generally however, the rules of language-games are deeply-embedded in a social activity or practice (giving orders, and obeying them, for example). Analyses of such games can therefore face the difficulty of being *post hoc*, certainty of definition relates only to how the game *was*, not to how it *is*.

Important parallels and divergencies exist between the concept of language-game and the concept of discursive practice *a la* Walkerdine [Walkerdine, 1988]. *Game* is from the Old English *geim* meaning amusement or sport and is also related to the Gothic *gaman* meaning

fellowship [Oxford English Dictionary]. Thus, not only does the word imply the existence of a contract, but it also points to an affect: the contract affords *gratification*. By contrast, *practice* is an affect neutral term, it refers solely to the action itself. On the other hand, however, both concepts divert attention away from pure thought and its objects and towards (social) activity. Moreover, both authors focus on the social - and thus context bound - activity of *language* as constituting what is perceived as real [*cf* Pimm, 1990, p129].

Three illustrations of Wittgenstein's remark follow. It is hoped that the treatment of these examples will provide a succinct introduction to many of the central concerns projected in this paper.

Before proceeding, however, brief reference should be made to the rhetorical devices of metaphor and metonymy, used below. Metaphor is a language-game whereby meaning is generated by the form of an utterance, the terms are free to be substituted for others. For example, in the two formally identical sentences - The sun is bright, The girl is bright - meaning is generated in the second sentence by substituting 'girl' for 'sun'. Metonymy, also a language-game, functions by generating meaning through a contiguous relationship of terms. For example the U.S. Administration is physically located in the White House, thus it is possible to say with meaning,"The White House denied the claim."

Example 1: Object permanence, developing the concept of conservation

In Piaget's theory the infant's discovery that objects continue to exist when no longer visible is treated as 'both intellectual and real'. However, I will argue that this treatment obscures an underlying dynamic of *contest*, a deeply embedded game structure. For Freud, the breaking of the infant's bond with the mother, and the altogether imposssible desire for it to be re-established, leads to the development of unconscious fantasy - it is only by fulfilling wishes in fantasy that the child is able to reconstitute relations of intimacy with the mother. Walkerdine[1988, p190] quotes Freud's example of how this might be achieved: a child *plays a game* with a cotton reel, saying 'da' (here) and 'fort' (gone) as it is rolled towards and away from him. This fantasy of wish-fulfillment operates, according to Freud, by a series of 'condensations and displacements'. In the mother-child relationship, the mother is displaced by the cotton reel; the rolling towards and the rolling away of the reel is therefore a condensation of the appearance and disappearence of the mother. The fantasy involved, is thus a fantasy of control over the mother: in an imaginary contest of the infant with the mother, her appearance and disappearance is made to conform to the infant's desire. As noted by Walkerdine[p191] Lacan modifies this Freudian account in crucial ways. For Lacan, the real can only be captured by fantasy, and fantasies are created in language ('da', 'fort', and so on). For instance, in the cotton reel fantasy above, Freud's concepts of condensation and displacement are thought of in linguistic terms as metonymy and metaphor. The child's relationship of contiguity with the cotton reel is a metonymy for a similar desired relationship with the mother. Likewise, the substitution of the cotton reel for the mother, and the rolling of the reel back and forth

establishes a metaphor for controlling the mother. What has been argued here is that what for Piaget is the infant's discovery in the realm of pure thought - cognition - might be better understood as the *gratification of desire within the context of an acivity, a (language) game.*

Example 2: Number facts, developing the concept of number

A common approach to teaching number concepts is to assemble concrete materials (*eg* MAB) in a particular way and construct a correponding linguistic reference. For example, five girls in a class are formed into subgroups of two and three, the teacher then forms the assembly of words [*cf* Walkerdine, p185]

two girls plus three girls equals five girls

The meaning in this string of words may be derived for a student from their juxtaposition. More fundamentally, however, meaning for the student relates to the transformation (employing *metonymy*) of the relationship 'student-assembly of girls' to the relationship 'student-assembly of words'. The teacher proceeds by further transforming this assembly, this time invoking the strategy of *metaphor*, to present

2 girls + 3 girls = 5 girls

and

$2 + 3 = 5.$

Given symbolic utterances such as these, the student is encouraged to generate others using the same metaphoric strategy *eg*

2 apples + 3 apples = 5 apples

In summary, the admissable utterances in the language-game depicted are generated by either metaphor or metonymy. Within this game not only the concepts of number, but also relationships between numbers emerge as merely admissable moves.

Example 3: Developing the algebraic concept of variable as a pattern generaliser

Within the language-game generated in Example 2, it is possible to assemble 7 chips on a desk and, involving metomymy, as described above, say or write

7 chips

Metaphoric transformation now allows the admissable move, or denotation

7

Similarly, an assembly of an unknown number of chips can be denoted

x chips

and thus, as above, the sympbol 'x' may also gain admissability within the language-game played. Likewise, the following utterances are also moves within the game

2 boxes of 7 chips each	2 X 7 chips	2 X 7
2 boxes of x chips each	2 X x chips	2 X x

Thus '2 X x' and, after a procedure of denotation is applied, '2x' gains admissability as a legitimate utterance within the language-game. It follows that the metonymic utterance

$2 + 3 = 5$

is in turn open to transformation by metaphor to the now admissable statement

$2x + 3x = 5x$

Thus, the concept of variable as pattern generaliser [Usiskin, 1988] emerges as a move within the playing of a game.

What has been attempted in these examples is to make plausible Wittgenstein's claim that mathematical concepts arise as admissable moves (moments) within specific language-games. An important conclusion to follow is that both *learning* mathematics and *doing* mathematics (*eg* solving a problem) involve playing the *games of mathematics* - indeed, there may be sense in regarding both of these activities (learning,doing) as themselves games. The next step in this paper is to consider the variety of such games within the school context.

Before doing so, however, it is important to emphasise that Example 1 registered the significance of both elements of language and game (as affording gratification) in the production of concepts as language-games. Examples 2 and 3, by comparision dwelt mainly with the role of language. An important sub-theme in this paper is the emergence of gratification (and its denial) as an important aspect of these purely linguistic transactions.

§ 3 Towards a typology of the games of school mathematics

Consider the following classroom situation in which a mathematical game [*cf* Bright, Harvey, Wheeler, 1985, p161] is embedded

The teacher gives the following instructions. "I'll throw 2 dice, you put the numbers into the triangles in the expression $\Delta x + \Delta = 5$ in order to form an equation. Solve the equation, then swap work and check the solution by substitution. If your solution was correct, add it to your progressive tally. If not, throw it away. After 10 throws of the dice the person with the highest tally wins."

The claim is that when the class plays this game, a great number of other games become involved. The gaming episode is thus characterised by the *interactive* play of a large number of companion games. In following this analysis it will be clearer if a distinction is made between the initiating game and the companion games which ghost this game, these latter will be called *metagames* of the game, the initiating game will be called the *root* game. A metagame is thus a game of games, a game whose outcomes will, typically, govern the progress, the *play*, the succession of moves, of other games *eg* the root game and other metagames. Admissable moves in a metagame are not necesarily admissable moves in the root game. In the following examples, metagames governing the play of the root game indicated above will be grouped into categories suggestive of a possible typology of metagames.

In playing this root game, each student in the class needs to make decisions concerning the extent/duration of their particpation in the game. The psycho-social dynamics of the classroom informs the process of these decisions according to variables of purpose (motive) and risk; these variables are constituted by relations between individuals and coalitions of

individuals both present and absent (*eg* parents, authority figures). Such relations include prestige, reprobation, success, failure - the affectual domain generally. The claim is that for each student these dynamics constitute a metagame in which each student must participate. However, each student may not play the same metagame, a variety of games exist, each with it's own characteristic objectives (winning moves). Moves in such games are characterised by psycho-social determinants, and could, perhaps, best be thought of as episodes, or incidents involving the student, other students and the teacher and are played out within the social space of the classrom and psychological space of the student.

For instance, in an exteme case, a student might make moves in a dynamic of *non-engagement* with the root game. Another student might limit participation with a metagame whose moves negotiate a *withdrawal* from the game ("It's too hard"). Another student may appear to persist, despite "not getting anywhere", yet be actively playing out a metagame whose moves corrupt or *vary* the root game. The objectives of the metagames mentioned so far are responses within the classroom situation which seek to directly *deny* the root game, and so by extension also deny to the (explicit) contest of the root game a legitimate claim over the student's attention. Other responses may, at first sight, appear more positive. For example a student may attempt, conscientiously, to play the root game out, see it through to completion. Another could take up play in such a way that the root game need never be played again. The metagame involved in the first of these might be called *appeasement*, in the second *perfection*. Later it will be argued that although these metagames do not directly deny legitimacy of the root game, they both *indirectly* involve denial of the contest implicit within the root game[*cf* Lyotard,1984, p10]. The theme of denial introduced here hints at a general understanding of how games may afford gratification.

Another category of metagames involve procedure. A common element in these games is involvement in the process of constructing an admissable move in the root game. In this case, this means taking the numbers shown by the dice (say 5 and 2) and forming a corresponding equation (say, $5x + 2 = 12$) and finding the solution ($x = 2$). The procedures a player would need to follow here involve language-games of denoting, performing *etc*. For instance, if the player is a novice, he/she will need to master the following games [*cf* Booth, 1988]

> concatenation (*ie* '2x' denotes 2 multiplied by x)
> equality as equivalence rather than the performative 'give the answer'
> deploying the legitimacy of non- closure

As indicated in Examples 2 and 3 language-games of metaphor and metonymy also figure strongly here [see also, Tahta, 1985; Pimm, 1990].

A further category of metagames involves strategy. Metagames within this category have as an aim the construction of a strategy for playing the root game. A 'strategy' is thought of as a decision procedure which imposes a structure on the succession of moves played in the

game. For instance, if the player chooses always to make the smaller of the two numbers the coefficient then the succession of ensuing moves will be related not only by the criteria of admissability within the root game, but also because these moves have satisfied *additional* requirements. Strategies can be generated by a variety of metagames. For instance, the above may arise by the systematic *comparision* of the solutions corresponding to equations of the form: $5x + 2 = 12$ and $2x + 5 = 12$ (solutions are $x=2$ and $x=3^1/2$, respectively). Metagames of *modelling* the root game (or elements of it) or *simplifying* the root game could, similarly, provide further examples.

Beyond these, another category of metagames exists - these games exert editorial control over the entire gaming episode, their common purpose is to decide questions concerning the optimality of strategies. Does the root game have an optimal strategy? If so, what is it? Which of the available strategies produces a better result in the root game? In order to address this question a player may involve a language-game of, say, *examing* alternative strategies. Such a higher-order metagame will, typically, involve strategic metagames (eg analysing and synthesising, hypothesising, modelling etc) as well as procedural metagames. For example, in considering the optimality of the strategy indicated above, a player may need to involve the strategy of translation, *eg* translating the equation, say $2x + 5=12$, into two linear equations $y = 2x + 5$ and $y = 12$ leading to the consideration of the corresponding graphs. From this the optimality of the nominated strategy follows. Note that moves in this higher-order metagame were constructed by metagames at the same or preceding levels. Further examples of metagames within each category are provided in the following table.

Category of metagames	Objective of characteristic metagames	Characterstic metagames
PARTICIPATION & AFFECT	Player etablises psycho-social relationship with root game	avoiding, negating, varying, postponing, completing, perfecting
PROCEDURE	Construction of admissable moves in root game	describing, denoting, interrogating, performing, recommending,copying, metaphor and metonymy
STRATEGY	Construction of strategy for playing root game	simplifying, analysing and synthesising, patterning, hypothesising, translating, concretising and abstracting, modelling, comparing, inferring
HIGHER-ORDER	Determining opimality of strategies	examining, considering, managing, editing, surveying, prescibing, rationalising

It is plausible to suggest that the 'depth' of play is a function of the complexity of metagames activated within the strategic and higher-order categories (the expert/novice literature

in cognitive psychology provides suggestive support, see Mayer, 1983, p323). Further research should make explicit the constructs of cognitive psychology hinted at here.

In the above, an account of the structure of playing a mathematical game has been attempted; the conclusions of §2 prompt the hope that coverage of this structure may extend to provide a future account of *learning* and *doing* mathematics which emphasises these structures of play.

§ 4 Conclusion

In this paper, Wittgenstein's notion of language-game has been introduced into an analysis of the discourse of mathematics classrooms. Examples have been given of Wittgenstein's claim that mathematical concepts are language-games. The strong emphasis consequently placed on the *textual* quality of learning processes is coincident with Walkerdine's analysis of school mathematics *qua* discursive practice. Unlike Walkerdine, however, the ordinary notions of 'game' and 'playing a game' are thought of as providing a model for (school) mathematical experience itself. An important step in filling out the theory here will be to clarify how *contest* functions within play as that dynamic in which pleasure and threat - correlates of Desire and Necessity [*cf* Jameson, p102] - are brought into mutual relations of production and exchange. Later work will need to spell out implications for the theory and practice of teaching and learning.

Bibliography

Booker, G. and Kanes, C.(1990). "The Social Construction of Mathematics through games and other heuristically structured activities" Paper presented to the Social Psychology Working Group, PME XIV, 1990

Booth, L. R., "Children's Difficulties in Beginning Algebra", Ideas of Algebra, K - 12, 1988 Yearbook,NCTM, 1988, pp20 - 32

Bright,G.W., Harvey,J.G., Wheeler,M.M., "Learning and Mathematics Games", Journal for Research in Mathematics Education, Monograph Number 1, NCTM, 1985

Jameson, F. The Political Unconscious, Methuen, 1981

Leroche, Bergeron, Herscovics (1984), "Research on the role of play and games in the learning of mathematics", Proceedings of the Sixth Annual Meeting, PME-NA, pp203 - 212

Lyotard,J-F., The Postmodern Condition, Manchester University Press, 1984

—— ,Thebaud,J-L., Just Gaming, Theory and History of Literature, Vol 20, University of Manchester Press, 1985

Mayer, R.E., Thinking, Problem Solving, Cognition, W.H. Freeman and Co.: New York, San Francisco, 1983

Pimm, D.(1990) "Certain metonymic aspects of mathematical discourse", Proceedings of Fourteenth PME Conference, vol 3, pp129-136

Tahta,D.(1985) "On notation", Mathematics Teaching, vol 112, pp 48-51

Usiskin, Z. "Conceptions of school algebra and uses of variables", Ideas of Algebra, K - 12, 1988 Yearbook,NCTM, 1988, pp 8 - 19

Walkerdine,V., The Mastery of Reason, University Printing House, Cambridge, 1988

Wittgenstein,L., Philosophical Investigations, Basil Blackwell and Mott, Ltd., 1958

Translating Cognitively Well-Organized Information Into a Formal Data Structure

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We examine a particular data modeling episode to trace the cognitive restructuring required to create a formal row-column record structure for a set of well understood data based on a survey. The three 8th grade students had prior experience with manipulating artificial data, which turned out not to be particularly helpful in the context of their own "real" data. Difficulties were related to individual/class confusions, perceptual similarity between the standard record structure and certain crosstabulations, and, most importantly, confusion regarding the processes of data translation and transformation and the related question of information preservation across translations and transformations.

Introduction and Overview

In (Hancock & Kaput, 1990), we introduced new software (the TableTop) that provides means by which students could create data models in a highly visual and interactive way. We also described a highly iterative and interweaved process of building and applying a data model. The overall structure of this process has been confirmed and elaborated in approximately 25 data modeling projects over the past year. However, it is our intent in this paper to describe in detail a single extended episode from a particular data modeling project that is especially revealing in the way it exposes certain cognitive restructurings required to create the standard row-column record structure that is normally the form in which data is input and that underlies computer data bases. We will examine these restructurings and their inverse relation to the way a data analysis tool transforms data from a record structure to forms that explicitly reveal relationships of interest.

We feel that the modeling issues are of intrinsic interest apart from the potential generalizability of the phenomena described, because they offer a glimpse into the difficulty that students may have in reconciling a rich, semantically organized understanding of some situation with the requirements associated with creating a formal data structure to describe that situation. This is another form of the more general question of how to structure our experience mathematically, how to build mathematical models. While interviews with experienced teachers of applied statistics have suggested that the difficulties experienced by the students examined in this paper are fairly common, we make no claims about the frequency of their occurrence on the basis of the data directly available to us. Indeed, careful examination of our students' experiences prior to and within the episode to be discussed reveals factors that contributed to their cognitive difficulties.

We shall first give some background to set the stage for the episode and subsequent analysis, and then provide some details in the form of annotated transcript segments. A larger paper with more data and a fuller discussion will be made available at the conference.

Background

The TableTop enables quick and easy descriptions of data by representing each item in a

database as an icon on the screen. The user can create Venn diagrams, scatterplots and related representations of the data by simple mouse-click commands, and the icons respond accordingly by moving about on the screen according to the constraints that the user provides. The starting point form of the data is the standard row-column record structure illustrated in Figure 1 for a simple "toy" database about 8 fast food restaurants, where fields, which give properties of restaurants based on hypothetical ratings on a 5 point scale, appear as columns, and records, corresponding to restaurants, appear as rows.

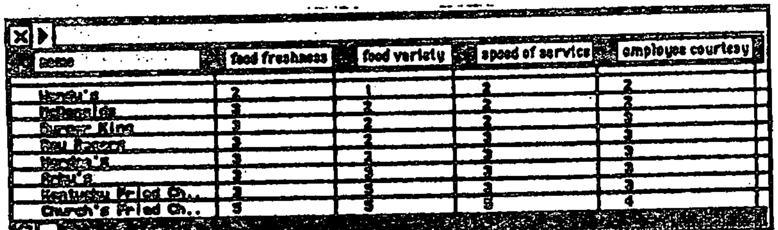

Figure 1

In Figure 2 is given a display relating speed of service to food variety using icons representing restaurants (the user clicks on the axes icon at the top of the screen and then clicks on the respective axes and chooses the properties to be displayed from a pop-up menu. The icons move into place accordingly after each action. In Figure 3 is the corresponding cross tabulation obtained by clicking on the compute facility of the system near the bottom of the screen, which counts the number of items in each cell (count is one among several compute options).

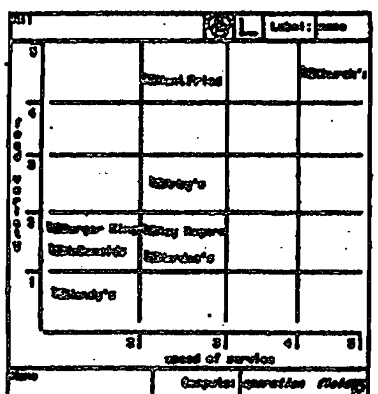

Figure 2

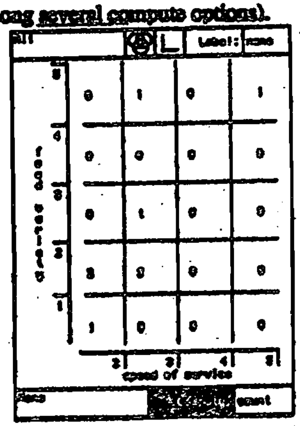

Figure 3

Three eighth grade boys (C, S, and N) of above average ability had been engaged in an investigation of possible relations between their peers' musical tastes and other personal and academic characteristics. They had met twice (although two members of the group had met once previously to work with the software system investigating an existing database). The first session was in two parts totaling one and a half hours, where the first 45 minutes was spent in finding and defining the problem and the second 45 minutes was used to familiarize them with the software,

including new features that even the two "veterans" had not seen before. As had been the pattern, there were no real difficulties in working with the TableTop to analyze simple data. In order to help the students anticipate data encoding and analysis, the second part of the first session was devoted to using the TableTop to analyze first the above Fast Food database. Towards the end of about 15 minutes of examining the database, to introduce some database editing features of the TableTop, the teacher suggested that they add a field that described the "healthiness" of the food at the respective restaurants. This provoked some laughter since all the restaurants were regarded as providing unhealthy food. Thus, by adding this field and then putting in arbitrary values for the different restaurants, the students gained a deeper sense that the data set was entirely arbitrary, a sense that appeared to have an impact later on what the students could learn from the experience. This "arbitrariness" factor, typically ignored in introducing data analysis activities, may prove to limit the value of the experience in helping students deal with "real" data, that is, data of the students' own making. For the last 15 minutes they examined a much larger database based on real information concerning the 24 largest industrial democracies.

The second session, slightly longer than an hour, was devoted to designing a rather complex pilot version of a survey instrument, an outline of which was produced by the end of the session. This pilot survey took the form of a series of preference questions regarding the respondent's favorite music and artists, with a fairly rich subcategory structure reflecting the students' extensive knowledge of popular music. For example, one set of questions asked the respondent's opinion of the main category Rock music (like it, dislike it, like it a little), followed by a five point preference question for each of the following subcategories of rock: "classic, softrock, metal, oldies, other." Other sets of questions inquired about Classic music and various subcategories, etc. These music preference questions were followed by a series of questions about academic and personal interests: how the respondents used spare time, music listening habits, whether they played a musical instrument and if so which, favorite radio station, whether and what they read outside of school, etc. The survey ended with a section filled out by the survey taker that involved descriptions of the person's clothes and hair styles. No contact with the computer was made in this survey-design session.

They each administered several copies of the pilot survey instrument before the third session two weeks later. It was in preparation for coding this pilot data into the computer during the two hour third session that the phenomena of interest occurred.

3. Description of Attempts at Data Restructuring

At the beginning of the third session, they discussed the results of the pilot survey and reconsidered the category structure for the music types. After 15 minutes, they turned to the computer, but discovered that portions of the program had been deleted and so were unable to get a version of the TableTop running. Thus they were not able to enter the data from the pilot form directly. This is where we pick up the action. We shall simply refer to the teacher (the first author listed) as "T" and the students by first initial. Our convention will be to include verbatim remarks in italics, except occasionally, where the use of parentheses is more appropriate, and include the descriptions in smaller font.

[Te]acher first recalls the earlier experience, especially the Fast-Food activity. The teacher introduced the term " in terms of the columns of the tabular database and used restaurants as the primary example. They recall adding the "healthiness" field.

T: *How would you go about setting this up for your data?*

C: *Same way as the other one.*

S: *No, no because we don't have really have any fields.*

C: *Yes we do.* (He lists the main music categories)

S: *Those are the main categories.*

T tries to get them to create rows and columns, asking specifically what the rows would be. They then launch into a description where (from N) the categories would be "across the top" (hence columns), and then C suggests doing the "grades down the side."

C is intent on a 2 dimensional classification scheme that is an attempt to do a cross-tabulation. S is convinced that it is a great idea. N buys in also. N proposes an extension to another "graph" that does grade levels instead of grades on the vertical axis, and then another involving musical instruments as columns. T tries to force the issue by asking how they they would enter the surveys that they had. Immediately, C and N mentioned "average."

Their first attempts are thus in terms of computer output of descriptive graphs rather than of computer input of records. The data-reduction strategy of using averages appeared repeatedly in the session as a way of capturing all the information in the survey data while also retaining the semantically defined structure of the crosstabulations.

T again asks: *How you gonna get from the data you collect on the form to the data in the computer?*

N asks the revealing question: *What do you mean?*

T opened an Excel spreadsheet as a starting point for a thought experiment, in which they would imagine how they would enter data in row-column form. Thus the spreadsheet offers the visual structure of a row-column data structure paralleling that of the TableTop in a data entry mode (and, in fact, most data analysis computer systems).

T: *You gotta decide what your columns are going to be.*

N (with agreement from others): *We put 1, 2, 3, 4, 5 across the top and A B C D F across the side.*

T: *What does the row, say number 4, stand for?*

No response. They all seemed confused, and attempt to improve the intelligibility of the array with attempts at labeling columns and entries indicating that the numbers are (unspecified) music preferences from the 5 point scale. T intervenes to suggest that the array is for their use only, not to be read by others and tries to force the issue by handing them a single filled out survey form and asking them to enter it into the computer.

They attempt to fill out a grade-by-music-category row-column array, without success.

T asks: *Suppose you finish that, and now how do you enter the next one?*

N: *What do you mean?* Three times, T repeats the question, and three times N says *What do you mean?*

S also gives some indication that he doesn't understand the task. He says *We should figure it all out first, and then have the typical A student, likes whatever first, ...*

N: *We just can't do it on this, T. It just screws us up.* More expressions of confusion.

T recalls the Fast-Food data and its content and then asks them to explain how it was set up. C says that it had the name of the restaurant and "all the stuff about it next to it" (i.e., a row for each restaurant). T asks what a column stood for (C: *all kinds of stuff*) and what did the rows stand for? (C: *the different restaurants.*)

T then asks them to interpret what a particular cell stands for in the C column, where C stands for "speed of service:" *We're in the 4th row, which stands for Church's Fried food, say. What goes in there?*

They agree that it stands for the rating it got, which, ambiguously, may be interpreted as some kind of average or aggregate data.

T then asks what all the "things in row 4 stand for," again trying to get across that a row is a record and vice-versa: *What do they all have to do with?*

S: *Church's chicken.* They all agree.

T reviews what the column means: *speed across all the restaurants.*

T then tries to transfer to their form, by asking what a row stands for in their layout.

They still feel that a row stands for a grade.

N has been waiting to make a suggestion, and suggests that they switch, and put music categories down the vertical axis and grades across the top: *And in each one we could write the average.*

Others immediately agree.

S: *Like C-students mostly listen to ..., all the B-students listen to ...*

N confronts the problem of having a set of entries for each student: *because we'd have 60 different graphs, for every single different one.*

Note the word "graph" - they seem to be thinking repeatedly not of raw data records, but of the scatterplots, etc., the transformed data. They want their semantically understood correlation represented in their initial data structure rather than derived from it via a computer transformation.

T again asks: *How a particular form gets put in?*

They reply that a particular form does not get put in.

N: *A particular form does not get put in. You average them.*

T points out that they are doing the work before they put the data into the computer rather than have the computer do the work and emphasizes that the data needs to be input before it gets transformed by the computer.

N: *I didn't know it could do that.*

N suggests, in the form of a question, that after getting it in the computer, they could "Ask the computer, what kind of people listen to classic rock rock, and it could tell us?"

T answers "yes" and they all seem pleased that this is possible.

They then discuss the possibility of creating Venn diagrams and using other TableTop tools, again with delight at the prospect that the computer could do so much of the work.

T points out: *But wasn't that exactly what you were doing in the hamburger case?* They agree.

But they obviously did not make a connection between the analyses that they had done with someone else's data and what they could do with their own data. The previous experience had remarkably little positive impact, despite its likely negative impact. This is a strong indication of the importance of the first stage in data analysis, where the students build the questions and the problems, make their own hypotheses to explore, etc. The experience of dealing with somebody else's data, especially hypothetical data, did not at all connect to the current problems.

With increasing frustration, N suggests vaguely that they pick a music category and put in all the data from all the forms for that category and then have the computer average them all up, go on to the next category, and so on. But then, he worries about the grades: *Then the A's, B's, C's would have nothing* (his emphasis) *to do with the music.*

C, who was staring at the Excel screen, is getting excited about an idea based on the fact that they put in a new field "healthfulness" as a column for the Fast Food database. His idea is to put in all the preference numbers for a music

; and then do it for grades, etc., as individual columns. But he does not seem to have incorporated the need to coordinate records.

N objects emphatically with gestures, using his two hands to represent different fields and raising and lowering them as on a balance scale - to show that he intends them to be coordinated - this conception is deeply in control, even of his gestures: *There'd be nothing to do with ... there'd be no connection between music and grades.*

N: *If we did that, there'd be, um ..., like the kind of music they listen to wouldn't be related to their grade. You'd just have an average grade. What good is that going to do you?*

C: Recalls the Venn diagrams and how they could use 2 sets - with gestures indicating an intersection isolating the students who like soft rock and with students who get A's ....

After asking how C would get this into the computer, he asks: *How is the computer going to know? The people that like reggae, how are they going to know what kind of grades those particular people get? You put those into the same category?*

In responding to N's continuing worry that relationships among data are not being preserved (e.g., between music preferences and grades), C suggests there would also be a grades field.

But N argues that then: *Grades wouldn't be related to the kind of music that they like.*

N: *You wouldn't have, like, names or anything to distinguish different people or anything. All you have is grade levels. There'd be too many things in one category. You couldn't do that.*

N is still trying to preserve his understood dependency relationship between the music preferences and student grades as part of the data structure to be input. He does not see that as something to be represented as the result of transformations mediated by the computer. For him (and probably the others), it seems that the data is already structured in that way and that that particular structure must be preserved as part of the input process. Of course, the record structure is organized not by semantic or causal relationships between fields, but rather by the needs to preserve identity of responses as individual records and the coordination of fields across records. This requires a large difference in conceptions of the data, but is at the heart of the data-record convention needed for traditional computer-based data analysis. C and S seem unable to deal with N's objections, because they still don't see the data as records organized by fields, but as some kind of average records, perhaps as with the Fast Food data. However, as N points out, this loses information about individuals.

They get discouraged, especially N, who is not impressed by the relevance of adding the "Healthiness" field. He notes: *We just made that up. We could have said that McDonalds is the healthiest restaurant in the world.*

N recognizes that the informational content of their data is not at all arbitrary, as opposed to the arbitrary and artificial data in the Fast Food database. Its semantic content was irrelevant as a constraint on data structure, so he was unable to appreciate the significance of its organization. This reveals a major potential weakness of activity based on artificial data.

C wants to use the database in the Venn diagram context to isolate respondents by grade and music preference, for example. C draws two sets, one for "soft rock and one for B's."

N asks if what appears in the sets is "the number of people ... like 50 people have B's?" and C agrees. C offers that there would be another number of people (38) who like soft rock, and there would also be a certain number of people who like soft rock and get B's.

All three express delight at this representation, as it appears to preserve structure that they are anxious to represent, but N continues to worry: *How do you get that into the computer?*

After another attempt by the students to deal with averages instead of records, T unsuccessfully tries to restrict them to talking about entering a <u>single</u> form. They continue to attempt to input aggregate data, both averages and frequencies, until N, with much gesturing and holding his fingers together to form a pair of rings, identifies what seems to an intersection of sets of icons as part of a system of crossed categories rather than as simple intersection of two sets - in effect the representation of Figure 2, but not a record structure.

<u>The Translation-of-Information View and the Solution</u>

After what seemed to be an impasse, T moves on to simplify the problem to 6 and, eventually, 4 responses and 3 pieces of information per form: (1) gender, (2) last math grade, (3) like/dislike music. T attempts to remove as much ambiguity as possible about the data to be entered, leaving pure, non-numeric categorical data which will not easily be averaged. The question was then restated, how to enter this? Lacking a suitable response, T gives a schematic representation of all 4 forms that had been filled out on a chalkboard (labeled with names for respondents) treating this as a translation-of-information task - from one place to another, where he also provided a 4X3 rectangular array of empty cells.

N numbers the rows 1-6 and then, ignoring objections from T about 4 the number of forms, lists the 5 possible math grades down the left side, A to F.

T asks: *Grade for whom?*

After N repeats the averaging idea, C lists the respondents' names across the top, and enters the grade information in what might be termed a binary contingency table - a check mark appears in a student's row under the grade obtained for that student. S says he likes the idea too.

After T asks how they would encode music preferences, they realize that they cannot encode all the information into a single table of this type.

T brings up the Fast Food again, recalling that each place had a rating on some aspect of its service.

Finally, after the group had spent about 10 minutes working with the simplification, N jumps up with excitement: *I got it!* (In reviewing the videotape with N a year later, N could not recall what prompted him to get this idea.)

N takes the chalk away from T and erases all the entries. Lists names down the side and then lists fields across the top, beginning with gender, grade and then music.

C begins to fill in the data beginning with the gender, in the standard record form. He immediately knew how to use it.

T first complicates the artificial data, changing the music question to a listing of students' favorite type of music, and eventually brings them back to their original pilot form. With a small amount of retrogression, they eventually consolidated their understanding of the record structure and went on to revise their form and complete the project.

Discussion

Why did these bright students have such difficulty in creating a record structure? We offer three hypotheses beyond the fact that the teacher-researcher was reluctant to offer direct assistance. The first two interact strongly with the third to produce the observed phenomena.

(1) The Fast Food database turned out to be an unfortunate example pedagogically for a variety of reasons: (a) It was inherently ambiguities regarding the status of the data. It was unspecified whether the restaurants listed represent classes of restaurants or individual restaurants. This yields another ambiguity, namely, whether the ratings are averages or not. Another ambiguity resided in the nature of the ratings themselves - were they rounded averages of multiple ratings across time, raters, or even restaurants? (b) Both the values of the fields and the relations among them were arbitrary and without real perceived semantic content - as would be expressed, for example, by expected dependency relations between ratings, either on an individual or aggregate basis. Since the goal of the subsequent activity was to restructure data with strong semantic content to a

semantically-free data record form, the learning-impact of the Fast Food experience turned out to be minimal. This circumstance may be quite prevalent where arbitrary data is used to introduce students to data modeling or the tools of data modeling.

(2) Another factor that seemed to have an influence on the three students' attempts to build a data structure was the perceptual similarity between the row-column view of traditional data records and crosstabulation types of graphs as can be seen between Figures 1 and 3 (note that about 3 weeks had elapsed since the students had worked with the computer, so they were operating on the basis of long term memory).

(3) We feel that the heart of the issue resides in processes of data translation and transformation, and the related question of information-preservation. As indicated towards the end of the session, one can regard the encoding process as one of information-preserving data translation - performed by the encoder. Once the data is translated into the computer, it can be transformed - by the computer - usually with loss of information, from its initial encoding into other representations, as when one forms the graph in Figure 2, which ignores all but three fields - the names and those used in the axes. The students were especially unclear as to the difference between the results of such transformations performed to expose some potential relationship embodied in the data, and the initial encodings (a fact partly revealed by their use of the word "graph" to cover all attempts at encodings. They seemed to have an implicit sense of the need to preserve information, however, and usually seemed aware when information was being lost. But their sense of information-preservation extended to the relationships that they felt existed in the data they were collecting (e.g., between music preference and grades). Virtually all their attempts at encoding were based on the need to preserve what they regarded as those important relationships. Coupled with the perceptual similarities mentioned above and the absence of a TableTop to provide feedback, they found great difficulty. The students were also unclear as to where the transformation can take place. They seemed rather surprised that the activity that they had performed in earlier sessions was actually being done by the computer - the actual computations were apparently so well concealed as to be out of mind as computations.

It may be possible to bypass the need to deal with row-column record structure directly, and we are beginning testing a version of the TableTop with a survey system that allows on-line data design and collection, and hence, automatic encoding. Nonetheless, we feel that an understanding of record structure, as a way of encoding information, prior to further transformation, is valuable in its own right. The questions raised in this paper are under current investigation, and we are also developing a systematic framework for discussing the matters of data structure, data representation, and translations and transformations. These will be the subject of a forthcoming paper.

Reference

Hancock, C. & Kaput, J. (1990) Computerized tools and the process of data modeling. In G. Booker, P. Cobb, T. de Mendicutti (Eds.) Proceedings of the 14th Meeting of the PME, Mexico, 165-72.

A Procedural-Structural Perspective on Algebra Research

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In reporting the seventh- and eleventh-grade results of the fourth mathematics assessment of the National Assessment of Educational Progress (NAEP), Brown et al. (1988) concluded that

> Secondary school students generally seem to have some knowledge of basic algebraic and geometric concepts and skills. However, the results of this assessment indicate, as the results of past assessments have, that students often are not able to apply this knowledge in problem-solving situations, nor do they appear to understand many of the structures underlying these mathematical concepts and skills. (pp. 346-347)

However, to cover their lack of understanding, students resort to memorizing rules and procedures that they eventually come to believe represents the essence of algebra. Brown et al. reported that a large majority of the students of the NAEP study felt that mathematics is rule based and about half of the students considered that learning mathematics is mostly memorizing.

What is it that compels many students to resort to memorizing the rules of algebra? What is it that makes the understanding of school algebra such a difficult task for the majority? Is it the content, or the way it is taught, or, yet, is it that students approach algebraic tasks in a way that is inappropriate for learning the subject material? In this paper, I review some of the existing research literature that bears on these questions and show how each contributes to the difficulties that students have in learning algebra. In particular, I argue that (a) algebra requires a structural understanding that students have not encountered before in their studies, (b) students have difficulty in moving from a procedural to a structural understanding of algebra, and (c) algebra instruction tends to be driven by the chapters in the textbooks, hence not focusing on these important points noted in (a) and (b). To support these points, I first describe a model of mathematical conceptual development (Sfard, in press), then briefly consider the content of school algebra from the perspective of this model, and finally synthesize findings from research on the teaching and learning of algebra.

A Procedural-Structural Model

The history of the development of algebraic symbolism bears witness to the gradual evolution of structural conceptions. However, until Vieta's irreversible invention of a truly symbolic algebra, the essence of rhetorical and syncopated algebra was the solving of certain kinds of problems by means of verbal prescriptions that involved a mixture of natural language and special characters; in other words, these prescriptions were basically descriptions of computational processes. Vieta's invention of an extremely condensed notation permitted algebra to be more than merely a procedural tool; it allowed for the symbolic forms to be used structurally as objects. Another example of a procedural-structural cycle in mathematical development is the evolution of the notion of function: from the 18th century procedural elaboration by Euler in terms of independent and dependent variables to the 19th century modification by Dirichlet that emphasized the arbitrary correspondence between real numbers to the 20th century structural definition by Bourbaki that defined function as a relation between two sets.

The existence of stages during which various mathematical concepts such as number and function evolved historically from procedural to structural has led Sfard (in press) to conclude that abstract mathematical notions can be conceived in two fundamentally different ways: "structurally" (as

objects) or "operationally" (as processes). (Note that, in the context of school algebra, my use of the term procedural is intended to mean the same as Sfard's use of the term operational.) She argues that the operational conception is, for most people, the first step in the acquisition of new mathematical notions. The transition from an operational conception to a structural conception is accomplished neither quickly nor without great difficulty. After they are fully developed, both conceptions are said to play important roles in mathematical activity.

Sfard contrasts the distinctions between the two approaches in the following way:

> There is a deep ontological gap between the structural and the operational approach. Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing--a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details. In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence only in a sequence of actions. Thus, whereas the structural conception is static and integrative, the operational is dynamic and detailed.

Sfard has created a three-phase model of conceptual development whose phases parallel the operational to structural stages in the historical evolution of mathematical learning. During the first phase, called interiorization, some process is performed on already familiar mathematical objects. The second phase, called condensation, is one in which the operation or process is squeezed into more manageable units. The condensation phase lasts as long as a new entity is conceived only operationally. The third phase, reification, involves the sudden ability to see something familiar in a new light. Whereas interiorization and condensation are lengthy sequences of gradual, quantitative rather than qualitative changes, reification seems to be a leap: A process solidifies into an object, into a static structure. The new entity is detached from the process that produced it. Sfard states, for example, that, "When the concept of function is reified, the person can be really proficient in solving equations in which 'unknowns' are functions (differential and functional equations, equations with parameters), talk about general properties of different processes performed on functions (such as composition and inversion), and eventually see a function as a not-necessarily computable set of ordered pairs."

School Algebra

Just as the historical development of algebra can be viewed as a cycle of procedural-structural evolution, the study of school algebra can be interpreted as a series of procedural-structural adjustments that must be made by students in coming to understand algebra. The way in which algebra is presented in most high school mathematics textbooks reflects the structural developments in algebra during the last century and a half. Some examples of the structural aspects of traditional high school algebra curricula are the following: a) simplifying and factoring expressions, b) solving equations by performing the same operation on both sides, and c) manipulating parameters of functional equations such as $y = y + (x - h)^3$ to yield families of functions. The introductory chapter of most algebra textbooks emphasizes links to arithmetic; algebraic representations are treated as generalized statements of the operations carried out in arithmetic, that is, in procedural terms whereby numerical values can be substituted into expressions to yield specific output values. However, as soon as this relatively smooth introduction is completed, algebraic representations are then presented as mathematical objects upon which certain structural operations can be carried out, such as combining literal terms, factoring, or subtracting the same literal term from both sides of an equation.

It is important to distinguish here the way in which the terms procedural and structural are being used in this paper. Procedural refers basically to arithmetic operations carried out on numbers to yield numbers. For example, if we take the algebraic expression, $3x + y$, and replace x and y by 4 and 5 respectively, the result is 17. Another example involves the solving of $2x + 5 = 11$ by substituting various values for x until finding the correct one. In these two examples, which look ostensibly algebraic, the objects that are operated on are not the algebraic expressions but their numerical instantiations. Furthermore, the operations that are carried out on these numbers are computational--they yield a numerical result. Thus, both of these examples illustrate a procedural perspective in algebra.

The term structural refers, on the other hand, to a different set of operations that are carried out, not on numbers, but on algebraic expressions. For example, if we take the algebraic expression, $3x + y + 8x$; this can be simplified to yield $11x + y$ or divided by z to yield $(11x + y)/z$. Equations such as $5x + 5 = 2x - 4$ can be solved by subtracting $2x$ from both sides to yield $5x - 2x + 5 = 2x - 2x - 4$, which can subsequently be simplified to $3x + 5 = -4$. In both of these examples, the objects that are operated on are the algebraic expressions, not some numerical instantiation. The operations that are carried out are not computational. Furthermore, the results are yet algebraic expressions.

As has been pointed out, most algebra textbooks attach a facade of procedural approaches onto their introduction to algebraic objects by providing a few exercises involving numerical substitution in algebraic expressions and various arithmetical solving techniques for algebraic equations--techniques that allow the students to, in a sense, bypass the algebraic symbolism. However, this pretense is soon dropped when expressions are to be simplified and equations are to be solved by formal methods. The implicit objectives of school algebra are structural. That students may attempt to circumvent or may not be prepared to handle the structural intent of the curriculum is soon to be discussed. The cognitive demands involved in operating on algebraic expressions as objects with operations that are quite unlike the operations of arithmetic are clearly reminiscent of the intellectual struggles that occurred during the historical development of algebra as procedural interpretations made way for structural ones.

Algebra Research

Since the length constraints of these proceedings do not permit an exhaustive treatment of algebra research, I will include only a few illustrative examples of findings for each of the main topics of the algebra curriculum. A more comprehensive synthesis appears in Kieran (in press).

Variables and Expressions

Booth (1984) has suggested that if students in elementary school do not recognize that the total number of items in two sets containing, say, 5 and 8 items respectively can be written as $5 + 8$ (rather than 13), it is highly unlikely that they will recognize that $a + b$ represents the total number of items in the sets containing a and b items--in other words, that being able to treat $a + b$ as an object in algebra has some intuitive precursor in arithmetic. Algebra demands that students recognize, for example, that $a + b - c$ is not the same as $a - b + c$ (unless $b = c$). Do students recognize these same structural constraints in arithmetic?

There is evidence to suggest that students are not aware of the underlying structure of arithmetic operations and of their properties. Chaiklin and Lesgold (1984) asked sixth graders to judge the equivalence (without computing the totals) of three-term arithmetic expressions with a subtraction and an addition operation (e.g., 685 - 492 + 947, 947 + 492 - 685, 947 – 685 + 492, 947 - 492 + 685). They found that students used several different methods of combining numerical terms, even within the same expression, depending on the expression with which it was being compared. They also

noted that students preferred to calculate in order to decide whether expressions were equivalent. This suggests that the pupils were not in a position to be able to judge equivalence without computing.

Collis (1974) used a task that involved finding the value of □ in, for example, (235 + □) + (679 - 122) = 235 + 679 to investigate whether or not students could recognize the relations among the various operations or would seek recourse to calculation. He used three formats: small numbers, large numbers (as shown), and letters. It was found that the younger students succeeded only with the small number items since their only available method was to calculate. Collis described the ability to work with expressions without reducing them by calculating as "Acceptance of Lack of Closure."

Students' understanding of the various ways in which letters can be used in algebraic expressions has been reported in detail by Küchemann (1981). He found that only a very small percentage of the 13- to 15-year-old pupils, who were tested in the 1976 Concepts in Secondary Mathematics and Science (CSMS) assessment, were able to consider the letter as a generalized number--despite classroom experience in representing number patterns as generalized statements. Even fewer were able to interpret letters as variables. The majority of students (73% of 13-year-olds, 59% of 14-year-olds, 53% of 15-year-olds) treated letters as concrete objects rather than as numbers of objects. Küchemann's findings suggest that most of the students tested had not yet begun to interpret literal expressions as numerical input-output procedures--an awareness that could be considered, according to Sfard's (in press) procedural-structural model, part of a first phase in developing a structural conception of algebraic expressions.

There is some evidence that long-term experience in Logo programming can assist students in developing such an understanding of variables and of algebraic expressions. One of the aims of the 3-year Logo Maths Project (LMP) (Hoyles, Sutherland, & Evans, 1985) was to develop and test materials designed to help 11- to 13-year-old students relate their understanding of variable in Logo to their understanding of variable in paper-and-pencil algebra. Sutherland (1987) reported that the LMP students performed substantially better than the students of the CSMS project on the same CSMS questions. However, it is to be noted that they were not able to succeed on the highest-level questions in which the letter represented a range of unspecified values and for which a systematic relationship existed between two such sets of values. That is, long-term programming experience had seemed not to equip these students to handle questions requiring a structural conception of literal terms and expressions.

Chalouh and Herscovics (1988) carried out a study designed to introduce Grade 6 and 7 students to algebraic expressions within the context of various geometric models. They reported that the students learned to interpret expressions such as $4x + 4y$ in terms of replacing x and y by numbers and carrying out the numerical operations, but that they believed these expressions to be incomplete. The students had to express them as part of an equality, such as "Area = $4x + 4y$" or as "$4x + 4y$ = <u>something</u>"--suggesting that a procedural interpretation of an algebraic expression requires that part of the representation indicate the result of carrying out the procedure. Similar findings were reported by Wagner, Rachlin, and Jensen (1984) who found that many algebra students tried to add "= 0" to any expressions they were asked to simplify. The need to transform expressions into equations was also illustrated by the results of a study by Kieran (1983) who found that some of the students could not assign any meaning to the expression $a + 3$ because the expression lacked an equal sign and right-hand member.

<u>Simplifying Expressions</u>

Students are asked fairly soon in their algebra classes to simplify expressions--an activity that, for

simple expressions, can be related initially to a numerical, procedural conception of expressions but which cannot for very long remain at that level. The complexity of the expressions, as well as the nature of the simplifications called for, quickly make such tasks undoable, unless the student is able to develop a sense of operating on the algebraic expression as a mathematical object in its own right. Such a structural conception involves applying properties not to numbers but to expressions.

Greeno (1982) carried out a study with beginning algebra students on tasks involving algebraic expressions and found that students' performance was quite haphazard, for a while at least. Their procedures were fraught with unsystematic errors, indicating an absence of knowledge of the structural features of algebra. Their confusion was evident from the way that they partitioned algebraic expressions into component parts. For example, they might simplify $4(6x-3y)+5x$ as $4(6x-3y+5x)$ on one occasion, but do something else on another occasion. Wenger (1987) has described some of the poor strategic decisions made by students with extensive algebra experience--decisions that result in their "going round in circles" while carrying out simplification transformations because they cannot seem to "see" the right things in algebraic expressions. Even older students who are successful at mastering the techniques of simplifying one type of expression, say, polynomials, have been found not to be able to transfer what they have learned to the next kind of simplification task involving, say, radicals; furthermore, they appear to perceive the two topics as separate (Rachlin, 1981).

Another aspect of learning the structure of algebraic expressions involves an awareness of the conventions of algebraic syntax. Bell, Malone, and Taylor (1987) report that beginning algebra students are often perplexed at being permitted to combine $2a + a + 15$ to $3a + 15$ but not $a + a + a \times 2$ to $3a \times 2$. Freudenthal (1973) points out that, if in ab the a is replaced by $-a$, it becomes $-ab$; but if b is replaced by $-b$, it does not become a-b but $a(-b)$. The student must learn where to add brackets and where not. By conscious bracketing, the text is structured. However, beginning algebra students often do not see the need for brackets (Kieran, 1979). Kirshner (1989) conducted a study of 400 students drawn from Grades 9 and 11 and a freshman calculus class in order to assess their reliance on visual cues in algebraic syntax. He found that for some students the surface features of ordinary notation provide necessary cues to successful syntactic decision.

<u>Equations</u>

One of the requirements for generating and adequately interpreting structural representations, such as equations, is a conception of the symmetric and transitive character of equality. Kieran (1981) carried out a study with seventh graders on constructing meaning for equations by "hiding numbers" in arithmetic identities. She found that it is possible to change beginning algebra students' unidirectional and answer-on-the-right-side perception of the equal sign and of arithmetic equalities into a procedural view of algebraic equations that includes the notion of (a) letters standing for numbers, (b) an equal sign representing the equivalence of left and right sides, and (c) a right-hand member not necessarily consisting of a single numerical term, but rather an algebraic expression--an awareness considered essential in the development of a structural conception of equations. (More research on generating equations is presented in the section on Word Problems and Functional Situations.)

<u>Solving Equations</u>

Operating on an equation as a mathematical object involves the formal solving procedure of performing the same operation on both sides of the equation. However, this is generally not the first method that is taught to students. "Guess and test" methods involving numerical substitution, as well as other informal techniques such as the cover-up method and working backwards, are often used as introductory approaches to equation solving (Bernard & Cohen, 1988). These are approaches that would appear to link well with procedural conceptions of equations. However, very soon afterwards,

students are taught the formal method.

Kieran (1988) found that seventh graders who had already learned in elementary school how to solve simple equations by substitution of numerical values were more receptive to learning the formal procedure of performing the same operation on both sides of the equal sign than were the students who had acquired in elementary school the skill of inversing (i.e., solving $2x+5=12$ by first subtracting 5 from 12 and then dividing by 2). Furthermore, the students who initially used substitution in solving the equations of the study seemed to possess a more developed notion of the balance between left and right sides of an equation and of the equivalence role of the equal sign than did the students who never used substitution.

Petitto (1979) noted that students who used a combination of formal and intuitive processes were more successful than those who used only one of these methods. Whitman (1982) pointed out that students who had been taught the cover-up equation-solving procedure followed by formal techniques were more successful at equation solving than the students who had learned only formal techniques.

In a previous section, it was noted that students often make poor strategic decisions when attempting to simplify algebraic expressions. Similar findings have been reported in studies that have investigated the solving of multi-operation equations (e.g., Carry, Lewis, & Bernard, 1980). Students, even older ones, have been found, in general, to lack the ability to generate and maintain a global overview of the features of an equation that should be attended to in deciding upon the next algebraic transformation to be carried out.

The effectiveness of concrete models in the teaching of formal equation-solving procedures has been researched by Filloy and Rojano (1984). In their teaching experiments they aimed at helping students create meaning for equations of the types $ax \pm b = cx$ and $ax \pm b = cx \pm d$ and for the algebraic operations used in solving these equations. Their principal approach was a geometric one, although they also used the balance model in some of their studies. Their findings provide further evidence that the transition from procedural to structural conceptions of algebraic equations is a difficult one for most students to achieve.

Still more evidence of the inability of students to distinguish structural features of equations is provided by Wagner, Rachlin, and Jensen (1984). They asked ninth grade students to solve the equation, $s/8 - 3 = 14$, and then to find the solution to the same equation after an alphabetic transformation of the variable had been effected--from s to t. In contrast to the subjects of an earlier Wagner study, most students knew immediately that the solution to the equation would not change. In the next task, the literal term t of the equation was changed to $t + 1$, and students were asked for the value of $t + 1$. The majority re-solved the equation, some solving for $t + 1$ directly, but most of them solving for t and then figuring out the value of $t + 1$. In a later task, the students were asked to solve for $(2t + 1)$ in $4(2t + 1) + 7 = 35$. Only one student solved directly for $2t + 1$. The findings of this study show that algebra students have trouble dealing with multiterm expressions as a single unit and suggest that students do not perceive that the basic surface structure of, for example, $4(2t + 1) + 7 = 35$ is the same as, say, $4x + 7 = 35$.

<u>Word Problems and Functional Situations</u>

Generating equations to represent the relationships found in word problems is well known to be a major area of difficulty for high school algebra students. Bell, Malone, and Taylor (1987) carried out a study that focused on the obstacles experienced by students when generating equations. They asked students to form equations for problems such as the following: "Imagine 3 piles of rocks, A, B, C, where B has 2 more than A, and C has 4 times as many rocks as A. The total number of rocks is 14 Find the number of rocks in each pile using x and do the problem in 3 'different ways'--i.e. using the x in three different positions." Bell, Malone, and Taylor reported that all students started with pile A as x, giving $x + 2$ and $4x$ for the other two piles. With pile B as x, students wrote $x - 2$ and $4x - 2$ for the remaining two; none used brackets for $4(x - 2)$. The resulting equation, $x - 2 + x + 4x - 2 = 14$, did not provide the same solution as before and consequently led to a discussion on the need for brackets. The final approach, with pile C as x, gave $1/4x$ and $2 + 1/4x$ for the other piles, which students mainly wrote as $x + 4 + 2 + x + 4 + x = 14$. After collecting $3x$ together, they wondered what to do with the numbers. Intervention helped the students learn certain techniques for dealing with the mechanics of forming equations. The researchers noted that the initial conceptual obstacle of how to express word problem statements (e.g., "15 more than x") were overcome; however, the second-order difficulties, that is, treating an algebraic expression as an object (e.g., coping with "15 more than $(x - 30)$"), were less fully resolved.

Filloy and Rojano (1984) have emphasized that a rupture occurs with problems that can be modeled by equations of the type $ax \pm b = cx \pm d$. Students must not only begin to think in terms of "forward" operations (Usiskin, 1988) in order to model these problems by equations, but also use a solving procedure that operates on both sides of the equation, that is, a process that operates on an algebraic object. Lesh, Post, and Behr (1987) have distinguished algebra problem solving from arithmetic problem solving by pointing out that, in algebra, the problem requires "first <u>describing</u> and then calculating" [italics added] (p. 657).

But even the activity of "describing" in algebra problem solving can be done in either procedural or structural terms. A good example of the difference between the two is provided by the set of studies carried out with university students by Soloway, Lochhead, and Clement (1982). The structural interpretation with respect to the now-classic Student-Professor problem (i.e., "Write an equation using the variables S and P to represent the following statement, 'There are six times as many students as professors at this university'; use S for the number of students and P for the number of professors.") requires generating an equation in which the letters are treated as variables and the equal sign is used to express an equivalence. A variation of this question, which allows for a procedural interpretation, uses the following form: "Given the following statement, 'There are six times as many students as professors at this university,' write a computer program in BASIC which will output the number of students when supplied (via user input at the terminal) with the number of professors. Use S for the number of students and P for the number of professors." One of the characteristics of the procedural approach for this situation is an input-output interpretation that involves numerical inputs and outputs. Thus, even though literal terms are involved in the computer program, students are able to interpret these in terms of numbers upon which a specific arithmetic operation is performed in order to produce a numerical output. Students have been found to be considerably more successful with a procedural approach, which specifies an algorithm for computing one magnitude by means of another, than with a structural approach, which specifies an equality relation among variables.

Sfard (1987) administered a questionnaire to sixty 16- and 18-year-olds, who were well-acquainted with the notion of function and with its formal structural definition, in an attempt to find out whether they conceived of functions procedurally or structurally. The majority of the pupils who were questioned conceived of functions as a process rather than as a static construct. In a second phase of the study involving ninety-six 14- to 17-year-olds, students were asked to translate four simple word problems into equations and also to provide verbal prescriptions (algorithms) for calculating the

solutions to similar problems. As with the Soloway et al. study, students succeeded much better with the verbal prescriptions than with the construction of equations.

The studies cited in this paper have shown that, despite an "object-oriented" approach to teaching algebra, a fully developed structural conception of algebraic objects seems rather rare in high school students. Sfard (1989) designed a study on functions in which students were taught initially by an operational (i.e., procedural) approach that was gradually transformed into a structural approach. She wanted to see whether a structural conception could be provoked in students by means of a teaching that adhered faithfully to such a sequence. Responses to a questionnaire administered at the end of the course showed that substantial progress toward structural conceptions had been achieved; nevertheless, Sfard notes that "our attempt to promote the structural conception cannot be regarded as fully successful" and conjectures that "reification is inherently so difficult that there may be students for whom the structural conception will remain practically out of reach" (p. 158).

Concluding Remarks

The overall picture that emerges from an examination of the findings of algebra research is that the majority of students do not acquire any real sense of the structural aspects of algebra. At worst, they memorize a pseudo-structural content; at best, they develop and continue to rely on procedural conceptions. Sfard (in press) has emphasized that a lengthy period of experience is required before procedural conceptions can become transformed into structural conceptions. Research indicates that procedural, input-output conceptions are accessible and that strength in this area could serve as an aid to making algebraic activity more understandable and more meaningful. This suggests, first, that greater effort ought to be invested in classroom instruction toward creating a solid base for developing the structural conceptions of algebra by spending considerably more time with procedural conceptions. Second, the transition from procedural to structural conceptions would seem to require more explicit attention than it currently receives in most textbooks. Since teachers tend to teach what is in the textbooks (Slovin, 1990), a step toward ensuring that procedural conceptions both develop and then evolve into structural conceptions would be to have textbooks reflect this stance. It is to be noted, however, that the acquisition of structural conceptions by which expressions, equations, and functions are conceived as objects and are operated on as objects does not eliminate the continued need for the procedural conception which served as a foundation for the construction of the structural conception; both play important roles in mathematical activity. However, very few studies have addressed the issue of the role and interaction of both conceptions in doing algebra. Lee and Wheeler (1989) have shown that students behave as if arithmetic and algebra were two closed systems. The challenge to classroom instruction is not only to build upon the arithmetic-to-algebra connection but also to keep alive the algebra-to-arithmetic connection, that is, to develop the abilities to move back and forth between the procedural and structural conceptions and to see the advantages of being able to choose one perspective or the other--depending on the task at hand.

References

Bell, A., Malone, J. A., & Taylor, P. C. (1987). Algebra--an exploratory teaching experiment. Nottingham, England: Shell Centre for Mathematical Education.

Bernard, J. E., & Cohen, M. P. (1988). An integration of equation-solving methods into a developmental learning sequence. In A. Coxford (Ed.), The ideas of algebra, K-12 (1988 Yearbook, pp. 97-111). Reston, VA: National Council of Teachers of Mathematics.

Booth, L. R. (1984). Algebra: Children's strategies and errors. Windsor, UK: NFER-Nelson.

Brown, C. A., Carpenter, T. P., Kouba, V. L., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Secondary school results for the fourth NAEP mathematics assessment: Algebra, geometry, mathematical methods, and attitudes. Mathematics Teacher, 81, 337-347.

Carry, L. R., Lewis, C., & Bernard, J. (1980). Psychology of equation solving: An information

processing study (Final Technical Report). Austin: Univ. of Texas at Austin, Dept. of C. & I.

Chaiklin, S., & Lesgold, S. (1984, April). Prealgebra students' knowledge of algebraic tasks with arithmetic expressions. Paper presented at annual meeting of AERA, New Orleans, LA.

Chalouh, L., & Herscovics, N. (1988). Teaching algebraic expressions in a meaningful way. In A. F Coxford (Ed.), The ideas of algebra, K-12 (1988 Yearbook, pp. 33-42). Reston, VA: NCTM.

Collis, K. F. (1974, June). Cognitive development and mathematics learning. Paper presented at the Psychology of Mathematics Workshop, Centre for Science Education, Chelsea College, London.

Filloy, E., & Rojano, T. (1984). From an arithmetical to an algebraic thought. In J. M. Moser (Ed.), Proceedings of the Sixth Annual Meeting of PME-NA (pp. 51-56). Madison: Univ. of Wisconsin.

Freudenthal, H. (1973). Mathematics as an educational task. Dordrecht, The Netherlands: D. Reidel.

Greeno, J. G. (1982, March). A cognitive learning analysis of algebra. Paper presented at the annual meeting of the American Educational Research Association, Boston, MA.

Hoyles, C., Sutherland, R., & Evans, J. (1985). The Logo Maths Project: A preliminary investigation of the pupil-centred approach to the learning of Logo in the secondary mathematics classroom, 1983-4. London: University of London, Institute of Education.

Kieran, C. (1979). Children's operational thinking within the context of bracketing and the order of operations. In D. Tall (Ed.), Proceedings of PME-3 (pp. 128-133). Coventry, UK: Warwick Univ.

Kieran, C. (1981). Concepts associated with the equality symbol. Educ. Stud. Math., 12, 317-326.

Kieran, C. (1983). Relationships between novices' views of algebraic letters and their use of symmetric and asymmetric equation-solving procedures. In J. C. Bergeron & N. Herscovics (Ed.), Proceedings of PME-NA-5 (Vol. 1, pp. 161-168). Montreal, Canada: Université de Montréal.

Kieran, C. (1988). Two different approaches among algebra learners. In A. F. Coxford (Ed.), The ideas of algebra, K-12 (1988 Yearbook, pp. 91-96). Reston, VA: NCTM.

Kieran, C. (in press). The learning and teaching of school algebra. In D. A. Grouws (Ed.), Handbook of research on the teaching and learning of mathematics. New York: Macmillan.

Kirshner, D. (1989). The visual syntax of algebra. Jour. Res. Math. Educ., 20, 274-287.

Küchemann, D. (1981). Algebra. In K. M. Hart (Ed.), Children's understanding of mathematics: 11-16 (pp. 102-119). London: John Murray.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. Educ. Stud. Math., 20, 41-54.

Lesh, R., Post, T., & Behr, M. (1987). Dienes revisited: Multiple embodiments in computer environments. In I. Wirszup & R. Streit (Eds.), Developments in school mathematics education around the world (pp. 647-680). Reston, VA: National Council of Teachers of Mathematics.

Petitto, A. (1979). The role of formal and non-formal thinking in doing algebra. Journal of Children's Mathematical Behavior, 2(2), 69-82.

Rachlin, S. L. (1981). Processes used by college students in understanding basic algebra. Unpublished doctoral dissertation, University of Georgia, Athens.

Sfard, A. (1987). Two conceptions of mathematical notions: Operational and structural. In J. C Bergeron, N. Herscovics, & C. Kieran (Eds.), Proceedings of PME-11 (Vol. III, pp. 162-169). Montreal, Canada: Université de Montreal.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), Proceedings of PME-13 (Vol. 3, pp. 151-158). Paris: Laboratoire PSYDEE.

Sfard, A. (in press). On the dual nature of mathematical conceptions: Theoretical reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics.

Slovin, H. (1990, April). A study of the effect of an in-service course in teaching algebra through a problem-solving process. Paper at NCTM meeting, Salt Lake City, UT.

Soloway, E., Lochhead, J., & Clement, J. (1982). Does computer programming enhance problem solving ability? Some positive evidence on algebra word problems. In R. J. Seidel, R. E Anderson, & B. Hunter (Eds.), Computer literacy. New York: Academic Press.

Sutherland, R. (1987). A study of the use and understanding of algebra related concepts within a Logo environment. In Proceedings of PME-11 (Vol. I, pp. 241-247). Montreal: Univ. de Montreal

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford (Ed.), The ideas of algebra, K-12 (1988 Yearbook, pp. 8-19). Reston, VA: NCTM.

Wagner, S., Rachlin, S. L., & Jensen, R. J. (1984). Algebra Learning Project: Final report. Athens University of Georgia, Department of Mathematics Education.

Wenger, R. H. (1987). Cognitive science and algebra learning. In A. H. Schoenfeld (Ed.), Cognitive science and mathematics education (pp. 217-251). Hilldale, NJ: Lawrence Erlbaum.

Whitman, B. S. (1982). Intuitive equation-solving skills. In L. Silvey (Ed.), Mathematics for the middle grades (5-9) (1982 Yearbook, pp. 199-204). Reston, VA: NCTM.

CONSEQUENCES OF A LOW LEVEL OF ACTING AND REFLECTING IN GEOMETRY LEARNING - FINDINGS OF INTERVIEWS ON THE CONCEPT OF ANGLE

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This paper presents a part of an empirical research project whose purpose was to work out individual intuitions, associations and ideas of pupils concerning the concept of angle. The analysis of an interview with a thirteen year old pupil shows some interesting difficulties in which he was involved. It is pointed out that a higher level of acting und reflecting could enrich pupils' concept formation enormously.

Background and methods

The following paper is mainly concerned with an empirical research project which has been carried out within the framework of a dissertation on the theme "Living Geometry" ("Lebendige Geometrie", Krainer 1990) in Austria. In this project 127 pupils in the 7th grade of grammar school were tested and two of them interviewed. The 12 to 13 year old pupils had learnt about the concept of angle in the 6th grade and could therefore refer to school-experiences with this concept. The primary aim of the research was not the analysis of misconceptions but the working out of pupils' individual conceptions of angle with all their intuitions, associations, ideas and difficulties.

The research comprised four phases:

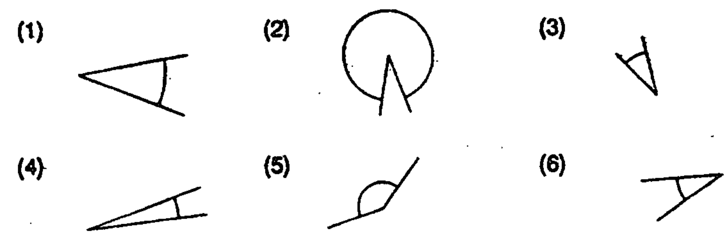
1. Written pilot-questionnaire, realization in one class (33 pupils)
2. Revised questionnaire, realization in three classes (94 pupils)
3. Detailed interviews with a boy and a girl of one of the classes
4. Summary of results, working out of didactical consequences.

The principal aim of this paper is the analysis of one of the two interviews (with some remarks to the second one). Each interview lasted for about half an hour. A set of tasks and questions was developed as a guide line for the interviewer. They were an expression of those open questions which came to light within the process of evaluation of the written questionnaire, using some ideas from the research of Close (1982).

Therefore the following tasks and questions neither correspond to the interviews' chronological course of events nor do they reflect the tasks and questions which were actually chosen:

- *What is easy/difficult for you in the concept of angle?*
- *What do you understand by "angle"?*
- *(Poss.: What do you understand by "distance"?)*
- *Draw the biggest angle you can!*
- *How would you design a protractor/set square by yourself?*
- *What does 1° mean?*
- *Do people have to measure angles in a particular direction?*

- *Do the sides of an angle have to have a particular position?*
- *Why do people often make an arc in connection with angles?*
- *Comment on the following sentences briefly:*
  - *An angle is a figure consisting of two half lines.*
  - *An angle is a distance between two half lines.*
  - *An angle is that magnitude by which one half line is rotated to reach a second half line.*
  - *An angle is the space between two half lines.*
- *Compare the following angles with respect to their magnitudes:*



One more remark on pupils' pre-knowledge: in the previous school year the teacher introduced the angle as the "*amount of deviation from a given direction*" and commented this with examples such as "*crossroads*" or "*compass-card*". The pupils were told to measure angles anti-clockwise. In the following school year angles were mainly treated in connection with left-oriented rotations.

A detailed analysis of the interviews with the two pupils (incl. a complete transcription) is given in Krainer (1990, p. 222 - 288 and 456 - 497).

Results

In the following analysis some relevant passages and situations from the interview with the pupil "beta" are discussed and some remarks to the interview with the pupil "chi" are made.

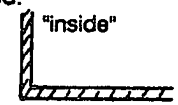
In the interview "beta" defined an angle as a "*distance between two lines which meet on a point.*" He pointed out that an angle could not consist of two half lines because endless lines don't fit on any sheet. By way of contrast he considered the rays of sunshine as endless rays. It is interesting that on the one hand he described limited physical phenomena (such as sunrays) as if they were unlimited and on the other hand he conceived mathematical objects as figures limited by concrete objects - such as a sheet in this case. One possible reason may be that he considered rays in comparison to lines - which pupils are able to draw - to be essentially longer. These considerations show that the sheet is not seen as a set of points but as a concrete object. This interpretation is substantiated by the pupil's use of the word "on" (see above): The formulation that two lines meet on one point probably expresses the idea that there are concrete things lying on each other. The vertex

is not seen as a point of an ideal plane but as a physical crossing-point of drawn lines.
All in all the examples show the influences of physical reality on the development of geometric concepts.
Returning to "beta's" understanding of an angle as a "*distance between two lines* ...": this conception leads to the presumption that the pupil is inclined to accept only angles smaller than 180°. In fact "beta" acted in this manner in most situations of the interview.
For example (fig.): "Beta" said that angle Nr. 5 is bigger than angle Nr. 2 because the sides (of angle Nr. 2) "*don't branch off so far*" as the sides of angle Nr. 5.


(2)

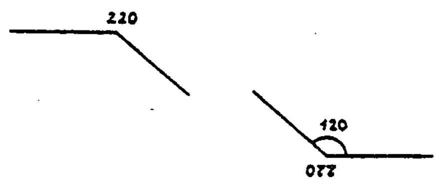

(5)

There are numerous explanations for this interpretation:
- In everyday life angles smaller than 180° dominate. Even in cases where another interpretation is possible, angles smaller than 180° are used.
  For example in the case of corners (fig.): we speak of right angles not only in the case of "inside" but also in the case of "outside".

  
  "inside"
  "outside"
- Even in the classroom mostly angles smaller than 180° are considered, in triangles e.g. only angles smaller than 180° exist.

It has already been mentioned that the angle was taught left-oriented. Therefore "beta" also knew that there are angles bigger than 180°. Together with his intuitive understanding of an angle as "*distance*" this led to interesting mixed conceptions. For example: as "beta" in the interview had only drawn or recognized angles smaller than 180°, the interviewer asked him to draw an angle bigger than 180°.

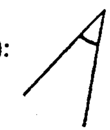
The pupil drew an angle (fig.) which he declared to be 220° (which actually was 240°). After this the interviewer turned the sheet about 180° (fig.) and remarked provocatively that the angle only had 120°.



The pupil replied: "*Yes, of course, from there it has 120° (turning the sheet back to the original position) ... but from here it was 220°.*"
Obviously the pupil was of the opinion that the assessment of this geometrical figure depended on its position with respect to the observer.
There were similar situations in the interview with the pupil "chi": she proposed e.g. to add a letter to the figure to indicate the "*right position*" of the sheet. (In another

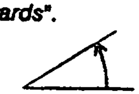
section of the interview she developed the intelligent proposal of marking the sides of the angle with letters in order to reach a decision.)

All these statements indicate that geometric relations, quantities, etc. were seen to be dependent on the observer, which nevertheless is inconsistent with congruence geometry. Here it is a matter of an objective, "position-independent" view of geometrical phenomena in which subjective (egocentric) views have no place. During the interviews there was frequent reference to special positions. Nearly all angles drawn by "beta" and "chi" had standard positions in so far as one side was drawn "horizontally". (The term "horizontal" is strictly speaking misleading because every line in an exercise-book is approximately in a horizontal position. In this case the horizontal line is running parallel to the lower edge of the sheet, which in turn is parallel to pupil's face.)
"Beta" was asked whether angles always have to have a side parallel to the lower edge of the sheet. He answered that we can also draw angles differently but that this would "*not be so beautiful*". After this he pointed to an angle in standard position and said: "*It is even better if we draw it like this.*"
The following situation is interesting:
the interviewer drew an angle which had the following position on the sheet (fig.):



The pupil drew an arrow "*pointing upwards*" (fig.). The arrow in no way fits into a left-oriented concept of angle although, from the pupil's egocentric view, the arrow was indeed pointing "upwards to the left".

In another section of the interview "beta" pointed out that the "*arrow always goes up and never down*". How deep "egocentric" concepts (see Piaget et al, 1974) such as "up/down" or "above/below" and "left/right" are anchored in pupils' heads is underlined by a statement from "chi": she stated that while measuring angles one has to turn the sheet in such a way that one side of the angle is pointing to the right. Then one has to "*measure from right to left going upwards*".
This shows that the learnt concept of "left-orientation" has been mixed with the every day understanding of "left". In standard positions (fig.), however, these two conceptions fit together.



These standard positions are mostly used in class. For this reason they become an paradigm (standard representative) whose generalization could become rather individual.
One of the highlights of the interview with "beta" began with the following dialogue:
I: Do we need to measure angles in a certain direction?
B: Sure, if there is a right angle, I have to draw a right angle; but if a left one is given, (I have to draw) a left (one).
I: Draw a right angle and a left angle!

The pupil made drawings like these (fig.):

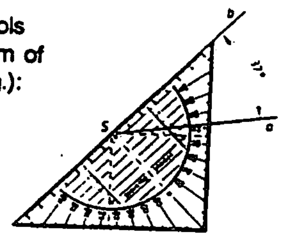

"*right angle*"

"*left angle*"

At this point the following idea should be stressed: It is too simple to characterize the answer of "beta" as being "false". In fact, he had a good idea and only had the bad luck that it did not fit our mathematical view.
We see that "right" can appear in at least three meanings (which throughout could be mixed in pupils' heads):

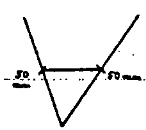
1. "Right" of me (as seen from the position of a particular person)
2. "Right" - oriented (clockwise)
3. "Right-angle" ("right"/correct when dividing a straight angle in two equal parts).

For "beta" an angle was closly linked with its specification in degrees. He saw - as mentioned above - an angle as a *"distance between two lines ..."*, therefore as something with which he could count. He stated that we have to know, *"how many degrees there are"*.
It should be mentioned that in Austrian schools angles are mainly measured by a special form of protractor/set square, called "Geodreieck" (fig.): (An advantage of this instrument is the fact that in many situations we can apply two functions at the same time, namely the drawing\measuring of an angle and the drawing/measuring of a straight line.)



"Beta" proved in the interview that he knew the basics of this technique. But he had no idea how to design a protractor by himself or to explain its internal construction. Indeed he saw 1° as a unit but only in the sense that an angle *"should be named by something"*. The fact that a round angle has 360° he saw as something incontestable, he therefore did not realize the arbitrariness in choosing the unit. Never did "beta" mention the idea of obtaining the quantity of an angle by subdividing a circular line. It was sufficient for him to point to the correctness of the protractor's scale: *"These distances here ... the angles are surely drawn correctly"*. "Beta" considered the description of an angle by using degrees as an essential property of angles. For him the protractor was a materialization of the idea of an instrument for measuring angles.

The pupil "chi" - likewise defining an angle as *"distance"* - also did not come to the idea of dividing the circular line. In trying to reconstruct the construction of a protractor with a drawing, she drew a right angle which she wanted to halve twice. When calculating this she got 22 1/2°: this result - which surely was arithmetically reasoned - led her to threw away the geometrical idea of halving the angle continuously (in getting real measure-units). In trying to divide a right angle into three parts of the same size she divided the corresponding chord.
In another section of the interview she even tried to measure the magnitude of an angle (fig.) with a ruler and to indicate the "distance" in millimetres.



Conclusions

The analysis of the interview with the pupil "beta" refers to some interesting aspects of two problems of geometry learning:

- The mixing of real world and theoretical conceptions
- The mixing of geometrical and arithmetical conceptions

a) <u>The mixing of real world and theoretical conceptions</u>
Detachment from egocentric views:
The pupil "beta" used real world concepts like "left", "right", "above", etc. to describe geometrical facts and stated in one case that the quantity of an angle depends on the angle's position with respect to the observer. Egocentric views make learning geometry more difficult, as it is not a matter of relations between a figure and an observer but a matter of relations between geometrical figures independent of any (particular) position of an observer. (Similar results concerning children's recognation of right-angled triangles in various positions are described in Cooper/Krainer 1990.)
Detachment from physical phenomena:
Some passages of the interview show that "beta" had a physical view of geometry. He did not know that geometrical objects - in contrast to concrete things from the world around us - are ideal objects. Pupils should realize that drawings can be used as interfaces between concrete things and geometrical objects. It is also important to know that phenomena such as "vertical" and "horizontal" are physical properties.
The complexity of the concept of angle:
There are many intuitive conceptions regarding the concept of angle, for example "angle as a figure", "angle as a space", "angle as an inclination", "angle as a rotation", etc. There are also many ways of defining an angle whereby it is not possible to include all intuitive conceptions within one definition (see e.g. Mitchelmore 1989, Krainer 1990). In the case of "beta" the conception of an *"angle as a distance"* was mixed within the learnt conception of a "left-oriented" angle.
Judicious theoretical learning in geometry education requires that pupils realize real world conceptions as such. In order that detachment (e.g. from egocentric views) can take place, the inclusion of the world around us (in geometry learning) is very important. For this, acting and reflecting by pupils are useful supports.

b) <u>The mixing of geometrical and arithmetical conceptions</u>
The strong link between angle and angular measurement:
Beta's conceptions can be described by the following "chain of associations": "angle" = "angular measure" = "angular measure in degrees". It indicates a dominant arithmetisation of geometrical conceptions. This arithmetisation contains the problem that interesting geometrical relations (e.g. the fact that the three angles of a triangle combined result in two right angles) are reduced to numbers and measures ("angular sum" = 180°).
The protractor as a "finished product":
"Beta" had no idea how to design a protractor. He considered a protractor as a black box making the necessary quantification of angular figures possible. Above all

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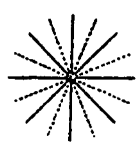
..e linear angular scales on the protractor may lead pupils to assume a strong connection between angular and linear measurement. Maybe the conception of an *"angle as a distance"* is a part of this view.
Of course acting and reflecting by pupils play an important role to counteract the mixing geometrical and arithmetical conceptions. For example: designing a protractor; working out alternative methods of measuring angles (e.g. ratio of sides); comparison of linear measurement and angular measurement.

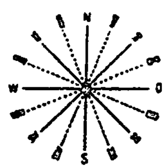
Outlook

The intuitions, associations and ideas of "beta" und "chi" are - from a mathematical point of view - unsatisfactory. It is assumed that the main reasons are in a low level of acting and reflecting in geometry education.
One possibility that pupils examine geometrical problems intensively, is the working on with "powerful" tasks. In this manner tasks are understood in which acting and reflecting by pupils are in the foreground. The pupils are seen as "producers" ("designers") and not as "consumers".
Within this paper it is only possible to introduce two tasks briefly:

Task 1: "Acting instructions for angles"
Discuss the following text: Mary and Paul have created an interesting game: one of them has the task to explain a particular concept to the other without using the corresponding word. The other one has the task to guess it. Then they discuss how to explain the concept better. After playing with the concepts "table" and "friend" Mary has the idea to choose a geometrical concept. She interprets it like this: "Stand up and think of two different directions! After that, look in one of these directions and turn yourself as fast as possible to the other."
Which of the four conceptions, which we learned in class, is explained in this text? Discuss this with your neighbour! What kind of instructions should be given to represent the other three conceptions of angle? Play the corresponding scences in groups! Finally formulate those aspects of angle which all four conceptions have in common!

Task 2: "We make a simple protractor"
Fold a sheet in this way that you get folding lines (fig.) as shown on the right. With these folding lines we have got an easy compass-card (fig.).





We can draw in it the most important cardinal points. With the help of this drawing it is also possible to measure angles. It is obvious to choose the sixteenth part of the round angle as a unit for measuring.
For instance, we can name it "siteen" (sixteenth).
How many "siteens" does this angle (fig.) have?

Draw once more angles in your exercise-book and let your neighbour measure them!
What changes if you take other units (e.g. "eighteens", "fouries", ...)?
Finally formulate what we can learn from this!

These two tasks are part of a "system of tasks" for the concept of angle (consisting 69 tasks) which is described in Krainer (1990). There tasks were seen as flexible units for the planning of learning processes of which acting and reflecting by pupils play an important role.
In didactical discussions the connection between tasks and acting (e.g. Christiansen/Otte 1986, Nohda/Becker 1990) is given more and more consideration, we should not underestimate the necessity of reflecting.

The stressing of acting and reflecting by pupils is an expression of the view to see mathematics education as a constructive learning process and not as a mapping from mathematics into pupils' heads. Mathematics should be learnt as a creative language which helps us to communicate better about (and with) the real world. In mathematics education - in comparison to foreign languages - too much of "Grammar", "history and geography of the country" and "learning of vocabulary" is done, but too little is done in communication.
Mathematics is a powerful means of becoming freer - we should not use it as means to restrict pupils' thinking.

References

Christiansen, B., Walther, G. (1986): *Task and activity*. In: Christiansen, B. et al (Eds.), Perspectives on Mathematics Education, D. Reidel Publishing Company, Dordrecht/Boston/Lancaster/Tokyo
Close, G. S. (1982): *Children's understanding of angle at the primary/secondary transfer stage*. Polytechnic of the South Bank, London
Cooper, M., Krainer, K. (1990): Children's recognition of right-angled-triangles in unlearned positions. *Proceedings of the 14th International Conference for the Psychology of Mathematics Education*, México, Vol. 2, 227 - 234
Krainer, K. (1990): *Lebendige Geometrie. Überlegungen zu einem integrativen Verständnis von Geometrieunterricht anhand des Winkelbegriffes*. Lang, Frankfurt a. Main/Bern/New York/Paris
Mitchelmore, M. (1989): The development of children's concept of angle. *Proceedings of the 13th International Conference for the Psychology of Mathematics Education*, Paris, Vol. 2, 304 - 311
Nohda, N., Becker, J.P. (1990): Paradigm of 'Open-Approach' Method in the Mathematics Classroom Activities - Focus on Mathematical Problem Solving. *Proceedings of the 14th International Conference for the Psychology of Mathematics Education*, Mexico, Vol. 2, 201 - 208
Piaget, J. et al (1974): *Die natürliche Geometrie des Kindes* (orig. La géométrie spontanée de l'enfant), Klett, Stuttgart

THE ANALYSIS OF SOCIAL INTERACTION IN AN 'INTERACTIVE' COMPUTER ENVIRONMENT

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The problematic relationship between developing a computerprogram for solving a mathematical problem and learning mathematics will be discussed. This relationship will be studied in the interaction among students when they are jointly to solve a given mathematical problem for which they are to develop a computer program. The thesis of this paper is, that it is the interactively accomplished translation between a program-related perspective and a mathematics-related perspective among coworking students in group situations that facilitates relatively sophisticated mathematical learning. An illustrative example taken from a research project clarifies how students in front of a computer commonly structure their interaction both among themselves and with the computer for producing a mathematics-related and a program-related understanding of the given problem.

1. Theoretical background and general empirical findings

Results of a project which is concerned with the peer-group activities in a programming-environment will be given and discussed. The project was funded for 2 years by the German Research Foundation "DFG" (see KRUMMHEUER 1989, 1988, 1987, 1986). Over a period of 5 weeks peer-groups have been observed and videotaped during their joint group-work at a computer. The main focus of the analysis is on the routine use of a computer with a special interest concerning the effectiveness of computer-use in mathematical learning situations.

The project is embedded in the theoretical approach of the interactional theory of learning and teaching mathematics as it has been developed by BAUERSFELD, VOIGT and KRUMMHEUER (1986), based on Symbolic Interactionism and Ethnomethodology (BLUMER 1969, CAZDEN 1986, GARFINKEL 1967). The main interest is the reconstruction of everyday structures in mathematics teaching-learning processes and the clarification of the relationship between these structures and the content-specific learning of the individual. For this reason the study is focussed on already relatively experienced classes in programming. All of them have attended a BASIC-programming course for at least 1½ years. Additionally the members of each group are accustomed to working together for at least half a year. The students are 8th to 10th graders, those in the presented episodes are 10th graders.

The general findings of the study can be summarized als follows:

(a) The possibility of learning was rather given when the students developed different interpretations of the problem which they further pursued in their cooperative problem-solving attempts.

(b) Learning was also facilitated when the students tried to compare these alternative interpretations argumentatively.

(c) Learning of the individual took place when these different interpretations generated a cognitive conflict and the comparative argumentations of the interaction helped him to cope with this internal conflict.

These findings will be illustrated more concretely in the following example. Three sections taken from a group activity give an insight into a cooperative problem-solving process, which in comparison with all other observed processes, seems to be the most sophisticated one. One reason for the relative success in this example seems to be the interactive production of a relationship involving both a mathematics-related and a program-related interpretation of the problem-solving situation (see KRUMMHEUER 1989). Whereby a "mathematics-related interpretation" means, that the students interpret the situation as one similar to their math class, accordingly "program-related interpretation" means, that the students interpret the situation as one of their computer class.

Other than this distinction, both kinds of interpretation also have common characteristics; they are firstly interpretations by the participants of the situations and not by external mathematics or computer experts and secondly they have already stabilized into patterns of interpretation, which have evolved in the commonly shared mathematics resp. computer practice of the students. With regard to GOFFMAN (1974) these kinds of routinized and patterned processes of interpretation are called 'framing'. Thus the relative success of the observed episode can now be formulated as the interactive production of a distinction between their mathematical framing and their computer-related framing.

2. An Example

The groups are given among others the following problem.

Take any number. Call it U(0). One can determine a sequence of numbers out of it, for example:

$$U(1) = 6/(5 - U(0))$$
$$U(2) = 6/(5 - U(1))$$
$$U(3) = 6/(5 - U(2))$$

etc.

Your tasks

1. Develop a program, with which one can calculate such a sequence of numbers.

2 Alternate the initial value U(0). Compare the different sequences of numbers. Can you find a rule behind it? (for this problem see for example OLDKNOW/SMITH 1983)

TRANSCRIPT - FIRST SECTION

After receiving the above problem the students of the here selected group wrote the following on a piece of paper:

```
    U(1)   U(0)
  6 = 6 :(5 - 4)
 -6 = 6 :(5 - 6)
 11 = 6 :(5 + 6)
```

After this the following dialogue arises:

1 D: Look here, this is U zero. This is that, there. And five minus four is one, right. Now, six divided by one equals six.

2 A: Well...agree

3 D: And six is U one, And now back here we are supposed to put U one. This means, the number goes here. Now, then the six here, this is five minus one

4 C: Yes.. the result is minus six

5 B: What, what is the result, then?

6 C: Minus six

7 D: six divided by minus one equals six. That's the way, that we are supposed to do it practically.

8 C: Yes, let's write a program

The first section takes place after several calculations with the use of paper and pencil, that is with mathematics-related means. D's words in 1 "look here" lead to a common focus on the three written equations. By pointing with his finger D demonstrates, how the calculated value U(1) gets the initial value for U(2) and so on. The iterative structure of the defintion of this number-sequence has been exemplified. C's "yes let's write a program" in 8 is interpreted here as the agreement of the whole group, that D's demonstration is acceptable for them. A commonly shared understanding of the iterative structure has been interactively emerged on the basis of their mathematical framing of the problem situation (see KRUMMHEUER 1989)

TRANSCRIPT - SECOND SECTION

After a short while the group continues with the following dialogue:

10 C: Please input U zero. Go, write INPUT. First you have to tell me

11 D: First of all, INPUT U zero. And then...

12 B: what do we call the variable?

13 D: ...FOR

14 C: From zero

15 D: FOR I equals one TO hundred FOR NEXT loop

16 C: You have to do the DIM...right

17 D: And, U I equals six divided by parenthesis open five minus U, parenthesis open I minus one, parenthesis, parenthesis closed. ...And NEXT I.

18 A: Sounds reasonable.

The developed program is:

```
99 DIM U(100)
100 INPUT "U VON 0"; U(0)
110 FOR I = 1 TO 100
120 U(I) = 6/(5 - U(I-1))
125 PRINT "    "U(I)
130 NEXT I
```

In this episode the dialogue of the students during their development of the program is presented. We can reconstruct, how the students transform the idea of starting with an initial value, like 4, into the BASIC-command INPUT U(0). Further we can reconstruct, how they transform their non-verbally based understanding of an iteration in the first section into the formal program-structure of a FOR-NEXT loop. Whereas in the first section the agreement was still based on the concret demonstration of the change of the calculated number into an initial number, here in the second section we can see, that the ideas of iteration are shaped into a rather formal BASIC-text.

Starting, for example, with the initial value U(0) = 4 the program produces the following output. (In order to save space the output here is structured in a slightly different way:)

4, 6, -6, .5454546, 1.346939, 1.642458, 1.787022, 1.887428, 1.915358, 1.94512, 1.964071,
1.976331, 1.984344, 1.989617, 1.993102, 1.995412, 1.996946, 1.997966, 1.998645, 1.999097, 1.999398,
1.999599, 1.999733, 1.999822, 1.999881, 1.999921, 1.999947, 1.999965, 1.999977, 1.999984, 1.99999,
1.999993, 1.999995, 1.999997, 1.999998, 1.999999, 1.999999, 1.9999992, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2

TRANSCRIPT - THIRD SECTION

After several runs with different inputs the group recognizes, that the sequences of numbers come close t the number 2. Now the question arises, if this is also mathematically correct. The group returns to its written notes (see first transcript-section) and starts discussing

50 D: Six divided by three must come. Therfore it needs a two here. ...Right, if there is the two once, then fi minus two appears all the time. Is three, six

51 C: Wait. Two, this is always the number three.

52 B: Right, why don't we look, if this is right even when the numbers are big. May be it is more, one million.

The students let the program run with U(0) = 1000000.

53 D: It comes to the number immediately.

54 C: If it is the two, divided by three, equals two. Thus it comes again and again.

55 D: Practically....

56 C: Sure, it's logical.

57 D: It has to be this way.

The boys had several RUNs and perceived, that despite different initial values, at the end always the number 2 appears. They are exited and astonished. They are looking for a validation or an explanation. They turn back to their previous mathematics-related activities. In 50 D knows from the computer that a 2 must be the result. Looking at the expression 6/(5-U) it is logical for him, that 5-U(n) needs to be 3. It follows that U(n) has to be 2.

Beyond this clear mathematical inference it is important to note, that D integrates the result of the computer in his argumentation and that he tries to verify whether those computer-results have a mathematical rationale. In 51 C reconfirmes D's argument. And in 52 B agrees, too. Another check with a big number convinces the group, that the convergence to the number 2 is logical. By the way the group later also examines the crucial initial values 3 and 5 (see KRUMMHEUER 1989).

3. Resumee

Empirically the most interesting part of this interaction seems to be, that

(a) the students can frame their activities in two alternative ways, in a mathematical and in a program-related way and that

(b) the students can signalize to each other, when and that they want to switch from one frame to the other one.

Empirical studies of computer-work in school-situations show that generally students do not establish such an elaborated culture of problem solving. The so often criticized "trial and error" or "spaghetti programming" style (PERKINS 1985) just describes this one-sided computer-fixed orientation in students'work at the computer (see also DEKKERS et al. 1983, KULIK et al. 1983, PEA/KURLAND 1984, WILLETT et al. 1983, WISE/OKEY 1983).

The students in the example establish conjointly two different kinds of framings, computer-related and mathematics-related. For them these two perspectives are qualitatively distinct from each other and they develop an argumentation that consists of parts from both perspectives, thus coordinating them. This kind of structuring an interaction is called a "working interim" (see KRUMMHEUER 1991, BAUERSFELD et al. 1988). As several other analyses about the discourse of regular math classes show, one condition for the posibilty of learning is the establishment of such a working interim. In recent research about designing computer software this issue of alternative perspectives has been emphasized as a means of facilitating learning, for example by presenting different embodiements on the screen and stimulating the students to coordinate them (SCHWARZ/ZEHAVI 1990).

The framing-difference in the regular classroom discourse is characterized by the asymmetrical discourse involving qualitatively differently framed activities of the teacher and the students. In the presented three episodes the students by themselves reassemble a similar working interim in their symmetrical discourse by means of their activities with a so-called "interactive" programm environment. It is not the computer technology that enables this transference, but the habitual practice of the students that transfers the regular patterns of classroom interaction into their interaction in front of the screen.

4. References

BAUERSFELD,H./KRUMMHEUER,G./VOIGT,J(1988): Interactional theory of learning and teaching mathematics and related microethnographical studies.in: STEINER,H.G./VERMANDEL,H.:(eds.): Foundations and methodology of the discipline mathematics education (didactics of mathematics). Antwerpen: University of Antwerp, 1988

BLUMER,H.(1969): Symbolic interactionism. Perspective and method. Englewood Cliffs, NJ: Prentice Hall

CAZDEN,C.(1986): Classroom discourse. in: WITTROCH,M.C. (1986, ed.): Handbook on teaching. New York: Macmillan

DEKKERS,J./DONATTI,S.(1983): The integration of research studies on the use of simulation as instructional strategy. in: Journal of Educational Research 74(1981)6, p. 425-427

GARFINKEL,H.(1967): Studies in ethnomethodology. Englewood Cliffs, NJ: Prentice Hall

GOFFMAN,E.(1974): Frame analysis. An essay on the organisation of experience. Cambridge: Harvard Univerity Press

KRUMMHEUER,G.(1983): Das Arbeitsinterim im Mathematikunterricht. in: BAUERSFELD,H./BUSSMANN,H./KRUMMHEUER,G./LORENZ,J.H./VOIGT,J.(1983): Lernen und Lehren von Mathematik. Analysen zum Unterrichtshandeln II. Köln: Aulis, 1983b
KRUMMHEUER,G.(1986): Interaktionsanalysen am Computer. in: Beiträge zum Mathematikunterricht. Bad Salzdetfurth: Franzbecker, 1986
KRUMMHEUER,G.(1987): "Technisierte" Kommunikation und Mathematiklernen. in: Beiträge zum Mathematikunterricht. Bad Salzdetfurth: Franzbecker
KRUMMHEUER,G.(1988): Technisierte Kommunikation im Kontext kooperativer Lösungsprozesse. in: FIEHLER,R./WEINGARTEN,R.(Hrsg.): Technisierte Kommunikation. Opladen: Westdeutscher Verlag
KRUMMHEUER,G.(1989): Die menschliche Seite am Computer. Studien zum gewohnheitsmäßigen Umgang mit Computern im Unterricht. Weinheim: Deutscher Studien Verlag
KULIK,J.A./BANGERT,R.L./WILLIAMS,G.W.(1983): Effects of computer-based teaching on secondary school students. in: Journal of Educational Psychology 75(1983)1, p. 19-26
LEHMANN,J./LAUTERBACH,R.(1985): Die Wirkung des Computers auf Wissen und Einstellung. in: LOG IN 5(1985)1, p. 24-27
OLDKNOW,A.J/SMITH,D.V.(1983): Learning mathematics with micros. Chichester: Ellis Horwood
PEA,R.D/KURLAND,M.(1984): On the cognitive effects of learning computer programming. in: New Ideas in Psychology 2(1984)2, p. 137-168
PERKINS,D.N.(1985): The fingertipp effect. How informationprocessing technology shapes thinking. in: Educational Researcher Aug/Sept. 1985, p. 11-17
SCHWARZ,B./ZEHAVI,N.(1990): A perfect student expert for function characteristics. in: Zentralblatt für Didaktik der Mathematik 1990, 5, p. 178-183
WILLETT,J.B./YAMASHITA,J.J.M./ANDERSON,R.D.(1983): A meta-analysis of instructional studies in science teaching. in: Journal of Research in Science Teaching 20(1983)5, p. 405-417
WISE,K.C.(OKEY,J.R.(1983): A meta-analysis ot the effects of various science teaching strategies on achievement. in: Journal of Research Science Teaching 20(1983)5, p. 419-435

CAN CHILDREN USE THE TURTLE METAPHOR TO EXTEND THEIR LEARNING TO INCLUDE NON - INTRINSIC GEOMETRY?

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ABSTRACT: *The results from the reported research are used to argue that the turtle metaphor is not necessarily a tool for the learning of Intrinsic geometry but also for Euclidean and Coordinate. Three case studies involving 12 year-old Logo-experienced children took place, the children working in pairs within Logo microwolds designed so that controling the turtle could involve the use of all three geometrical systems. The analysis shows that when the children used geometrical ideas, they did not seem to find qualitative differences related to the geometrical system the ideas came from and that they developed a meaningful use for the turtle metaphor in understanding ideas from all three systems.*

THEORETICAL FRAMEWORK

Analyses of the geometrical identity of Turtle Geometry have characterised it as Intrinsic and Differential (Abelson and DiSessa, 1981, Harvey, 1985), i.e. as a geometry built on the idea of locality where a given geometrical state is fully defined by its relation to the immediately previous state (Loethe, 1985). However, little attention has been given to the nature of the geometrical content children can actually learn with the turtle metaphor. That is, there has been limited explicit analysis of the conceptual framework (Vergnaud, 1982) of geometrical notions which children might use while engaged in turtle geometric activities. Do these notions belong to Intrinsic geometry only?

Little research has been carried out concerning the nature of the process by which children identify with the turtle to drive it on the screen, and in particular, the nature of "body - syntonic" learning. Papert argued that identifying with the turtle enables children to bring their knowledge about their bodies and how they move into the work of learning formal geometry, stating that the turtle metaphor enables children to make sense of an idea (Papert, 1980). As research suggests, however, although children employ the turtle metaphor, they do not seem to be doing a lot of geometry, relying rather on their visual perception to control the turtle (Kynigos, in press).

Equally we know very little about children's understandings of the differences and the relations involved in constructing geometrical figures by means of Turtle geometry (and its intrinsic nature) and by methods requiring non - intrinsic points of reference such as, for instance, the centre of a circle or the origin of the Coordinate plane. After an in-depth study of a six year old's thinking in turtle geometric and coordinate environments, Lawler argued that the understanding of Turtle geometry and Coordinate geometry depended on disparate fragments of knowledge which respectively have their roots in a child's very early locomotion and visual experiences (Lawler, 1985). Moreover, he argued that

forming connections between such fragments of different descent is by no means a trivial task for the child. In the author's opinion, Lawler's in - depth probing of Miriam's thinking offered limited, in the sense of generalisability, but very precise evidence of a very close correspondence between the content differences separating two distinct geometrical systems -the Coordinate and the Intrinsic- and the respective fragments of knowledge the child applied to understand them. Lawler regarded these fragments as descendents of intuitive ideas, deriving from early experiences with motion for Turtle geometry and with vision for Coordinate geometry. More recent studies, support this point, by indicating that children find it hard to relate geometrical ideas understood with the turtle metaphor, to the same ideas in "static" environments (see for example, Hoyles and Sutherland, 1990).

Within the above framework, this report is about research carried out to investigate the potential for 11 to 12 year - old children to use the Logo Turtle as a vehicle with which to form understandings of a rather wide span of geometrical ideas; that is, ideas belonging not only to Intrinsic geometry, but also to other geometrical systems such as the Euclidean and the Coordinate.

OBJECTIVES

The question posed by the title of this report is a specific issue emerging from a wider research study, carried out to investigate the process and content of children's geometrical understandings, by closely observing pairs of children working collaboratively within three turtle geometric microworlds especially designed to embed geometrical ideas not only from Intrinsic, but also from Euclidean and Coordinate geometry (Kynigos, 1989a).

A few words about how these geometrical systems were defined in the context of the three microworlds are due here. Embedding geometrical ideas in a microworld involved the careful design of the microworld's "conceptual field" (in the sense of Vergnaud, 1982) so that it would incorporate the above ideas. The prevailing characteristic of the conceptual field of all three microworlds of the research was that they retained the turtle and its state of position and heading as their mathematical entity while embedding ideas from Intrinsic and Coordinate, or Intrinsic and Euclidean geometry.

Firstly, the Turtle in the Coordinate Plane (T.C.P.) microworld embedded intrinsic and coordinatre ideas, equipping the turtle with the ability to measure distances and direction differences based on coordinate locations and directions. The second microworld incorporated intrinsic and Euclidean ideas, also equipping the turtle with distance and direction measuring instuments, but based in this case on plane locations relative to the turtle's path (Post Distance Direction - P.D.D. microworld, adapted from Lawler, 1985). Finally, the Circle microworld was based on constructing geometrical figures given the choice amongst four circle procedures constructed by the children themselves and then used as primitives. These procedures used differing combinations of intrinsic and Euclidean ideas in the method by which the circle was constructed.

The way in which ideas from the three systems were incorporated in each of the three microworlds was by certain additional primitives providing the children with a choice of method by which to change the turtle's state. Changing the turtle's state would thus involve using an Intrinsic or a non - Intrinsic (Euclidean or Coordinate) idea. For instance, deciding on an input to a state - change, without referring to places outside the turtle's immediate vicinity, involved the use of intrinsic ideas (e.g. RT 45). Making such a decision in the T.C.P. microworld by referring to a distant point in the plane by means of an absolute system of reference involved using Coordinate ideas (e.g. SETH TOWARDS -20 20). In the P.D.D. microworld, deciding on an input by referring to a point of a geometrical figure away from the turtle's position, but not using an absolute reference system, involved the use of Euclidean ideas (e.g. RT DIRECTION :A, where A signifies a point on the plane, previously "posted" by the turtle). In the Circle microworld, the four circle procedures differ in whether the linear input signifies the side of the polygon approximating the curvature or the radius. They also differ on the state of transparency being on the curvature or at the centre of the circle. The final procedure simulates a Euclidean construction of the circle through turtle "leaps" (in PENUP mode) from the centre to mark dots at equal distances.

By observing children working within the three microworlds, the objective was to investigate the potential for them to use their turtle metaphor to develop understandings of Intrinsic, Euclidean and Coordinate notions. Four aspects of the problem were investigated:

a) the nature of the schema children form when they identify with the turtle in order to change its state on the screen (T.C.P. microworld);
b) whether it is possible for them to use the schema to gain insights into certain basic geometrical principles of Coordinate geometry (T.C.P. microworld);
c) how they might use the schema to form understandings of Euclidean geometry developed inductively from specific experiences (P.D.D. microworld);
d) the criteria they develop for choosing between Intrinsic and Euclidean representations of geometrical ideas (Circle microworld).

METHOD

Ten 11 to 12 year - old children from a Greek primary school in Athens participated in the research. They had previously had 40 to 50 hours of experience with Turtle geometry, working amongst a total of twenty children in groups of two or three in an informal setup with their classroom teacher. The research involved three case - studies of pairs of children engaging in cooperative activities, each case - study within one of the three geometrical Logo microworlds. Three pairs worked for a total of 14 hours in the T.C.P. microworld study, one pair worked for 15 hours in the P.D.D. microworld study and one pair worked for 25 hours in the Circle microworld study. The collected data included hard copies of everything that was said, typed and written. Verbatim transcriptions from audio tape, dribble files and written notes were used respectively. Each case - study involved a design balance between open - ended and task - oriented activities for the children, the researcher intervening accordingly. The latter activities were designed so as to allow the pupils to

choose the strategy used to achieve the set goal. The researcher's interest was focused on the process by which the children solved a task rather than on the actual goal that was set.

RESULTS

The analysis of the data indicates that the children's use of the turtle metaphor did not necessarily entail the use of geometrical notions, i.e. the ideas embedded in the microworlds. This argument, however, does not in turn imply that the children did not use geometrical ideas before and during the research. For instance, the predominent schema for controling the turtle (termed "intrinsic schema" or "turtle schema" for convenience) which the children had formed prior to the research, in conventional Logo, involved two main ideas; that of a turtle action and its quantity and that of sequentiality, i.e. the linear sequencing of Logo commands. Furthermore, the analysis indicates that their increasing use of analytical cues (Hillel and Kieran, 1987) was accompanied by an increasing use of the embedded geometrical notions (Kynigos, in press). The main aim of this section is to describe how the children employed the turtle metaphor in using geometrical ideas from the three systems.

Using intrinsic and coordinate notions.

In the later stages of the T.C.P. microworld study (Kynigos, 1988), all three pairs of children were given the task to use the turtle to join three unlabelled points on the coordinate grid and measure the lengths and angles of the formed triangle. The following example will illustrate different ways in which the children used Intrinsic and/or Coordinate ideas on a specific occasion during the task. On that occasion all three pairs coincidentially found themselves having to measure the internal angle at point -80 -40, while the turtle was in the state shown in figure 1.

Natassa and Ioanna used Coordinate notions in changing the turtle's heading to face point -20 90. Natassa used a Coordinate notion when she changed the turtle's heading with reference to a location described by means of the Cartesian Coordinates, i.e. by typing in: SETH TOWARDS -20 90.

N: "It should look upwards this..."
(she types SETH TOWARDS -20 90)
"Now we'll tell the protractor..."
I: "We'll find the angle now..."
(they type PR DIRECTION 70 -70)

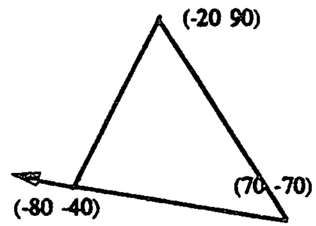

Figure 1
Measuring an internal angle in the T.C.P. microworld

She therefore changed the heading with a non action - quantity method: what causes a heading change with the SETH TOWARDS command is a description of a location on the Cartesian plane. The children then performed a measurement

without seeming to relate it to the quantity of a turtle action *("...now we'll tell the protractor...")*. Ioanna's comment indicates that the children perceived the measurement independently of their action - quantity schema.

Anna and Loukia's strategy also consisted of two parts: changing the heading and then measuring the angle. To change the turtle's heading, however, they seemed to combine an Intrinsic and a Cartesian idea; by typing - in RT DIRECTION [-20 90], Anna seemed to perceive the change of heading as a turtle turn. On the other hand, what determined the quantity of that turn (i.e. the input to RT) was a reference to a plane location by means of the Cartesian Coordinates. Anna's verbal explanation to her peer supports the argument:

A: "We'll turn this way... (she means towards -20 90) *and then from here we'll ask it... we'll say PR DIRECTION... 70 -70 and she'll tell us."*

Instead of saying something like: "We'll turn this much..." (i.e. action - quantity), she said: *"We'll turn this way..."* (i.e. action - direction); furthermore, the latter part of the strategy, was expressed as a non - action measurement *("we'll ask it... and she'll tell us")*. In general, although the children used both types of geometrical ideas, they found it hard to relate one type to the other. This would support Lawler's view that Intrinsic and Coordinate ideas are based on different types of intuitions.

Using intrinsic and Euclidean notions.

The schema the children formed in order to use Euclidean notions did not involve such drastic changes to their intrinsic schema, such as abandoning the notions of action - quantity and sequentiality. The Euclidean notions were used in deciding on quantities and in referring to parts of a figure away from the turtle's position. Here too, however, the common basis for using Intrinsic and Euclidean notions was the ability to use the turtle schema, i.e. to think in terms of the turtle changing its state on the screen. An example of an episode illustrating children's use of their turtle schema and Euclidean ideas will be taken from the P.D.D. microworld study (Kynigos, 1989b) where this issue was in focus.

The episode in question took place during the final stages of the case - study, where the children were asked to carry out a project of their own choosing. They decided to make a "triangle with a stripe" and used a procedure for an isosceles triangle (LASER :N :P) which they had written earlier. The first input to LASER signified the length of one of the equal sides of the triangle and the second signified the size of one of the equal internal angles. How the children came to construct the procedure and what this meant for their learning can be found in (Kynigos, 1989b). Relevant here is not the procedure itself, but the fact that the children knew that the triangle they made was isosceles and that the input :P they gave to the procedure signified the internal angle HIM (or HMI, fig. 2).

The episode in question refers to the process by which Nikos decided to turn the turtle from state "a" to state "b" as shown in figure 2, illustrated by the way he explained it to his peer, Philip. Rather than using the DIRECTION command (i.e.

RT DIRECTION :M) , he added the two angular components of the respective right turn, thus using Euclidean angular properties in order to perform an intrinsically - oriented action and its quantity in degrees by typing RT 230.

(turtle in state a, Nikos addressing Philip)

N: "RT... 230. You know why I'm putting 230? Because 230... look... from here to here it's 180 isn't it? (P. agrees) This angle, isn't it 50? (P. agrees again) O.K., 180 and 50 doesn't it make us... for the turtle to go woop, woop and it does us 230?"

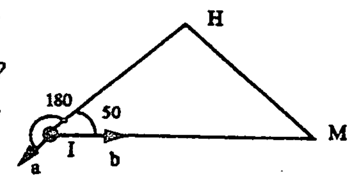

Figure 2
Adding two components of an angle to turn the turtle

Notice how Nikos made an angle calculation *("180 and 50 doesn't it make us...")*, seemed to pause to think and finally seemed to change to talking about the outcome of the calculation as if it was the quantity of a turtle turn *("...for the turtle to go... and it does us 230.")*.

In this context of employing the intrinsic schema to use ideas from the three geometrical systems, the analysis of the data from all three studies showed that it is possible for children to generate learning environments akin to the ones generated in studies of children using conventional Logo. This issue, which was in research focus during the P.D.D. microworld study in the context of Euclidean geometry, supports the argument that turtle geometric environments need not be restricted to Intrinsic geometry if they are to preserve their dynamic characteristics.

Choosing between intrinsic and Euclidean notions.
The above conclusion is re-inforced by the Circle microworld study which indicated that the children did not find inherent qualitative differences between Euclicean and Intrinsic notions used while employing their turtle schema. That is, they did not find one type of notion harder to understand than the other (Kynigos, 1989c).

To illustrate this point, an example will be given from the late stages of the case - study, discussing Valentini's and Alexandros' final solution of a structured task consisting of a figure of three circles whose centres were placed at the vertices of an equilateral triangle (as shown in figure 3). The children had four circle procedures to choose from called CIR4 :S, CIR9 :R, CIR19 :R and TC :R. The first simulated the classical intrinsic construction of a circle, the input being the length of the side of the polygon approximating the curvature. The second and third constructed the circle in the same manner, but differed in that the input signified the length of the circle's radius. Moreover, in the CIR19 :R procedure, the turtle started and finished at the centre of the circle. The TC :R procedure constructed the circle by repeating back-and-forth equidistant leaps in PENUP

mode, from the same spot, making a visible dot at the end of each leap and a small turn at the centre spot.

The children's solution indicates a coherent use of a combination of Intrinsic and Euclidean ideas; the latter involve the use of the CIR19 circle procedure, where the input and the state of transparency refer to the length of the circle's radius and to its centre (both notions "unkown" to an intrinsic turtle); they also involve the use of the notion of a circle's centre in order to explain the connection among the positions of the three circles. Valentini's written explanation of why she used that particular circle procedure (CIR19) illustrates the above point:

V: "I used the CIR19, because at the place where one of the lines of the triangle ends and the other one starts, is the point which is in the middle of the circle."

```
TO TR :S
  RT 30
  REPEAT 3 [CIR19 :S FD 50 RT 120]
END
```

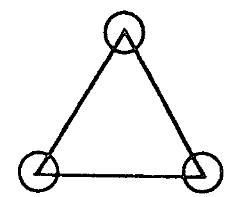

Figure 3
Using Intrinsic and Euclidean ideas while solving a structured task

In explaining to her peer the turtle's turn of 120 degrees in the TRI procedure shown in figure 3, however, Valentini used the Intrinsic notion of partitioning a total turn of the turtle in three equal turns (the stucturing of the procedure was achieved through her initiative); referring to the turtle, she said that *"there were three turns which were... 360 and therefore dividing 360 by 3 gives 120 degrees for each turn..."*. Furthermore, in explaining why the total turn was 360 in this case, she coherently used as an example the turtle's total turn in the children's TC procedure, i.e. the procedure embodying the Euclidean definition of the circle as the set of points equidistant to the centre point!

So, as in Valentini's use of both Euclidean and Intrinsic ideas in the same task, the children alternately employed ideas from the two systems without seeming to find one type harder to understand than the other. The Circle microworld study thus indicates that children may find the intrinsic schema equally useful in learning Intrinsic and Euclidean ideas.

CONCLUSIONS

In general, the research has elaborated that the children did not seem to find qualitative differences in understanding Intrinsic, Euclidean or Coordinate notions to control the turtle in the respective microworld environments. Employing the intrinsic schema did not necessarily imply that the children were aware of using geometrical notions. Furthermore, when geometrical notions were used, they were not necessarily Intrinsic. This finding would suggest that the turtle metaphor need

invite children to learn only intrinsic geometrical ideas, as implied by Papert.

In conventional Logo, the turtle metaphor seems designed to invite children to employ experiences of bodily motion to understand intrinsic ideas. In the microworlds of the present study, however, there were indications that the children drew upon their own experiences of movement in the real world in their use of the metaphor of a turtle equipped with the means to refer to distant points on the plane. What is more, it is suggested that this turtle metaphor invited the children to use a wider span of geometrical ideas (as shown in the specific findings of the three case - studies) and made equal, if not more, sense than the strictly intrinsic turtle; after all, children acting out a turtle path would not stumble over a chair that was in their way!

REFERENCES

Abelson H. and DiSessa A. 1981: *Turtle Geometry: The Computer as a Medium for Exploring Mathematics*, MIT Press.

Hillel J. and Kieran C. 1987: 'Schemas Used by 12 year-olds in Solving Selected Turtle Geometry Tasks', *Recherches en Didactique des Mathematiques*. p. vol. 8, no 12, p. 61-103.

Hillel J. 1986: 'Procedural Thinking by Children Aged 8-12 Using Turtle Geometry', *Proceedings of the Tenth International Conference for the Psychology of Mathematics Education*, p.433-438.

Hoyles C. and Sutherland R. 1990: *Logo Mathematics in the Classroom*, Routledge.

Kynigos C. 1989a: *From Intrinsic to non - Intrinsic Geometry: A Study of Children's Understandings in Logo - Based Microworlds*, unpublished Doctoral Thesis, Institute of Education University of London.

Kynigos C. (in press): The Turtle Metaphor as a Tool for Children's Geometry, in Learning and Mathematics. Hoyles, C. and Noss, R. eds, M.I.T. press.

Kynigos C. 1989b: "Can Children use the Turtle to Develop Understandings of Euclidean Ideas in an Inductive Way?" *Proceedings of the Fourth International Conference for Logo and Mathematics Education*, The Israeli Logo Centre, Dept. of Education in Science and Technology, Technion - Israel Institute of Technology, editors: U. Leron, N. Krumholtz.

Kynigos C. 1988: "Constructing Bridges from Intrinsic to Cartesian Geometry", *Proceedings of the Twelfth International Conference for the Psychology of Mathematics Education*, Veszprem, Hungary, editor: A. Borbas.

Kynigos C. 1989c: "Intrinsic Versus Euclidean Geometry: Is the distinction important to Children learning with the Turtle?" *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education*, vol. 2, Editor Prof. G. Vergnaud, J, Rogalski M. Artigue G.R. DIDACTIQUE CNRS Paris V, Laboratoire PSYDEE. Paris, France.

Lawler R.W. 1985: *Computer Experience and Cognitive Development. A Child's Learning in a Computer Culture*, Ellis Horwood.

Loethe H. 1985: 'Geometrical problems for a Turtle with Direction and Distance Finder', *Proceedings of the First Logo and Mathematics Education Conference*, p.123-129.

Papert S. 1980: *Mindstorms. Children, Computers and Powerful Ideas*, Harvester Press.

Vergnaud G. 1982: 'Cognitive and Developmental Psychology and Research in Mathematics Education: Some Theoretical and Methodololgical Issues', *For the Learning of Mathematics*, 3., 2., p.31-41.

PRE-SCHOOLERS, PROBLEM SOLVING, AND MATHEMATICS.

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Abstract

How pre-schoolers solve a novel mathematical task presented to them in a play setting is the focus of this paper. The degree to which the students' attempts conformed to the hierarchy predicted by the Structure of the Observed Learning Outcome (SOLO) taxonomy is examined. Whether their efforts were influenced by their partner is also explored.

Introduction

It is now widely accepted that children have acquired considerable mathematical knowledge before they commence formal schooling. The relevant literature reveals that many have an understanding of numerical invariance when they start school, that they are able to make relative number judgements, are able to count meaningfully, are able to recognize small arrays of objects as number patterns without counting, use appropriately terms like more and less, and that they are able to handle simple addition, subtraction, and spatial tasks (see, e.g., Mansfield & Scott, 1990; Young-Loveridge, 1987). As well, many children are able to devise and use their own written representations of small groups of objects to solve simple mathematics problems (Hughes, 1986; Leder, 1989).

The research reported here focuses on the strategies adopted by a sample of pre-schoolers who were given a mathematical task in a play setting. Of particular concern is the degree to which the students' attempts, or modes of

functioning, conformed to the hierarchy predicted by the Structure of the Observed Learning Outcome (SOLO) taxonomy. Whether their efforts were influenced by the approach and explanations used by the (changing) partner with whom they were working is also of interest.

The SOLO taxonomy

A detailed description of the SOLO taxonomy can be found in Biggs & Collis (1982). The model has been used not only to classify responses, but also to classify items or questions. It has been widely used in mathematics.

Briefly, the taxonomy allows for five levels of response structure: prestructural, unistructural, multistructural, relational, and extended abstract. Prestructural responses are those which consist of a single piece of irrelevant or inappropriate information. A response in which only one relevant piece of information is used is deemed to be unistructural. A multistructural response reflects the use, typically in sequence, of several relevant pieces of information given in the stimulus. Responses which indicate the integration of all the relevant information given are considered relational, while those which in addition generalize to abstract concepts and principles are defined as extended abstract.

Method

Subjects

The sample comprised seven students, four boys and three girls, who attended a pre-school in the metropolitan area of Melbourne four half days each week. The students were in the 'four-year-old' group, i.e., they all turned four before June

30th that year. While detailed data were gathered for all members of the group, the results reported here focus primarily on one student.

The setting

In an information booklet given to parents the staff indicated that they particularly valued the following: " . a positive self concept; tolerance and fair treatment of other people; the development of language as an effective means of communication; involvement in music, movement, mathematics, science and art experiences, for the enormous learning capacity within these areas and for their aesthetic value".

The mathematical activities were typically developed in a small group setting. Such sessions generally lasted between 15 and 30 minutes and tended to be teacher-lead.

Procedure

The tasks discussed here were carried out during the second half of the academic year. By this time Jenny, a member of the project team, had become a very familiar figure to the students.

An office area had been created in one corner of the kindergarten, well before the data gathering sessions. It was a popular play area, with its typewriters, paper, crayons, and letter box. The tasks set built on the children's familiarity with, and liking for, this equipment. No formal teaching took place during the sessions described in this paper. While Jenny was an integral part of each three-member group she was neutrally accepting of all responses made by each child.

e task

Working in groups of three, two children and Jenny, the task was to distribute some letters (either specified or chosen by the child) to each member of the group, post the letters in the letter box, guess how many letters were inside the box, confirm this by checking how many letters it indeed contained, and finally "put something on paper" about the game that had just been played. The children performed all the steps, except that the first time the game was played with a particular group Jenny distributed the letters. She gave each person the same number of letters (say two, or three), and placed the remainder aside.

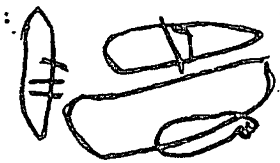
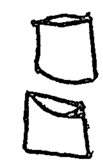
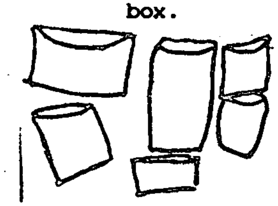
The task was a novel one for the children and was quite different from the way in which the office material had previously been used by them. Asking for written representations of mathematical situations was also unique to the game being played. No such request was ever made by the teachers who taught the pre-school program.

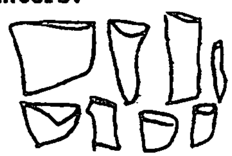
The data presented trace the actions and descriptions of one student, Chris, as he worked with his different partners on four separate occasions over several weeks. Some reference is also made to his interactions with his partners.

Results

An overview of the results obtained is presented in Table 1. While for most problems there was a good correspondence between Chris's verbal explanations and written representations this was not the case for each child. Some drew pictures which were quite unrelated to the task they were given.

Table 1: Summary of Chris's results for the letter posting game.

Problem (Group) *(1)	N(letters distributed)	N(letters in post box) Guess	N(letters in post box) Confirm	Written representation
1. (Jenny, Chris, Pat)	2, 2, 2.	Two. Changes this to six when partner says six.	Counts: 1,2,3, 4,5,6.	
2. (Jenny, Chris, Pat)	5, 6, 1.	Seven. Changes this to 10 when partner says 10.	" 'cos I think there is ! Counts 1, 2, ... 11,12, there's 12.	
3. (Jenny, Chris Pat)	2, 2, 2.	I think (counting to himself) 1, 2, 7, 8.	Empties the box. Counts 1, 2, ...5, 6.	This shows I posted two. 
4. (Jenny, Janet, (Chris)	2, 2, 2.	Counts quickly to himself: 6.	Counts the letters as he takes them out of the box: 1, 2, ...6.	This shows how many letters I've got.  And this shows how many there are in the box. 

Problem (Group) *(1)	N(letters distributed)	N(letters in post box) Guess	Confirm	Written representation
5. (Jenny, Janet Chris)	3, 3, 3.	10	Watches partner count: 1, 2, ... 8, 9/ Says: 'cos, 'cos, three plus one three plus two three more =, (holds up nine fingers)	Counts as he draws. Mutters, at one stage: I have to draw three more. 
6. (Jenny, Janet Chris)	4, 4.	counts to himself: 8	counts the letters. 1, 2, ... 8 Justifies his answer with 'cos we're good counters.	
7. (Jenny, Jeff, Chris)	2, 2, 2.	6	two & two make four. two & two & two makes six. When asked how he knows this, Chris replies 'cos I do and I've counted them.	I need to draw six 

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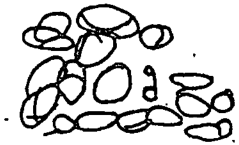
(Meanwhile Chris's partner is also drawing busily) when Chris notices Jeff's attempt, he comments: "Oh, that's more than six".



Problem (Group) *(1)	N(letters distributed)	N(letters in post box) Guess	Confirm	Written representation
8. (Jenny, Jeff, Chris)	4, 4, 4,	Counts to himself 1, 2,.. 12. Partner comments: "I think it is 12". Chris: It is 12.	Because it is 12. I counted them.	Counts to himself as he draws 
9. (Jenny, Richard, Chris)	4, 4, 4	Chris: 12 Partner: 13	Chris: It's 12 Count them: 1, 2, ... 11,12. I'm drawing 12 letters... And 'cos 'cos 4 & 4 & 4 make 12.	

(Initially Chris draws nine letters. When he stops to count them and reaches nine, his partner says "Nine, there was 9. You were right." Chris responds "No! I have to draw more." He proceeds to draw another three letters.

Problem (Group) *(1)	N(letters distributed)	N(letters in post box) Guess	Confirm	Written representation
10. (Jenny, Richard, Chris)	5, 5.	Counts softly: 10	I know it's 10 because I counted them.	

*(1) The name underlined indicates who distributed the letters.

Conclusions

Because of space restrictions, the implications of the results presented are not explored in detail here. Rather, a number of noteworthy conclusions are listed.

1. There are striking differences between Chris's solutions to the early and later tasks. On the first problems his attempts (both verbal and written) are at the prestructural level. By tasks 7 and 9 his verbal explanations are relational.

2. Chris's written representations for tasks 7 and 9 are multistructural and do not (yet?) reflect the trend towards relational reasoning apparent in his verbal explanations.

3. Chris's written elaborations for problem 4 illustrate responses at both the unistructural and multistructural levels: one which shows "how many letters I've got", followed by one which shows all the letters in the box.

4. While initially Chris is influenced by the responses of his partner (e.g. problems 1 and 2) once he has selected a strategy about which he feels confident his solutions are resistant to inappropriate suggestions from others. Thus in problems 7 and 9 he rejects the solutions offered by Jeff and Richard respectively, using multistructural and relational reasoning to do so.

5. The data confirm that children are capable of quite sophisticated mathematical reasoning, before the commencement of their formal schooling. The SOLO taxonomy is a useful mechanism for describing the spontaneous solutions offered by them to mathematical problems presented in a familiar play setting.

Note: I wish to express my sincere thanks to Jenny Hoff for her help in the collection of these data and to the Australian Research Council for their financial assistance.

References

Biggs, J.B., & Collis, K.F. (1982). Evaluating the quality of learning: the SOLO taxonomy. New York: Academic Press

Hughes, M. (1986). Children and number: difficulties in learning mathematics. Oxford, England: Basil Blackwell

Leder, G.C. (1989). Number concepts of preschool children. Perceptual and Motor Skills, 69, 1048-1050

Mansfield, H., & Scott, J. (1990). Young children solving spatial problems. Proceedings 14th PME conference (vol.2), Mexico, 275-282

Young-Loveridge, J.M. (1987). Learning mathematics. British Journal of Developmental Psychology, 5, 155-167

LA FUSEE FRACTION: UNE EXPLORATION INUSITEE DES NOTIONS D'EQUIVALENCE ET D'ORDRE[1]

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Abstract: This paper reports observations made on children from a multi-aged class (4th, 5th and 6th grades) experimenting a micro-world specifically constructed to allow the exploration, from an unusual perspective, of the notions of equivalence and order in fractions. The "rocket fraction" is one of the several micro-worlds in a new software entitled "Les deux tortues". The purpose of the rocket fraction micro-world is to allow children, first, to construct and explore the concepts of equivalence and order, and, second to formulate rules and laws that describe and explain phenomenons related to these concepts. Children working alternately with paper and pencil and on the computer is an important pedagogical aspect of the micro-world. The children's reactions and levels of involvement are described and discussed in the light of the pedagogical goals of the rocket fraction.

Le micro-monde de la fusée fraction: contexte général

Le micro-monde de la fusée-fraction qui fera l'objet de notre étude est tiré d'un ensemble didactique nommé "Les deux Tortues" (Côté, 1989; 1990; sous presse). L'ensemble didactique "les deux tortues", vise à travers différents scénarios d'activités à intégrer l'utilisation de l'ordinateur dans le contexte de l'enseignement des mathématiques au second cycle du primaire et au premier cycle du secondaire. Il comprend un logiciel, un guide pour l'enseignant et des fiches de travail pour l'élève. Les activités proposées se rapportent à des thèmes tantôt associés aux systèmes de nombres, à la mesure, à la géométrie ou aux débuts de l'algèbre. Le logiciel "Les deux tortues" tout en s'inspirant de la tortue propose une redéfinition des commandes de base qui la différencie et l'éloigne des versions usuelles de programmation Logo.

L'ensemble didactique "les deux tortues" apparaît comme une réponse au besoin souvent souligné de mieux faire le lien entre l'ordinateur (en l'occurence, la tortue) et la mathématique. De tous temps le lien entre Logo et la mathématique a été un point important souligné par les concepteurs et adeptes de Logo (que cela soit dans les écrits initiaux de Papert -- Feurzeig et al. 1969; Papert 1971 -- ou dans des écrits plus récents de d'autres auteurs -- Hillel et al. 1989; Côté 1990). Or selon plusieurs chercheurs, ce lien se trouvent trop souvent camouflé dans le cadre d'une utilisation usuelle de

[1] Although this paper is written in French for purposes of publication in the PME proceedings, the lecture itself will be delivered in English at the PME meeting.

Logo (projets Logo). Déjà plusieurs auteurs (Gurtner, sous presse; Lemerise, 1990; Hoyles & Noss 1987) ont discuté du besoin de "mathématiser" Logo, compte tenu de la tendance des enfants à escamoter la recherche de la compréhension mathématique au profit d'une réalisation purement empirique du projet choisi ou suggéré. "Les deux tortues" ne constitue certes qu'une forme, parmi plusieurs possibles, de réponse à ce besoin. Notre étude d'un des micro-mondes (la fusée fraction) présents dans le logiciel permettra de voir comment le pont entre la tortue et la mathématique fut ici réalisé.

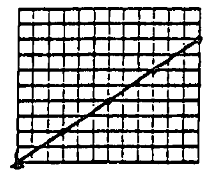
Avant de présenter les caractéristiques spécifiques de la fusée fraction, il conviendrait de souligner deux caractéristiques générales du micro-monde. Le première réfère à une manipulation de la tortue qui ne sera toujours qu'en mode directe, e.i. sans référence à la programmation. La seconde caractéristique réside dans l'alternance constante qui est proposée entre le travail sur papier et le travail sur ordinateur. La simulation de la tortue sur papier apparait tout aussi nécessaire que la simulation à l'ordinateur pour supporter la découverte, la compréhension et la formulation des régles et lois régissant les phénomènes chaque fois abordés.

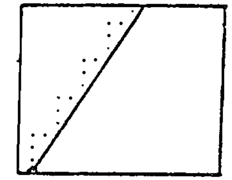
Le micro-monde de la fusée fraction: contexte spécifique

Le micro-monde de la fusée fraction vise 1) à aborder la notion de fraction non plus comme une portion d'un tout, mais comme une relation entre deux nombres et 2) à explorer et comprendre, dans le cadre de cette représentation concrète, les aspects suivants: a) les rôles respectifs du numérateur et du dénominateur, b) les ensembles de fractions équivalentes et c) la mise en ordre de série de fractions (ou rapports).

La représentation concrète utilisée pour ce micro-monde est celle d'une fusée que l'on lance et dont le trajet peut être plus ou moins incliné. La pente de ce trajet est caractérisée par un rapport entre deux nombres que l'on appelle HAUT et DROITE. Pour construire un trajet sur papier quadrillé, on commence par identifier des points appartenant à ce trajet pour ensuite les réunir en traçant un trait avec la régle. Par exemple, pour construire le trajet 2/3, on compte du point de départ de la fusée 2 carreaux vers le haut pour 3 vers la droite, ce qui donne le premier point. On continue de cette façon, 2 vers le haut pour 3 vers la droite, pour trouver les second, troisième et autres points. En réunissant ces points, on obtient le trajet illustré à gauche ci-dessous. De façon similaire, à l'ordinateur, on tape le mot FUSEE (ou simplement F), puis on spécifie les valeurs correspondant à HAUT et DROITE. On observe alors des petits points qui apparaissent correspondant au comptage de carreaux vers le haut et la droite, puis le tracé comme tel.

Par exemple, le trajet 3/2 va correspondre sur l'écran à ce qui est illustré ci-dessous dans le carré de droite.





La fusée fraction est basée sur le fait que la notion de pente rend observable la valeur d'une fraction et par conséquent permet de déterminer visuellement l'équivalence ou la grandeur relative de deux ou plusieurs fractions (ou rapports). Si on trace les trajets 2/3 et 4/6, on constate que c'est exactement le même trajet. Si, par ailleurs, on trace les trajets 3/7 et 4/9, on voit qu'ils sont différents et on peut identifier lequel a la pente la plus forte. On a donc une représentation concrète nous permettant de savoir si deux rapports donnés sont équivalents ou pas, et s'ils ne le sont pas, lequel est le plus incliné, par exemple. On est alors en mesure d'explorer la logique des relations d'équivalence et d'ordre dans les rapports proposés et ce à travers les rôles respectifs du numérateur et du dénominateur. Différentes activités proposées par Côté 1989 ont été pré-expérimentées auprès d'un groupe d'enfants du primaire. La sélection des activités fut faite ici en fonction de nos priorités thématiques (l'équivalence et l'ordre) et en fonction du niveau de scolarisation des participants (second cycle du primaire).

Scénario d'activités proposées

Le scénario comprend tantôt des activités de construction où l'enfant doit procéder à la construction pas à pas d'un phénomème donné (d'abord avec papier crayon, puis à l'aide de l'ordinateur), tantôt des activités d'exploration et de réflexion, où il s'agit surtout de comprendre les phénomèmes rencontrés et, si possible, de formuler des règles ou des lois permettant de les prédire, de les expliciter. Le présent scénario comprend une introduction au monde de la fusée fraction et six activités. Les activités choisies se rattachent principalement aux notions d'équivalence et d'ordre. Dans ce cadre de trajets de fusée, la notion d'équivalence est abordée par le biais d'une recherche des différents rapports produisant un seul et même trajet; la notion d'ordre, pour sa part, est abordée à travers l'identification des facteurs ou lois régissant la plus ou moins forte inclinaison de différents trajets. Une description plus détaillée des activités sera fournie ci-dessous, lors de la présentation des données d'observation.

Caractéristiques des éléves et du contexte d'intervention

Les enfants à qui fut proposé le scénario d'activités proviennent d'un classe multi-âge. Au total, 18 enfants ont expérimenté l'ensemble des activités (4 enfants de 4ième; 8 enfants de 5ième et 6 enfants de 6ième). Les âges varient entre 9 et 11 ans; le groupe se compose de 13 filles et 5 garçons. Les enfants ne connaissent pas Logo; leur initiation à la notion de fraction tient surtout à des expériences informelles vécues en contexte alternatif d'éducation.

Un ordinateur (Macintosh) est installé dans un coin de la classe. Les enfants viennent travailler au projet fraction deux à la fois; entre 3 et 4 séances, d'une vingtaine de minutes chacune, furent nécessaires pour la réalisation des 6 activités proposées. Règle générale, les enfants travaillent individuellement (activité 6 exceptée), mais les échanges et discussions entre partenaires sont toujours hautement recommandés et favorisés.

Cette première application du micro-monde fraction en contexte éducatif est pris en charge par une personne ressource. Un protocle de travail est remis à chaque enfant explicitant les différentes activités et fournissant le matériel nécessaire (papier quadrillé; feuilles d'activité, etc.). Les questions, discussions et commentaires des enfants sont enregistrés sur bandes sonores. Un dossier est constitué pour chaque enfant permettant de conserver brouillons de travail, feuilles-réponses, observations de l'intervenante et verbatim retranscrit à partir des cassettes audio. Le rôle de l'intervenante est essentiellement de tenter à travers questions et commentaires d'expliciter les activités, de participer aux discussions entre enfants et de tenter de favoriser la formulation de réponses le plus explicite possible. Au début, des interventions furent nécessaires pour inciter les enfants à ne pas utiliser l'ordinateur dès le point de départ, mais très tôt le rythme proposé (travail papier crayon, hypothèse/prédiction, travail sur ordinateur, formulation de règle) fut adopté par la majorité des enfants.

Données d'observation pour chacune des tâches

Un brève description des activités précède la présentation des principales observations recueillies à chacune d'elles.

<u>Introduction à la fusée fraction</u>: Suite à une démonstration du mode de construction du trajet de la fusée, l'élève doit lui-même construire, sur papier quadrillé, deux trajets différents. Deux types d'habiletés doivent être exercées ici: la coordination "physique" des actions permettant de relier efficacement les points de départ et d'arrivée et la coordination mentale des opérations permettant d'établir le lien entre le trajet fusée et le couple d'actions (Haut: et Droite:)

Quatre enfants éprouvent des difficultés à bien coordonner leurs actions physiques (faible coordination des déplacements verticaux et horizontaux; confusion dans le dénombrement des cases, etc.). Ces mêmes enfants et quelques autres, semblent avoir de la difficulté à se représenter le rôle exact de la paire d'actions (H: et D:) dans la détermination du trajet suivi par la fusée (centration sur la construction point par point de "l'escalier" plutôt que sur les points finaux délimitant le trajet de la fusée).

Des expériences différentes du micro-monde de la fusée sont aussi observées à travers l'analyse des tracés réalisés. Sept enfants s'en tiennent aux deux trajets demandés: règle générale, ces trajets sont très près l'un de l'autre (4/2, 5/3) limitant l'exploration des trajets possibles à un très faible éventail de possibilités. Les onze autres enfants ont succombé à la tentation de construire trois, ou quatre trajets, explorant ainsi un large éventail de types d'inclinaison des trajets. Enfin, il convient de noter que la reproduction des tracés à l'ordinateur favorise, pour plusieurs, une meilleure compréhension du processus de construction du trajet de la fusée.

<u>Activité 1: Initiation au phénomène de l'équivalence</u>. L'activité demande d'abord que l'élève identifie quelles commandes H: et D: ont pu être utilisées pour le tracé de trois trajets reproduits sur papier quadrillé. Dans un second temps, il doit trouver une façon alternative d'identifier chacun des trois trajets.

Les enfants étudient les tracés, font des hypothèses, les vérifient (certains les vérifiant pour l'ensemble de tous les points, d'autres s'arrêtant à la première confirmation de leur hypothèse). Deux des enfants éprouvent des difficultés de coordination des actions et ne parviendront pas seuls à identifier les tracés. La demande d'identification alternative pour un même trajet surprend beaucoup les enfants. Certains nieront longtemps toute possibilité de le faire; d'autres, se référant au trajet unité (nommé 1/1,3/3, ou autres) inclus dans les trajets proposés, parviennent à saisir le sens des rapports équivalents et peuvent, de façon encore tout empirique, se débrouiller pour appliquer leur nouvelle découverte à d'autres types de trajets. Un travail laborieux et systématique est ici fait sur le papier quadrillé. L'ordinateur viendra confirmer l'adéquacité des trajets alternatifs proposés. La confirmation ou l'infirmation des réponses par l'ordinateur entraîne toujours des réactions très expressives de la part des enfants. De plus, le travail à l'ordinateur facilite pour plusieurs l'identification des relations entre les différentes fractions proposées pour un même trajet (relation additive, relation multiplicative) et permet souvent de tester de nouvelles alternatives (exercices de généralisation de la règle).

<u>Activité 2: Discussion relative à l'équivalence des fractions.</u> La relation entre 1/2 et 2/4 est travaillée et discutée alors qu'appliquée d'abord à une entité, puis à une collection, et enfin à une série d'évènements (trajets).

Cette activité est intéressante pour ceux qui ont récemment découvert la loi "du double" (pour les trajets) -- elle permet de faire des liens avec le monde connu des fractions traditionnelles -- et pour ceux qui, ne l'ayant pas encore découverte, y parviennent en fin d'activité. Certains, par ailleurs, profiteront moins directement de l'activité; convaincus de la non équivalence des trajets 1/2 et 2/4, ils tenteront par tous les moyens possibles d'illustrer qu'ils correspondent à des trajets différents!

<u>Activité 3: Exercice d'identification de rapports (fractions) équivalents (es).</u> Deux séries de six fractions (rapports) sont présentées aux enfants; ces derniers doivent identifier lesquelles produisent des trajets identiques. Travail sur papier quadrillé d'abord; l'ordinateur est recommandée pour vérification d'hypothèses sérieuses seulement.

Deux grandes stratégies de travail sont observées. Une première stratégie, où les enfants font des hypothèses (tantôt basées sur une règle adéquate, tantôt non), les vérifient (minimalement ou exhaustivement) sur papier quadrillé, et si confirmées, les revérifient à l'ordinateur. Paire par paire, toutes les fractions sont ainsi analysées. L'autre stratégie observée est plus empirique (ne part pas d'hypothèse); elle nécessite de reproduire sur papier quadrillé chacun des trajets et d'identifier lesquels sont équivalents. L'ordinateur fournira la confirmation finale des efforts de travail. Il arrive que le travail à l'ordinateur éclaire l'enfant sur des relations entre trajets et lui permette de passer à une stratégie basée sur des hypothèses.

<u>Activité 4: Formulation et application d'une règle relative aux déterminets de trajets identiques.</u> Les enfants ont ici à formuler une loi leur permettant d'identifier rapidement une série de fractions ou rapports sous-tendant des trajets identiques.

A ce stade-ci, trois enfants seulement ont besoin de recourir au travail sur papier quadrillé. La majorité peuvent désormais anticiper quels trajets seront équivalents. Certains tiennent encore à vérifer à l'ordinateur, d'autres déclareront que ce n'est pas nécessaire. Quatorze des dix-huit enfants formulent la règle des multiples ("si l'on multiplie par deux"; "si l'on additionne chaque chiffre à lui-même"); neuf élèves peuvent la généraliser à tous les multiples (en multipliant le numérateur et le dénominateur par 2, 3, 4 ou 5, etc.) Deux enfants, parmi les plus jeunes, refusent toute formulation de loi et maintiennent que c'est en le faisant qu'on voit si les trajets sont équivalents. Deux enfants formulent des règles qui ne sont pas nécessairement adéquates (ex: il y a équivalence des trajets quand les chiffres sont très près: 11/12 et 10/11!).

<u>Activité 5: Prédiction et explication de l'allure de différentes séries de trajets.</u> Six séries de quatre trajets chacune sont proposées. Chaque série a sa propre règle de construction (ex: 2/3, 2/4, 2/5, 2/8; ou encore 2/4, 3/3, 4/2, 5/1). La demande est de prédire comment se comporteront les trajets successifs et de formuler une règle expliquant le pourquoi de tels "comportements". Ici encore, il est recommandé que l'ordinateur soit utilisé en dernier recours. Les comparaisons mentales sont privilégiées, et le travail sur papier quadrillé suggéré pour des besoins très ponctuels.

Pour quatre enfants la prédiction s'avère trop difficile; ils doivent recourir à la construction trajet par trajet. Les explications alors fournies seront surtout descriptives. Les quatorze autres élèves parviennent à prédire et à justifier correctement l'allure des séries de trajets. Six d'entre eux ne recoureront ni au papier ni à l'ordinateur et justifieront les allures anticipées en fonction des relations observées entre numérateurs et dénominateurs; certains sont plus explicites que d'autres, mais essentiellement les relations s'avèrent bien comprises. Huit élèves voudront vérifier, à l'occasion sur papier ou sur ordinateur la justesse de leurs anticipations. Le travail demandé est exigeant ici, mais chez les plus vieux (6ième et 5ième) des discussions animées entre partenaires parviennent à grandement alléger le poids du travail à fournir.

<u>Activité 6: Le jeu de l'entre deux.</u> En alternance chaque partenaire reproduit à l'écran deux trajets; l'autre partenaire doit alors trouver un trajet "entre deux". Cette activité rejoint la notion d'ordre entre fractions; elle favorise, dans un contexte ludique, la manipulation des règles en jeu dans la détermination d'une plus ou moins forte inclinaison des trajets.

Malheureusement, peu d'observations spécifiques purent être recueillies ici. En effet, toute la classe s'impliquait à chaque fois, soufflant les réponses aux uns et aux autres; suggérant des trajets super fous (trop faciles ou trop difficiles). Les sujets cibles étaient souvent trop "excités" pour référer à des utilisations songées des principes découverts en cours d'activités. Le plaisir des enfants était tellement grand en cette veille d'un long congé scolaire, qu'il fut choisi ici de sacrifier le sérieux de la recherche au plaisir évident des enfants et de reporter à un moment ultérieur l'analyse de cette tâche clairement très riche en potentiel d'apprentissage.

Discussion

Nos observations permettent de confirmer que les deux objectifs sous-jacents au micro-monde de la fusée fraction furent atteints. La majorité des enfants ont réussi, à travers leurs actions, réflexions et formulations, à élargir et approfondir la notion de fraction et à dégager certaines règles et lois régissant l'équivalence et l'ordre dans cet univers notionel, vu sous un

angle tout à fait inusité. Certains enfants, surtout les plus jeunes, tentent de se maintenir à des niveaux purement empiriques, et gagneraient à explorer, dans des délais raisonnables, plus à fond ce micro-monde. La fusée fraction s'avère une excellente façon de faire le lien entre la tortue et la mathématique au programme: elle promouvoit l'expression d'habiletés fondamentales, telles la construction et l'exploration, tout en favorisant une réflexion de qualité sur les phénomènes abordés (identification des règles, formulation de lois). Finalement, il importe de souligner que certaines caractéristiques de l'environnement pédagogique inhérent à ce micro-monde (l'alternance entre du travail papier crayon et du travail à l'ordinateur, par exemple) s'avèrent hautement compatibles avec la présence dans la classe d'un seul ordinateur; l'ordinateur n'a ici qu'un rôle complémentaire et ponctuel et différentes modalités de travail peuvent alors être imaginées par les professeurs ayant choisi d'utiliser le micro-monde de la fusée fraction.

Références

Côté, B. 1989. Les deux tortues: logiciel de construction et d'exploration en mathématiques pour le primaire et le secondaire. Document inédit. Département de mathématique et d'informatique, Université du Québec à Montréal.

Côté, B. Sous presse. La tortue revue et corrigée. La Bus. (printemps 1991).

Coté, B. 1990. Activités de base et exploration mathématiques avec "Les deux tortues". Communication présentée, 8ième congrès "L'ordinateur et l'éducation", Décembre, Montréal: University McGill.

Feurzeig, W. et al. 1969. Programming languages as a conceptual framework for teaching mathematics. Report no C615. Bolt, Beranek and Newman. Camb, Mass.

Gurtner, J.L. Sous presse. Between Logo and mathematics: a road of tunnels and bridges. In : C. Hoyles &R. Noss (eds). *Learning mathematics and Logo.* Cambridge, MIT Press.

Hillel, J., Kieran, C., Gurtner, J.L. 1989. Solving structured geometric tasks on the computer: the role of feddback in gnerating strategies. Educational Studies in Mathematics, 20, 1-39.

Hoyles, C., & Noss, R. 1987. Synthesizing mathematical conceptions and their formalization through the construction of a Logo-absed school mathematics' Technology, 18, 581-595.

Lemerise, T. 1990. How to integrate Logo into the regular mathematic's curriculum and still preserve the Logo spirit? For the learning of mathematics, 10, 3.

Papert, S. 1971. Teaching children to be mathematician vs. teaching about mathematics. AI Memo 247 and Logo Memo 2. Cambridge, Mass: MIT.

'Critical Incidents' in classroom learning - their role in developing reflective practice

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Abstract In this paper, we discuss the development of the skills of reflective practice during an in-service masters course in mathematics education, through a focus by the students on critical incidents in their classrooms. Our starting point was the notion of a reflective practitioner as described by Schön when he claimed that the teacher applies reflection-in-action, which is partly a conscious process. However, we extend this notion to that of an action researcher, whose attention and awareness of children's learning is developed and fully conscious, and we describe the function of the focus on critical incidents in enhancing this development. The paper concentrates on the work of one student, describing incidents that were critical for him, and what actions he took, as well as his reflections about the way in which the task affected and developed his own learning. We recognise that our own learning of teacher education, in an analogous way, is developed by focusing on critical incidents in interaction with those students, but this aspect will be described more fully at the Working Group on Teachers as Researchers in Mathematics Education.

Reflective Practice

A recurring theme in the literature concerning both pre-service and in-service teacher education is the development of reflective practice. This theme is also increasingly found in work concerning research in teacher education [Jaworski 1988, Lerman 1990], in relation to teacher educators developing skills in reflective practice. Schön [1984] maintains that teaching is reflective practice, in his discussion of the unsatisfactory distinction drawn between theoreticians and practitioners.

In describing his notion of reflection-in-action, Schön (1984 p. 5) starts from what he calls the practitioner's knowledge-in-action:

> "It can be seen as consisting of strategies of action, understanding of phenomena, ways of framing the problematic situations encountered in day-to-day experience."

When surprises occur, leading to "uncertainty, uniqueness, value-conflict", the practitioner calls on what Schön terms reflection-in-action, a questioning and criticising function, leading to on the spot decision-making, which is "at least in some degree conscious".

In our situation, we would suggest that we are extending the idea of the reflective practitioner. We would agree with Schön that recognising that a teacher is much more than a practitioner is essential, both for teachers themselves (and ourselves), and for teacher educators, administrators and others. However, when we use the term reflective practitioner, we are also describing metacognitive processes of, for instance, recording those special incidents for later evaluation and self-criticism, leading to action

research; consciously sharpening one's attention in order to notice more incidents; finding one's experiences resonating with others, and/or the literature, and so on. We are concerned with the transition from the reflective practitioner in Schön's sense to the researcher (Scott-Hodgetts 1990), who has a develc, critical attention, noticing interesting and significant incidents, and turning these into research questions. By analogy with some recent views of the nature of mathematics (e.g. Lerman 1986, Scott-Hodgetts 1987), 'Mathematics Education' is most usefully seen, not as a body of external knowledge, recorded in articles, papers and books, that one reads and uses, but as an accumulation of work upon which a teacher can critically draw, to engage with those questions that concern and interest her/him (Scott-Hodgetts 1991).

Critical Classroom Incidents

In our view, described in more detail in other papers (e.g. Blundell, Scott-Hodgetts and Lerman 1989, Lerman 1989, Scott-Hodgetts and Lerman 1990), a radical constructivist theory of learning, and its implications for the nature of knowledge, bring together theory and practice as an integrated process of theory-construction within language games or discursive practices. As such, we view the classroom and the practices of the teacher as the central site of activity for teachers, through which to develop the awareness necessary to becoming reflective practitioners in the sense described above. For us, a valuable and successful focus has therefore been critical classroom incidents. Critical incident analysis has formed the basis of earlier research studies in mathematics education (see, for example, Hoyles, 1982), and our strategy has been to encourage pre-service and in-service teachers to use such incidents as starting points for the development of new teaching strategies, investigations and small research studies (Scott-Hodgetts, 1991). Central to this development is the interaction between theory and practice; within our context, such links are facilitated by students reflecting upon their reading and upon the theoretical inputs at South Bank, and by sharing ideas with other students and tutors. We believe that the focus on critical incidents has the potential to stimulate the development of skills of reflective practice in a rich and fruitful way. Indeed, we use this focus ourselves as a primary method of improving our own teaching practices (Lerman 1991).

The particular group of students on which this study is based is made up of mature students with a number of years of experience of teaching mathematics at secondary (11-18) level in the London area. They are engaged on a Master's Course in Mathematical Education, the first unit of which is entitled "Psychological Perspectives of Learning and Teaching Mathematics". The theme of the coursework for this unit, and the major focus of the activities during the course, is the identification and analysis of critical incidents in the classroom, and in this paper we will discuss the effect of this focus on the development of reflective practice by the

students, as seen through their journal writing and formal coursework submission. In what follows, we concentrate on illustrating that learning process by the new students on the masters programme exemplifying this by the work of one such student, Richard.

Richard's Reflections

One principle which underlies not just the featured course, but all of our work with teachers, intending teachers and our mathematics undergraduates is that it is important for a student to be able to review her/his own progress (or lack of it!). To this end, we encourage students to keep a record of their initial thoughts and feelings as they undertake a new area of study, and to update these as appropriate over time. Some of Richard's thoughts at the beginning of the work on Psychological Perspectives are given below:

At the start of this course I noted down some ideas I had about how mathematical learning takes place. I found this difficult to do with clarity. From the wide array of incidents, stored away in my mind, where I had seen someone learning something, it was difficult to generalise. I came up with the following:

Mathematical learning takes place when people:

1. work on tasks which present problems which require connections and categorisations to take place;
2. play around with mathematical ideas with which they are already familiar.
3. talk to each other about things which require precise expression;
4. spot patterns and make connections

Point 2 is expressing the idea that mathematical knowledge is built brick by brick like a wall and that each layer must be in place before a new layer is possible.

In points 1 and 4 I have in mind that, given suitable tasks and problems, people make connections and categorisations in their minds, and this can lead to mathematics. Some categorisations, though, are not as useful as others and this can be a starting point for some fruitful discussion in the classroom

We would suggest that Richard's starting point for mathematical work is a very enlightened one. Having set up the way that problems are given to pupils, however the following extract demonstrates Richard's growing awareness that ultimately 'context cannot be set by the teacher only by the individual (Walkerdine 1989)

Whilst working around the related areas of proportion and percentage, I gave the class the following question (Year 11, low attainers)

> 'In a village of 1500 people there are 750 ginger haired people, 450 black and brown haired people and 300 blonde haired people. Describe a typical group of 100 people from the village.'

I expected to have to do some discussion work about the meaning of the word typical, but I had not expected this: one pupil, Penny, insisted that it depended on who had misbehaved in the village. It took me a few moments of confusion before I suddenly

pictured one of Penny's elders, upon seeing her do something naughty, saying, "That's typical!".

Initially, then, I think I have recognised some problems associated with mathematical learning, and from this point I can do some research which involves me confronting these problems and trying to uproot them.

Rather than give an account of the full range of incidents, actions and reflections recorded by Richard, we will focus our discussion on the progression of ideas motivated by one particular incident, involving a student named Claire:

With one pupil, Claire, I was reviewing a mental arithmetic test in which she had had to calculate 1/3 of 12. Claire is in year 11 (aged 15) and working on the SMP 11-16 Graduated Assessment scheme. The level to which she has graduated so far should imply that finding 1/3 of 12 is within her capabilities. She had given the answer 2 She explained how:

Claire (C): Halve it into 3.
Teacher (T): How?
C: Half 12. 6.
T: Yes.
C: Then half again. 3.
T: Yes.
C: Then half again, 2. Oh no, it doesn't work. Can't halve 3.

One personal characteristic apparent from Richard's description of his classroom interactions is his sensitivity to the feelings of his pupils. We see this demonstrated by his reaction to Claire's response to attempts to discuss the problem she has encountered:

After Claire realised that her method would not work to find 1/3 of 12, she became very confused in further discussion, seeming to panic, blurting out unreasonable answers, seemingly unable to think straight. It would have been counter-productive to pursue the problem at this stage but I realised that I could not end the interview without causing myself considerable problems in future discussions with Claire. I did not want to have Claire believe that talking to her mathematics teacher could only be an ordeal.

He therefore set her a new problem, which he hoped would take her forward:

I then gave Claire four 2 pence pieces and four 1 penny pieces and proceeded:

T: How much is that? (Gives money)
C: 12p.
T: Here are three people. (Draws three stick figures) Share the money equally between them.
C: (Places money as shown).

C: There's two left.
T: Okay, how much do they each have?
C: (Pointing correctly) He has 4p, he has 3p and he has 3p.

T: What about this two pence then? (Points to remaining pennies)
C: I can't split that, it wouldn't be fair.

Richard recalled that this type of problem had been discussed at a seminar at South Bank, and recognised that Claire had shared the coins as objects, so that each person had the same number of coins rather than the same amount of money. He therefore rephrased the question:

T: Okay. Take the money back. Now, your task is to share all the money between the three people so that each person has the same amount in pence and there is no money left over.

Unfortunately, Claire, despite several attempts, was unable to solve the problem before the end of lesson. Again, Richard picked up on her feelings, realising that she felt frustrated, "even aggressive", and responded by giving her the reassurance that there was no trick involved, and that this particular problem did have a solution.

Next lesson, Claire strode in, declaring her success, and showed me the correct solution. What amazed me was not the fact that she had solved the problem, somehow I had never doubted that she would, but that she had the capacity to be such a confident and assertive person. I had never seen this in her before.

Richard used Claire's solution as a starting point, explaining that, because there were three equal amounts of money, each pile was one third of 12 pence:

I was guessing that this was a problem of language because Claire had thought that to find a third of something she had to halve it three times; because she had created an incorrect algorithm to find a third of something; and because her idea of what a third of something was attached to the algorithm she used. I explained to Claire that to find a third of anything, it had to be divided into three equal parts (I was using suitable gesticulations to try to give concrete meaning to my words) and that each part would then be a third of the whole.

On testing Claire's understanding by giving another abstract problem (What is 1/3 of 18?), Richard discovered that his explanation had not been enough to dissuade Claire from applying her original algorithm within this context.

Richard's attention to Claire's language, both verbal and physical, is at this stage fully conscious and sharply focused. In the next extract we can see him begin to make generalisations and pose research questions from his reflections:

Starting from this point: we have a problem set in a familiar context, with the pupil having a firm grasp of the concepts, which form the basis of the problem as stated: we are offering the opportunity for the pupil to develop their thinking. This contrasts with my initial approach with Claire, which entailed me trying to explain to her what a third was. So my reflections led me to this point. I began to believe that if the didactic approach was not good enough for Claire, it was not good enough for any pupil; that instruction and explanation by the teacher had its place in outlining the problem, but not somehow giving mathematical ideas. So I began to understand better the practical

implications of the constructionist (sic) point of view (hypothesis 1):

> "Knowledge is actively constructed by the cognising subject, not passively received from the environment" (Kilpatrick 1987).

Reflection-in-action is certainly a characterisation of the behaviour of teachers who, if they noticed Claire's reactions and had sufficient time to probe further, would have attempted to offer alternative activities or explanations to her. Richard's noticing is of a different order. He searches for ways of engaging with Claire's constructions, whilst being aware both of the effect that he has on her as the teacher, and also that which mathematics as an entity has on her. He constructs conjectures about her meanings and probes further. This then leads him to a view about learning in general, which resonates powerfully with work done on the masters course, and he is ready to take some action in his classroom to make changes, which he will again monitor and evaluate.

Richard discusses, in some detail, Claire's anxiety about mathematics, in the context of some of the literature about mathophobia. By recognising and legitimising Claire's insecurities, he finds that his own behaviour can help her.

> On consequent occasions I found that simply giving Claire the option to say, "I am beginning to panic" meant that the problem could be alleviated . . . Once Claire had a problem which she understood and once I knew that she was solving the problem that I had set, her whole persona changed. Solving the problem with the money might have been the first time Claire has done any mathematics in ages.

Richard next points to a problem we have discussed elsewhere (Blundell, Scott-Hodgetts and Lerman 1989), that of notions of what constitutes "real maths". He outlines what he sees as the characteristics of the money problem, including its accessibility, the lack of time limit etc., and then comments:

> Strangely, all these characteristics are contrary to Claire's and many other peoples' view of what mathematics is. Consequently Claire is reluctant to work on satisfying problems like this, preferring instead to return to her text book and "real maths". Ironically, it is Claire's belief that maths is important that continues to motivate her to struggle through meaningless tasks. Claire is just one example of a group of pupils which, when given meaningful tasks, will not accept that they are mathematical in nature.

Richard's Conclusions

> The work I have done has opened my eyes to the rich resource I have at my disposal in my own classroom, which I think I must use to develop my ideas further. I feel that I have improved my understanding of what it is to be learning mathematics; and clarified my ideas about the work I do as a teacher. I have had a glimpse at the potential benefits to my teaching of doing research of my own and I feel I can begin to improve my research methods from this starting point.

Our Conclusions

Richard's descriptions of his experiences are particularly articulate and insightful, but are not untypical in nature and content. We find

that, like Richard, most of our students go further than simply uncovering particular aspects of their pupils' learning, to a recognition that working in this way can benefit them if they make it an integral, on-going feature of their practice:

> Ben: "One of the knock-on effects of doing this course has been the change in my attitude to each and every lesson. I use every opportunity to find out how and why pupils respond the way they do."

It is precisely this shift of perception, to seeing research as a natural part of their professional role, that we hoped to engender in our students by the introduction of the sorts of activities to which we have referred within this paper. In the spirit of what we have written, we continue to reflect upon our experiences of this work, and attempt to extend and develop our strategies in the light of new critical incidents that we encounter.

References

Blundell D., Scott-Hodgetts R. & Lerman S. 1989 "Mathematising the Hard Stuff" *For the Learning of Mathematics* Vol. 9 No. 2

Hoyles C. 1982 "The Pupil's View of Mathematics Learning" *Educational Studies in Mathematics* Vol. 13 p. 340-372

Jaworski B. 1988 "One Mathematics Teacher" Proceedings of the Twelfth Conference of the International Group for the Psychology of Mathematics Education, Hungary, Vol. II p. 425-432

Kilpatrick J. 1987 "What Constructivism might be in Mathematics Education" Proceedings of the Eleventh Conference of the International Group for the Psychology of Mathematics Education, Montreal, Vol. 1 p. 3-27

Lerman S. 1986 "Alternative perspectives of the nature of mathematics and their possible influence on the teaching of mathematics" unpublished PhD dissertation, King's College London

Lerman S. 1989 "Constructivism, Mathematics and Mathematics Education" *Educational Studies in Mathematics* Vol. 20 No. 2 p. 211-223

Lerman S. 1990 "The Role of Research in the Practice of Mathematics Education" *For the Learning of Mathematics* Vol.. 10 No. 2 p. 25-28

Lerman S. 1991 "Shifting Perspectives" Paper to be presented at Working Group "Teachers as Researchers in Mathematics Education" Fifteenth Conference of the International Group for the Psychology of Mathematics Education, Assissi, Italy.

Schön D. A. 1984 "Educating for Reflection-in-Action" Paper presented at Harvard Business School Anniversary Colloquium on Teaching by the Case Method.

Scott-Hodgetts R. 1987 "Preferred learning strategies and educational mathematical philosophies: An holistic study" Proceedings of the Eleventh Conference of the International Group for the Psychology of Mathematics Education, Montreal Vol.3 p. 339-345.

Scott-Hodgetts R. & Lerman S. 1990 "Philosophical/Psychological Aspects of Mathematical Activity: Does Theory Influence Practice?" Proceedings of Fourteenth Conference of the International Group for the Psychology of Mathematical Education, Mexico Vol. 1 p. 199-206.

Scott-Hodgetts R. 1990 "Qualified, Qualitative Research: The Best Hope in Our Quest for Quantifiable Improvements" Paper presented at the Fourth Conference on Systematic Co-operation Between Theory and Practice in Mathematics Education, Brakel, Germany.

Scott-Hodgetts R. 1991 "Mathematics Learners and Learning" Open University Press.

Walkerdine V. 1989 "Mastery of Reason" Routledge.

HUMAN SIMULATION OF COMPUTER TUTORS: LESSONS LEARNED IN A TEN-WEEK STUDY OF TWENTY HUMAN MATHEMATICS TUTORS

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This study investigated the understandings, assumptions, and procedures that teachers who were various levels of expertise actually used in tutoring, and observed the evolution of these tutoring abilities over time, as tutoring effectiveness improved. Results included: (i) Evaluations of tutori quality were highly correlated with (a) the teachers' ability to recall information during post-sessi debriefing interviews, (b) the depth and sophistication of descriptions that they gave during post-session interviews, and (c) their ability to make accurate predictions about how the student woul perform during post-session tests. (ii) The quality of tutoring increased significantly during the t week project along a number of dimensions. (iii) During poor sessions the teachers operated (implicitly) like AI-based tutors; whereas, during better sessions, the preceding principles tended be explicitly rejected, and replaced by principles informed by "constructivist" perspectives.

The tutoring project described in this paper was part of a long-term and continuing series of research projects whose central goals have been: (i) to clarify the nature of strategic conceptual models and reasoning patterns which underlie students' elementary mathematical knowledge (e.g Lesh, Post, & Behr,1989), (ii) to investigate the foundation-level understandings and processes needed to construct and use elementary mathematical models in everyday problem-solving situations (e.g., Lesh & Akerstrom, 1982; Lesh & Zawojewski, 1987), and (iii) to describe mechanisms that encourage the development of these models and reasoning patterns (Lesh, 1990

Throughout the preceding projects, we continually faced the dilemma that, for a majority of students, many of the mathematical concepts that we wanted to study did not develop beyond

primitive levels unless an artificially rich environment was provided to stimulate and focus development. But, this means that the capabilities we observe are partly the results of a student's existing cognitive structures, and partly a result of the situations we create. In other words, we are studying the interactions of students with their environments; and, it becomes as important to generate and test models to explain the environments we create as it is to generate and test models of students' interpretation frameworks. In particular, when we study the way a given concept develops, it is as relevant to investigate how excellent teachers make abstract ideas concrete as it is to investigate how children develop ideas from concrete to abstract.

To help clarify the nature of productive learning environments, the approach that we took was to provide opportunities for experienced teachers to observe, mold, and shape students' mathematical models and reasoning patterns – so that we in turn could use these opportunities to observe, mold, and shape the models and reasoning patterns that teachers used to interpret, explain, and inform their tutor decision-making activities. That is, we wanted to observe the development of teachers as they gradually evolved into more effective tutors; and, one useful way to accomplish this goal was to create a tutoring environment in which human teachers provided the intelligence behind a computer-based tutoring system. Then, a combination of peer group discussions and brief staff-directed presentations were used to provide mechanisms to encourage teachers' development toward their own conception of "excellence" in tutoring (see Lesh and Kelly (1991).

The current study focused on a ten-week project in which human tutors provided the intelligence behind a computer-based tutoring system. The goal of this study was to investigate the understandings, assumptions, and procedures that teachers who were at various levels of expertise truly used in tutoring, and to observe the evolution of these tutoring abilities over time, as tutoring effectiveness improved. For a more extended report of this study, see Lesh and Kelly (1991).

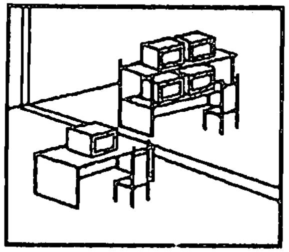
(a) the tutors: The study involved two groups of ten typical middle school teachers. Each week during a ten-week period, each teacher participated in two 60-minute tutoring sessions that were held on either Tuesday and Thursday, or Monday and Wednesday. Then, on Friday of each week, each group of ten teachers met independently to give brief written-and-oral reports about their two tutoring experiences for the week. Also, during these Friday sessions, the group discussed the pros and cons of various tutoring approaches and refined their collective conceptions about the nature of (i) "good" tutoring techniques, (ii) "good" problems, (iii) "good" follow-up questions or feedbacks, and (iv) "good" helps or hints. The main goal of the study was to investigate the ways in which our teachers' tutoring capabilities would evolve during the ten-week period. In particular, we were interested in changes that might occur concerning: (i) the problems and follow-up questions that would be emphasized, (ii) the kinds of answer-checking, feedbacks,

hints, and helps that would be used, and (iii) the rules that would be used to determine when tl preceding tutor inputs would be initiated.

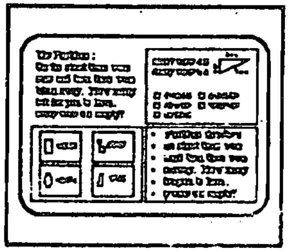
(b) the students: Forty students participated in the study (8 fifth graders, 12 sixth graders seventh graders, and 12 ninth graders), with each student participating in one tutoring session week for five consecutive weeks. That is, each teacher selected four students from his or her class, with selections being based on the teacher's' judgements about children who might profi most from one-to-one tutoring. Therefore, nearly all of the students who were selected were viewed as being "average or below average," but as "having potential" to improve.

(c) the laboratory setting: During the tutoring sessions, the students were told that they wer interacting with a smart computer; and, indeed, they were! However, the main components of computer's intelligence were provided by their teacher, who was able to view the student's activities from behind a one-way mirror (or using closed circuit TV monitors, one focused on tl student's face, and one focused on the student's pencil-and-paper activities, see Figure 1). In other words, teacher-student pairs worked together on two linked PCs in which special "windc on the student's screen enabled teachers to communicate with students using a "local E-mail" system to transmit: (i) brief messages that were written by the teacher prior to the on-line tutorir session, or (ii) simple graphics that the teacher could create at run-time (see Figures 5-6). In addition to being able to view the student's behaviors through a one-way mirror, or using close circuit TV monitors, the teacher was also able to view two computer screens: (i) one showed ar exact duplicate of the student's screen, and (ii) the other showed a menu of problems, feedback helps, and hints that had been by the teacher prior to the on-line tutoring session (as shown in tl figure below).

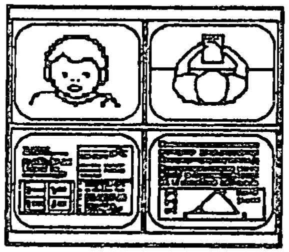
: Laboratory Setting



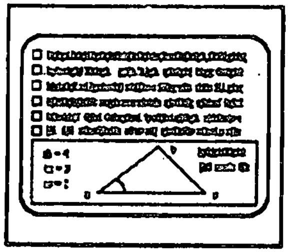
The Student's Computer Screen



The Teacher's Monitors



The Teacher's Computer Screen



(d) the teacher's pre-tutoring assignments: On the day before each tutoring episode was to occur, each teacher was given one of the following two assignments designed to help them prepare for the up-coming tutoring session. The assignment was somewhat different on odd numbered weeks and on even numbered weeks:

- During odd numbered weeks, on the day before the tutoring session, each teacher was given a scored ten-item worksheet that the student had completed, and on which the student needed tutoring. Then, the teacher was asked to write a set of problems that they could use during the up-coming tutoring session, with the aim of helping students prepare for a post-tutoring test (similar to the pre-tutoring worksheet) that would be given to the student after the tutoring session.
- On even numbered weeks, the teachers were given the problems that would be used during the tutoring session. That is, each teacher was given a ten-item problem set that included problems about a single topic area (such as "similar figures" or "proportional reasoning"). These problems were explicitly constructed to include a variety of types ranging from routine word problems to more innovative problems such as those in which several levels or types of correct answers, or those that included "too much" and/or "not enough" information, or those which

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were designed to capitalize on the fact that calculators were available. (See below for an example of a problem that was used.)

[Consider the following 40-minute problem. This particular problem is typical of those we had used in an earlier project about Using Mathematics in Everyday Situations (Lesh & Akerstrom 1982; Lesh, 1985). It is a problem that was design-ed for use with average ability seventh graders who worked in situations that were as much like realistic everyday experiences as we could make them in a school environment.

Near a school in a Chicago suburb, a research assistant (with a map of Chicago in hand) walks up to a group of 2-3 students and asks, *"What's the best way to get from here to O'Hare Airport?"*

Note: The term "best" was left undefined because we wanted to go beyond investigating answer-giving abilities to investigate processes involved in problem formulation, information interpretation, and trial solution evaluation. A map was available, but no other suggestions were given about whether "best" was intended to mean: *shortest, quickest, safest, simplest, least confusing, most convenient, least expensive*, or some other possibility. Also, no suggestions were made about whether a car was available – or whether a bus, taxi, limousine, or train might be needed. However, these kinds of information were available if students requested it.
]

Also, in both of the preceding two situations, the teachers used a take-home PC to write feedbacks, follow-up questions, hints, and helps for each problem that would be used during the tutoring session. The goal was to anticipate which kinds of questions, follow-ups, feedbacks, hints, and helps might be most useful to students. Then, prior to the tutoring session, the preceding teacher-generated items were up-loaded into an easily accessible menu of options on the computer system that would be used during the tutoring session. In this way, during the tutoring session, the teachers were able to simply "click" on problems, feed-backs, follow-ups, hints, and helps, in order to send them as "E-mailed" messages to the student.

(e) the Friday discussion groups: On Friday of each week, each group of ten teachers met together independently to give brief written-and-oral reports about their two tutoring experiences for the week, and to discuss the usefulness and productivity of various types of problems, follow-ups, feedbacks, helps, and hints. We also introduced examples of new problem types and

"techniques that teachers at another site have used" that we wanted our teachers to consider.

Finally, we spent a small amount of time describing students' solutions to realistic everyday problems such as the "getting to the airport problem" in which finding adequate ways to think about the problem involves constructing new (to the student) conceptual models for interpreting givens, goals, and possible solution paths. In general, we wanted our teachers to at least consider the possibility of using tutoring techniques that focus on processes similar to those that students themselves use to improve their own conceptualizations of such problems.

(f) the post-session debriefing sessions: On even-numbered weeks, a tape recorded post-tutoring interview was conducted with half of the teachers. In these sessions, teachers were asked to recall and describe the first tutoring session that they had conducted during the preceding odd-numbered week. They were also were shown a problem set similar to the one that had been used during the tutoring session, and were asked to predict how the student performed (in terms of accuracy, insights, and errors) when the given problem set was used as a posttest following the tutoring session. (Note: Teachers were not shown actual results from the post-tutoring tests until after the post-tutoring interviews had been completed, and predictions had been made.) Our goal was to investigate possible relationships between: (i) the quality of tutoring sessions and (ii) the completeness and accuracy of recalled and predicted information.

(g) the tutoring quality evaluations: On even numbered weeks, information about each teacher's tutoring effectiveness was gathered from four sources: (i) improvements, or lack of improvements, noted in pre-to-post-session assessments of students' capabilities, (ii) teachers' self assessments of their own tutoring effectiveness for the week compared with the effectiveness of peers, and compared with their own effectiveness during previous weeks, (iii) teachers' retrospective re-evaluations of tutoring effectiveness during previous weeks, (iv) teachers' assessments of their peers' tutoring effectiveness for the week, (v) evaluations by three "experts" on our project staff concerning each teacher's tutoring effectiveness for the week, and (vi) experts' retrospective re-evaluations of tutoring effectiveness during previous weeks. These latter assessments were based on: (i) analyses of print-outs and oral reports describing the types of problems, follow-ups, hints, and helps that were used, (ii) observations about the insightful use of these teacher-inputs during tutoring sessions, and (iii) observations of insights gained by students during the tutoring sessions.

An Overview of Results:

General results from the preceding study included the following. (i) For a given tutoring session, pooled evaluations of tutoring quality were highly correlated with the teacher's ability to recall rich, detailed, and accurate information during post-session debriefing interviews; it was highly correlated with the depth and sophistication of descriptions that teachers were able to give during post-session interviews based on videotapes of selected sessions; and, it was highly

correlated with the teacher's ability to make accurate predictions about how the student would perform during post-session tests. (ii) The quality of tutoring increased significantly during the ten-week project -- in which quality assessments were based on a combination of students' learning gains, teachers' self-assessments, teachers' peer-assessments, and assessments by "expert tutors" on the project's staff. (iii) During sessions when the tutoring was least effective, the teachers generally appeared to operate (implicitly) using rules of the type that have characterized procedurally-oriented types of AI-based tutors; whereas, during sessions when tutoring was most effective, the preceding principles tended to be explicitly rejected, and replaced by principles informed by "constructivist" perspectives. As our human tutors developed progressively greater expertise:

- Relatively little time was spent trying to diagnose or correct students' procedural "bugs."
- Multiple linked representations were emphasized as powerful instructional devices; and, "representational fluency" was encouraged on the part of students.
- Tutor's questions were often "fuzzy" in the sense that they encouraged multiple types and levels or interpretations and responses -- and therefore tended to be self-adjusting in difficulty.
- Students were encourage to use computer-based procedure-executing tools (such as graphics-linked calculators and modeling tools) to "leap frog" over unproductive procedural details.
- Within pre-planned and carefully structured exploration environments, tutors focused on following and facilitating students' thought processes, rather than the other way around.
- Students' errors were often simply ignored; or, in other cases, they were actually induced in order to confront students with the need for conceptual reorganizations.
- The objectives that were emphasized shifted away from "basic facts, rules, and skills" toward "deeper, broader, and higher-order understandings and processes."
- The ***idealized model of potential student knowledge*** that our teachers used to direct their tutoring activities was not considered to be a description of the state of knowledge for a given student, nor was it a description of the state of knowledge of an expert in the given conceptual neighborhood. Instead, it represented a deep understanding of an elementary topic area.

The approach of the tutors was more consistent with the kinds of conceptions of mathematics, mathematics learning, and mathematics problem solving that are being expressed in documents such as the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989), or the Mathematical Sciences Education Board's *Reshaping School Mathematics* (MSEB, 1990), or the American Association for the Advancement of Sciences' *Project 2061* (American Association for the Advancement of Science, 1989).

References

American Association for the Advancement of Science. (1989) *Project 2061: What science is most worth knowing?* Washington, D.C.: American Association for the Advancement of Science.

Lesh, R. (1985). Processes, skills and abilities needed to use mathematics in everyday situations: *Education and Urban Society*, 17 (4), 439-446.

Lesh, R. (1990). Computer-based assessment of higher-order understandings and processes in elementary mathematics. In Gerald Kulm (Ed.). *Assessing higher order thinking in mathematics*. Washington, D. C.: American Association for the Advancement of Science.

Lesh, R., & Akerstrom, M. (1982). Applied problem solving: Priorities for mathematics education research. In F. Lester & J. Garofalo (Eds.), *Mathematical Problem Solving: Issues in Research* (pp. 117-129). Philadelphia: Franklin Institute Press.

Lesh, R. & Kelly, A. E. (1991). *Foundations for AI-Based Tutoring: Obedient Butlers Versus All-Knowing Gurus*. Technical Report. Educational Testing Service. Princeton, NJ.

Lesh, R., Post, T., & Behr, M. (1987) Dienes revisited: Multiple embodiments in computer environments. In I. Wirszup & R. Streit (Eds.), *Developments in School Mathematics Education Around the World*. Reston, VA: National Council of Teachers of Mathematics.

Lesh, R., & Zawojewski, J. (1987). Problem solving. In T. Post (Ed.). *Teaching mathematics in grades K-8: Research-based methods*. Boston: Allyn and Bacon.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Mathematical Sciences Education Board (1990). *Reshaping School Mathematics: a philosophy and framework for curriculum*. National Research Council, Washington, D.D. National Academy Press.

ADVANCED PROPORTIONAL REASONING

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ABSTRACT

Four levels of understanding hierarchy in simple ratio was estalished by one of our previous work. This work developed three more advanced understanding levels beyond those four levels. Among the variety domains of advanced proportional problems, multiple ratio, ie. $a \propto bc$, and non-linear ratio, ie. $y \propto kx^2/kx^3$, were chosen as the concept domains for this study. 1539 subjects who are representative of Taiwan senior high school students of aged 16 and 17 were attended in this study.

These three advanced hierarchical levels are linked nicely with the four levels of understanding in simple ratio. They are characterized strongly by the different kind of relationships between quantities appeared within the tasks, such as inverse ratio, composition of two positive ratios, composition of positive and inverse ratio, quadratic relation and cubic relation. The reasoning strategies used by students and the meaning of understanding levels were analysed in this paper.

INTRODUCTION

One of our previous work was investigating levels of understanding in ratio and proportion at junior secondary

level (aged 13~15) in Taiwan (Lin 1987). It was one of a replicated study of the Concepts in Secondary Mathematics and Science (CSMS) programme in England (Hart,1981). A hierarchy including four levels of understanding in simple ratio tasks was developed. A student who is at the highest of "understanding of level four" can apply correct multiplicative strategies, such as ratio formula, multiplier method or unitary method to solve simple ratio tasks with non-integer rates in different settings (Lin,1987). Almost all researches on ratio in the literature dealt with the concept of ratio to the depth of this level (c.f.Tournaier & Pulos,1985). In 1988, I proposed a question: do there still exist developmental hierarchy of understanding ratio beyond the understanding of level four? At the same time, I had shown a tentative results of an investigation on this question, and suggested that the answer of this question might be affirmative (Lin,1988).

A two-year research project was founded in Taiwan to investigate this question systematically. During the first year of investigation, five domains of advanced ratio and proportional tasks were developed, namely:

(1) The parameter of ratio.
(2) The duality between <u>examples</u>,<u>concepts</u> and <u>theorems</u> of proportional relations.
(3) The linear interpolation method with respect to different representations of a function.
(4) Multiple ratio, ie. $a \propto b*c$, where a,b,c are variables.
(5) Non-linear ratio, ie. $y \propto x^n$, where x,y are variables.

Tasks in domains (1)~(4) are all concerned with linear proportionality. The non-linear ratio was developed due to the following considerations.

(a) From symbolic generalization point of view, the relation $y=kx^n$, n=2,3, is a natural generalization of the relation $y=kx$.
(b) Non-linear reasoning is used in everyday life. There are situations in everyday life in which the change of one dimensional measurement is used to represent the change of two or three dimensional measurement, eg. the size of a round cake is represented by its diameter rather than by its volume. To make different size of cakes according to a given recipe, it is necessary to apply the non-linear proportional relation.
(c) The reasoning process of non-linear and linear ratio are parallel.

Mathematics understanding is multi-dimensional (Usiskin,1987). Understanding levels is meaningful within the same dimension only.

Among the above five domains of advanced proportional tasks, which shall be the candidates such that the understanding of tasks in that domains and the tasks of simple ratio used in our previous study could be structured as an uni-dimension?

To solve tasks in the domains of multiple-ratio and non-linear ratio, one has to recognize either one more relation or a more general relation then solving simple ratio tasks. Moreover, our interview showed that many strategies used by students in solving multiple and non-linear ratio tasks can be linked to their strategies in solving simple ratio tasks. Therefore, multiple ratio and non-linear ratio are combined to form one test, and was called "Advanced Proportional Reasoning" test. This test was conjectured to be an uni-dimensional understanding test. This conjecture will be examined in this paper. The studies on the other three domains will be reported elsewhere.

METHODOLOGY

The aims of this study are:

— to examine and describe the hierarchical levels of understanding in simple ratio, multiple ratio and non-linear ratio.
— to describe the methods Taiwan students use to solve multiple ratio and non-linear ratio problems, and the errors they made.

There are 25 items in the test. Six/nine/nine of them are simple/multiple/non-linear ratio items respectively. Five simple ratio items are the items used to describe "understanding of level four" in the previous study.

Each of new developed items is determined using the following two procedures:
— interviews with about 40 students to check its meaning is well understood and to find out the strategies students used;
— pilot testings two classes to check that the objective of item tested was achieved.

A two-stage stratified proportional sampling was applied. It includes the following procedures.
— All 160 senior high schools in Taiwan are stratified into five strata according to the performance of each school in a national study "Survey Study on Senior High School Students' Understanding of Basic Science" (Yang,1988).
— Simple random sampling some numbers of schools and students from each stratum according to the students population. Whenever a school was selected, 4 to 8 students from each class in this school would be selected.

As a result, 11 schools and 1539 (about 1.1% of the totle sample) subjects, aged 16~17, were selected. The tests were administered by at least one project researcher in each school and some teachers from that school. Each test last 110 minutes. To each item, all subjects were asked to write down their process in solving the item.

TASK ANALYSIS

Tasks of multiple ratio and non-linear ratio which are developed for this study are described briefly in Table 1.

Table 1 **Task Analysis**

Context	Variable	Relations *	Remark
Walking	s:distance t:time v:velocity	$V \propto \frac{1}{t}$	. S is a constant which is given implicitely
Motorcycling	s,t,v	$T \propto \frac{1}{v}$	. Percentage is included
Bicycling	s,t,v r:radius of wheel n:number of rota-tions of wheel	(1) $R \propto \frac{1}{n}$ (2) $V \propto \frac{s}{n_1} : \frac{t}{n_2}$	. dynamic measurement . s and t are constant
Lighting	p:power consump-tion n:number of elec-tric bulb t:time	(1) $P \propto nt$ (2) $T \propto \frac{n}{p}$	. the changing rate of each variable is given.
Heating	g:volume of gas w:volume of water t:temperature of water	(1) $G \propto wt$ (2) $T \propto \frac{g}{w}$	. the changing rate of each variable must be found.
Recipe of Cake	i:ingredient d:diameter of cake	(1) $I \propto d^2$ (2) $D \propto \sqrt{I}$ (3) comparison of D_1 & D_2, given two ingredients i_1 & i_2	. the answer is an irrelational number
New Train Test	s,t,v_0= 0 a:acceleration	(1) $s \propto t^2$ (2) $T \propto \sqrt{s}$	
Sand Clock	h:height of sand t:time	(1) $T \propto h^3$ (2) $H \propto \sqrt[3]{t}$	

* The unknown in each item is represented by capital letter.

With respect to different relations induced from multiple ratio $a \propto b*c$, an item is called of type I when a is a constant, type II when its unknown is a, and type III when its unknown is b or c.

Results

The method of analysis used on the data in order to obtain groups of items was an adoption of the method used in CSMS study (Bart,1981,p.7).

The facility range of all 25 items in this study is between 18.8%～95.7%. The facilities of four out of five items used in our previous study are between 92.6%～95.7%. They are grouped as 'level four' items in this study. Three more groups of items which are beyond level four are formed. The facility range for each group is shown in Table 2.

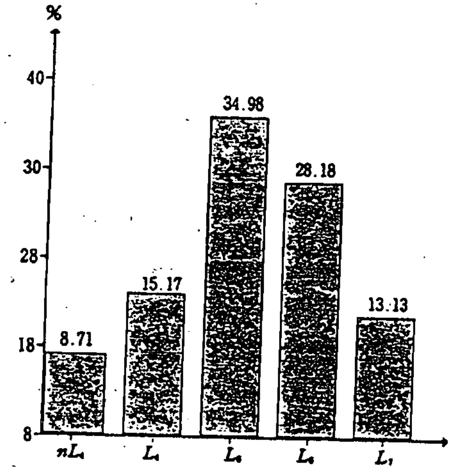
Table 2 Levels of Understanding in Ratio

Level	Facility Range	Pass Mark
4	92.6～95.7%	3/4
5	67.3～82.2%	3/5
6	47.5～65.6%	5/7
7	18.8～38.6%	5/7

The frequency of Taiwan students who achieved each level is shown in Table 3. The description of levels composed two components, ie.mathematics structure and students' abilities of each level which are shown in Table 4 and 5 respectively.

To simplify, in Table 5 the correct strategies students used are grouped into two classes,ie. economic strategy and step-by-step strategy. A students who directly apply ratio relations between variables in solving problem can skip some steps in his(her) reasoning process. He(She) is using an economic strategy. A student who present all necessary steps in his(her) paper is using a step-by-step strategy.

Table 3 Frequency of students who achieved each level
Total Sample 1539
Valid Sample 1478
(including error sample 3.45%)



Values of Φ coefficient in each level

	Mean	Median
L5	.183	.162
L6	.391	.303
L7	.363	.307

Table 4 Mathematics Structure of Levels

Level	Mathematics Structure
5	. Relation between variables easy to recognize . Continued ratio (demanding reduction and enlargment of rates) . Multiple ratio: type Ⅰ and Ⅱ . One step reasoning
6	. The changing rate of each variable must be found. . Multiple ratio: Type Ⅰ ⊕ %, type Ⅱ and type Ⅲ . Non-linear ratio: $y \propto x^2$ (awareness of the relation is a common sense) . One step reasoning
7	. Relations between variable are hard to recognize . Multiple ratio: type Ⅲ (variable was described indirectly in equation form) . non-linear ratio: $y \propto x^3$ and $y \propto x^2$ (awareness of the relation $y \propto x^2$ demand physics knowledge) . Comparison of rates . Two steps reasoning

Table 5 Students' Abilities

Level		Correct Strategy: Type I	Type II	Type III	$y \propto x^2$	$y \propto x^3$	Weakness
5	*	24 %	43 %	23 %	7 %	2.5 %	. Solving cubic relation item, 50% apply linear relation 11% apply quadratic relation. . Solving quadratic relation item, 70% apply linear relation. . Only 51~56% can cope with multiple ratio: type III. . Only 17~33% can cope with comparison of rates. . 10% has difficulty in understanding percentage.
	**	34 ~ 42 %	27%	19 ~ 24 %	4 ~ 5.6 %	2.7~ 5.6 %	
6	*	21 %	50 %	38 %	36 %	4.3 %	. Solving cubic relation item, 17% apply linear relation 32% apply quadratic relation. . Only 36~57% can cope with comparison of rates. . 11~16% are affected by complicated computation eg. decimal and irrelational number.
	**	40~ 68 %	32.6~ 51.8 %	31.5~ 54.1 %	26.6~ 60.8 %	6.1~ 11.9 %	
7	*	32.6 %	56.5 %	50 %	60 %	29 %	. Solving cubic relation item, 2% apply linear relation, 10% apply quadratic relation.
	**	42~ 62 %	32.6~ 41 %	31~ 39.5 %	25.4~ 37.3 %	32.7~ 39.9 %	

* denotes the percentage of students in each level who used an economic strategy.
** denotes the percentage of students in each level who used a step-by-step strategy.

REFERENCES

Hart, K.M. (ed) (1981): Children's Understanding of Mathematics: 11~16, John Murray.

Lin, P.L. (1987): Describing Level of Understanding: Ratio, Proceedings of P.M.E.-XI, VII 416~423.

Lin, P.L. (1988): Beyond Ratio Formula, Proceeding of P.M.E.-XII, VI 320~325.

Tourniaire F. & S. Pulos (1985): Proportional Reasoning: A Review of the Literature, Educational Studies in Mathematics V.16, 181~204.

Usiskin Z. (1987): The Real True Understanding of Mathematics of Mathematics, paper presented at Fifth Annual Hellman Lecture, Long Island University, Jan.28.

Yang, J.S. (1988): Survey Study on Senior High School Students' Understanding of Basic Science, Science Education Center, Taiwan Normal University (in Chinese).

RULES WITHOUT REASONS AS PROCESSES WITHOUT OBJECTS — THE CASE OF EQUATIONS AND INEQUALITIES

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Following the claims about the operational/structural duality of mathematical conceptions, ([2], [4]) we notice that the majority of mathematical notions draw their meaning from two kinds of processes: the primary processes, namely the processes from which the given notion originated, and secondary processes — those for which instances of this notion serve as an input. Abstract objects act as a link between these two kinds of processes, thus seem to be crucial for our understanding of the corresponding notions. Pseudostructural conceptions are the conceptions which develop when the student, unable to think in the terms of abstract objects, uses symbols as things in themselves and, as a result, remains unaware of the relations between the secondary and primary processes. In the case of equation (or inequality), which in this paper is used as an illustration for the above claims, the primary processes are the arithmetic operations encoded in its component formulae, the secondary processes are those which one must perform on the equation in order to solve it, and the abstract objects behind the symbols its truth-set. Our empirical study carried out among secondary school pupils has shown that in the algebra, pseudostructural conceptions may be more widely spread than suspected.

While introducing in 1976 the term instrumental understanding, Skemp ([5]) explained that he was referring to "having rules without reasons" — to the kind of comprehension which expressed itself in not more than technical proficiency in executing various mathematical procedures. The more advanced type of understanding was named relational and was described as an ability to connect "the rules" with the previously developed system of concepts in a meaningful way. In the present paper, we shall use the thesis of operational/structural duality of mathematical conceptions, promoted in [2] and [4], to take these ideas a little further. The question we shall try to answer regards the nature of the links needed to relate a new mathematical notion or technique to those which developed before. To put it in a slightly different language, we shall be dealing here with those interconceptual ties which bring about the relational understanding, with the ties in the absence of which the instrumental understanding remains the best alternative one may ever hope for. All the theoretical claims will be illustrated with observations regarding secondary students' conceptions about equations and inequalities. It should be emphasised that this article is not more than working report. Since it is only the first in the series of similar communications, we shall confine ourselves to the theoretical framework of our research and to a concise account of one small part of the ongoing empirical study.

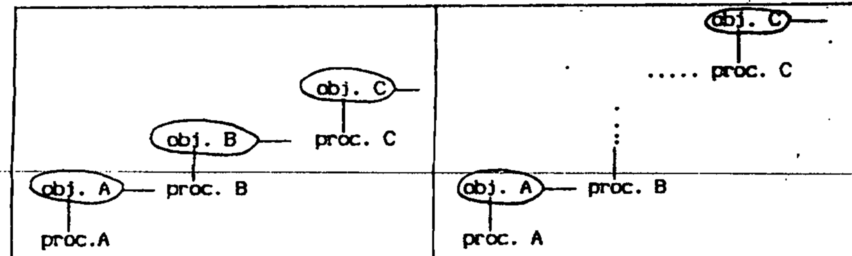
1. INTRODUCTION: PSEUDOSTRUCTURAL CONCEPTIONS

In [2], [3] and [4] an idea of operational/structural duality of mathematical thinking has been promoted. according to which majority of mathematical notions may be conceived in two complementary ways: as processes (operational conception) or as abstract objects (structural conception). In [3] and [4] several examples have been brought to show that a constant pattern repeats itself time and again both in history of mathematics and in individual learning: computational procedures are converted into object-like entities to serve as inputs for certain higher-level processes (Fig.1a). According to this model of concept formation, mathematics as a whole may be viewed as a hierarchy, in which what is conceived operationally on one level, should be conceived structurally on a higher level.

To put it in a slightly different language, abstract objects emerge at these junctions in knowledge development where some well-known processes (in this context, let us call them primary) meet with higher-level (secondary) processes. For instance, the concept of function originates in numerical computations, but fully crystallizes only when one becomes able to manipulate computational procedures as self-sustained, permanent wholes. Thus, an abstract object mediates between the primary and secondary processes, and acts as a link by help of which new knowledge is connected to the previously developed concepts. From now on, we shall concentrate in this paper on the possible "holes" and discontinuities in student's conceptual system and on their influence on the learner's mathematical understanding.

It should be clear by now that as long as no mention is made of any secondary procedure, there is also no need for the abstract object. Structural approach, however. becomes necessary when an attempt is made to proceed toward a higher-level theory. Indeed, the old and the new knowledge can only be tied together through abstract objects, thus reification of the primary processes (turning them into fully-fledged objects) is, in this case, indispensable for relational understanding. In [2] and in [4] it was argued, however, that reification is inherently very difficult; so, difficult, in fact, that at a certain level, structural approach may become practically out of reach for some students. Once the developmental chain has been broken (see Fig.1b), the process of learning is doomed to collapse: without the abstract objects, the secondary processes will remain "dangling in the air" — they will have to be executed... on nothing. Unable to imagine the intangible entities (functions, complex numbers, sets) which he is expected to manipulate, the student would use pictures and symbols as a

FIG.1: Development of mathematical concepts as a sequence of transformations from operational to structural conceptions



a. a "healthy" developmental chain

b. a broken developmental chain: process B has not been reified, so there is no link between B and C.

substitute: a graph of a function or an algebraic formula, a name of a number, the letters ø and x — each of these signs will turn into a thing in itself, not standing for anything else. In such case we shall say that the learner developed a pseudostructural conception. Lacking operational underpinnings, this kind of conception would leave the new knowledge detached from the previously developed system of concepts, and the secondary processes would seem totally arbitrary. The student may still be able to perform these processes, but his understanding will remain instrumental.

2. THE AIMS AND THE METHODS OF THE EMPIRICAL STUDY

In [4], the above claims were applied to the concept of function. In the present study we shall concentrate on the notion of equation and inequality. In a series of experiments, one of which will be reported in the sequel, we tried to have a deeper insight into the ways students confer meaning upon algebraic expressions. Special effort was made to trace down pseudostructural conceptions.

Let us begin with a closer look at the way in which equations and inequalities are introduced to the students. In Israeli schools, modern approach is applied, according to which equations and inequalities are two different, but closely related, instances of a single mathematical notion: propositional formula (PF, from now on). This universal construct is defined as "a combination of symbols (names of numbers, letters, operators, predicates, and brackets) that turns into a proposition when names of numbers are substituted instead of the letters". The idea of PF is introduced as early as seventh grade, and equations and inequalities are then brought and dealt with simultaneously. Every PF has its truth-set (TS, for short), namely the set of all the substitutions that turn this PF into a

true proposition. Any two PFs with the same truth sets are calleed equivalent. Solving equation or inequality means finding its TS. As a consequence of this approach, even the solving procedures are described in set-theoretic terms: to solve, say, an equation E, one must find the simplest possible PF which is equivalent to E. The basic steps which may be taken to transform an equation into an equivalent FF are called elementary (permitted) operations. To summarize, this is a good example of a structural approach: a mathematical notion (PF) is explained in terms of abstract objects (truth-sets). An alternative operational approach would relate the equations to underlying processes rather than to such imaginary entities: the two component formulae of an equation would be considered as a concise description of computational procedures, and solving the equation would mean deciding for which inputs both procedures would yield the same results.

The first way of dealing with the subject is certainly very attractive due to its mathematical elegance, consistency, and universality. Nevertheless, from the didactic point of view, introducing the student to a new notion through such an uncompromisingly structural approach is not necessarily the best move. The students, forced to begin with the abstract truth sets instead of safely proceeding from operational to structural approach, will be only too likely to develop pseudostructural conceptions: unable to see the abstract objects hiding behind the symbols, they would have considerable difficulty with relating the "permitted" operations (the secondary processes) to the arithmetic procedures encoded in algebraic formulae (the primary processes). To put it in a different language, the learner would be unaware of the rationale behind such operation like adding the same number to both sides of an PF. He would often prefer to talk about it in terms of transferring from side to side rather than of balancing, he would only have a vague idea why this particular manipulation has been declared as "permitted" and what are the criteria according to which one should check whether it works. In consequence, the pupil would resort to treating the symbolic expression as a thing in itself, for which the secondary processes would be an independent and the only source of meaning.

Since we knew that diagnosing pseudostructural conceptions was not going to be an easy task, it seemed only natural to try several tools rather than to lean on a single method. We decided to use various kinds of questionnaires, followed by interviews, classroom observations, and finally maybe even by a teaching experiment. By combining and comparing the results of all this moves, we hoped to identify some significant common tendencies. Due to space limitations, only the first of the several steps that have already been taken will be reported in this paper.

3. THE QUESTIONNAIRE ON EQUIVALENCE

Since our objective was to track down pseudostructural conceptions, it seemed the right move to draw the bead on the concept of equivalence. Although no procedure is mentioned in its definition, we expected that in order to decide whether two PFs are equivalent, some students would look for transformations by which one of the PFs could be turned into the other. By itself, this tendency cannot yet be regarded as an evidence for pseudostructural conceptions. Such conclusion would become justified only if we could show that the student uses the criterion of transformation automatically and never returns to the underlying processes and abstract objects in order to verify his conclusions. We decided, therefore, that as a tool for spotting pseudostructural conceptions we should use non-standard pairs of PFs which, in the case of such automatic behaviour, would lead to inconsistency with the definition of equivalence.

To construct the set of items presented in Fig. 2, we looked for four pairs of equations (and four pairs of inequalities) that would represent all the possible combinations of two parameters: equivalence according to the structural definition on one hand, and, on the other hand, possibility to transform one of the PFs in into the other by help of symbolic manipulations. Let us have a closer look at each of the categories.

While (E,T) and (-E,-T) (items a,b and g,h, respectively) consist of quite standard examples (pairs which either satisfy or do not satisfy both the requirement of equivalence and that of formal transformability), the remaining two groups were expected to pose a difficulty for some pupils.

In (-E,T) (items e,f), the PFs in a pair are not equivalent in spite of the fact that one of them may be formally transformed into the other. Clearly, there is a contradiction between these two conditions, so at least one of them must only seem to be satisfied. Indeed, neither the division of both sides by $3x-1$ (example e), nor the extraction of the square root from both sides of inequality (f) is a permitted operation. Nevertheless, our experience as teachers taught us that some students do use this kinds of operations without the necessary precautions. Such behaviour can be interpreted as an indication of student's inability to go back to the primary processes in order to verify their decisions.

The category (E,-T) (items c,d) is also non-standard, and to some people may seem counterintuitive: the PFs are equivalent according to the criterion of equal truth-sets, but no sequence of elementary operations would transform one of them into the other.

For a researcher, curiosities and non-standard examples create rare

FIG.2: The questionnaire and its results

	TRANSFORMABLE (T)				NON-TRANSFORMABLE (-T)			
	no	item	IA	NA	no	item	IA	NA
EQUI-VALENT (E)	a	$4x-11=2x-7$ $4x=2x+4$	9	2	c	$4x-11=2x-7$ $(x-2)^2=0$	68	15
	b	$5x+4<11(x+2)$ $4<6x+22$	18	11	d	$5x+4<11(x+2)$ $4x+5>x-4$	54	19
NOT EQUI-VALENT (-E)	e	$(3x-1)(2x-5)=x(3x-1)$ $2x+5=x$	28	17	g	$7x+2=3x+1$ $4x=5$	8	17
	f	$4x^2>9$ $2x>3$	45	43	h	$3x+2<1-7x$ $5(x-1)>6$	9	35

IA: % of answers which are inconsistent with the definition of eqivalence
NA: % of students who gave no answer

opportunity for probing student's understanding of different concepts. By exposing the student to such deceptive examples like those in category (-E,T) and to such unexpected (some would say unnatural) ones like those in (E,-T), we hoped to assess their readiness to go beyond standard procedures and to think in terms of the underlying processes and the abstract objects hiding behind the symbols. In this context, their answers to the question about equivalence were less important than the verbal explanation they were required to give in order to justify their decisions.

It is important to stress that although in this paper we shall confine ourselves to the collection presented in Fig. 2, there were other two parts in the questionnaire which was used in the study. In our investigation we did not overlook the exploratory potential of another kind of non-standard situations: those which arise when the student is confronted with a singular equation or inequality (tautological or contradictory PF). This kind of PFs appeared in the second part of our questionnaire, while its last part contained direct questions about the meaning of such concepts like equivalence, elementary operation and a solution of an equation.

4. THE SAMPLE AND THE FINDINGS

The study was carried out in three secondary schools in Jerusalem. 280 students of different ages (15-17) and of diverse abilities answered our questionnaire. Although there were some subtle differences between the results obtained in various subgroups, all the findings clearly indicated the same tendency. Because of this, and because of space limitations, we shall report here only the general results, leaving their analysis according to age and ability to another paper.

FIG.3: Arguments given by the respondents to support their answers

	EQUATIONS						INEQUALITIES					
	item	T	S	F	OOF	NA	item	T	S	F	OOF	NA
(E,T)	a	30	22	11	5	31	b	20	28	9	2	41
(E,-T)	c	52	29	4	4	12	d	47	19	4	3	27
(-E,T)	e	40	16	9	2	33	f	41	16	9	7	26
(-E,-T)	g	20	23	16	3	38	h	20	18	6	3	53

T: argument based on an attempt to formally TRANSFORM one PF into the other
S: argument based on full SOLUTION of both PFs and comparison of the results
F: argument based on the similarity or differences in the FORM of both PFs
OOF: out-of-focus response
NA: no argument

By the time the study was carried out, all the pupils have already had quite a long experience with the topics on which our questions were focused. For all of them, solving equations and inequalities was a basic skill, an indispensable ingredient of their everyday mathematical activity. Even so, in items c, d, f, and g relatively high percentages of the respondents gave answers which were inconsistent with the definition of equivalence (see Fig. 2). Thus, according to our expectations, the students' behaviour in categories (-E,T) and (E,-T) indicated that for many of them, the formal transformability was practically the only criterion for equivalence. Moreover, the answers to the questions e and f showed that the decisions whether a given transformation is "permitted" or not had often been quite arbitrary, and certainly had not been based on any requirements regarding underlying processes and objects (it should be noticed that in item e, the percentage of answers inconsistent with the definitions was substantially lower than in c, d, and f; this can probably be explained by the fact that careless dividing of both sides by an expression containing the variable is one of those common mistakes against which teachers repeatedly warn their students in advance).

The arguments with which the students justified their answers are summarized in Fig. 3. The findings seem to reinforce the impression that for many respondents, an equation or inequality was nothing more than a string of symbols which can be manipulated according to certain arbitrary rules. Of those pupils who did explain their decisions, the majority used the transformability as a criterion. Many others leaned on purely external features of the PFs, such as partial similarity and partial difference between their component formulae. Under the title "out-off-focus arguments" we have collected all the responses in which different mathematical entities have been confused (for example, two sides of the same equation has been compared in order to answer the question about the equivalence between this equation and another). Although some of the above arguments could be given

also by a student who fully adopted the structural approach, in majority of cases they may only be interpreted as indicative of pseudostructural conceptions. Indeed, more often than not, they have been brought to support an incorrect claim about equivalence of two PFs. The possibility that student's understanding was merely instrumental cannot be dismissed even in those cases in which the respondents solved both equations and compared the solutions. Although this is exactly what should be done according to the definition of equivalence, the respondent's actions could sometimes be dictated by a habit rather than by the deep relational comprehension.

5. SOME TENTATIVE CONCLUSIONS

Our little study brought new evidence to what has already been noticed by many researchers (see [1] for the review of relevant literature; see [6] for more findings about equivalence): in spite of all the attention devoted in secondary schools to algebra and to symbolic notation, students' command of the formal language of mathematics is often instrumental rather than relational. In this paper, the pseudostructural conceptions were made responsible for this state of affairs, and the mechanism which leads to the development of such conceptions has been analyzed.

Our final remarks regard the possible ways of preventing and fighting pseudostructural conceptions. Although the teaching method does not seem to be the only factor which determines the kind of conceptions developed by the students (see [5]), much can probably be done to improve understanding of algebra just by replacing structural approach with operational while introducing the subject. Indeed, the operational approach has already proved its effectiveness in several topics (see [3],[4]), so it is certainly worth trying. Teaching experiment in which the method will be implemented is planned as one of the next steps in our research.

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[1] Kieran, C.: 1990, The learning and teaching of school algebra, to appear in D. Grouws (ed.), Handbook of research on Mathematicss Teaching and Learning.

[2] Sfard, A.: 1988, Operational vs structural method of teaching mathematics, in Proceedings of PME XII.

[3] Sfard, A.: 1990, On the dual nature of mathematical conceptions, to appear in Educational Studies in Mathematics.

[4] Sfard, A.: 1991, Operational origins of mathematical objects and the quandary of reification, to appear in Dubinsky, E. and Harel, G. (eds), Learning and teaching the concept of function.

[5] Skemp, R.R.: 1976, Relational understanding and instrumental understanding, Mathematics Teaching, 77, 20-26.

[6] Steinberg, R.R., Sleeman, D. and Katorza, D.: 1990, Transformations and equivalence of equations in high school algebra, to appear in Journal for Research in Mathematics Education.

Everyday Knowledge in Studies of Teaching and Learning Mathematics in School.

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This paper suggests an analysis of a selected number of recent studies of teaching and learning in school which pay attention towards the relationships between individuals' everyday knowledge and the acquisition of mathematical concept. Three kinds of studies are considered : "Before and After" studies, "Rationality" studies and "Interaction" studies. Terminology and conceptions of Everyday Knowledge are finally discussed.

I will present the analysis by considering the following questions:

1. how do the works contribute and relate to what we might call "traditional topics" in the didactics of mathematics :
-the construction of knowledge of instructional approaches and methods of teaching
-the construction of knowledge of the consequenses of mathematics education: This is partly general questions about transfer, and partly specific questions about various results of education of persons of various background such as gender, race, social class or cultural background.
-and the choice of what should constitute the curriculum, [1]

2. how can "the geography" of the works be described:
- with regard to whether the main interest in the works is in the years before school, the years when they go to school, the years after school
- with regard to the main level-interest: from kindergarten to university and vocational schools
3. how can the proposals in the works towards implementation in school be described.
4. what is the nature of everyday knowledge mentioned in the studies: is it knowledge of phenomena in the everyday world, is it knowledge of how to solve problems, or is it rather meta-knowledge of what mathematics really is, and what learning mathematics really is.

<u>Introduction</u>
Shift of focus.

In the ICMI study "Mathematics and cognition" (1990) ,p.136 Nicolas Balacheff et al summarize the shifts of focus in the work done by PME members:
1. Focus has shifted from understanding mathematics learning as

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[1] Studies in school cannot suggest specifications of change of curriculum. Neither can studies of cognition in practice (like Jean Lave's). In my opinion, however, a variety of studies outside the shool setting exist from which it is possible to draw such conclusions: Studies of ethnomathematics and studies of innumeracy and critical citizenship.

the acquisition of competance and skills to understanding it in terms of thinking processes.
2. Focus has shifted from studies of students' difficulties to studies of students' knowledge which underlies the difficulties.
3. The approach to observed students' errors has shifted: one does not any longer thinks of these errors as facts that can be counted and classified; one stresses hunting for explanations of the origins of the errors.
4.These shifts of focus are mirrored in a tendency to a shift in the use of metaphors from terms such as "misconceptions" to terms such as "students' conceptions" or "concept image".

I think everyone will agree that all these shifts place preknowledge and everyday knowledge in a profoundly central position when studying the acquisition of mathematical concepts. Nevertheless, as I will demonstrate in the following review, we have not arrived at any agreement on how everyday knowledge and preknowledge interact with the acquisition of mathematical concepts, how we can design further study of this interaction or what educational conclusions we may draw. Even the definitions of everyday knowledge and preknowledge differ.

I have found it appropriate to establish three clusters of studies, each characterised by its certain design:

-"Before and After" studies, in which students' answers to mathematical exercises and problems are compared before and after a mathematical course.
-"Rationality" studies, in which one is emphasizing that students' errors and students' conceptions can be interpreted as something logical and rational and one is exploring the possible reasons for students' observed errors and behaviour.
-"Interaction" studies, in which descriptions and understanding of various kinds of interaction in the classroom are constructed.

The following review will demonstrate how research design determines what specific matter in regard to everyday knowledge you can call attention to.

"Before and After" studies.

The first kind of studies that I will exemplify is "before and after" studies. In these studies you compare students' answers to mathematical exercises and problems, before and after these students attend a specific mathematical course. According to everyday knowledge the purpose is to analyze what kind of everyday knowledge exists before instruction that enables students to cope with mathematical issues, and how instruction influences these abilities positively or negatively.

Only one example will be mentioned here. It is Claes Alexandersson's study "Stability and Change" (1985). The study is about a group of adults dealing with percentages. The study shows that before the course the adults actually had some everyday knowledge of conceptual nature and of how to go on working that helped students to cope with some of the exercises. After the course the adults' answers to some of the same exercises and problems were slightly less correct than before the course.
So it seems as if the mathematical instruction about percentages

had destroyed some valuable previous knowledge and had caused a change in attitude towards a problem that is reflected in a mechanical applications of algorithms also to problems which cannot be solved by these techniques.

The study stresses the need of research on general questions such as:
-the instructional question of how to help students to algorithmetise knowledge when the students know a good deal before they start.
-and the question of transfer, in which we do not normally think of negative transfer where school knowledge destroy previous capabilities.

According to proposals towards implementation in school we can only conclude from such "before and after" studies that instruction ought to be changed: so far as this kind of study does not further investigate the actual instruction we cannot come to any conclusion as to how the instruction ought to be changed.

"Rationality" studies

The second kind of study of teaching and learning in school I have called "the studies of rationality". Unlike "before and after" studies, "rationality" studies analyse the very thinking processes and knowledge which underlie students' difficulties. "Rationality" studies cope with the questions of how to characterize the nature of students' conceptions and how to characterize their development.
The reason why I chose the term "rationality" studies is my wish to emphasize that students' errors and students' conceptions can be interpreted as something logically and rationally determined by students' everyday knowledge, attitudes and previous mathematical conceptions. In my opinion, this kind of study really gives one an opportunity to explore the possible reasons for students' observed errors and behaviour.

From error-analysis it has long been known that different presentations of the same mathematical core calling for the same mathematical operation invoke very different student behaviour. Most studies are about young children.

For example young children's handling of additive word problems is thoroughly examined. It is found that students' thinking and acting are highly dependent on the specific semantic form of additive word problems. It is also found that additive word problems can be categorized into a small number of semantic categories. (In e.g Mary Riley's interpretation there are four categories: change, equalise, combine, compare.)
According to everyday knowledge the studies cope with a here and now question and with a dynamic question:
A. The here and now question whether everyday knowledge exists that enables students to give meaning to mathematical symbols and to do some exercises in the context of real-life situations ?
The answer in several studies of whole number arithmetic is yes. The answers in studies of fractions differ, and most studies of fractions focus on misconceptions. Only a few e.g. Nancy K.Mack's work, show students coming for instruction with a rich store of informal knowledge about fractions which helps them to perform

operations on fractions when problems are presented in the context of real-life situations.But a close match between problems represented symbolically and those that draw on their informal knowledge is necessary.

B. The dynamic question whether everyday knowledge exists upon which students can build new school knowledge ? Some studies investigate this dynamic effect. Some stress the limiting effects (Behr, Kerslake, here from Mack p.29), others that everyday knowledge can provide a basis for understanding mathematical symbols and procedures, but once again the connection between the informal knowledge and the fraction symbols must be reasonably clear. (Hiebert 88, Mack 90).

The semantic categories form a description of how students interpret the word problems and of the kind of everyday knowledge students reactivate in problem solving. But obviously the kind of everyday knowledge one can catch sight of in these studies is the kind that the specific word problems in school mathematics call for, as research design is located inside the school.

The pedagogical implications of this research are that if teachers interpret students' success or failure in one semantic category as a general success or failure in additive word problems, they are making a real mistake. Bergeron and Herscovics et al (M&CF p 52) describe the pedagogical implications of this research as numerous and conclude that it can convince teachers that students possess much greater knowledge than usually believed.
Nevertheless, as I see it, the more we realize how much students do themselves independently of teaching, the more crucial and necessary teachers' personal invention and communication would seem to be.

Most "rationality" studies concern younger children and additive word problems. Some exist concerning children and fractions. Only few studies analyse "older" students and more advanced mathematics. It seems to me that these studies stress the great importance of meta-knowledge of what mathematics really is, so the specific everyday knowledge involved in these studies is rather of what it means to understand mathematics, than it is of phenomena and methods.

Robert B. Davis' classic study of Lucy, a 15 year old student at a course in calculus, interprets Lucy's behaviour as automatic, as "following the leader" behaviour, as if she fails in synthesizing enough frames, and as if she is rarely engaged in meta-analysing her own actions. What is to be done about such learning difficulties in instructions ? More time will not necessarily solve Lucy's problem, Davis claims. The difficulty is rather that she did not know what it meant to understand a piece of mathematics.
This moves the object of investigation to new questions:
- A. the question of how to characterise students' notions of what it means to understand and what it means to profit by a piece of mathematics,
- B. the question of what determines these notions.

I have proposed some answers to question A. I think we have a

variety of notions among students. One notion I have found, among students in the Danish gymnasium, is that understanding mathematics and profiting from it is but doing the exercises the teacher asks you to do. Another notion is that it is knowing something about "the meaning" of the piece of mathematics: why it has ever been constructed, or why it is in the curriculum, or how to describe its usefulness. A third notion is that it is using the piece of mathematics to solve some problems which also are of extra-mathematical interest. A fourth notion is that you understand and profit simply when you feel you are succeedingg in relation to the requirements established by the teacher.
Question B as to what may determine such notions cannot be answered if we restrict ourselves either to looking at previous school experience or to looking outside school. I think the notions are determined by students' everyday knowledge created outside as well as inside the school setting. It is everyday knowledge not about specific phenomena or processes, but about broad issues such as the nature of knowledge and of mathematical knowledge, and such as the role played by school, education and mathematics in society. This kind of everyday knowledge develops in the individual in connection with the the development of personal identity.

"Interaction" studies

The third kind of study of teaching and learning in school is "interaction" studies. The goal of "interaction" studies is to construct valid descriptions and informative understanding of interaction in the classroom. Some focus on interaction between teacher and student, some focus on interaction between teacher and the epistemological h structure of mathematical knowledge. Indeed, these studies suggest new explanations and new kinds of mirroring the events in the classroom and in the teaching of mathematics. What is especially interesting according to this review: some of these studies include teachers' and students' everyday knowledge in their explanations for reasons of observed interactions.

Only one example will be mentioned here: Helga Jungwirths study of gender-specific modifications of interactions. Two kinds of everyday knowledge are considered in the study :
a. everyday knowledge of the subject-matter one actually applies mathematics to.
b. everyday knowledge and everyday experience of specific aspects in situations, such as the lack of sufficient information and knowledge.

In answer to question a. she suggests that mathematics educational approach to extra-mathematical issues suits boys more than girls: girls would associate differently and girls would prefer different methods to be applied to the extra-mathematical issues.(p.124)
In answer to question b. Helga Jungwirth examines how the teacher manege the conversation in the lesson: they pose questions and they finely adjust the students' answers, and she examines how the students cope with this manegement: they develop special methods of participation. From these examinations she suggests a new understanding of the reason why some teachers assess girls boys to be less good at mathematics than boys:

"It may be said that the methods which enable girls and boys to succeed in taking part in the lessons based on development through questioning have become more a routine for the boys than for the girls. As long as you do not "look behind the scenes" of this teaching method, something which happens only very rarely, it looks as if the boys were mathematically more competent than the girls" (p.ix). To develop the explanation a little further, we can conclude that the everyday knowledge relevant here originates from experiences with ambiguous questions, with new kinds of questions or with circumstances in which one does not have all the knowledge one needs.

Concluding remarks on terminology and conception of everyday knowledge.

The different kinds of studies of teaching and learning in school do not compete. The studies supplement each other, as the different kinds of studies examine different questions. The terminology and the conception of everyday knowledge, however, could be more clear and sharp, in my opinion. Several terms are used: mathematical conception, subjective image, informal mathematics, intuitive knowledge, situated knowledge, prior knowledge and informal knowledge. And in this paper I have suggested the term "everyday knowledge". But to what extent are the terms reflecting the same phenomena and the same understanding ? How are they alike ? How do they differ ? I feel the need of having these questions much further examined.

It is important to stress that we do not search for any explicit specification of what in general could characterize everyday knowledge, as the relation between everyday knowledge and mathematics are quite different from that of other subjects. It is appropriate to build the specification upon our knowledge of mathematics educational issues. Besides, if we want to adress also such kind of knowledge, which is important in acquisition of advanced mathematics, we have to construct a broader concept that includes meta-issues too.

In my actual understanding I have but some few ideas of what should be reflected by the term :
-everyday knowledge is sensed by the individuals as factual knowledge. In this regard the term "everyday knowledge" is in opposition to the term of fantasy. What we sense as factual knowledge we do not question or further investigate: we simply trust it and build on it, sometimes as tacit knowledge without being aware of it. The interesting point is that some of this everyday knowledge could produce learning difficulties for the students and could produce problems in communication between teacher and student.
-the origin of everyday knowledge lies outside what is on the agenda of the formalized schooling.
-some everyday knowledge relates to phenomena in the world, some to problem-solving methods, and others to "meta-issues" such as what is math, really, and what are education and learning, really.
-some everyday knowledge about phenomena is in accordance with the core of the mathematical concepts, and in these cases pupils can build directly on their everyday knowledge. (Children's everyday knowledge of sharing pizzas, for instance). Some everyday knowledge, however, forms a contrast to mathematical concepts. (This can be the case with probability. Some of my own students are certain that when you throw dice the six has a lower frequency than the others.)
-some everyday knowledge of problem-solving methods and of "meta-issues" are appropriate to the actual teaching of mathematics, others are not.

Literature.

Alexandersson, Claes (1985) "Stability and Change. An empirical study of the relation between knowledge acquired in school and in everyday life", Studies in Educational Sciences: 53, Acta Universitatis Gothoburgensis, Gøteborg, Sweden.

Balacheff, Nicolas et al (1990) "Future Perspectives for Research in the Psychology of Mathematics Education" in ICMI Study 'Mathematics and Cognition', p.135-148.

Beattys, Candice & Herscovics, Nicolas & Nantais, Nicole (1990) "Children's Pre-Concept of Multiplication: Procedural Understanding", Fourteenth Conference for the Psychology of Mathematics Education, Vol III, 183-190, Mexico.

Behr,M.J., Lesh,R., Post,T.R., Silver, E.A. (1983) "Rational Number Concepts" in Lesh,R. and Landau,M. 'Acquisition of Mathematics Concepts and Processes', Academic Press, Orlando.

Brown, Stephen I. (1986) "The Logic of Problem Generation: from Morality and Solving to De-posing and Rebellion" in Burton, Leone (ed.) 'Girls into Maths Can Go', Cassel education, London.

Davis, Robert B. (1985) "Learning Mathematical Concepts: The Case of Lucy." in The Journal of Mathematical Behavior 4, 135-153.

Frankenstein, Marilyn (1989) "Relearning Mathematics. A Different Third R -Radical Maths". Free Association Books, London.

Ginsburg, Herbert P. & Allardice, Barbara S. (1984) "Children's Difficulties with School Mathematics" in Rogoff, Barbara & Lave, Jean 'Everyday Cognition', Harvard College.

Greer, Brian (1987) "Understanding of arithmetical operations as models of situations" in Sloboda, John A. & Rogers, Don 'Cognitive Processes in Mathematics', Oxford University Press, New York.

Harris, Mary (1987) "An Example of Traditional Women's Work as a Mathematics Resource." For the Learning of Mathematics 7,3.

Harris, Mary (1989) "Common Threads". Department of Mathematics, University of Education, London.

Hiebert,James & Wearne, Diana (1988) "Instruction and Cognitive Change in Mathematics", Educational Psychologist, 23(2), 105-117.

Jungwirth, Helga (1990) "Madchen und Buben im Mathematikunterricht. Eine Studie uber geschlechtsspezifische

modifikationen der Interaktionsstrukturen." Bundesministerium fur Unterricht, Kunst und Sport, Wien.

Lave, Jean (1988) "Cognition in Practice. Mind, mathematics and culture in everyday life". Cambridge University Press, Cambridge.

Leinhardt, Gaea (1988) "Getting to Know: Tracing Mathematical Knowledge From Intuition to Competance". Educational Psychologist, 23(2), 119-144.

Lindenskov, Lena (1990) "Mathematical Concept Formation in the Individual". Fourteenth Conference for the Psychology of Mathematics Education, Vol I, 61-68, Mexico.

Lindenskov, Lena (1990) "The Pupils' Curriculum, Rationales of Learning and Learningstories". First Conference for Political Dimensions of Mathematics Education: Action & Critique, Institute of Education, University of London.

Mack, Nancy K. (1990) "Learning Fractions with Understanding Building on Informal Knowledge". Journal for Research in Mathematics Education, vol 21, no 1, 16-32.

Mellin-Olsen, Stieg (1987) "The Politics of Mathematics Education", Reidel Publ.Compagny, Dordrecht.

Mukhopadhyay, Swapna, L.B. Renick, L. Schauble (1990) "Social sense-making in Mathematics: Childrens Ideas of Negative Numbers". Fourteenth Conference for the Psychology of Mathematics Education, Vol III, 281-288, Mexico.

Nantais, Nicole, Herscovics, Nicolas (1990) "Children's Pre-Concept of Multiplication: Logico-Physical Abstraction", Fourteenth Conference for the Psychology of Mathematics Education, Vol III, 289-296, Mexico.

Nesher, Pearla & Kilpatrick, Jeremy (eds) (1990) "Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education", ICMI STudy Series, Cambridge University Press.

Olivier, Alwyn, H.Murray, P.Human (1990) "Building on young children's Informal Arithmetical Knowledge". Fourteenth Conference for the Psychology of Mathematics Education, Vol III, 297-304, Mexico.

Resnick, Lauren B. (1983) "A Developmental Theory of Number Understanding" in Ginsburg, H.P.(ed.) 'The Development of Mathematical Thinking', Academic Press, London.

Resnick, Lauren B. & Cauzinelle-Marmèche, Evelyne & Mathieu, Jacques (1987) "Understanding Algebra" in Sloboda, John A., Rogers, Don 'Cognitive Processes in Mathematics', Oxford University Press, New York.

Resnick, Lauren B (1990) "From Protoquantities to Number Sense", Fourteenth Conference for the Psychology of Mathematics Education, Vol III, 305-311. Mexico.

Seeger; Falk (1990) "Die Analyse von Interaktion und Wissen im Mathematik-Unterricht und die Grenzen der Lehrbarkeit", Journal fur Mathematik-Didaktik, 2, 129-158.

THE KNOWLEDGE ABOUT UNITY IN FRACTION TASKS OF PROSPECTIVE ELEMENTARY TEACHERS.

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This study reports an analysis of the pedagogical knowledge about fractions of 26 prospective elementary teachers. Individual interview questions probed the prospective teachers pedagogical content knowledge about instructional representations systems (chips, non contextualized drawings and symbols) of the concept of fraction. The double analysis of the interviews illustrated the incapacity of many preservice elementary teachers to identify the unity, to represent some fractions with chips, and to work with fractions bigger than 1.

El conocimiento y la comprensión del profesor de y sobre la materia que enseña han generado diversas cuestiones de investigación. Algunas de estas se han centrado en el análisis de la comprensión de los profesores sobre determinadas nociones matemáticas. (Ball,D.L., 1990; Civil,M. 1989; Graeber,A.O., Tirosh,D. y Glover,R., 1989; Post,T., Harel,G.,Behr,M. y Lesh,R. 1988).

Otras analizan diferentes aspectos del conocimiento de contenido pedagógico del profesor (Carpenter,T., Fennema,E., Peterson,P. y Carey,D. 1988) caracterizado por L. Shulman (1986). En esta caracterización se destacaron dos ideas: (i) El conocimiento del profesor de diferentes sistemas de representación instruccional del contenido a enseñar, y (ii) la manera en que son utilizados durante la instrucción. El análisis de la forma en que el profesor utiliza diferentes sistemas de representaciones instruccionales en el aula, para ayudar al aprendizaje de diferentes nociones matemáticas por parte de los alumnos, ha constituido el objetivo de estas investigaciones (Baxter,J. y Stein,M.K. 1990; Bromme,R. y Steinbring, 1990).

En este contexto, el estudio realizado examina algunos aspectos del conocimiento de contenido pedagógico que sobre las fracciones poseen estudiantes para profesor en el programa de formación inicial. En general el análisis se centra en el conocimiento y comprensión de los estudiantes para profesor sobre (i) la conexión de diferentes referentes concretos al símbolo a/b, y (ii) el uso de

referentes concretos para justificar el "funcionamiento" de algunos procedimientos matemáticos. Las cuestiones generadas en la investigación desarrollada se apoyaban en la hipótesis de que los estudiantes para profesores poseen un conocimiento previo sobre los contenidos matemáticos, fruto de su aprendizaje como estudiantes cuando estaban en la escuela.

METODO

* Participantes.

Veintiseís estudiantes para profesores de primaria, matriculados en segundo y tercer curso en el programa de Formación de Profesores de Primaria de la Universidad de Sevilla, fueron entrvistados. Estos estudiantes habían recibido un curso de contenido matemático - centrado primordialmente en contenidos algebraícos como teoría de conjuntos y estructuras algebraícas de los conjuntos numéricos - durante su primer año en el programa de formación. Lo que estos estudiantes para profesor conocían sobre las fracciones reflejaba la comprensión que habían desarrollado durante sus estudios de Matemáticas anteriores a dicho programa.

* Instrumento.

En la entrevista se plantearon diferentes cuestiones sobre las fracciones. Se utilizaron tres diferentes sistemas de representación descontextualizados (dibujos geométricos, fichas, y recta numérica) y cuatro tipos de tareas (representar diferentes fracciones, reconstruir la unidad, buscar fracciones equivalentes y determinar la relación de orden entre dos fracciones). Al estudiante para profesor se le pedía explicar sus representaciones y decisiones realizadas. El entrevistador, aunque poseía un guión de la entrevista, podía plantear cuestiones adicionales para obtener una mejor descripción de las concepciones de los estudiantes para profesor. Estas cuestiones estaban dirigidas a aclarar los argumentos utilizados al resolver la tarea propuesta, en particular las características del conocimiento sobre las fracciones empleado en las explicaciones.

* Procedimiento.

Las entrevistas fueron grabadas en audio y posteriormente transcritas integramente. Se realizaron dos tipos de análisis de las transcripciones obtenidas. En primer lugar se realizó un cuidadoso análisis descriptivo de los diferentes procedimientos utilizados por los estudiantes para

profesor. Este análisis descriptivo inicial generó un conjunto de categorías de las respuestas producidas. Posteriormente se realizo un segundo análisis conceptual tomando como marco de referencia la teoría de Hiebert (1988) sobre el desarrollo en la competencia en el manejo de los símbolos matemáticos.

Mientras en el primer análisis las categorías se generaron a partir de las transcripciones de las entrevistas, en el segundo análisis se intentó dotar de una explicación - comprender - dichas categorías a la luz de las hipótesis descritas por Hiebert en la Teoría propuesta.

El informe presentado aquí es una parte de la investigación realizada, y describe los resultados obtenidos en relación a la comprensión de los estudiantes para profesor de este estudio sobre la noción de unidad en dos tipos de tareas (reconstruir la unidad y representar una fracción) con dos sistemas de representación instruccional (fichas y dibujos geométricos).

RESULTADOS.

Según Hiebert (1988) el significado para los símbolos individuales se crea cuando se establen las conexiones entre éstos y las cantidades/acciones que representan. Este proceso de conexión viene caracterizado por la naturaleza de esta correspondencia. La conexión entre el símbolo y aspectos irrelevantes de los referentes puede interferir en la construcción apropiada del significado.

Para obtener información sobre estos aspectos del conocimiento de la noción de fracción en los estudiantes para profesores se propuso la siguiente tarea

> Se colocan seís fichas sobre la mesa
>
> O O O O O O
>
> "Si estas seís fichas son los 3/2 de la unidad. ¿Cuántas fichas forman la unidad?"

De forma general, los estudiantes para profesor utilizaron un vocabulario vinculado a la idea "parte-todo" en sus respuestas. Estas aproximaciones consistían en intentar "hacer partes de un todo y coger algunas". Los procedimientos que tendían a aplicar sugerían una noción de fracción como una "acción sobre algo" (sentido directo de la idea de operador). En segundo lugar esta idea

implicaba "coger algunas partes", lo que inducía a pensar en fracciones menores que la unidad. Ninguna de estas dos características coincidía con la tarea propuesta. Resolver dicha tarea exigía una reelaboración del significado que los estudiantes para profesor asociaban a la noción de fracción. La no coherencia entre los rasgos característicos de la tarea (reconstrucción de la unidad y fracción mayor que uno) con el significado asociado a la idea "parte-todo" generó toda una serie de interferencias. Estas interferencias se manifiestan en los diferentes procedimientos utilizados por los estudiantes para profesor al resolver la tarea.

Tres de los estudiantes para profesor (EP) fueron incapaces de centrarse en la tarea en el nivel concreto. No podían ver las fracciones representando cantidades y acciones sobre cantidades. Las fracciones sólo eran vistas como números con los que se puede operar. Mientras dos de ellos veía la fracción como un decimal (1'5) o como un número mixto (1+1/2) y eran incapaces de resolver la tarea, el tecero de ellos trasladaba la tarea propuesta a la expresión símbólica 3/2.x=6, señalando que

> R.2.14.: ... yo la fracción ... no soy capaz de pasarla a figuras ni entendería como una cosa real ... siempre me limito a éso, a hacer sumas, restas, multiplicación, división de fracciones. O a buscar los tres medios de un número determinado ...
> (entre.2)

Sólo 14 de los EP desarrollaron procedimientos apropiados, en el nivel de los concretos, que les permitieron resolver la tarea. Por ejemplo, Manuel, después de separar las 6 fichas en tres grupos indicaba,

> M.1.15.: ... divido en tres grupos las 6 fichas. Son dos por grupo. Y son tres mitades ... como un total tiene dos mitades nada más ... pués tengo que eliminar una (mitad). Entonces me quedo con cuatro (fichas)(entre.5)

Generalmente los procedimientos utilizados consideraban la unidad formada por n/n. En el caso particular de esta tarea la unidad era vista como 2-GRUPOS, cada grupo recibia el nombre de MEDIO en función del número de grupos de la unidad. Asi, la fracción 3/2 era vista como 3-

GRUPOS. El número de grupos en que se dividía la unidad lo proporcionaba el denominador.

Por otra parte, 7 de los 24[1] EP eran incapaces de resolver la tarea, no desarrollando ningún procedimiento identificable. Las dificultades planteadas en la identificación de la unidad en esta tarea se generaron por el significado restringido asociado a la idea de fracción. Cuando la tarea no encaja con el significado adscrito se produce la incapacidad para resolverla. Los siguientes protocolos ilustran dichas dificultades

> F.4.6. : ... al no tener el total no puedo asociarlo.
> (entre.18)
>
> M.2.1. ... es que lo estoy haciendo a ver si me sale pero arbitrario (ensayo y error).
> (entre.17)

Con la modificación de algunas características de la tarea se pretendía obtener información sobre la naturaleza de la conexión entre el significado adscrito a la idea de fracción y las características del referente. Así, ante la tarea

> Se colocan seis fichas sobre la mesa
>
> O O O O O O
>
> "Si estas 6 fichas son los 2/3 de la unidad. ¿Cuántas fichas forman la unidad?"

16 de los 24[2] EP resolvieron esta tarea en el nivel de los concretos aplicando procedimientos vinculados a la idea parte-todo en los que la unidad era vista como 3-TERCIOS. Por ejemplo

> F.2.2.: ... como son dos tercios (separa las 6 fichas en dos grupos de tres), es decir, tres (fichas) son un tercio ... tres son otro tercio, que son dos tercios ... y si añado tres fichas más, son tres tercios que es la unidad.
> (entre.3)

Cuatro de los EP tradujeron la situación al nivel de símbolos aunque sólo dos de ellos fueron capaces de obtener un resultado correcto. Sólo cuatro de los EP no fueron capaces de resolver esta

tarea en el nivel concreto. Aunque utilizaban un vocabulario vinculado a la idea "parte-todo" no era posible identificar ninguna estrategia coherente. Por ejemplo

> C.4.5. de tres cojo dos. El total serían dos, ¿no?
> C.4.9.. ... de tres partes cojo dos. Pero .. ¿cómo es éso?
> C.5.1.. ... ya no estoy segura de nada.
> (entre.15)

	nivel símbolos		nivel concreto	
	NR	R	NR	R
D; 3/2 de ? = 6	2	1	7	14
D; 2/3 de ? = 6	2	2	4	16

Cuadro 1. Resolución de la tarea por los EP
NR= no resuelve; R= resuelve; D= contexto discreto (fichas)

Otra de las tareas propuestas pedía la representación de una fracción (5/4 y 2/3) utilizando fichas y dibujos geométricos (círculo o rectángulo). La tarea de representar 2/3 mediante fichas indicando la unidad considerada fue realizado correctamente por todos los EP excepto uno (sobre 25 EP). La unidad considerada fue en unos casos 3, 6, 9, 12, o 15 fichas. Representar dicha fracción mediante figuras geométricas también fue resuelto por la mayoría de los EP, aunque en algunos casos algunos mostraron ciertas dificultades en representar partes iguales en los círculos (tres de ellos).

Sin embargo, la representación de la fracción 5/4 mediante fichas planteó mayores dificultades. Mientras sólo 18 de los EP identificaban un grupo de fichas idóneo como unidad, 8 no fueron capaces de identificar una unidad utilizable. Con los dibujos geométricos 19 de los 26 EP realizaron una representación correcta, tomando como unidad un rectángulo o un círculo. Dos EP invirtieron los términos de la fracción para representar 4/5, y uno consideró como unidad los dos rectángulos dibujados. Con las fichas la posibilidad de ver la unidad compuesta por fracciones unitarias (3-MEDIOS implicaba una unidad formada por 2-MEDIOS) y estas como grupos de fichas permitió a los EP concebir un grupo de fichas adecuado en cada caso para ser la unidad (considerando ficha=fracción unitaria).

	2/3 a f	2/3 a d	5/4 a f	5/4 a d
R	24	26	18	19
NR	1	-	8	7

Tabla 2. Resolución de la tarea "representación de la fracción".
R= resuelve; NR= no resuelve; f= fichas; d= dibujo geométrico.

DISCUSION

El informe presentado se centra en la conceptualización de la noción de unidad en tareas de "representar fracciones" y "reconstruir la unidad" por parte de 26 EP de primaria. Las mayores dificultades se presentaron en las tareas con fracciones mayores que la unidad. La noción de fracción, para los EP de la investigación, generalmente estaba vinculada a la idea "parte-todo"; es decir hacer partes de un todo y coger algunas. Esta noción venía caracterizada por (i) ver la fracción como una acción directa sobre el todo, (ii) identificar el todo, (iii) coger algunas de las partes en que se divide el todo. Las dificultades se generaron por la incapacidad de identificar una unidad adecuada al utilizar fichas como sistema de representación. Mientras que con los dibujos geométricos (círculo y rectángulos) la unidad viene implícitamente establecida por la propia figura geométrica (representación), con las fichas el número de éstas que forman la unidad viene determinado por la propia fracción (es decir la tarea). Así, mientras en las tareas de representar las fracciones 5/4 ó 2/3 el número de fichas que constiuía la unidad podía ser variable (sólo se le exigía poder hacer 4 grupos o tres grupos de fichas en cada caso), en las tareas de reconstruir la unidad, el número de fichas que constituían ésta era determinado por la propia tarea. En este contexto, los procedimientos empleados por los EP utilizaban las fracciones unitarias (1/n) para concebir la unidad.

Asumiendo que el aprendizaje de las nociones matemáticas y el proceso de dotar de significado a los símbolos matemáticos escritos se fundamenta en el uso adecuado de diferentes sistemas de representación, el conocimiento de los profesores (EP) del papel desempeñado por dichas representaciones se constituye en un elemento clave de la calidad de la instrucción. En este

contexto, la naturaleza de la comprensión de los EP de la vinculación entre diferentes concretos y los símbolos matemáticos constituye una información interesante para diseñar intervenciones en la formación de profesores de Matemáticas.

(*) Miembros de GRUPO DE INVESTIGACION DIDACTICA de la Unv. Sevilla
(1) Esta tarea sólo la realizaron 24 EP
(2) Esta tarea solo la realizaron 24 EP

REFERENCIAS.

Ball, D.L. (1990): Prospective elementary and secondary teachers' understanding of division. Journal of Research in Mathematics Education, vol. 21, nº 2, 132-144.

Carpenter, T., Fennema, E., Peterson, P. y Carey, D. (1988): Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. Journal for Research in Mathematics Education, 19(5), 385-401

Civil, M. (1989): Prospective elementary teachers' conceptions about the teaching and learning of mathematics in the context of working with ratios. Proceedings of the XI Annual Conference of the North American Chapter of the International Groups for the Psychology of Mathematics Education. New Brunswick, New Jersey.

Baxter, J.A. y Stein, M.K. (1990): Linking Teaching and Learning Trough Representations: A case from Functions and Graphing. Paper presented at the Annual Meeting of the American Educational Ressearch Association, Boston.

Bromme, R. y Steinbring, H. (1990): Interactive Development of Subject Matter within Instruction in the Calssroom. Paper of Institut für Didaktik der Mathematik. Univerität Bielfeld.

Graeber, A., Tirosh, D. y Glover, R. (1989): Preservice teachers' misconceptions in solving verbal problems in multiplication and division. Journal for Research inMathematics Education, 20, 95-102.

Hiebert, J. (1988): A Theory of Developing competence with written mathematical symbols. Educational Studies in Mathematics, 19, 333-355.

Post, T, Harel, G., Behr, M. y Lesh, R. (1988): Intermediate teachers' knowledge of rational number concepts. En E. Fenema, T. Carpenter y S. Lamon (Eds.) Integrating research on teaching and learning mathematics. Madison, WI: Wisconsin Center for Educational Research.

Shulman, L. (1986): Those who understand: Knowledge Growth in Teaching. Educational Researcher, February, 4-14.

DESCRIBING GEOMETRIC DIAGRAMS AS A STIMULUS FOR GROUP DISCUSSION

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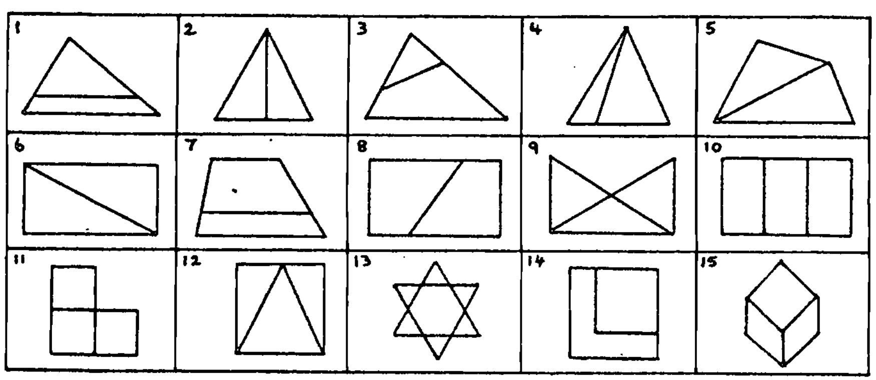
Groups of Secondary school pupils, University students, and Teachers were asked to write short descriptions of 15 geometric diagrams. These were then used as the basis for discussion with some of the above groups. The number of misconceptions that arose from the descriptions suggests that this procedure can serve as a useful diagnostic instrument and an ideal stimulus for discussion. Two aspects in particular are reported here. First, problems associated with words having both an everyday meaning and a specific mathematical meaning. Second, the relationship between meaning and the contextual and linguistic framework used to illustrate it.

INTRODUCTION

Classes of 3rd and 4th year secondary school pupils were asked to write a short description for each diagram shown below. On completing the task they went through the paper again and tried to write an alternative description (if possible) for each diagram. It was stressed that there are no "right" answers since each person's descriptions would simply be a record of their way of seeing it.

After the initial testing three things were quickly apparent. First, the majority of pupils were unable to provide genuinely alternative descriptions. (Often the second description was simply a re-wording of the first). Second, the rich variety of language used for the descriptions



suggested that the exercise could form an ideal stimulus for discussing concepts with a class. Third, certain misconceptions could often be identified especially when comparing the same child's descriptions of different diagrams. This suggested that the exercise could also have a diagnostic use. The analysis of alternative descriptions is part of an ongoing research project. Below is a report on the other two aspects, in particular on the discussions arising from the descriptions.

MISCONCEPTIONS

There were many cases of words being used correctly in one instance but subsequently misused by the same student for a later diagram.

Consider the following examples :

Student A. Diag.2 : This shows a triangle with a perpendicular from the top to 1/2 of the base.

Diag.4 : This shows a triangle with a perpendicular from the top to 1/4 of the base.

Student B. Diag.1 & Diag.3 : One triangle is enlarged to another by a scale factor from the vertex.

Student C. Diag.1 & Diag.3 : This shows a triangle on top of a trapezium

In all the above cases the description quoted for the first diagram is perfectly accurate and could lead the reader to assume an adequate understanding of the terms being employed. However, some of the later misuses illustrate the problems associated with words which have both everyday meanings and mathematical meanings (e.g. Similar and Enlargement) especially when the everyday usage subsumes the mathematical. The word similar was in fact the most commonly misused (mathematically speaking) in all the descriptions and this is discussed further at a later stage.

RANGE OF DESCRIPTIONS

The richness of vocabulary used by the pupils and the possibility of exploiting this in subsequent discussion cannot be fully illustrated within the limits of this article. However, a flavour can be given by

taking just one of the diagrams as an example and quoting some of the range of descriptions given for that particular case. These have been chosen not because they always include a mistake but because they each contain a point of interest which could be developed in discussion. The following are all descriptions of Diagram 1 :

i) It is a triangle and a parallelogram.
ii) It is a triangle with 3 sharp edges and 2 parallel lines.
iii) Two triangles with one same angle.
iv) 2 triangles with same base and height of different measurement.
v) It has a double adjacent.
vi) A picture of hills
vii) 2 triangles with one common vertex.
viii) To find the area of the shape must add the area of the triangle to the area of the trapezium.
ix) A triangle is been enlarged with scale 1.
x) It is a triangle with a vertical line in it.
xi) It is a triangle with one side on the ground.
xii) From a 3-D view, a triangle-based pyramid with the top part horizontally cut off.
xiii) The ratio of the big triangle to the small one is 2:1 .
xiv) If you cut according to the line you will find that the cross-section is still a triangle.
xv) Altogether there are 2 triangles. It has no rotational symmetry.

Some brief observations:

Non-mathematical descriptions such as (vi) were fairly unusual although Diagram 9 often elicited an "envelope" description.

Apart from the obvious example of Diagram 15, other diagrams were sometimes interpreted as 3-dimensional, as in (xii) here. In particular, Diagram 4 was quite often seen as a pyramid.

Example (viii) illustrates a description in terms of operational instructions. Diagram 14 also produced such descriptions.

It was not unusual for a negative statement to be introduced (e.g. xv). invariably related to the most recent topic studied by the class.

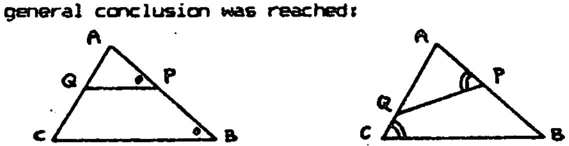
DISCUSSING THE DESCRIPTIONS

Because so many interesting points seemed to be emerging from the descriptions I decided to look at the test with groups of B.Ed. students and with teachers on In-service courses. This was done in two ways. With some groups we considered the descriptions given by school pupils and discussed aspects arising from these. With other groups they themselves wrote descriptions and these were then discussed together. In all cases

the authors of the descriptions being discussed remained anonymous unless they chose to identify themselves. I want now to consider a few of the interactions in more detail. The first concerns the following description by a Secondary teacher for Diagram 3.

"This shows two similar triangles, one enclosed inside the other"

This was put to the In-service group concerned for comments. The initial reaction was an immediate rejection with suggestions that the writer should look at Diagram 1 for a clear example of similarity. The question was then raised whether it was possible that the two triangles were similar (assuming that the angles might not be drawn accurately). This focussed attention on the angles themselves rather than the positions of the triangles. Some members of the group continued to assert that it was not possible while others now tried to draw examples with particular angles. This quickly led to the assertion that there were indeed some cases at least where it was possible. After further discussion the following general conclusion was reached:



'Angle A is common to both triangles. We need to copy angles B and C in the interior triangle. If we take a point P on AB we can copy either angle B (giving the parallel case of Diagram 1) or angle C (giving the alternative case of Diagram 3). This is always possible (provided P is chosen so that PQ intersects AC)'

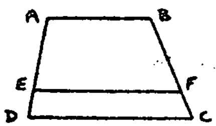
The group were now asked to consider this pupil description of Diagram 7 :

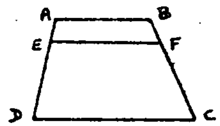
"Here are two similar trapeziums"

Now the initial reaction was the opposite of the previous example, with the whole group agreeing that the statement was accurate. Asked to justify this, a typical response was "It's just like Diagram 1". In order to seek clarification I asked for the similar trapezia to be identified. Using the

notation below (7a) they were described as ABFE and ABCD. (The analogy with Diagram 1 is implicit in these referants).

7a





7b

They were then asked to consider the case shown in 7b. There was general agreement that the two trapezia previously referred to no longer "looked" similar but some of the teachers clearly felt a little uneasy about this and still wanted to use the analogy with Diagram 1 as a justification. (One or two suggested that now ABCD and EFCD were similar). It was only at this point that the ratios of corresponding sides was raised by members of the group, eventually leading to the realisation that since AB remained fixed it was impossible for the trapezia to be similar.

The importance of these discussions was that it led the group to investigate the concept of similarity for <u>themselves</u> (some of the teachers using Geostrip as a concrete aid). In particular they were able to clarify their understanding of the necessary and sufficient conditions for similarity in the special case of triangles and the more general case of polygons and other shapes. At a later stage, another description was presented for discussion :

Diag.14 : "2 similar squares, one in the top corner of the other"

Now it was quickly asserted that the description was not wrong but that the word similar, as a qualifier, was redundant since <u>all</u> squares are similar. I am sure that this kind of insight would not have occurred without the discussion raised by the previous descriptions.

Turning to a different example, I want to discuss one particular description that led to wider considerations of language and concepts.

Diagram 9 : "This shows 3 equal triangles"

One group of Primary school teachers was very evenly split between those claiming that the description was accurate and those saying it was false.

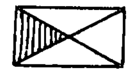
Two typical observations were:

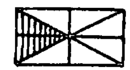
i) "The 2 opposite triangles are the same but the bottom one is different so they are not all equal"

ii) "If you complete the diagram then the rectangle is cut into quarters so they must be equal"

Following up the latter case, the question "What fraction of this rectangle is shaded?" (see 9a below) was now considered.

9a



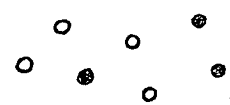


9b

A few of the "Unequal" group were now unsure since they recalled seeing such a diagram in a maths textbook. I now drew in 2 extra lines (see 9b) and again asked what fraction was shaded. At this point there was nervous laughter and confusion from some of the group since there was general agreement that the fraction was 2/8 and therefore must be 1/4 after all.

This led to a great deal more discussion about fractions and the meaning of "Equal". I want to consider now the main points that arose. A fundamental aspect involved in the concept of fractions is that of equal parts. But what precisely does the "equal" refer to? In fact it has different meanings in different contexts but this is rarely made explicit. In the "Parts of a Whole" paradigm, use is often made of geometric diagrams and shading, as in the example above. Many of the examples given to children show a shape divided into congruent parts but the implicit meaning of "equal" is related to areas. Thus each of the following shows 1/4 of a rectangle shaded:

However, when we turn to the "Subset" paradigm we have a different meaning implied. Again, the early examples will often show congruent pieces as in (A) below. But in what sense are we referring to equal parts in (B) ?



(A) What fraction of the counters are white?

(B) What fraction of the group are women?

The relative sizes of the individuals in (B) are irrelevant now because the implied meaning is numerical equality. Thus we are saying there are 5 discrete entities, 2 of them are women, so the fraction is 2/5. But unless the meaning of the word "equal" is made clear in the different contexts in which it is used, it is not surprising that many children have difficulty coming to a full understanding of fractions. Consider the following problem: A ragged corner is broken off a bar of chocolate. What fraction of the bar was broken off ? Using the "Parts of a Whole" model, what is involved here is precisely the meaning and interpretation of "equal parts". Moreover, it is possible to construct a concrete model to produce two different answers to a problem. Consider a simple jigsaw puzzle which has been painted in 3 different colours as shown. If we ask the question "What fraction of the puzzle is red?" then we are back to the chocolate bar problem and must solve it in terms of area (or mass).

However, suppose the puzzle is taken apart and the pieces spread out. If we now ask "What fraction of the pieces are red?" we are using the Subset paradigm and the answer is 3/9 (or 1/3).

Is it the case that whenever we use models to illustrate a concept then the meaning is relative to the model used ? I believe it is. And in the above situation the structure of the language used is highly significant. Taken as abstract nouns it seems obvious that "one-fifth" is singular and "two-fifths" is plural. However, consider the following sentences:

i) One-fifth of the rectangle is shaded. ii) Two-fifths of the rectangle is shaded. iii) One-fifth of Secondary school Principals are women. iv) Two-fifths of Secondary school Principals are women.

The verb used in each case is a precise indicator of the implied model.

To take a historical example, the Elements of Euclid begins with a set of axioms which are described as self-evident truths. One of these is the following: "If equals be added to equals, the wholes are equal". But when

applied to a specific situation the truth of the statement depends on what "equals" refers to. Suppose we consider equal volumes. If we add equal volumes of salt and sand to equal volumes of water it is <u>not</u> true that the resulting volumes are equal. The meaning is dependent on the context.

SUMMARY

There are two main aspects to the work described above. The first is that meaning always resides in the context. In one sense this is obvious and needs no comment. For example, the descriptions quoted earlier of shapes being similar are only wrong when interpreted in a mathematical context. Words like Similar and Enlarge are part of everyday usage but have a wider field of application than the mathematical. The key considerations are the intentions of the speaker and the interpretations of the listener. In most fields of discourse these are taken for granted but particular care needs to be taken by the mathematics teacher with words such as those discussed above. But perhaps more importantly, I have argued that there are subtle differences of meaning within mathematics especially when different models are used to illustrate a concept. It is in these situations that the teacher needs to be even more aware so that such differences can be made explicit to the pupil. The second aspect is that of the discussions themselves. The descriptions were simply used as a stimulus, the usual method being to read out a description and ask for comments, agreement, disagreement etc. Of course, some descriptions were more fruitful than others. Also there were occasions when a particular description generated real discussion with one group but elicited very little from another. The most successful occasions were when there was strong disagreement within the group to begin with. But I firmly believe that all the students and teachers involved in this work benefited enormously from it. They were often forced to re-examine their own understanding of concepts and also had to articulate their arguments to their peers, and this was especially valuable for teachers whose mother tongue is not English.

PUPILS' PERCEPTIONS OF ASSESSMENT CRITERIA IN AN INNOVATIVE MATHEMATICS PROJECT

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Abstract

As part of the system of examinations at aged 16 in the UK, a group of teachers has devised a scheme based entirely on assessing the work carried out by the pupils during the mathematics course, rather than by a written final examination. In this paper we aim to examine the pupils' perceptions of their assessment, and especially how far it relates to the teachers' intentions and their explicit criteria for assessment. We have interviewed groups of pupils from each of the project schools (along with their teachers), and from their responses attempted to draw out themes which are significant to them. Hence we hope to form a picture of how the pupils construe the work they do in this mathematics course.

The project and its assessment.

In 1988 a new system of assessment at age 16, known as the General Certificate of Secondary Education (GCSE), was introduced into schools in England and Wales. This system is based on a mixture of teacher assessment of coursework and end-of course written examinations. It is administered by five examining groups who are each responsible for several mathematics schemes. A partnership was formed between the Association of Teachers of Mathematics (ATM) and the Southern Examining Group (SEG) to implement a proposal by a group of experienced secondary mathematics teachers for an alternative form of mathematics assessment based *solely* on coursework. As this group says, (ATM GCSE Group, 1990) '[we] met and developed a syllabus which would enhance and encourage active learning.' Thus, although the project was set up within the new system of *assessment*, the teachers' purpose was to devise a *curriculum* which accorded with their views of learning mathematics. The proposal was run as a pilot project in a limited number of schools.

All GCSE assessment schemes must satisfy national criteria (Secondary Examinations Council (SEC), 1985a,b), including the requirements for coursework. In general, 'coursework' means work carried out in normal lessons, or at home, under the supervision of the teacher and assessed by the teacher. The *National Criteria for Mathematics* simply stated that 'coursework may take a variety of forms including practical and investigational work; tasks should be appropriate to candidate's individual levels of ability.' In many schemes coursework was seen as the means for assessing strategic aspects of mathematics, with the written examination assessing mathematical content. Such process-based coursework varied from project work in which pupils had a large degree of choice and control over the questions worked on, the ways of working and of presenting that work to well-defined and highly structured tasks done under controlled conditions.

For the project group coursework was not a separate component, either of what was worked on or of assessment; but rather, the whole of what was done in, and as a result of, mathematics lessons. The entire range of phenomena in the mathematics classroom was thus potentially available for assessment. The project teachers gave extended tasks to the pupils who typically worked on them for 2 to 3 weeks. These tasks were based in mathematical content areas, but had a strong emphasis on pupils exploring the situation, formulating problems, and deciding the lines of approach for themselves. One teacher in the group described her practice:

> I introduce a variety of ways of learning. Computers, calculators, textbooks, worksheets, practical situations, mental imagery and open ended situations all have their place. With this syllabus I can challenge pupils constantly, but in return I can value everything they do because any piece of work can contribute to a GCSE grade. (ATM GCSE, 1990)

At the end of each task the pupil was expected to hand in a written account of their work which would not simply be a list of results or their working, but a report indicating the choices they had made, with their reasons for these and any conclusions reached. Their work is assessed through this written report, but also as it is carried out.

> An extremely important feature of the way that we work with students is the quality of the discussions that take place with students as the work progresses from starting with a problem to the final written up piece of work. Once completed, the teacher will write a comment back to the student in order to indicate what is good about the work and where appropriate point out aspects of the work where improvements might have been made. (ATM GCSE 1990)

The group of teachers identified seven domains on which the work would be assessed: a *Mathematical knowledge* domain, which was largely determined externally by the National Criteria, and 6 others: *Communication, Implementation, Interpretation, Evaluation, Mathematical attitude, and Autonomy*. There were no established referents for these other domains and little, if any, formal precedent on which to base their use.

Our enquiries

Although expressed informally, the notions of active learning and constant and varied challenge seem consistent with a constructivist view of knowledge on the part of the initiating group of teachers. Furthermore the emphasis on verbal feedback, both spoken and written, can be in accordance with a view of the social construction of knowledge through classroom discourse. The teachers believed that we were sympathetic to such views and we were invited to evaluate the project from the perspective of the participating schools and teachers. The main questions we faced were:

- Is the project enabling the schools to work in the way intended?
- Could the project be extended to involve other schools and groups of teachers?

investigating these questions we found that we needed to examine how the intentions of the initiating teachers were translated into forms required by both SEG and SEC. and how both intentions and formal requirements were shared with other teachers in the participating schools, who were not directly involved in the steering group. A natural corollary of the enquiry was to ask whether the teachers considered it appropriate or important to share these intentions with the pupils taking the course and to explore the extent to which pupils were aware of the assessment criteria. One assessment domain, that of 'evaluation', requires pupils to assess their own work - not to grade it, but to discuss how far it has achieved their intentions:

> the candidate's awareness of the progress of his/her own work is assessed. This could be part of an ongoing process throughout a piece of work or occur as a reflective activity at the end. Both of these affect what the candidate does in the future.(SEG 1987)

One of the bases pupils have for evaluating the quality of their work is the way in which it will be assessed by the teacher. One possible interpretation is that the existence of this domain implies that the assessment criteria and their domain structure should be shared with pupils. The pupils perceptions of these assessment categories provide the focus for the present paper.

Our approach

Much work has been done in recent years on the social construction of knowledge through classroom discourse (Edwards and Mercer 1987, Walkerdine 1988). Such studies have been based on analysis of observed classroom interactions and assumed or asserted teacher intentions of providing learning experiences through which children acquire specific subject knowledge and implicitly meta-knowledge.

Our evidence is gained through semi-structured interviews with teachers and pupils. We have focused on the testimony of pupils from particular classes within those schools. Thus our evidence is of what has been construed rather than of the process of construing. We have also taken 'classroom discourse' to include the provision of written feedback by the teachers although this may be produced outside the classroom.

Furthermore we are looking at an area of 'knowledge' which has finite expression and currency. There is a more or less closed system consisting of those members of the examining group and project teachers who form the steering group and the teachers and pupils who are participating in the project. Members of the steering group had the initial responsibility for drawing up the verbal definitions which reflected their personal and perceived common intentions whilst meeting the National Criteria. Teachers in the schools then had the task of operationalising these definitions by working out possible meanings in practice. There were three main sources of feedback for them in this task:

- their colleagues in the same school with whom they worked to produce the classroom

  tasks and to cross-mark pupils' written work;
- teachers from the other project schools with whom similar meetings were held;
- their pupils' actions and responses as they worked on the tasks, through which teachers discovered the effectiveness of the criteria in shaping and describing what was happening.

Teachers at each stage in the chain of communication were likely to be confronted by the need to make explicit their beliefs about mathematics teaching. Studies have uncovered marked mismatches in the beliefs and practices of mathematics teachers (Thompson,1984) and it is likely that such mismatches were present in this group. Through processes of repeated mutual adjustment individual construals were modified to permit the cooperative work at the heart of the project.

A consequence of the limited currency of the project is that pupils in the project schools can *only* have constructed knowledge of the criteria through classroom discourse. However the details of those constructions will depend largely on how their mathematics teachers have shared the domains with them, though mediated by existing individual perceptions of related ideas to do with mathematics, teaching and learning and assessment.

The pupils' perceptions.

In what follows we have confined our selection of pupil responses to comments by pupils from two of the schools. These pupils were interviewed in groups of four. Their two teachers, Mr R in School A and Miss S in School B, were both fully involved in implementing the project from the outset, but neither has been on the Steering Group.

Awareness of the domains.

In both schools the teachers are explicit with their pupils about the domains, but there is considerable difference in the procedures they use to make them explicit. At School A each piece of work is assessed on a sheet which lists the domains; the teacher carries out the assessment with the pupils as the work is being done. His method is to work with the domain categories explicitly.

> I'm writing on [the assessment sheet] as I'm discussing with the pupils. That's just a reminder to the kids, or even a reminder to me, where they're at, how they got there, what they will doinext. I will sometimes home in on a particular domain ... [The pupil] may get lost so I will only perhaps consider 'interpretation' for a particular piece of work, for that particular kid. I'll try, if I can, to home in on the particular [domain] assessment that I want either to improve or the kid has done particularly well in.

The domains are thus made explicit to the pupils as the means of assessment. At School B the assessment is done at the end of the course in quite a different way:

> We just look at the [total collection] as a whole and assess it from there rather than assessing each individual piece of work. ... We think of some projects that would probably emphasise communication or interpretation or autonomy,

The pupils, then, do not have the same direct discussion of domains in connection with each piece of work. The domains are made explicit by other means:

> We keep emphasising it; we have posters up [with the domains listed in informal language]. We give them questions to look at under each domain so that they know a couple of things they are doing.

The reaction of the pupils to this apparent lack of information is ambivalent. One says

> I would have preferred it if I'd had some kind of set list and they'd told me how to go about it. Those sheets we've recently been given out with evaluation, communication and stuff like that, I think actually I would have preferred it earlier on.

Another echoes this,

> Well Miss S has been through it a few times but she doesn't explain it that much but she's told us what we have to do, what we are aiming to get down on paper, but apart from that, we haven't been sufficiently told and how to, and taught the skills of that certain section.

However, another pupil:

> I don't see how [knowing] would have helped that much, because some of the things, you just have a list there, you're just writing it all down so you can just remember it. I mean, you'd be looking at them and you'd be thinking how can I put this into my project. It would be really false, it wouldn't flow like it normally does.

The pupils attention is drawn to the domains, but they are still relatively unaware of the specific domains.

Understanding and using the domains

There are difficulties for both teachers and pupils in understanding the domains. Teacher R comments on his attempts to help pupils.

> I will sometimes home in on a particular [domain]. I have difficulties at the beginning of the 4th year because of course they haven't got the material for understanding what the domains mean. If you try to explain that to a 15 year old ... well, it's difficult enough with professional colleagues. As they get further into the 5th year they're getting more and more understanding of the domains, so you can be more specific as to why that was an E instead of a D.

There are different problems for the various domains. For many the least problematic was *Communication*.

> I think you get [Communication] in the project anyway - you've got to explain what you're doing. You explain from this experiment you're doing, you say how you get this answer from your set of results and things, ... so you have to say that anyway.
>
> I think if you all worked in a class then you're all doing the same thing, at the same place, the only one that could really change out of this category is the communication. It's going to be different every time you write it down.

The pupils identified *Communication* with the written report they were required to hand in, in spite of the explicit inclusion of oral communication in the form of discussions withe teacher

Evaluation was more difficult. as teacher S remarked:

> I find [evaluation] impossible almost - I find it very difficult to evaluate my own work, never mind getting the children to evaluate what they're doing, especially when they're in ? grades. Some of them might say what they do as they go along, whereas if they're a grade A they've got to reflect on a piece of work as a whole.

The *autonomy* domain, which was used to assess the degree to which the pupils' work was their own, or communally produced, needed intimate knowledge of the class to operate Teacher S comments that the grade would be

> very much the teacher's decision. Unless you've actually taught that pupil earlier on, and you know what kind of pupil they are and how they behave. So we do try to make notes on the work - 'This is your own idea', or 'you worked with so-and-so to produce this'

This was the domain of which the pupils were the least aware. One pupil echoed others in asserting: "Autonomy. I don't really know what that means."

Although the domains are made more explicit in School A, there was little difference in the perceptions of pupils from the two schools about how useful the domains were when working mathematically. For many the way to produce a 'good' piece of work is to work at the mathematics and ignore the domains.

> I just explain what I'm doing, write down what I've found out. Then when he marks it if I haven't included that, I haven't included it. I just do what I think I've got to do and that's it.
>
> When you're doing it, you want just to do the maths, but if you [used the domains] you'd be worrying so much about how you should get, say, autonomy into it. And then 'communication', how shall I communicate that I want to do this

How far do they try to use the categories? The following exchange between a group of pupils at school B is revealing of their perceptions of the *Evaluation* domain.

> A: The evaluation bit at the end, evaluating the project, I don't understand that at all.
>
> B: I manage the evaluation bit. I put it in my conclusions.

A: I haven't. I wasn't aware of that at all.

C: But I suppose in your conclusions you include the evaluation bit. You say what I thought I could have done better.

A: Well, of course you can say something like "If I had more time, I'd have gone on and extended this in certain place ... "

The general laughter that followed this indicated the pupils' sceptical awareness of the routinisation of producing work for assessment. The didactic tension (Brousseau,1981) has been used to describe a problem for the teacher when pupils produce behaviour rather than learning; this exchange indicates that pupils can also be aware of the gap between what will formally satisfy the teacher's needs and what is really intended. They could make evaluative sounding statements, knowing that they would not be required to substantiate them.

One pupil at School B, D, was helped by his general orientation to cope with the uncertainty of the nature of domains. He had a lot to say about communication and presentation which was revealing about his perceptions of mathematics and his strong evaluative streak.

> ... when I first started out I just thought that maths was boring ... it wasn't stimulating. It was just [things] like numbers, angles. ... It was just like "If x=y, what do you do, blah, blah ..."
>
> I just take things very light-hearted. Miss S said I was very self-critical; and I'll be critical when we do conclusions. I'll have a go at the system of maths: what its like, what I feel like. If I find something really silly, which is going on, instead of getting on with it, I'll say 'This is not on. It's really stupid. Why?' Because if I can do things like this, it'll stimulate me to do things better, and often there are things that I'll actually want to do. ...to do because I can draw and write and things. But, still, to make it fall in line with maths. It's not just drawing pretty pictures, its part of maths.

So how did D. know what to do?

> I've been very lucky because I've always done my [coursework] like essays and things. Which is why I wasn't really caught out [by the domains being revealed] as much as some of the other people. So I got away with that pretty easy. But at the same time its very easy to do that if you're not told, and I probably would have done it just like that. Its the way I've been doing it which has saved me from a fate worse than death.

Issues

There is a tension between making explicit to pupils what is required of them and eliciting that behaviour through providing appropriate learning contexts. Described by Love 1988 and OU 1989. There seemed to be a difference between the groups of pupils in the two schools in the extent to which they believed that the assessment criteria had been made explicit to them. Pupils in school A thought they were well aware of what the criteria were, while those in School B had recently been surprised by their existence. Both teachers claimed to have

used ways to share them though in School A a process of continual grading operated while in School B grades were not given until the final year. It is of interest that although knowing the criteria, School A pupils claimed not to make use of them as they worked. However it is possible to use knowledge to obey quite complex sets of rules of behaviour and communication (Edwards and Mercer) without being able to articulate those rules and indeed without even being aware of those rules for much of the time. As in all human interaction, failure to 'obey' undeclared rules is much more obvious than compliance. Hypotheses to acount for this might be either that the pupils in School A were obeying the rules without explicit awareness or that they were simply more relaxed about the agenda, having no suspicion that any part of it was hidden from them. They seemed in general much more confident of their own power to improve the grades they had been awarded. However D. at School B remains a notable exception to many of these observations. D. had an intuitive sense of communication and had spotted a golden opportunity to exercise some of his skills in this area.

References

ATM GCSE Group, 1990, *No right or wrong*, The Times Educational Supplement, 4-5-90.

Brousseau,G, 1981,Problemes de didactique des decimaux, *Recherches en didactique des mathematiques*, 2.1, pp 37-125..

Edwards D.,and Mercer N., 1987, *Common Knowledge : the development of understanding in the classroom*, Methuen, London.

Love E. 1988, 'Evaluating Mathematical Activity', in Pimm D. *Mathematics, Teachers and Children*, Hodder and Stoughton, London.

Mason J.,1989, 'What to do when you are stuck', ME234 - *Using Mathematical Thinking*, Unit 3, Open University.

Secondary Examinations Council 1985a, *General Criteria, GCSE*, HMSO, London

Secondary Examinations Council 1985b, *National Criteria for Mathematics*, HMSO, London

SEG 1987*Coursework Guidance*,

Thompson, A.B. 1984, 'The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice', *Educational Studies in Mathematics*, 15, pp.105-127.

Underhill, R.G. and Jaworski, B,1991,*Constructivism and mathematics education: a discussion*, Centre for Mathematics Education, Open University, Milton Keynes..

Walkerdine V.,1988 *The Mastery of Reason*, Routledge, London.

DEVELOPING A MAP OF CHILDREN'S CONCEPTIONS OF ANGLE

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This paper describes the design of a study aimed at developing a map of children's conceptions of angle by analysing their responses in a variety of situations and under different conditions. A selection of activities is presented together with some of the interesting early findings.

If geometry is viewed as the act of grasping the spacc in which children live, breathe and move (Freudenthal, 1973) then it is reasonable to hypothesize that children's understanding of one geometric concept, namely angle, arises at least in part from their own experience obtained through interaction with the environment. By adopting this essentially constructivist position, we agree with the Piagetian point of view which suggests that knowledge is neither subject nor object — rather it is in the interaction between them. For Piaget, objects can only be known by a series of successive approximations constructed by the subject through various activities (Piaget, 1971). Piaget claims that the knowledge arises from logical operations which are constructed in the course of cognitive growth, and he defends the position that cognitive growth is represented by structures, mental and biological, which are developed through four factors: maturation, the physical environment, the social environment, and equilibration. He asserts that the social environment influences development, but his focus of attention is upon development rather than on the social medium. Thus, for Piagetian theory, fragmentation of knowledge means either horizontal or vertical decalage.

In the present study, we maintain a Piagetian approach but give attention to the social environment, and the possibility of fragmentation of knowledge, following the work of Lawler (1985) and Lave (1988). In particular, we wish to explore the influence on response of the *semantic situation* — a place that has meaning for the child (Carraher, T.N., personal communication). Thus, combining a contextualised approach with a constructivist position we intend to investigate three questions:

1) From the vantage point of cognitive development, how does a child acquire the concept of angle?
2) How does a child make sense of angle within different settings?
3) How do settings structure a child's understanding of angle?

The aim of the study is to develop a map of children's (aged 6 to 15 years) conceptions of angle by analysing the ways they deal with angles presented in different situations and under different conditions.

The Study

The universe of study is composed of five nested sets: perspectives, contexts, settings, arenas, and activities.

Perspective refers to the two ways by which an angle can be understood — the static and the dynamic. We use the definitions found in Close (1982): For the *static perspective;* the angle is the portion of a plane included between two straight lines in thc plane which meet in a point. For the *dynamic perspective;* the angle is the quantity or amount (or measure) of the rotation necessary to bring one of its sides from its own position to that of the other side without moving out of the plane containing both sides.

Drawing mainly from the work of Lave (1988), we see the *arena* as a place which is physically organised in space and time, the *setting* as a personally ordered view of the arena and the *context* as the relationship between arena and setting. In this study, the dynamic perspective is instantiated in two different contexts — Navigation and Rotation — and the static perspective in one context — Comparison. We have chosen three settings: everyday[1], paper and pencil, and Logo. Within each setting we consider four arenas: for the everyday setting; mini city, turnstile, watch, and stick game; and for the paper & pencil and Logo settings; map, spirals, arrow, and open figures of angles. Finally specific *activities* are designed within each arena. We have set up activities which are matched in terms of angles and the requirements of the task in order that comparisons can be made of the pupil responses; for example navigation in the mini city and navigation around a map in Logo; angles presented on watches and the same angles drawn on paper & pencil; arrows rotated in Logo and similar angle turns in paper & pencil. To summarise, *perspective* is the starting point of the study and *activities* are the end point, and each *activity* and its relationship with other activities within and between arenas, settings and contexts will have importance for the construction of the map of children's conception of angle.

Having described the situations to be explored, we distinguish three conditions to be investigated: *recognition*, *action*, and *articulation*. These three clusters of conditions relate to three different moments in thinking about angle: to perceive similarities and differences between figures using angle as an invariant (*recognition*); to build figures considering their angles (*action*) (this has two aspects, construction and prediction); and to clarify if and how angle has been taken into account (*articulation*) — this again has two aspects, description and explanation. Table 1 summarises the elements of the universe of study with reference to the three conditions.

1 The everyday setting derives from a particular culture, the northeast of Brazil and is instantiated by simulating an everyday life situation rather than acting within the everyday situation itself.

Perspective	DYNAMIC						STATIC		
Context	NAVIGATION			ROTATION			COMPARISON		
Setting / Condition	Everyday	P & P	Logo	Everyday	P & P	Logo	Everyday	P & P	Logo
RECOGNITION	Mini City	Map	Map	Mini City Watch	Spirals Arrow	Spirals Arrow Watch	Mini City Watch Stick Game	Angles	Angles
ACTION	Mini City	Map	Map	Watch Mini City Turn stile	Arrow	Watch Arrow	Mini City Stick Game	Angles	Angles
ARTICULATION	Mini City	Map	Map	Watch Mini City Turn stile	Spirals Arrow	Spirals Arrow	Mini City Watch Stick Game	Angles	Angles

Table 1: The Distribution of arenas according to setting, context, perspective and conditions of the study.

We now briefly describe each arena and give an example of an associated activity or activities.

Mini City is a model of an everyday setting and consists of a section of a schematic map of a city drawn on a large card. Miniature cars can be "driven" around the city and two routes A and B are marked. Mini City is used in all contexts. One activity is as follows:

Recognition: Did you turn 90^0 at any point along route A? Where?
Did you turn 90^0 at any point along route B? Where?
Name the largest turn you made along route A?
Articulation: Explain your answer
Recognition: Name the largest turn you made along route B?
Articulation: Explain your answer
Recognition: Comparing your answer to the largest turn along route A with your answer to the largest turn along route B, which of them is largest?
Articulation: Why?

The *Map* arena is similar to Mini City but is drawn on a sheet of paper (paper & pencil setting) and on the computer screen in Logo. The sequence of questions is exactly the same as that for Mini City.

The *Watch* arena consists of four circular watches, in two different sizes each made of card. There are no numbers on the watches in order to discourage children from focussing on the times that the watches are showing. All the watches will however have a reference mark at the 12 o'clock position. One activity involves the comparison of a large and small watch summarised as follows:

Recognition: Who has worked the longer if both A and B start work at 12.0 but A finishes at 12.45 (time shown on the small watch) and B finishes at 12.25 (time shown on large watch)? (See Figure 1).

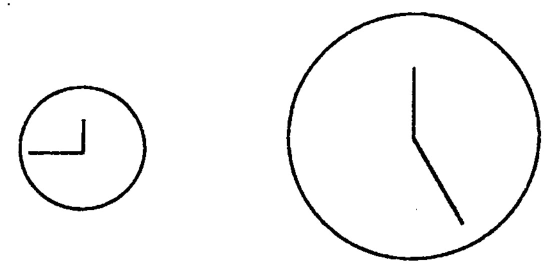

Figure 1: Comparison of angles in the Watch Arena

The *Spirals* arena consists of three pairs of spirals (circular and square) drawn on paper and in Logo. In each setting the child is asked to look at the spirals in pairs and compare the angle through which each spiral has turned as illustrated in Figure 2.

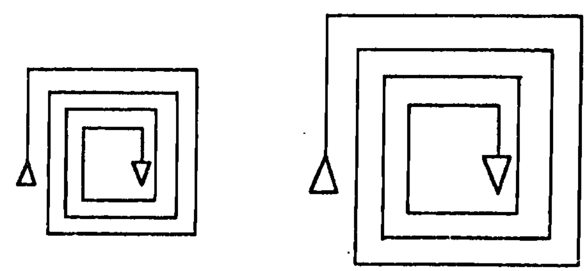

Figure 2: Comparison of Spirals

The *Turnstile* consists of a miniature turnstile made in strong wire material, and placed at the entrance of a miniature zoo. An arrow made of coloured card marks the starting point of any turn. One activity is:

Action: If the turnstile has turned through one and half complete turns, how many people can go into the zoo?
Articulation: Why?

The *Arrow* arena consists of a 3 cm. arrow fixed at its base and drawn on a 7 cm^2 square (paper & pencil setting) or on the computer screen in the Logo setting. One activity is:

Action: Draw the position of the arrow after it has turned 90^0 clockwise — that is, a quarter turn to the right.
Articulation: Explain why you came to your answer?

The *Stick Game* is played throughout Brazil, particularly in the Northeastern area. It is most popular among children between 6-12 years old. Children play it in pairs and the perfect place to play it is in the back yard of the house. Each child has a tiny stick which they throw into the sand and then draw a straight line joining the original and final stick positions. The game finishes when one child's lines surround the lines of the other child. As a model of this game, we use a geoboard - fifteen rows of fifteen nails fixed on a square board — and the lines are made by using an elastic band.

Two activities are illustrated in Figure 3 below:

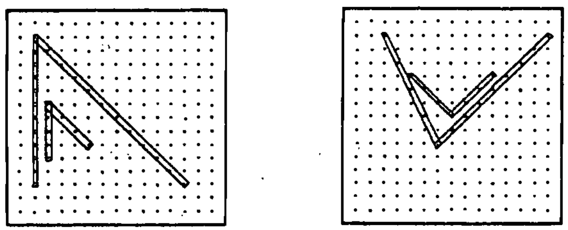

Figure 3: Comparison of Angles in the Stick Game

Recognition: Are these angles the same?
Articulation: Why?

Within the *Angles* arena either two or four angles are compared. In one activity, a pair of angles is drawn on paper or in Logo and the child has to decide if the angles are the same (or not) and why. The angles presented are the same as those of the Stick Game.

Another activity involves the presentation of four different angles on paper and in Logo and the child must decide which of them is the biggest (or smallest) angle and why. The angles used are the same as those in the Watch arena.

One activity is illustrated is Figure 4 below.

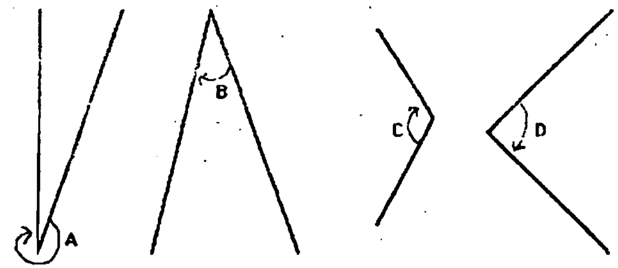

Figure 4: Comparison of Four Angles

Recognition: Compare the size of angles A, B, C and D. Which of them is the smallest (largest)?
Articulation: Explain why you came to your answer?

The Pilot Study
For pragmatic reasons, the pilot study was undertaken in two parts: the everyday setting in Brazil with 32 children (18 from public school, 14 from private school); the other two settings, Logo and paper & pencil, in England with 6-11-years-old children of differing attainment levels. The aim of the pilot study was to test out the activities (with a view to modification) rather than provide representative responses. Thus, although all the tasks were tested out,they were not applied as a set for each child as is the intention of the main study; the children were submitted to a maximum of two settings. Nevertheless, we discuss some findings which we consider worthy of attention.

The Influence of Setting
We give two examples which serve to indicate the influence of setting on pupil response:
1) When children were navigating a car in the Mini City or the turtle around the Map, the most difficult turn for them to recognise, implement and predict was *less than* 90°. This contrasted with their responses in paper & pencil which were more in line with previous research (for example Close, 1982); that is that children find acute angles comparatively easy but have greater problems in recognising angles greater than 180°. We suggest an interpretation based upon 'figure and ground'. In a paper & pencil setting which has little semantic sense it is hard to perceive an angle bigger than 180° because it becomes the *background of* the smaller angle. In arenas such as Mini City and the Map, the *turn in the path* is the focus of attention. For the main study we intend to explore this phenomenon and include turns of <90°, between 90° and 180°, and >90° in both arenas.
2) Our results suggested that children performed better in the Logo setting, than the paper & pencil setting for all the activities. Two interpretations can

given to this; first, the Logo activities happened after those with paper & pencil, so children could have learnt something more about turn prior to their Logo work; second, the children developed their ideas of angle *during* their interactions with Logo that is, *while they were doing the tasks*. This will be investigated further in the main study.

The Influence of Presentation

The recognition of similar angles in different orientations with rays of the same size was *not* a problem for most of the children after 4th grade in all three settings. When the size of the rays varied, this activity became the most difficult for English children whether in paper & pencil or Logo settings; while Brazilian children, after 3rd grade, did not exhibit the same difficulty. There were however differences in the way in which this situation was presented to the children which may be significant: for the English children, the angles were presented one *beside* the other (in both paper & pencil and Logo settings), while for Brazilian children they were presented one *inside* the other. These differences in presentation will be further investigated.

Strategies

Amongst English children, the most common way to make a comparison between a pair of angles was by looking at the "openness" of the angle at the end of its sides. This was similarly the case within the Brazilian sample. Also of interest here was that although our Brazilian sample had 16 children who were studying at high school level, only three children mentioned angles explicitly in the activities presented in the everyday settings.

The Main Study

The main study will be carried out in Northeastern Brazil with 60 children, aged 6-15 years. The sample will be divided into 10 groups of six children, each group refering to a specific age and grade. Half the sample within each group will come from private school, and the other half from public school. No child will have had any previous Logo experience and all the children therefore will receive a training Logo session before undertaking the research activities. All the children will be submitted, individually, to the whole set of activities over a number sessions. The first clusters of activities will be those included in the Logo setting, the second the everyday setting, and the third the paper & pencil setting. There will be an interval of two weeks between the administration of each cluster of activities. Data collection is to be carried out during Jan-April 1991 and preliminary findings will be presented at the conference.

References

Close, G. *"Children's understanding of Angle at the Primary/Secondary Transfer Stage"*, M.Sc. Thesis, Polytechnic of South Bank, London, 1982

Freudenthal, H. *"Mathematics as an Educational Task"*, D. Reidel Publishing Co. Holland, 1973.

Lave, J., *"Cognition in Practice"*, Cambridge University Press, 1988.

Lawler, R. *"Computer Experience and Cognitive Development"*, Ellis Warwood Ed. 1985.

Piaget, J. *"Biology and Knowledge"*, Edinburgh University Press, 1971

THE CONSTRUCTION OF MATHEMATICAL KNOWLEDGE BY INDIVIDUAL CHILDREN WORKING IN GROUPS

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A problem task was given to a class of second-grade children and again five-months later when they were beginning third graders. Detailed analysis of the written work and videotape transcripts of six of the children provide a description of how their representations are built and modified over time. It provides insight into the power of the students' own mental representations and the logic of their thought processes. The videotape segments provide insight into the process by which the children first built their representations and how, in cooperative group problem-solving, children's initial representations are influenced by the representations of others. In this report we consider two children, Stephanie and Dana.

According to Davis (1984), thinking about a mathematical situation involves cycling through a series of steps. First, it involves building a representation of the input data. From this *data* representation, memory searches are carried out to construct a representation of relevant knowledge that can be used in solving or trying to solve the problem. A mapping between the *data* representation and the *knowledge* representation is constructed, checked out, and if satisfactory, associated with the knowledge representation to solve the problem. Moving from a data representation to a knowledge representation that might be useful may not be automatic. As the learner attempts to map the representations, checks are made along the way and other knowledge may be entered. In the process, some representations are rejected and/or modified. When the constructions appear satisfactory, other techniques associated with the knowledge representation may be applied to carry out the solution to the problem. Examples illustrating how the "data representation" of the problem statement are gradually constructed by individual students are given in Davis and Maher (1990). In this paper, we provide a description and analysis of some of the processes by which young children gradually build up their initial representations of a problem situation as they think about it from the perspective of the Davis model. In particular, it describes how the original

ideas of two children are modified and refined as they are influenced by each other's thinking.

Method

As a part of the regular mathematics activity, the following problem was administered in two sessions, Spring 1990 to second-graders, and again in Fall 1990 to the same class (now third graders).

> **Stephen has a white shirt, a blue shirt and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?**

In the context of considering six children over both sessions of the problem administration, limitations in space require us to focus on the problem solving behavior of two children, Stephanie and Dana, who were in the same group for both sessions. During the first problem-administration, the second graders worked in groups of three to build a solution. Dana, Stephanie and Michael comprised one group. During the second problem administration, the children (who were now third graders), worked in pairs, with Dana and Stephanie as one of the pairs. Each class activity was organized to encourage individuals and groups of children to work at their own pace and without teacher intervention.

Results

<u>Grade Two Problem Solving - May 30, 1990</u>: The children begin to build a representation by focusing on pieces of data that deal with the problem situation. As relevant information, they focus on type of clothing and color.

D: I'm just gonna draw a shirt...that's all we have to do...and then put like. [*Dana draws shirts*]
S: I'm gonna make a shirt...and put white...wait..."W" for white. [*She draws a white shirt*].

Their ideas are further refined as they enter information about the number of items of clothing.

S: Ok, blue and then a yellow shirt. [*Stephanie draws blue and yellow shirts*] He has a pair of blue jeans and a pair of white jeans. [*She draws two pictures of jeans.*]
How many different outfits can he make? Well.. [*Dana looks at Stephanie's paper and draws blue and white jeans.*]

Notice how the ideas become further refined as they decide that *differences* in the kinds of outfits become relevant. Stephanie cycles back at this point and rereads the problem checking that the input data representation is consistent with her knowledge representation.

S: Well, no [*reading*] how many different outfits ...he can make a lot of different outfits. Look, he can make white and white...
D: He can make all three of these shirts with that outfit. [*Dana points to her three shirts, Stephanie interrupts her.*]

At this point Dana clearly has the key idea but might not yet be fully aware of how to use it. Stephanie used her diagram to develop a coding strategy to form her combinations:

S: I'm gonna make a shirt [*Stephanie begins to draw.*] and put white...wait...**W** for white. [*She then labels the shirt by writing* **W** *inside its traced outline.*]

Stephanie then illustrated each distinct outfit with a pair of letters, the first for the shirt and the second for the jeans. She recorded each outfit and kept track of them by numbering each combination. As she was recording the first outfit, she turned to Dana and said:

S: You can make it different ways too. You can make white and white, that's one...W and W [*She draws a* 1 *and* W *over* W.] Two could be blue..blue jeans and a white shirt...blue,W. [*She draws* B *over* W]

Dana found a notation that enabled her to make use of her original idea and spontaneously drew lines that connected each of her W and B shirts to each of her B and W jeans and her Y shirt to her B jeans, concluding that there were five different outfits (See Figure 1).

D: Yeah, well just put white with blue [*Dana draws* 5 *connecting lines between the rows of shirts and pants.*]

Stephanie continued to list pairs of letters for her outfits and concluded, also, that there were five combinations in total.

S: Ssshhh...Ok, yellow shirt...number three can be a yellow shirt. [*She draws* 3 *and* Y *over* W.]
D: It can't...yellow can't go with the white.

Dana's drawing indicates that she didn't draw a line between the yellow shirt and white jeans. The entering data may be constrained by her views about the way one makes outfits. Notice that Stephanie points out to her that the outfit doesn't have to match.

S: No...how many outfits can it make?...
M: I'm doing white shirt and white pants, blue pants, blue shirt.
S: It doesn't matter if it doesn't match as long as it can make outfits. It doesn't have to go with each other Dana.

Dana taps with her pencil counting five connecting lines.

D: Four outfits it can be...
S: It can be more if you put them mixed up. Just watch. I'm on my third one right here, number four. It could be a blue shirt and a blue pants. [*She draws 4 and B over B*] Number five can be...a white shirt...[*She draws 5 and W*]... Wait..[*She erases the W*] OK. .a blue shirt...Wait!...Did I do blue and white?

Dana never went back to include in her diagram the yellow shirt and white jean combination even after Stephanie commented on taste. On the other hand, Stephanie's diagram does not include the white shirt and blue pants outfit. For her fifth entry, Stephanie's worksheet shows a W between the letters Y and B used to record the YB outfit (See Figure 2). Analysis of the videotape indicates that Stephanie first wrote the W, erased it, and then wrote a Y over the B. Dana then recorded Stephanie's solution on her paper beneath the line diagram.

S: Two is blue shirt and white pants...a blue shirt and yel...wait... a yellow shirt...did I do yellow and white? A yellow shirt and blue pants. [*Stephanie draws Y over B.*]
D: A yellow shirt and blue pants.
S: Do you know what? There's five combinations...there's only five combinations. Cause look you can do a white shirt with white pants..

A consequence of the group sharing portion of the lesson was that no singular strategy was adopted for use by the children and no agreement evolved as to the correct number of outfits. Some of the answers reported by different groups were 3, 5, 6, and 7 outfits.

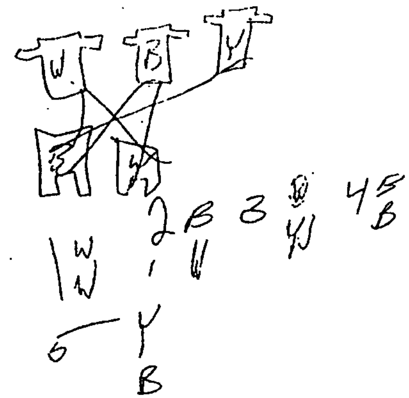

Figure 1 - Dana's 1st Drawing

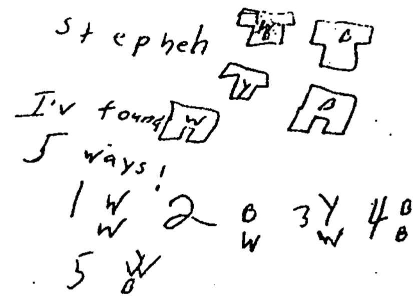

Figure 2 - Stephanie's 1st Drawing

Generally, the children were pleased to share their representation of the problem solution and talk about how it was derived. Although numerous paths of solution were shared, the apparent result was that no one solution was widely accepted by the students. The next opportunity to explore this problem came in the month of October when the children entered third grade.

<u>Grade Three Problem Solving: October 11, 1990.</u> During the five months between the first and second administration of the problems, no class discussion occurred about the problem or about the strategies that were employed. The shirts and pants problem was again presented to the students and Dana and Stephanie worked together on the problem. The following dialogue shows Dana and Stephanie beginning to build a representation for the input data. Dana begins by reading the problem aloud and Stephanie responds by suggesting that they begin by drawing a picture:

D: Stephen has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

S: We..why don't we draw a picture?

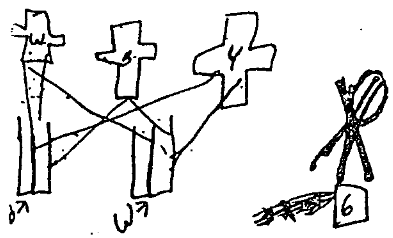

Figure 3 - Dana's 2nd Drawing

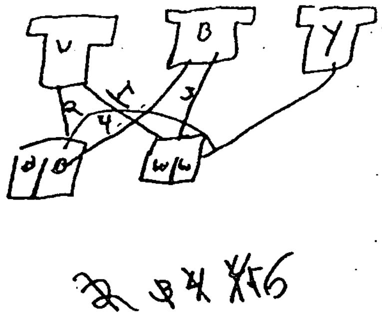

Figure 4 - Stephanie's 2nd Drawing

As Dana and Stephanie drew their pictures (See Figures 3 and 4), we see them focusing on the pieces of data that deal with numbers of shirts and pants and their colors. In so doing, they search for a way to map their knowledge representation to the data representation of the problem:

D: OK..He had a white shirt (the girls draw pictures of shirts)
S: So, I'll make a white shirt. [*Notice the mapping of the that Stephanie is checking her representation as she draws her picture to match the problem data.*]
D: A blue shirt...
S: I think I'll have to use the big marker for this one..you know, color it in blue [*Stephanie makes explicit reference to shading the shirt blue to match the problem data.*]
D: And a yellow shirt [*The girls draw another shirt.*]

At this point Stephanie suggests that the data be coded by assigning the first letter of the color rather coloring the piece of clothing. Notice that she does not write what she says:

S: Why don't we just draw a Y, a B and a Y [sic] instead of coloring it in?
D: That's what I'm doing.
S: W, B, Y [*Stephanie places a letter in each diagram of the shirt to represent its color.*] OK, he has.
D: A blue.

Stephanie rereads the problem as she works to build a representation of the problem.

Dana, monitoring Stephanie's thinking, completes the construction:

D: And a pair of white jeans.

Stephanie indicates that she has begun to construct a representatic of the relevant knowledge and proceeds to attack the problem:

S: All right. Let's find out how many different outfits you can make. Well, you can make white and white so that would be one. I'm just going to draw a line.

Later, Dana and Stephanie were asked by the instructor why they used connecting lines. Stephanie replied:

S: So we could make sure that we were...so that we didn't do that again [*referring to repeating a combination*] and say that was 7,8,9,10 [*referring, again, to repeated combinations*] we drew lines...so then we could count our lines and say, "Oh, we can't do that again!" or so we could know if we already matched that...so we don't go... "Oh, OK, that's two [*drawing the lines with her finger on the desk top*], that's four...and we'd get more than we were supposed to.

Stephanie, in her justification of the use of the line strategy, indicates a shift from working with the representation of the problem data to working instead with a representation of the *process* by which she solved the problem. Her reflection on that process shows a development to another level of awareness, a shift to a meta level. Whereas at first her pictures represented direct translations of problem data (pictures and colors of shirts and pants), she now invents notation to monitor her own behavior. Notice that her diagram indicates a number label attached to her connecting lines which enables her to keep track of each combination (See figure 4). In this session, neither girl used a coded listing strategy; they simply drew the three shirts and two pairs of pants, connected shirts to pants with lines and counted their lines. Stephanie explains her preference for the use of lines because they indicate the number of combinations.

Final Comments

The dialogue between the two girls apparently made them more aware of the question of "what goes with what" to make an outfit. In fact, this difference (tasteful combinations you may actually be willing to wear versus all possible combinations) may have led to a heightened awareness of the possibilities of seeing combinations instead of seeing shirts and seeing pants. At this point Dana indicates the central idea that each shirt can go with one pair of pants. While she has the key to the solution, she might not as yet be aware of it. In fact, in this case her procedure may have preceded the idea. It's interesting to note that the language the girls use shifts from talking about what they are doing to planning ahead what they will do next. Each girl is building her own representation in her own mind. As comments from her partner impinge on this, some notice is taken of them but they are not immediately and directly included. Eventually, the ideas of others are integrated into their own schemas, but not always immediately. Note, for example, Stephanie's later use of Dana's device of connecting lines. Here we see Piaget's assimilation and accommodation at work.We also see meta analysis at work in the final session when Stephanie shifts from a concern with shirts and pants to a concern with effective methods of keeping track, a process of monitoring one's own behavior. It is clear from observations of these two problem-solving sessions that as Stephanie and Dana repeatedly cycle through their constructions, the input data change and the representations become more powerful.

References

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* New Jersey: Ablex Publishing.

Davis, R. B. & Maher, C.A. (1990). What do we do when we learn mathematics? In R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics* (53-69). Reston, Va.: NCTM.

THE TABLE AS A WORKING TOOL FOR PUPILS AND AS A MEANS FOR MONITORING THEIR THOUGHT PROCESSES IN PROBLEMS INVOLVING THE TRANSFER OF ALGORITHMS TO THE COMPUTER *

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This research stems from a project designed to introduce 11-14 year old pupils to computer studies in interaction with mathematics. In this context the "table" is used as a bridge between the two phases of the process: the construction and/or analysis of an algorithm and its transposition to the computer, by means of a mode of didactic management which enables the pupil to make increasing personalised use of the table depending on the requirements of the task in hand. This research has highlighted how such didactic management when using the "table" to solve algorithmic problems enables two important objectives to be reached: the acquisition by the pupil of an aid for his intellectual work, and for the researcher the collection of significant information on the thought processes of the pupil when engaged in this type of activity.

1. INTRODUCTION

The table is a commonly-used working instrument in various areas of mathematics: it is useful in sinthesising and visualising information to be processed (statistical research, logical and/or arithmetical problems), in carrying out comparisons (e.g. between real and simulated data in mathematical models) in examining the succession of values taken on by a set of variables as an algorithm is being run. For this last application, in particular, the table proves itself to be an instrument of mediation of thought processes that enables the values taken on by the variables to be coordinated step-by-step, to examine the transition from one set of values to another, to highlight (by following the progress of the algorithm) any regularities and either to confirm them or disprove them. This last aspect led us to consider the table as a suitable means of assisting the pupils when approaching the task of transferring an algorithm to the computer. With this type of problem one of the major difficulties encountered by the pupil, once a procedure has been established, is that of seeing the problem in two different lights: they have to try to imagine the step-by-step workings of the machine (internal collocation) and they have to visualise the procedure in its entirety, understand its mode of operation, forsee the results and correct any deficiencies (external collocation). The problem, then, was to adopt an approach that enabled these thinking processes to be "defined", an approach that was in some way connected to the *representation and processing systems* (Hoc and Nguyen-Xuan, 1989) which have already been grasped by the pupils. Consequently we endeavoured to construct and try out a succession of teaching situations whose aim was to get the pupils to make a progressively more autonomous use of the table: firstly as an instrument for analysing a procedure, subsequently for designing procedures and lastly for reflecting upon one's own thought processes and those of others. The intention of this approach was to verify the potential of an appropriate use of the table as an instrument in relation to the learning difficulties mentioned earlier.

The research, having been initiated with this aim in mind, subsequently brought to light other possibilities for such an instrument. The table, as well as being a useful tool for the pupil, proved to be a useful means of gaining an understanding of their behaviour and the difficulties they experience when confronting this type of problem. Analysis of the protocols has shown a

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...le variety of individual elaboration of the table depending on the particular approach adopted and has, furthermore, on examination of the graphics used in connection with these adaptations, provided some interesting indications on the thought processes followed by various pupils.

2. CONTEXT

The research stems from a project designed to introduce 11 - 14 year old pupils to computer studies in interaction with mathematics by making use of certain problem-situations, each of which enables the pupil to use the computer in a mathematical laboratory situation without having prior knowledge of any programming language - a situation which would otherwise not have proved possible or, if possible, not incisive. At the same time the pupil begins to get some idea of what a structured programming language is, of the kind of responses the computer can provide and in which way to behave with the computer (Pellegrino, 1989). From a mathematical point of view, the tasks set are easy to understand and solvable with relatively simple, though not banal, algorithms; they are set in such a way as to stress that the choice of procedure used is entirely up to the person solving the problem and such that the quantity or the order of magnitude of data involved is not a secondary consideration in the choice of algorithm to be transferred to the computer.

The information which forms the basis of this research project was obtained from two parallel classes of 20 and 24 pupils, respectively, taken by the same teacher from the first to the third year of the comprehensive school and took up on average 20 hours of class time out of a total of 120 dedicated to mathematics. The type of work carried out was based on a "didactic contract" (Brousseau, 1986 p.65) which stressed - to an ever-increasing degree throughout the three years and in relation to the pupils' development - the externalisation of their reasoning processes, the use of language (verbal/written) as an instrument in developing awareness of their thoughts and interaction with others in the group.

The sequence of situations relating to the introduction to and the use of tables was as follows:

- *Odometer*, where the problems of changes in base of numbering systems were approached by analysing the workings of the Odometer and its simulation in LOGO, while at the same time experimenting a "naïve" approach to recursivity;
- *Russian multiplication* involving the analysis of the algorithm for multiplying natural numbers with the method once used by Russian farmers, the algorithm was transferred to the computer and its inverse algorithm was identified (which turned out to be the so-called Egyptian division);
- *Sum of the first hundred natural numbers*, where three different methods of finding the solution were taken into consideration, depending on the method used: pencil and paper, pocket calculator and computer.

As regards the teaching method employed, which in certain aspects varied considerably, there were some "constant" factors:

- preparation of the situation, carried out with particular attention in order to gain the pupils' attention, achieved by reading stories, with the use of well-known characters or every-day events;
- group discussions carried out to compare and reflect on the results obtained by the pupils and/or to review, expand on and consolidate the knowledge built up.
- the teacher's strategy was that of interpreter between the pupils and the computer by giving a translation of the proposed procedure as "literally" as possible in the programming language, so that the pupils could identify the important aspects of the phrase and compare them with their normal language;
- use of tables. For the *Odometer* episode the table was provided by the teacher for the operational analysis of the procedure, after it had been tried on the computer; with the *Russian multiplication* the teacher suggested to the students that they make use of a table to design the procedure to be transferred to the computer; with the *Sum of the first hundred natural numbers* they were spontaneously used by the pupils when considering which method to use to solve the problem with the pocket calculator.


```
PER SUCCESSIVO :NUM
                                    SE :NUM = "9 RIPORTA "0
SE ULT :NUM = 0  RIPORTA PAROLA MENULT :NUM 1
SE ULT :NUM = 1  RIPORTA PAROLA MENULT :NUM 2
SE ULT :NUM = 2  RIPORTA PAROLA MENULT :NUM 3
SE ULT :NUM = 3  RIPORTA PAROLA MENULT :NUM 4
SE ULT :NUM = 4  RIPORTA PAROLA MENULT :NUM 5
SE ULT :NUM = 5  RIPORTA PAROLA MENULT :NUM 6
SE ULT :NUM = 6  RIPORTA PAROLA MENULT :NUM 7
SE ULT :NUM = 7  RIPORTA PAROLA MENULT :NUM 8
SE ULT :NUM = 8  RIPORTA PAROLA MENULT :NUM 9
SE ULT :NUM = 9  RIPORTA PAROLA SUCCESSIVO MENULT :NUM 0
FINE
```

Fig. 1

3. PUPILS' PROTOCOLS IN RELATION TO THE USE OF TABLES

3.1 Tables as an instrument for analysing procedures.

The following is concerned with the first part of Odometer - described in its entirety in Pellegrino and Garuti (1989) - where the naïve approach to the subject of recursivity was handled with great caution in order to avoid, where possible, the known teaching/learning difficulties associated with it (see, for example, Rouchier, 1987, or Rogalsky and Samurçay, 1989). At this juncture, after having examined the mechanism by which the odometer progresses from one number to the next, the pupils were made to describe this progression without resorting to the use of counting or addition in order to obtain an algorithm with a recursive procedure that was easily transferred to the computer. After the procedure had been written, in LOGO (Fig. 1), and tested on the computer, the pupils were made to examine its mode of operation more closely in order to reveal its recursive nature and to understand its hidden operating mechanism. In order to facilitate the pupils' analysis of the procedure the teacher, once the procedure had been completed, represented the steps taken by the computer to arrive at the next number in a table (denominated "how it works") - see Fig.2 - and asked the pupils to write a description of these steps.

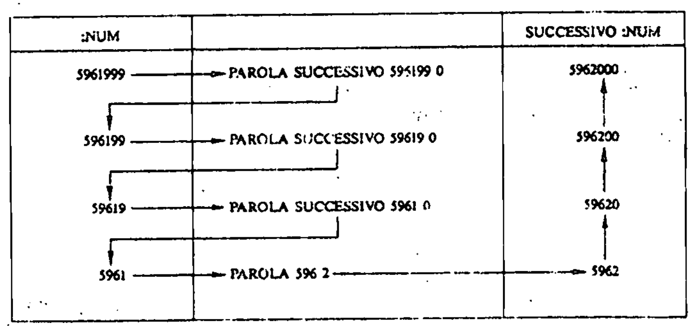


Fig. 2

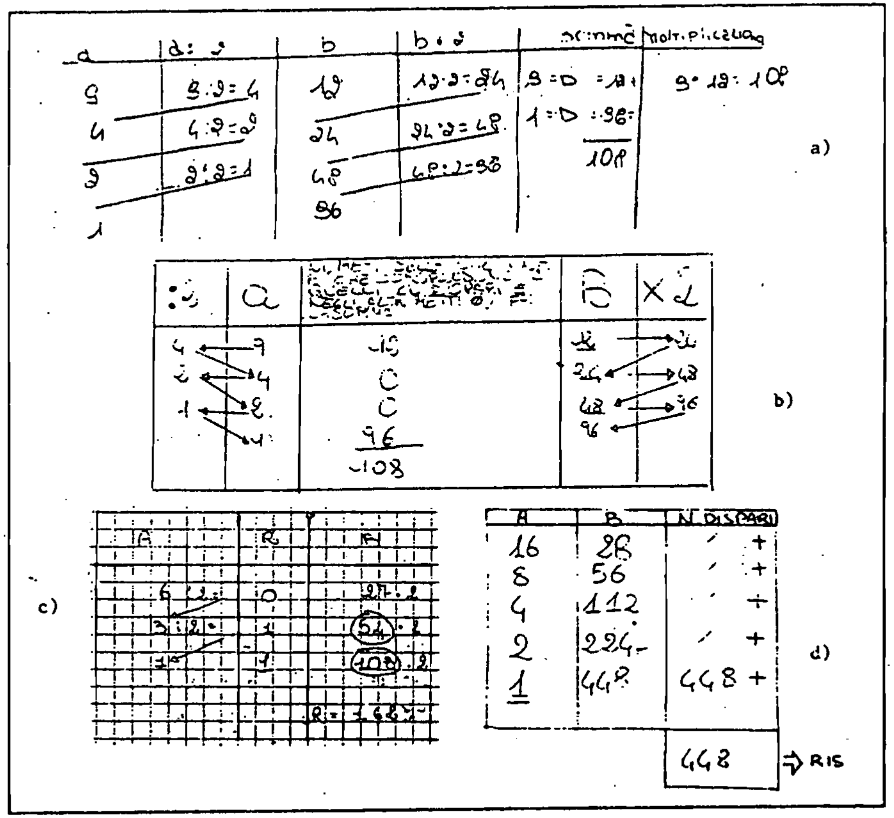
The advantage of the use of the table in this situation, as has already been said, is that it allows the problem to be seen from two different points of view at the same time: on the one hand, since it sets out the various steps taken by the computer, it induces one to identify with the computer and so perform the selfsame steps; on the other hand, as the table "freezes" the processes followed by the computer, it allows the procedure to be seen in its entirety. The

pupils' protocols reflect this double view even though they are mostly concerned with describing the step-by-step procedure. Many, however, make the observation that *the computer always works in the same way, but with ever smaller numbers* [fewer digits] *until it reaches a number less than nine*. Moreover, for many, the way the computer keeps the steps it has to take in its memory remains a mystery. Erika writes: *...there is one thing that strikes me though: I wonder how the computer can remember all the steps it takes, and their successions, without writing anything down.* This confirms one of the cognitive difficulties identified by Rogalsky (1989): *programming activity operates on a physical machine which may not be transparent in its functioning for learners.* The use of the table caused some of the pupils notice and remark on the difference, in this situation, between the mode of operation of the computer, which repeats the checking sequence every time, and that of a human, who checks the last number and acts accordingly without needing to carry out any further checks. From the point of view of computer literacy, these are very significant observations as they make the student aware that checks which we take for granted are not automatically carried out by the computer. It is to be noted that on this occasion the pupils' responses were limited by their linguistic abilities: the task of describing the way the procedure works, even though simulated with the table, was by no means a simple one. When the students elaborates their own tables, a clearer, and more varied, indications was obtained of their thought processes.

3.2 Tables as an instrument for designing procedures

In the case of ***Russian multiplication***, the function of the table was inverted: not as an instrument for analysing previously-formulated procedures but as an instrument for designing procedures still to be devised. The work was initiated with a description with examples of the method Russian farmers used to calculate the product of two numbers. As is well known, the product is found as follows: the first number is divided by two and its whole quotient is written beneath it, the second number is doubled and written beneath it; this procedure is repeated until the first number, continually halved, becomes 1; at this point the required product is given by the sum of the numbers in the second column that correspond to the odd numbers in the first column. After carrying out a number examples to confirm the method's effectiveness, the pupils were asked to analyse the method in order to "teach" it to the computer and to try to find out why it works. It was suggested that they describe, with an example and using a table, the steps which the computer ought to follow to find the product of two numbers using this method. No further indications were given. The pupils proceeded to use the table as a working tool to create a model of an algorithm with the operating logic of the computer, or rather, what they took to be its operating logic. They started with the knowledge of an algorithm that could or could not be suitable for the computer and had to invent a way of making the algorithm compatible given the inherent constraints of the computer. Analysis of the pupils' protocols, which illustrate each pupil's personalized approach to the use of the table, highlights both the personal requirements of the pupils (as can be seen in the different ways the tables were organised) and their objective difficulties (introduction of variables, checks to be made) in transferring this algorithm to the computer. In order to demostrate the mental processes followed by the pupils Fig.3 contains a few of their protocols showing prototype tables they produced. In protocols a) and b) the halving and doubling procedures are indicated to show how the initial values of the two numbers are progressively modified. This indication is achieved by leaving an appropriate space and with the use of a symbol, an arrow,as a general indicator of the operations (note that the pupils were not accustomed to using arrows as operators). The arrow is an efficient way of emphasizing the progressive change in values of the variables, which somehow animates the intrinsic static nature of the tables, and in particular reveals the table to be a personal instrument of the pupil. As regards this point, for example, in protocol a) the arrow is used to indicate that the result of each operation be taken as the value of a certain variable and is therefore in fact used to indicate a transformation; in protocol b), on the other hand, it is used both in this sense and in

Fig. 3



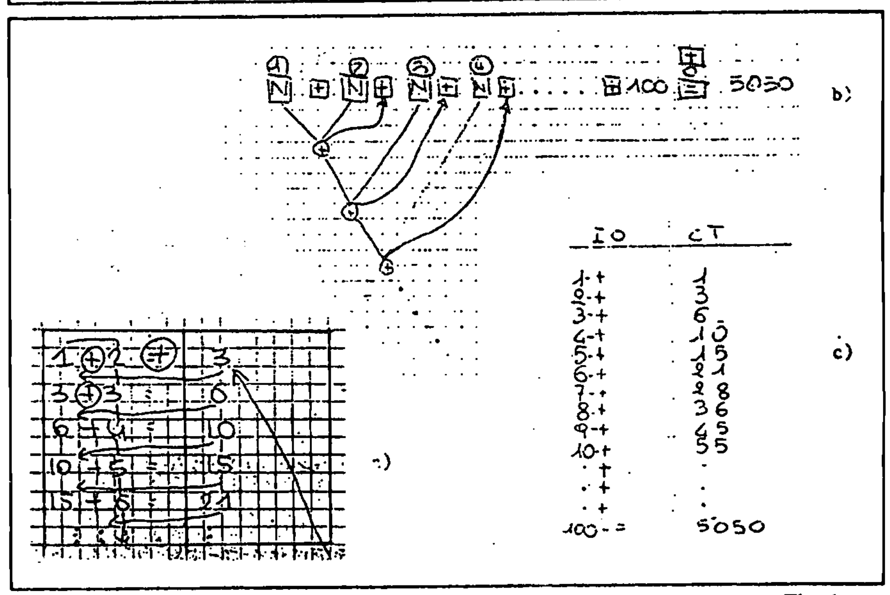


Fig. 4

that of operator (it is interesting to note the vertical symmetry of this table). Evidently the pupils have made use of the arrows to visualise the procedure in order to clarify the steps taken.

Unlike the others, protocol c) contains a column of remainders. This pupil and few others considered the need to tell the computer when a number is even or odd, but above all they attempted to find the means of doing so that was suitable for the computer. Normally to distinguish between an odd or an even number one checks the parity of the last digit, this is, however, a complicated procedure to transfer to the computer. These pupils realised as much and overcame the difficulty by using the remainder function, thereby enabling them to insert the check in the table, or more precisely, to insert the result of the check. Many others were not able to circumvent this obstacle and it is here that the differences between human modes of thought and those of the computer are evident and where it becomes clear - as pointed out by Ershov (1988, p.55) - that having to communicate with the computer forces one to think about the concepts involved and brings about an appropriation of those concepts. Incidentally, the use of the column of remainders, whose sequence corresponds in base 2 to the first of the factors taken into consideration, enabled the students to understand the mathematical key of the algorithm.

Protocol d) is representative of many pupils' reactions to another obstacle encountered in transferring this algorithm to the computer: how to tell the computer to perform the sum of the selected values to obtain the product. Once again we see the difference in procedure when using pencil and paper, where firstly the halving and doubling operations are carried out and secondly the sum of the selected values is carried out, and the procedure with the computer, where it is convenient to perform these operations simultaneously. Nearly all pupils, even though they included in their tables a column of values to be summed, did not realise that in this case not only was it necessary to control a number of different variables simultaneously but it was also necessary to introduce a variable of accumulation. In fact, as shown by Samurçay (1985, p.146) this is one of the difficulties inherent in this type of problem. Only one pupil in her written comments about her table mentioned this aspect of the problem, pointing out the necessity to tell the computer the way in which it should sum these values. She is also the only pupil in making those observations who expressed her thoughts from a point of view which described the manner in which she proceeded when devising her table. This seems to confirm the well-known difficulties of attempting the solution to a problem while at the same time contemplating the procedures one is using (see, for example, Becker 1990 p.112-113). In order to encourage self-observation and the externalisation of thought processes in the pupils, use was made of algorithmic problems that were not excessively complex and which were suited to this purpose; this enabled the pupils to give good metacognitive performances.

3.3 Tables as an instrument for reflection.

In the case of the *Sum of the first hundred natural numbers* - a problem which was by no means beyond the pupils' capabilities - the intention was to get the pupils to consider the methods of solution according to the instrument used to perform it, as well as to reflect on their approach to each of the methods; the teaching situation was expressly one of research and comparison: the point of the exercise was not so much the solution to the problem but the manner in which the solution was arrived at. We shall concentrate here on the activities that took place in relation to obtaining the solution with the pocket calculator (PC). For this section the pupils spontaneously made use of tables to record their thoughts about the procedure adopted for the solution, adding clarifying notes on the meaning and functions of the symbols used. Figure 4 shows some of these tables. The most interesting aspect of these are the adaptations that each pupil makes to the instrument "table" depending on what the pupil wishes to highlight and the depth of thougth achieved.

Table a) in Fig. 4 is similar to those devised in the case described in the previous paragraph, the steps of the algorithm and the expression of the values of the variables involved are clearly set out. It reflects sufficiently well the author's thinking on the procedure involved in operating the algorithm with the PC. The pupil is aware of the necessity of linking the sequence of natural numbers perhaps in order to distinguish them from the sequence of running totals. This distinction is also noted in the comments written, *the sum of the first two numbers gives 3, to this is added 3 which is the number that follows 1 and 2* (this specification probably arises from frequent errors made in selecting the desired number on the the PC). Table b) in Fig.4 contains extensive modifications: it is in fact a graph (instrument which had not been introduced to these classes). The author, as is explained in the accompanying notes, makes use of two different symbols to distinguish between the functions of the + key from what appears after having pressed this key: ⊕ *represents the result obtained each time I press the* ⊞ *key*. He feels it necessary to point out that the same key has differing functions and writes: *+ is the same as = after adding two numbers* (anyone who uses a PC soon realises this). Another interesting aspect is the attempt at generalisation: he uses the letter N to represent the general term in the sequence of numbers, and the symbol ⊕ to represent the general term in the sequence of running totals. Furthermore, the sequence [N]+[N]+[N]+... can be seen as a representation of the algorithm for the PC, and in this context [N] represents the command INPUT N.

In both tables, the concept of addition as a binary operation is explicit, a concept which is not evident when using a PC but which came to light as a result of thinking about the problem. Furthermore, the arrows used in both these tables give the sequence of operations a sense of time and motion and express the connection between the succession of natural numbers and the accumulation of the running totals. Realising this distinction is a significant result, as specified by Capponi and Balacheff (1989, p.199) in writing about the same problem, resolved however, with the electronic spread-sheet, *the greatest difficulty is in creating the relationship between these two groups of numbers, in particular the re-use of the preceding accumulator and the relationship between the content of the accumulator and the whole number generator.*

Protocol c) in Fig.4, on the other hand, expresses a different point of view adopted by some of the pupils: to determine the difference between the actions of the PC and the actions of the operator. Adopting this point of view, however, precludes the use of tables as an instrument for reflecting upon the analysis of the algorithm.

Having reviewed the classes' protocols, the teacher suggested that the pupils should evaluate these three tables in order that they might compare them with their own and also that they should try to understand which aspects each pupil had wished to emphasize. In this way the table becomes an instrument for reflecting upon others' actions and, by comparison, upon their own. This enabled the pupils to effect an interpretation by assigning meaning to the symbols used, to identify similarities in meaning in different symbols and differences in meaning with the same symbol, and so to arrive at a first experience of how, in man-machine interactions, *it is necessary to keep a conscious control of both the semantics and the syntax of the symbols used* (Martinelli-Boero-Garuti, 1990).

Finally, as well as using the table for the purpose described, it was employed as a design tool and the activities carried out concerning the PC constituted the base from which the pupils undertook the costruction of the procedure for the solution to the problem using the computer.

4. CONCLUSION

In the light of our experiences, the most relevant aspects of which we have analysed, we feel able to assert that the table is a valid tool in semplifying typical difficulties associated with working with the computer. It enables the pupils to identify themselves more easily with the system with which they have to communicate. It should be noted, however, that the table in itself is not significant but that the effectiveness of the instrument depends on the overall situation in which it is used. Encouraging an autonomous and personalised use of the table ensures, in our opinion, it is productive. These personal adaptations also enable one to see how the pupil reacts when confronting the difficulties in transferring algorithms to the computer. It is well known that in this type of problem it is necessary to formulate (or re-formulate) the

algorithm in such a way as to take into account how or in which sequence the computer will be able to carry out the steps; it follows that, when devising the procedure, it is necessary to consider both the timing and the appropriate memory allocations associated with these actions. As we have seen, the use of the table in designing algorithms or for examining them enables, by way of the graphic lay-out used (the insertion of relevant spaces for checks, the use of appropriate symbols to indicate operations or the temporal sequence of actions) information to be obtained regarding the thought processes followed by the pupils. This aspect highlights the potential of this instrument in a new field of research: an anlysis of *mental space-time models* created by the pupils to help solve complex problems (see Chiappini - Lemut - Martinelli, 1990, p.114).

Before concluding, we wish to point out that in each of three cases described the use of tables was always in conjunction with the use of the spoken word. During the course of the three years the relationship between these two instruments became increasingly close and inter-dependent: The pupils progressed gradually from describing, if not always in a relevant manner, the analysis of algorithms using the table, to describing the methods of designing a table and finally to expressing their thoughts on their actions and that of others. The table thus becomes an instrument that complements and supports the spoken word in metacognitive activities.

In conclusion, our research would seem to show how a specific form of didactic management in the use of tables for solving algorithmic problems effectively enables two important and differing objectives to be reached: the acquisition by the pupil of an aid for his intellectual work and the appropriation of significant information on thought processes during algorithmic problem-solving activity of a certain complexity.

REFERENCES

Becker G., 1990, Self-control in analyzing problem solving strategies, *Proceedings PME XIV*, Mexico,vol.1, 111-118.

Brousseau G., 1986, Fondaments et Méthodes de la Didactique des Mathématiques, *Recherches en Didactique des Mathemathiques*, vol.7, n.2,. 33-115.

Capponi B., Balacheff N., 1989, Tableur et calcul algebrique, *Educational Studies in Mathematics*, 20, 179-210.

Chiappini G., Lemut E., Martinelli A.M.,1990, On the problem solving with the computer: analysis of difficulties and requirements, in McDouglas A. and Dowling C. (ed.), *Computers in Education*, Elsevier Science Pubblishers B.V. (North-Holland).409-414

Ershov A., 1988, Computerization of Schools and Mathematical Education, *Proceedings ICME VI*, Ann & Keith Hirst (eds), J. Bolyai Mathematical Society, Budapest 49-65.

Hoc J.M., Nguyen-Xuan A., 1989, Language Semantics, Mental Models, and Analogy, to appear in Hoc J.M. et al., *Psycology of Programming*, Academic Press, London

Martinelli A.M., Boero P., Garuti R., 1990, The figure of the teacher as promoter and organizer of his students' metacognition, to appear in *Proceedings CIEAEM 42*, Szczyrk (Poland) July 23-30, 1990

Pellegrino C.,1989, Computer Science in the teaching of Mathematics to pupils aged 11-14, in Bazzini L. and Steiner H.G. (ed.) *Proceedings Italo-German Symposium on teaching of Mathematics*, Pavia, 367-384

Pellegrino C., Garuti R.,1989, Dall'avvio alla ricorsività ai cambiamenti di base nei sistemi di numerazione attraverso la simulazione in LOGO del contachilometri: descrizione di una esperienza realizzata in una scuola media, *La Matematica e la sua Didattica*, IV, n.1, 1990, 19-30

Rogalski J., Samurçay R., 1989, Acquisition of programming knowledge and skills, to appear in Hoc J.M. et al., *Psycology of Programming*, Academic Press, London

Rouchier A., 1987, L'ecriture et l'interprétation de procedures récursives en LOGO, *Psychologie Française*, tome 32-4, 281-285

Samurçay R., 1985, Signification et fonctionnement du concept de variable informatique chez des eleves debutants, *Educational Studies in Mathematics*, 16, 143-161

INTERRELATIONS BETWEEN DIFFERENT LEVELS OF DIDACTIC ANALYSIS ABOUT ELEMENTARY ALGEBRA

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Summary

In this paper I distinguish several levels of observation. The micro-didactic level involves a single problem-situation and deals with the observation of a specific lesson in the classroom. The meso-didactic level involves some problems about the same mathematical concept and deals with the observation of a series of lessons in the classroom as well as with the analysis of a chapter of a school-book. Finally, the macro-didactic level involves all the concepts and situations in the same conceptual field and deals with the study of academic course programme, as well as with epistemological analysis and investigation about students conceptions.

In the framework of elementary algebra, the analysis I have undertaken led me to believe that there are basic interrelations among these three points of view, which point out the current tendency to exclude any algebraic finality.[1]

This paper took place into the discussion on the teaching and learning of elementary algebra (Kaput 1987, Kieran 1989). I would start from an analysis of the functionality of solving equations in algebra, and use the result of this analysis to interpret some observations of the didactic system.

1. Elementary algebra as a tool

Most of the basic notions of elementary algebra have a particular character which is rather unusual: I mean the *tool character*. It is possible to give one or several adequate definitions of a circle, even at an elementary level, but not of the term "factorization" or of the statement "solve an equation". In fact the result of factorization or equation-solving depends on the goal which is set. Régine Douady (1985, p35) declares: "We say that a mathematical concept is a *tool* when our interest is focused on the use to which it is put in solving problems. *By object* we mean the cultural object which has a place in the body of scientific knowledge [...]. We consider as an object every mathematical notion presented by its definition, and possibly examples, counterexamples, structural description."

[1] From "The concise Oxford Dictionary": Finality: Principle of final cause viewed as operative in the universe; being final; belief that something is final; final act, state, or utterance.

If it were a question of constructing one or several didactic sequences for elementary algebra, I would therefore look for *the problems or classes of problems which give meaning to these mathematical tools*. But in the context of the *observation of a specific didactic system* (in France at the end of the 1980's) the exclusive character of elementary algebra as a *tool* poses specific problems in teaching. In fact, Régine Douady describes the current pedagogical practice as an "object/tool mechanism". In the absence of an *object* character, how can such an approach work? A tool, even if it is explicit, cannot be *defined* in mathematical terms.

Observing how didactic system tries to solve such problems necessarily draws our attention to one of the key questions in mathematical didactics, i.e. that of *finality*. A tool has meaning in a specific *context of use*, in particular *situations*. A tool raises questions of effectiveness, i.e. poses problems of *validation*.

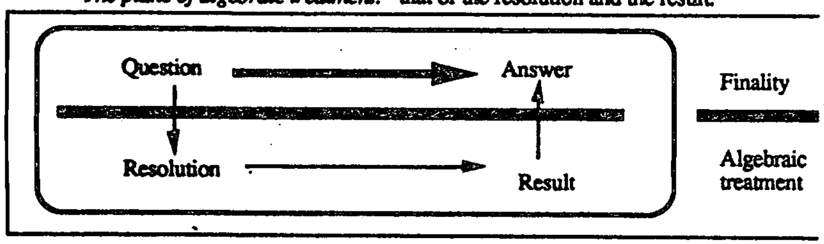
1.1. What does "solving an equation" mean?

Solving an equation means, first of all, *asking a question*. It involves getting a better grasp of all the numbers which correspond to the conditions defined by the equation. This question lead to a *resolution* that can be more or less easy according to the type of equation and to the knowledge of the person who is trying to solve it.

Trying to find an answer to the question will finalize the equation-solving activity. But this later step may involve an *algebraic treatment* (solving a literal equation, replacing the initial numerical equation by equivalent numerical equation, etc.). Here, the question may, for a certain length of time be, not forgotten, but suspended until there are enough elements to answer it. *As soon as we think we have these elements*, the *resolution* determines a *result*, which may possible to provide an *answer*. What I am here calling a *result* may qualitatively be very different from an *answer*, even if the question is purely algebraic, with no reference to a "concrete-problem".

They are therefore two planes with different meanings:

- *The plane of finality* that of the question and of the answer;
- *The plane of algebraic treatment:* that of the resolution and the result.



The difference between these two planes does not lie in the "less formal" nature of the first opposed to the second but in a *relation of subordination* from the point of view of meaning. The algebraic treatment is *finalized* in the giving of the answer, in a particular situation. The plane of finality also involves the *control process*. By control process, I

mean the anticipation of the validation. This control process is guiding the equation-solving, even through the working out of algebraic treatment.

1.2. Rapid analysis of equation-solving

Question/Answer The link between question and answer is complex. It involves, as does a great number of communicative operations, the implicit element bound to *speech situations*. What is acceptable as an answer in one situation can well not be adequate in another. Regarding the solving-equation, according to the context, the question can demand *a* solution or *the form* of the solutions, etc. An "answer" to a question is a speech-act in which the speaker attempts to *reduce the degree of uncertainty* contained in the question. The more this answer reduces the degree of uncertainty, the better it is. That is why the answer S={x [R /3x+4=7x-12} to the question *Solve in R the equation 3x+4=7x-12* is not acceptable, since it gives no information. On the other hand, S={(x;y) [R^2 / y=2x} is an adequate answer to the question *Solve in R^2 the system (y=2x and 2y=4x)* as it gives additional information about where the solutions can be found (and moreover, the best possible).

Question/Resolution To enter into a resolution implies a choice, even in elementary algebra. For example, when faced with the problem *Solve in R the equation (2x+3)/5 = 7*, someone who is used to working out algebraic problems would not find himself in the same situation if the equation were isolated, as he would if it were in a series of equations to be solved in the form (ax+b)/c=d. In the second case, he should solve the literal equation, and then directly apply the result (cd-b)/a to all subsequent equation.

Resolution/Result *What I call result is the final step in the resolution.* In other words, the result is the moment at which we think we have enough elements to provide an answer.

Result/Answer The most important distinction in this scheme is that between the result and the answer. The result is determined by the answer we are attempting to find to a given question, in a given situation. But the formulation of the answer based on the result depends on the whole equation-solving and, more particularly, on the method used.

Example:

The question is: *Solve in R the equation 3(x-1)+4(2x-3)=11x+1*

Resolution 1:
3x-3+8x-12=11x+1
<=>11x-15=11x+1
<=>-15=1.
Result 1:-15=1
Answer 1:S=ø

Resolution 2:
3x-3+8x-12=11x+1
3x+8x-11x=1+3+12
0x=16
x=16/0
x=∞
Result 2:x=∞
Answer 2: "there is no solution, the equation is impossible"

In the first way of solving, the result $-15=1$, produces the answer $S=\emptyset$ because of a "ad absurdum" reasoning, since the initial equation is equivalent to an equation which is always false.

In the second way of solving, the result $x=\infty$, produces the answer "there is no solution, the equation is impossible" upon the examination of a table of results and answers in an algebra book (here an algebra school book written by the great mathematician Emile Borel in 1920; the resolution has been copied from examples he gives).

1.3. Synthesis

The analyses I have here briefly presented have made it possible to show a certain number of variables which enable us to determine the way in which the two planes of meaning are present in the didactic system under study. These analyses are based on the study of a number of French algebra school-books (4th French class, medium level of school, students aged 13-14) and on the reports of experimental questioning of students (observation of pairs without the intervention of the experimenter) I did in a French 2nd class (high-school, students aged 15-16). I can present here only the results of this observations.

2. Elimination of the plane of finality

2.1. In French school books

In the recently-published French algebra school-books, one notices the disappearance of what established the plane of finality in opposition to the plane of algebraic treatment (in accordance with Tonnelle 1979 where it is shown that the teaching of algebra is only "formal" and never "functional"). The elements which are from my criteria significant for validation and control are nowadays most of the time absent from the school-books. The algebraic part of the equation-solving is in fact the learning of some *rules of transformation*. The meaning is supposed to be for outside the context of algebra, in concrete problems.

The algorithmisation which is the consequence of the lack of notions coming from the plane of finality is apparent not only in the course but also in the exercises. The almost complete absence of exercises other than numerical equation-solving and, in particular, the absence of parametric equations shows (in accordance with Schneider, 1979) this algorithmisation process.

It does not seem to me that the appearance of concrete problems could be interpreted as a tendency to work in the plane of finality since the elements that determined the control process of the algebraic treatment are lacking in the school-books (cf also Sutherland 1990, algebra as "hidden curriculum"). This phenomenon has therefore to be attributed to the process of the didactic transposition of algebra and it

seems to me to be an indication of the cultural "pejoration" of algebra of which Yves Chevallard speaks (1985, 1989a, 1989b).

This analysis is characteristic of the meso-didactic level (see summary) and seems to correspond to the macro-analysis done by Chevallard and the "Marseille research group".

2.2. In students' protocols

Rapid description of the task given to the students

I can give here only the general framework of the task. On the first page of the worksheet, the text read: *"A student gave the following solution* (S) *to this exercise* (text of E)". S was proposed as usual in classwork ($S=\{2\}$ or $S=\Delta$, etc.). E was an equation of the first degree, with one or two unknowns. I then asked: *"Without solving the exercise, can you say whether the solution is false, probably false, probably true, true, or "I'm unable to say"?"* The students had room on their papers to make the necessary calculations, and I pointed it out when presenting the task. On the second page (given out only after they had answered the questions on the first page) were the following instructions: *"The student worked out the problem below. His way of going about it may be incorrect; it may also be correct. Look for possible mistakes in his work. Note where they are located with your pen."* Then there was the complete equation-solving which took up several lines. At the end of the page, I asked the same questions as on the first page.

The students were disoriented by this questions. Therefore, they expressed some reactions that we never saw in usual situations.

Results of the observations

The difficulties of the students in front of the unusual problem they have to deal with show, by contrast, the adequacy of the personal relationship of these students to usual problems. What one sees here is a good reproduction of the official relationship as we have noticed it in the school-books (for official and personal relationship, see Chevallard 1989b).

The difficulties of the students to express themselves and their use of words (calculate, solve and verify) as an implicit lever in the negotiation with their partner are similar to the school-books on this point. The vocabulary is not explained, it is shown. It is clear that when you "verify" you do something else than when you "solve" only because: (a) there are two different words to say it, and (b) two different moments for these actions in the usual chronological development of the solving-equations in the class-room.

In the plane of algebraic treatment, the students are more comfortable and no longer express the trouble that one can see in the plane of finality. This work is usual. Here again, the personal relationship is to be seen consistent with the official relationship conveyed by the school-books.

The calculations achieved by the students and the mathematical properties that they use are very rarely verified, even when it could have been done easily. This suggests us that the students reproduce the sharing of responsibilities in the class, that is to say the exclusive part of the teacher in terms of evaluation and correction. In any case, the difficulties of the students are by no means attributable to a lack of knowledge about technics of rectification, since when we explicitly asked them to rectify a written piece of reasoning they succeeded quite easily.

The change of frame (especially the graphic frame in the case of systems of equations) has been evoked, but very rarely carried out. One have seen in the school books that the possibility of using alternative methods is always pointed out, but it is not the task of the student to decide to use them or not, as far as the school book or the teacher always give an indication if it is needed.

The students are reproducing what Chevallard (1985) calls the *topogenesis* of knowledge. The personal relationship seem to us perfectly consistent with the official relationship. *In that sense*, the teaching of elementary algebra is perfectly successful...

3. Usual class management and algebraic finality

The usual running of a class corresponds to what I have called "textualizing": knowledge is considered as a *text*, that the teacher has to read. The *implicit* aspects of the tool are to be transmitted through the implicit elements of the didactic communication i.e. through an implicit didactic contract. The didactic situation is thus a speech situation in which the request to solve the equation is a closed question asked by the teacher, and to which he expects an answer. I will develop the *natural consequences* of the "textualizing" on the equation-solving.

Result/Answer The teacher *will show* what "solve" means, by working out a problem algebraically. In the same way, he *will show* what he will accept as an answer. He *will* do so, because he posed a closed question, not finalized. Implicit communication elements are a necessary lever for him in this context. *He therefore will not say clearly the aim of algebraic treatment: search the answer.*

Question/Resolution The most economical way for the teacher to render apparent its interpretation for the question is to show his own resolution. The teacher will have a better chance of being understood if he proposes *few choices* among possible resolutions. Limiting the resolution choices reduces the work of the teacher and that of the student. Moreover, if he doesn't limit the choices, the teacher may run the risk of losing control of the evaluation. *He therefore will eliminate the control on the choice of resolution.*

Resolution/Result The end of the resolution will also be shown. For the same economical reasons as above, the teacher will show *only one possible end*, that which he considers as being the best adapted, given his conception of solving-equations and the level of the students. It is necessary to standardize the end of the resolution, because

otherwise evaluation becomes impossible. If a student can stop "where he wants" in the solving process, any and all types of cheating are possible. *He therefore will eliminate the control on the end of the resolution.*

Result/Answer If the teacher is showing everything, he has no reason to make a distinction between result and answer, because it is not economical in this context. It would even be artificial for the teacher to show that he could have answered before the usual result. He has no reason to emphasize the work of interpretation. *He will therefore eliminate the difference between result and answer.*

Synthesis

We can therefore say that in the case of textualizing, economical principles incite the teacher to avoid showing his choices, so as to give more importance to the demonstration of his visible actions. In this way, he simplifies his task and momentarily that of the student. The student is in fact only informed of algebraic treatment, and so the plane of finality is eliminated. This micro-didactic analysis confirms the analyses of the other levels presented above.

This analysis lead me to think that a change in the teaching of elementary algebra should affect not only course programmes, but also the "textualizing" the teacher use as habits in the classroom. Very different results are obtained if there is a situational treatment, i.e. if the teacher can finalize the activity (cf. Margolinas 1989, also according to Chevallard 1989a about the use of modelisation in teaching algebra)

4. Conclusion

Taught algebra does not therefore bear the characteristics of a tool for mathematical activity, because a tool does not exist as such (object), but is defined according to its finality. The question here, which we will leave open, is that of the adequacy of such a relationship to the meaning of elementary algebra (even if we limit ourselves to a strictly academic perspective which is the use of algebra in high-schools). At the high-school level, algebra becomes a working tool for the student, especially in analysis (Chevallard, 1985-88-89). In this context, knowing how to solve an equation cannot be a goal in itself, but a means to find something: for example the zeros of a function. The necessary relationship to the equation-solving therefore undergoes a change. Algebraic treatment depends on the plane of finality, and the question here takes on a mathematical status, independent of the didactic contract.

But there is no specific teaching of elementary algebra at the high-school level. The necessary personal reorientation is not provided by the school system. It is the responsibility of the student, who is supposed to have already learned what he needs to solve the algebraic problems he meets at this level. The difficulty of this change of relationship to algebra is not recognized by the didactic system, but it can explain many difficulties at this level.

The interpretation which I give to this absence confirms the empiricist common conception for learning. According to this conception, the student, by the contact with the activity of problem-solving, is supposed to have integrated, not only the rules of transformation, but also the meaning of this activity. From this point of view, the meaning of knowledge is not the result of a genesis but derives directly from reality: the mere confrontation suffices to provide adequate meaning.

Thus, the introduction of "concrete problems" corresponds to an attempt on the part of the didactic system to remedy the problems caused by the absence of meaning that students encounter. But this introduction does not organise the meaning-giving task, it is but another aspect of the empiricist model. It remains an attempt to obtain, with no resources in sight, and thus with no guarantee, what the educational system cannot attain without a significant change in its didactic choices, and in particular in its micro-didactic choices.

The problems which are posed are therefore: 1) what is the relation between knowledge thus acquired and mathematical knowledge? 2) even from a strictly scholastic point of view, is the empiricist method effective?

References

DOUADY R., 1985, The interplay between different settings. Tool-object dialectic in the extension of mathematic ability, *proceedings of the ninth international conference for the psychology of mathematics education*, vol. 2, pp33-54.

CHEVALLARD Yves, 1985-88-89, Le passage de l'arithmétique à l'algèbre dans l'enseignement des mathématiques au collège *petit x*, ed. IREM de Grenoble.

CHEVALLARD Yves, 1989a, Arithmétique, algèbre, modélisation, *IREM d'Aix Marseille.*

CHEVALLARD Yves, 1989b, Le concept de rapport au savoir. Rapport personnel, rapport institutionnel, rapport officiel. *Sémainaire de didactique des mathématiques et de l'informatique de Grenoble*, LSD2-IMAG.

KAPUT J.James, 1987, PME XI algebra papers: a representational framework, *proceedings of the eleventh international conference for the psychology of mathematics education*, pp 345-354

KIERAN Carolin, 1989, A perspective on algebraic thinking, *proceedings of the thirteenth international conference for the psychology of mathematics education*, vol 2, pp163-171

MARGOLINAS Claire 1989, *Le point de vue de la validation: essai de synthèse et d'analyse en didactique des mathématiques*, Thèse de doctorat de l'Université Joseph Fourier, LSD2-IMAG

SUTHERLAND Rosamund, 1990, Developments in the teaching of algebra in the U.K.: the potential of computer-based environments, *Sémainaire de didactique des mathématiques et de l'informatique de Grenoble*, LSD2-IMAG.

Age variant and invariant elements in the solution of unfolding problems

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Abstract. At different age levels, variant and invariant elements emerge from the analysis of pupils' performances in the solution of problems regarding the folding and unfolding of elementary solids.
Particularly, 10-11/13-14/17-18 year old pupils were considered, during individual standardized interviews, and the analysis was carried out as regards the stability of mental images, the influence of the figural aspect, the interference of standard figures and the use of symmetry. The reference frame is a wide research project concerning mental processes underlying the solution of a geometrical problem.

1. Introduction

The objective of this report is the discussion of peculiar aspects of pupils' performances highlighted by their solution strategies of a particular geometrical problem.
The analysis of a rich collection of protocols suggested the presence of age invariant elements that this paper is devoted to illustrate.
The experimental data on which our considerations are grounded, represent the results of a wide experimental research project on geometrical reasoning. A first account of this research project was presented on PME 13, but it is still in progress.
As regards the details of the general experimental design I refer to the description given in [2], but I just make a short remark on some aspects, which are relevant to the following discussion.
The main interest consists in studying the process of interaction between mental images and concepts in geometrical reasoning. Even if general references about mental processes involved in the elaboration of mental images were taken into account (such as [3][4][5]), the main objective remains that of providing results meaningful from the point of view of mathematical education.
The reference frame of my research has its central point in the idea of Figural Concept [1] and its main objective in the study of the mental process underlying the solution of a geometrical problem. The basic hypothesis lies in considering the dialectics between the figural and the conceptual aspect as characteristic of the geometrical reasoning, thus the aims of the experimental research are the observation and the description of the modalities of this process. The collection of data is organized around the problem of folding / unfolding of a solid, while the method of inquiry is that of standardised interviews.
The choice of the folding/unfolding problems has revealed its advantages as concerns the analysis of variant and invariant aspects according to age.
Even if this problem is not simple to treat rigourously from a mathematical point of view, it is easy to understand and it does not require any specific school knowledge, so that it can be proposed at different age levels, without changing its presentation. This fact presented the opportunity to observe the solution strategies and generally the approaches to the problem along the time dimension. The study was carried out at three

different age levels and with pupils attending different schools in different social contexts (among them some Israeli pupils). Particularly, three different age levels were inquired, following the same experimental design: the end of the Primary School (pupils aged 10-11), the end of the Middle School (pupils aged 13-14) and the end of the High School (pupils aged 17- 18). Pupils were selected at random, I just made sure that they had not had a specific instruction in folding / unfolding problems.

As regards the method of observation, a particular kind of interview was carried out. Usually, the richness and complexity of information coming from the protocols of the interviews make their analysis difficult. Thus to make the comparison of the protocols easier, a particular kind of standardized interview was carried out: that is to say, the schedule of the interview is fixed once and for all and the same list of questions was proposed to all the subjects. In this way each one was asked the same questions, but at the same time the subject was left completely free to use his own strategy of solution and to comment it. On the other hand, the flexibility of the method of interviews gave the possibility of observing and making most of the process underlying the solution explicit. When the answer became interesting and much more information seemed to be required to understand what was going on , specific questions arose, which stimulated the pupil to illustrate his solution.

The result is a comprehensive view of the solving strategies, which enables the comparison of methods and mental processes both at different age levels and among pupils of the same age, but with different school/curriculum experience.

Besides other general results, the protocol analysis focuses some elements which can be considered as "invariant"; it means that those elements can be found at different age levels and in different school contexts, leading to very similar performances in the solution of the same geometrical problem. This will be the subject of the following discussion.

2. Stability of the mental image

According to the two dimensions of comparison, mentioned above, <u>the stability of the mental images emerges as an age invariant element</u>.

Let us consider the following example. The pupil is asked to count the number of the edges of a cube mentally, that is to say that the cube is not available and its image is to be recalled. The common (correct) answer sounds like the following: "... *12, 4 on the top, 4 on the bottom and 4 on the side ...*" Similarly, in the case of the vertices: "... *4 on the top, and 4 on the bottom...*".

Examples of this kind can be found at each of the different age levels. - Cristina (10:10 year old)- Iuri (13:10 year old) - Leonardo (17:6 year old) (see the protocols A1-A2-A3 enclosed) clearly show that the counting resorted to a spatial reference frame. It is interesting to remark the example of Leonardo, because he uses a particular organization referring to the *depth* (from the front to the back).

However, the main characteristic is the presence of certain systematization, that signals the transition to a level of abstraction where the image turns out to be organized, usually according to a spatial reference frame.

The mental organization enables the subject to elaborate the mental image with great ease. On the contrary, the lack of organization leads to failures. Surprisingly, it turns out to be very difficult to count the number of the edges of a cube without any order.

But, probably, the most interesting fact is that sometimes the presence of the object hinders this kind of organization and often the counting on the object fails. Usually, pupils count in their mind grouping the edges "per faces", but when the object is available they normally count the edges one at a time, turning the object in their hands and losing the thread. According to their report, during the counting "in the mind", the mental image of the cube is motionless, on the contrary, once they have the object in their hands the counting strategy changes. Let us give an example: Cristina (10:8 year old), she correctly counted in her mind, then I give her the cube. She begins to count, she turns the object in her hands and counts, ...*12*, ...*16*. I ask her how she can be sure. She answers "*I should keep it still.*" Then she comes back to the count and "...*12 [grouping by 4]*"(see protocol A1 enclosed).

Similarly, the organization of the image can play a part in the solution of the unfolding problem. For example, let us consider the task of drawing the unfolding of a Triangular Prism. The object is not available, it is hidden, thus the subject must recall it in his mind (" *the faces were ... 2 squares, 2 triangles and a rectangle ...*"). The image is organized according to those elements and their relations, so that the resulting image is consistent / adequate to its use to draw the unfolding. The following protocol is a good example.

Anna (10:3 year old). [Unfolding task, without the object - triangular prism]. *I think it is like this* [she has drawn a correct unfolding]. How did you think of it? Let me see...*I counted the square faces and then the triangular ones and then, I thought how I should put them.* To think where to put them, did you think of reconstructing or opening...*I don't know* ...

Similar examples can be found at the other age levels.

3. Influence of the figural aspect

The analysis of the protocols clearly reveals the influence of the figural aspect on the solution of the problem . It means that the solving process is affected by particular features, related to the concrete experience both of the object - as it perceptibly appears - and of the transformation/deformation which is necessary to attain the unfolding. Obviously, <u>this influence decreases with age</u>: it means that the oldest subjects can better control the effect of the figural aspect; but it is interesting to remark the fact that <u>the influence of the figural aspect never disappears completely</u>, as shown by the following examples (see fig.1).

The effect of the figural aspect can be clearly found in these unfoldings, drawn during the interviews. Let me point out the similarity of the three drawings (fig.1 a,b,c), given in the case of the regular Tetrahedron and referring to different age levels.

Moreover, a well spread opinion emerges, according to which there is only one possible unfolding for each solid. This fact can be due to the strength of the strategy associated to the transformation mentally evoked to unfold the object.

For example, the subject is asked about the possibility of drawing more than one unfolding of the same solid. <u>At different age levels</u>, it is very common to obtain the following response: "...

I am going to try ... [and after some attempts]...*It is always the same!"*
Similarly, it is meaningful to remark the fact that the number of those who are convinced that there is only one possible unfolding is percentually more in the case of the cube than in the case of the parallelepiped. Again the influence of the figural aspect can provide an explanation. Let us consider a Parallelepiped and the following unfolding strategy: "turn up the top and free the lateral faces". Different unfoldings of the Parallelepiped can be drawn according to this strategy, only just changing the base; on the contrary, let us consider the cube, because of its symmetry, changing the base will not effect the resulting unfolding (provided the same strategy is applied). Quite a similar phenomenon occurs in the case of the Triangular Prism and the regular Tetrahedron.
Further more, the strength of a certain strategy, related to the particular transformation of one object, leads to some difficulties and different mistakes. In fact, often, the subjects are not able to adapt their strategy to a new situation. For instance, the strategy consisting in the unfolding of the lateral surface ("like a ring") and then adding the top and the bottom, naturally applies to the cube, but it does not directly adapt to the unfolding of the Tetrahedron (see protocol B1 enclosed).

4.Interference of standard images.

Looking at the protocols of the interviews a very common phenomenon can be observed; frequently and again at different age levels, the mental image of an object is assimilated to a standard image. Usually this occurs when unfamiliar objects are considered and, however, when the task of memorising the details is not explicitly given.
Let us give an example. During the interview, the regular Tetrahedron is shown and then it is hidden, the pupil is asked to count the faces, the vertices and the edges . Many times, the answer sounds like: *"5 faces, 5 vertices, 8 edges"*. In other words the regular Tetrahedron is assimilated to the Square Pyramid and the responses are consistent with the substitute image. Furthermore, sometimes, actually very often, the assimilation is not complete and the resulting image appears like a paradox, leading to an inconsistent response. For instance, the answer sounds like: *"7 edges"*, together with the explanation, *"... three on the side, four on the base"*. This response reveals the presence of an "inconceivable" mental image, something like an hybrid coming from the Tetrahedron and the square pyramid.
This kind of phenomenon is very common at different age levels: according to our estimate, at the end of the Middle School, that is at the age of 14, the percentage of those referring to a Square Pyramid during the counting task for the Tetrahedron is about 70%! But, overall it is very persistent, it means that even if the pupil has been corrected on the counting task, the interference can appear again on the unfolding task (see fig.2).
Obviously, looking at the object, it is very easy to realize that one has made a mistake, but it not so easy to correct it - many examples of this fact can be provided.(see protocols C1, C2 enclosed).
However, it is meaningful to remark that this kind of phenomenon besides being very common, can also strongly affect the solution

PROTOCOLS

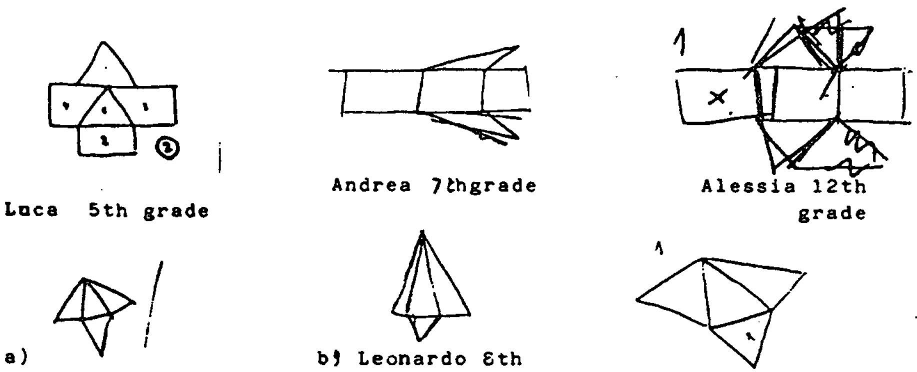
A1. CRISTINA (10:10 year old)[the cube was shown and then it was hidden] How many faces has a cube? *[...] 6.* Did you count them? Yes How did you count them. Could you describe how? *First I counted those on the side, that is those like this, and then those over them.* All right, now let us count the vertices, the tips. *[...] 8.* Again, how did you count them? *I did 1,2,3,4, counting those on the top and those on the bottom.* Fine. the edges. *Then the edges ...they should be* .... Those "spines". *[...]4.* How did you count them? *I mean, here is the cube [actually the cube is still hidden], and I did, and here ... doing* ... . Don't you think that they are not enough? *4* . Only 4 !? *But, as for the vertices, do you mean...* All of them ...if I turn it ... . *They are 4.* You think 4 O.K..Then, now let us count again in the same way, with the object. Now, first the faces. let me see how you counted them. *I did ...1,2,3,4,5 and 6.* Fine. I understand. Now let us see the tips. How would you count them? *The tips? 1,2,3,4...5,6,7,8.* O.K.. Now let us consider the spines. *These ones?* Yes. *1,2,3,4.* Listen, and this one? *Here it is! 5, 8. Because 1,2,3,4 ...5,6,7,8,.* Let us count again. *1,2,3,4,...5,6,7,8.* And these on the bottom? *10, [just a moment of interruption] 12.* Do you think there are any more? And if I go on turning? How can I be sure that they are 4,8,12,16, ...? *By keeping it still.* That's a good idea. Now, let us keep it still and count them, but all of them, right.? *1,2,3,4,...12 [she counted grouping by 4].*

A2. IURI (13:10 year old)[the cube was shown and the it was hidden] How many faces has a cube? *6* . How did you do it? *I mean, I...did, I...said, ...let's say, the lateral faces, those around are 4, and the bases , I mean at the bottom... one on the top, one at the bottom...* And, the vertices? *[...]the vertices...are...8.* How did you do it? *There is, 4 on the top and 4 at the bottom.* And the edges? *[...] the edges ...emh ...12.* How did you manage to count them? *Because, I mean ...if the faces are 6, I said ... well each face has two of them and ... multiplying* .... Each face has two of them !? No! You cannot remember how you counted them, can you? *No, I can.* Now I'll give you the cube. Can you count again, on the cube? *These are the lateral faces [he touches each one with his finger], plus two, then the vertices are 8: I said 4 on the top and these 4 and then the edges ... the edges ... I mean, now that I can see ... I see that, I mean, 4 ... 4 at the bottom ... the others... I mean these... plus the 4 ... on the side ... [he is uncertain, he turned the object].* 16 ? What is it that's not right? *I mean, those...I mean , that I have already seen and those that... I have to see, practically now ...these 4 ..:4 at the bottom and the 4 on the side [he takes control , organising the counting well].*

A3.LEONARDO(17:6)[without the object, counting the faces] *6, playing dice.* What about the vertices? *8. Those ...the cube in front, the first 4 on the front face and one, and then I looked the depth, and the other 4 on the back.* What about the edges? *12. Something like before, I looked on the front, 4 of the first [square] ... looking in depth and they go to the last square on the back ... the other 4 of the next square ... they [the second 4] are those that make the figure three-dimensional.*

Final remarks

As a final remark, let me come back to the general reference frame of my research, that is to say the interaction between figural and conceptual aspects in geometrical reasoning.
What was illustrated above, focuses the basic role played by mental images in the solution of a geometrical problem. But likewise, the great complexity of the processes involved emerges: particularly, the fact that conflictual aspects of this interaction can be present at any age level.
As a consequence, further investigations are required with the objective of finding a common explanatory principle to interpret the complex of phenomena observed. On the other hand it is necessary to clarify the possible relation between pupils' performances and their past experiences at school, specifically as regards teaching/learning activities in the geometrical domain.
However, from the educational point of view this kind of results represent a contribution about the general features of the mental processes involved in the geometrical reasoning, that teachers could take into account when a didactical situation is planned.



Luca 5th grade — Andrea 7thgrade — Alessia 12th grade

a) Paolo 5th grade — b) Leonardo 8th grade — c) Martine 12th grade

fig. 1

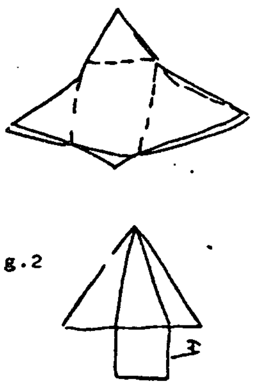
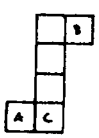


fig.2



fig;3a

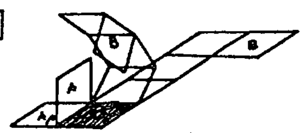


fig.3b

of the problem, even when the task is a very simple one such as that of counting faces, vertices and edges.

5. The problem of symmetry.

Let us consider the mathematical idea of symmetry; it represents a very important general concept, mainly from the point of view of geometry. More generally, we use "symmetry" as a tool to understand, to describe , to cope with the complexity of a task. Actually, symmetry is a schema of thinking and in this sense must be acquired. But sometimes, the possibility of grasping this idea is over-estimated. The analysis of the protocols provides good examples.
Let us consider the task of folding the solid starting from the drawing of a particular unfolding (see fig.3a). The subject is asked to identify the couples of segments on the perimeter that will join on the solid, after the reconstruction.
From a mathematical point of view, it seems natural/simple to use the possibility of reasoning in terms of symmetry. But, actually this does not come straight away; the operation of folding requires passage from the 2-D space, where the unfolding is drawn, to the 3-D space, where the object is reconstructed. To use the "symmetry of the figure" means to interpret the symmetric features of the two-dimensional drawing in terms of the three-dimensional operation of folding. This fact can imply radical change of the structural order of the reconstruction, a new order can be achieved only after an appropriate / proper rational elaboration of the situation. In other words, looking at the drawing, as a particular 2-D figure, specific symmetrical features can be highlighted, but their interpretation, at the level of the reconstruction process, requires the great effort of relieving the mental image from the constraints of the concrete operation of reconstructing.
For instance in the case of the unfolding of figure 3a, when one of the faces is fixed as the 'base' of the solid to be folded, the symmetry of the drawing, as a plane figure, will disappear. This is to say, for instance, that in the reconstruction, fixing face C as the base, faces A and B will be transformed differently (fig 3b). As a consequence, following this process of reconstruction, symmetry cannot be directly used - put into action - to accomplish the task. In fact, in this case "reasoning by symmetry" would mean to consider the operation of reconstruction as independent from the face chosen as the base. In this way the specific properties of the transformation do not come straight from the experience, from a mental simulation of folding, but from the rationalization of the process.
As for as this problem is concerned, <u>the behaviour of the pupils seems to be variable, according to individual characteristics, rather than to age levels</u>. Many, nearly the majority of the subjects interviewed, at any age level, were not able to refer to symmetry, or at least they did not take it into account. Of course, sometimes the possibility of changing the base is recognized and the symmetry of the situation applied; but the high percentage of failures(87% at the primary level, 65% at the level of the Middle School) shows the unavoidable difficulty. The subject is able to interpret the symmetry of the unfolding in terms of the symmetry of the reconstruction only if, operating on the drawing, a conceptual control takes place on the figural aspect.

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B1. FEDERICO (13:10 year old) As regards the unfolding he always followed the strategy "as a ring". When he has to unfold the regular Tetrahedron he gives the following drawing (fig.a)

No! Why not? *No ... this ... [he moves one of the triangles (fig.b)] ... but there is something wrong ...* [for a long time he remains uncertain, only after miming the two reconstructions he seems to be convinced]

C1 SERENA(10:10 year old) Let's begin to count on the Tetrahedron. This one [showing the regular Tetrahedron] Do you know what it's called ...? *A Pyramid, I think so.* Yes a Pyramid. So, this Pyramid [hiding the object] How many faces are there? *4* ... How did you count them, these 4? *Now, before the 3 on the top and the one at the bottom.* And the vertices? the tips? *The tips, yes ...[...] 4.* How did you count them? *Now, first the tip like this, then the three ones at the bottom.* Fine! Now the edges, the spines. *Then ... 7.* How did you count them? *Then, first those on the top, three on the top and the 4 at the bottom.* Could you show me? [giving the object] Then let us start with the faces. *Now, 1,2,3,4.* .... The counting goes on correctly and she realizes that she has made a mistake. Which was your mistake, what do you think? *I thought that there were 4 ..., here ... That the base was ...Yes, I was wrong.* That the base was a square.[She nods. I ask her to draw the unfolding and Serena draws rapidly, the object had been hidden] Let's check. How did you think of it, let's hear?[the object is still hidden]. *Now, this comes vertically, this one too, this one too and then this one, we can say that it is the base [she points the square at the middle] and then they come together ... and it would come out...* Are you sure that it is right? *But, There is a hole left.* There is a hole left. Can I put another face there? *I don't think so. However, let me tray.* Why don't you think so? *I mean, because it has three faces ...* On the top? *Yes, on the top. And then a fourth, otherwise they would be 5.* The discussion goes on, but she is not able to resolve the conflict. I give her the Tetrahedron. Serena checks the 3 triangles, she trys to draw again, then, finally, she raises the object and looks at the base ...

C2. ALESSIO (14 year old) He behaves in the same way: he fails the counting without the object, he corrects himself. But, when he has to draw the unfolding without the object, he puts a square base again.

References

[1] E. Fischbein , *Conceptele figurale*, Ed. Acad. Rep. Pop. Romine, (1963)
[2] M.A.Mariotti, Mental images: some problems related to the development of solids, *Actes de la 13e Conférence Int. PME* (1989)
[3] Shepard,R.N. & Cooper,L.A.(1982) *Mental images and their transformations*, The MIT Press, Cambridge, Massachusetts
[4] Shepard,R.N. & Feng,C. (1972) A cronometric study of mental paper folding, *Cognitive Psychology* ,3,228-243. Reprinted in Shepard & Cooper (1982).[5] Piaget, J, Inhelder,B.(1947). *La representation de l'espace chez l'enfant.*

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