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ABSTRACT

Research into children's measurement conceptions is discussed, presenting a review of the literature in the field that includes studies conducted by J. Piaget and his colleagues and training programs that have been designed to support the development of measurement skills. This literature is used as a background to argue that new investigations might include accounting for students' development in terms of social and cultural influences, coordinating research analyses with instructional development, and using psychological analyses to develop the instructional intent of further investigations. A teaching experiment that was conducted with 16 first graders is described to highlight a way to support mathematical development in the classroom context. It was clear that in this classroom students' participation in the mathematical processes involved the use of many measurement tools. It is suggested that students' activities with these tools profoundly influenced their development of mathematical reasoning as interviews with the students demonstrated. Students developed ways of acting with the measurement strip and stick they were given that had numerical significance for them, and they were able to be relatively effective with these tools. They demonstrated intellectual capabilities they had developed during the experiment. (Contains 12 figures and 68 references.) (SLD)

# Children's conceptions of measurement: 35 years after Piaget

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## Introduction

The skills involved in measuring play a prominent role in both in and out of school practices. Since measuring lengths of objects are common practices in a wide range of out of school contexts, instruction in linear measurement is written into most elementary curricula. Accordingly, the National Council of Teachers of Mathematics Curriculum and Evaluation Standards for School Mathematics (1989) emphasize the importance of establishing a firm foundation in the basic underlying concepts and skills of measurement. These Standards stress that children need to "understand the attribute to be measured as well as what it means to measure" (p. 51). A large body of literature on children's conceptions of measurement has amassed over the past three decades with undoubtedly the most influential work being that of Jean Piaget and his colleagues (Piaget, Inhelder, Szeminska, 1960). Piaget et al. identified developmental stages that they claimed children pass through as they learn to measure. As a result of Piaget's analysis, many researchers have tried to isolate the ages at which children develop certain measurement concepts. Other researchers devised training programs in order to increase the acquisition rate of measurement concepts. However, few studies have been done which address social and cultural influences on children's development.

The purpose of this paper is three-fold. First, we will present an extensive literature review describing prior research on children's measurement conceptions. This will include descriptions of both studies conducted by Piaget and his colleagues and training programs which were designed to support children's development of measurement skills. Second, we will use this literature as background to argue that new investigations might include: (1) accounting for students' development in terms of social and cultural influences, (2) coordinating research analyses with instructional development, and (3) using psychological analyses to develop the instructional intent of further investigations. Finally, we will describe one teaching experiment on measurement conducted with a class of first graders in 1996. The focus here is not to provide an exhaustive analysis of the conceptions of individual students but rather to highlight one way of supporting students' mathematical development as they participate in the social context of the classroom. As this paper is a work in progress, we will not document the evolution of the mathematical practices established by the community. Rather, we will present the mathematically significant issues that arose as a measurement sequence was enacted and describe how the learning trajectory of this group of students was realized during the course of the teaching experiment

## Measurement Investigations

Most of the investigations into children's conceptions of measurement have been based upon studies that were published in the 50's and 60's by Piaget and his colleagues (e.g., Piaget et al., 1960; Piaget and Inhelder, 1956). Researchers have set out to either substantiate or disprove the claims that Piaget made about children's development. Since the majority of the literature on children's conceptions of measurement focus on Piaget's stage theory, it is helpful to begin this section by discussing Piaget's measurement investigations.

### Piaget's Studies

Piaget defined measurement as a synthesis of change of position and subdivision (Piaget et al., 1960). More precisely measurement for Piaget involves 1) understanding space or the length of an object as being partitionable (subdivision), and 2) partitioning off a unit from an object and iterating that unit without overlap or empty intervals (change of position) (Sinclair, 1970). It is the coordination of these two notions along with the understanding that these continuous units form inclusions (i.e., the first length measured is included in the length that comprises two units, etc.) that Piaget argued leads to a full understanding of measurement. The claim that conservation of length develops as a child learns to measure is implicit in this definition. Conservation of length means that as a child moves an object, that object's length does not change. Conservation of length is not equivalent to the concept of measurement but develops as the child learns to measure (Inhelder, Sinclair, and Bovet, 1974).

For Piaget, measurement required not only that a child can apply the procedures/skills of measuring, but that (s)he must also be able to carry out these activities without abjectly following specific, formalized measuring procedures. In other words, measurement for Piaget was not merely performing the "steps" required to measure. Rather, children who can measure in his terms are almost immediately able to use the standard measure with insight, not by trial and error (Piaget et al., 1960). Consequently, a child who can use a ruler correctly to measure would not necessarily be measuring according to Piaget's definition.

Piaget et al. (1960) categorized children's development of measurement conceptions into either 3 or 4 stages depending upon which aspect of measurement is being investigated. For example, the development of the conservation of length falls into 3 distinct stages with various substages in between the three main stages. What follows is a brief, integrative description of some of the key stages in Piaget's account of children's measurement development. For a more detailed inspection of the various stages of development see Piaget et al. (1960), Carpenter (1976), Copeland (1974), and Sinclair (1970).

At the initial stages of development (this includes various substages), children conserve neither length nor distance. Piaget characterized distance as being the empty space between two objects such as the distance between two trees. Length, for Piaget, denoted the length of an object such as a stick. Children classified at early stages of development rely strictly on visual perception for judgments about length. For instance, Piaget asked several children in clinical interviews about two strips of equal length (see Figure 1). He placed them in direct alignment with each other (A) and asked the child which strip was longer. Once equality of length was acknowledged by the child, the interviewer moved the bottom strip a few centimeters (B) and again asked which one was longer.

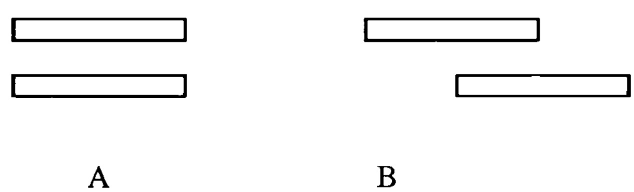



Figure 1. Interview task focusing on conservation of length.

Those children classified as in the early stages of development concluded that the bottom strip was now longer than the top strip because the bottom strip extended beyond the top strip. Piaget explained that children who argued that the length of an object grew as it moved were relying on perceptual judgments centering on the position of the endpoints. He argued that operational measurement is not possible for children classified as in early stages because space is not viewed as a common medium containing objects with well-defined spatial relations between them. According to Piaget, children with such an uncoordinated view of space do not understand that the distance from A to B is the same as the distance from B to A. Also, for children at early stages, the empty space between two objects or points is disturbed when another point/object is placed between the two points. Therefore, if a house, which was not originally present, is now placed between two trees, the distance between the two trees is smaller because the house takes up room.

Piaget argued that children that he classified as in the early stages are not able to reason transitively (i.e., if the length of object 1 is equal to the length of object 2 and object 2 is the same length as object 3, then object 1 is the same length as object 3). Being able to reason transitively is crucial for measurement and involves taking a stick, for instance, and using it as an instrument to judge whether two immovable towers are the same size. A child who can reason in this manner can take a third or middle item (the stick) as a referent by which to compare the heights or lengths of other objects. He argued that this is impossible for children who do not conserve lengths because once they move the middle object (the stick), it is possible, in their view, for the length of the stick to change.

Piaget asserted that children classified as in the early stages do not have an understanding of either subdivision or displacement. This can be seen as children measure the length of an item with a smaller unit. Typically, these children either run the smaller unit alongside the length of the item without making partitions or they make iterations of unequal spaces, sometimes neglecting gaps or overlapping areas. Their thinking at these early stages is said to be intuitive and irreversible (i.e., they are unable to go backwards; for instance, A to B is not the same as B to A).

Piaget identified transitional stages between these early stages and the final stages of operational measurement. Children in these transitional stages (about 6-7 years old) begin to reason transitively by using their body as a middle referent. Later, at about 7-8 years of age, they can reason transitively with other objects and begin to conserve length. However, Piaget asserted that children classified as in a transitional stage have not yet coordinated change of position and subdivision but possess each of the notions separately. They understand that changing the position of an item does not alter quantity and that the whole is the sum of its parts, but there is no coordination of the two.

Finally, the last stage, operational measurement, is attained when the child has synthesized displacement with subdivision (about 8 to 10 years old). It is important to differentiate this last stage from the mere ability to perform the skills of conventional measuring. To be operational, the actions of the measurement process must be interiorized into conceptual acts that are an integral part of an organized structure. Children at this stage can compare units lengths and discover that a small unit is a third or half of another.

With this brief overview of some of the most important aspects of Piaget's developmental stage theory in mind, we will now turn to some of the later literature on children's conceptions of measurement.

Reactions to Piaget

Researchers sparked by Piaget's analysis of children's conceptions of measurement focused on two different sets of issues. One interpretation of Piaget's developmental model was that certain measurement concepts developed in fixed order. In other words, the conservation of length developed before transitive reasoning. Also, the ages at which children construct particular measurement concepts were interpreted as being relatively fixed. It is interesting that very few researchers categorized children into the stages outlined by Piaget, but rather took these as given. It was more common for researchers to test the order in which measurement concepts developed or debate the ages at which children reasoned transitively. As a consequence of this *stage theory* interpretation, many researchers sought to substantiate or discredit Piaget's earlier claims.

A second set of investigations based on Piaget's analysis focused on increasing the acquisition rate of certain measurement concepts (Beilin, 1971). For example, researchers attempted to train children to conserve length by using a variety of training techniques. The primary focus of the *training studies* was to increase the rate at which children learn the conservation and transitivity of length using various training techniques. Therefore, there were many different experimental methods being used.

These two issues, *testing stage theory* and *training studies*, appear throughout most of the literature on measurement. However, not every researcher investigated children's conceptions of operational measurement. In fact, most of the experiments sparked by Piaget's analysis focused on children's understanding of the conservation and transitivity of length. These two concepts can be described as pre-measurement concepts since Piaget argued that the development of the conservation and transitivity of length occur in the process of constructing a full understanding of measurement. Thus, the literature on pre-measurement and measurement concepts can be organized into three main sections: conservation studies, transitivity studies, and measurement studies. Within both premeasurement and measurement studies, the research can be divided into the two different sets of interpretations mentioned previously, *testing stage theory* and *training studies*.

In the following sections, we will try to synthesize the majority of the studies based on Piaget's analysis. However, because the researchers made a variety of different and sometimes conflicting theoretical assumptions, it is very difficult to draw definitive conclusions about many of their findings. Thus, it is helpful to briefly describe the most common theoretical perspectives that guided the research. Our intent is not to supply an exhaustive account of the research but to provide an overview of the topics that were studied and debated.

Conservation Studies.

There were two main types of studies devoted to understanding the development of the conservation of length. The first body of literature reports experimental studies that tested Piaget's developmental stage theory. The second body of literature concerned accelerating students' ability to conserve length.

*Testing stage theory*. As mentioned above, few researchers characterized children according to the conservation of length stages identified by Piaget. Rather, researchers tested the order of the pre-measurement concepts that Piaget identified by using different experimental methods and tasks. For instance, McManis (1969) argued that the development of the conservation of length appeared before the development of transitivity. Similarly, Shantz and Smock (1966) conducted a study investigating whether the development of the conservation of length occurs before the development of a spatial coordinate system. Their findings supported Piaget and Inhelder's (1956) theory that the development of the conservation of length (occurring around age seven) precipitates the development of a spatial coordinate system which occurs at age nine. Generally, research on the age at which conservation develops and the order in which the development of conservation occurs with respect to other pre-measurement concepts supported Piaget and Inhelder's claims.

Other studies on the development of the conservation of length questioned Piaget's criteria for judging if a child actually conserved (Murray, 1965, 1968; Gelman, 1969; Carpenter, 1975; Shantz and Smock, 1966). Piaget argued that a child does not conserve length if (s)he incorrectly concluded that the bottom strip in Figure 1 was longer than the top strip. Other researchers reasoned that the nature of the interview tasks might cause children to attend to irrelevant perceptual cues. For example, Murray (1965) and Gelman (1969) hypothesized that when Piaget asked questions during interviews concerning length, the subjects may have thought the interviewer was referring to the position of an object's endpoints and consequently, made judgments based on perception rather than quantity. In order to eliminate the endpoint confusion, Murray showed children a set of two parallel dotted lines of equal lengths. Once equality was acknowledged by the subject, the interviewer distorted the two original lines, rather than physically moving them, in order to create the perceptual illusion that one of the lines had grown. Murray concluded that children were probably not interpreting the word length as referring *only* to the relative position of the lines' endpoints. These results supported Piaget's conclusions that conservation of length typically occurs around seven years of age. On the other hand, Gelman (1969) argued that teaching children to ignore irrelevant perceptual cues decreased the age at which the conservation of length develops. However, Gelman trained her subjects for a number of weeks which probably accounted for children's increased performance on conservation tasks. In the next subsection we will discuss training studies in more detail.

*Training studies.* The second body of literature dealing with conservation came about as a reaction to Piaget's naturalistic slant on learning. Piaget developed a general theory of conceptual development for transitions between stages, but most of this research focused on individual children performing tasks presented in clinical interviews. There was very little empirical evidence that supported his theories about the process and ways of supporting conceptual development. Consequently, training studies centered on the *acquisition* of cognitive structures by considering the effect of different types of training procedures.

It is interesting that Piaget described training studies as an attempt to rush development (Rogoff, 1990), and is generally interpreted as doubting that training has an effect on the development of children's conceptions. Nevertheless, many researchers based their training techniques on the basic components of Piaget's *equilibrium model*. Equilibration is best described as a dynamic, *self-organized* balance between assimilation and accommodation. Assimilation is the process of organizing new experiences in terms of prior understanding. When new experiences cannot be adequately organized in this way or when they give rise to contradictions, an accommodation typically occurs. An accommodation involves reflective, integrative processes that create new understanding in order to restore equilibrium. Piaget conjectured that disequilibrium is brought about by experiences that generate cognitive perturbations (von Glasersfeld, 1995). Many of the training programs used aspects of the equilibrium model as the assumptions behind their training methods. Researchers tried to induce cognitive conflict in children in order to bring about disequilibrium (Brainerd, 1974; Smedslund, 1961; Overbeck and Schwartz, 1970; Murray, 1968). For example, Brainerd (1974) attempted to induce disequilibrium in subjects by providing positive and negative feedback during training sessions on conservation. He found that providing positive and negative feedback improved children's abilities to conserve length on typical Piagetian conservation tasks.

Other researchers argued that reversibility might play an important role in decreasing the age at which conservation of length developed in children (Smith, 1968; Brison, 1966; Murray, 1968). For instance, one of the experimental treatments in Smith's (1968) training program involved asking children to make judgments about the length of an object after adding and subtracting pieces from the end of it. In this way, the child was "shown" that reversing operations can help determine equality of objects.

Although all of the training studies devoted to conservation discussed thus far used aspects of Piaget's equilibrium model to determine training methods, we argue that training goals differed from Piaget's goals. In our view, Piaget was concerned with identifying children's understanding of and meanings associated with conservation. In contrast, training studies only focused on whether or not children could conserve length by the end of the training sessions. Their primary concern was on children's *ability* to conserve length, and this ability was determined by correct answers to conservation tasks. In other words, evidence that a child conserved length was determined by his/her performance on conservation tasks. As a consequence, training studies on conservation, and training studies in general, focused on children's premeasurement or measurement *skills*.

Transfer of training studies provide the most compelling illustration of researchers' attention to conservation skills rather than conceptual understanding. Many researchers investigated whether children who are trained to conserve length would transfer these skills to conserve weight or area (Beilin, 1965; Gruen, 1965; Brainerd, 1974; Brison, 1966). For instance, one of the more recent studies on training for conservation (Tomic, Kingma, and Tenvergert, 1993) used Obuchova's (1972) training method in order to examine whether children transferred conservation skills from one set of tasks to another. If a child gave correct answers to questions testing conservation that used stimuli different from the stimuli used in training, then the child was said to have transferred their conservation skills to different but related tasks. The researchers concluded that Obuchova's training method successfully trained non-conservers to conserve length. Generally, researchers found that training for the conservation of length using various techniques transferred to other domains (cf. Brainerd, 1974; Beilin, 1965). For a more in depth review of the literature concerning transfer of training see Osborne (1976). Studies concerning the transfer of conservation skills, in general, seem to diverge from Piaget's orientation since operational measurement for Piaget was more than demonstrating skills.

In other methods for training conservation skills, researchers deserted the equilibrium model and challenged the importance attributed to reversibility. Kingsley and Hall (1967) argued that training had not been overly successful because researchers had not been taking students' prior experiences into account. They countered that rather than focusing on the maturational unfolding of internal structures, researchers should examine each student's prior experiences related to concepts of measurement and begin training at each child's level of experience. For Piaget, children's experiences consisted of situations in which reorganizations take place in reaction to a new experience. In other words, Piaget would explain prior experiences in terms of cognitive reorganizations. It seems that Kingsley and Hall would describe experience in terms of prior knowledge or skills that a child can perform. Thus, using Gagne's theory of learning sets (Gagne, 1965), Kingsley and Hall compiled a list of five items that, in their view, children should know in order to have a conceptual understanding of measurement. Then, they used children's prior experience/knowledge as a place to begin training on these five items. After a training period, children were retested on Piagetian-type conservation tasks. The results suggested that Kingsley and Hall's training through the use of learning sets, significantly improved children's performance on conservation tasks. However, this study again illustrates a focus on children's skills related to measuring, rather than documenting their conceptual understanding.

In summary, there seemed to be two interpretations of Piaget's developmental stage theory. Some researchers sought to substantiate Piaget's stage theory by adopting the equilibrium model to explain conceptual development. They conducted experiments to support the order in which certain concepts develop in children (e.g., conservation appears before transitivity). Other researchers were concerned with the acquisition rate at which children could be trained to conserve length. It was here that many different experimental methods were used and findings from training studies yielded conclusions that seemingly contradicted Piaget. However, Piaget might have argued that training studies focused on the acquisition of conservation skills rather than conceptual development. Therefore, children may have learned to conserve length at an earlier age, but their conceptual understanding of conservation might be in question.

Transitivity Studies.

*Testing stage theory.* Transitivity studies were the subject of so many methodological variations that researchers argued that the ages at which transitivity developed in children ran from as early as 4 years to as late as 8 years of age. As a result, a debate ensued between Braine (1964) and Smedslund (1963, 1965) concerning assessment techniques and the need to define more exactly what was meant by evidence of transitivity. Both Braine and Smedslund accounted for learning using Piaget's equilibrium model, but disagreed on the ages at which transitivity occurred. Braine found evidence that children as young as 4 to 5 years exhibited signs of transitivity which was two years younger than the ages Piaget gave. However, Smedslund argued that Braine's experimental techniques allowed children who may not have reasoned transitively to solve problems correctly. For instance, Smedslund developed an interview task with three strips of paper (labeled A, B, and C) where A was longer than B and B was longer than C. If children responded correctly to each of A>B, B>C and A>C, then those children's response was called a system response. He argued, however, that the absence or presence of a system response did not necessarily indicate nontransitivity or transitivity, respectively. He also asserted that some children may have guessed the correct answer (i.e., A>C) or could have simply compared the lengths of A and C directly. Children may have also relied on the knowledge that A is longer than B (the first comparison) to successfully conclude A>C without making the intermediate comparison. Adjusting for these possible misinterpretations of the presence of transitivity, Smedslund found that the age at which transitivity appears in more than half of the children interviewed was 8 years old. Braine (1964) countered Smedslund's critique by arguing that verbal cues given by the interviewer could have distorted Smedslund's data.

*Training studies*. Experimental methods continued to be the topic of debate between these two researchers while others focused on Piaget and Inhelder's earlier claims that transitivity and conservation develop simultaneously. Some researchers (McManis, 1969; Steffe and Carey, 1972; Brainerd, 1974; Lovell and Ogilvie, 1961; Smedslund, 1961) performed experiments that negated this claim. Smedslund (1961) used aspects of Piaget's equilibrium model to show that the development of the conservation of length occurred prior to transitivity. Using positive and negative feedback in training sessions, Brainerd (1974) claimed that transitivity developed before the conservation of length. However, Smedslund had argued earlier that giving feedback was not consistent with Piaget's equilibrium model since "practice is not assumed to act through external reinforcements but by a process of mutual influence of the child's activities on each other" (1961, p. 13). Brainerd countered that his method was indeed consistent with the equilibrium model in that negative feedback could induce cognitive perturbations and stood by his conclusions. Brainerd's conclusions are consistent with Lovell and Ogilvie (1961) who argued that 53% of their subjects reasoned transitively but did not conserve length. However, McManis (1969), following Smedslund's earlier arguments with Braine on what constitutes evidence for transitivity, charged that subjects in Lovell and Ogilvie's study may not have been reasoning transitively even when giving correct answers. Correcting for what he thought were methodological errors, McManis claimed that conservation developed before transitivity. It is clear from the studies reviewed that the type and nature of the questions an experimenter employs can profoundly affect the quality of the answers given by subjects, thus rendering different conclusions about both conservation and transitivity studies.

Measurement Studies.

As mentioned earlier, most researchers were interested in the development of pre-measurement concepts. The relatively few studies on children's conceptions of measurement (i.e., a coordination of subdivision and change of position) can be divided into two main types. Both types used Piaget's equilibrium model as their underlying theory. The first type of study attempted to substantiate Piaget's stage theory.

*Testing stage theory*. Lovell, Healey and Rowland (1962) asked children typical Piagetian measurement tasks and categorized subjects into each of the stages that Piaget had outlined with regard to measurement. Lovell et al. listed the criteria for placing children into each of three stages:

> I and IIA: The construction of a unit of measure is impossible. Subdivision is sometimes used without change of position or change of position at the expense of subdivision.
>
> IIB: Transitivity is just beginning to be understood by children.
>
> III: Subdivision is generalized. Subjects can measure the lines with 3 cm and 6 cm cards and express the latter length in terms of the former. (Lovell et al., 1962)

Their findings generally substantiated Piaget's stage categorizations. Also, the ages at which full measurement develops in children were found to be similar to Piaget's estimate of nine years.

Another group of studies on children's conceptions of measurement was conducted from an information-processing perspective. Studies had been conducted in the 70's by researchers trying to identify the information-processing demands of tasks that dealt with the conservation of length (Klahr and Wallace, 1970; Baylor, Gascon, Lemoyne and Pothier, 1973; Hiebert, 1981). The computer was the dominant metaphor with which to describe children's mathematical learning.

> It was because the activities of the computer itself seemed in some ways akin to cognitive processes. Computers accept information, manipulate symbols, store items in "memory" and retrieve them again, classify inputs, recognize patterns and so on...(Gardner, 1987, p. 119)

Human thought was conceptualized as an information-processing system and the goal was to develop computational models whose input-output relations matched those of children's observed performance (Cobb, 1990). In some cases, researchers actually attempted to write computer programs that matched their observations of children's strategies (e.g., Baylor et al., 1973). This computational model seemed to leave out primary aspects of human thought, that is, meaning-making. According to these researchers, many mathematics tasks required the ability to process several pieces of information simultaneously, and students with low short term memory capacities were assumed to perform less well than children with high capacities. Therefore, researchers took into account the information-processing requirements of a task as they analyzed students' solutions to problems. Hiebert (1981) and Baylor et al. (1973) studied the relationship between information-processing capacities and children's ability to learn concepts of linear measurement and found that the information-processing capacity had no detectable influences. Nonetheless, Baylor et al. (1973) concluded from their study that using the notion of information-processing capacity to explain Piagetian operations was a valid way to analyze children's conceptions of measurement. Hiebert (1981) agreed with Baylor et al., but added that the usefulness of this perspective in educational settings requires more research to specify an information-processing unit of analysis that takes into account both the child's capacity and the demands of an instructional task so that these two aspects can be meaningfully compared (see also Klahr and Wallace, 1970; Inhelder, 1972).

*Training studies*. The second type of study that concerned the development of measurement concepts in children can be categorized as training studies (Inhelder and Sinclair, 1969; Inhelder, Sinclair, and Bovet, 1974; Bailey, 1974). Many significant training studies were conducted by researchers at the University of Geneva. These researchers found that conflict induced by measuring with different sized units of measure facilitated measurement concepts. Inhelder et al. (1974), following Piaget, categorized the relation between children's conceptions of measurement and numerical conceptions into stages. At the beginning stages, children are influenced by the number of items that make up a given length of an object. For instance, Inhelder et al. conducted interviews in which they showed children two rows of matchsticks (see Figure 2). The matches placed in row A were longer than the matches used in row B.


Figure 2. Matchsticks used in an interview task.

When children were asked which row was longer, children categorized at the early stages argued that the bottom row of sticks was longer than the top row because the bottom row contained more matchsticks. Inhelder et al. (1974) concluded that length differs from numerical quantity both by its continuous nature and by the fact that it has to be quantified through the use of units. Children categorized at the early stages make judgments of length based upon discrete numerical quantity rather than continuous, quantified units (Inhelder et al., 1974).

This study seems to be different from most training studies on the premeasurement concepts in that preliminary experiments were conducted to understand children's conceptions of the relationship between number and length and to identify stages of development pertaining to this relationship. After the stages were identified, training sessions were conducted and several of the children's responses during the session were analyzed in order to identify the psychological processes involved in coordinating discrete units with continuous length. Recall that the training studies discussed previously were concerned with the results of pre- and post-tests. Inhelder et al. were also concerned that their training sessions yielded better results, but the majority of their analysis centered on psychological analyses of children's solutions. This kind of study reflects Piaget's position more than the other training studies; however, the analysis is still primarily individualistic.

## Reconceptualizing Measurement Investigations

In light of the methodological debates and the concentration on studying measurement as meaningful activity and teaching children skills, we believe it is necessary to reconceptualize measurement investigations. The classroom teaching experiment put forth in this proposal is one attempt to extend investigations as well as support students' development of measurement concepts. Recall that three main issues emerged from the studies described in the literature review. In this section we will develop our theoretical position on each of the issues by elaborating the following: social and cultural issues, developmental research, and instructional intent.

Social and Cultural Issues

Social and psychological perspectives. All of the studies discussed in the literature review, including the training studies, involved a primarily individualistic, psychological perspective. Recall that Piaget's main concern was to identify the stages through which children develop when learning how to measure. Although he acknowledged that social interaction was a primary source of perturbations, the main focus of his analysis was on students' meanings associated with their answers to pre-structured interview tasks rather than describing the interview as a social situation in which meanings of tasks are negotiated between the participants. Even training studies and the few longitudinal studies that were analyzed (cf. Almy, 1971) were individualistic in nature. Training sessions usually involved one-on-one teaching sessions with an interviewer posing problems to a child and post-tests to assess the success of the training technique. The social interactions occurring in these one-on-one training sessions and assessments were not considered when analyzing the children's activity. This was also the case for the few classroom-based studies that were conducted. Interactions between the teacher and students or interactions among students were not considered as part of the analyses of students' development. The main focus of these studies was instead on students' performance on pre- and post-interviews.

In our view, learning is both an individual *and* social accomplishment with neither taking primacy over the other. Therefore, further investigations of measurement, from our point of view, must necessarily be framed in terms of individuals' participation in the social practices of the classroom community or interview situations. In recent years, the role of social and cultural processes in students' development has been acknowledged (cf. Lave, 1988; Saxe, 1991; Rogoff, 1990, Cobb, 1995). As a consequence, the view that learning occurs in a socially-situated context is increasingly common. Although Piaget and his successors would not deny that social interactions are an important source of cognitive conflicts, actual analyses focus on cognitive on cognitive development without acknowledging the role that interactions with peers or interviewers play in the process. The relationship between social and psychological processes has become a central issue in recent years and is defined differently in contrasting theoretical perspectives. Socioculturalists, on the one hand, give primacy to social and cultural influences. They would argue that there is a relatively *direct* link between social interactions and psychological processes (Cobb and Yackel, in press). In other words, the quality of students' thinking is generated by or derived from the social processes in which they participate. In contrast, emergent theorists would characterize the link between collective and individual processes as *indirect* in that participation enables and constrains individual learning, but does not determine it. In this viewpoint, participation in the collective mathematical practices is said to constitute the immediate social situation of learning and to provide opportunities for the possibility of learning (Cobb and Yackel, in press). Learning is both an individual and social process with neither taking primacy over the other. The relationship between social and psychological processes is rather strong in that the two cannot be separated; they coexist.

Despite the differences between the emergent and sociocultural perspectives, proponents of both theoretical perspectives claim that students' development cannot be adequately explained in cognitive terms alone; social and cultural processes must be acknowledged when explaining psychological development. It must be noted that sociocultural and emergent theories do not discount psychological analyses conducted in interviews. However, theorists from both sociocultural and emergent perspectives would argue that traditional psychological analyses characterize students' conceptual understanding independently of situation and purpose (Cobb and Yackel, in press). Interviews are viewed from an emergent perspective as social events where the interviewer/child system is the unit of analysis.

Taken together, these theoretical developments indicate the need to reconceptualize how we would investigate and analyze students' development of measurement understandings. From the emergent perspective, clinical interviews are analyzed to understand individual students' meanings. However, during analyses it is essential to keep in mind that a student and an adult are interacting and negotiating the interview tasks. Further, interviews are not the sole means of assessing students' conceptions. The classroom teaching experiment involves accounting for individuals' mathematical development as they participate in the social and cultural practices of the classroom community (Cobb, in press; Yackel, 1995). Individual students' development is analyzed in terms of their participation in and contribution to the emerging, communal mathematical practices. Further, students are seen to contribute to the evolution of the mathematical practices as they reorganize their activity while participating in these practices. In essence, the classroom teaching experiment is one site for investigating students' understandings of measurement that takes into account students' participation in social practices.

Cultural tools. In addition to their primarily individualistic focus, most studies of students' measurement activity attribute a limited role to their use of cultural tools. In our view, tools can be characterized more specifically as physical materials, tables, pictures, computer graphs and icons, and both conventional and non-standard symbol systems. The measurement analyses previously discussed throughout this paper (with the exception of Piaget et al., 1960) do not take into account students' activity with cultural tools. Vygotsky (1978) emphasized that semiotic mediation is integrally involved in students' development. Currently, the nature of this involvement is still subject to debate. Most theorists contend that conceptual development and symbol creation are reflexively related. In other words, the creation of symbols and meanings develop simultaneously. Psychological constructivists view learning as a process in which individuals reorganize their prior activity. This process is often facilitated by the use of symbols to aid in the construction of mathematical objects.

The sociocultural perspective developed in reaction to mainstream psychology's attention to individual activity. Instead, they emphasize the complexity of learning within a social environment and suggest that tools have a direct influence on how individuals learn. The notion of distributed intelligences, which draws heavily upon sociocultural theories, suggests that students may use cultural tools as scaffolds to their internalization process (Pea, 1994). Pea describes tool-use as a reorganizer, rather than amplifier, of understanding. Socioculturalists characterize tool-use as a process of internalizing a cultural tool so that it becomes a tool for thinking (Davydov & Radzikhovskii, 1985) or of appropriating it to one's own activity (Newman, Griffin, & Cole, 1989). They often argue that tools are the primary vehicles for enculturation into the cultural practices of the wider community because tools are carriers of meanings from one generation to the next (van Oers, 1996). In this way, students are said to inherit the cultural meanings involved in using a particular tool.

The emergent perspective also views tools as a constituent part of individuals' activities. Further, they would speak of students *reasoning with* physical materials, pictures, etc. (Cobb and Yackel, in press). An emergent perspective describing how learning occurs as students act with cultural tools is based on the assumption that the symbolic and conceptual meanings that evolved from the students' activity with tools do not develop apart from the mathematical practices in which the students participate. One difference between sociocultural and emergent views of tool-use concerns the negotiation of meanings involving students' activity with tools. Negotiation, from the sociocultural perspective, is a process of mutual appropriation in that the teacher and students continually co-opt or use each others' contributions (Cobb, 1994). Thus, the teacher's role is both to insert culturally-approved insights that students can appropriate and to co-opt students' contributions into the wider system of mathematical practices (Cobb, Perlwitz & Underwood, in press). Further, students are said to appropriate the culturally-approved ways of acting with a particular tool. In contrast, emergent theorists view negotiation as a process of adaptations that give rise to shifts and slides of meaning as the teacher and students engage in mathematical discussions. Along with Miera (1995), we contend that the meanings involved in acting with a particular tool evolve as the students are engaged in goal-directed activity with the tool. The teacher's role is to initiate and guide the development of individual's mathematical activity (in particular, tool activity) as well as the mathematical practices so that these practices become more compatible with those of the wider society (Cobb, 1994).

Although ways of analyzing and describing tool-use is still being investigated, it seems clear that attending to the role of semiotic mediation in measurement is of great importance. Cultural tools were used to a large extent in training students to conserve length, to reason transitively, and to learn to measure. Also, a number of different tools were used to support students' mathematical development in the first-grade teaching experiment. The question remains as to how students' activity with these tools changed how they came to view measuring.

As a matter of pragmatics, it should be stressed here that neither the sociocultural theory nor the emergent theories is more "right" than the other. Cobb (1996) describes the notion of theoretical pragmatism in which a researcher chooses a particular theory based upon the questions (s)he wants to answer because no one theory can claim to describe the world as it really is all the time. The implication of this approach is to "consider what various perspectives might have to offer relative to the problems or issues at hand" (p. 46). There are instances, for example, when a sociocultural theory would be a more useful tool for describing a situation. On the other hand, when analyzing students' participation in and contribution to the mathematical practices of the classroom as well as describing the collective meaning associated with using tools, it is our belief that the emergent perspective is the "right tool for this job."

Developmental Research

Another issue that emerges out of the prior literature is the distinction between studies attempting to analyze children's activity and studies designed to proactively support students' development. Piaget's analysis and others of that type differ from the training studies in that Piaget et al. (1960) analyzed the process of children's cognitive development. In contrast, researchers who conducted training studies attempted to support students' skill development, sometimes drawing on Piaget's analyses. These two distinct ways of researching students' development suggests that researchers address the separation between theory and instructional design. Generally, this relationship is one-way in that theory informs instructional design. Rarely does the designer's intent feedback to inform ongoing analyses of children's development in social context.

One way to examine the relationship between research/theory and instructional design is to view the two aspects as reflexively related. This reflexivity suggests that findings from previous research inform designers as they create and revise instructional activities. Further, the designer's intent can inform the ways in which researchers analyze how the activities are realized in classroom context (see Figure 3). This reflexive relationship is consistent with the tenets of developmental research (Gravemeijer, 1994).

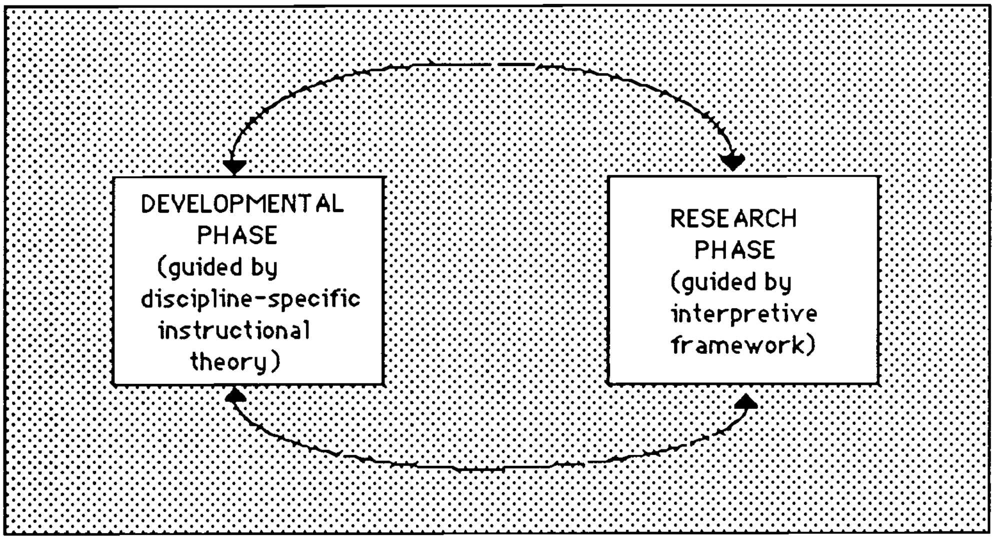



Figure 3. Phases of the development research cycle.

In Gravemeijer's account, the developer first carries out an initial thought experiment in which he envisions both how the proposed instructional activities might be realized in interaction and what students might learn as they participate in them. As Gravemeijer notes, in conducting this thought experiment (Developmental Phase of Figure 3), the developer formulates conjectures about both the students' mathematical development and the means of supporting it. These conjectures about the teaching and learning process constitute a hypothesized learning trajectory or conjectured developmental route of the classroom community.

Once the designer envisions the developmental route of the classroom community, he carries out a teaching experiment in which he tests his initial conjectures. This constitutes the Research Phase of the Developmental Research Cycle in Figure 3. Gravemeijer (1994) notes that an instructional sequence based solely in terms of observable behaviors is undesirable; thus, the teaching experiment conducted in the research phase is guided by an interpretive framework. Ongoing analyses of students' activity is necessarily guided by an interpretive framework that focuses on the meaning-making of the community. The classroom teaching experiment upon which this study is based was guided by such an interpretive framework as well as hypothesized learning trajectory.

The actual learning trajectory that is realized in the classroom typically diverges from the learning trajectory that was initially proposed. This divergence is due to the pedagogical judgments made while the teaching experiment is in progress. An example of this concerns how whole-class discussions are planned. In classroom teaching experiments, the mathematics class usually begins with students working in pairs or individually on a task set by the teacher at the beginning of class. While students work independently or with their peers, the teacher and members of the research team circulate around the room to document students' diverse interpretations of the task. Towards the end of individual work, the teacher and researchers briefly discuss observations and plan for subsequent whole-class discussion. The purpose of these meetings is to identify students' activity and interpretations that might lead to productive mathematical discussions. A classroom discussion is justifiable only if the issues that emerge as topics of conversation are mathematically significant. These decisions are made against the backdrop of the conjectured learning trajectory. The learning trajectory that is realized necessarily diverges from the anticipated learning trajectory since it is not always possible to predict 1) how students might interpret planned instructional activities and 2) the mathematically significant issues that may emerge as the sequence is enacted. As a consequence, researchers make and test conjectures on a daily basis and adjust instructional activities based upon students' mathematical activity. Thus, both the students' interpretations and the daily conjectures and decisions constitute the realization of a learning trajectory. As Cobb (1996, December) describes it, the teacher and the students interactively constitute an *actual* learning trajectory. Cobb goes on to say that

> [This] does not mean that classroom activities drift aimlessly. At any point, there is both an overall instructional intent and an envisioned means of achieving it. However, both the intent and the conjectured trajectory are subject to continual revision. Thus, to pursue Varela, Thompson, and Rosch's (1991) metaphor, the path is laid down by walking even though, at each point in the journey, there is some idea of a destination and of a route that might lead there (p. 14).

Thus, pedagogical decisions are made in terms of an end goal which itself can change or shift during the teaching experiment depending upon how the instructional sequence is interactively constituted.

This way of proceeding differs from previous studies conducted on measurement in that a more detailed account of children's developmental processes as they occur in social context can be documented. Also, the means of supporting development are carefully tested and revised in action as the teaching experiment progresses. Training studies that attempted to support the acquisition of measurement skills did not test and revise conjectures about students' developmental paths during training periods.

We have thus far stressed that a classroom discussion is justifiable only if the issues that are discussed contribute to the mathematical intent of the sequence. In the next section, we will describe the mathematical intent of the measurement sequence that was enacted in the first-grade classroom.

Instructional Intent

When preparing for a teaching experiment, it is important to clarify the global intent of the instructional sequences that are being developed. To this end, it is helpful to cast the outline of the instructional intent in terms of Greeno's (1991) environmental metaphor. In doing so, researchers "attempt to articulate the nature of the mathematical environment in which we hope students will eventually come to act" (Cobb, 1996, December b, p. 7). In specifying the nature of the mathematical environment, we[1] try to elaborate the potential mathematically significant issues that may emerge as topics of conversation as students act in this emerging mathematical environment.

Recall that training studies focused more on children's acquisition of measurement *skills* rather than the *meanings* involved in learning to measure. We see more value in Piaget's analyses which focus on the mathematical meanings and interpretations associated with particular premeasurement and measurement understandings. These types of analyses are consistent with current mathematical reform efforts. For example, the NCTM Curriculum and

Evaluation Standards (1989) stress that mathematics should be a meaningful activity rather than a set of learned rules and procedures for calculating. In preparing for the first-grade classroom teaching experiment, it was necessary to formulate what the we wanted measuring to be for students. To this end the instructional intent of the measurement sequence was formulated such that measuring was an activity in which students are physically or mentally acting on space. We wanted the activity of measuring to signify the partitioning of space into units that could be used to find a measure (change of position). Thus, measuring would not merely be a matter of iterating a ruler or some other measurement unit and verbalizing the number obtained when the measurement unit was iterated for the last time. We would want the number that results from the last iteration to signify not simply the last iteration itself but rather the result of the accumulation of the distances iterated (i.e., Piaget's notion that the whole is the sum of its parts). In other words, our primary concern was that students would come to interpret their activity of measuring as the accumulation of distance (cf. Thompson & Thompson, 1996). For example, if students were measuring by pacing heel to toe, it was hoped that the number words they said as they paced would each come to signify the measure of the distance paced thus far rather than the single pace that they made as they said a number word. It was also hoped that the students would come to act in an environment of quantities that could be structured in different ways. For example, we wanted students to be able to interpret their measuring activity as not only a space measuring 25 feet but also five distances of five feet or two distances of ten feet and a distance of five feet, as the need arose.

## Data Corpus

The first-grade classroom was one of four first-grade classrooms at a private school in Nashville. The class consisted of 16 children, 7 girls and 9 boys. The majority of the students were from upper-middle class backgrounds. Although not a Christian school, morals and values were part of the responsibility of schooling and children regularly participated in spiritual activities. The classroom teaching experiment took place over four months from February to

June, 1996. Just prior to the teaching experiment, the teacher and the students had been engaged in instruction on two-digit addition and subtraction.

The teacher was an active member of the research team and taught according to the reform guidelines of the NCTM Professional standards for teaching mathematics (1991). She had been involved with the project members for 3 years and originally sought help from experts in the mathematics education field because she had become dissatisfied with the available textbooks. The research team had developed a close, professional relationship with this teacher. The relationship was such that members of the team could interject clarifying questions at any point during a classroom session. The children responded to these interjections as if the researcher were a person who did not understand some of the content being discussed during class. At times children would offer explanations in order to help the researcher understand better because they believed (s)he did not understand.

The primary focus of this teaching experiment was to develop two closely related instructional sequences. The instructional sequences for supporting children's early number concepts (Steffe, Cobb, and von Glasersfeld, 1988) and measurement concepts were designed in collaboration with Koeno Gravemeijer from the Freudenthal Institute in the Netherlands and were compatible with the aims of reform described in the NCTM Curriculum and Evaluation Standards (1989). The first sequence involved supporting students' increasingly sophisticated understanding of measuring and the second built on the measurement sequence to support students' construction of mental computation and estimation strategies for reasoning with numbers up to 100. In the case of the measurement sequence, our hope was that students would come to reason mathematically about measuring rather than become proficient with measuring procedures. Further, we sequenced instructional activities so that problems built upon one another in such a way that students' measuring activity served as the underlying basis for mental computation and estimation strategies. The mathematical intent of the measurement sequence and the mental computation and estimation sequence has already been elaborated in an earlier section entitled Instructional Intent.

At the beginning of the teaching experiment, interviews were conducted to assess students' mathematical understanding so that the instructional sequence would build on their current ways of knowing. A comparison of pre- and post-interviews indicates that most of the students had developed sophisticated thinking strategies for solving two-digit addition and subtraction problems over the course of the teaching experiment. Also, post-interviews indicate that, for most students, space signified an object that could be partitioned, and measuring signified the accumulation of distance. In the remainder of this paper, we will document the measurement sequence as it was realized in this classroom by elaborating the mathematically significant issues that arose during the course of the teaching experiment. We will do this by describing the nature of the instructional activities that were used, the problems or issues that emerged out of students' participation in these activities, and the subsequent whole-class discussions in which the teacher attempted to support students' mathematical development.

The Instructional Sequence

King's foot. The instructional activities used in the teaching experiment were typically posed in the context of an ongoing narrative. To accomplish this, the teacher engaged the students in a story in which the characters encountered various problems that the students were asked to solve. The narratives served both to ground the students' activity in imagery and provided a point of reference as they explained their reasoning. In addition, the problems were sequenced within the narratives so that the students developed increasingly effective measurement tools with the teacher's support. Further, the narrative supported the emergence of tools out of students' problem solving activity.

The teacher began the first narrative by relating a story about a king who decided to measure items in his kingdom with his foot. However, the king did not know how he could accomplish this and requested the students' ideas. After students had made various contributions, the class decided that the king could measure objects by pacing heel-to-toe. Subsequent instructional activities included having pairs of students, each taking turns acting as king, pace the length of objects located in the classroom to determine their measure. As the students measured objects, the research team walked around the room observing students' activity.[2] We learned that the students had two ways of measuring the length of different objects, such as a rug. Some students placed one foot at the beginning of the rug and counted "one" with the placement of their second foot (see Figure 4a). Other students placed their foot at the beginning of the same rug and counted it as "one" (see Figure 4b).

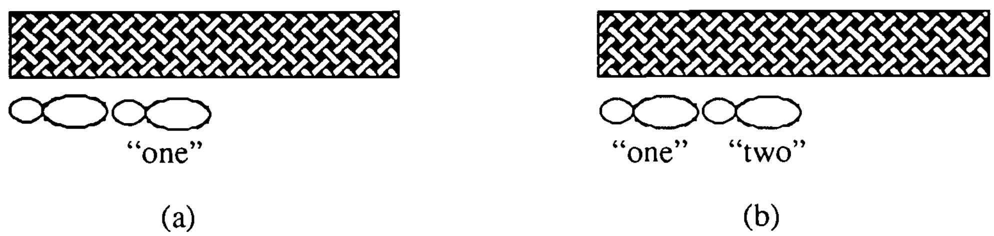



Figure 4. Two methods of counting as students paced the length of a rug.

We felt that these two ways of counting their paces were mathematically significant because students counting in the first manner (see Figure 4a) did not appear to be filling space as they paced. As an aside, it is at this point in the teaching experiment that we were conducting informal psychological analyses in order to find mathematically significant issues which could serve as topics for whole-class discussion. We felt that if the students discussed these two different ways of counting as they paced, filling space might become a topic of conversation. It was at this point, during the whole-class discussions, that we were taking a social perspective. It should be noted that the intent of these whole-class discussions was not to make students count their paces "the right way." Rather, we felt that if the students participated in a discussion about these two different ways of counting their paces, they would have an opportunity to reorganize their understanding about what it means to measure.

After discussing the students' two ways of counting their paces in the "huddle", a whole-class discussion ensued. The teacher asked two students , whom we had observed counting their paces in the alternative ways described above, to pace the length of the rug for the rest of the students. Then, the teacher asked what was common or different about the two students' activity. Many students commented that the student who had counted her paces in the first manner (see Figure 4a) was not measuring part of the rug. In order to support reasoning about pacing as filling space and facilitate whole-class discussion, the teacher marked the students' footsteps with tape alongside the rug so that the class would have a record of their paces. As a consequence, many students pointed to the space between the first and second piece of tape and argued that the students reasoning in the first manner (see Figure 4a) were not counting that step. Thus, a portion of the rug was not being measured. Subsequent instructional activities involved asking students to measure items outside the classroom. Our observations indicated that most students counted their paces according to the second way (Figure 4b). Throughout the remainder of this paper, when we claim that the class began to reason a particular way, we are not claiming that *all* students interpreted their activity in the same manner. For example, a few students continued to say "one" with their second foot, but the majority of the class was measuring in the second manner. As stated before, it was not our goal to correct these few students who were counting their paces in the first manner, but to provide opportunities in which they might reorganize their understanding.

A second issue that arose during the course of the first day's activities and remained an issue throughout the teaching experiment was the notion of the $n^{th}$ step. Recall that as part of the instructional intent, we wanted students to interpret their measuring activity in terms of accumulating distances. Thus, as students were pacing, we intervened and asked where, for example, "the four" was. To answer this question, most students pointed to the fourth foot rather than the first four feet they had paced. For us, the mathematically significant issue was the difference between measuring signifying the accumulation of distance and measuring as the act of pacing. Although pointing to the fourth foot was a reasonable response to the question, "Where is the four?", we eventually wanted students to reason about "the four" as an accumulation of four feet rather than the fourth step. In order to begin to support an accumulation of distance interpretation, the teacher marked the result of each pace with tape. Then, as she redescribed students' solutions, she would gesture with her hands in a very particular way. For instance, to show four feet, the teacher placed one hand at the beginning of the rug and the other hand at the end of the first foot and said, "one." Then, leaving her hand at the beginning of the rug, she moved her other hand to the end of the second foot and said, "two." She continued in this manner until she had shown the accumulation of four feet. In contrast, the teacher *could* have gestured by placing one hand at the beginning of the rug and the other hand at the end of the first foot saying "one." Then, the teacher could have moved both hands so that they were positioned at the beginning and end of the second foot, and so on. This way of gesturing, in our view, would not have supported measuring as an accumulation of distance. When talking about the accumulation of four feet, rather than the fourth foot, students began to refer to "the whole four" by making sweeping motions with their hands over four feet. Not everyone interpreted their measuring activity as an accumulation of distance, and as a consequence, this issue continued to emerge throughout the teaching experiment.

Two days later a final issue arose concerning how to account for an amount of space remaining at the end of the rug that did not make a whole foot. Often time, students' feet did not measure the length of the rug in an even number of feet. During observations of their activity, we noticed that many students accounted for the extra space by turning their foot sideways in order to stay within the space bounded by the rug (see Figure 5).

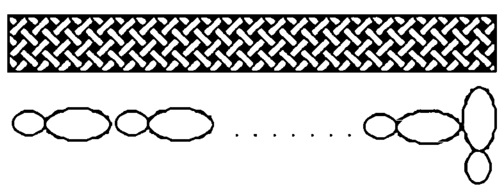


Figure 5. Students' solution to an extra amount of rug not covered by the last whole foot.

For these students, it seemed that pacing was an activity of filling or defining the space to be measured. In other words, they did not extend the measurement unit past the space to be measured. Other students, rather than turning their last foot sideways, simply counted the whole foot (e.g., 18 instead of 17 1/2). During subsequent whole-class discussion, the teacher told the students that she saw some people counting the extra space at the end of the rug as a whole foot. She asked if they thought that this was a good way to measure the extra space at the end of the rug. One student paced the length of the rug for the class according to the way the teacher had just described. Many students argued that they could not count the last foot as a whole foot because only part of the foot extended beyond the rug. Thus, they argued that the remainder signified about a half of a foot. This issue was not fully resolved until it again became the focus of discussion during the next phase of the instructional sequence.

King's footstrip and standard footstrip. Five days after the introduction of the king's foot scenario, the teacher explained that the king could not be everywhere at one time. Thus, the students' job was to think about ways that the king could use his foot to measure items in his kingdom without doing all the measuring himself. After various contributions, the teacher explained that one of the king's advisors suggested that the king trace his feet on pieces of paper. The teacher asked pairs of students to create their own footstrips composed of five of the king's feet; one student in each pair served as the king. Some interesting issues arose as students created their footstrips. Many students constructed footstrips which had gaps between the feet. Also, the teacher had unintentionally cut strips of paper that were shorter than five of the students' feet. Consequently, some students turned their last foot sideways in order for all five feet to fit on the strip rather than get more paper. Thus, the same "filling space" interpretation arose as the students were creating strips of five feet. In whole-class discussion, the teacher asked the students to measure the length of the rug with the footstrips they had created. It was during this discussion that the students who had not drawn their feet heel to toe realized they could not measure with their strips as they were. As a consequence, they revised their strips so that there were no gaps between feet.

Subsequent instructional activities involved students measuring lengths of objects on the playground with their footstrips. As we conducted observations of students' work, it was unclear to us how students were interpreting the remaining space when the end of an object extended past their footstrip. Consequently, in the whole-class setting the teacher asked one pair of students to measure the length of a cabinet that was situated along one of the walls in the classroom. They measured 5 full footstrips (25 feet) but did not know what to do with the remainder of cabinet that extended past the five footstrips. The end of the cabinet was bounded by one of the walls of the classroom so that the footstrip could not be laid flat along the remaining part of cabinet without running into the wall. As a consequence, most students simply slid the footstrip backwards until it reached the wall. This indicated to us that measuring for them was the act of filling the space to be measured, i.e., their activity was bounded by the space that was being measured (a tension between filling space and measuring it). As the class discussed how to measure the remaining cabinet, they decided that they could extend the footstrip up the wall and count only those feet alongside the cabinet (as opposed to counting the whole five feet in the footstrip even though some feet went up the wall). Several students commented that they just pretended "in their mind" to cut the footstrip where it measured the end of the cabinet and count only those feet.

A second issue emerged the same day as we observed students measuring items with their footstrips on the playground. As students were measuring, we intervened when they had counted, say 25 feet, and asked them questions such as, "how many footstrips would that be?" Many students could not answer that type of question which led us to believe that many students were having difficulty coordinating units. In other words, it had been our explicit goal at the outset of the teaching experiment that students come to act in an environment of quantities structured in various ways. Hence, during the subsequent whole-class discussion concerning the length of the cabinet (discussed in the last example), the teacher asked the class how many footstrips long the cabinet was (25 feet). In order to support this type of structuring activity, the teacher placed tape at the beginning and end of each footstrip that had been placed down (5 altogether). Students, then argued that 25 feet was the same as 5 footstrips. They reasoned that they could have measured it in terms of feet or footstrips, but that either way was the same. From an observer's point of view, the students were discussing the invariance of quantities under transformations. Please note that not everyone reorganized their activity as a consequence of this discussion such that they could readily structure the units in different ways.

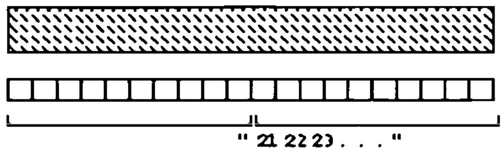
As part of the ongoing narrative, the following day the teacher explained to the students that the king wanted every footstrip to have the same feet on it so that people in his kingdom would obtain the same number when they measured objects. She gave each pair of students a Standard Footstrip which resembled the students' previous footstrips except that every standard footstrip was made with a standard sized foot.

Typical instructional activities with the standard footstrip involved asking students to use string to signify the lengths of different-sized fish that the king had caught on a fishing trip. Students were given a roll of string and asked to cut a piece that would signify a fish 8 1/2 feet, for example. Our observations of students' activity indicated that there were two different interpretations of 1/2. Many students felt that 8 1/2 referred to 8 feet and 1/2 of the ninth foot. In contrast, other students insisted that 8 1/2 meant 1/2 of the 8th foot or the distance up to half of the 8th foot. This was an important distinction in that we hoped that students would interpret the 8 as the accumulation of 8 feet. During subsequent whole-class discussions, the teacher asked students to show the "whole 8." She felt that most students interpreted 8 as the distance from the beginning of string to the end of the 8th foot. The problem from her point of view was the students' interpretation of 1/2. For those students that understood "the whole 8" as the distance covered by 8 full feet but reasoned that 8 1/2 was half of the 8th foot, we felt that the "half problem" was simply a matter of convention. In other words, the 8 signified a whole unit of 8, but they reasonably decided that 1/2 could mean half of the 8th foot. Thus, the teacher established the convention that 8 1/2 means a "whole 8" and 1/2 more. A few students continued to measure 8 1/2 as half of the 8th foot but adjusted their activity when reminded of the convention by the teacher or other students.

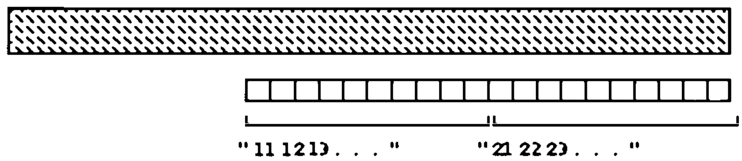
Smurfs. Eleven days after the beginning of the measurement sequence, the teacher began a second narrative about a community of smurfs that lived in mushroom houses. The students took on the role of smurfs and measured various items by using unifix cubes as substitutes for the food cans. It should be noted that this new context built on the prior activity of the students' measuring with their own feet and the footstrips. In the king's foot scenario, students were measuring with their feet, a physical extension of their bodies. In this new scenario, students placed unifix cubes, an "external" unit rather than a part of their body, end to end to measure lengths of objects. In the teacher's scenario, the smurfs sometimes needed to measure the length or height of particular objects by stacking their food cans. Initial activities with individual food cans included measuring paper strips cut from adding machine tape that signified both animals in the smurf village and the animals' pens. Typically, students placed long lines of cubes end to end along the full length of the object to find its measure.

As the narrative of the smurfs continued, the teacher explained to the students that the smurfs did not want to take an unlimited number of cans with them when they measured. After students suggested various ways in which the smurfs could resolve this problem, the teacher and students decided that the smurfs would only carry 10 food cans with them. This bar of 10 unifix cubes became known as a *smurf bar* because the teacher had related to the students during this narrative that smurfs were about 10 cans tall.

One main issue involving the accumulation of distance seemed to arise during the instructional activities with the smurf bar. During observations of students' activity, we noticed that some students were interpreting a measure such as 20 as the second decade rather than the distance covered by 20 cans. The distinction between 23 in the second decade and 23 as 23 individual cans emerged as the students were asked to cut strips of paper that were 23 cans long (see Figure 6).




(a) Twenty three in the second decade




(b) Twenty three as 23 individual cans

Figure 6. Two ways of measuring 23 with the smurf bar.

It seems that students reasoning in the first manner were thinking about the act of putting down the smurf unit the second time as defining a decade called the 20s. Thus, within the "20s decade" (the second decade) lie the numbers in that decade (21, 22, 23, etc.). During whole-class discussions, the teacher stopped students while they were measuring and asked “Where is the 20?” or “Where does 20 begin and end?” As students explained that 20 extended from 0 to the end of the $20^{th}$ cube they would show “the whole 20” by sweeping the distance with their hands. Also, several students counted 20 individual cubes to verify 20 cans long. Gesturing and counting individual cubes seemed to support students’ reorganization of their prior activity such that many students began to interpret 20 as the distance covered by 20 cubes. However, for many students this issue was not completely resolved at this point and arose again during another phase of the instructional sequence.

Measurement strip. As the narrative of the smurfs continued, the students measured various objects which the smurfs used or needed in their village. One of these activities involved asking students to measure and cut adding machine tape of different lengths that signified pieces of wood to be used for rafts. Many students would mark each iteration of ten with a marker on the adding machine tape as they found the length they needed. Building on this activity, the teacher asked the students if they could think of another way to measure without having to take the smurf bar with them. Several students suggested that, like marking on the adding machine tape to show the lengths of planks, the smurfs could measure a piece of paper that was 10 cans long and use it instead of the smurf bar to measure different items. After further discussion, the class agreed that it would be easier for the smurfs to measure if the strip of paper were 50 cans long rather than only 10 cans long. In pairs, the students used unifix cubes to construct measurement strips 50 cans long. However, the students simply marked iterations of 10 without marking individual cans within the decades (see Figure 7).

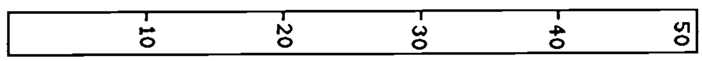



Figure 7. Initial constructions of the measurement strip.

Initially, it surprised us that none of the students drew lines between the units of 10 to show individual food cans. Upon further reflection, we conjectured that these constructions emerged out of students' prior participation in the mathematical practice of iterating with the smurf bar. Most students explained that they could use this measurement strip to measure any item as long as they could carry a few extra cans with them to measure the part of the object that extended beyond a decade. Our goal was to have a measurement strip 100 cans long emerge out of the students' constructions. However, since the students initially did not mark single cubes on their 50-strips, we decided to begin again with the construction of ten-strips. Thus, the next day the teacher again asked students to make a strip ten cans long that they could use to measure objects *exactly*. Again, most students' ten-strips had no markings of individual cans on them. Again, the teacher asked students to measure objects in the classroom with their modified ten-strips. A whole-class discussion ensued in which students demonstrated how they had used their new strips. One pair of students, who were measuring the white board, showed how they made marks at the end of every unit of 10 as they measured in order to keep track of their activity. As the discussion continued, the students explained that the marks at the end of the decades did not help them to find other measures on the board which were within a decade (e.g., 25). They subsequently suggested that individual marks would have to be drawn within a decade to find a specific measure. Then, the teacher taped 10 modified ten-strips on the board end to end, making a strip 100 cans long. Eventually, the class' measurement strip looked like the following (see Figure 8):

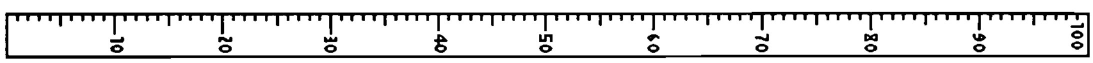



Figure 8. The standard measurement strip.

Initial activities with the new measurement strip included having students measure various items around the classroom. These activities seemed fairly routine for the students at this point, so the teacher introduced another type of instructional activity involving the use of the strip. Since one of our goals was to support the development of strategies for mental computation of two-digit numbers, we presented problems that might require students to reason with the measurement strip about quantities to 100.

One type of task which the teacher presented to the students in order to support reasoning with the stick involved the difference in the heights of sunflowers which were treated with a special formula invented by one of the smurfs. For example, one problem involved the students finding the difference in the heights of sunflowers that were 51 cans tall and 45 cans tall. The nature of the activity here is different from that involving simply measuring an item at hand because there were no actual sunflowers for students to measure. In addition to finding the height of the two sunflowers on the strip, the students had to reason with the strip in order to find the difference between the two heights.

It was during activities that supported students' *reasoning* with the measurement strip rather than *measuring* with the strip that mathematically significant issues arose. One issue which emerged concerned whether to count the lines or the spaces when using the strip to find the difference between the lengths of objects. For example, in the activity concerning finding the difference in the lengths of sunflowers which were 45 cans tall and 51 cans tall, we observed students solving this problem in two basic ways. In general, the students would find the 51st line on the strip and the 45th line and then count either the spaces/cubes in between the two lines or count the number of lines in between 45 and 51 including both lines for 45 and 51. Children reasoning in the second manner argued that the longer sunflower was 7 cans taller than the shorter sunflower because they counted 7 lines in between and including the lines beside 45 and 51. The students who were counting cubes or spaces argued that the longer sunflower was 6 cans taller than the other sunflower. For students who counted the lines, the line at 51 may have signified the instance of iterating the 51st cube rather than the end of a space that was an object with 51 as its measure. The distinction between a counting interpretation and a measuring interpretation again emerged as an issue. For this reason, we continued to ask questions regarding the "whole 51." During explanations of their solutions, several students used a smurf bar to iterate along the strip to find 51. They also used a smurf bar to find the difference between two lengths such as 45 and 51 by breaking the smurf bar into 4 cubes and 6 cubes and placing the 6 cubes between the lines for 45 and 51. Additionally, as the teacher attempted to support the students' reasoning through notations of their solutions, she notated their explanations on the a piece of white paper placed beside the measurement strip. As an example, consider the notation which the teacher did as students were explaining their solution to a problem involving the difference between a length of 35 cans and a length of 45 cans. To show 35, the teacher drew a line from the bottom of the strip to the top of the 35th cube and marked 35 to the left of this line. She then drew a line from 35 to the top of the 45th cube and labeled it 10. In order to support 45 as an accumulation of 35 cubes and 10 cubes, the teacher drew a second line from the bottom of the strip to the top of the 45th cube and labeled it 45 cubes (see Figure 9).

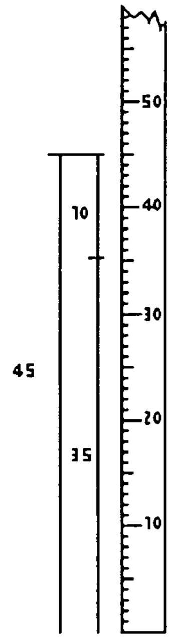



Figure 9. Teacher's way of notating the difference between 35 and 45 cans.

It was hoped that this way of notating would support part/whole reasoning as well. Also, the students had a trace of their activity to which they could refer as they were explaining the way in which they found the difference between the two lengths. Through these notations and through the use of the smurf bar in their discussions, the explanations of several of the students who initially had a counting interpretation shifted such that they began to talk in terms of the top of the 35th can rather than in terms of the 35th can itself. However, this issue arose again during the next phase of the sequence.

Measurement stick. Eleven days after the introduction of the measurement strip, the teacher described a situation about Sailor Smurf who was sailing his boat on a river of different depths. One thing he needed to know as he traveled was how deep the water was so that his boat did not scrape the bottom of the river and get stuck. The teacher explained that a measuring tool would be helpful if Sailor Smurf could put the tool in the water and read off water depths. To this end, she introduced the measurement stick (see Figure 10) and asked the students how Sailor Smurf might use this particular measurement tool.

Figure 10. The measurement stick used for determining water depths.

The left side of the measurement stick above re-presented 100 food cans each alternating in color from red to white. Further, the right side of the stick was marked with alternating blocks of red and white to indicate groups of 10 food cans, or smurf bars.

When the teacher asked the class how Sailor Smurf might use this measurement stick, the students responded that along the right side were long blocks that marked ten single cans on the left. Thus, the introduction of the measurement stick built on the students' prior activities with the smurf bars. Because the measurement stick was introduced in this manner, the students began to reason with it relatively easily. Initial activities consisted of marking on the stick how high the water would be if the river were, say, 31 cans deep.

It was on the very first day of using the measurement stick that the issue of counting lines versus counting spaces re-emerged as a topic of conversation. During whole-class discussion, students brought up the question of what was meant by 10 on the strip. This question arose out of their observations that some students, in their explanations, referred to 10 by pointing to the middle of the space for the $10^{th}$ cube while others pointed to the line at the end of the $10^{th}$ cube.

In order to support the resolution of this issue, we specifically asked the students to show how high the water was when it was 10 cans deep. Several of the students acknowledged that they meant that the water extended to the top of the $10^{th}$ can even when pointing to the middle of the $10^{th}$ can because that was the method they used to count individual cans. Therefore, the class decided they would point to the *top* of the $10^{th}$ cube when referring to the "whole 10" even if they had pointed to the *middle* of the cans to count them individually.

A second issue which arose as the students used the measurement stick involved the interpretation of the 20s as the second decade rather than 20 as 20 individual cans. During observations of students' activity on the second day of using the stick, we noticed that some students who were marking a depth of, say, 23 cans on the stick would actually mark at a depth of 13 cans. As these students were measuring, they would generally count "10" for the first decade, "20" for the second decade, and then count at the beginning of the second decade for 21 (at 11), 22 (at 12), and 23 (at 13). We felt that this was a mathematically significant issue that needed to be addressed in whole-class discussion. As such, we specifically asked students in whole class to demonstrate how they would find a depth of 23 cans. Through challenges from other students who either counted individual cans by ones from the bottom of the stick or who used the smurf bar to show where a depth of 23 cans would be marked, most of the students seemed to reorganize their activity such that they interpreted 23 as the distance covered by the accumulation of 23 cubes.

As the students continued to work with the measurement stick, pairs of students were given a laminated measurement stick and a transparency marker so that they could show a record of their activity on the stick. Since the measurement stick was designed to support the development of strategies for two-digit addition and subtraction, many activities with the stick consisted of showing a particular water depth (e.g., 27 cans) and marking a record of a series of changes in the height of the water (e.g., water rises 10 cans, 10 more cans, 10 more, etc.). Children were encouraged to find patterns and to reason about why these patterns occurred. For example, for 27 + 10 = 37; 37 + 10 = 47; etc., the pattern is that the new number always ends in a seven when adding by ten. Some students explained that the number plus three more from ten would always be a decade, and then there was always seven left from that ten to be added.

It was during these types of activities with the stick that a third mathematically significant issue arose. As we observed students working individually, we noticed that several of the students consistently counted from the bottom of the stick in order to find large numbers such as 93 rather than going backwards 7 from the top of the stick. We conjectured that this could be a result of participating in the prior mathematical practice of iterating with the smurf bar. In other words, finding the lengths of objects by iterating from the beginning of the item and saying, "ten, twenty, . . ." had become an established mathematical practice. However, when using the measurement stick, we wanted to see if the students could interpret the stick as a composite of 100 cubes and reason with it. In order to support this, we placed a piece of paper signifying a rock over the bottom of the stick to see if the students would reason backwards from 100. While most students did work backwards from 100, some drew the bottom of the stick over the rock so that they could count from the bottom of the stick to find a given depth of water.

Throughout the use of the measurement stick, it should be noted that there were various other ways in which we encouraged students to reason with the stick rather than always counting from the bottom of the stick to find water depths. For example, during whole-class discussions, the teacher would often ask the students to mark a depth of, say, 60 cans on the stick. Then, she would encourage the students to use that marked depth of 60 cans to find a depth of 85 cans. To support this, she would notate on white paper placed under the stick the ways in which students found the second height. In notating the explanation of one student, the teacher drew a jump of 10 from 60 to 70, another jump of 10 from 70 to 80, and a jump of 5 from 80 to 85 (see Figure 11).

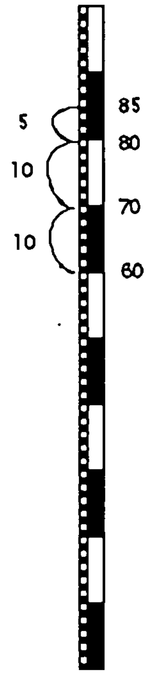



Figure 11. The teacher's way of notating the difference between 60 and 85 cans.

Thus, the student explained that it would take a jump of 25 from 60 to reach 85.

Preliminary analyses of post-interviews indicate that students developed a variety of sophisticated strategies for solving sums and differences between numbers up to 100 using the measurement stick. For instance, students were given a task with 84 marked on the stick and were asked to find 51. Observations of students' individual activity revealed at least three ways of solving this problem. These ways included students making jumps from 84 down to 51 and then counting the number of cubes signified by their jumps (see Figure 12).

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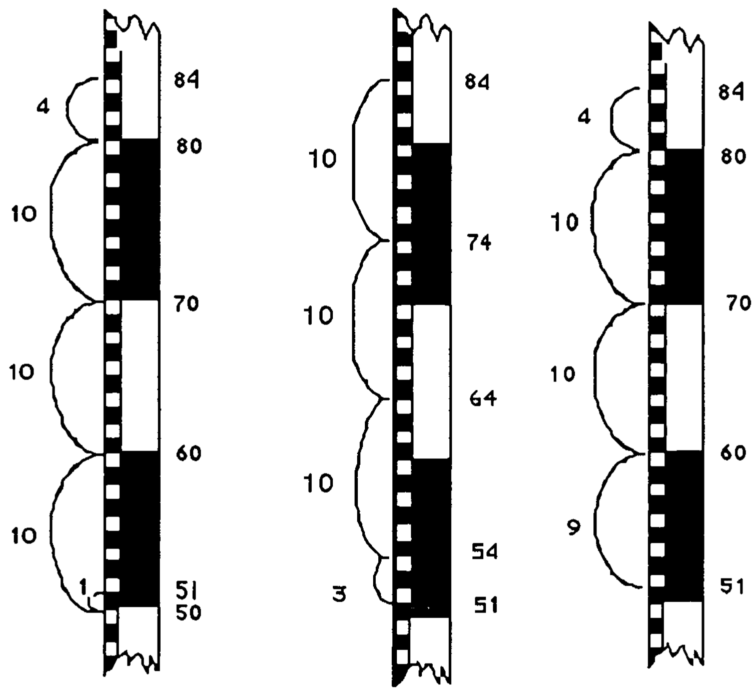


Figure 12. Students' various ways for finding 51 from 84.

Only one or two students continued to draw the remainder of the measurement stick over the rock that hid the bottom of the stick. Thus, they iterated up from the bottom of the stick by tens until they found 51. We reasoned that these students could not take 84 as a composite unit and use it to find 51. In other words, they could not "see" the 51 embedded in the 84 cubes; they were unable to reason about part/whole quantities (Steffe, 1992a; 1992b; 1994).

## Conclusion

In the account we have given of the measurement sequence as it was realized in this first-grade classroom, it is clear that the students' participation in the mathematical practices involved the use of many measurement tools. We believe that the students' activity with these various measuring devices profoundly influenced their development of mathematical reasoning as documented in the post-interviews. The students had developed ways of acting with the measurement strip and stick that had numerical significance for them and could be relatively effective with these tools. Their view of the jumps they drew as signifying composite units reflected intellectual capabilities that they had developed as they contributed to the emergence of the mathematical reality. In our view, further micro-analyses on the use of these tools in helping students reorganize their activity would contribute to understanding the development of this group of students . One such analysis might include documenting the evolution of measuring tools serving first as models of their informal mathematical activity to becoming models for their arithmetical reasoning. Also, we have attempted to coordinate social and psychological perspectives in our analysis by describing how informal psychological analyses of students' activities informed us as we planned for whole-class discussions. A further coordination would involve documenting the evolution of the mathematical practices while locating individual students' various interpretations and activity within these practices. This paper is a work in progress and such an analysis is forthcoming.

We have also attempted to describe our way of research that is consistent with the tenets of developmental or transformational research. In doing so, the notion of the learning trajectory as a way to anticipate and support students' mathematical development was essential. Before conducting the teaching experiment, the anticipated learning trajectory included, as its end goal, mental computation with two-digit numbers. With this goal in mind, we made local decisions and judgments in the classroom on a daily basis building on the students' activity each day. In this way, the students contributed to the realization of the learning trajectory. Therefore, the learning trajectory that was realized took a much different shape than initially anticipated. A metaphor that has proven useful for describing the emergence of a learning trajectory is that we laid the path as we were walking (Simon, 1995). In other words, we made decisions on a daily basis that shaped how the learning trajectory was realized and this realization took different paths as the experiment progressed. However, the local decisions were made against the background of a proposed learning trajectory and the instructional intent of the sequence. Hence, decisions are made in terms of the *big picture,* but the route to the end goal can take various shapes depending on how the trajectory is realized *in interaction*. This analysis was informed by the designer's intent of the measurement sequence and will serve to inform the designer of any modifications that can be made in the sequence for future teaching experiments.

[1] The research team whose countless hours of work made the teaching experiment a success includes: Paul Cobb, Kay McClain, Maggie McMagatha, Beth Petty and myself. Erna Yackel and Koeno Gravemeijer also collaborated on this project but were unable to participate on a daily basis.

[2] A typical math instruction period was structured in the following way. The teacher began class by setting up the task(s) for the day. Then, students worked in pairs or individually to solve the problems that the teacher had posed. As students worked on the problems, we would walk around the room and observe students' various solution methods. When it was clear that students were almost finished with their work, the teacher and researchers "huddled" in the room to share observations of students activity. The purpose of these huddles was to identify students' activity that might lead to productive mathematical discussions that were consistent with the mathematical intent of the instructional sequences. After these huddles, the teacher led whole-class discussion which was guided by the issues highlighted during the huddle.

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