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ABSTRACT

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CENTER FOR THE DEVELOPMENT OF TEACHING
PAPER SERIES

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Learning Mathematics for Teaching

Lessons in/from the Domain of Fractions[1]

Deborah Schifter

It is widely recognized that developing a successful teaching practice, one that is grounded in the principles that guide the current effort to reform mathematics education, requires a qualitatively different and significantly richer understanding of mathematics than most teachers in the 1990s possess. However, it is not as clear how teachers' mathematical understandings develop and how those understandings affect instruction. This paper explores two avenues for K–6 teachers' mathematical investigations—inquiry into mathematics itself, and inquiry into children's mathematical thinking—and illustrates how they arise in elementary teaching situations and how they can be explored in a professional development setting.

The year Anne Marie O'Reilly began teaching sixth grade, she was working to develop a practice that built from her students' ideas. She had studied reform documents, read Dewey and Piaget, among others, and thought hard about the implications for her classroom of a constructivist view of learning. But, despite all her preparation—she had even taken several workshops and courses about learning and teaching mathematics—that first year teaching sixth grade was difficult and unsatisfying. O'Reilly illustrated her struggles in this vignette drawn from a lesson on fractions:

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> The class had no problem telling me how to write the fraction for "one-half" on the board—"1/2." Then I asked my students what the top and bottom numbers in that fraction stood for. I was not prepared for David's answer:
>
> 1 thing
> — cut into
> 2 equal pieces

David made it clear, as he drew the horizontal bar in the air with his finger, that the line took the place of the words "cut into."

> I had done my homework. I had consulted what I considered to be a reliable resource for helping me to understand the mathematics involved in what I was teaching. . . . What I wanted to hear was something to the effect that the top number tells how many things we have and the bottom number tells what fractional part is being counted. I was stunned and at a loss as to how I should challenge David to help him say what I wanted to hear. I looked to his classmates to offer a challenge and asked if anyone else wanted to give an explanation. . . To my dismay, most people agreed with David. (O'Reilly, 1996, pp. 71–72).

Not knowing how to go on, O'Reilly ended the lesson and, next day, started in on a new set of activities. Several weeks later, reflecting on that event, she confessed, "It was only after I had acknowledged my difficulties in unraveling the mathematics I was trying to teach . . . that I suddenly began to identify instances where my limitations were getting in the way of my listening and teaching for understanding" (p. 73).

In recent years, new standards for the reform of K–12 mathematics education have set an ambitious agenda for mathematics instruction: "teaching mathematics for understanding" (NCTM 1989, 1991, 1995). Recognizing that such teaching cannot be merely a matter of presenting math facts and demonstrating algorithms, and that learning for understanding cannot be merely a matter of memorizing those facts and procedures, reform documents describe a vision in which teachers' and students' roles in the classroom are drastically revised. Teaching is now seen as a matter of engaging students in significant problems and facilitating discussions about them, and learning as a process of formulating conjectures, testing out ideas, and exploring alternative approaches. Intellectual authority no longer resides exclusively in teacher and textbook, but is dispersed among the members of the classroom community who offer defensible arguments (Ball, 1993; Cohen et al., 1993; Confrey, 1990; Lampert, 1988; Mokros et al., 1995).

However, in a practice in which student thinking takes center stage, classroom process becomes much harder to manage and much less predictable (Ball, 1993; Hammer, 1996; Lampert, 1988; Sassi & Goldsmith, 1996; Warren & Ogonowski, in press). Indeed, with increasing numbers of teachers constructing such a practice for themselves, quandaries like O'Reilly's are likely to become familiar experiences. As David's response to O'Reilly's question shows, "students' ideas can be as puzzling and oblique as they are inventive and insightful" (Cohen & Barnes, 1993, p. 244). What is a teacher to do who is working to establish a classroom culture in which intellectual authority is shared, when the students propose ideas that are surprising, incorrect, or even, it may seem, incomprehensible?

While dilemmas like O'Reilly's can never be anticipated in their specificity, teachers *can* become better equipped to deal with them. If they are to build their instruction around children's thinking, the skills teachers need to acquire include interpreting students' mathematical ideas, analyzing how those ideas are related to the mathematics of the curriculum, and challenging students to extend or revise those ideas in order to become more powerful mathematical thinkers.

It is by now widely recognized that developing a successful practice grounded in the principles that guide the current mathematics education reform effort requires a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess (Ball, 1989, 1996; Cohen et al., 1990; Even & Lappan, 1994; Schifter, 1993; Schifter & Fosnot, 1993; Thompson et al., 1994). But, although there is a growing body of literature documenting the changes teachers go through as they begin to realize the vision of mathematics instruction proposed in the NCTM *Standards* (c.f. Fennema & Nelson, in press; Friel & Bright, in press; Knapp & Peterson, 1995; Nelson, 1995; Russell & Corwin, 1994; Simon & Schifter, 1991;

Wood et al.,1991), teachers' developing *mathematical* understandings—and how those understandings affect instruction—have not received sustained attention from researchers and teacher educators. What kinds of understandings are required in teachers who are working to enact the new pedagogy? How are these understandings reflected in practice? How can one help teachers learn to think about mathematics in terms of underlying conceptual issues and not merely as an *ad hoc* sequence of topics?

Teaching to the Big Ideas (TBI) is a four-year teacher-enhancement project, sponsored by the National Science Foundation and jointly conducted by EDC, TERC, and SummerMath for Teachers (Schifter et al., in press). In its first two years, 6 staff members and 36 elementary teachers came together in two-week summer institutes, biweekly after-school seminars, and one-on-one biweekly classroom visits in order to examine the mathematics of the elementary classroom when teaching is focused on children's thinking. In its final two years, the project has been oriented toward developing teacher leadership: first, through producing written materials in which teachers communicate what they have learned, and second, through implementing the staff-development initiatives that TBI teachers are now conducting in their schools. The project began in the summer of 1993 and is currently in its fourth year.

TBI has also provided a context for project staff to research participating teachers' developing mathematical understandings and their effect on instruction as these teachers work to transform their practice. By visiting teachers' classrooms, members of the staff were able to identify which mathematical issues to put on the seminar agenda and, reciprocally, to evaluate how seminar learnings were being called upon in the classroom. Collected research data include classroom field notes, audiotaped interviews, and papers and journals authored by the teachers.

Teachers engaged in the project pursued their mathematical investigations along two avenues. The first took place in explorations of disciplinary content led by project staff, and usually involved topics from the elementary curriculum. Although it has often been assumed that elementary-school disciplinary content is simple and that any educated adult knows enough to teach it, this is simply not the case. The mathematics is, in fact, conceptually complex, but those who have been educated in the system needing reform have not, themselves, dealt with that implicit complexity.

The second avenue along which TBI teachers pursued their mathematical investigations was the examination of student thinking. Now, a number of projects aligned with current reform efforts have recognized the importance of "student-centered" instruction. Some emphasize teachers coming to see learning as a process of construction (Fosnot, 1989); others teach teachers to conduct interviews of children (Ginsburg & Kaplan, 1988); some present teachers with the findings of researchers who study children's mathematical ideas (Fennema et al., 1993); and still others call attention to students' mathematical talk (Russell & Corwin, 1993; Schifter & Simon, 1992). However, there has been insufficient appreciation of how the analysis of student thinking can itself become a powerful site for teachers' further *mathematical* development.

This paper explores these two avenues for teachers' mathematical investigations, illustrating how they arise in elementary teaching situations and how they can be addressed in a professional development setting. We consider teachers both as learners of mathematics—identifying how new understandings are reflected in practice—and as investigators into *children's* mathematical thinking, thereby examining how understandings so derived can also shape practice.

Inquiry into Mathematics

"Teachers teach as they have been taught" is a maxim frequently heard. So, too, is its familiar corollary, "In order for teachers to learn to teach differently, they must be taught differently." Widely accepted though they may be, their implications for teacher development are not so clear. What can teachers learn (about teaching mathematics in new ways) by becoming mathematics students themselves? How might professional development opportunities be structured to foster such learning?

In order to address these questions, we offer the case of Theresa. First we take a careful look at one lesson in Theresa's sixth-grade classroom, then we step back in time to consider some TBI mathematics lessons in which she participated. What is the mathematics at issue in Theresa's class, and how do her understandings come into play? What aspects of her learnings as a student does Theresa bring back to her teaching?

Sixth-Graders Confront a Question about Fractions

Theresa teaches in a small school district in a rural community. These days, she organizes her mathematics teaching so that most class sessions involve students working in pairs on word problems and presenting their solutions to the whole group. A question period follows each presentation as Theresa and her class work together to understand the presenters' thinking. The atmosphere is friendly and supportive, with students intent upon understanding one another. Questions about strategies and diagrams are seen as efforts to make sense of the presentation—not as indicators of poor work or of a mistaken result. While right answers are valued, much class time is spent sharing methods and diagrams. Conversations about problems continue long after solutions are known.

Creating such a classroom is important to Theresa not only as a goal by itself, but also because she realizes that environments such as this allow students to explore mathematical ideas and develop the kinds of meaningful understandings that she herself seeks.

Theresa's unit on fractions consisted of a series of problems designed to provide her students opportunities to make sense of this new kind of number. According to Theresa,

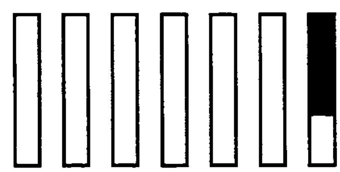
> Fractions and decimals for sixth graders are often counter-intuitive. The whole numbers that the students have recently become comfortable with (or are still not entirely comfortable with) are now used in different ways and mean different things. Getting comfortable with fraction and decimal numbers and developing a "fraction/decimal" number sense is an important aspect of our mathematics classes this year.[2]

For example, in one lesson (observed by a TBI staff member) that occurred several weeks into explorations of fractions,[3] Theresa posed this task:

> Solve the following problem by drawing a large, detailed, labeled picture. Write a number sentence (equation). Write the answer in a complete sentence.
>
> *Nancy has 6 2/3 meters of material. It takes 5/6 of a meter to make her fabulous fancy hair ribbons. How many fabulous fancy hair ribbons can she make?*

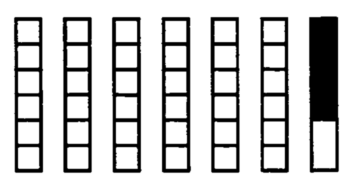
After distributing the worksheet, Theresa reminded her students to work with their partners. In a few minutes the class was fully engaged. Some pairs were trying to draw accurate diagrams using graph paper and rulers; others drew freehand. The students decided fairly quickly that the answer was 8 and spent most of their work time refining their pictures and explanations.

The work of Kathleen and Elizabeth is typical. Kathleen drew seven rectangles and shaded in two-thirds of one of them:



> Kathleen: These are the six things and the two-thirds. Okay?
>
> Elizabeth: Yes. And now we make six in each.

They drew lines dividing each of six rectangles into six parts. They did not draw any lines in either segment of the seventh rectangle:



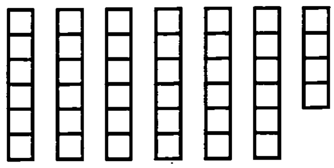
> Elizabeth: Five of each makes one.

Elizabeth pointed with her fingers to show that each meter would make one ribbon with one piece left over.

> Kathleen: Then there are six left, and four makes another two. So the answer is eight.

As she talked, Kathleen pointed to the diagram to show she was collecting the left-over piece from each meter and combining those six pieces with the four pieces from the two-thirds meter to make two additional ribbons. Kathleen wrote: "You can make 8 fancy fabulous hair ribbons." They then set to work to draw a neater diagram.

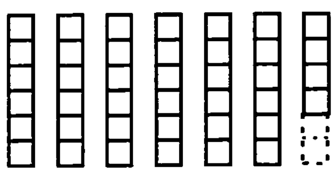
Theresa soon called the class to attention and asked for volunteers to go to the board to show their work. Most children raised their hands. Theresa invited Nicholas and Irene to come forward, and they drew this diagram:



While the whole group studied it, Nancy asked about the two-thirds piece.

> Nancy: I thought the end was 2/3. If she cut it into three, how can you have four?
>
> Nathan: There are six in the whole thing; 4/6 is the same as 2/3.
>
> Annie: Make the dotted lines so they can see the rest of it.
>
> Theresa (to Irene, who was drawing): Can you make it so we can see how it is 2/3?

Irene added a dotted section to her diagram:

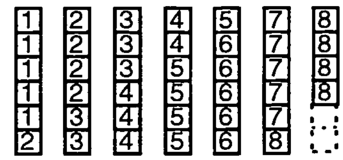


> Nancy: So it's not fourths. It's sixths because of the missing part.

Nancy's question and the responses it provoked among her classmates illustrate this class' concern with unit—the whole to which the fraction refers: Looking at the rectangle at the far right, one section is 1/6 of the whole meter, but is 1/4 of the segment drawn in unbroken lines. Satisfied with this conversation for now, Theresa and her students moved on to consider the arithmetic statement that matched their work.

> Theresa: So what is the sentence?
>
> Annie: 6 2/3 ÷ 5/6 = 8.
>
> Irene: 6 2/3 meters. You need 5/6 meters to make one ribbon. There is 8 of them, so the answer would be 8.

Kevin then came to the board and offered a variation of the diagram already presented:



> Kevin: You have the 6 and you have to divide that up and it makes 7 and you have one left and it—put with the piece of the meter—becomes 8. 6 2/3 is how much material you have. It takes 5/6 to make a ribbon, so 6 2/3 divided by 5/6 tells you how many ribbons.[4]

The class was quiet for a while as they thought through Kevin's comments. Kevin continued by writing his statements on the board: "6 ÷ 5/6 = 7. 1/6 + 4/6 = 5/6 = 1. So it's 8."

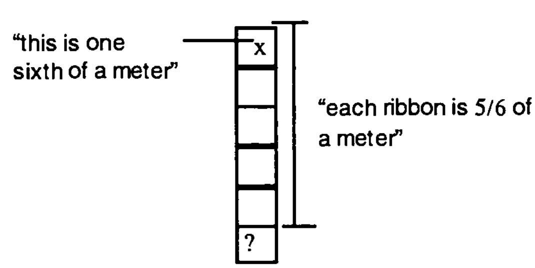
Kevin's oral and written statements *do* describe the process he went through to solve the problem but obscure the issue of the changing unit. Some of his equations are conventionally correct, e.g., 1/6 + 4/6 = 5/6. However, his statement, 5/6 = 1, for example, transgresses rules of conventional notation.

> TBI Visitor: I thought 6/6 was 1. How can 5/6 be 1?
>
> Kevin: They are two different things.
>
> Elizabeth: 5/6 is 1 ribbon. 6/6 is just 1.
>
> Kathleen: 5/6 of the material is 1 ribbon.

Elizabeth and Kathleen were apparently able to make sense of Kevin's claims by referring back to the problem context and so didn't recognize inconsistencies in his statements. Irene went to the board to explain the situation.

> Irene: 5/6 is how much you need to make a ribbon. It's not a whole meter.

She drew the following diagram and read her labels aloud:



The class seemed to accept Irene's labels. However, a disagreement broke out concerning what to call the "extra" piece, labeled here with a question mark.

Students: It's 1/5 of a ribbon.

No. 1/6 of a meter.

1/6. It's just 1/6, period.

Theresa recognized the significance of this disagreement and invited a few minutes of small-group discussion about what to call that extra piece. While her students debated, Theresa prepared an impromptu demonstration. She pulled out a long strip of paper, which she now held in front of her and called the class to attention.

Theresa: This is a meter. Okay?

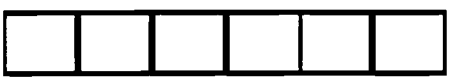
The students realized what she was doing and joined in by calling out directions to her.

Students: That is one meter.

Now make it in half.

Fold up each piece.

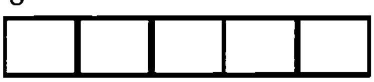
Theresa folded the paper in half and then folded each half into thirds to make sixths:



Students: Now it's in sixths.

Fold one piece back.

Theresa folded back one section of the paper so that the class could now see only the five pieces making one ribbon:



Theresa: So how long is this?

Students: 5/6 of a meter.

Theresa: And this is one ribbon?

Students: Yes.

After establishing that they understood the model, Theresa returned to the original question.

Theresa: What name can I give this section then?

Nancy: The pieces are still 1/6. There are five of them but they are sixths.

Theresa: What part of the hair ribbon am I holding onto?

Brenda: One-sixth.

Both Nancy and Brenda were still considering the whole meter as the unit, but Theresa wanted to shake up that conviction.

Theresa: What if I brought this ribbon to the teacher next door?

Irene: Well it is a fifth if you didn't tell her. I mean, she'd think it was a fifth if you didn't tell her how we made it.

Theresa: So she would think this section is a fifth?

Frances: You are holding a fifth. There are five equal pieces. That is what it would look like to her.

Brenda: Yeah, but it really is a sixth.

Irene returned to the board and wrote, "One ribbon 5/6 meter. One meter 6/6."

Nathan: One whole hair ribbon is not one whole meter.

Charles: A meter is a hair ribbon and a sixth—no—a fifth of a hair ribbon.

Theresa: For homework tonight I want you to write about this. What names can you give to this section?

By the end of the lesson, Nancy and Brenda were adamant that you must call the section 1/6, while Nathan, Charles, Frances, and Irene were willing to consider that it might depend on which unit you are thinking about, the ribbon or the meter. Theresa would learn from their writing where other students stood on the debate.

In an after-class conversation with the visiting TBI staff member, Theresa said the lesson had taken an unexpected turn. She had been hoping the ribbon problem would lead to an exploration of division algorithms and was encouraged by Charles' remarks near the end of class.

Charles was really close to saying that a meter was one-and-one-fifth of a ribbon. This is what happens in division, first one thing is one and then another thing is one.[5]

She thought she could use Charles's idea to help the class see that both $6\ 2/3 \times 6/5$, as well as $6\ 2/3 \div 5/6$, described the solution.

However, Theresa had recognized that she needed to set aside her agenda because the students were working on a different, although closely related, issue: how to label fractional pieces. Comments from Nancy, Kevin, and Brenda made it clear that the class was working hard to make sense of fraction names. Algorithms for dividing fractions would have to wait.

From Mathematics Student to Mathematics Teacher

Initially, readers of our description of Theresa's lesson might be struck by how little was heard from the teacher! Can one even assess a teacher's mathematical understandings when so much attention is focused on her students? In fact, this episode reveals a great deal about Theresa's mathematics understandings; understandings, furthermore, that were developed in the context of her professional development work.

By Theresa's own account, her mathematics teaching has been changing dramatically. She used to teach the way she was taught in sixth grade herself, moving page by page through the textbook, explaining the procedures the children were to learn. It was a result of being a mathematics learner in a new kind of classroom that Theresa began to develop a sense of enlarged possibilities—for mathematics, for herself, and for her students.

A major component of the TBI project involves teachers as mathematics learners in lessons designed by project staff. From as few as four to as many as ten 1- to 3-hour sessions are committed to activities in each of the following areas: number systems and whole-number operations, data analysis, geometry, fractions and decimals, ratios, combinatorics, probability, variables, and functions. Even as these topics are listed, it must be noted that such traditional labels do not convey the interconnections and relationships between topics that are highlighted in TBI lessons. Once a mathematical area is introduced, it is never left behind, but rather is invoked again and again in other contexts. For example, a unit was set up to allow teachers to explore the concept of ratio in the context of geometric shape. Later, learnings from these lessons were recalled and elaborated in TBI's work on probability, data analysis, and functions.

A mathematics lesson in TBI generally begins with the introduction of a task, often (especially for explorations of arithmetic), though not always, in the form of word problems. As teachers work in small groups, staff members listen in, sometimes posing additional questions to probe and stimulate teacher thinking. Staff members then bring the group together to discuss the task and to use the various groups' findings to develop the mathematical ideas further.

Alhough these lessons address the content of elementary-school mathematics, they are directed toward mathematical issues that challenge adult learners. Through their work in groups, teachers can identify their past understandings and confusions, pose further questions for themselves and their colleagues, and reach toward new levels of understanding; through their journals, they can continue to explore their ideas individually and communicate them on a one-on-one basis with a member of project staff.

For many teachers, participating in these mathematics lessons calls into question their past conceptions of the very nature of mathematics. When they are challenged at their own levels of mathematics competence and confronted with mathematical concepts and problems they never before encountered, they experience mathematics, often for the first time, as an activity of construction, rather than as a finished body of results to be accepted, accumulated, and reproduced. Thus, as participants explore mathematics content, the mathematics lessons themselves provide grist for reflection: What is the nature of the mathematics they are learning? How is mathematical validity assessed? What does it mean to "think mathematically"?

Furthermore, within the context of these mathematics lessons, many teachers become aware, for the first time, that they, themselves, can initiate mathematical thought—that they can offer conjectures, test them, become curious about a mathematical question, and make their own way through problems they did not expect to be able to solve. Again, their own experience leads to further reflection, now to consider the process of learning: What was their experience of putting these ideas together? Can they trace their lines of thought? What were the emotions involved? What aspects of the classroom environment promoted or inhibited your learning?

It is in such a setting that Theresa began to develop a sense that doing mathematics is a social process. Recognizing herself and her students as initiators of mathematical thought, she realized that she, too, could organize her own classroom around communal inquiry into mathematical ideas. She also knew that, in order to do so, her students would need her help to see mathematics as a social enterprise. Indeed, half-way through the year she observed, "Creating a cohesive group of students who listen to each other and work well with each other has taken time."[6]

As Theresa became aware of the power of dialogue in making sense of mathematics, and of the possibility that limited understanding might lurk behind correct solutions, she developed classroom procedures that would make discussion of her students' methods—and of the conceptual issues raised by those methods—the central experience of her mathematics lessons.

In exercising these classroom processes, Theresa was making herself vulnerable in the way Anne Marie O'Reilly's vignette illustrates. By organizing her practice around her students' mathematical dialogue, she could no longer predict when a student might express an idea that would derail her lesson.

Indeed, it is not difficult to imagine how a teacher who, having given her class the fabulous-fancy-hair-ribbon problem to work on, might be stymied by the debate over what to call Irene's extra piece—"It's 1/5 of a ribbon. No, 1/6 of a meter. 1/6, it's just 1/6, period." Nor is it difficult to imagine that that teacher, flustered by her own confusion, would quickly move to end the discussion. In Theresa's case, however, her own explorations of fractions in TBI anticipated just this debate among her students, and so allowed her to help them work through their conceptual confusions, rather than abandoning the lesson.

When TBI began its unit on fractions, the group had already been through an intensive two-week institute and one semester of bi-weekly seminars. During that time, a culture was developed in which teachers understood that in their mathematical explorations, they would be given problems *without* first being shown how to solve prototypes; it would be their task to think them through together, in small groups. They had also come to see that inquiry does not end with a correct solution—one can always dig deeper, make further connections. And they discovered that learning mathematics—doing mathematics—involved posing their own questions to get at the "meat" of the conceptual issues. While individual teachers differed in the degree to which they rose to the challenges, these were the mores that were coming to define the group's work together.

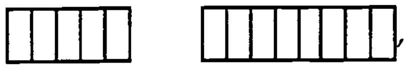
On February 3, 1994, the TBI group began a unit on fractions that continued in meetings on February 17, March 10, and March 24. In each of these sessions, the group explored fractions for one to two hours.

When staff designed these sessions, it was assumed that most teachers' experiences with fractions had been limited to memorizing algorithms for computation. (Although some teachers already understood fractions more deeply, the assumption that most did not was grounded in a decade's work with teachers.) Thus, lessons were structured to help participants develop tools that would allow them to draw on their own powers of reasoning as they confronted new issues.

Most work on fractions involved exploring arithmetic models of situations from daily life. By examining the quantitative relationships instantiated in familiar contexts, learners could pursue deeper mathematical insights. The lessons also emphasized representing fractions and their operations with area or linear diagrams and manipulatives. Staff expected that, through these activities, the teachers would begin to pose their own questions, make new connections, and, when confronted with surprising results, would work through them.

An underlying theme in every lesson involved keeping track of units. Work with children as well as adults has shown that, as learners move from the realm of whole numbers to that of fractions, the role of *unit* takes on greater complexity (cf. Hiebert & Behr, 1988; Lamon, 1993; Mack, 1993). "How can that piece of cake be 1/2 and 1/4 at the same time?" is the kind of question frequently heard. Hence, it was important for teachers repeatedly to confront apparently paradoxical results that could only be resolved by sorting out the whole to which the fractions refer.

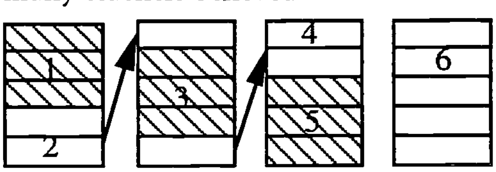
For example, at the beginning of the fractions unit, when teachers were asked to draw area models to compare 3/5 and 3/7, many came up with a representation,



which appears to show that 3/5 = 3/7. They were given the opportunity to explore the implications of such a result in order to grasp why comparison of fractions presupposes that they have the same reference whole.

In the next lesson, teachers were asked to consider why the following problem is not an addition problem: *In Maureen's class, 1/5 of the boys are absent and 2/5 of the girls are absent. What fraction of the class is absent?* After all, the analogous problem, *Find the number of absent children in the class if 1 boy and 2 girls are absent,* is solved by addition. Confronted with these questions, the group needed to think through why, when adding or subtracting fractions, the reference whole must stay constant.

And in a later session, they were asked to solve such problems as, *Wanda really likes cake. She has decided that a serving should be 3/5 of a cake. If she orders four cakes, how many servings can she make?* When they first diagrammed a solution, many teachers believed the answer to be 6 2/5.



2/3 or 2/5?

However, the conventional algorithm for solving 4 ÷ 3/5 yielded 6 2/3. Now teachers needed to consider the different units used for the dividend, the divisor, and the quotient. As teachers worked through these problems, they learned that a great deal of confusion can be resolved by asking the question: What is the unit to which this fraction refers? (Activity sheets from which these problems were drawn are included in the Appendix.)

Although questions about units were raised in each session, they were not the exclusive focus. Among the other issues the group worked on were, Why define fractions in terms of equal parts? What is *really* going on with equivalent fractions? Why is "1/3 of 15" a multiplication problem when the use of division can solve it? How can it be that a single problem (*I eat 2/3 of a cup of cottage cheese each day for lunch. I have 2 and 2/3 cups of cottage cheese in my refrigerator. How long will that last?*) can be solved using addition, subtraction, multiplication, or division?

Like many of her colleagues, Theresa came to the seminar sessions on fractions able to perform traditional algorithms. At the same time, she was aware that, while she knew *what* to do to compute with fractions, she did not understand the mathematics behind the sequence of steps. Throughout the seminar she worked to make sense of fractions and operations with fractions in a way that was new to her. In particular, she found that using diagrams to represent problem situations helped her to develop a sense of what each operation involved.

> I wasn't taught math like this, drawing pictures and visualizing and making models. Now when I think about this work, I see it inside my head. I have a picture. That picture is very powerful for me. [I'm] developing an internal sense of what is going on.[7]

For example, the seminar problem about Wanda's cakes (see above) provided Theresa the opportunity to examine the mathematics embedded in a procedure that was familiar to her:

> For the first time ever I was able to know where that inverted number in "invert and multiply" came from.... When I did repeated subtraction I got an answer of 6 and 2/5 left over. When I did division [applied the invert and multiply algorithm] I got an answer of 6 and 2/3 left over. Every time I looked at the picture to try to explain the different answers I could justify them both. That didn't seem to make sense. And then I saw it!
>
> When I do the repeated subtraction (reality might be cutting the cake for each person that came up and asked for cake) the 2/5 left over is *2/5 of the whole cake* left over and that is right. When I do division (real life—cut all the cake and make all the portions at one time) the 2/3 left over is *2/3 of a serving of cake*. When I started looking at servings vs. left over cake I then could see the 5/3 (4 divided by 3/5 becomes 4 multiplied by 5/3). The 5/3 is the number of servings in a cake! [Theresa's emphasis.][8]

This journal entry shows Theresa able not only to justify why 4 × 5/3 produces the same answer as 4 ÷ 3/5, but also to identify actions that would support each interpretation—what she calls "real life" scenarios. Readers can see the mathematical ideas that Theresa was working on as well as the depth of understanding that she was seeking. Until she could articulate how both expressions (4 × 5/3 and 4 ÷ 3/5) were represented in a single diagram and connect each arithmetic statement with a set of physical actions, she was dissatisfied with her level of understanding.

> It has only taken me how many years to get that invert and multiply thing???? And that is a perfect example of fleeting knowledge and also the fact that it did not impair my ability to DO invert and multiply. I could DO the procedure. This is where the monumental philosophical differences about what it is to "DO mathematics" comes in. I know that there is a difference between doing the invert and multiply procedure and knowing what all the fractions mean.[9]

To develop this kind of understanding, to develop an "internal sense of what is going on," now became Theresa's goal for her students—and she employed strategies learned in the seminar to that end:

> Insisting that problems be solved using manipulatives, pictures and verbal explanations has been painstaking work. With these things in place, as it seems to be now, everything flows so smoothly.[10]

Theresa's objective for the lesson presented in this paper was to begin work on division of fractions with the goal of thinking through the logic of the division algorithm. Thus, she gave her students a word problem with the assignment to come up with a pictorial representation as well as a number sentence.

At the same time, she realized that a major component of understanding fractions is noticing the unit to which the fraction refers and she worked to be sure her problems would help her students engage with this idea. When asked how she chose the problems she assigned, she reflected, "I have a sense that they [the problems] will promote good discussion. That they would bring up the changing whole."[11]

Indeed, we can see from this lesson in December that her students had learned to work with word problems and to create diagrams, as well as conventional arithmetic expressions, to use in exploring the mathematics. Following the students' thinking in this lesson reveals that they were struggling with the conceptual issue Theresa identified: Given a single meter, 5/6 of which is needed to make a ribbon, what is the name of the left-over portion—1/5 or 1/6? While some students could see that it is both—1/5 of a ribbon and 1/6 of a meter—many students could not, feeling sure that it must be either one or the other.

Having wrestled with this issue herself, Theresa recognized the significance of this disagreement and invited the children to talk at their tables for a few minutes about what to call the extra piece. As her students debated, Theresa prepared an impromptu demonstration, pulling out a long strip of paper which she held in front of her and called to their attention. Her demonstration did not resolve the question for her students, but it did clarify what the question was. Nor did she expect that her class would find resolution quickly:

> [The seminar] allowed me to understand that it [takes time] . . . to put this idea together—the changing whole and the division of fractions. It comes up again and again and again.[12]

Theresa began her fractions unit aware that her students would have to revisit the idea of the changing whole several times before they would feel comfortable with it. Even while she sought to develop the connection between problem-situation and division-of-fractions algorithms, she was able to register that her students needed work on a prior idea—for, each time she posed a problem which ought to have led to a discussion of computation, they engaged, instead, in debate about labeling fractional pieces. Theresa's acknowledgment of the time it took *her* to put these ideas together, and her knowledge of the relatedness of these ideas, made such class discussions possible.

Thus, Theresa's actions in this unit show how she called upon new beliefs about the nature of mathematics and mathematical engagement, and new understandings of specific content and how it is learned. In her meeting with a TBI staff member after this class, Theresa reflected on her students' experience:

> It reminded me of what I went through with the cake problem. Because of the changing whole. How can one object have two names? How can one piece be one-fifth and one-sixth at the same time?[13]

Then she added that she planned to continue the discussion the next day, this time in the context of Wanda's cake problem.

Investigating Student Thinking

Theresa's case demonstrates how a teacher's own work as a mathematics learner opens up new possibilities for her. Specifically, her mathematical studies help her to understand how she can organize her teaching around communal inquiry into mathematical ideas. Her studies influence her goals for student learning and the methods she employs to reach those goals, and allow her to recognize the complex mathematical issues her students need to confront. These turned out to be the same issues she needed to confront and work through herself.

However, other aspects of Theresa's practice can also be identified that are less likely to be fostered by mathematics lessons led by project staff—the skills of listening to students' words, interpreting the mathematical ideas they express, and identifying the relationship between student thinking and the mathematics on her agenda. Yet, precisely because her students' confusions so closely matched those that Theresa, herself, needed to wrestle with, it is difficult to perceive these other skills she must exercise. For this purpose, a visit to Jaimie's classroom is more revealing.

Third- and Fourth-Graders Confront a Question about Fractions

Jaimie teaches a combined third- and fourth-grade class in an independent school in the suburbs of Boston. For the last two years, she has organized her mathematics teaching around what she calls the "question of the day" (or "qod"). As the children enter the classroom first thing in the morning, they read the qod written on the white board and, settling in for the day, get out their notebooks. One finds the children scattered throughout the room—lying on the floor, sitting at one of several tables, or slouched on the sofa—working on the problem. Some work together, some work alone—as they choose. The children feel free to ask one another or any adults in the room to help them think it through.

When Jaimie judges that the children have had enough time to get into the problem, she calls them together for a group discussion. Children and teacher sit in a circle on the floor to share their thoughts.

In addition to the qod, Jaimie employs other devices to support the development of her students' mathematics. The children are given individual weekly assignment packages (for mathematics as well as other subjects), which they work through during daily "independent" periods. At times she organizes small-group problem solving, each group working on a set of problems particularly selected to challenge them at their level.

To get a sense of what Jaimie values in her students' mathematical engagement, consider the following excerpt from her journal:

> I . . . watch Betsy and Evan while they work and note the tremendous amount of attention they give their work, the extraordinary ability they have to lean into the discomfort of thinking. There are a number of children in the class who exhibit this same power. . . . These children are successful thinkers. . . . It is significant to me that persistence is so crucial to creative and deep thinking in math. Look at Betsy's "question" for the second part of her qod. She spent nearly 40 minutes writing this question. There are a lot of ingredients to her final product: persistence, facility with language, freedom of time, a way of knowing (and I don't really know what I mean by that except to say that there is an inner dialogue that goes on for these children; there is someone inside them who communicates with their questioning self) and of course a fundamental notion that number is intrinsically connected with language and idea.[14]

Jaimie's own recent work as a mathematics student has taught *her* the value of "leaning into the discomfort of thinking." She, too, has persisted—at times through tears—learning to tolerate the frustration of confusion when seeking clarity. She now thinks of mathematics as ideas that can be deepened and extended as she engages with her colleagues or even her own "questioning self." Having brought this knowledge of what mathematics can be into her own teaching, she now recognizes the power of her students' mathematical processes.

Jaimie began working on fractions with her students in March, while she was studying fractions in the TBI seminar. Although she had taught this age group the previous year, her new perspectives on mathematics made her feel as if she were venturing into uncharted territory. What did her students know about fractions and what did they need to learn? These were the questions she posed to herself as she opened up the topic to her class. They were not questions she expected a textbook or resource guide to raise. Rather, she was looking for the major conceptual issues that these children needed to work through in order to develop as mathematical thinkers, and she expected to find them by listening to her students. A few days into the unit, she reflected in her journal:

> They have been "doing fractions" since they started sharing things and it has been a minimal but integral part of conversations for a long time—since two years old maybe. But I am finding that they know next to nothing about how to express their ideas about smaller than one. They haven't any information about fractions smaller than 1/2 or 1/4 for the most part and so all this stuff is brand new and intriguing to me.[15]

The following dialogue occurred shortly after Jaimie's class initiated their fractions unit, in the middle of a discussion on a day a TBI staff member was visiting.[16] Although the lesson began, as usual, with the qod, the whole-group discussion quickly ventured away from its particulars:

> Jonathan: I sometimes don't understand the word "fraction."
>
> Jaimie: Yeah.
>
> Jonathan: I think it seems to be hard to explain.
>
> Jaimie: Because you've asked people and they haven't been able to give you a very clear idea of it?
>
> Jonathan: Right.
>
> Jaimie: Let's ask everybody. Betsy, what is a fraction? You were going to give a definition. And Kyle, maybe if you think about this, you might want to contribute also to what is a fraction. Betsy?
>
> Betsy: Well, I think of a fraction as less than a whole. It isn't a whole; it isn't a half, and it isn't a quarter, usually.

The discussion continued, with various thoughts about what fractions are. For example,

> Jonathan: Sometimes the way I understand the word "fraction" is "different parts." Like two fractions would be a half. Four fractions would be a quarter. . . .
>
> Kyle: Well, if you think about a fraction—anyway, it's not a whole but it's something *of* a whole. Let's say it's one fourth of 20. Well, it's 5, because you have 5, 10, 15, 20. And you're dividing 20 into 4, and [it's] one of those sections, one of those groups. . . .

Harriet got out some cubes and arranged six of them into three groups of two.

> Harriet: A fraction is when things are split up into even groups. So I've now arranged them into three groups. So you know what this fraction would be called?
>
> Jonathan: Three fractions?
>
> Harriet: No. It would be called a third. . . .
>
> Jonathan: Why are they called a third?
>
> Harriet: Because there are three of them.
>
> Jonathan: Why do you call it a third? I mean, like, uh, it could just as well be three quarters.

Still later in the discussion, Jonathan posed his question another way:

> Jonathan: The question I'm asking is, if things are a fraction, why don't they say the thing instead of a fraction?

The visiting TBI staff member listened to the conversation, along with Jaimie, both of them trying to sort out different students' understandings. And along with Jaimie, she was working hard to figure out just what Jonathan was asking. Some minutes later, after the class had returned to the qod to consider a proposed answer, 14/24, the staff member offered a question:

> TBI visitor: The fraction was 14/24. And so the question is, sort of, what's the difference between the answer, 14, and the answer, 14/24?
>
> Jonathan: Right.

In the consultation between Jaimie and the staff member after the lesson, Jaimie began to identify the idea her students needed to "wrap around":

> What a new place it is for them to think of a fraction as an idea. Which is why when you were talking about, towards the end of our meeting, about the 2 and the 1/3 [or 14 and 14/24] . . . —for these kids to sort of wrap around that thought, that it's an idea—[that a fraction is] a really different thing than whole number.[17]

Most of the children in the class still needed to sort out the notion that their classmate, Kyle, had begun to articulate ("It's not a whole but it's something *of* a whole"). Given Harriet's arrangement of 6 cubes into 3 groups of 2, the children needed to think of the 2 cubes not only as *2*, but also as *2 in relation to* the 6. And at the same time that each cube is a unit, so do the 6 cubes make up a unit, and the 2 cubes make up 2/6 or 1/3 of that unit or that one whole.

Recognizing the importance of this issue, Jaimie decided to structure her qods specifically to point the class toward it. When the staff member returned a few days later, Jaimie had presented her students with the following problem:[18]

> *Maria, Angelo, and Christina were sharing a giant round tortilla covered with queso (cheese) and salsa. The tortilla was divided into 12 slices. Write a question for which the answer is 3. Write a question for which the answer is 3/12.*
>
> **Bonus: Write a question for which the answer is 6/24.*

As children worked on their own, the staff member heard Jonathan ask a classmate, "What is 3/12? What does it mean?" And he brought the question up again in the whole-group discussion:

> Jonathan: I still don't get what 3/12 is.
>
> Jaimie: Right. . . But you know what, Jonathan? I read a lot of questions that people wrote in response to this exact problem—when do you get to call it "3" and when do you get to call it "3/12"? And there are a large number of people in here who have the exact same question, but haven't figured out how to phrase it just the way you did. Which is, "I don't understand where '3/12' comes from. I don't know why you name it '3/12' as opposed to '3.'"
>
> Jonathan: Yeah.
>
> Jaimie: Okay. This will continue to be a question for you. And I want you to know that it is a question for a lot of people in here.

Before continuing the discussion of fractions, Jaimie asked the children to share the questions they wrote whose answer is 3. Very quickly, however, their attention turned back to the meaning of 3/12.

> Adam: 3/12 is 3 out of 12, not just 3. 3 is not like out of anything. It's just a whole number.
>
> Jaimie: So you could . . . open the refrigerator door. And you can find 3 oranges in there. But if you went to the market and you bought a dozen oranges and you ate 3 of them on the way home, that would be 3 out of 12 as opposed to just 3 oranges?
>
> Adam: Yep.
>
> Carla: 3/12.
>
> Jaimie: That would be 3/12?
>
> Joan: If you split them up, that's a fraction.
>
> Jaimie: If you don't have to split them up you leave them as whole numbers?
>
> Jonathan: But if you take it from the market and you buy it, and you have 3 of them, that's 3 oranges, and there could be more oranges at the market.
>
> Fred: So that could be 3 out of 12. . . .
>
> Harriet: You took 12 from the market and you eat 3 out of those 12; it's 3/12.
>
> Carla: The 12 that you buy is the whole number of oranges. And then the 3 out of the 12 oranges is the fraction. Say you bought the 12 oranges at the market. So you've got 12 oranges. And on the way home you eat 3 of them. Now erase from your mind all the other oranges in the market.
>
> Jonathan: That would be zero.
>
> Carla: No. . . . You keep the ones in the car that you have, but erase from your mind the others. They don't mean anything. Then you eat the 3 from the 12 and the other oranges at the market don't have anything to do with what you need to find out.
>
> Jonathan: But that still doesn't answer my question.
>
> Jaimie: You know what I would like—something that I noticed, Jonathan, about what you're talking about. Is that the 3 oranges that you eat are the same 3 oranges that you eat whether there's 3 of them or 3/12 of them.

> Jonathan: Yeah.
>
> Jaimie: They're the same oranges. And so we're talking about the next level of how to describe them. But you're absolutely right. There's no difference between 3 oranges and 3 oranges.

In the days between the staff member's visits, the children's ideas about fractions had continued to develop. Different components of the concept were coming together for different children, and the explanations offered about what fractions *are* were becoming more sophisticated. However, Jonathan, for one, still seemed stuck on the same question: Why call it "3/12" when you can see it's 3?

Jaimie appreciated Jonathan's willingness to be public with his question and to try to articulate his confusion. For it was with Jonathan's help that she was able to identify a major conceptual hurdle her students needed to work through. In order to track the development of various students' conceptions, she decided to write a set of qods similar to the "giant round tortilla" question (e.g., *Ana and Jose were sharing a pizza. The pizza was divided into 8 slices. #1) Write a question for which the answer is 2. #2) Write a question for which the answer is 2/8. Challenge: #3) Write a question for which the answer is 4/16*). She gave the class different versions of the same problem several days apart and sometimes collected the children's written work, carefully studying their responses in order to understand just what they were thinking; other days she listened to the whole-group discussion to follow how the children's ideas were developing. After several weeks, she shared her observations:

> [There's an idea that's] sort of floating through the classroom. [There are] kids who weren't even really aware that [Jonathan's question] was a question. . . . Last week it was Keith and Harriet and Maura who hadn't formulated the idea that 2 and 2/8 would be different ideas. . . . My understanding is that Maura is probably much clearer on the defining factors of those things. Carla is certainly much clearer. And Adam knows what's going on, and there are a few other children who are. And now there's this whole other crop of kids who have decided that 2 and 2/8 are the same. It's sort of like they're a week behind. They didn't really have even enough going to make a decision about it last week, to even really make a decision about the fact that 2 and 2/8 were the same or weren't the same. And Andy is one of them and I think he's now saying, the question is good enough for both of those answers. And so he's sort of a week behind the last crew, but it's really interesting to me to note that he wasn't even aware of this dilemma at all last week, that Jonathan was way ahead of him in even asking the question. Paul hasn't even gotten to this decision that they're the same thing yet. . . . It's like there's this whole group of kids who are sort of having the same community conversation and yet they are definitely coming at it from whatever they have in their own little suitcase and hearing pieces of it and not hearing other pieces of it and coming the following week to this same thing.[19]

While Jaimie had identified a major conceptual issue her students needed to work through, she knew that she could not understand it *for* them. She *could* work to create situations in which the children would have opportunities to confront it. As she provided such opportunities, she watched closely to see which children seemed to have moved through that issue; which children were turning to face it, but hadn't yet found their way through; and which didn't even realize it was there. Her classroom environment was respectful of the children wherever they happened to be, but she saw it as *her* challenge to keep them all actively thinking and learning, building stronger mathematical conceptions.

Learning to Listen

As with Theresa, so, too, did Jaimie bring to her classroom a new understanding of what it means to learn and to teach mathematics. For, while the notion that mathematics is about ideas was not new to Jaimie, the possibility of teaching to mathematical ideas was. In fact, one year into TBI, she came to view engagement with ideas as the real goal of her mathematics teaching:

> I'm convinced that what was missing from the qod last year was talking about numbers as a whole way of [thinking]. . . . This year has much more to do with the behavior of numbers and . . . last year it had to do with answers to the questions. . . . To engage in strategy alone—which is, I think, how I thought of math and the qod [last year]. . . that's like a dead end. I mean, it's certainly worth doing and kids may try somebody else's, but at some point there's

> not much there. . . . And so I think to have something to really mull over—a dilemma—is really much more interesting.[20]

In her own work as a mathematics student, Jaimie had already learned that mathematics is not merely a set of procedures to be memorized and mechanically applied. Rather, she—and her students—could confront problems never seen before and apply their ingenuity to work them through. Furthermore, she was learning that mathematics was more than solving novel problems and sharing solution strategies. Instead, the problems had become a means to exploration of mathematical concepts.

However, the possibility of engaging children in an exploration of ideas presupposes being able to identify the concepts that are at issue for them. But this is no simple task, since their ideas are necessarily expressed in the words of young beginners. Furthermore, in contrast to Theresa's lesson, the issues that Jaimie's students needed to work through had been faced, in one way or another, by their teacher when *she* was a child—and had not been called to mind since.

As Jonathan presented his confusions—"Why are they called a third?" "Why don't they say the thing instead of a fraction?" "I still don't get what 3/12 is"—Anne Marie O'Reilly's predicament is again called to mind. For O'Reilly, David's surprising interpretation of the fraction 1/2 was a lesson-stopper. Today (four years later), O'Reilly is prepared—as was Jaimie with Jonathan's puzzlement over the very justification for there being fractions at all—to dig under the children's words to find the sense in their perplexities.

Retrieving a child's perspective on a concept, many years after one's own initial acquaintance with that concept, can offer powerful mathematical insight. Aspects of an idea, perhaps long since buried, perhaps never before noted, can come into view.

For example, adults are generally unable to recall the time when their concept of "number" was confined to the experience of counting whole numbers. Yet, listening to children being introduced to the idea of "fraction" and realizing how these children's very notion of "number" is now being challenged, offers adults an opportunity to think through how the concept of "number" expands as one moves from the domain of whole numbers to that of fractions. It is no longer merely a matter of counting units. Instead, one must now count the number of units in one quantity, count the number of units in a second quantity, and derive a third number—a new *kind* of number—that indicates the first quantity *in relation to* the second.

Though most adults, including most elementary teachers, learned years ago how to operate with fractions, few developed an articulated sense of the conceptual distinctions between whole numbers and fractions—something that Jaimie's students were now working toward. Thus, attempts by Jaimie and her TBI colleagues to make sense of Jonathan's question—"Why don't they say the thing instead of a fraction?"—and to come up with answers for themselves, deepened their own understandings of what fractions are and how they differ from whole numbers.

An important part of the work of the TBI seminar involved learning how to use children's perplexities to make such deep conceptual issues visible to both teachers and students.

The TBI teachers began their investigations into children's mathematical thinking by analyzing other teachers' students, studying videotapes of clinical interviews and classroom discourse as well as written materials illustrating student work. These analyses were supplemented with readings of assigned research articles.

For example, during the fractions unit, the group viewed a videotape from Deborah Ball's classroom, depicting her third-grade class discussing which is larger, 4/4 or 4/8. Accompanying documentation included excerpts both from notebooks in which students reflected on their classmates' conjectures and arguments, as well as from their teacher's journal (Ball, 1990). Using this rich set of materials, teachers analyzed how Ball's eight- and nine-year-olds confronted the reference whole of a fraction.

At the same time, assigned readings raised issues about children's work with mathematics in classroom contexts. "Magical Hopes: Manipulatives and the Reform of Math Education," by Deborah Ball (1992), challenges the commonly held belief that manipulatives provide *the* key to promoting students' mathematical understanding. To make the point, Ball describes how Jerome puzzles over fraction bars to address his teacher's question, "Which is more—three-thirds or five-fifths?" "Jerome is struggling to figure out what he should pay attention to about the fraction models—is it the number of pieces that are shaded? the size of the pieces that are shaded? how much of the bar is shaded? the length of the bar itself?" (p. 17). "Halves, Pieces, and Twoths: Constructing and Using Representational Contexts in Teaching Fractions," also by Ball (1993), first offers a short overview of the research literature and how it connects to planning a third-grade unit on fractions, and then takes readers into her third-grade classroom to explore how representational contexts are co-constructed by members of the class and their teacher. In "Of-ing Fractions," Joanne Moynahan (1996) describes her sixth-graders' attempts to decide which arithmetic sentences model such problems as, "1/3 of the 15 people at the picnic were Davises. How many Davises were at the picnic?" Moynahan's paper goes on to raise further questions about the relationship between a mathematical representation and the situation it models, as well as about what constitutes a valid argument.

However, the teaching practice that TBI was working to help these teachers construct requires facility that goes beyond understanding basic principles of children's mathematical thinking *in general*. Rather, it involves developing a new ear, one that is attuned to hearing the mathematical ideas of *one's own* students. This latter point is expressed in the words of teachers as they began to listen to their students in new ways:

> I listened for right answers, confirmation that the students understood what they had been taught. I was accustomed to listening for specific indicators that a student was following my line of thinking.
>
> It is almost as if I heard them previously, but I had my next statements already planned. I attempted to adjust their thinking to what I planned to say next, instead of analyzing what they said to determine what I should ask, say, or do next.

In a practice that puts students' ideas at the center of instruction, teachers now listen to students not only to assess the extent to which those students' ideas match their own, but also to understand these ideas in their own right. What is it that this child is communicating? What is the sense in this child's idea? What does he/she understand? What is the child confused about? What is the issue that the child is working on?

> I'm [becoming] able to see how individual kids are thinking and see what concepts are troublesome for kids to make sense of. . . . I feel like I'm getting more skilled at finding out what kids do "get" rather than just thinking "they don't get it."

Within TBI, teachers' own classrooms became a major resource for learning about student thinking. One mechanism developed for such investigations is "episode writing." Twice in the first year, and as a regular monthly assignment in the second and third, teachers wrote scenarios—episodes—from their own teaching. The assignment was to write a 2- to 5-page narrative that captured some aspect of the mathematical thinking of one or more students, using transcriptions of classroom dialogue or samples of students' written work.

TBI teachers have described what they learned from this writing assignment:

> I find that sometimes I make connections—between two kids' thinking, or two things one kid said, or between what happens on two different days—that I might not make if I wasn't spending the time and effort to write episodes. I also appreciate the chance to think about the math and the kids with permission to sometimes ignore myself. What did the kids say/do? Not, what should I have said?
>
> Writing the episodes became a window into my students' thinking. I've talked about listening to my students in the past as a personal goal and responsibility of mine, but the episodes forced me to really listen to them. What were their words? What did I make of their explanations? What does it tell me about their mathematical thinking?
>
> Knowing that I will be writing episodes definitely makes me pay much closer attention to what my students are saying and doing. Be-

> cause I know I will be writing about what I think they're thinking, I examine their work and words carefully and turn it over in my mind. I keep better records of their ideas also. I save work that gives me insight into their thinking. ... After a while, I begin to do the whole process automatically, even when I do not have to write.

Theresa herself wrote,

> Writing episodes forces me to listen more carefully. I must understand what students are saying and thinking to be able to write the episode. My writing preserves details that I forget with the passage of time. The episodes capture moments of confusion and deep thinking that have resulted in students becoming more open and articulate.

Equipped with illustrations from their own classrooms, teachers met in small groups during seminar sessions to share and discuss their work, explaining what they saw in one another's episodes, and airing their own questions. Episodes were then read by project staff, who worked to figure out what was revealed about student thinking and wrote responses to highlight *this* for teachers. In general, staff did not comment on the teaching, but posed questions about the ideas the students were working on.

Finally, members of the staff selected sets of episodes to bring back to the whole group in order to look for themes common across the represented classrooms. For example, the seminar might have grouped three episodes, each telling a story about a child, or group of children, working to sort out base-ten relationships while exploring ways to add multi-digit numbers. Or it might have considered a set of episodes in which groups of children are solving problems that involve adding fractions. During the third year of the project, teachers were assigned to focus groups which convened over the course of the semester, to focus episode-writing and discussion on particular topic areas.

In addition to the episodes, staff visits to participants' classrooms provided another mechanism for investigating children's thinking. These visits took place twice each month during the first two years of the project. The *pedagogical* purpose (as opposed to the *research* purpose) of these visits was to provide support to teachers as they worked to develop a classroom culture promoting significant mathematical activity and dialogue. Each meeting consisted of a classroom visit followed by a half-hour, one-on-one discussion. At the beginning of the year, teachers chose their own goals as a starting point for transforming their instruction. During half-hour discussions, visiting staff members did not focus on classroom logistics, but worked to pose questions that would continue the process of reflection: What do you want your students to learn? How will you find out what they already know? What activities will you choose and how will those activities help your students learn? How will you know what they have learned? As teachers became oriented to their new practice, the one-on-one discussions moved toward close study of student thinking: teacher and staff analyzed dialogue and written work together, looking for the big ideas students were confronting.

Jaimie's activity during her fractions unit shows what is involved as teachers learn to listen to their students' mathematical discussions, interpreting their words to identify the issues they are working on, and choosing tasks to support their development in those areas.

At the start, Jaimie was unclear about what her students needed to learn, but she did know it was her task to figure that out as she listened to their discussions. And so she posed questions for the class to work on and listened hard.

In the articles assigned to TBI teachers, Jaimie had read about young children working to make sense of what a fraction is. For example, she read about how Jerome, inspecting that fractions bar, didn't know if he should be paying attention to the number of pieces shaded, the size of the pieces shaded, the amount of the bar that is shaded, or the length of the bar itself (Ball, 1992, p. 17). However, as she listened to her own students, she was not clear about how their questions related to what she had read or, indeed, to her own understanding of fractions.

When one listens to Jonathan's third-grade articulations of his question—"Why do you call it a third? I mean, it could just as well be three quarters," or "Why don't they say the thing instead of a fraction?"—it is not immediately obvious just what he is asking. But by considering his remarks in relation to his classmates' demonstrations and arguments, as well as to readings about children's mathematical thinking, Jaimie, with the help of her TBI visitor, identified the concept he and his classmates might have been struggling with. As the two adults posed further questions to the children, it seemed the difficulty was with the idea that a fraction represents a given quantity *in relation to* another.

Although the children's conceptual issues were not the same as those the teachers in the seminar worked through, Jaimie was reminded of a problem posed to her colleagues: *Jorge has 2 pizzas, one pepperoni and one cheese. Each pizza is cut into 8 slices. He eats 1 slice of the pepperoni and 2 slices of the cheese pizza. Write a word problem about Jorge's pizza eating so that the answer is 3. Write a word problem about Jorge's pizza eating so that the answer is 3/8. Write a word problem about Jorge's pizza eating so that the answer is 3/16.* And so she used variations of this problem as her assessment tool.

By examining written responses to the giant-tortilla problem, Jaimie considered the children individually: which students were struggling with the difference in meaning between 3 and 3/12, which students had largely worked it through, and which students were not yet ready to approach it. And by giving them similar problems over the next few weeks, she could track how their conceptions developed.

After the fractions unit ended, Jaimie remarked that she expected her fractions lessons next year to be better, that it would be much easier "to go deeper into it earlier on for kids. . . . I do think it will have made a difference next year doing fractions, obviously knowing much more about them myself." And when asked about how her learning took place, she said that it was through both doing mathematics herself and analyzing student thinking.

> The point of view of someone who has never been taught how to do all of that stuff really enriches my own thinking about fractions. And of course will show me much more where the stumbling blocks are or where the things to trip over are—more than my own work in math. So I'm thinking for me it's very, very important for both of those things to be happening at the same time.[21]

Particularly important, she said, was the careful analysis she did on the children's written responses to two of her qods. "I needed . . . not just the listening to what they were doing, but I really needed to look at and think about their answers."

CONCLUSIONS

This paper opened with a scene from Anne Marie O'Reilly's classroom. Stumped when one of her students offered an unexpected and puzzling idea, she didn't know how to respond. The next day, she reported, "I decided to put the geoboards aside for a while and have the students try some activities with fraction strips" (O'Reilly, 1996, p. 72).

Teachers who choose to develop a practice like the one to which O'Reilly aspires will almost certainly face similar situations. When teaching is designed to elicit student thinking, there are likely to be surprises. However, as O'Reilly herself has pointed out, professional development projects can help teachers become better prepared to confront them. With a richer understanding of the territory she and her class were exploring, O'Reilly might have recognized other, more productive options than changing the subject. With more experience listening to students in order to sort out the mathematical issues they are working on, she might have been able to pose additional questions to David and his classmates, probing to understand how their thinking connected with conventional meanings for fractions. Indeed, in the four years since O'Reilly originally wrote the anecdote told here, she *has* continued to take advantage of professional development opportunities and reports that she *can* engage her students in mathematical discourse with much greater confidence and fluency.

In this paper, the cases of Theresa and Jaimie have been used to illustrate how teachers call upon learnings from a professional development program—learnings about mathematics and about children's mathematical thinking—as they engage their students in a study of fractions. However, TBI has had the luxury of working intensively with a group of 36 teachers over an extended period of time. This kind of intensive and extensive work has supported exploration of the kinds of mathematical understandings required of teachers working to enact the new pedagogy. As staff visited teachers' classrooms, they were able to identify issues to bring back to the seminar and, reciprocally, to observe how seminar learnings showed up in the classroom. Given that the resources necessary to support this kind of teacher development effort on a large scale are now simply unavailable, how can TBI's results be made relevant to other settings?

At an advisory-board meeting 14 months into TBI, project staff were challenged to reach beyond *reporting* on the group's learning, to develop materials that would help others have discussions like those taking place within the project. At that same meeting, the board suggested that the episodes that were being produced for within-project consumption could serve that purpose. Indeed, as the teachers developed an ear for their students' mathematical thinking, staff recognized the strength of teachers' episodes as a foundation for case discussions among other groups of educators. Furthermore, as *their* episodes began to accumulate, project staff saw that they could be grouped and ordered to track the development of particular mathematical ideas from kindergarten through sixth grade.

The advisory-board meeting initiated a period of experimentation in which project staff explored what kinds of cases are particularly evocative and what kinds of prompts stimulate productive discussion, both of mathematics and of student thinking. This experimentation led to the development of a curriculum for elementary- and middle-school teachers, called *Developing Mathematical Ideas* (DMI). The curriculum is intended to provide one possible context for helping teachers to deepen their mathematical understandings, encouraging them to move back and forth reflectively between the teacher-education seminar and the mathematical experience of their classroom.

At the heart of DMI are sets of classroom episodes that illustrate student thinking as described by their teachers. Through these episodes, the two avenues of approach to K–6 mathematics pursued in TBI—investigations of the mathematics itself, and investigations of children's mathematical thinking—become more closely integrated.

For example, consider excerpts from an episode in which a fifth-grade class is introduced to the expression, $5 \div 39$, and is trying to think through what it could possibly mean:

> Jack: I think that you will end up with a fraction of a number because, well, because, 5 and 39—you can't divide 5 into 39 equally. I think it's going to be a number below 0. . . .
>
> Al: I agree with Anthony but not with Jack. We had some story problems where the answers were decimals but they were not below 0. I think we could say that $39 \div 5$ could be a decimal number. [Al goes to the board and shows how he solved for a quotient of 7.8.] . . .
>
> Darrel: I think 39 can't go into 5. I mean it can go into it, but it's going to be a fraction, it's got to be a fraction. A larger number into a smaller number—5 can go into 39, but there's a remainder. No, it's not a remainder; it's not a number.

Teachers working from this episode think through what the children are saying as they try to make sense of the idea that a smaller number can be divided by a larger one. What is it that each of these children is saying? In what ways are their ideas correct? What are their confusions and errors? Are there ways in which their incorrect ideas might make sense from the perspective of the child? In sorting out these children's perspectives, teachers' own understandings of fractions become enriched. The episode provides teachers the kind of opportunity Jaimie had—namely, to illuminate basic notions that young children must construct, but that are likely to be invisible to adults, who are used to manipulating fractional expressions.

Later in the episode, one pair of children comes up with a context—5 pizzas to be shared among 39 kids—in order to think through the arithmetic of $5 \div 39$, and another offers a diagram—5 rectangles each divided into 8 pieces. Given the context and the diagram, the class poses the following questions:

- What is the last piece called? Is it 1/8 or 1/40?
- What's the "whole"?
- What happens if the last piece is divided into 39 pieces? What if it is divided into 40 pieces?

- Do we know what 5/39 means? Is "slightly more than 1/8" a better answer than 5/39 because it's clearer even though it's less exact? What about 1/8 and 1/39 of 1/8? What about 1/8 and 1/39 of 1/40?
- When we say 1/8, it's 1/8 of what? When we say 1/40, it's 1/40 of what?
- If we cut the last piece into fortieths, each person gets 1/8 and 1/40 of 1/8. What happens to the extra fortieth? Do we keep on dividing it? (This is where we discussed the "piece," the "sliver," and the "crumb.")

As teachers engage with this episode, the children's questions become their own. That is, beginning with a careful reading of the questions these fifth-graders pose, the teachers, too, are led to mathematical inquiry. The episode brings them to explore the same concept that Theresa and her colleagues wrestled with in their seminar.

In addition to case discussions and in order to support teachers' further engagement with mathematics, the DMI curriculum includes mathematics activities to be led by facilitators—for example, exploring problems such as those presented to TBI teachers and described earlier in this paper. The study of children's thinking is supplemented by video cases, an overview of related research, and a series of activities designed to support teachers' inquiries into *their own* students' thinking: sharing and discussing samples of *their* students' work; planning, conducting, and analyzing mathematics interviews of one of *their own* students; and writing their own cases.

While the materials are intended to help teachers work through particular content areas—with the dual focus of engaging in the mathematics for themselves and examining children's mathematical thinking—they are also meant to help teachers develop a disposition to inquiry. The new understandings teachers must develop, and the teaching situations they must negotiate, are too varied, complex, and context-dependent to be anticipated in one or even several courses. Teachers must become investigators—of mathematics and of student thinking—in their own classrooms (Ball, 1996; Heaton, 1996; Featherstone et al., 1993; Russell et al., 1995). As exemplified by Jaimie in this paper, this imperative is not simply a matter of remediation. Rather, it is an aspect of the very nature of the mathematics of the "reformed" classroom. When mathematics is understood as a network of connections among various concepts, their representations, and the contexts in which they are embedded, then no piece of mathematics can ever be fully understood; with any mathematical concept, there are always new ways to look at it and more connections to make. Similarly, while there are patterns in children's engagement with particular mathematical concepts, the individual experiences that a single child or group of children may bring to a particular question are unlimited. Thus, teachers must come not only to expect, but to *seek* situations in their own teaching in which they, themselves, can view the mathematics in new ways, especially through the perspectives that their students bring to the work.

Because pilot tests of the DMI curriculum are underway, several questions are now being investigated: Does this curriculum indeed provide a context in which teachers can deepen their understandings of mathematics content and of children's mathematical thinking—and if so, in what ways? In what ways do teachers learn to pose their own questions and develop a disposition toward inquiry? What changes in mathematics teaching practice can be seen as teachers work through the curriculum? How does this curriculum interact with implementation of innovative curriculum for elementary mathematics classrooms (e.g., *Investigations in Number, Data, and Space*; *Everyday Mathematics*; *Math Land*)? How might DMI be adapted for various contexts and formats or for different audiences? And how do other teacher educators, staff developers, and teacher leaders learn to use it?

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APPENDIX: Worksheets

Worksheet distributed on 2/3/94

"CONSTRUCT YOUR OWN RULES"[22]

I. Using DIAGRAMS, determine which fraction in each pair is the larger:

1. 5/8, 7/8
2. 3/5, 3/7
3. 5/6, 7/8
4. 3/8, 2/9

II. What rules about the relative sizes of fractions can you develop from these examples? You might need to try some additional examples to test and extend your rules. Write your rules in a way you can share them with the class.

III. For each pair of fractions, explain how you can tell which fraction is larger.

Use diagrams as needed. Continue adding, testing, and modifying your rules.

5. 4/5, 4/7
6. 2/5, 1/2
7. 11/12, 21/11
8. 11/12, 13/14
9. 11/24, 17/30
10. 22/35, 11/18
11. 7/8, 13/12
12. 12/17, 11/18

Worksheet distributed on 2/17/94

"ADDITIONAL FRACTION PROBLEMS"

1. Consider the following problem:

In Maureen's class, 1/5 of the boys are absent and 2/5 of the girls are absent. What fraction of the class is absent?

a. *Why doesn't it work to add the fractions in this problem? That is, why isn't the answer 3/5?*
b. *What can you find out about the answer to the question? That is, can you calculate a range of possibilities for the answer?*
c. *What other information would you need to determine the particular answer to the question?*

2. Consider this problem:

Harry has two packs of gum. A five pack of spearmint and a five pack of fruit flavored gum. He eats 2 sticks of the spearmint and one stick of the fruit gum. Harry says he ate:

$$\frac{2}{5} + \frac{1}{5} = \frac{2+1}{5+5} = \frac{3}{10} \text{ of the gum.}$$

a. *Explain this new way to add fractions. When does it work? Why does it work?*
b. *How is this problem similar to number 1? How is it different?*

3. Consider this situation:

Jorge has 2 pizzas, one pepperoni and one cheese. Each pizza is cut into 8 slices. He eats 1 slice of the pepperoni and 2 slices of the cheese pizza.

a. *Write a word problem about Jorge's pizza eating so that the answer is 3.*
b. *Write a word problem about Jorge's pizza eating so that the answer is 3/8.*
c. *Write a word problem about Jorge's pizza eating so that the answer is 3/16*

Worksheet distributed on 3/10/94

"EATING AND DRINKING"

Draw a DIAGRAM to solve each problem. Then solve each one arithmetically.

Explain how your arithmetic matches your diagram OR write the arithmetic you do find in the diagram solution.

1. I eat 2/3 of a cup of cottage cheese each day for lunch. I have 2 and 2/3 cups of cottage cheese in my refrigerator. How long will that last?

2. I put 2 and 2/3 gallons of gas into my empty lawn mower. I noticed that it was 2/3 filled. What is the capacity of my gas tank?

3. Mountain spring water comes in containers which hold 6 and 3/4 liters. Last week we used 2/3 of the container.

a. *How much is left?*
b. *How many glasses (capacity 3/8 liter) can be filled with what remains?*

4. Fourteen sandwiches are to be shared equally among eight people. How much will each person get?

5. Wanda really likes cake. She has decided that a serving should be 3/5 of a cake. If she orders four cakes, how many servings can she make?

a. After 6 portions are eaten, what's left?

b. Why is the answer to the division problem 6 2/3 rather than 6 2/5?

c. Look for a representation of 4 times 5/3 in your diagram.

d. How does this problem explain the "flip and multiply" rule?

Notes

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[2]2/16/95 reflection paper.

[3]12/12/94 observation.

[4]The staff member present, who saw Kevin's gestures and heard his pauses and intonation, interpreted his words as follows: "You have the 6 [meters] and you divide that up and it makes 7 [ribbons]. And you have one [small section] left [from the 6 meters] and [you] put [it] with the [2/3] piece of the meter. [Now it] becomes 8 [ribbons]. 6 2/3 [meters] is how much material you have. It takes 5/6 [meter] to make a ribbon, so 6 2/3 divided by 5/6 tells you how many ribbons."

[5]12/12/94 consultation.

[6]2/10/94 journal entry.

[7]3/12/95 consultation.

[8]3/15/94 journal entry.

[9]3/4/94 journal entry.

[10]2/10/94 journal entry.

[11]3/12/95 consultation.

[12]3/12/95 consultation.

[13]12/12/94 consultation.

[14]4/5/94 journal entry.

[15]3/21/94 journal entry.

[16]3/23/94 observation.

[17]3/23/94 consultation.

[18]4/1/94 observation.

[19]4/13/94 consultation.

[20]4/1/94 consultation.

[21]5/4/94 consultation.

[22]The original version of this worksheet was devised by Martin A. Simon.

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