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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following research papers: "A Model of Understanding Two-Digit Numeration and Computation" (H. Murray & A. Olivier); "The Computer Produces a Special Graphic Situation of Learning the Change of Coordinate System" (S. Nadot); "Epistemological Analysis of Early Multiplication" (N. Nantais & N. Herscovics); "Are the Van Hiele Levels Applicable to Transformation Geometry?" (L. Nasser); "Intuitive and Formal Learning of Ratio Concepts" (P. Nesher & M. Sukenik); "Early Conceptions of Subtraction" (D. Neuman); "Computational, Estimation Performance and Strategies Used by Select Fifth and Eighth Grade Japanese Students" (N. Nohda, J. Ishida, K. Shimizu, S. Yoshikawa, R.E. Reys, & B.J. Reys); "Associations Among High School Students' Interactions with Logo and Mathematical Thinking" (J. Olive); "Graphic Constructions with Computer to Learn 3D Reference Systems" (I. Osta); "Applied Problem Solving in Intuitive Geometry" (J.P. Pace); "L'incidence de l'Environment sur la Perception et la Representation d'Objets Geometriques" (R. Pallascio, L. Talbot, R. Allaire, & P. Mongeau); "Angles et Pixels - Quelle Synergie a 9 Ans?" (C. Parmentier); "Interaction by Open Discussion and 'Scientific Debate' in a Class of 12-Years Old Pupils" (T. Patronis); "Formal and Informal Sources of Mental Models for Negative Numbers" (I. Peled, S. Mukhopadhyay, & L.B. Resnick); "Inverse Procedures: The Influence of a Didactic Proposal on Pupils' Strategies" (A. Pesci); "Through the Recursive Eye: Mathematical Understanding as a Dynamic Phenomenon" (S. Pirie & T. Kieren); "Cognitive Aspects of the Learning of Mathematics in a Multicultural School" (N. Presmeg & A. Frank); "Qualitative and Quantitative Predictions as Determinants of System Control" (M. Reiss); "Transfer between Function Representations: A Computational Model" (B. Schwarz & T. Dreyfus); "Transition from Operational to Structural Conception: The Notion of Function Revisited" (A. Sfard); "Supercalculators and Research on Learning" (R. Shumway); "How and When Attitudes Towards Mathematics and Infinity Become Constituted into Obstacles in Students?" (A. Sierpinska & M. Viwegier); "Learning Y-Intercept: Assembling the Pieces of an 'Atomic' Concept" (J. Smith, A. Arcavi, & A.H. Schoenfeld); "Computers, Video, Both or Neither: Which is Better for Teaching Geometry?" (N. Snir, Z. Mevarech, & N. Movshovitz-Hadar); "Vocational Mathematics Teachers' Cognition of Mathematical and Vocational Knowledge" (R. Strasser & R. Bromme); "Training

Elementary Teachers in Problem Solving Strategies: Impact on Their Students' Performance" (J.K. Stonewater); "Developing Algebraic Understanding: The Potential of a Computer Based Environment" (R. Sutherland); "Verbal Evidence for Versatile Understanding of Variables in a Computer Environment" (M. Thomas & D. Tall); "Conceptual Adjustments in Progressing From Real to Complex Numbers" (D. Tirosh & N. Almog); "Does the Semantic Structure of Word Problems Affect Second Graders' Eye-Movements?" (L. Vershaffel, E. de Corte, & A. Pauwels); "The Lesson - A Preconceptional Stage" (S. Vinner); "An Analysis of the Emotional Acts of Young Children While Learning Mathematics" (E. Yackel, P. Cobb, & T. Wood); "The Use of Graphs as Visual Interactive Feedback While Carrying Out Algebraic Transformations" (M. Yerushalmy); "Images of Geometrical Transformations: From Euclid to the Turtle and Back" (R. Zazkis & U. Leron); and "A Knowledge-Base of Student Reasoning about Characteristics of Functions" (N. Zehavi & B. Schwarz). Includes a listing of author addresses. (MKR)

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A MODEL OF UNDERSTANDING TWO-DIGIT NUMERATION AND COMPUTATION

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This paper suggests that a full understanding of place value is not a prerequisite for many powerful, pliable computational strategies; that these strategies are formulated and widely used by young students; and that the use of these computational strategies facilitates a full understanding of place value. Based on an analysis of the computational strategies employed by young children, a model is proposed for the development of children's understanding of two-digit numbers. This model holds serious implications for both when and how to introduce two-digit numbers, and also for the role assigned to standard written algorithms in the junior school.

INTRODUCTION

Various models have been proposed to describe the development of young children's understanding of place value, e.g. Resnick (1983), Kamii (1985), Ross (1985). The general consensus among these researchers and also authors such as Richards and Carter (1982) seems to be that a full understanding of place value necessitates the conceptualization of *ten* as a new abstracted repeatable (iterable) unit which can be used as a unit to construct other numbers. It is also clear that this abstraction is quite difficult and that many third and fourth grade children have not attained this understanding in spite of many years of intensive teaching about place value.

We have available taped and transcribed protocols of interviews conducted at the *start* of the school year in 1987 with all 140 third grade pupils (aged eight to nine) of two fairly representative white schools. During each interview the student was presented with context-free addition problems involving whole numbers of increasing size, first orally, then set out horizontally, and finally set out vertically with the digits correctly aligned. The student was encouraged to solve each problem in whatever way he chose to, and asked to describe his solution strategy for every problem. These students had all had at least nine months' intensive instruction in place value and the standard vertical algorithm for addition. Analysis of the protocols shows, however, that these children use the taught algorithm infrequently, but rather prefer informal (untaught) computational strategies. The data also show vast qualitative differences in understanding of two-digit numbers, that is evidenced by the different types of computational strategies utilized by different students. The data have led us to postulate four levels of understanding of two-digit numbers, each level easily identified by the computational strategies employed to perform context-free computations. The levels that *precede* a full understanding of place value are probably important and far more useful than has been realized before, and function as vital developmental stepping stones towards the place value concept.

DIFFERENT RANGES OF NUMBERS

Steffe, Von Glasersfeld, Richards and Cobb (1983) describe children's understanding of the single-digit numbers as progressing through different levels of abstraction until the number is constituted as an abstract unit item with a meaning independent of physical objects or counting acts. This implies acquiring the *numerosities* ("how-manyness") of the numbers. Whereas children who have not yet acquired the numerosities of the range of numbers with which they have to perform computations must necessarily utilize pre-numerical strategies like counting all, children who *have* acquired the numerosities of the numbers have the capacity to use numerical strategies like counting on from first, counting on from larger, etc.

One could expect, maybe, that children's use of numerical strategies in computations with small numbers will transfer to computations with larger numbers. It has, however, frequently been documented that children find problems involving smaller numbers easier than those involving larger numbers (e.g. Carpenter and Moser, 1982), and that children change their behaviour when the sizes of the numbers in a given situation change (Cooper, 1984). The following are clear examples from our research showing children regressing to more primitive strategies or completely senseless juggling of symbols when they have to compute with larger numbers. (These problems were all presented orally. A summary of the child's strategy is given next to the problem.)

Elsa – a regression to the pre-numerical strategy of counting all

$7 + 5 = 12$	$5 + 6 = 11$ and add 1
$9 + 2 = 11$	$9 + 1 = 10$; $+ 1 = 11$

but

$11 + 17 = 27$	draws 11 small circles, then 17 small circles, then counts all
$37 + 5 = 42$	draws 37 small circles, then 5 small circles, then counts all

Marlene – a regression to meaningless manipulation of digits

$9 + 2 = 11$	$9 + 1 = 10$; $9 + 2 = 11$
$9 + 6 = 15$	$9 + 1 = 10$; $+ 5 = 15$

but

$29 + 4 = 96$	writes $29 + 4$, then: $2 + 4 = 6$ and puts 9 next to the 6
$23 + 12 = 53$	writes $23 + 12$, then: $2 + 3 = 5$ and $1 + 2 = 3$
$25 + 8 = 87$	writes $25 + 8$, then: $2 + 5 = 7$ and puts the 8 next to the 7

We argue that this regression is explained by the fact that these numbers are outside the children's range of constructed numerosities (and in Marlene's case, coupled with a perspective of mathematics as meaningless manipulation of meaningless symbols). When a child has acquired the numerosities of the smaller numbers, e.g. up to nine or twelve, he has not necessarily acquired the numerosities of the two-digit numbers as well, e.g. it is clear that Elsa's lack of "feeling" for 37 forces her to *recreate* 37 by means of circles which can be counted from the beginning. Although a child may therefore be able to employ numerical strategies within a *certain* range of numbers, the numerosities of numbers beyond this range have also to be acquired before he is capable of using numerical strategies when computing in a range of larger numbers.

DIFFERENT TYPES OF COMPUTATIONAL STRATEGIES

When children work with smaller numbers, their computational strategies fall into two broad classes: the *pre-numerical* strategies where the child has to count all because he has not yet acquired the numerosities of the numbers he is using, and the *numerical* strategies like counting on or bridging through ten. In computations with two-digit numbers, the pre-numerical/numerical distinction between strategies of course still exists. We can also distinguish different types of *numerical* strategies.

One type of numerical strategy is counting on. Another type, not based on counting, Carpenter (1980) calls heuristic strategies. Heuristic strategies often involve the decomposition of one or more of the numbers in a problem in order to transform the given problem to an easier problem or series of problems, e.g.

$$36 + 27 = 36 + 4 + 23 = 40 + 23 = 40 + 20 + 3 = 60 + 3.$$

Peter solves 36 + 27 as: "Three *tens* and two *tens* gives fifty, and six and then seven, which gives 63," whereas Marietjie solves the same problem by saying "*Thirty* and *twenty* gives fifty, then add six and seven." Although seemingly the same strategy, we see the different naming as manifestation of different understandings of two-digit numeration.

THE MODEL

We hypothesize that there is a relationship between children's understanding of two-digit numbers and the computational strategies that they use. It is not necessarily a linear relationship, because children do not consistently use their optimal computational strategies; at best we can say that the use of a certain type of computational strategy "defines" a certain minimal understanding of number and numeration. Based on our research data and a theoretical conceptual analysis, we have formulated a theoretical model describing four increasingly abstract levels of types of computational strategies with two-digit numbers in a given range, each type associated with its prerequisite understanding of number and numeration.

The first level

At the first level the child has not yet acquired the numerosities of two-digit numbers in a given range, and can therefore only use the pre-numerical strategy of counting all for computations in this range. The child knows the number names of the two-digit numbers and their associated numerals, and associates the whole numeral with the number it represents, but assigns no meaning to the individual digits. At this level the symbol group 63 can be regarded as a way of "spelling" the number name. A common error is to interchange the digits (e.g. writing 36 for sixty-three), yet often this has no adverse effect on the child's understanding of the number itself, as evidenced by Johan. To solve 30 + 4 (presented orally) he writes:

te e
03+4=34

and says "thirty plus four is thirty-four." His incorrect use of the t and e symbols (for "tens" units" in Afrikaans) in no way affects his computation, because the "tens" and "units" no meaning for him yet.

Results of previous research support the conclusion that the understanding of the whole numeral precedes understanding of the individual digits (e.g. Barr, 1978; Kamii, 1986).

The second level

At this level the child has acquired the numerosities of the two-digit numbers in a given range, which implies that he can utilize numerical computational strategies like counting on for computations in that range.

Whereas it is sufficient for a child to use only counting on strategies for smaller numbers, counting on becomes very tedious and also prone to error when used with larger two-digit addends.

The third level

At this level the child sees a two-digit number as a composite unit, and can decompose or partition the number into other numbers that are more convenient to compute with, e.g. to replace 34 with 30 and 4. This provides the child with the conceptual basis to use heuristic strategies.

The heuristic strategies used by students in our research are almost always based on *decimal* decomposition, i.e. a decomposition into a multiple of ten and some units, e.g. 67 as 60 + 7. But the tens are most emphatically *not* treated as "so many tens"; they are called by their *full number names*, e.g. sixty-seven becomes "a sixty and a seven", *not* "six *tens* and seven *units*." Students then use their knowledge of adding multiples of ten to obtain answers, e.g. Chris does 23 + 12 by saying: "Take the three away, add the twelve to the twenty, then add the three again", partitioning 23 into twenty and three. If both numbers are large, he partitions both: 36 + 27 is solved as "take the six and the seven away, thirty plus twenty is fifty; now add six, then add seven."

We have identified ten different heuristic strategies based on decimal decomposition. These strategies are very powerful and completely trustworthy: we have not found children applying them incorrectly. If the answer is incorrect, it has always been because the child has failed to add the units correctly.

The fourth level

At this level the child is truly able to think of a two-digit number as consisting of *groups* of tens and some units, i.e. the child can conceptualize *ten* as a new iterable unit, without losing the meaning of the number as a number. Whereas at level 3 the child works with ten as a *number*, that is no different than any other number, at level 4 he is able to work with ten as an *iterable unit*, a thing that can be counted as a unit, so that e.g. the number 23 is conceptualized as "two tens and three ones." Richards and Carter (1982) make this distinction clear:

> "Seeing ten as iterable is distinct form (*sic*) being able, say, to add ten and ten to make twenty. Seeing twenty as built up out of two units of ten is conceptually different from simply being able to add ten and ten to get twenty. In this sense, 'Ten and Ten' are distinct from 'Two Tens'. The former is not different from taking any pair of numbers..." (p. 61)

Concerning computation, level 4 understanding of numeration facilitates a progressive schematization ("shortening") and abstraction of the level 3 heuristic strategies. Here are some examples of this type of "groups of ten" thinking as opposed to the "tens part as a complete number" thinking of the previous level: Annemarie solves 26 + 37 (presented vertically) by saying: "Six plus seven is thirteen. Five tens plus one ten is sixty. Sixty plus three is sixty-three." For 36 + 27 she says: "Thirty plus two tens, that's fifty. Six plus seven is thirteen, that's sixty-three." Very few of the children we interviewed use this conceptualization of ten as a new "unit"; even Annemarie frequently prefers the level 3 method:

$$39 + 14 \rightarrow 30 + 10 = 40;\ 4 + 9 = 13;\ 40 + 10 + 3 = 53.$$

Level 4 understanding of numeration is a prerequisite for the *meaningful* execution of the standard written algorithms. A further abstraction allows one to operate on the *digits* of numbers—e.g. in the number 56, the meaning of the five as fifty or five tens can *temporarily* be suspended to work with 5 as a digit for the sake of convenience and the further progressive schematization of computational strategies.

When executing the standard algorithm has become automatic, it is difficult to deduce from the child's behaviour his understanding of the procedure and the underlying numeration concepts. However, it is clear from our research that many children *seem* to think of "groups of tens" in the correct (meaningful) way, because they *talk* about tens and units while they are computing, yet closer examination reveals completely superficial use of the terms "tens" and "units", with no possible evaluation of the numbers involved and the acceptability of the answers obtained. Sonja shows a proficiency with the standard algorithm for vertical addition, which is yet *not* based on a true understanding of the number symbols. She computes 34 + 17 and even 26 + 37 successfully by means of the standard vertical addition algorithm, but 5 + 37 (also written vertically with the digits aligned correctly) as 5 + 3 + 7 = 15, and 5 + 23 becomes 55 (the first 5 becomes the tens of the answer, and the units of the answer are the sum of 2 and 3). A superficial facility in executing the standard written algorithms may therefore hide serious deficiencies in the understanding of numbers and place value.

We have not come across a single child who operates with level 3 strategies ("the tens part as a *number*," e.g. "sixty") showing confusion of the above kind, probably because the mathematics underlying the level 3 strategies can never be hidden from the child: it is impossible to employ a level 3 strategy without understanding what you are doing, but it is extremely easy to implement a standard written algorithm in rote fashion. When the standard written algorithm is routinely employed, one operates on the *syntactic* level, manipulating the symbols *directly* as 'concrete' objects of thought according to certain rules, totally removed from their meanings as *numbers*. The level 3 and 4 heuristic strategies are, however, on the *semantic* level: One deals with the symbols by referring back to their meaning, i.e. in 23 the 2 refers to 20 or two tens. Many students who falter using the standard algorithm either do not have the necessary level 3 semantic knowledge to monitor their syntactic rules, or their syntactic and semantic knowledge appear to co-exist completely unconnected.

SOME RESULTS

The following table represents a summary of a preliminary analysis of the protocols of a few selected computations:

Percentage of students using types of strategies[a] [b]

Computation		Levels 1 & 2 Counting strategies	Levels 3 & 4 Heuristic strategies	Standard algorithm[c]
25 + 8	(set orally)	39 (65)	41 (81)	9 (46)
21 + 8	(set horizontally)	34 (67)	44 (82)	11 (25)
27 + 6	(set vertically)	31 (81)	36 (92)	22 (16)
34 + 21	(set orally)	16 (22)	49 (77)	16 (55)
34 + 23	(set horizontally)	11 (27)	56 (90)	19 (63)
36 + 27	(set orally)	10 (7)	42 (73)	22 (19)
26 + 37	(set vertically)	4 (17)	34 (87)	21 (31)

[a]Students not included in this summary either 'did not know', 'guessed', 'knew', or were not asked, because they failed or persevered with similar strategies in similar problems.

[b]Numbers in parenthesis represent the percentage of students who used a particular type of strategy that solved the problem correctly.

[c]A student was coded as using the standard algorithm if he gave direct written or verbal evidence of computing ones and tens separately as digits, from right to left.

The data clearly show to what extent students prefer heuristic strategies, and the high success rate of these strategies. In contrast, the data also show how few students actually employ the standard taught algorithm, as well as the low success rate in using the algorithm. The data also show, however, that a large number of students could not cope with the computations at all (e.g. in the last two categories a maximum of 26% and 41% respectively).

DISCUSSION

If our model provides an accurate description of the development of children's understanding of two-digit numeration, and if one believes that instruction should be based on the developmental sequences observed in children, then the model and our data have serious implications for the *teaching* of two-digit numeration and computation.

We stress that our subjects have had intensive instruction in "tens and units" place value and in the standard written algorithm for addition. While it is acknowledged that this type of instruction had contributed to the facility of many students with heuristic strategies (that were *not explicitly taught*, and that they preferred to the standard algorithm), this type of instruction also contributed to some students regressing to primitive (but to them meaningful) counting strategies for computing with larger numbers, to students' poor grasp of the standard algorithm when they chose to use it, and the helplessness of many others.

The near universal method of introducing two-digit numeration is by quantifying sets of objects by groupings of tens and ones and learning the numeral and number name associated with the sets of tens and ones. This approach is based on an *a priori* logical analysis of the concepts and has a great deal of intuitive appeal (to teachers) because of the understanding that (supposedly) precedes the symbolization. Yet, this approach does not consider the psychological nature of children's learning: understanding of two-digit numbers as groups of tens and ones is at level 4 and can therefore be expected to be too abstract for students who are operating at level 1, 2, or 3. We have ample evidence that it is not successful to teach children about the tens and ones meaning of the symbols in the symbol groups *before* they have become accustomed to a symbol group as representing a single number (level 1). The child has to work with 63 as a way of writing "sixty-three" for a *long* time before he becomes ready to understand 63 as 6 tens and 3 ones. Similarly, level 2 and level 3 thinking are necessary prerequisites for children to understand the sophistication of two-digit numeration and computation (cf. Murray, 1988).

There is some evidence that the compositional structure of numbers arises first in the context of oral counting. Kamii (1985, 1986) attributes children's difficulty with place-value partly to the teaching of standard procedures and outlines a teaching sequence based on counting, and reading and writing numerals without groups of tens, and on children inventing their own procedures to add 2–, 3– and 4-digit numbers. In a teaching experiment Barr (1978) found that kindergarten children who were introduced to two-digit numeration through counting, and reading and writing numerals before grouping exercises designed to provide understanding, did better than those who did the grouping exercises first.

It it seems that when students' level 1 and 2 counting strategies become too cumbersome for computation with larger numbers, teachers "help" children by introducing the standard algorithms as necessary (the only) computational tools. Some teachers may try to build a conceptual basis for the algorithms (level 4), but such efforts seem ill-fated if level 2 and 3 understandings are bypassed. Other teachers introduce the standard algorithms at the syntactic level, thereby undermining the development of adequate number concepts and fostering a perspective of mathematics as instrumental understanding. Rather than trying to discourage counting, teachers should help children to become efficient and accurate counters, and help develop level 3 understanding of numeration and computation, i.e. give *much more emphasis* to the first three levels of understanding. Level 3 understanding provides sufficiently powerful computational strategies, so that the introduction of the standard written algorithms may be delayed, if they should be taught at all.

The influence of computing technology necessitates a re-orientation of goals of elementary school mathematics, especially regarding the role of pencil-and-paper computation. There is a call for de-emphasizing standard written algorithms and integrating the calculator into the curriculum as the primary computational tool, accompanied by an increased emphasis on mental methods, estimation, understanding of number and algorithmic thinking as a mathematical process (eg. Olivier, 1988). It seems that the level 1 to 3 understanding of numbers and computational strategies are exactly those that are necessary for developing the skills of mental methods, estimation, and flexible computational pro- ...res and the understanding of number and numeration. It must be stressed that the ...istic strategies are not necessarily mental methods, because some children prefer to

record at least some portions of their computations. However, the types of strategies formulated by these third graders themselves correspond very closely to the "mental methods" and "street mathematics" described by authors such as Plunkett (1979) and Carraher (1988).

We have outlined a model describing the development of children's understanding of two-digit numeration and computation. Such a model should be complemented by a teaching program to facilitate transition through the different levels of understanding. We are at present implementing an experimental syllabus based on these ideas in eight schools. We shall report the results of the experiment in due course.

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THE COMPUTER PRODUCES A SPECIAL GRAPHIC SITUATION OF LEARNING THE CHANGE OF COORDINATE SYSTEM.

Suzon NADOT

GRECO DIDACTIQUE - PARIS

Summary: Articulated by the visual, this research is situated at the junction of informatic and mathématics. It takes its roots both in a technical conjuncture where the picture and the informatic develop a new communication and a social conjuncture where the didactic comes back to improve the pedagogy by analysing the knowledge and the rules which run the transmission of situations. The working on the computer introduces a real problematic which will make second form pupils think about the change of coordinate system and the change of variable.

INTRODUCTION

A lesson of maths about functions also deals with the outline of a curve so as to illustrate, explain and give a solution. This drawing, both géométrical and schématic is distinctly defined as an activity which must occupy an important place in the different parts of analysis programm, being specified by its language and its representative process; it's a real significant which can lead to a direct vision of things.

On the overland, the development of the graphic possibilities of the computer, must seduce the authors of didacticiels who conceived automatic graphic treatments: the imagiciels. As an automatic treatment, the imagiciel gives to the utilizer a real short cut in the executive tasks, it gives the possibility of going beyond the conventionnal and singular visions thanks to ways of juxtaposition, superposition and transformation. In a parallel direction, the learning of informatic language joins the mathematics notions linked to the functions. The writing of a simple imagiciel, the tracer, is based already in its conception upon the notion of function and rises all the questions of coordinate system.

By studing both contraint of nowadays ways of teaching

in our second form and the possibility offered by the computer, we've looked for an original situation of learning which also allowed us to see the procedures developped by the pupils.

TODAY'S WAYS OF TEACHING

After analysing different school books, the graphic activities have been divided into four categories of aims:

- the learning of graphic language where the accent is put on the explicitation of the translation algebra graphic.

- the writing of graphics where the student must give in a diagramm all the informations he has got by studying the function.

- the reading of graphics, the aim of which is to verify, conjecture, even solve in extremes cases, that is to say cases the pupils can't solve in another way. Contrary to the previous activity, the pupil must sketch what he perceives visually in an algebric way.

- Combined activity where graphics and algebra are mixed to give and treat information.

learning	writing	reading	combined
11%	84%	23%	16%

The main activity is indoubtly the writing of the curve, the final point of the algorithm of the study from which it stands as the instrument of coherence. We can also state the permanence in all the exercices of the "given coordinate system". The graphic representations refers on a triple (coordinate system, function, drawing), one of the elements being fixed, we must interpret the relation existing between the two others:

1st case: given the coordinate system, then we work on the relation function - drawing; it's the normal situation.

2nd case:given the function, we work on the relation coordinate system - drawing, by modifying the system, we can see different caracters of the function.

3rd case: given the drawing: we work on the

coordinate system - function, by modifying the coordinate system, we represent another function, it's the change of coordinate system.

THE KNOWLEDGE OF THE PUPILS

Following a double set of questions given to four second form last february then june, we should know the abilities of the pupils.

The item c-1 (february) refered to the interpretation of the graphic; a drawing being given to him, can the pupil read the adequate pieces of information in it about the function it represents.

c-1: By using the graphic representation of the g function below determine by explaining what you do, g(1), then the values of x as g(x)=2? g(x)<2?

The item c-2 (june) resumed the same question.

Parallel to the item c-1 was asked the item c'-1 in which the same problem was proposed to the pupils in a different language.

We consider the function f definited by $f(x) = x^2 + 1$. Compute f(2). Determine x so as f(x)=10 ? f(x) <10 ? Justify your answer.

% for the pupils who complete this item successfully:

item	f(a)	f (b)	f([c,d])	Σ
c'-1	88,3%	79,6%	6,6%	137
c -1	26,3%	12,4%	3,6%	137
c -2	56,2%	48,2%	38,7%	137

The reading of a graphic and its interpretation in another system of signs is not completely unterstand by all the pupils, even of we can see progress all along the school year.

The way of of interpreting graphic informations is less unterstan than treating them algebrically.

The three questions we asked belong to the same field of competence, however there's a hierarchy between them which remains after the familiarisation.

The items b-1 and b-2 refer to the simultaneity of the coordinate system: a naturel, the one of the squared piece of paper where the unities are spontaneously related to the square and so the one of the mathematics universe in which have to be represented clearly.

b-1 and b-2: If you can use a big size small squared piece of paper to situate the points, the coordinate of which are given below, how would you choose the units ? Situate the points is not asked.

x	- 67	-60	- 52	- 37	-30	- 15	0	15	30	37	52	60	67
y	-0,7	1	0,7	0,7	1	0	1	0	1	0,7	0,7	1	-0,7

For each of the two items, the percentages give the right, unfinished, wrong and the missing answers.

item	right	unfinish	wrong	missing	Σ
b-1	22,6%	13,1%	21,2%	43,1%	137
b-2	54 %	10,2%	23,4%	12,4%	137

The success remains feeble about this activity which seems elementary. The choice of the unities refers a numeric problem and a theorical problem: the one of the double coordinate system. Observations made in other time showed that the change of coordinate system is a blind point among the majority of students, it stands in all cases as a senseless point.

We have chosen among a lot of possibilities to set a didactic situation to introduce the change of coordinate system.

THE DIDACTIC SITUATION

Our situation being settled on an experimental approach, the computer created the problem and allowed an observation of the procedures of individual resolution.

1)Description of the situation

We asked the pupils to set out a part of curve on the screen of the computer by logo graphic programming. For the programmation of a setting out the curve, the confrontation of tl coordinate system is continuous; on one hand the logo

graphic universe set, always the same: the origin is in the centre of a squaring, the 64000 points of which are coded from −160 to 159 and from −100 to 99, the instruction point (x,y) lights a point on the screen having (x,y) as coordinates; On the other hand the mathematic universe conditioned by the part of the sisnusoïd to reproduce on the screen.

2) The population

We have experimented pupils in a second form learning informatics, for four weeks (four times two hours).

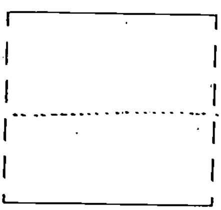
c) The development

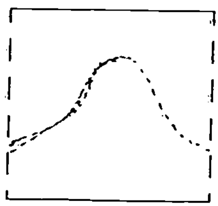
The pupils had to write a logo programm which drew a part of a curve representing the cosine function on the successive intervals: [−160°, 159°]; [−320, 318]; [−60,259]; [−160, 318].

The holding of the problem was immediate, the pupils had a reproduction of the sisusoïd settled by them during a previous work, they had already used imagiciels and therefore know what a computer could produce. They know the aim to reach and could control the rightness of the facts all along the working of the machine.

d) The procedures

Immediately in the case equality, they have programmed the algorithm of construction: point (x,cosx) for x which varies from 1 to 1 between −160 and 159. The reaction was one of perplexity, waiting was not successfull (d−1), so they came back to the programm, changed it but failed, then called us and facing our behaviour, no syntactic mistake, they have tried to find and they have seen the illegibility of the logo unity after several simulations, have pointed (x, 80cosx) (d−2).

(d−1)


(d−2)


In the case of the zoom, the first operation being don that of change interval of x, modifying −160 into −320

and facing a error message, two procedures emerged:

1rst type: point (x/2,80cosx) with x varying from -320 to 318. The procedure which developped itself as soon as the discovery of the relation of the amplitudes of the intervals and which proved its rightness by showing the right shape of the curved.

2nd type: point (x, 80cos2x) with x varying from -160 to 159.

To solve [-60,259] has been the most difficult case. The pupils saying: "we see what is happening, we can see why it doesn't work right, but we don't know how demarrage, we shouldn't find the origin in the middle. At last the two previous procedures reappeared.

Success of the group (two or three pupils in a group) among the 7 groups of the form.

1st week	2nd week	3rd week	4th week	item
6/7 85,7%	7/7 100%			drawing on [-160,159] equality
4/7 57,1%	6/7 85,7%	7/7 100%		drawing on [-320,318] zoom
0/7 0%	1/7 14,3%	2/7 28,6%	5/7 71,4%	drawing on [-60,259] travelling
0/7	1/7 14,3%	2/7 28,6%	3/7 42,9%	drawing on [-320,318] combined

ABOUT THE OBSERVATIONS

a) From a mathematical point of view, two main questions have been raised. We have put the accent on different coordinate system: "we can assert we change of function but we can also say we change of coordinate system". The change of ... e: the ordered pair (x/2, cosx) and (x, cos2x) are

different and however we get the same graph ? The answer has been more undecided, but it's precising numerically, the data in every cases that the pupils have been convinced that cosx and cos2x can be the result of the same process of calculus owing to the change of variables.

b) From a cognitive point of view the problem of the treatment of change of reference marks revealed itself very difficult particulary the case of a travelling, the decentring raises a bigger difficulty, the treatment of the simultaneity add - substract is less under control than that of multiply divid, and the visual signs which helped to make the extending are helpness when there is a decentring.

schema

c) From a point of view of the didactic situation, the decontextualisation is not easy. If the experimental generates some efficient practises and basic questions, a passage remains to be accomplished; It remains the passage leading to the decontextualisation. That's on this particular point we nowadays continue our work.

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Epistemological analysis of early multiplication

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Prior to launching a three-year study on the understanding of early multiplication in primary schools, an epistemological analysis of this conceptual scheme is essential. The approach used in this paper relies on criteria identified in the elaboration of a two-tier model of understanding, the first tier describing the understanding of preliminary physical concepts, the second tier describing the understanding of the emerging mathematical concept. At the physical level, a situation is perceived as being multiplicative when the whole is viewed as resulting from the repeated iteration of a one-to-one or a one-to-many correspondence. Three distinct levels of understanding can be identified with multiplicative situations. The emerging concept of arithmetic multiplication can also be described in terms of three complementary aspects of understanding.

In the last fifteen years, research on additive structures has been quite extensive and the results have been rather significant. More recently, several PME and PME-NA papers have dealt with the concept of multiplication of real numbers. However, hardly any studies have been concerned with the early beginnings of multiplication of natural numbers. An investigation of the acquisition of this conceptual scheme by primary school children will be carried out over a three year period at the University of Sherbrooke. The objective of the present communication is to open the discussion on the proposed conceptual framework used in this project.

The different meanings of multiplication in N

If we ask any teacher what is the meaning of multiplication of natural numbers, one usually gets as a response: "multiplication is repeated addition". The description here refers essentially to the arithmetic procedure needed to find the answer. Curiously, the other three arithmetic operations can be identified as the arithmetic reflection and quantification of physical procedures : **addition** refers to the quantification of either

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augmenting a given set or of putting together two given sets; **subtraction** usually refers to the quantification of the set remaining after some elements have been **taken away** from an initial set; **division** refers to the quantification of the result of **exhaustive equi-partitioning** giving either the number of parts, or the number of elements per part. But when it comes to multiplication, one is hard put to identify a physical action corresponding to it.

In a survey of the different meanings used for the introduction of multiplication at the primary level, Herscovics et al. (1983) found that some textbooks presented it by counting jumps on the number line, while others referred to the notion of a Cartesian product (number of blouses x number of skirts = number of outfits). In their critique of these different models, Herscovics et al. showed that they involved concept more advanced than the quantification of discrete sets and thus, that they did not constitute a good intuitive basis for the initial construction of multiplication. They were unlikely to tap the natural emergence of the multiplication scheme in the young child.

Piaget and Szeminska (1941/1967) are the ones who came closest to identifying multiplication with a physical operation when they described it as the iteration of a one-to-one correspondence between several sets: "From a psychological point of view, this simply means that setting up a one-to-one correspondence is an implicit multiplication: hence, such a correspondence established between several collections, and not only between two of them, will sooner or later lead the subject to become aware of this multiplication and establish it as an explicit operation" (Piaget &Szeminska,1967, p.262). In their evaluation of Piaget's model, Herscovics et al.(1983) pointed out that preliminary results obtained when young children are asked to use a deck of cards to make four piles of three cards, they are more more likely to achieve this through the iteration of a one-to-many correspondence than through the more difficult iteration of a one-to-one correspondence. However, both procedures are possible and hence must be accepted as actions corresponding to the generation of a multiplicative situation.

That children can generate quite early various multiplicative situations is not too surprising. But can one claim that by iterating a one-to-one or a one-to-many correspondence they are actually aware of the situation as being multiplicative? Of course not. This claim can only be made when they perceive the whole set as resulting from the iteration of such correspondences. Using this last criterion as a working definition of multiplication, one is then in a position to perform an epistemological analysis of this notion. The term 'epistemological analysis' refers to the analysis of a conceptual scheme along likely patterns of construction by the learner. The particular

method of analysis that is proposed here is based on criteria developed by Herscovics & Bergeron (1988) in the elaboration of their Extended Model of Understanding.

An Extended Model of Understanding.

At the last meeting of PME-NA, Herscovics & Bergeron (1988) have suggested that the construction of some mathematical concepts might be well described within a framework of a two-tier model of understanding, the first tier describing the understanding of preliminary physical concepts, and the second tier identifying the understanding of the emerging mathematical concept. In this model, the **understanding of preliminary physical concepts** involves three levels of understanding:

> **intuitive understanding** which refers to a global perception of the notion at hand; it results from a type of thinking based essentially on visual perception; it provides rough non-numerical approximations;
>
> **procedural understanding** refers to the acquisition of logico-physical procedures (dealing with physical objects) which the learners can relate to their intuitive knowledge and use appropriately;
>
> **logico-physical abstraction** refers to the construction of logico-physical invariants, the reversibility and composition of logico-physical transformations and generalizations about them.

The understanding of the **emerging mathematical concept** can be described in terms of three components of understanding:

> **procedural understanding** refers to the acquisition of explicit logico-mathematical procedures which the learner can relate to the underlying preliminary physical concepts and use appropriately;
>
> **logico-mathematical abstraction** refers to the construction of logico-mathematical invariants together with the relevant logico-physical invariants, the reversibility and composition of logico-mathematical transformations and operations, and their generalization;
>
> **formalization** refers to its usual interpretations, that of axiomatization and formal proof which at the elementary level could be viewed as the discovery of axioms and the elaboration of logical mathematical justifications. Two additional meaning are assigned to formalization: that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior procedural understanding or abstraction already exist to some degree.

This model suggests a distinction between on one hand logico-physical understanding which results from thinking about procedures applied to physical objects and about

spatio-physical transformations of these objects, and on the other hand logico-mathematical understanding which results from thinking applied to procedures and transformations dealing with mathematical objects. In this framework, one can contend with reflective abstraction of actions operating in the physical realm without necessarily describing it as somehow having to be mathematical. We will now use this model to describe the understanding of early multiplication, that is, products of numbers not exceeding 9, for any discussion of the larger products would also require consideration of the multiplication algorithms.

The understanding of preliminary physical concepts.

Let us recall that we are identifying conceptualization at the preliminary physical tier according to the following criterion: **A situation is perceived as being multiplicative when the whole is viewed as resulting from the repeated iteration of a one-to-one or a one-to-many correspondence.** Using this as a working definition, one can then attempt to classify various knowledge related to this conceptual scheme according to the different levels of understanding.

Intuitive understanding. A first criterion of intuitive understanding might be the ability to perceive visually the difference between a situation that is multiplicative and a situation that is not . For instance a set consisting of several equal subsets might be compared to a set consisting of unequal subsets. Since rectangular arrays are so useful in illustrating multiplicative situations, a second criterion might establish if the rows or columns can be viewed as equal subsets. A third criterion might involve the visual comparison of two multiplicative situations in which one of the "factors" is different. For instance, without knowing the total number of objects present, one could compare 4 sets of 5 chips with 4 sets of 6 chips or 4 sets of 5 chips with 3 sets of 5 chips and decide where there are more. A fourth task might involve various configurations of 9 subsets of 7 objects. The total number would be large enough to discourage enumeration but bring out the fact that if the number of subsets and the number of elements in the subsets are the same, the whole sets must have the same cardinality.

Procedural understanding. We are looking here for the generation of multiplicative situations calling on logico-physical procedures based on the iteration of 1:1 and 1:n correspondences. A first criterion might the child's ability to transform a additive situation (in which all the subsets are not equal) into a multiplicative one, by a redistribution of some of the elements. Another task might involve the covering of a

rectangle by equal strips which would correspond to either columns or rows. A third task might verify if the child is able to relate a multiplicative situation generated by a 1:n correspondence to a multiplicative situation based on a 1:1 correspondence. A fourth task might assess the the child's awareness of the fact that some quantities of chips can be arranged into rectangular arrays of two or more rows whereas some quantities cannot. This type of activity leads to the eventual notions of prime and composite numbers.

Logico-physical abstraction. The initial problems we are looking for involve the invariance of the whole with respect to some irrelevant spatio-physical transformations. A first criterion of logico-physical abstraction might be the invariance of the whole with respect to various configuration. For instance, a set of 12 chips can be arranged into subsets of 2, 3, 4 and 6 elements respectively. A second criterion might involve notion of commutatitvity. This can easily be established by rotating a rectangular array through 90°. A third task might aim at verifying the equivalence of certain factors thrcugh a redistribution of the elements. For instance, a set subdivided into 4 subsets of 3 might be transformed into 2 subsets of 6. This is somewhat different from the first activity since it starts from an existing multiplicative configuration. A fourth criterion might involve the notion of distributivity. For instance, a 4 by 5 array and a 4 by 6 array both represent two multiplicative situations. However, when they are combined along the rows, the resulting 4 by 11 array is again a multiplicative situation which illustrates the distributivity axiom.

The understanding of the emerging mathematical concept

Procedural understanding. By procedural understanding we mean the appropriate use of explicit arithmetical procedures. Initially, when young children in grade 2 are asked "How much is three times four?", many will respond by saying that they have not learned it yet. Some will model the problem by making three sets of four and count them starting from 1. While simple enumeration provides an answer, it cannot be considered as a multiplicative procedure since it does not take into account the existence of the subsets. The most primitive procedure that can be considered as being somewhat multiplicative must provide such evidence. This is reflected when the child manages to skip count on a number line: 4,8,...12. If no number line is available, the child may remember the first part and produce "4,.....8,9,10,11,12. A more advanced procedure involves repeated addition: 4 + 4 = 8 and 8 + 4 = 12. Gradually, by grades 4 and 5, children learn to memorize some number facts which they can use in deriving larger products as for example the product

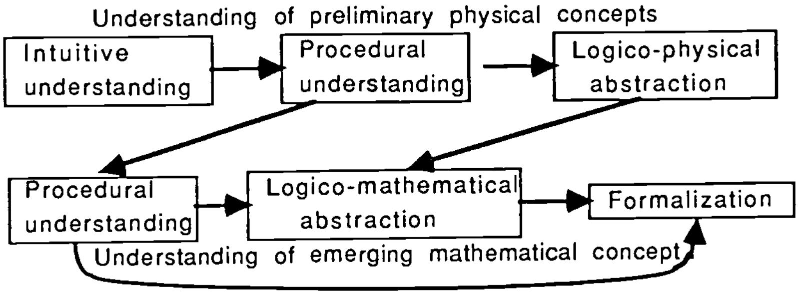
4 x 6 which may be obtained by the smaller product 2 x 6 =12 and then the sum 12+12 = 24.

Logico-mathematical abstraction. Gradually, as the child's procedural knowledge evolves, the reversibility of the operations and the perception of some mathematical invariants becomes possible. For instance, the child no longer needs concrete material to break a number down to its factors. This inevitably leads to the perception of these factors as also being divisors and thus the operation becomes reversible. Knowledge of the multiplication table also enables the child to perceive the equivalence of various products with respect to a given number without having to depend on their different configurations. In terms of axiomatizations and generalizations, the commutativity of multiplication becomes self-evident and somewhat later, so does the distributivity of multiplication over addition.

Formalization. Interpreting formalization in terms of the symbolic representation of the learner's previously acquired knowledge, children first learn the usual notation for multiplication and can interpret 4 x 3 as meaning four sets of three objects. They also can recognize an appropriate additive situation as being multiplicative by expressing the sum as a product (e.g. 3 + 3 + 3 + 3 = 4 x 3). On the other hand, when this arithmetic equation is read from right to left, it expresses a form of procedural understanding since it symbolizes repeated addition. Interpreting formalization in terms of axiomatization, the axioms of commutativity and distributivity can be crystallized in various notations, a simple one being $\square \times \bigcirc = \bigcirc \times \square$ and $\Delta (\square + \bigcirc) = \Delta \times \square + \Delta \times \bigcirc$. The use of letters might create some difficulties initially.

By way of conclusion.

It should be noted that the three levels of understanding included in the first tier are linear. Without prior intuitive understanding, the acquisition of concrete procedures could hardly qualify as understanding. Similarly, one cannot expect the child to achieve any logico-physical abstraction without being able to reflect on the procedures used to generate multiplicative situations. Neveretheless, the model as a whole is not linear. The aspects of understanding identified in the second tier need not await the completion of the physical tier. Well before they achieve logico-physical abstraction, children can start acquiring the various relevant arithmetic procedures by the quantification of problems introduced in the first tier. The formalization of multiplication need not await the completion of logico-mathematical abstraction; the formalization of the arithmetic procedures will occur much earlier than formalization of the axioms. The following diagram illustrates this non-linearity:



This work has some interesting pedagogical implications. It suggests an alternative to the age-oldt tendency of introducing multiplication merely as repeated addition. Instead, it suggests that prior to the introduction of this arithmetic operation, one might present children with didactical situations in which they could recognize and generate a great variety of multiplicative problems. Indeed, corresponding to the different criteria used for the different levels of understanding in the first tier, one can develop a broad sequence of activities. The stress on work at the concrete level should not be interpreted as an attempt to diminish the importance of the traditional work on explicit arithmetic procedures. But the prior introduction of multiplicative situations will provide some motivation and relevance.

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ARE THE VAN HIELE LEVELS APPLICABLE TO TRANSFORMATION GEOMETRY?

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Abstract. *To improve the teaching of Geometry in Brazil, a study has been started to investigate possible changes. This article reports an attempt to analyse school Transformation Geometry according to the van Hiele levels of thinking. Van Hiele levels were established for Transformation Geometry and a pilot study was carried out to check their validity, as well as the relation between the van Hiele levels in traditional Euclidean Geometry and in Transformation Geometry.*

The teaching and learning of Geometry in Brazilian secondary schools is problematic. It has not changed for many years, having a Euclidean approach, based on axioms, theorems and proofs. Whenever teacher training courses are offered, the most popular subject is Geometry, suggesting how insecure teachers feel about it. On the other hand, the students do not like Geometry, since they cannot grasp its abstraction and the meaning of the demonstrations.

An overview of the most used textbooks in Brazilian secondary schools shows that:

- the study of Geometry starts with point, line and plane, treated as concepts that do not admit definitions;
- almost no concrete materials are used;
- the Geometry content is concentrated in the final parts of the two most advanced books. As the time is often not enough to completely cover the books, a great part of the geometry is missed.

*This study is part of a Ph D degree at King's College, University of London, supervised by Professor K Hart.

A change in the teaching of Geometry is, then, urgent, both in its content and in the way it is taught. To be effective, this "new" Geometry teaching must be based on research evidence. But there is a lack of research about the teaching and learning of Geometry, as pointed out by Bishop (1983). Most of the research papers on Geometry presented at the last PME meetings have been about the van Hiele levels of thinking.

The van Hiele Levels of Thinking:

In the late 50's, Pierre van Hiele and his wife Dina van Hiele-Geldof, worried about their secondary students' performance in geometry in The Netherlands and so, dedicated their Ph D studies to this problem. In 1957, P van Hiele presented his paper: "La pensee de l'enfant et la geometrie" (van Hiele, 1959) at a Mathematics Education conference in Sevres, France. In this article, van Hiele established a model of thinking in Geometry based on five levels and on five phases.

The van Hiele levels are summarized by Hoffer (1981, 1983) as:

Level 0 (Recognition) : students recognize figures by their global appearance, but they do not explicitly identify their properties;

Level 1 (Analysis) : students analyse properties of figures, but they do not explicitly interrelate figures or properties;

Level 2 (Ordering) : students relate figures and their properties, but they do not organize sequences of statements to justify observations;

Level 3 (Deduction) : students develop sequences of statements to deduce one statement from another, but they do not recognize the need for rigor;

Level 4 (Rigor) : students understand the importance of precision in demonstrations and analyse various deductive systems.

To progress from one level to the next, students must experience the following 'phases' : inquiry, direct orientation, explanation, free orientation and integration (van Hiele, 1959).

The main characteristics of the van Hiele model were summarized by Fuys, Geddes and Tischler (1988) as follows:

(a) the levels are sequential;
(b) each level has its own language, set of symbols and network of relations;
(c) what is implicit at one level becomes explicit at the next level;
(d) material taught to students above their level is subject to a reduction of level;
(e) progress from one level to the next is more dependent on instructional experience than on age or maturation; and
(f) one goes through various 'phases' in proceeding from one level to the next.

The British Experience:

An attempt to improve the teaching of Geometry was made in Great Britain in the late 60's, replacing Euclidean geometry by Transformation Geometry in the secondary school syllabus. According to Küchemann (1981), the reasons for this change were:

(a) the fact that Euclidean geometry was not appropriate for the majority of the students; it was taught in a deductive way and learned by rote;
(b) the hope that students would discover general rules about the combination of transformations, providing insights into mathematical structure;
(c) the belief that Transformation Geometry would provide a coherent embodiment of matrix algebra, giving the students an idea of the unity of Mathematics.

After more than ten years of school use and influenced by the results of the CSMS :t (Hart, 1981), Küchemann (1980) stated: 'Unfortunately, it has become

increasingly clear that these aims (of the introduction of Transformation Geometry) are as inaccessible to many children as was the deductive geometry that the transformations replaced, and it is doubtful whether their central role in courses for 11 - 16 year olds can any longer be justified".

This suggests the need for further research to find out whether Transformation Geometry can really be a solution to the challenge of reforming the teaching of Geometry.

The Present Research:

In this work, levels corresponding to those established by van Hiele for Euclidean Geometry are suggested for Transformation Geometry. Further, an investigation was carried out in order to:

(a) check the validity of these levels, i.e., if they form a hierarchy; and

(b) find out if there is a relation between the levels attained in traditional Geometry and in Transformation Geometry.

The levels considered for Transformation Geometry are:

Basic level: students recognize and identify the transformations (reflection, rotation, translation and enlargement);

Level 1 : students identify and analyse the properties of the transformations, as: mirror-line (reflection), centre and angle of turning (rotation), scale factor of enlargement;

Level 2 : students recognize combinations and inverses of transformations;

Level 3 : students understand the significance of deduction, the converse of a theorem and the necessary and sufficient conditions;

Level 4 : students make formal demonstrations of properties and establish transformations in different systems.

According to van Hiele, it is very difficult to achieve level 4 in secondary school. Actually, Usiskin (1982) stated that "level 4 either does not exist or is not testable". For this reason, this investigation concerns only the levels below level 4. A test on Transformation Geometry was devised, to the same pattern as Usiskin's van Hiele test for traditional Geometry i.e., sets of multiple-choice questions, each set corresponding to a van Hiele level. So, five questions were selected from each set of Usiskin's test (excluding set five), to match with the five questions in each one of the four sets in the Transformation Geometry test (corresponding to levels: Basic, 1, 2 and 3).

As a pilot study, both tests were given to 24 15 - year old British students from a comprehensive school. The tests were marked according to the following criterion: if the student scored three or more in a set of five, s/he was considered as attaining the corresponding level.

It can be stated that the Transformation Geometry levels form a hierarchy, since only two students (8.3%) attained a higher level without attaining a lower one (both of them attained level 2 and not level 1). These two students were excluded from the sample, as well as another student whose response showed the same type of discrepancy in the traditional Geometry test.

For the sample of 21 students, the relation between the levels obtained is shown in Table 1. Table 2 shows the number of students that attained different levels in the tests.

There is a correspondence between the van Hiele levels in traditional Geometry and Transformation Geometry, as shown by this small sample. However, it is not as strong as one might hope. All the children in the sample have learned their

Geometry mainly through transformations. It is interesting to observe that Mayberry (1983) and Gutierrez & Jaime (1987) have found no correlation between children's van Hiele levels on different geometric concepts such as triangles, quadrilaterals, angles, etc.

Level	No of students
Basic	0
1	9
2	2
3	1

Table 1 : Same level in both tests (57.1%)

Level in Transformation Geometry	Level in Traditional Geometry	No. of students
3	2	1
3	1	2
2	1	5
1	2	1

Table 2: Different levels

Comments

The analysis of the traditional Geometry test (adapted from Usiskin) showed that, when the question at the Basic Level required that a square be recognized as a rectangle, all the sample failed. When a similar question was asked at Level 2, only three students out of the six who otherwise attained this level succeeded.

On the other hand, in the Transformation Geometry test, only one student seemed to know the meaning of "congruent triangles". In question 2.5, 13 students have ticked the first and the third options, showing the knowledge of the properties of translations. Nevertheless, they did not tick the option that mentioned congruent triangles.

2.5 - Tick which are true for a translation:
- ☐ The image of a flag is a flag of the same length
- ☐ The image of a horizontal flag is a vertical flag
- ☐ The image of a horizontal flag is a horizontal flag
- ☐ The image of a triangle is a congruent triangle
- ☐ None of these is true

Question 2.5 - Transformation Geometry Test

From these observations, it can be seen that there is a gap in the knowledge of some concepts and properties of shapes. One possible reason for this could be the fact that these children had a Transformation Geometry approach, with no emphasis on Euclidean Geometry. But this is a point that requires further research.

The fact that the majority of those shown in Table 2 attained a higher level in Transformation Geometry can be explained by their greater familiarity with the terms and pictures in this, rather than with those in traditional Geometry.

The pilot study has demonstrated that it is possible to categorize transformation geometric concepts according to van Hiele levels and that these levels appear to be hierarchical.

The continuation of the research will be concerned with the most effective ways of teaching Geometry in the classroom.

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Intuitive and Formal learning of Ratio Concepts

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The effect of the formal presentation of blue and yellow color mixtures as rational numbers on students' ability to solve ratio problems of comparing these mixtures was examined. Children in grades 7, 8, and 9 were first given comparison of mixtures problems with an opportunity to watch the actual outcome and check their predictions (Task 1). They were then asked to solve similar problems, after being introduced to the representation of the mixtures as rational numbers (Task 2), and finally tested about their ability to reach a general solution to problems of the same nature (Task 3). Overall results' analysis showed that a substantial number of students had benefited from both tasks 1 and 2, but the actual contribution of each one of them still remains to be tested.

The concepts of ratio and proportion and their development and acquisition in children of different ages have been the target of many research studies. (See Tourniaire and Pulos, 1985 for a partial review of the literature). Most of these studies have manipulated one or more of the following variables: The context of the problem (e.g. balance beam, fish and food sugar and water, etc.), the numerical values appearing in the problems,(e.g. presence of an integer ratio, presence of 1, etc.), and the kind of task (comparison or missing value problems). The procedure was usually consisted of administering a test of ratio problems, (given with or without illustrations, presented either in written or oral form), and analyzing the subjects' responses in terms of the strategies used to answer these problems. The results of most of these studies have revealed that a substantial number of secondary school students did not use proportion successfully to solve even simple ratio tasks. One of the dominant erroneous strategies used by children of all ages is the additive strategy, that is, viewing the relationship within the ratios as the difference between terms, instead of realizing that it is a multiplicative one (e.g. Karplus et al., 1974; Hart,1978; Quintero, 1987).

The main focus of the present research was not on the way children solve problems involving ratios, but on the tension between the intuitive judgement as demonstrated in the experimentation of

something that calls for ratio and its formal representation in the form of a/b. We were interested in the question whether children perceive those problems as ones involving ratios at all. If not, then we were interested in the effect of concrete experiencing with the problems, accompanied by, and explicitly relating to formal presentation of ratios. Would this experience make the children realize the need for using correct strategies, and improve their performance on those problems.

This study differs from previous ones in several aspects: (1) The ratios used were intensive quantities (mixtures of blue and yellow colors that result in a green color shade), which could be objectively perceived and judged by the students, thus enabling them to make concrete comparisons between pairs of ratios. (2) The exact numerical values of the ratios were not controlled, however they were obtained so as to induce either a multiplicative or an additive strategy. (3) In order to see if the intensive quantities were perceived by the children as ratios, Ss were asked while solving the problems, to write down the quantities they dealt with. We were interested to find out how many of the Ss will spontaneously use the ratio formal notation. (4) Subjects were getting immediate feedback to their responses, by confronting their judgments with actual results, thus giving them the opportunity to change strategies accordingly. (5) After concrete and intuitive experiencing with the intensive quantities, their formal representation was presented to the students prior to their judgement. This was done in order to examine the effect of the formal representation on their strategies. The subjects' ability to generalize their learning, in terms of comparing any pair of intensive quantities, as a result of the formal representation, was also tested. The major hypothesis was that formal representation enhances the understanding of ratio and proportion concepts.

Method

60 subjects participated in the experiment, 20 of each grade 7, 8, and 9 selected randomly from a junior high school in Haifa. They were all individually interviewed for a period of about thirty minutes each. The subjects were told they are going to take part in an experiment which deals with children's knowledge about mixing colors. Each session consisted of three main tasks: (1) Ss had to compare and predict the resulting shade of two mixtures of different paint quantities in terms of

same or different color-shade, and then observe the result. (2) Ss were introduced to the formal presentation of the paint quantities as rational numbers (ratios), and then had to compare and predict as in task one. (3) Ss were given a short written test involving generalizations about mixing colors. Following is a detailed description of each task:

Task 1: Intuitive Judgement

The experimenter presented two containers filled with blue and yellow colored water, and poured with a pipette a specified number of blue drops (B1) into a cup (cup 1). The subject was first asked to predict the shade of color if a specified number of yellow drops (Y1) was added to the same cup. After observing the results, another empty cup (cup 2) was introduced. The subject was told that a specified number of blue drops (B2) and of yellow drops (Y2) would be added into cup 2 and was asked to judge whether the color of both cups (cup 1 and cup 2) would be the same or different, He was also asked to explain his answer. The experimenter then made the mixture of cup 2, and let the student watch the result, and judge whether his prediction was realized. The color obtained in all mixtures was demonstrated by dipping a cotton swab into the mixture, so that the judgement of the color in each cup would not be influenced or biased by the total amount of liquid in it, and so that the difference or sameness of color in both cups could be clearly seen. If the subject's judgement, after watching the results, about the actual colors obtained, did not coincide with his former prediction, he was asked to try to explain the discrepancy. All the above procedure was repeated seven times with varying values for B1, Y1, B2, and Y2. These values were selected so as to induce either a multiplicative or additive judgement strategy. For example, the pair B1=7 Y1=2 (in cup 1), and B2=14 Y2=4 (in cup 2) was assumed to induce a multiplicative strategy, while, on the other hand, the pair B1=7 Y1=2, and B2=17 Y2=12, would call for an additive strategy. In addition, the quantities were checked experimentally to make sure that the comparison between the two mixtures' shades would not be ambiguous. (See appendix A for the exact quantities). All subjects received the same quantities in the same order, except for the first two pairs of mixtures, in which the order was interchanged between subjects. The procedure of the last pair did not include the actual mixing, and

the subjects had only to predict the outcome. Before each prediction of the outcome was given, and the pair of mixtures was made, the subject was encouraged to write down the quantities specified by the experimenter, in any way he wishes to do so, for future comparisons of tryouts.

Task 2: Formal Representation

After predicting and observing the results of seven pairs of color mixtures, the subject was introduced to a way of representing the specified quantities in the form of rational numbers (or ratios), i.e. B1/Y1 in cup 1 and B2/Y2 in cup 2. The values of B1, Y1, B2 and Y2 were obtained as described in task 1 (See appendix A for the exact quantities). He was then asked to judge whether the color-shade of both mixtures would be the same or different and to justify his judgement by supplying reasons. All children were given the same quantities in the same order. In some cases the experimenter made the actual mixing, but usually this task did not involve observing the outcome. Each student had to make three to five judgments, depending on his initial responses.

Task 3: Generalization

A short written test was administered at the end of the session consisted of five items. The subject was first presented with a given mixture of 2 yellow drops to 3 blue drops presented as 2/3, and was asked to suggest the number of yellow and blue drops needed in order to get a mixture with lighter color than the reference, and a mixture with the same color but in larger quantities. Another question asked for the number of yellow drops needed to get the same color (2/3) if 12 blue drops were used. Finally, the student had to suggest a general rule that would specify the conditions for obtaining different mixtures that have the same color. The last item asked whether the whole experiment reminded the subject of anything related to mathematics.

Results and Discussion

The Ss' performance in each of the three tasks described above, was analyzed, considering mainly the strategies used to answer each of the mixture problems, rather than the correctness per se. After analyzing these strategies, one of the following scores was given: In task 1 (Experience), a plus (+) indicated the use of correct strategies on all problems, a minus-plus (-+) was given to those who during the concrete experiencing used both correct and incorrect strategies, and a minus (-) indicated the use of incorrect strategies on all problems. In tasks 2 and 3 (Formalism and Generalization), which included less steps, only two scores were given, either a plus (+), when the formalism was realized and used correctly by the subject, or a minus (-), when this was not the case. According to these scores, 12 patterns of performance were possible, as described in table 1:

Table 1: Possible patterns of replies:

	Task 1 Concrete Activity	Task 2 Formalism	Task 3 Generalization
1)	+	+	+
2)	+	+	-
3)	+	-	+
4)	+	-	-
5)	-+	+	+
6)	-+	+	-
7)	-+	-	+
8)	-+	-	-
9)	-	+	+
10)	-	+	-
11)	-	-	+
12)	-	-	-

The number of children exhibiting each of the above patterns would suggest the answer to the main question raised in this study. Children falling in the patterns (1) and (12) were not affected by the experiment. Children falling in category (9) improved their performance as manifested by correct generalizations, due to the formal presentation. Pattern (5) includes children who had improved in the middle of the concrete experimentation, supported by the formal tasks. We should emphasize that the

concrete experimentation had also a formal touch. While working on the mixing colors task, the children wrote down the numbers involved with each mixture which probably increased their tendency to reason about the numbers. Ss' performance on patterns (4), and (8) show decrease in the children's performance due to the formal representation and will contradict the major hypothesis of this study. Patterns (2) and (6) show no achievement on the more general level. All other patterns would indicate inconsistent and unreasonable performance, and would require special attention. Each subject was assigned one of the above patterns, according to his overall performance. Table 2 presents a summary of the number of subjects of each age level, manifesting the various patterns (The empty patterns were omitted):

Table 2: Distribution of Ss among the patterns:

	Pattern	Grade 7	Grade 8	Grade 9	Total
1)	+ + +	2	5	7	14
5)	-+ + +	10	6	10	26
8)	-+ - -	2	2	0	4
9)	_+ + +	0	2	3	5
11)	- - +	0	2	0	2
12)	- - -	6	3	0	9
		20	20	20	60

To summarize these results even further, it can be seen that patterns 1 and 12 which are irrelevant for the present analysis, include 23 children distributed as expected among the age groups and demonstrating a developmental trend of learning. As for patterns that support the hypothesis about the contribution of the formal component, pattern (9) consists of only 5 Ss while pattern (5) which consists of some learning at the concrete experimentation as well as in the formal task includes 26 Ss. Four children in pattern (8) indicate a situation in which their performance was hindered by the formal activity. We cannot explain Pattern 11 (2 Ss). In the limitation of the present study we could not examine the generalization achieved without being exposed to the formal learning (task 2). This remained to be studied next.

A question raised by our experiment is, how can children know that the mixture of colors is a

specific application of ratio and proportion? As noted in previous studies many of the children started with qualitative reasoning about colors and with additive strategy. It was interesting to note that since the children received immediate feedback they could revise their prior hypotheses, as in the following protocol: "It looks as though only if the ratio is in multiplication, then the colors are the same. That is, if it's twice between the blue and the blue, and the yellow to the yellow is also twice, then the colors will be the same, but if it is in addition - then it is not" (as he thought before). The struggle between the intuitions about colors and the relevant formal model, caused in many cases some tension as evidenced from the following protocol. In this case the student was not convinced by the formalism. She was given in task 2 both mixtures (a) and (b) in their formal representation: a=1/4, b=3/12, and when asked to decide if their color shades were the same or different, she replied: " B will be darker. (Why?) Oh, no. It will be the same color. Because 3/12 equals a quarter if we reduce fractions". The examiner then makes the actual mixtures and the student says: "They look the same. On second thought, it's not so related, the reducing of the fractions and the drops mixture". (Why?) "Because here (with the drops) you don't ask to reduce, you simply say there are 3 blue and 12 yellow, and if you will reduce you won't put 1 drop of blue and 3 of yellow. So the reason is wrong, but the result is right."

This experiment has demonstrated to some extent the effect of the formal representation of ratio in an intuitive based experiment. It should be noted, however, that the present experiment did not enable to control separately and independently the impact of the two variables -- concrete activity and formal presentation. Therefore it is difficult to infer about the unique influence of each one of these variables on the final performance.

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Appendix A:

The Quantities of Blue and Yellow Drops Used in Tasks 1 and 2

	B1	Y1	B2	Y2
1)	1	3	7	9
2)	1	3	6	18
3)	7	2	17	12
4)	7	2	14	4
5)	2	8	4	16
6)	2	8	10	16
7)	2	5	3	7
8)	1	4	3	12
9)	1	4	4	7
10)	2	5	6	15
11)	2	5	5	8
12)	2	4	3	6

B1 = Number of blue drops in cup 1.
Y1 = Number of yellow drops in cup 1.
B2 = Number of blue drops in cup 2.
Y2 = Number of yellow drops in cup 2.
Steps 1-7 were administered in task 1.
Steps 8-12 were administered in task 2.

EARLY CONCEPTIONS OF SUBTRACTION

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Abstract
In a phenomenographic study 105 Swedish school starters were interviewed in order to map out, among other conceptions, the conceptions of how to grasp the number in the unknown part in subtraction tasks within the number range 1-10. The children were given some easy word problems immedately after school start, before any teaching in addition and subtraction yet had started and were observed when they solved the problems, and interviewed about the ways in which they had done this. It was possible to categorize even very early conceptions. These are the ones that will be in focus in this presentation.

BACKGROUND

Phenomenographic investigations aim at mapping out existing conceptions of different phenomena in the world around us (Marton, 1988). In a phenomenographic study (Neuman, 1987) 105 7-year-old Swedish school starters were interviewed in order to find out about their different conceptions of numbers and of how they get hold of the number asked for in verbally given problems. One of several intentions with the study was to find out if there might be some logic in unusual answers to simple addition and subtraction tasks which had been observed among children in special education lessons in the first grades. The part of the study which concerned early conceptions of subtraction ending up in those answers will be in focus in this presentation.

METHOD

82 of the 105 school starters who were interviewed were all of the pupils from four classes, while 23 were chosen from five other classes. 13 of these 23 pupils were interviewed because they seemed to have very rudamental assumptions of how to get hold of the number asked for in quantitative problems, according to a preliminary test given to all children before the interviews were carried out. Nearly as many of the pupils in the four classes where all were interviewed seemed to have as little understanding of counting as these 13 children.

One of the interview questions was a game, where the child was asked to put up as many buttons as a figure - 9 - written on a card. All the

pupils were able to do that. The interviewer then hid the 9 buttons in two boxes, and asked the child to guess how many of them were hidden in each box. The pupils were allowed to make five guesses.

Beside the guessing game two addition tasks and four subtraction tasks were given. Only two of the latter were given to all pupils. Of these the first was a missing addend task and the second a take away task:

Q 1. If you have 3 kronor and you want to buy a comic for 7 kronor, have you got enough money? - (No!) - How many more kronor do you need?

Q 2. If you have 10 kronor in your purse and lose 7 of them, how many have you got left?

These two tasks together with the guessing game were the ones that best elucidated the early conceptions of subtraction.

In a phenomenographic study you first separate the answers from the individuals who answered, categorizing the answers per se according to some characteristic. After that you again relate them to the individuals looking for if a group of individuals giving a specific answer to one question gave an answer illustrating the same kind of thinking to some of the other questions. The findings in these two analyses are after that interpreted and the conceptions they are thought to be expressions of are "labelled" and described.

In the present study a quantitative evaluation was also carried out in order to have some idea of the background knowledge existing among children expressing different conceptions. In this study the following kinds of task were given: "How far can you count?"; "Can you add one to eight (subtract one from seven)?"; "Can you count backwards from 10 to 1?"; "How many fingers do you have altogether on your two hands?". Beside that the pupils were given some Piaget inspired problems, e g to put up as many bricks as the 17 which were placed in a row on the table, to answer the question if these bricks were more or less or as many as before when the interviewer by chance had pushed the bricks in one row so they were spread out, and to seriate 15 sticks of different lengths. One or more points were given to ...f the tasks (e g two if the pupils could count to 30, 1 if they could

count to 25, and none if they could not even do that). Beside that one point was given for any correct answer to the eleven analysed tasks (the guessing game and the six questions). Together it was possible to obtain 26 points in this evaluation. The median for the children in the four classes where all pupils were interviewed was 21.5.

FINDINGS

Five different conceptions, one of them expressed through two different strategies, were mapped out. The two earliest conceptions of how to find an answer to a quantitative problem could rarely be viewed as realated to conceptions of addition, subtraction or even of number. However, the third conception observed seemed to be a conception of early ordinal, and the fourth one of early cardinal kind. In the fifth conception the early ordinal and cardinal qualities of number finally had become integrated. The five conceptions will be described below.

1. Movements. The most unusual answers to the guessing game were that there were 9+11, 11+13, 13+18, 10+11 etc buttons in the two boxes. 11 such answers were given altogether by 6 pupils. These answers were observed only in the guessing game. One of the children who guessed so three times had no points in the quantitative study, and the other two who guessed so more than once had 7 and 9 points respectively. These two children also expressed some other early conceptions.

The concrete counting which was carried out when the buttons were placed on the card by these children, at least by the one who had 0 p in the quantitative evaluation, seemed to have been carried out as a script where the counting words rather were related to the movements in a movement game than to the buttons placed on the card. In the guesses she just seemed to choose a couple of number words by chance. This conception only seemed to be a conception of "how to behave" in the way the adult expects when the question "How many" occurs.

2. Fair shares. Another kind of odd guesses were that there "had to be the half" or "the same" in each box as two of the children explained it. 16 gave altogether 35 answers of this kind. In the guessing game they

guessed e g "6+6, 9+9, 2+2 or even 1+1, and to q 2 three of them answered "7". (Only one of the 89 children who did not express this conception in the guessing game gave this answer to q 2.) These answers seemed to reflect an experience of "fair sharing" and the belief that the unknown part must be one of the equal parts. In this conception the counting words seemed to have been related to the objects counted. The counting words used were within the number range 1-9. However, the parts were not yet related to the whole. This is easy to understand if it reflects an experience of fair sharing. The whole is rarely counted before a fair sharing is carried out (cf Miller, 1984). This conception cannot be viewed as related to a conception of number since no part-whole relations seem to exist.

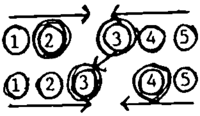
The median in the quantitative study for the 8 pupils who expressed this conception twice or more was 8.

3. Names. In the third conception mapped out the guesses were of the kind "3+9, 5+9, 7+9 or 1+9". 67 answers of this kind were given by 32 children. The answer to q 1 was "7" (given by 12 children in this group but only by 2 of the other 73 children) and the answer to q 2 "0" or "1" (given by 3 children in this group but only by one of the other children). The median in the quantitative evaluation for the 14 children who had answerd in this way at least twice was 7. The children who had given such answers might in some situation, e g in a "fair sharing", have been aware of partly that each object got its own numerical "name" in the sharing procedure, partly that the further in the sequence the number name of the last distributed object is situated, the larger the number of sweets, marbles etc, delivered. The counting word sequence seemed to have become a kind of "felt" or imagined "measuring tape" (cf "the mental number line" described by Resnick, 1983). The children seemed to describe the figure on their imagined "measuring tape" to which the buttons in the boxes reached, if each button would be related to one of its numerals. The buttons in the last box then always must end with the button related to the numeral "9". This "limit name" was used to communicate the number of the last part as well as of the Here the parts seemed to constitute the whole. In take away tasks, i

e in q 2, and sometimes in guesses (where the children either could think of the buttons in the last box as taken away from the whole or as a missing addend added to the first part) the pupils further seemed to express the conception that take away tasks should be thought of backwards (cf Carpenter and Mosers, 1982, findings pointing to that small children use different strategies for problems they experience as additive and subtractive respectively). Thus they seemed to set out from ten and to think backwards down to "7" on their imagined "measuring tape" and after that to think also of the last part backwards to the last limit name, stating that there were "0" or "1" left if seven were dropped.

However, there was also another strategy related to this conception, a strategy where the inner limit names were used to describe the size of the parts. The children using this strategy guessed e g "2+3, 7+8, 3+4 or 1+2. Four of them answered "6" to q 2. (Only three children outside this group gave that answer to q 2.) 24 children have used this strategy 44 times. The median in the group answering in this way at least twice (n=14) was 15, thus much higher than the median for the group of children who used the earlier strategy related to the conception "Names". Also the children who used this latter strategy seemed to point to figurs on their "minds measuring tape". And they seemed to experience take away as something which should be carried out through thinking backwards, exactly as the children answering "0" or "1" to q 2 had done. It was possible to interpret these strategies e g from the way in which a couple of pupils enumerated the "names" of the coins in the "left part" in q 2: "Seven dropped ... then there are six, five, four, three, two, one ... six left". The coins "named" "10, 9, 8, 7", thus the coins down to and including "7" on the "measuring tape", seemed to be the "dropped" ones. One child used this strategy in a very elucidating way in a following up task where five buttons were hidden in the two boxes. First he guessed that there were three in one box and two in the other, explaining that he knew because, as he said, "it can't be four". When the interviewer pointed out that she of course might have hidden four in that box, and asked

how many it then could be in the other, he answered: "Well, then there's three", explaining further: "I knew ... you just take away one" (fig 1).



"Three and two"

"Four and three" Fig 1.

If the button named "four" would be the last one thought to be in the second box, where the buttons were taken away from the whole, and thought of backwards, then the "3-button" must have been "taken away" from this box and moved to the first one, where it then would have become the last button thought of forwards. Thus there had to be three in this box (fig 1).

In these two strategies the children seemed to use an early ordinal conception of the counting words only: the limit names, which described where parts and whole ended, thought of either forwards or backwards. They seemed to measure the number, not to count the units.

4. Extensions. In the next kind of answers on the other hand, an early cardinal conception solely seemed to be used and the counting words seemed to mean "a little", "much/many" or "something inbetween". One child e g answered q 2 by saying: "Then I've got four left .. or two .. four or two ... you can't be sure ..". When the interviewer asked if there couldn't be eight left the child answered: "Eight left!? ... if you lost that much, it can't possibly be that much!" "A little + "rather much" could be "much", but not "much" + "rather much". However, if all the words "two, three and four" means just "a little", "you can't be sure" of which one you should use. The extension covered by a number of units on the "measuring tape" seemed to be what the children expressing this conception had in mind when they estimated the unknown part, while the separate units within this extension did not seem to be of any interest.

60 children gave together 109 answers of this kind. 13 of these children gave the answer "3 or 4 or 5" (or only one of the counting words "3" or "4") to q 1. Only one child outside this group gave this answer. 7 of

them gave the answer "2 or 3 or 4" (or only one of the counting words "2" or "4")) to q 2. Only 2 other children answered so.

The median for the children answering twice in this way was 16.

5. <u>Finger numbers</u> and <u>Counted numbers</u>. First in the last of the early conceptions the children seemed to divide up the extension into units, representing them by one finger each or with one counting word each only, counting e g "1,2 ... 3,4,5,6,7,8,9 ... 7 or 8 missing .." (q 1). Here the early ordinal and the early cardinal aspects of numbers had become integrated. Yet, the children still had to estimate the number in the part which was thought to be added or subtracted if it was larger than three. However, now it was the number of units which was estimated, not an "extension" only. When the conception "Extension" was expressed no fingers were used and no countingwords were enumerated.

Even when "Finger numbers and "Counted numbers" were used it was possible to observe how the strategies used in q 1 and q 2 were the same as the ones used in the guessing game and the other questions. 23 children expressed altogether 38 times the conception that the numbers should be "Finger numbers" or "Counted numbers", but had not yet developed strategies allowing them to find the correct number in the added or subtracted part if it was larger than 3. The median among the children expressing the conception more than once (n=9) was 18.

DISCUSSION

One after the other the different aspects of numbers seemed to be separated from the whole for closer investigation, and after that integrated again, changing the quality of the conception and making it more and more functional. In the quantitative evaluation it was possible to see how the median became higher in the groups of children expressing more developed conceptions or the more developed strategy within the conception "Names".

The children seemed to express one of these conceptions in many answers, even if they might "fall back" to earlier conceptions in difficult situations. This is illustrated by the ways in which the answers to q 1 and related to the guesses and the other questions in the interview.

The conceptions following these earlier ones in the map of conceptions created through the analyses, revealed the ways in which these early conceptions gradually changed into more functional ones. The fingers used in a way where the numbers became simultaneously ordinal and cardinal "Finger numbers" seemed to play the most important role in this developmental process. Used for "keeping track", however, the fingers seemed to obstruct the way towards abstract arithmetic thinking.

The children who used their fingers in the first way did not divide up the first hand if one of the parts was five or bigger. Thus they could solve a "missing addend" of the kind 2+_=9 by taking away the two last fingers, and a take away task of the kind 9-7=_ by folding the seven first ones. The strategy "Choice" (Resnick, 1983) seemed to be concretely created in this way.

If the two parts were less than five the thumb - or the thumb and the forefinger of the first hand - was moved over to the second hand. In this way a "Transformation" strategy (Neuman, 1987) changing the parts within the whole (e g 5+2 to 4+3) was concretely created.

The conception that "finger numbers" could be used in these ways in order to grasp the unknown part in subtraction tasks, was the most frequently expressed conception in the study. Gradually these ten "finger numbers" with their strategies became visualized, or just "felt" - "body-anchored". In the end they seemed to become thinking strategies related to abstract numbers. 9 pupils illustrated in different ways how they "thought with their hands" and about 1/4 of the 105 pupil used the strategy "Transformation" and/or "Choice" as thinking strategies.

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COMPUTATIONAL ESTIMATION PERFORMANCE AND STRATEGIES USED BY SELECT FIFTH AND EIGHTH GRADE JAPANESE STUDENTS

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(Abstract)

Identification and characterization of the computational estimation skills and strategies possessed by Japanese students were primary purposes of this project. Second purpose was to contribute further to the successful development of a general framework charactering the thinking processes of student's computational estimation. More specifically, this research was designed to identify and describe computational estimation processes used by the best estimators in grade 5 and 8, and to characterize their thinking strategies and techniques. Twelve different Japanese schools (7 elementary and 5 junior high) and a total of 279 fifth graders and 187 eighth graders participated in this research.

Background

The mathematics curriculum plan released by the Japanse Ministry of Education identifies estimation as a topic that should be specifically taught within school mathematics programs (Course of Study, Arithmetic, 1989). The report issued by the Japanese Ministry of Education is certain to promote change in new mathematics textbooks written for Japanese students. Thus some progress is being made to focus curricular and instructional attention toward computational estimation.

Computational estimation has not received much attention in Japanese school mathematics curriculum but it has been included in some of national assessments. For example, the problem reported in Table 1 was included in a Japanese national assessment report (Total Assessment in Mathematics, 1984). This assessment was designed to obtain the "calculation"(e.g., method) used by the ...nts to arrive at their answer in addition to the answer itself.

Table 1

Japanese National Assessment # 5-B-7 Item on Estimation of Arithmetic

There is a number which was rounded to an integer after the calculation of 304×18.73. Do rough calculation to find a correct answer, and choose a correct one from following items; 1. 570 2. 5697 3. 56967 4. 569673 And write how to do rough calculation:________________

Although 62 percent of students chose the correct answer, only 19 percent actually estimated. The majority reported calcultating the exact answer then matching it to the closest corresponding foil. Although the students were "very good at solving straight computation problems", this assessment reported that they were "rather poor at doing estimation problems".

Previous research has provided the beginning for a theory about how good estimators in grades 7 through 12 as well as adults actually make estimates (Reys, Bestgen, Rybolt and Wyatt, 1981). Three global cognitive processes identified among these good estimators (Reys, et al, 1980) These processes are translation (changing the equation or mathematical structure of the problem to a more mentally manageable form); reformulation (changing the numerical data into a more mentally manageable form); and compensation (adjustments made in the intial or intermediate estimate to exact answer).

Purpose

Identification and characterization of the computational estimation skills and strategies possessed by Japanese students were primary purposes of this project. Second purpose was to contribute further toward the development of a general framework charactering the thinking processes successfully used by students doing computational estimation. More specifically, this research was designed to identify and describe computational estimation processes used by the best estimators in grade 5 and 8, and to characterize the thinking strategies

and techniques these students used when estimating.

Sample

Twelve different Japanese schools (7 elementary and 5 junior high) and a total of 279 fifth graders and 187 eighth graders participated in this research. The schools were selected to represent a range of school and economic backgrounds. Eight of the schools were selected within Tsukuba, a city of approximately 135,000 popuration located about 60 km from Tokyo. In order to insure a broader representation of schools, four rural schools outside of, within 50 km to Tsukuba were also involved. One fifth and eighth grade class in each school was selected by the principal to be tested. The class size ranged from 33 to 47 students in both the fifth and eighth grade classes. Students in all classes were heterogeneously grouped as is the traditional custom in elementary and junior high schools in Japan.

The Screening Test

The 39 open-ended item screening test used in this research contained 25 items from the ACE (Assessing Computational Estimation) test (Reys, Bestgen, Rybolt and Wyatt, 1981), Some items from the ACE were modified slightly to make them appropriate for Japanese students and several other items were added which the researchers thought might be particularly interesting, such as the 12/13 + 7/8 items from the Third National Assessment of Educational Progress.

Each of the 39 items was produced on a 35-mm slide with the items shown sequentially using a corousel slide projector. This format allowed for group administration and controlled the amount of response times (10-15 seconds) for each item. The test included 25 straight computation items (those containing only numerical data) and 19 application items (those containing numerical data embedded in a physical context) all deigned to be relevant to Japanese students and presented in their naitive language. All four operations were included but the majority of items involved the operations of multiplication and division. A few items involved fractions and decimals, but the majority involved whole ers.

A standard set of instructions for the screening test were used in each school by the test administrator. Students were told that this was an estimation test. Since there is not a direct Japanese translation for estimate, the term "rough calculation" was used in the directions. They were told that each problem would be timed, and that they would have between 10 and 15 seconds to make and record their estimate. The students were also told "Not to copy the problem but to do the work in your head".

The interview

An individual interview was done with 21 (10 fifth and 11 eighth graders) of the best estimators in an effort to learn what strategies and processes each of the subjects used in solving differents estimation problems. Students were asked to describe as fully as possible the strategies and thought processes they used to arrive at their estimates.

Students were available for only a limited time, usually one class period, so only a few estimation problems could be posed. Since the researchers were also interested in learning how consistent students were in the estimates produced on the screening test were also used in the interview. A total for 13 estimation problems, 6 straight computation and 7 applied computation involving a multistep multiplication and division problem was used with the eighth grade students.

To supplement the interview problems, specific probes were developed to provide consistency among the four Japanese interviewers as well as to focus more carefully on specific characteristic hypothesized to be common among good estimators. A summary packet which highlighted these strategies and processes which students might use, was prepared for each interview problem. Training sessions were held among the four Japanese researchers doing the interviews. Following the training, the interviewers practiced doing several interviews before any reported interviews were conducted.

Screening Test

[A] 39-item screening test for all fifth grade and eighth grade students was

done respectively. As the results, the scores were widely distributed, ranging from 0 to 24 for fifth and 0 to 26 for eighth grade, with 7.44 and 11.18 being the mean number of acceptable estimates on the screening test for fifth and eighth grades respectively.

At the conclusion of the screening test, all students were asked "Are you a good estimator?" and their responses suggest an interesting paradox. Table 4 reports that whereas about three-fourths of students at each grade level said estimation is important, only a few of them (12 percent of fifth and 4 percent of the eighth) rated themselves as a good estimator. It is also interesting that about two-thirds of the students said they were not a good estimatos, and this self assessment parallels very closely the generally low performance on the computational estimation test.

Table 2

Japanese Students Self Assessment on two Estimation Statements

Statement	Gr. 5			Gr. 8		
	Number of M.	F.	Total	Number of M.	F.	Total
Are you a good estimator?	150	129	278	96	91	187
Yes	18.0	5.4	12.2 %	7.4	1.1	4.3 %
No	57.3	64.3	60.6 %	66.3	73.6	70.1 %
Not Sure	24.7	30.2	27.2 %	26.3	25.3	25.7 %
Do you think estimation is important?						
Yes	76.7	74.4	75.6 %	72.6	69.2	70.6 %
No	6.0	3.1	4.7 %	13.7	6.6	10.7 %
Not Sure	17.3	22.5	19.7 %	13.7	24.2	18.7 %

An examination of Table 2 confirms that the screening test produced a wide range of scores, particularly on the upper end from which the "good estimators" were selected for the interview. It took a score in the top 5% of students in ir respective grade level and thereby become a candidate for an individual

interview. Although there was considerable variability among students within each of the participating schools there was at least one student in each school who scored high enough to be interviewed.

An examination of the screening tests for the good estimators showed that 2 of the fifth graders and 4 of the eighth graders made an acceptable estimate on the screening test, whereas Table 7 shows that nearly all of them answered it correctly during the interview. Although a majority of the students at each grade level recognized "12/13" and "7/8" as being close to one, this observation was not always immediate. For example, one fifth grade student first found a common denominator of 104 and then after one minute observed that each fraction "was near one". On the other hand another fifth grader responded in two second that the sum was about two. She said, "12/13 is only 1/3 smaller than one and 7/8 is only 1/8 smaller than 1, so their sum is almost 2".

Table 3

Summary of Strategies Used on Exercise Involving

Estimating the Sum of Two Fractions

Exercise 12/13 + 7/8	Frequency	
Strategies	Gr. 5	Gr. 8
· Recognition that each fraction is near one, so sum is near one	5	8
· Use of Common Denominator		
// 12/13 is about 9/10, 7/8 is about 9/10, so 18/10	1	0
// 12/13 is 24/26, 7/8 is 21/24, so the sum is about 24/25 + 21/25 or 45/25	1	2
// 12/13 is about 10/10, 7/8 is about 8/10, so 18/10	0	1
// 12/13 is about 12/10, 7/8 is about 7/10, so 19/10	0	1
· Computed exact answer using mental algorithm	1	1

An examination of the strategies highlighted in Table 3 shows a heavey

reliance on algorithms. Most of the students at each level tried to perform a written algorithm on this problem, and in the process provided some interesting insights into their thinking. Some students were successful in using algorithmic techniques to produce an acceptable estimate. For example, one fifth grade girl was able to perform the exact coputation mentally and produce an "estimate of 124/114" in less than half a minute. This is another reminder of the challenge of getting a valid measure of estimation, and it also gives some indication as to her ability to do mental computation very quickly and accurately. Another fifth grader produced an acceptable estimate, by reporting that each fraction is about 9/10 and then added them together and reported an "estimate of 18/10". However, this student gave no indication that he understood that his estimate of 18/10 was near 2. Although some students were successful using the addition algorithm mentally, other were unsuccessful. For example, a fifth grade boy added numerators and got 19. He then said "13 ×2=26 and 3×8=24, and the average of 26 and 24 is 25, so my estimate is 19/25".

Discussion

Interviews with the highest scoring students led to the identification of some specific estimation techniques and strategies. Students at both grade levels, but the fifth graders in paticular, tended to apply learned algorithemic computational procedures. Their tendency to use paper/pencil procedures mentally often tended to interfere with the estimation process and made it more a mental computation task. Such procedures were not only inappropriate but inefficient as well.

An area which tended to be a strength with the Japanese sample was their knowledge and use of place value techniques. Few order-of-magnitude errors were observed in the interviews when large numbers (i.e. values greater than ten thousand) were involved. This may be due in part to the Japanese use of the "mang" (word for 10,000) as a strong reference as oppose to the American reference of 1,000. It may also reflect the Japanese monitary system which provides regular opportunities for working with large whole numbers.

Model building is important. Our hope is that research into how students estimate will lead to a learning model that not only helps describe the learning process but can provide direction for the development of appropriate instructional experiences which help all children their computational estimation skills. We fell that this research contributes further toward the development of a general framework which describes the estimation processes used by good estimators. This study has confirmed that despite the tendency of many students to mentally apply previously learned paper/pencil algorithmic teckniques, the earlier hypotheseized cognitive processes of translation, reformation and compensation were evident among Japanese students. Not all students used all the processes all the time. However, we found that each of students used one or more of these processes during the interview. Similarly, not all of the characteristics of good estimators were evident in any one student, but each was frequently found among the Japanese students interviewed.

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ASSOCIATIONS AMONG HIGH SCHOOL STUDENTS' INTERACTIONS WITH LOGO AND MATHEMATICAL THINKING

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Even though many claims have been made for the potential of Logo there is still a critical need to demonstrate what, if any, impact students' interaction with Logo has on their mathematical thinking. Three years ago we began an investigation to determine such impact. This investigation was based on current models of mathematical thinking which led to the development of qualitative measurement strategies for assessing students' interactions with Logo, their geometric understanding and related thought processes. It is evident that an integrated review of results from each qualitative measurement strategy and from measures of mathematical achievement can serve to strengthen knowledge of how students construct their understandings. Therefore, this paper explores the associations among the models and related data sources.

Design of the Study

A course in Turtle geometry was developed and implemented for ninth grade students in two inner-city high schools during the 1985-86 academic year. The focus of the course was on informal explorations of topics such as polygons, circles, and transformations. High school mathematics teachers were trained to implement a Logo learning environment, based on a guided discovery approach. The Turtle geometry course was taught to one class each semester (eighteen weeks per class). Students were enrolled in Algebra I, and were on track to take geometry in tenth grade. At one school the students were all black, at the other they were evenly distributed, black and white. Each class was held in a lab setting with fourteen microcomputers, and students generally worked in pairs. Students' interactions with Logo were recorded in dribble files on disk.

This report documents an analysis of the complexity of students' responses, mode of working, and general problem solving approaches across a sequence of four tasks given to all students in the second semester Logo classes held in the Spring of 1986 (n=30). The results of this analysis are then compared with individual students' math grades. A simple method of hypothesis testing has been employed which enables the investigator to pose and answer questions concerning relationships among the different measures.

Analyses of Students' Dribble Files

Data on students' interaction with the Logo environment were collected via dribble files on disk and classroom observation notes. Students' dribble files were analyzed for the following four Logo tasks:

1. CHECKPOINT 2 assessed students' facility with debugging simple shape procedures. This was basically a Logo programming activity given after four weeks of work with Logo.
2. MIDTERM was a more complicated debugging task involving the use of super and sub-procedures in a HOUSE design. This task was also focussed on the programming structure.
3. PARALLELOGRAMS, explorations with a generalized parallelogram procedure, was focussed on the

geometric properties of parallelograms, especially class inclusion of special parallelograms. Students were also challenged to create a three-dimensional box structure using the parallelogram procedure.

4. RHOMBUS MADNESS, the final exam project, required students to create parts of a flower structure using a generalized rhombus procedure, and then put the parts together to form the flower. This was both a programming and geometric challenge for the students.

Teacher instructions and student handouts for the four tasks can be found in the Final Report of the Atlanta-Emory Logo Project (Olive, Lankenau and Scally, 1988).

The focus of the dribble file analysis was on the structural complexity of a student's response to the Logo tasks and indications of relational or instrumental understanding of relationships (both Logo and geometric) which may have emerged from this analysis. The criteria for this complex analysis procedure was based on a synthesis of the SOLO taxonomy with Skemp's (1979) model of mathematical understanding (Olive, 1985 & 1986).

The SOLO Taxonomy (Biggs & Collis, 1982) was designed primarily as a tool for the evaluation of the quality of student responses to a task. The Taxonomy consists of five levels: Prestructural, Unistructural, Multistructural, Relating and Extended Abstract which can describe how a student uses different kinds of Logo objects (primitive commands, fixed procedures, variable procedures, etc.) with respect to both the Logo task and the internal structure of the object itself. The following general guidelines were used for assigning SOLO levels to students' Logo responses:

Prestructural (P): The Logo object is not used appropriately or the student does not use an available object when it would be appropriate to do so.

Unistructural (U): The object is used by itself. Immediate feedback is required on the effects of its use before any other Logo commands are used (inability to withhold closure).

Multistructural (M): Objects are used in combination with other objects or commands on the same line or within a procedure (ability to withhold closure), but the objects are not related correctly (with respect to the task).

Relating (R): The objects are related together in order to accomplish the task. The relating operations (relationships) are dependent on the nature of the task and the structure of the objects. The Logo objects are used as building blocks.

Extended Abstract (E): Objects are related together to create a new object which is more generalized, more abstract than its parts; or a generalized procedure is used effectively to create specific objects with which to build and accomplish a task.

By applying these criteria to the dribble records of each student's work on the four Logo tasks, patterns emerged which often determined the quality of learning: instrumental or relational (Skemp, 1976), and which (in some cases) gave some indication of the student's van Hiele level of thinking (van Hiele, 1986). A student was assigned "visual" if s/he appeared to make decisions based on the visual feedback from the screen and was dependent on the visual feedback, while ignoring the syntactic structure of the Logo commands. A student was assigned "descriptive" if s/he appeared to work primarily with the syntactic structure, often not requiring

immediate visual feedback in order to make programming decisions. It should be noted that one of the attractions of Logo is its appeal to a visual mode of working and thinking. [A prior report on relationships among students' van Hiele levels of geometric thinking (determined through clinical interviews) and experience in the Logo classes has been given by Scally, 1987.] The detailed results of the dribble file analyses across the four tasks for each school can be found in the Final Report of the Project (Olive et al, 1988).

The first two Logo tasks (CHECKPOINT 2 and MIDTERM) dealt primarily with facility with Logo programming concepts (debugging and structured procedures), whereas the last two tasks (PARALLELOGRAMS and RHOMBUS MADNESS) involved work with geometric concepts as well as some programming facility. A summary of the dribble analyses was performed based on all four tasks, but categorized in terms of a student's responses to Logo PROGRAMMING challenges and the GEOMETRIC CONCEPTS involved in the tasks. This summary categorized responses in terms of the SOLO taxonomy, van Hiele level of approach to a task (visual or descriptive) and Skemp's quality of understanding (relational or instrumental).

Relationships Among the Different Measures

The summary data from the dribble analysis were combined in a simple data base with the math grades of all 30 Logo students. These data were used to test hypotheses concerning relationships among the different measures. By simply selecting cases on the basis of stated criteria concerning the measures, or simply arranging the data on the basis of one particular measure, questions concerning these measures could be posed and (in many cases) answered. No statistical tests were applied to the following hypotheses. The supporting evidence can be obtained by inspection of selected subsets of the data or by rearranging the data. The Tables generated from Table 1 for each hypothesis can be found in the Final Report (cited). They have been omitted from this paper because of lack of space. The data in Table 1 have been arranged by students' grades in their final math course.

Relationships emerging from the dribble data

1. Most students who approached the geometric tasks with some descriptive level thought (van Hiele contains d) also approached the programming tasks with some descriptive level thought, but the converse did not appear to be true.

2. Most students who achieved Relating SOLO level responses to the geometric tasks also achieved Relating SOLO level responses to the programming tasks, but the converse did not appear to be true.

These first two results suggest the following hypothesis: sophistication in Logo programming is necessary but not sufficient for success in Logo geometric tasks.

3. Lack of Relating SOLO level responses on geometric tasks corresponds to a visual approach to the geometric concepts for most of the Logo students (van Hiele = v).

4. Instrumental understanding of either programming or geometric concepts (Skemp = I) appears to correspond to visual level thinking (van Hiele = v).

5. Relational understanding of either programming or geometric concepts (Skemp = R) corresponds to Relating SOLO level responses (SOLO contains R) but not necessarily to decriptive level thinking (van Hiele contains d).

TABLE 1: SOLO, van Hiele and Skemp categories of response with math grades

		LOGO PROGRAMMING			GEOMETRIC CONCEPTS			GRADES			
STUDENT	SCH	SOLO	VAN H	SKEMP	SOLO	VAN H	SKEMP	ALG11	ALG12	GEOM1	GEOM2
VH	1	U,M,R	d	R	M,R,E	d	R	B	A	A	B
SF	1	U,M,R		R	M,R		R	A	B	B	B
OS	1	U,M,R		R	U,M,R		R	B	B	B	B
DZ	2	U,M,R	v	R	M	v		B	B	B	B
DS	1	U,M,R		R	U,M	v		B	B	C	B
JY	2	U,M-R	v	R	M-R	v	R	C	C	C	B
CH	2	R	v-d	R	M,R	v-d	R	B	B	B	C
DB	1	U,M,R	v-d		U,M	v	I	C	B	B	C
TM	1	U,M,R	v	R	U,M,R	v	R	C	C	B	C
KY	1	U,M-R	v	I	M	v	I	C	C	B	C
KR	1	U,M,R		R	M,R		R	B	C	C	C
AF	1	M,R		M				C	C	C	C
SB	2	U-M,M-R,	v-d	R	M-R	v-d	R	C	C	C	C
LO	1	U,M-R	v	I	U,M-R	v		C	C	D	C
AR	2	U,M m/ed	m/ed	m/ed	m/ed	m/ed	m/ed	C	F	D	C
AD	2	U,M	v	I	M	v-d	I	C	C	C	D
MJ	1	U,M,R	v-d	R	U,M	v		C	B	D	D
QA	1	P,U,M	v	I	U,M	v	I	C	C	D	D
JR	2	M(m/ed)	m/ed	m/ed	M(m/ed)	m/ed	m/ed	C	C	D	D
SH	1	U,M,R	v		U,M	v	I	C	F	D	D
RT	1	U,M			U,M			C	F	D	D
RG	2	M,R,R-E	v-d	R	M-R	v-d	R	A	C	F	D
TN	2	M-R	v		P,U,M	v	I	A	A	C	F
AJ	2	U,M,R	v-d	R	M	v	I	B	B	C	F
CM	1	U,M	v	I	P,M,R	v	I	C	F	C	F
SC	1	U,M	v	I	P,U,M	v	I	C	F	F	F
SW	2	M,R	v	I	M	v	I	A	F	N	N
LC	1	U,M,R			U,M,R			C	F	N	N
SL	1	M,R	v-d		U,M	v	I	C	B	C	W
RM	1	U,M	v-d		U,M			C	C	W	W

NOTE: "m/ed" indicates that work in the Logo editor was missing from these dribble files An "N" grade indicates that the student did not take the course. A "W" indicates that the student withdrew from school.

Relationships among qualitative measures and math grades in each course

1. ALG11: Algebra 1, part 1 (pre-Logo)

Almost all students with grades better than "C" in this pre-Logo course appeared to have Relational understanding of Logo programming and (to a lesser extent) of the geometric concepts (both SKEMP categories show "R"). Also, all students with grades of "A" or "B" obtained Relating level SOLO responses on the programming tasks. Most students with a "C" grade appeared to have only Instrumental understanding of geometric concepts (SKEMP = I). There appears to be no relationship between Algebra I grades pre Logo and students preferred van Hiele level of working in Logo.

2. ALG12: Algebra 1, part 2 (concurrent with Logo)

All students with grades better than a "C" in this course obtained Relating level SOLO reponses and appeared to have Relational understanding of the programming concepts. Also, six out of the eight students who were assigned a van Hiele level indicated at least a transition towards a descriptive level of working with Logo. It is also important to note that no student who failed the Algebra course was assessed as having relational understanding of either programming or geometric concepts, nor working at even a transitional descriptive level.

3. GEOM1: First Semester Geometry

Again, the strongest relationship emerging from the data on this course concerns students' quality of understanding of both Logo programming and geometric concepts. All except one of the students with grades better than "C" on first semester geometry were assessed as having relational understanding. Most of the students with grades lower than a "B" were assessed as having Instrumental understanding of the geometric concepts on the Logo tasks.

It is interesting to note that one of the students (RG) who failed this course was assessed as having relational understanding of both programming and geometric concepts, reached a transition towards extended abstract responses on the programming tasks and relating SOLO levels on the geometric tasks, and appeared to work somewhat descriptively on both programming and geometric tasks. He also obtained an "A" on the first algebra course. As can be seen in the Table, RG stayed in the geometry sequence and passed the second semester course!

4. GEOM2: Second Semester Geometry

The relationship between relational understanding of both programming and geometric concepts and success in the geometry course also holds for the second semester course. All students with better than a "C" grade in this course had relational understanding of programming, and no student was assessed as having instrumental understanding of geometric concepts; whereas, only one student (RG) with a grade less than "C" was assessed as having relational understanding of the geometric concepts. No student who failed the course had relational understanding of the geometric concepts, and only one appeared to have relational understanding of Logo programming.

The above comparisons looked at each math course separately. The following three Tables pose questions concerning students' grades across all three math classes post Logo. They were generated from the data in Table 1 by specifying certain selection criteria.

Table 2 indicates that five students obtained consistently low grades post Logo. All failed algebra at the end of ninth grade, even though one (SW) obtained an "A" in the first algebra course. None of these students were assessed as having relational understanding of either Logo programming or geometric concepts. None indicated work at even a transition toward descriptive level thinking on either programming or geometric concepts. However, three of the five students did obtain relating SOLO levels on programming tasks.

TABLE 2: Which Logo students obtained consistently low grades post Logo?
Selection: ALG12, GEOM1 and GEOM2 are greater than C

		LOGO PROGRAMMING			GEOMETRIC CONCEPTS				GRADES		
STUDENT	SCH	SOLO	VAN H	SKEMP	SOLO	VAN H	SKEMP	ALG11	ALG12	GEOM1	GEOM2
SH	1	U,M,R	v		U,M	v	I	C	F	D	D
RT	1	U,M			U,M			C	F	D	D
SC	1	U,M	v	I	P,U,M	v	I	C	F	F	F
SW	2	M,R		I	M	v	I	A	F	N	N
LC	1	U,M,R			U,M,R			C	F	N	N

TABLE 3: Which Logo students obtained consistently good grades post Logo?
Selection: ALG12, GEOM1 and GEOM2 is less than D

		LOGO PROGRAMMING			GEOMETRIC CONCEPTS				GRADES		
STUDENT	SCH	SOLO	VAN H	SKEMP	SOLO	VAN H	SKEMP	ALG11	ALG12	GEOM1	GEOM2
VH	1	U,M,R	d	R	M,R,E	d	R	B	A	A	B
SF	1	U,M,R		R	M,R		R	A	B	B	B
OS	1	U,M,R		R	U,M,R		R	B	B	B	B
DZ	2	U,M,R	v	R	M	v		B	B	B	B
DS	1	U,M,R		R	U,M	v		B	B	C	B
JY	2	U,M-R	v	R	M-R	v	R	C	C	C	B
CH	2	R	v-d	R	M,R	v-d	R	B	B	B	C
DB	1	U,M,R	v-d		U,M	v	I	C	B	B	C
TM	1	U,M,R	v	R	U,M,R	v	R	C	C	B	C
KY	1	U,M-R	v	I	M	v	I	C	C	B	C
KR	1	U,M,R		R	M,R		R	B	C	C	C
AF	1	.M,R			M			C	C	C	C
SB	2	U-M,M-R,	v-d	R	M-R	v-d	R	C	C	C	C

All students in Table 3 obtained at least a transition towards a relating SOLO level on Logo programming. Only two students were assessed as having instrumental understanding of programming or geometric concepts.

The last Table provides further evidence for the relationship between Relational Understanding and success in the math courses which has emerged from all of the comparisons in this set.

TABLE 4: Selection: both SKEMP categories contain R

		LOGO PROGRAMMING			GEOMETRIC CONCEPTS			GRADES			
STUDENT	SCH	SOLO	VAN H	SKEMP	SOLO	VAN H	SKEMP	ALG11	ALG12	GEOM1	GEOM2
VH	1	U,M,R	d	R	M,R,E	d	R	B	A	A	B
SF	1	U,M,R		R	M,R		R	A	B	B	B
OS	1	U,M,R		R	U,M,R		R	B	B	B	B
JY	2	U,M-R	v	R	M-R	v	R	C	C	C	B
CH	2	R	v-d	R	M,R	v-d	R	B	B	B	C
TM	1	U,M,R	v	R	U,M,R	v	R	C	C	B	C
KR	1	U,M,R		R	M,R		R	B	C	C	C
SB	2	U-M,M-R,	v-d	R	M-R	v-d	R	C	C	C	C
RG	2	M,R,R-E	v-d	R	M-R	v-d	R	A	C	F	D

With one exception (RG) all students assessed as having relational understanding of both Logo programming and geometric concepts obtained a grade of "C" or better in all of the math courses.

Conclusions and recommendations

A most important outcome of this study is the refinement and synthesis of the three major theoretical models which were used as a basis for the design of the study and the analyses of qualitative data. The links established among students' levels of thinking (van Hiele model), the structural complexity of student responses to Logo tasks (SOLO taxonomy), and the quality of understanding (Skemp's model), have laid the ground work for an integrated model of Teaching and Learning. Some initial development of this integrated model has emerged from two detailed case studies (Olive and Scally, 1987) which looked at the relationship between students' learning processes in the Logo environment and their progress in geometric thinking as determined via the van Hiele interviews.

This current report demonstrates that relational understanding of both Logo programming concepts and geometric concepts is linked to success in math courses. It also suggests that the more successful math students pre-Logo were more likely to reach that level of understanding within the Logo course.

Although some students did appear to achieve relational understanding while working at a predominantly visual level of thinking, a transition towards descriptive level thinking appears to be indicative of success in algebra but not necessarily in geometry courses. Perhaps this result is evidence that the Logo environment can help students whose predominant level of thinking is visual, to effectively use a visual approach to solving geometric problems. On the other hand, the ability to use Logo in a purely visual way may have inhibited some students' movement towards a descriptive level of thinking and working.

The Logo course developed during this project attempted to use Logo programming as a vehicle for

mathematical exploration. The above results indicate that success in the programming experiences was necessary in order to grasp the geometric concepts. However, some students who were successful with the programming tasks did not appear to grasp the geometric concepts, thus programming success was not sufficient for understanding the mathematics. This result strongly suggests that, for some students, a non-programming use of Logo may be more beneficial for exploring and constructing mathematical concepts. The use of Logo microworlds, specifically designed for the exploration of particular mathematical concepts, integrated into the regular math classes, is a major recommendation of the Project.

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GRAPHIC CONSTRUCTIONS WITH COMPUTER TO LEARN 3D REFERENCE SYSTEM

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Can computer help to overcome problems raised by representations of spatial configurations, in the context of 3D geometry teaching ? can it contribute to restate the meaning link between spatial and nuymerical frameworks in learning 3D analytic geometry ?

Adopting the hypothesis of an interaction between the mastery of graphic representations and the construction of 3D geometric knowledge, we constructed a teaching sequence having the 3D reference system as object, based on the production and transformation of plane representations of 3D configurations, by using computer (two CAD softwares) as a tool. The dynamic treatment of graphic representations must use specific geometric knowledge; it must, also, extend and make evolve this knowledge.

By observing the school-books and the teaching practice, we can notice that the *reference system* notion is introduced, since the complementary level, as an established fact. It isn't constructed as a solution to specific problems necessitating to organize and to structure physical space.

As this notion is introduced, pupils are suddenly projected in *analytic geometry* . A new language, a new system of symbolic representations are used, fixed by the teacher, but not constructed by the pupils, on the base of their geometric knowledge. Algebraic relations are defined and used to replace geometric relations between the elements of a spatial configuration. Then, the geometric activity is transformed into a calculatory activity, in which the interpretation at the geometric level is neglected. With this modification of the nature of "geometric" activity, *graphic representations* almost disappear. The *meaning link* between spatial and algebraic frameworks risks then to be suddenly broken.

In *3D geometry* , such a problem is even more accurate: there is a strict separation between concrete activities of manipulation and observation on one hand, and activities using abstractions, theorizations and concepts on the other. One of the main reasons of such an aggravation is the difficult access to spatial situations. It can essentially be done through plane graphic representations; some of the characteristics of the spatial configuration are then absent or modified; It's necessary to make explicit a code of interpretation and production of representations. Although it is necessary, such a code is not sufficient to overcome difficulties related to coordination of viewpoints or construction of relations between graphical and physical spaces.

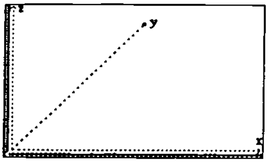
We suppose that using computer as aid tool permits a new approach of the above problems. Situated in the frame of a constructivist theory of knowledge, our method is to construct several "situations" linked to each other, aiming to a learning process, and based on using graphic softwares. Computer intervenes in these situations as an aid tool in teaching. Teaching computer science is not our aim, neither is training to use softwares. Nevertheless, we adopt the hypothesis that there exists a close interaction between the acquisition by the pupils of the concerned geometric knowledge and their construction of the functioning mode of used softwares. Our research aims to study, in the context of resolution of problems with specific softwares, the processes of adaptation and evolution of pupils' strategies, taking in account their confrontation to the constraints of these softwares, based on different systems of information treatment.

Teaching sequences have been constructed and realized, in the context of a computer workshop in a french school, with pupils of the forth complementary class (14-16 years). pupils work by pairs. During this experience, we recorded the steps of pupils' work (as computer files); we recorded also their dialogues; we used this data for a clinical analysis of the evolution of their strategies.

In this paper, we present one of the situations of this teaching sequence (for more details about the other situations of the sequence, see Osta 1988, chap.I). The problem consists in constructing the graphic representation of a spatial configuration, by using Mac Space.

Conceptual analysis of Mac Space :

Mac Space is a conversational graphic editor, it works on Macintosh. It helps user to construct representations of 3D objects by constructing the three views (top, face and side views). Treatment is possible in the three windows of orthogonal views; the representation in perspective appears progressively in the 3D window that is only a control window. During the treatment, modifications on one of the three views are translated on the other, as well as on the perspective which translates spatial transformations on the represented object. In the space of Mac Space, the basic geometric element is the polygonal facet: any treatment of a representation (creation, elimination or transformation) can only be executed on facets; it's then impossible to trace segments or isolated points.



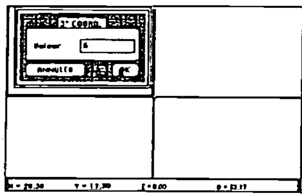
Analysis of the sujacent reference system: The space of the software is controlled by an implicit orthonormal reference system, composed by three non-materialized axes, two by two perpendicular: Ox, Oy, Oz. Here's a simulation, in the 3D window, of the three virtual axes of this system.

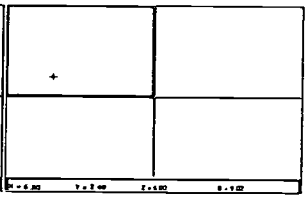
In the interface, the coordinates system is apparent in a communication window where, since the selection of a graphical tool, are displayed the coordinates of the current point represented by the cursor. The space is represented in all windows. It is considered, in each one of them, as an addition of a privileged direction, perpendicular to a privileged plane. It is isomorph in each window to a non-associative product of three unidimensional spaces: (Ox.Oy).Oz in the window of the top view, Ox.(Oy.Oz) in that of the side view and (Oz.Ox).Oy in that of the face view.

One point of the space is characterized, in each window, by:

* its coordinate along the privileged direction of this window. At the practical level, this coordinate is communicated to the machine in a static way, by using the command "3° coord." of the menu "curseur". In default of such an operation, this coordinate is equal to zero.

* the coordinates of its projection on the privileged plane of this window. At the practical level, these coordinates are communicated to the machine in a dynamic way, by moving the cursor in the window.





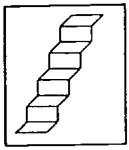
When introduced, the 3°coord. with respect to one window won't be influenced, as the two other coordinates, by the displacement of the cursor.

Proposed task and objectives :

At this moment of the teaching sequence, pupils had acquired some aspects of the reference system controlling the software: the bidimensional system of the privileged plane in each window. They had, also, constructed correspondances between the displacements of the cursor along the principal directions of each window on one hand, and the variations of the coordonates values on the other.

This task aims to overcome and extend this knowledge, toward the 3D reference system. In each of the treatment windows, pupils have to construct the functioning mode of

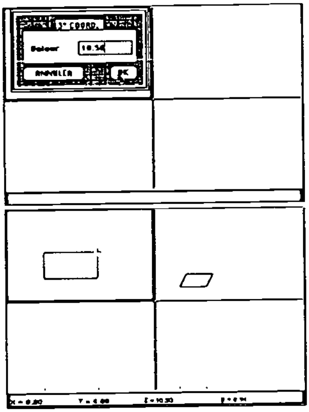
referring along a 3° dimension; they have also to coordinate it with the bidimensional reference system in the privileged plane of the corresponding window. Without such a construction, it's impossible to realize the task.



Assignment: By using the software Mac Space, construct the graphic representation of a surface in steps, having the following characteristics: the dimensions of one step are 10 and 7 units of measure; the height of one counter-step is 5 units.

A priori analysis of the task :

To construct a rectangle with Mac Space: The rectangular facet is the fundamental object in this task. This analysis will only take in account the rectangular facets parallel to the planes of the rectangular trihedron of Mac Space.



A rectangle constructed with Mac Space is determined, in the corresponding window, by:

* the value of the "3°coord.", coordinate of the plane of this rectangle with respect to the privileged plane of the window; this "reference-value" determinates the adequate processing level for the facet construction;
* the absolute coordinates of the first validated vertex;
* the dimensions of the rectangle that are relative coordinates of the 2° validated vertex with respect to the first one.

At the practical level, the value of the "3° coord." having been introduced, the rectangle is constructed in the corresponding window by the validation of two opposite vertices.

As longer as the graphism takes place in one window, the three other views of the rectangle (between which the perspective one) are reproduced in the other windows. The displacement of the cursor is accompanied by a dynamic dislay of the coordinates (x,y,z) of the current point it represents.

Peticular structuration of space: The objects concerned by this situation are sufficiently known by the pupils; their familiar structure makes possible to the pupils to have for these objects an internal representation according to the different principal observation directions. The planes of the component facets are separated by a constant "step" which is not necessarily the same in the three directions.

Constructing the reference system of the software: To construct each one of the consecutive facets, pupils have to communicate to the machine the right values of the parameters defining this facet. At every operation, they have to re-invest the value of the corresponding "step" to determine the position of the new component; then, they have to explicitely communicate to the machine the numeric value that determines this position, by using a mode of representation fixed by the machine.

If the pupil didn't explicitate this value, he obtains an undesired graphic result; there is then awareness that it is necessary to communicate to the machine an adequate numeric value, that of the "step" between two successive facets. So, he is engaged in a research process, searching for a function or a command that permits to communicate this value.

Some results :

In the following, we most present results about the important interaction we elucidated between the spatial and numeric frames. We care about the two following exigencies of the task:

* to communicate to the machine the numeric data of the problem: using which mode of representation? by affectation to which parameters?....

* to assure, between the component facets of the scale, adequate geometric relations (especially connexity and relative positions); to construct one peticular facet: what position parameters have to be determined? with respect to which other facet? with respect to which system?.....

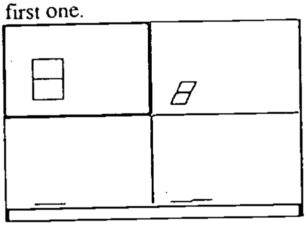
The following results demonstrate the importance of experimentation possibility given by the computer; they show the retroactions in the pupils' intellectual activity: an undesired graphic result incites to a research for the reason of error; this reason can induce an other representation of the solution, based on other conceptions. The realization of the new representation infers a new experience and a new graphic result, which consolidates or devaluates these conceptions.... and so on.

During the pupils' activity, we noticed a close interaction between the evolution of their construction of reference system and that of the significations they progressively attributed to the "3°coord.". The motors of such an evolution are: the exigencies of the task, the interaction with the reference system of the software, and the interaction between spatial, numeric and graphic frames.

The "3°coord." as "3°measure": This signification appeared after the construction of the first step. Two given numeric values (10 and 7, dimensions of the step) having been

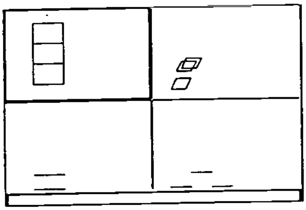
communicated to the machine, the pupils' main preoccupation was to find a means to communicate the third given value (5, height of a counter-step): "*we've got to tell him that the height is 5*". Such a signification reveals that the aspect "measure" of the "3° coord." is predominant with regard to the aspect "reference".

<u>The "3°coord." as a "reference-value" to situate the departure point of the new facet</u>: Some pupils showed this signification at the moment they wanted to construct the 2° step which, without any indication, was situated at the same level as the first one.



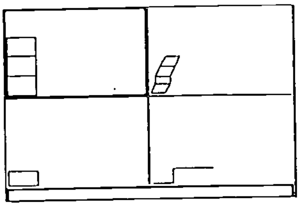
For them, there is problem "*because we've got to 'say' to the computer where we want to put the step*". After several trials, they found that the "3°coord." is the way to do it. Such a signification was accompanied by a dynamic conception of referring. Having obtained a graphical result as that of this fig., pupils searched for a means to "*push*", or to "*pull*" the 2° step.

<u>The "3°coord." as the permanent value of the "displacement step"</u>: Such a signification appeared after the pupils discovered the command "3°coord." as a solution; It's related to a specific representation of the functionning mode. This signification appeared at the moment pupils wanted to construct the 3° step: "*no, we don't have to type the 3° coord., it's already there...*"



This significcation supposes that the effect of an affectation of a value to the "3°coord." stays valid for every later operation. It seems that the "3°coord." is more considered as a parameter for the whole problem than a parameter for the construction of one facet. The fig. gives the resulting drawing.

<u>The "3°coord." as a temporary value of the "displacement step":</u> The undesired graphic result destroys the last conception, about the functioning mode of the software: "*it returns to the same place!*". Pupils discover that the value of the "3°coord." returns to zero at the end of each operation; nevertheless, they don't give up the meaning of "3°coord." as the "displacement step", between two consecutive parallel facets. Again, they affect the value 5.

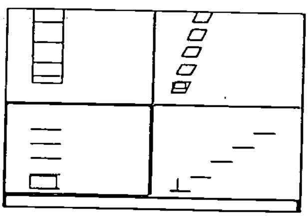


This signification is related to an internal representation of the construction as a continuous process, working by connexity, and based on a principle postulating that: "a new construction begins where the last one reached".

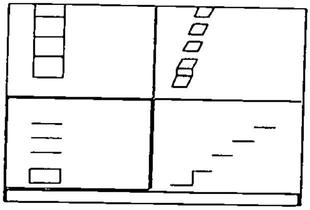
<u>The "3°coord." as a value determining the position with respect to the first facet of the same window</u>: This new signification shows an important evolution in the process of discovering and constructing the refernce system of the software: at least in the current processing window, it reveals an awareness of the existence of a inique referential, with respect to which are referred all the components to construct in this window; this referential is linked to the object . On the other hand, it consolidates the conception of the "3°coord." as a reference-value and eliminates all remainers of its conception as a measure-value. With this signification, pupils overcome the exclusive relation between "3°coord." and the value 5, by affecting to it other values.

<u>The "3°coord." as a value determining the position with respect to a referential independant from the object:</u> We'll develop the evolution toward this signification by using the example of the pair O.&S., one of the rare who reached this signification.

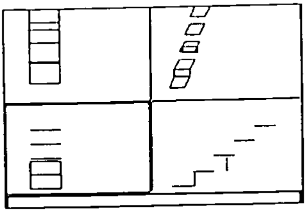
After the steps were constructed in the top view window, pupils chose the window of the top view, for the construction of the counter-steps. With this first window changing, the exclusive relation pupils constructed between "3°coord." and the top view window had to be broken. The exigencies of the task and the constraints of the software occasioned an evolution of this relation toward its extension. Recognizing this command as a solution to the same problem into an other window hasn't been automatic. It's been preceded by several strategies, revealing an opposition to generalize its effect. In this paper, we cannot develop these strategies (voir Osta 1988).



To construct the first counter-step, pupils tried the value 7 as "3°coord.". But the departure point of the first step had initial coordinates unequal to zero; they obtained a non-accepted graphic result. In fact, their trial reveals a representation of the solution based on the relations:(R1: $v0 = 7$ and R2: $v_i = v_{i-1}+7$): the first value of "3°coord." is 7, each one of the following values being obtained by adding 7 to the previous value.



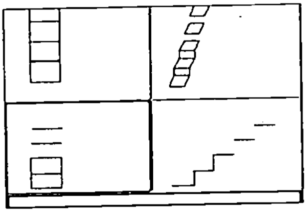
A conflict is created by the contradiction between this mental construction and the result of experience; the graphic result is not compatible with R1, at least. They decide to approach the right position of the facet by adequately modifying the value of "3°coord.", at each trial. To validate their result, their means of control is perception. After several trials, the value 11.5 assures the connexity.



For the pupils, it's the value 7 that gives all their meaning to relations R1 & R2. For them, this value instates, in fact, a relation between the relations R1 & R2. When the value 7 was devaluated, this relation between R1&R2 stayed valid. It's assured by the value 11.5. The two relations become:

(v0 = 11.5) & (vi = vi-1 + 11.5).

So, for the 2° counter-step, value of 3°coord. = 11.5 + 11.5 = 23. The graphic result being not acceptable, they try again to approach the right position:



S: ... Let's try 21 & a half, now

O: no, it's clearly less.... you try....18

S: 18 & a half?

After several trials, experience showed that the value 18.5 is the right one.

O: since it's 18.5, we had to know that we have to add seven at each time...

S: oh yes,7... that's obvious, the width is 7.... don't you think it's logical, you?... it will be 7 by 7, because the width is7.... it grows 7 by 7.

The two relations become: (R1: v0 = 11.5 & R2: vi = vi-1 + 7). The relation R2 takes again its meaning as assuring the "displacement step" pupils have to add, at each step, to the previous value.

Having found this intelligible relation between all the elements implicated in this problem, pupils have even succeded in interpreting the meaning of the numeric relation R1, and in linking it to its correspondant in the geometric framework:

O: we've got to look, from the beginning, at the coordinates

S: or simply begin at zero... I think we understand now.... we've got to begin in the corner... at zero... so we couldn't have problems

This dialogue reveals the acquisition by the pupils of the whole logic controlling the functioning of the software for this problem (and for all those of the same type).

Conclusion :

By the pupils' activity, this situation gave us informations that helped us to elucidate the processes of construction of their knowledge, in the context of the used software. The pupils' activity (esecially at the end) showed an evolution toward an organization and a structuration of the software space. Such a structuration is surely based on a non-isotrope representation of space, considering the construction of one peticular facet, because the communication to the machine of informations concerning its position cannot be done in the ...ne way for all these informations. But, from a global point of view, this situation infers an

isotrope representation of space, which means that a perfectly analogue treatment must be done in the two windows: that of the top view and that of the face view.

This situation also gave to the analysis some phenomena concerning the evolution of the construction, by the pupils, of the reference system controlling the space of the software. The analysis showed an interaction and a concomitancy between such a construction and the evolution of the signification of "3°coord.". This evolution is related to progressive abstraction and generalization of the meaning of "3°coord.", and of its status with respect to the space of each window. From the "3°coord." as a practical method to fix the position of a facet in a one window, by using peticular values, there is evolution toward the "3°coord." as a concept, independant from a peticular window, value or direction.

On the other hand, the important interaction that took place between geometric and numeric frameworks was a guaranty for the construction of the meaning of analytic knowledge as a link between these two frameworks: pupils constructed correspondances between successive positions of facets into one window and the numeric values attributed to "3°coord.". Even more, we found indicia of construction of correspondances between the displacements of a facet in one orientation or another and the algebraic operations (augmentation or diminution of the value of "3°coord.").

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APPLIED PROBLEM SOLVING IN INTUITIVE GEOMETRY

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Essex County College, Rutgers University

A recent study with 67 adult community college developmental arithmetic students has established a well-defined model for teaching and learning the basic concepts of area and perimeter. A pedagogical perspective, some theoretical background, the research design and a sketch of the results are described.

> If thinking does not imply a purpose or goal, discovery would be a blind guess, communication a miracle and a twice told tale the shattered accents of an echo. (Hook, 1927, p. 56)

In the million year gap between hominid and present human, the conscious development of abstract ideas that can be characterized as mathematical in nature is a relatively new phenomenon. Primoridal man, overburdened by the difficulties of surviving the harshness of a world marked by irrepressible scarcity, could ill-afford sustained abstract speculation concerning shape or quantity. Even in most of the ancient agrarian civilizations of more recent millenia, we find mathematic notions so intertwined with a human struggle to survive as to make them emerge more as an aspect of broader cultural development than as some separate well-defined collection of ideas. The ultimate disassociation of mathematical ideas; their objectification from the broader cultural context, *per se*, is an exceedingly modern interpretation of what is in the nature of mathematical subject matter. This objectification of mathematics, especially in the most recent of times, has helped lead to a vast development of our advanced mathematical knowledge. However, whether in the name of deductive efficiency or otherwise, the ahistorical precipitation of mathematical concepts out of any apparent cultural context can nearly erase any association of mathematics to human interest and pursuit. Especially to new learners, the extraction of mathematical concepts from a suitable identifiable human context can seem to sacrifice all sense of logical purpose and direction within mathematical studies. Surprisingly, one

example of a mathematical subject that, from the learners' point of view, can seem to make such a sacrifice concerns the study of area and perimeter with simple polygons; work preliminary to the study of Euclidean geometry.

The edifice of logic that is today's school geometry rests upon a firm but obscured foundation that has beeen poured by ages of human experience. In fact, geometry represents an informed and concisely symbolic casting of one part of mans' knowledge. To borrow the sense and terminology of John Dewey, like any subject, geometry is a curricular "reconstruction" (1966, p. 76) of some various parts of the human experience. But this reconstruction at which we have arrived; this geometry, is not only a logical set of axioms and theorems; the assumptions and the derived rules and regulations by which deductions may be correctly realized, it is also a formal product; one devoid of all but the faintest hint of the centuries of historical process by which, or any of the variety of intended purposes for which, it came to be created and developed.

In classes where topics such as area and perimeter; topics preliminary to the study of Euclidean geometry, are being developed, there may be little recognition by students of the possible purposes or ultimate consequences of such studies. They could not be expected to be aware of the historical context of the development of such activity, may often question the intended purposes of their work and, at times, even doubt that the results of their labors signify *anything at all*.

In an attempt to explicitly address what can be seen as the purposefulness of such geometric topics, and in conjunction with what is an axiom that describes humans essentially as creatures seeking meaning in their actions, we designed an applied problem solving research study. The purpose of the study was to develop and realize a well-defined applied problem solving model for learning and teaching the concepts of area and perimeter by adult students in an arithmetic class.

Population

The 67 predominately Black and Hispanic adults students who

participated in this study were enrolled in 4 sections of a remedial mathematics course, (2 sections during Fall and 2 during Spring of 1987-88) at Essex County College; located in New Jersey's largest city, Newark.

Design of the Study

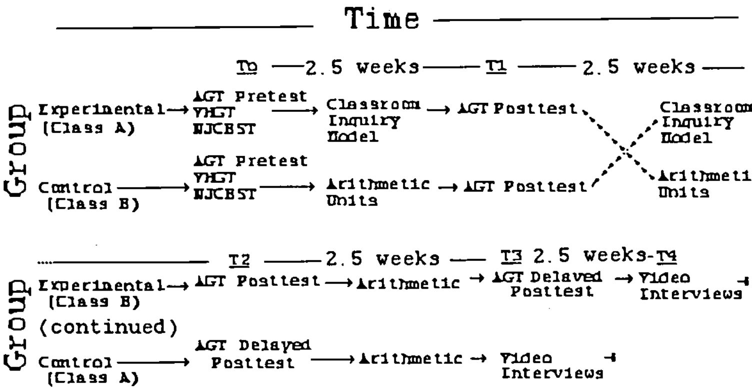


Figure 1. Outline of the experimental design.

The reseach design is given in Figure 1. Briefly, at time T_0, students in each of two classes were administered two assessment tests, the Van Hiele Geometry Test (VHT) (Usiskin, 1982), and the New Jersey College Basic Skills Placement Test (NJCBSPT); and a content specific pretest, entitled the Applied Geometry Test (AGT). Following the testing, for a 2 1/2 week period, the first class (Class A) participated in a classroom inquiry model; one primarily involving applied geometry problem solving situations. Meanwhile, the second class (Class B) was taught basic topics of arithmetic computation; topics unrelated to geometry. At time T_1, the classes were then retested with a different form of the AGT. During the next 2 1/2 weeks, the treatments were reversed for each class, and at time T_3, a third form of the AGT was administered. During the next 2 1/2 week period, both classes were taught arithmetic. At time T_4, videotaped interviews were conducted with randomly selected

students from Class A, while students in Class B were tested again with the AGT. Finally, at time T_5, randomly selected students from the Class B were video-interviewed.

Some Theory

Knowledge is constructed when human minds are actively engaged in some persuit.

> The roots of thought must be sought in action, and operational schemata derive directly from action schemata. . .Generally speaking, logico-mathematical structures are extracted from the general coordinations of action. . . (Piaget, 1971, p. 181)

It requires little valor to agree with Jean Piaget. Nonetheless, in our research, it was still required that we operationalize that agreement. We had to consider the serious and thorny problem of just what student "action" would provide the fertile ground for "the roots of thought". It was action we sought to incorporate, but not just any action would do. Activity, in and of itself, is motion; sheer sensory motor dynamics, and,

"Mere engagement in activities will not facilitate learning, of course, if those activities are not appropriate to the students' needs" (Brophy, 1986, p. 327).

And the "needs" that we saw for students were precisely those which would be met by the kinds of actions that would most likely lead to the "general coordinations" that Piaget describes above. These actions are typically not so easily specified. Thus, while Piaget's epistemology of constructivism may provide a viable model for the genesis of human thought, there yet seems an unanswered question as to a specified mechanism that will cause an engagement of the constructive process. For Piaget (1971), this "engagement" issue may be moot.

"Life is essentially autoregulation" (p. 26), and while the organism exists, the process of equilibration actively "compensates against outside perturbations" (p. 25), and "the organism as a whole preserves its autonomy and, at the same time, resists entropic decay" (p.13).

Thus, it seems, for Piaget, "Cogito ergo sum" (I think therefore I am) is a biconditional statement; i.e., "Sum ergo cogito", as well. However, while life itself may imply the autoregulated functioning of thought, it does not necessarily imply that the content of that thought will be mathematically rich. Thus, after even so compelling a description fo the epigenesis of knowledge as Piaget demonstrates, for the mathematics educator, there still must always remain the question of how to engage students' constructive processes in mathematically significant concepts. For one answer we turned to John Dewey (1980).

> The weakness of ordinary lessons in observation, calculated to train the senses, is that they have no outlet beyond themselves, and hence no necessary motive. Now in the natural life of the individual and the race there is always a reason for sense-observation. There is always some need, coming from an end to be reached, that makes one look about to discover and discriminate whatever will assist him. Normal sensations operate as clues, as aids, as stimuli, in directing activity in what has to be done; they are not ends in themselves. Separated from real needs and motives, sense-training becomes a mere gymnastic and easily degenerates into acquiring what are hardly more then knacks or tricks in observation. . . (p.93)

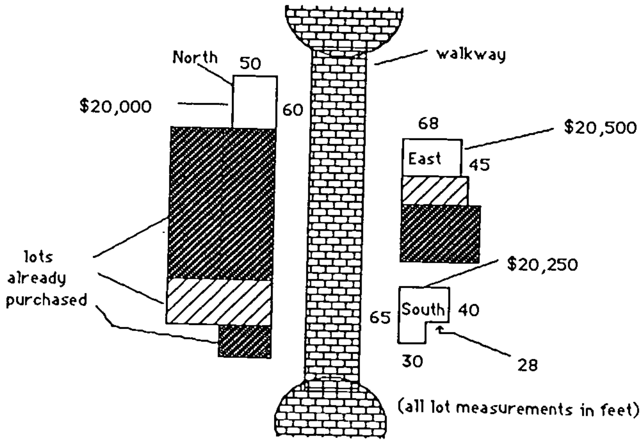
Dewey's comments imply the notion of purposeful inquiry on the learner's part. Briefly, it is a subject's actions, on the basis her/his self-felt purposes, within mathematically and conceptually rich domains that were the ensembles that this research sought to promote. One way of providing purpose, or "necessary motive", is through the use of applied problems (Lesh, 1981).

Generally and briefly, our model tried to blend to constructivism of Piaget, the purposeful instrumentalism of Dewey and the small group applied problem solving of Lesh.

A Treatment Example

During the experimental treatment, students were asked to

consider a variety of applied problems. These applied problems (Lesh, 1982) were constructed as lifelike situations in which mathematics was used as a major element in the resolution of some difficulty; or might have assisted in making some evaluative judgement between a variety of alternatives.



You and two other members of this class have pooled your money and convinced a bank to grant you a significant loan in order to open a small business in one of Newark's soon-to-be revitalized downtown commercial neig.borhoods. You are presently considering likely locations for your business. In the figure you see three possible alternati-e and parcel selections, labeled North, East and South, and the price of each. Make a choice for purchase that you all agree on, and give the reasons why you chose as you did.

Figure 2. The Shopping Mall Problem

One such prototypical problem (see Figure 2) concerned small groups of students working together to choose a potential business site from among alternative land parcels. This problem was one that we felt could potentially be of interest to adult students; especially as we were located within the physical and social context of contemporary urban redevelopment in the city of Newark. The task, which was designed as a vehicle for

developing area and perimeter concepts, sought to actively involve students in a problem situation where geometrical concepts were "lurking close by" in a fairly natural and inescapable manner.

Results Sketched

The full technical results of this research are detailed in a forthcoming publication (Pace, 1989). However, briefly, through a number of single and multivariate, stepwise, linear regression models whose parameters were estimated by the computer program, Regress II (Madigan & Lawrence, 1983), we established that, by the measure of geometric achievement utilized, the experimental program of teaching was susccessful in both short and long term cases. Summaries of the videotaped interviews supplemented the quantitative findings with both critical support for and dissent from the major findings.

It may be neither surprising nor particularly impressive that students were taught and therefore they demonstrated achievement and retained growth. After all, this research makes no claim that this particular experimental model is significantly *better* than other methods of teaching area and perimeter concepts. Any such claims of pedgogical superiority ultimately require a discussion of how one defines superiority; in terms of explicit educational values (Pace, 1988) and goals. On the other hand, what is claimed is that this research offers a well-defined model; one theoretically justified and operationalized, one that exists and can be known as a possible alternative to whatever other approaches may exist. Following the traditional methodology of mathematical reseach, this research has established through its results, the *existence* of a particular method. Any demonstrated *uniqueness* of results of that method; i.e., in the sense of superiority to other methods, remains to be shown.

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L'INCIDENCE DE L'ENVIRONNEMENT SUR LA PERCEPTION ET LA REPRÉSENTATION D'OBJETS GÉOMÉTRIQUES

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Abstract

First, we discuss the basis of a new typology for classifying the spatial abilities. Next, we present the results obtained from interviews with 10- and 11-year-old children, functioning in various types of space. Some interesting contrasts arise from these findings, allowing us to question some elements of Piaget's theory and the interventional model used actually for the teaching of geometry in schools.

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Introduction

Après avoir identifié les facteurs composant l'habileté à percevoir l'espace et examiné divers moyens de la développer, nous nous sommes intéressés à l'apport de l'environnement dans le développement organisé des habiletés perceptives et représentatives d'objets géométriques, situés dans un micro-espace. A cette fin, nous avons élaboré une typologie des habiletés spatiales, que nous présentons dans les paragraphes qui suivent.

L'objet de l'expérimentation relatée ci-après traite des relations entre ces habiletés spatiales et les types d'espace qui environne le sujet. L'intérêt de cette démarche est de provoquer éventuellement une diversification des interventions didactiques dans l'enseignement de la géométrie et de toutes autres disciplines touchant à la maîtrise de l'environnement, comme les arts graphiques, qui tiendrait compte des types d'espace qui environne les sujets.

La perception structurale de l'espace

Piaget avait mis en lumière la nécessité de dissocier l'espace perceptif de l'espace représentatif, afin de bien comprendre l'ordre dans l'appropriation des propriétés géométriques: la notion de voisinage intervenant avant les autres axiomes euclidiens, l'intuition des dimensions fondée sur l'intériorité et l'extériorité intervenant avant l'abstraction d'un volume euclidien...[1] L'espace peut aussi se caractériser de plusieurs points de vue: physique, social, géométrique, etc. Notre recherche s'est intéressée à la perception d'un espace géométrique. Cette perception peut enfin s'examiner sous un angle formel ou structural.

Alors qu'une perception formelle consiste en l'intériorisation[2] quantitative d'un modèle spatial par l'analyse et la synthèse de ses propriétés en termes de rapports, de proportions, de mesures et de coordonnées, la perception structurale considère plutôt l'intériorisation qualitative d'un modèle spatial par l'analyse et la synthèse de ses propriétés topologiques, projectives, affines et métriques. Nous prévilégions dans notre étude cette dernière approche. "La représentation spatiale est une action intériorisée et non pas simplement l'imagination d'un donné extérieur quelconque."[3]

La typologie des habiletés spatiales

La typologie que nous avons développée (Baracs, Pallascio, Mongeau), est définie sur la base d'un tableau à triple entrée. Une de ces entrées est définie par cinq (5) habiletés hiérarchisées, une deuxième entrée est définie sur les quatre (4) niveaux géométriques, alors qu'une dernière entrée distingue les deux (2) plans, perceptif et opératoire (ou représentatif). Le tableau contient donc quarante (ou 5 x4 x2) intersections, correspondant potentiellement à autant de degrés d'habileté spatiale où pourrait se situer un individu.

Les habiletés spatiales sont respectivement la transposition, la structuration, la détermination, la classification et la génération. La **transposition** est l'habileté à établir les correspondances, les équivalences, et à effectuer le passage entre les différents modes de représentation (physique, linguistique, algébrique et géométrique) et niveaux géométriques. La **structuration** est l'habileté à identifier les propriétés et la combinatoire géométriques d'une structure spatiale. La **détermination** est l'habileté à délimiter les éléments ou les paramètres définis par des contraintes géométriques sur une structure spatiale. La **classification** est l'habileté à grouper des structures spatiales selon un choix de propriétés ou paramètres géométriques communs.

Enfin la **génération** est l'habileté à produire ou modifier une structure spatiale de façon à ce que cette structure réponde à certains critères géométriques prédéterminés.

Les niveaux géométriques sont les niveaux topologique, projectif, affine et métrique. Le niveau **topologique** correspond principalement à l'étude des propriétés d'adjacence et de connexité des structures spatiales, propriétés qui sont conservées suite à une ou des déformations continues, telles que l'étirement, le rétrécissement, le pliage ou la torsion. Le niveau **projectif** correspond principalement à l'étude des propriétés d'incidence et de platitude, qui sont conservées suite à une projection centrale. Le niveau **affine** correspond principalement à l'étude des propriétés de parallélisme et de convexité, qui sont conservées suite à une projection parallèle. Enfin le niveau **métrique** correspond principalement à l'étude des propriétés de distance et d'angulation.

En dernière analyse, le plan **perceptif** est constitutif d'une action mentale de reconnaissance des formes, alors que le plan **représentatif** est constitutif d'une action concrète de transformation des formes.

Les types d'espace

Alors que le **micro-espace** est le lieu de la manipulation de petits objets où il est facile pour le sujet de changer de points de vue par rapport à l'objet, et que le **méso-espace** est l'espace des déplacements du sujet dans un domaine contrôlé par la vue et qui s'obtient par le recollement de micro-espaces connexes, le **macro-espace** est celui qui nécessite une représentation implicite des mouvements relatifs de plusieurs systèmes de références, que l'on pourrait imager par un "recollement de cartes", selon l'expression de Guy Brousseau (1986).

Nous avons cherché à déterminer les relations et les incidences qu'il pouvait y avoir entre un environnement donné et les habiletés perceptives et opératoires appliquées à un micro-espace, comme celui des formes géométriques utilisées dans un test-entrevue élaboré antérieurement pour valider partiellement notre typologie. Pour ce faire, nous avons choisi et comparé deux groupes de sujets, dont l'environnement spatial est radicalement différent: un groupe d'enfants vivant dans un environnement rural du sud du Québec et un groupe du même âge vivant dans un village Inuit du nord du Québec.

Au niveau du **micro-espace**, les enfants du sud, en milieu rural ou urbain, sont davantage initiés au dessin imaginatif ou figuratif, plutôt qu'au modelage de formes tridimensionnelles, alors que les enfants Inuit sont initiés très jeunes à la sculpture dans la pierre à savon, tandis que le papier demeure une denrée plus rare (les arbres sont loin!).

Au niveau **méso-spatial**, l'environnement visuel varie sensiblement d'un milieu à l'autre. Alors qu'en milieu rural, les habitations sont des prismes rectangulaires allongés, étendus ou pyramidés (fermes, demeures isolées...) et qu'en milieu urbain les édifices sont essentiellement des prismes rectangulaires, les habitations traditionnelles des Inuit, les Igloo (qui signifie "maison" en inuttitut), que les enfants apprennent encore à construire lors de sorties familiales pour la chasse ou la pêche, sont formées de pyramides tronquées, où le parallélisme ne domine pas.

Enfin, au niveau **macro-spatial**, alors que les dénivellations sont variables en milieu rural et fortes en milieu urbain (métro, stationnement souterrain, édifices à plusieurs étages...), c'est plutôt un espace bidimensionnel qui s'ouvre à l'horizon de l'Inuit qui doit compter sur des accidents de terrain épars pour se repérer dans la toundra.

La méthodologie

Le test utilisé, administré par entrevue individuelle, était composé d'une douzaine de tâches ou problèmes à résoudre, couvrant nécessairement une partie seulement de la typologie, à savoir sept (7) des (40) modules, correspondant à l'un ou l'autre des niveaux topologique ou projectif, à l'une ou l'autre des habiletés et dans l'ordre du plan perceptif, constitutif d'une action mentale de reconnaissance des formes, ou du plan opératoire (représentatif), constitutif d'un action concrète de transformation des formes.[4]

Les deux groupes d'élèves comparés étaient composés de 16 enfants. Un premier groupe (du sud) était formé de 8 garçons et 8 filles, alors que le second groupe (du nord) était formé de 12 garçons et 4 filles, tous et toutes des Inuits, sauf un jeune Amérindien du peuple Cree. Le test, limité à 13 tâches, a été administré au printemps 1988.

Les résultats

Nous observons que les deux groupes s'opposent radicalement au niveau des **plans perceptif et représentatif**, au niveau des **propriétés géométriques**, topologiques et projectives, et au niveau des habiletés perceptives et représentatives qui dominaient, soit les activités du **début de la typologie**, soit celles de la **fin**. Le **sexe** des sujets n'intervient pas, ni à l'intérieur des groupes, ni globalement.

Tableau - *Comparaison globale*

Groupe du sud	*Groupe du nord*
Plan perceptif	Plan représentatif
Reconnaissance des formes	Transformation des formes
Géométrie topologique	Géométrie projective
Transposition, structuration	Génération, détermination et classification
Début de la typologie	Fin de la typologie

Bien sûr, les espaces différents qui définissent les environnements des deux groupes de sujets ne sont probablement pas la cause unique des différences observées dans la perception et la représentation des objets géométriques micro-spatiaux. Au niveau méso-spatial, par exemple, certaines constructions coutumières chez les Inuit leur font manipuler des objets aux propriétés davantage projectives qu'affines (p.e.: les blocs de neige servant à la construction d'un igloo sont des pyramides quadrilatérales tronquées et disposées en spirale, et non des parallélipipèdes). Mais les relations et les incidences que nous avons identifiées sont suffisantes pour nous questionner sur la nécessité d'établir des parcours différentiés dans le développement des habiletés spatiales, objet de nos prochaines recherches.

Notes

1 La représentation de l'espace chez l'enfant, Piaget, J. et B. Inhelder, PUF, 1948, p. 535.

2 Par intériorisation, nous entendons un détachement graduel de la réalité permettant aux états de devenir des représentations de classes d'objets et permettant aux actions de se transformer en opérations mentales.

3 Piaget, J., Id., p. 539.

4 La validation de l'ensemble de la typologie (40 modules) se poursuit actuellement auprès de groupes de sujets plus nombreux et d'âge divers: environ 200 sujets, enfants, adolescents, étudiants universitaires et adultes.

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ANGLES & PIXELS Quelle synergie à 9 ans ?

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RESUME

Nos travaux de recherche sont orientés vers la prise en compte d'un enseignement de la programmation en LOGO graphique afin d'en constater l'efficacité lors de transferts et d'interactions avec d'autres domaines de connaissance. Dans cet article, les résultats obtenus à l'issue de trois Tests rendent compte de l'évolution des compétences lors de mesures d'angles pour deux groupes appareillés. Leur analyse apporte des éléments sur la structuration du concept d'angle grâce à la comparaison de deux traitements pédagogiques utilisant des supports de représentation différents: avec écran versus sans écran.

INTRODUCTION

Depuis les intuitions de S. Papert, de nombreux travaux ont été conduits sur les rapports entre le concept d'angle et son approche en LOGO graphique. L'analyse des champs conceptuels menée par A. Rouchier <1>, J. Hillel <2> ou les recherches concernant la structuration de ces notions auprès d'élèves menées par C. Hoyles <3> n'en sont que d'excellents exemples. Il convient toutefois, pour comprendre les résultats présentés d'ajouter deux précisions :

- Logo est un système sans unité ou les ordres sont conférés ainsi : AV 30 TD 30. L'élève doit assimiler que l'ordre comprend en lui même la notion d'unité. Le plus petit élément géométriquement traitable sur feuille est le point, sur écran c'est le pixel. Les différentes définitions d'écran déterminent alors la valeur réelle de l'unité ce qui constitue un référent complexe pour un élève, et conduit par exemple à pouvoir établir la distinction à priori entre une figure représentant un polygone de 25 côtés et un cercle en fonction de la surface et de la qualité de l'écran.

- Le système de mesure des angles Logo est basé sur la division du cercle en 360°. Il suppose donc une certaine représentation de la division que n'ont pas tous les élèves en début de cycle moyen. Enfin, les effets de la primitive TOURNE peuvent se combiner, par opérations, dans un système de base 360, dans lequel, 0 et 360 ont un effet identique. Si cette logique rappelle celle du cadran horaire, elle constitue toutefois un système assez nouveau pour les élèves au début du cycle moyen.

Pour cette recherche, des épreuves ont été construites et vérifiées auprès d'un échantillon représentatif,elles [illegible]ttent de rendre compte de l'état des compétences lors du calcul de périmètres ou de mesures d'angles. Par

ailleurs, elles vérifient l'évolution de certains pré-requis concernant l'aptitude à itérer une suite ou à discriminer droite et gauche sur un plan orienté. Enfin, elles évaluent certains «savoir-faire-faire» en Logo. A la suite d'une première passation, deux classes de CM 1 sont respectivement scindées en 2 groupes rendus homogènes : A et B. Ils vont alors découvrir concepts et outils en suivant simultanément deux progressions préalablement définies de 12h. A l'issue de chacune, ils repassent les épreuves. Les résultats obtenus par 24 élèves découvrant implicitement certains concepts grâce à l'apprentissage de la programmation en Logo graphique, puis explicitement sans ordinateur, sont comparés à ceux des 24 autres découvrant ces mêmes concepts dans un ordre inversé selon le plan suivant:

	Test 1	entraînement	Test 2	entraînement	Test 3
groupe A		LOGO		géométrie	
groupe B		géométrie		LOGO	

La mise en place des progressions s'inscrit dans le cadre général de l'enseignement des mathématiques et plus particulièrement de la géométrie à l'école élémentaire. A ce sujet, M. Blanc, <4> dégage trois périodes dans l'enseignement des mathématiques à l'école et au collège :

- de 1945 à 1970 : Les problèmes de mesurage sont au centre des préoccupations;
- de 1970 à 1977 : Mathématiques modernes; les angles ne figurent plus au programme;
- depuis 1977 : L'idée de situation problème s'impose. L'angle apparait comme un élément pertinent dans la construction de figures et comme un invariant lors de certaines transformations géométriques.

Résolument axés sur les acquis de notre époque, les conceptions issues de la troisième période ont marqué l'élaboration des progressions construites en équipe afin de s'inscrire dans le cadre scolaire. La progression utilisant le LOGO graphique propose après 2 séances de découverte des primitives fondamentales de faire dessiner sur l'écran des figures choisies dans un corpus ordonné selon des difficultés croissantes. L'agencement de celles-ci permet, l'acquisition de l'itération, puis, la définition de procédures. Les figures proposées sont des polygones, des figures composées de polygones, de segments, d'angles droits ou non-droits. Des difficultés pédagogiques sont soulevées par la mise en oeuvre de cette progression <5>. La progression de géométrie sans ordinateur, conduit les élèves à découvrir l'angle pour l'intégrer comme élément lors de situations de description. Ce descripteur est alors retenu et combiné à d'autres afin d'élaborer une classification des polygones et figures inscrites au programme.

Les résultats obtenus à l'issue des trois Tests : T1, T2, T3 font l'objet de traitements statistiques éclairés et complétés par l'analyse des justifications apportées par chaque élève. Délaissant momentanément les résultats concernant ... périmètre qui ont déjà en partie fait l'objet de présentations <7> les résultats rapportés concernent la

mesure d'angles.

EPREUVE DES ANGLES (cf. annexe)

Lors du rodage des épreuves et des progressions, nous avons fait passer cette épreuve à 85 élèves, en début de cycle moyen, répartis sur plusieurs classes dont celles des enseignants qui ont participé à l'expérience l'année suivante. A deux exceptions près les élèves n'ont rien répondu. De ce fait cette épreuve n'a pas été passée au T1 les résultats nuls étant considérés comme acquis.

A l'issue du T2

Préalablement à toute constatation partant des résultats, l'observation des réactions des élèves lors de la passation de cette épreuve est riche d'enseignements. Elle permet de remarquer que plusieurs individus du groupe A demandent à aller chercher leur règle alors que certains du groupe B réclament leur rapporteur. Poursuivant cette piste, nous constatons à la lecture des résultats que 8 élèves du groupe A fournissent toutes leurs réponses sans unité. Seuls 2 élèves du groupe B réagissent ainsi. Ces différences relevées peuvent en partie s'expliquer par le statut déjà mentionné qu'occupe l'unité dans la géométrie LOGO. L'absence de référent explicite et d'unité discriminante en géométrie LOGO ne permet pas, aussi facilement que pour la progression non-Logo, à l'élève débutant, de dissocier l'assimilation souvent constatée entre longueur et écartement des cotés.

Sur le tableau suivant où sont successivement représente en ligne : la somme (SX) des réponses justes obtenues puis la moyenne (MOY) pour chacun des groupes A et B, la valeur calculée au «T de Student» pour la comparaison des ces deux moyennes, enfin le seuil de signification de celle-ci en considérant 46 d.d.l. et ne retenant comme significatives que les valeurs inférieures à .10. En colonne, chaque item est représenté par la réponse attendue, et la dernière colonne représentée par un «T» est le résultat en ligne tous items confondus.

AN.T2	90°	180	60°	320	45°	45°	60°	30°	60°	120	T
SX	37	36	29	22	16	13	6	2	2	1	164
MOY A	0,79	0,54	0,50	0,42	0,42	0,42	0,25	——	——	——	3,50
MOY B	0,75	0,95	0,70	0,50	0,25	0,12	——	——	——	——	3,33
t cal	0,33	3,72	1,47	0,56	1,21	2,35	2,76	——	——	——	0,25
SEUIL	N.S.	<.01	<.10	N.S.	N.S.	<.05	<.01	——	——	——	N.S.

On obtient 164 réponses justes sur 480 attendues soit 34 % de bonnes réponses. Ces réponses sont également [illegible]ties sur les deux groupes donnant des moyennes presque identiques. On peut déjà considérer que du point

de vue des angles, les élèves du Groupe B ont appris quelque chose, on pouvait s'y attendre;mais ceux du Groupe A ont également progressé... et ailleurs qu'en LOGO. Item par item la comparaison des moyennes permet de savoir si un groupe progresse plus que l'autre.

Il n'y a pas de différence significative entre les deux groupes pour l'item concernant l'angle de 90°. C'est l'item le mieux réussit alors que l'angle de 90° est ici intégré dans une figure problème. Ceci montre en outre que les élèves ont pu lire, rester attentifs jusqu'au 5° item et repérer dans cette situatinn une connaissance acquise.

Le groupe B réussit mieux les items 2 et 3 (180 et 60°). L'addition d'angles pour obtenir 180° est l'item qui marque le plus la différence entre les deux groupes. Les élèves du groupe B justifient les résultats obtenus à l'item 180 par une addition de même à l'item 60° ils utilisent souvent la soustraction : 180 - 120 = 60. S'il y a une réelle influence de la méthode sur ces deux items, ce qu'il ne sera possible de vérifier qu'au T3, alors une recherche des causes sera entreprise afin de justifier ce constat.

Pour l'item 1 de 320° (complément à 360° de 40°) la différence entre les deux groupes est non-significative. Toutefois cet item est mois bien réussi que ceux concernant l'angle plat. On peut penser que la disposition des items sur la feuille induit quelques bonnes réponses en série pour les items 2 et 3. Il convient toutefois de remarquer que le concept d'angle semble se construire à partir de l'angle de 90° pour s'élargir à l'angle de 180° puis à l'angle de 360°. Il s'agit pour des élèves ne maîtrisant pas encore le mécanisme opératoire de la division d'une approche par fraction (moitié, quart). Cette observation, sera détaillée par l'analyse du protocole de Farid.

L'observation des résultats obtenus aux items 5-A et 5-B des deux angles de 45° du demi-carré montre, que tous les élèves du Groupe A réussissant un item réussissent l'autre. Il en n'est pas de même pour le groupe B : trois élèves réussissent un item et pas l'autre. Pour eux ces deux items ne sont [illegible] identiques. Il s'agit là d'un mauvais repérage des invariants lors de la constitution de la représentation d'une classe d'équivalence.

Les élèves du groupe A réussissant aux trois items du rectangle 6 ne sont capables de justifier de leur réponse ni à l'écrit à l'aide d'une opération, ni même à l'oral.

L'angle de l'hexagone de l'item 4 est pour l'ensemble des élèves encore inaccessible. Aucun transfert ne se fait pour le groupe A à partir du «Théorème du Trajet Total de la Tortue» (<1>).

La mesure de la dispersion s'impose ensuite. Elle permet d'obtenir une valeur de 2,64 pour l'écart type du Groupe A contre 1,62 pour celle du groupe B. Le groupe A est donc beaucoup plus dispersé. Il y a une influence de la méthode sur cet indice. Toutefois, il conviendra d'affiner et éventuellement de confirmer ce constat d'une part [illegible] observations similaires menées à l'issue du T3 d'autre part en analysant la répartition des bonnes

réponses suivant les individus.

Enfin, quelques explications relevées parmi celles fournies par des élèves corroborent certaines observations:
- Farid (groupe A) justifie toutes ses bonnes réponses par des affirmations utilisant des fractions connues : le quart ou la moitié. Ainsi à l'item 3 il propose : PARCE QUE : «120 est les trois quart de 180» et à l'item 5-B concernant un angle de 45°: PARCE QUE : «45° est le demi quart de 360°». Cette logique basée sur des fractions connues ne lui permet alors d'accéder ni à l'hexagone ni au rectangle de l'item 6. Elle montre toutefois comment un élève encore incapable de poser une division à relié cette opération au concept d'angle. L'utilisation de la division lors de conjectures sur les angles devrait permettre à Farid de passer de l'angle droit, traitable, comme il le fait, par les fractions de quart et de moitié à l'angle de 90°. Il pourrait, en intégrant toutes les divisions du cercle de 360°, améliorer ses performances.
- Tagati (groupe B) justifie sa bonne réponse de 90° au 5-A ainsi : PARCE QUE : «c'est un angle droit alors l'angle droit mesure 90°». La liaison logique qu'elle établit par l'utilisation de «alors» montre que la corrélation entre une classe d'équivalence et son signifié n'est pas encore établie de façon très logique et sure.

Si l'on rapproche les conclusions de cette observation de celle du protocole précédent alors on constate que l'association : angle droit, angle de 90° se construit et à une signification. Elle marque le passage d'une représentation rendue fonctionnelle par l'utilisation de fractions à une représentation devenue fonctionnelle et opérationnalisable.

A l'issue du T3

Les résultats obtenus sont présentés sur le tableau suivant dont la présentation est identique au précédent.

AN.T3	90°	180	60°	320	45°	45°	60°	30°	60°	120	T
SX	40	44	33	37	27	25	4	5	1	4	221
MOY A	0,83	0,88	0,67	0,79	0,54	0,46	0,04	0,08	0,00	0,04	4,38
MOY B	0,83	0,96	0,71	0,75	0,58	0,58	0,13	0,13	0,04	0,13	4,83
t cal	0,00	1,03	0,38	0,36	0,38	0,19	0,15	0,32	1,00	0,15	0,73
SEUIL	N.S.	N.S.	N.S.	N.S.	N.S.	N.S.	N.S.	N.S.	N.S.	N.S	N.S.

Tout d'abord, 221 réponses exactes, soit 46 % de bonnes réponses, sont obtenues. Cela correspond à une progression de 12 points par rapport au pourcentage obtenu au T2 (36 %). Ce progrès est moins important que celui constaté entre le T1 et le T2. Cela semble logique si l'on estime que les notions les plus faciles d'accès sont ...ies en priorité.

Les deux groupes n'étaient plus homogènes après le T2. Leur permutation a eu pour conséquence de niveler les différences révélées par les résultats. La répétition des effets vient donc, confirmer les hypothèses présentées à l'issue du T2. La plus générale était issue de l'observation des écarts types. Au T3 on relève une valeur de 2,00 pour l'écart type du groupe A et de 2,25 pour celle du groupe B. Compte tenu qu'il y a progression et rattrapage des différences après inversion des traitements pédagogiques, on est maintenant en mesure d'affirmer que du point de vue des résultats obtenus à l'épreuve des angles, la géométrie LOGO telle qu'elle a été pratiquée engendre plus d'écart entre les élèves. L'analyse des flux entre les deux Tests permet même de montrer que la géométrie non-LOGO mise en place est plus démocratisante. Il convient à nouveau de conduire l'analyse item par item.

Pour l'angle de 90° il n'y a aucune différence significative entre les deux groupes. Mais, l'ordre des réussites s'est modifié : Les élèves, au T3, réussissent mieux l'item 3 concernant l'addition des valeurs pour arriver à 180°, que celui de 90°. Additionner des valeurs représentant des angles est devenu plus simple qu'identifier un angle de 90° sur une figure.

Aux items 180 et 60° une différence significative entre les deux groupes à l'issue du T2 a été constatée. Puisque cette différence n'est plus significative après l'inversion des traitements pédagogiques,.on peut conclure à un meilleur effet de la progression non-LOGO pour ces deux items. Les élèves doivent, pour réussir ces deux items développer des conjectures utilisant la supplémentarité. Celles-ci sont liées dans un cas à l'addition dans l'autre à la soustraction. L'acquisition de ces mécanismes en LOGO a déjà été étudiée par D. Mendelson <7>. Par la progression non-Logo la découverte de la complémentarité et de la supplémentarité avait été abordée par l'observation de plans symbolisant des ouvertures de portes dans la ligne des travaux proposés par ERMEL concernant les fausses équerres et faux compas <8>. Les situations évoquées par les portes correspondent très exactement à ces deux items. La meilleure efficacité d'une progression s'explique alors par la nature de la métaphore employée qui permet plus facilement le transfert des compétences opératoires.

Pour les deux angles de 45°, on constate à nouveau l'inversion des tendances décrites, concernant la répartition des résultats entre les deux groupes à l'issue du T2. Maintenant, tous les élèves du Groupe B répondant juste à l'un des items concernant l'angle de 45° répondent juste à l'autre. Deux élèves du groupe A présentent un pattern de réponse différent pour ces deux items. Le travail en LOGO favorise la constitution des invariants nécessaires à la formation de classes d'équivalence pour les angles.

Pour les items 6, la différence entre les groupes est non- significative. A l'issue du T2 elle n'avait pu être testée puisqu'aucun élève du groupe B ne répondait à ces items. Toutefois l'observation des distributions par groupe des erreurs avait conduit à constater que les élèves du groupe A essayaient plus facilement ces items. Cette tendance s'inverse à nouveau.

L'angle de l'hexagone n'est plus l'item le plus difficile à réussir. 2 élèves ayant débuté par le LOGO sont capables de répondre justement.

CONCLUSIONS

La première conclusion qui s'impose concerne l'efficacité de l'épreuve construite qui permet une discrimination assez fine des élèves en enregistrant tout de même des effets plafonds (90°) et planchers (rectangle). Il manque toutefois à ce corpus quelques items conduisant à comparer des angles. Par un plan expérimental adapté, les résultats obtenus à l'issue de traitements pédagogiques différents montrent que les élèves ayant utilisé LOGO ont non seulement appris ce langage mais qu'ils ont également acquis des connaissances sur les objets manipulés. Ce constat n'autorise toutefois pas à plaidoyer en faveur de l'apprentissage de la programmation à l'école sans qu'une réflexion pédagogique soit entreprise à partir de l'ensemble des conclusions. En effet, les représentations du concept d'angle établies à partir de la progression LOGO sont moins fiables que celles établies à partir de la progrèssion non-Logo. Par ailleurs, si Logo a développé un esprit d'analyse parfois performant pour des figures complexes, sa portée, expérimentée en dehors de tout cadre conceptuel a été restreinte lorsqu'une conceptualisation modélisante a été entreprise à posteriori. Enfin et surtout, la progression utilisant LOGO, introduite sans distinction auprès de tous les élèves de deux classes de CM 1, comparée à partir de critères institutionnels à une progression n'employant pas cet outil, a eu, sur cette réussite, des effets élitistes.

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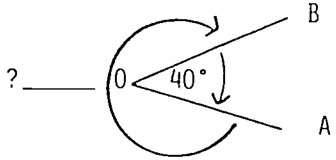
ANNEXE :

FIGURE À COMPLÈTER | EXPLICATIONS



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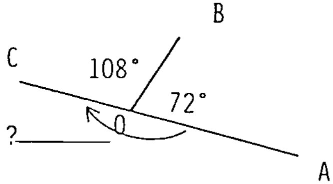

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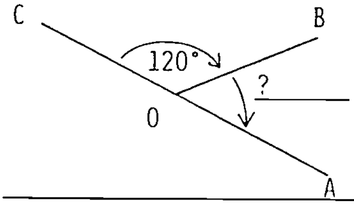

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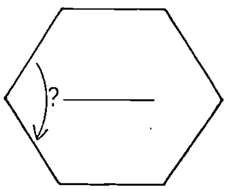

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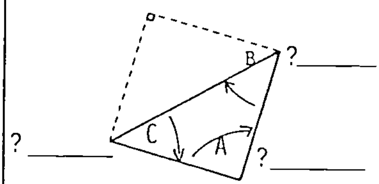

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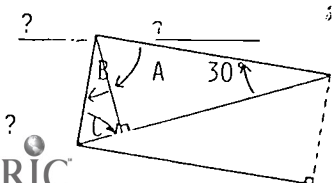

A PARCEQUE :

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C PARCEQUE :

INTERACTION BY OPEN DISCUSSION AND "SCIENTIFIC DEBATE" IN A CLASS OF 12-YEARS OLD PUPILS

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Department of Mathematics

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Presentation: T. Patronis

Summary: This paper is a report and analysis of an experience which is an instance from one year of experimentation with a class of 12-13 years old pupils. The general object of study is the development of thought in solving "open-ended generating problems" and the corresponding interaction between solvers. In the experience reported here there was a problem given, where it was asked from the pupils to compare the numbers of trees planted in three different ways in rectangular fields of same dimensions. In analysing the results of this experimentation, we discuss the role of a socio-cognitive conflict during the development of the process of solving the above problem in the classroom, in a situation of open discussion and debate.

1. The general context of the research

The experience that follows is only an instance in one year of experimentation with a class of 30 young pupils of 12-13 years of age. The general object of study is the development of thought in solving "open-ended generating problems" and the corresponding interaction, between the solvers, in a situation of open discussion in the classroom.

By an open-ended generating problem we mean a problem, formulated not in the usual mathematical language, but in a natural language familiar to the solvers, which leads potentially to some specific mathematical concepts (or models), provided that, either these concepts or the way of their ap-

plication to the solution of the problem are not known in advance to the solvers.

By <u>open discussion</u> in the classroom we mean any discussion on some problem and/or the process of the problem's solution, the formulation, the models used etc., provided that in this discussion the following conditions are satisfied:

- every student (or pupil) has already obtain some autonomy of action, and he (she) keeps this autonomy;
- any point of view, any ideas or conceptions are respectable and can be expressed in the discussion;
- there are several intentions in the discussion, but none of them (in particular neither those of the teacher) is considered as dominant a priori.

<u>Interaction by open discussion</u> may give rise to psychological and conceptual (socio-cognitive) conflicts which are important for the construction and elaboration of mathematical concepts and ideas, but it corresponds to a "didactical contract" - in the sense of G. BROUSSEAU (1986) - which, in general, cannot be easily realized in practice (see for example M. LEGRAND and his group (1987) f .. an organization of "scientific debate" in classes at the university level). Although situations leading to interaction by open discussion and "scientific debate" in the classroom can be provoked and stimulated by suitable (open-ended) generating problems, actually the larger part of such a discussion is spontaneous and the whole process cannot be predicted in advance, neither it can be evaluated by tests etc.

Starting from these remarks, we adopted the following organization of the discussion and method of observation: The class is divided into small groups, with 3-6 pupils in

each one of them. Discussion of a problem in these groups, developing of an idea (or plan) and carrying it out constitute the first phase of the process of interaction. The second phase comprises communication of results, open discussion and criticism in the classroom. In case of a lasting debate (as the one reported below) there is also a third phase, in which representatives from "opponent" groups are called to form a new group where the discussion continues. During all these phases each member of our research group undertakes a role of participant observer in one of the groups of the pupils and gives a report at the end of each meeting. Meanwhile all discussions have been tape-recorded; the tapes are compared to the children's notes or drawings and to the "local" reports of the observers, so that an analysis from a "global" viewpoint becomes possible.

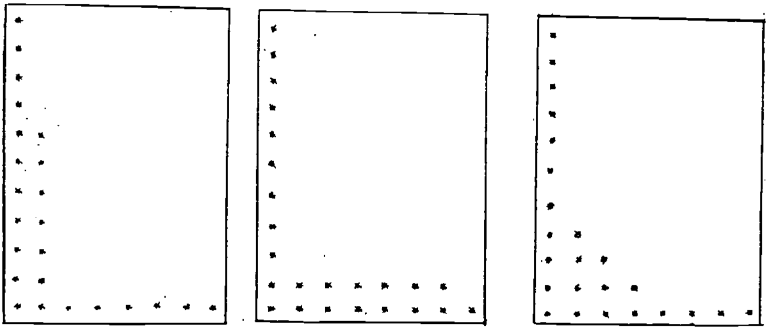
2. The experience and its analysis

A problem was given in the form of a dialogue between a father-farmer and his three sons, as follows.

FATHER: Boys, I have to go to the city this morning. I've just started planting those olive trees in the three fields you saw yesterday. You must continue now; each of you will take care of one field...
SON A : I'll get the smallest one!
FATHER: But they are all the same, you know that. Come and see them once more!
SON B : (Seeing the fields and the trees already planted-Fig.1): The three fields may be the same, but as you have put the trees in each-one of them, father, it seems to me that there are more trees to be planted in one of the fields and less in the others.
SON C : Oh, we have to check this by paper and pencil...
Son A : Okey...(whispering:) Anyway, I'll plant the fewest trees myself!...

It was asked from the pupils to continue this dialogue and help the three brothers in their trouble.

Fig. 1



During the first phase (interaction in small groups) and also for a long time during the next phases, a group of 6 pupils - which from now on will be called "Group A" - was trying to apply the formula of the area of a rectangle

Area = Base x Altitude

to the given figures, in order to evaluate from this the total number of trees that would finally exist in each rectangular field. The same approach has been followed also for a while by some isolated pupils outside of Group A. But these pupils were soon discouraged by the reaction of the rest members of their groups.

Meanwhile, the rest of the class (about the 4/5 of it, as it results from the children's own notes and drawings) had proceeded in a more direct and natural way: In each field there would be a final number of trees equal to

8 rows x 11 trees/row = 88 trees.

Subtracting the number of trees already planted (which is common for the three fields: 24) one gets 64 trees that have to be planted in each field. We shall consider the pupils who followed this approach as belonging to "Group B".

On the other hand, the result obtained by some of the

pupils of Group A at the beginning of their efforts was different: The final number of trees in each field was evaluated as 7 x 10 instead of 8 x 11. This result follows from a conception of the problem according to a scheme that appeared in the drawings of Group A (Fig. 2a). According to this scheme, the external rows of trees form a rectangle (with 4 trees at its vertices); if the distance between two consecutive trees in a row is, say, 2 units, then the area of the rectangle is

(7 x 2) x (10 x 2) area units.

Let us call this scheme "Scheme AI". Later this scheme changed and developed into "Scheme AII" (Fig. 2b).

Fig. 2a

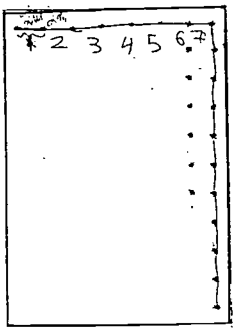
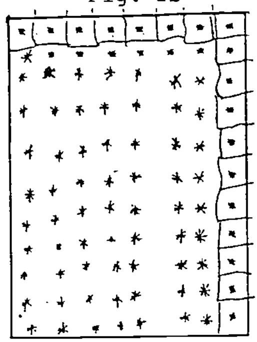


Fig. 2b



In the second phase of interaction (open discussion in the classroom) there was a strong opposition between Groups A and B, but in fact only a part of Group B was engaged in this debate (unfortunately most of the girls were not). The criticism of Group B was directed mainly against the method used by Group A. The main argument against this method was the following, as expressed by a boy of Group B:

> "How much is the distance between two trees? Is it 1m? It cannot be so, for if we have an area of, say, 284m², then we need not have 284 trees! For the same reason the distance cannot be 2m or 3m..."

In replying to this argument, the members of Group A produced their answer according to Scheme AII, which they defended by the following words:

> "Each tree corresponds to a unit of area. If the distance between two (consecutive) trees was bigger, then this area unit would be bigger too. The distance of trees determines the unit of area."

> "Let x be the distance between two consecutive trees in a row. I take this as a unit of length and I call it "tree-unit" (!). Then the area of the rectangular field will be equal to
>
> $$(8 \cdot x) \cdot (11 \cdot x).$$ "

But then a new objection was raised:

> "The distance between two consecutive trees need not be the same with the distance of extreme trees from the edge of the field (Fig. 1). So the true area of the field is different from that you are talking about."
> (The boy was addressed to Group A.)

As it has already been mentioned in the introduction, there was also a third phase in this experience. Some representatives of the two opponent "parties" were called to form a new group and continue the discussion. In this last phase a boy from Group A explained his point of view with the following words:

> "Look here...in order to unde--tand of what I am talking about: May be the area is not convenient, but I have used it in order to make things easier for me and for you ...because the area of the field and the number of the trees which will be planted in it may be related a little: the field contains as many trees as much is its surface area, and vice-versa; finally it's the same thing."

In replying to this, the children of Group B repeated their arguments, without anything new.

For an analysis and an interpretation of the results exposed above we took into consideration the views of social genetic psychology and epistemology, according to which the

socio-cognitive conflict is a conflict of communication rather than an internal conflict of the individual; it is by the interaction and common activity of partners that the subject is led to the construction (or co-elaboration with some partner) of new operational schemes during the cognitive development (F.CARUGATI and G. MUGNY, 1985). On the other hand, according to G. BROUSSEAU (1988), in a situation of cognitive conflict the subject has either to choose among two alternative schemes of action or models of formulation that are (or appear to the subject to be) incompatible, or to make these two alternatives compatible by modifying one of them.

In our case the formula for the area of a rectangle, which was known to subjects of Group A from the elementary school, offered to them a mathematical context, a model of formulation (in the sense of G. BROUSSEAU) for the solution of the problem. However, at the beginning this model was not well adapted to the problem itself.

In the present experience, as well as in many other instances of our experimentation, this kind of behavior was typical: In solving a problem, some pupils tend to apply those mathematical methods and tools (familiar to them from the previous "successful" mathematical school experience) which apparently "fit" the situation. Usually these pupils do not examine whether their method is relevant to the given problem. Under the conditions of the usual didactical contract, this behavior becomes easily stereotyped and it is generally accepted without any comments or reaction from the teacher and the other pupils. But in a situation of open discussion there is some reaction, which may be expressed in several ways. In our case this reaction was expressed at the cognitive level, on the mathematical

content and method used; thus it took the form of a "scientific debate".

This debate is not superficial. The initial approach of Group A to the given problem evoked a conflict, which has two principal, complementary aspects:

(i) It is an "internal" conflict for the subjects of Group A: The model they used being not well adapted to the problem, the solution initially obtained does not agree with that expected from experience (Scheme AI).

(ii) (Social aspect of the conflict:) The rest of the pupils of the class, having solved the problem in a more direct and natural way, do not accept the method used by Group A and some of them produce arguments against it. This situation has an immediate effect on the subjects of Group A, because these subjects are now led to justify their approach; but in doing so, they need to revise and reorganize it according to a new scheme (besides, this was necessary from (i), since their solution did not agree with empirical facts).

From this debate emerged a new formulation of the problem by a suitable modification of the initial scheme. The crucial step was to establish a natural correspondence between trees and units of area (Scheme AII).

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FORMAL AND INFORMAL SOURCES OF MENTAL MODELS FOR NEGATIVE NUMBERS

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Summary

Interviews with children prior to instruction on negative numbers reveal a progression from a model of number without negatives to models in which all of the integers, positive and negative, are ordered in a "mental number line." In the Divided Number Line model, two symmetric strings of numbers are joined at zero; children compute in terms of moves toward and away from zero, using special partitioning procedures to cross zero. In the more mathematically coherent Continuous Number Line model, they compute as if the number line were continuous, going "up" for addition and "down" for subtraction. These models are abstract and do not refer to practical situations such as debts and assets. They appear to be elaborations of children's knowledge of positive integers, which have become mental objects in their own right, without necessary external reference.

Introduction

Children's concepts of the positive integers can be shown to develop out of their early experience with the ways in which the physical material of the world composes and decomposes, together with their mastery of the formalism of counting (Resnick, in press). From the fact that physical material adds in systematic ways, and from their experience in quantifying amounts of material through counting, children arrive at a basic mathematical principle of <u>additive composition</u> of number. This, in turn, entails properties such as <u>commutativity</u> of addition and <u>complementarity</u> of addition and subtraction.

Beyond the positive integers, it is not so clear that mathematical knowledge can be directly rooted in physical experience. When negative numbers are added to the integer system, for example, there is no way that children can experience the quantification (e.g., through counting) of a "negative set." Does this mean that negative numbers can be learned only as a formal system? Or do children develop intuitions prior to formal instruction that they can use in understanding the formal system, much as younger children use their intuitions about physical quantity as they learn about the integer number system? Put another way, what mental models of negative numbers and of operations on them do children have prior to formal instruction?

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Method

To address this question, we studied children in a private girls' school. In this school, negative numbers were first formally introduced in Grades six and seven. Children in first, third, fifth, seventh, and ninth grades were given a written test of negative number knowledge. Following this, six children in each grade were interviewed using a clinical interview method that probed for their explanations and justifications of problems.

Summary of Test Results

Results of the test showed a clear effect of both age and instruction. First graders were totally unable to do arithmetic on signed numbers. For example, only one of them recognized that -4 is a larger number than -6. In third and fifth grades, up to half of the children were able to solve many of the operation problems, and almost all fifth graders knew that -4 is larger than -6, suggesting that they had constructed a mental number line that included negative numbers. By seventh grade, a year after instruction on the negative number system began, almost all students could do all of the problems; and by ninth grade, performance was perfect. Typical errors on the arithmetic operations problems among the first graders suggested that they had no conception that negative numbers might exist. Their answers were always positive integers. To arrive at these answers, they inverted numbers freely (e.g., 5 - 7 was treated as 7 - 5) or ignored signs (e.g., -5 +8 was treated as 5 + 8). Several made it clear that they thought there were no negative numbers by saying that a small number minus a larger one (e.g., 5 - 7) yielded zero. Third and fifth graders were more likely to generate negative numbers as answers, showing that they believed in their existence. However, they applied idiosyncratic rules that did not respect the conventions of negative number notation. For example, they treated -5+8 as if it were -(5+8), yielding -13 as an answer.

Interview Re...

Mental Models of Negative Number

Interview results provide a view of the mental models underlying the children's test performances. Generally, we saw a progression from a model of number in which negative numbers (numbers falling below zero) essentially do not exist to a model in which all of the integers, positive and negative, are ordered in a "mental number line" with a symmetrical organization of numbers around zero. Based on their number ordering performances, several first and third graders could be seen to have no representation of negatives as falling below zero. They either placed negatives next to the corresponding positives or treated them as all equivalent to zero. At the next level of development, one first grader knew that negatives fell "on the other side" of zero, but did not represent the symmetry of

the numbers around zero. Most third graders and all fifth graders did represent this symmetry and showed an ordering consonant with the number line.

Studying the children's responses to the general question of what they knew about negative numbers and their responses to the operations problems reveals that there are two forms of number line representation. In the most advanced (the Continuous Number Line or CNL model), children represent the numbers as ordered along a single continuum from smaller (the negatives) to larger (the positives):

Larger →

-3 -2 -1 0 +1 +2 +3

Children with this CNL model need no special rules for "counting across" zero. Children with this model might mention a division of the number line at zero, but they mostly computed as if the number line were continuous, going "up" for addition and "down" for subtraction.

A less mathematically coherent number line model joins two symmetric strings of numbers at zero and stresses movements toward and away from zero rather than just up and down. We call this a Divided Number Line (or DNL) model:

← Smaller Larger →

-3 -2 -1 0 +1 +2 +3

This model requires special rules for crossing zero, usually in the form of a partition of the number to be added or subtracted. The typical child using this model would partition the number to be added or subtracted into the amount needed to reach zero and then continue counting off "the rest" on the other side of zero. It is characteristic of children using a DNL model that, on problems in which it is not necessary to cross zero, they talked of doing addition or subtraction "on the negative side."

Constructing this mental number line model is not an all-or-none or an all-at-once matter for children. Several children could describe number line models but not use them effectively. Interwoven with problems of constructing a coherent mental model that includes negative numbers is the problem children face of learning the conventions of signed number notation. Several errors in the protocols seem to derive from not knowing how to encode certain notations. This occurs especially when plus signs are understood but not shown. It also occurs, however, when a child's mental model cannot handle a problem that is presented. This is the case when a negative number must be subtracted from a positive number (e.g., [+4]-[-2]). Neither the CNL nor the DNL model can coherently represent this problem. A typical response is either to mentally delete one of the minus signs (yielding the answer [+2] to our example problem) or to mentally exchange the positions of the plus sign and the negative operator sign (converting the problem to [-4]+[-2] and yielding the

answer [-6]).

Conclusion

This study, although only exploratory in nature, provides clear evidence that many children construct mental models that include negative numbers before school instruction on this topic is offered. Some become quite facile in doing arithmetic operations on the basis of this model, clearly drawing on their models of the positive numbers to do this. This is most apparent in the partitioning strategies (go to zero, then finish the rest on the other side) that children with the DNL model use.

It is striking that the two number line models that our subjects developed were quite abstract. We had expected to find them thinking about debts and assets--having numbers of things and owing amounts to others. A few children mentioned such conceptions: for example, one child said she had seen her mother's budget sheets at work and knew that negative numbers stood for how many more hours someone had to work to get paid; another said negative numbers were "bad marks" that balanced good ones. Debts and assets are thought to have played a role in the historical introduction of negative numbers in Western mathematics; negatives were needed for the bookkeeping systems that developed as commerce expanded in the Renaissance. Yet, although they mentioned them, none of our children actually used debts and assets in their reasoning. If they could reason about negative numbers at all, they did so in terms of the mental number line models we have described.

What are the possible origins of the mental number line model? First of all, there is good evidence that a mental number line for the positive numbers is established by most children even before school entry (Resnick, 1983). They initially use this representation to compare the relative sizes of numbers. It is reasonable to suppose that over the first years of school they gradually relate this number line representation to the operations of addition and subtraction. Children's general experience with symmetry (some even mentioned mirrors in discussing what negative numbers might be) is a likely source of the divided number line idea, once the existence of numbers with minus signs have been noticed and thought about. What would remain would be to find a means of crossing the zero when doing calculations on the mental number line. Here it seems that children were applying well-developed ideas of additive composition (cf. Resnick, 1986) to produce the partitioning strategy that we observed among many children. In sum, children seem able to develop pre-instructional intuitions about purely mathematical entities (the negative numbers) by elaborating previously developed ideas about number (additive composition and partitioning) that were originally rooted in physical experience but have, through practice, become so familiar as to become intuitions in their own right.

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INVERSE PROCEDURES: THE INFLUENCE OF A DIDACTIC PROPOSAL ON PUPILS' STRATEGIES (*) (**)

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SUMMARY

With this research we intend to study the influence of a didactic proposal for the ages 11-12 on pupils' strategies to solve problems with inverse procedures.

The didactic proposal includes the concepts of relation between two sets, inverse relation, composition of relation and inverse of a composite relation; both in and out of mathematical contexts and with the aid of visualisation with arrows.

The proposal was presented to an experimental group of 33 pupils. The experimental group and another of control (21 pupils) were tested by 3 questionnaires, the results of which are described.

1. Introduction

One of the principle aims of mathematics teaching to students 12-14 years old is the acquisition of proportional reasoning. But, every teacher notes that such acquisition is still very unstable in students of upper secondary schools. Moreover, I would also say that for many adults, lacking the help of scholastic habits, the solution, for example, of inverse multiplicative problems still constitutes an insurmountable obstacle.

This justifies the vast amount of literature on the theme in question.

In particular, referring to inverse procedures, it has been observed by Mariotti et al. (see Ref. 7) that errors can be due to the fact that, in common didactic practice, addition and subtraction, but even moreso multiplication and division, are not considered as two aspects of the same structure (additive and multiplicative respectively).

At the first presentation of these operations, at the

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(**) The psycologist M.G. Grossi collaborated in this research.

elementary level, the difference of meaning between addition and subtraction and respectively multiplication and division is underlined. But, after, the unified vision of these operations is not given.

In this perspective it seems important to us to build the more general idea of relation and inverse relation between two numerical sets.

Again in agreement with them, we maintain that, in reference to difficulties linked to the dimensional aspect, it is important to progressively lead the students to "free" themselves from the chain of dimension, so that they can work, more easily, with pure numbers. It is therefore essential to underline the structural analogies of the various situations.

The most favourable period for working towards these goals seems to be from 11 to 14 years old. With these premises we elaborated a didactic proposal for the ages 11-12 and we are studying its influence on the strategies used by the same students to face problems with inverse procedures.

2. The Didactic Proposal

The didactic plan, discussed with the teachers of our group, has the following order:

- examples suitable to emphasize the concept of relation (with expressions like "...was born in the month of...", "...is a fan of...", "...is preceded by...", "...is the double of...");
- discussion of the various types of representation of a relation (tables, graphs, list of pairs, etc.);
- the four arithmetic operations as relations;
- the importance of the order between two elements linked by a relation (ordered pair) and the concept of inverse relation;
- the choice of the language of arrows as the most powerful;
- the operations addition and subtraction (and respectively multiplication and division) as inverse relations of each other;
- the composition of relations in real situations ("...is the son of the son of...", "...is the son of the daughter of...") and in arithmetic type situations ("add 2 and multiply by 3", "add 1 and subtract 7", etc.) with the aid of

the language of arrows;

the problem of returning to the starting point in the composition of two or more arithmetic operations: the inversion of the composition of relations.

The main objective is to use the concept and the visualisation of relation and inverse relation for facing the usual inverse problems in arithmetic, geometry and daily life.

3. The Questionnaires

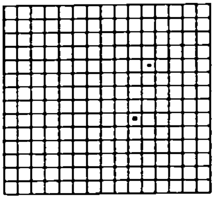
The first questionnaire with 8 problems (4 direct as "distractors" and 4 inverse) was presented in three classes (pupils aged 11-12). Two of these classes are experimental (S_1 and S_2) in the sense that, after the first questionnaire, the didactic proposal described above was presented. The third class (C) is a control group.

The problems on the questionnaires are the following:

1. A pencil, which is 14.5cm long, measures 2.4cm less than another. How long is the second pencil?
2. Consider the following game.
   A number is chosen and then 8 is subtracted, the result is divided by 5 and to that result 18 is added. Alessandra's answer was 30. With which number did she begin?
3. 1m of fabric costs £it 15000. How much does 0.65m cost?
4. A TV program lasts 90 minutes and 1/5 of it is publicity. How much time is dedicated to the program?
5. Marco calculates 3/4 of a number, then he adds 54 to the result and his answer is 144.
   With which number did he begin?
6. In an art book of 630 pages, 2/5 are illustrated; 1/3 of these are in colour.
   How many illustrated pages are in colour?
7. With a calculator I did 2/5 of a number and then I multiplied the result by 7, getting 266.
   With which number did I begin?
8. On a chessboard there are two pieces, ● and ■ , that can be moved only in the following ways:
   the move of ● : 2 squares up (↑) and then 1 square to the right (→)
   the move of ■ : 1 square down (↓) and 3 squares to the

right (⟶).

Look at the following situation:



You know that the pieces ● and ■ HAVE ALREADY DONE three moves each.

Trace their routes till you find where they began.

Problems 1, 3, 4 and 6 are direct. Problems 1 and 3 were selected from those given in the experience described by Deri et al.(see Ref.4). Moreover, problem 3 turned out to be difficult enough not only for the ages 11-12 but for the ages 14-15, too.

Inverse problems 2, 5 and 7 deal with the inversion of the composition of two or three arithmetic operations. Inverse problem 8, which turned out to be the most difficult, requires the successive application of the inverse of a composite relation.

It must be noticed that the problematic situations in 2, 5 and 7 are similar to those presented by the teacher in the didactic activity. But, the context is different.

As far as problem 8 is concerned, it deals with the more complex situation described above, which has never been presented in the class. But, also the context is unusual. The results of this questionnaire are in section 4.

After the administration of the first questionnaire, the teacher developed the established didactic proposal (for a total of 12-13 hours per class) in each of the two experimental classes S_1 and S_2. The developing of the work was recorded, by hand, by two final-year undergraduates in Mathematics (one in each class).

Just after the conclusion of the didactic unit previously ibed, another questionnaire was proposed in the three

classes S_1, S_2 and C. It also consists of 8 problems, almost identical to those of the first one, however, problem 8 is identical. This second test gives the initial indications of the influence of the proposed activity.

The results and the comparison to those of the first questionnaire are in section 4.

The plan for classes S_1 and S_2 included the recalling of the concepts and the resolving strategies; not systematically, but when it is necessary. So we want to consolidate and develop what has been given in the didactic plan.

A final questionnaire (with the same 8 problems as the first one) will be proposed, in the three classes, at the conclusion of the scholastic year. We think it could be indicative of the consolidation reached.

4. The Results

The experimental classes S_1 and S_2 are composed of 16 and 17 pupils respectively. The control class C has 21 pupils. The questionnaire was evaluated attributing 0 points for every wrong or omitted problem and 1 point for correct problems. The calculation errors were not considered.

In Table 1 the percentages of the correct inverse problems and the average scores obtained, for all three classes, in the first and second questionnaires respectively are reported.

In Table 3 there are the percentage variations of the average scores of the second questionnaire with respect to the first for all three classes. These ar r-lative to the direct and inverse problems.

Table 1

$S = S_1 + S_2$ Inverse Problems

		2	5	7	8	Average Score
1st Q.	S	36%	18%	18%	0	0.73
	C	48%	24%	33%	0	1.05
2nd Q.	S	51%	30%	30%	9%	1.18
	C	48%	28%	33%	5%	1.14

Table 2 Direct Problems

		1	3	4	6	Average Score
1st Q.	S	51%	36%	54%	39%	1.82
	C	71%	71%	52%	48%	2.43
2nd Q.	S	54%	24%	57%	39%	1.79
	C	52%	57%	57%	62%	2.28

Table 3

	Direct Prob.	Inverse Prob.
S	-2%	+62%
C	-6%	+9%

It is interesting to analyse the protocols and to examine the typology of the errors in the incorrect problems. But, here we will limit ourselves to some general observations.

It seems to us that the numerical data, especially in Table 3, show the positive influence of the didactic unit on the solution of the proposed inverse problems. On the other hand, as already observed, problems 2, 5 and 7 repropose problematic situations encountered in the didactic proposal.

Problem 8 merits a separate discussion. As Table 1 shows, this problem was revealed to be the most difficult. First of all, we recall that its context, of a non-arithmetic type, was not presented during the didactic activity. Moreover, it not only requires that the procedure of inversion of a composite operation is known but also that such procedure is applied three times in a consecutive way. If the procedure of inversion is not internalised, the successive application becomes difficult. To internalise a procedure means not only to understand its significance but, also to have placed its formal and generalisable scheme into long-term memory (see Ref.3).

In this sense we maintain that problem 8 can be considered indicative of the effective internalisation of the proposed inverse procedures.

5. Final Observations

First of all, it should be noted that the results obtained

are to be held only as indicative, taking account of the scarce number of pupils tested.

- The most significant contexts for the inverse procedures (proportionality in problems of similitude, percentages, etc.) cannot yet be proposed to pupils aged 11-12 because they are not included in the Government Programs.
- It should be said that the teachers were presenting the didactic unit for the first time. For that reason, their didactic procedure was not very well consolidated or efficacious.

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THROUGH THE RECURSIVE EYE: MATHEMATICAL UNDERSTANDING AS A DYNAMIC PHENOMENON

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ABSTRACT

Over the last couple of decades, attempts have been made to categorize different kinds of understanding. Rather than considering understanding as a single (or multiple) acquisition we offer here an overview of a new theory of understanding as a complex, dynamic process. It can be characterised as a levelled but non-linear, recursive phenomemon, each level being self-referencing but not the same as the preceeding level. This view of understanding as TRANCENDENT RECURSION allows us to see the way in which any given level is both dependent on the previous level for its initiating conditions and constrained by the nature of the succeeding level. Clearly this has implications for the teaching of mathematics.

"Everything said is said by an observer", Maturana,1980

**The experiencing organism now turns into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives.. among which is the never ending problem of consistent organization (of such structures) that we call <u>understanding</u>.
von Glasersfeld, 1987.**

Over the past 20 years or so there has been a continuing dialog, much of it through PME, on what it means for a person to understand mathematics. One of the features of this dialogue has been the theoretical identification of different kinds of understanding principally instrumental and relational understanding but also concrete, procedural, symbolic and formal understanding. Pirie (1988) has suggested that thus describing different kinds of understanding is inadequate as a means of differentiating children's performances exhibiting mathematical understanding. She claims, and illustrates from extensive taped interactions of children doing mathematics, that mathematical understanding is a complex phenomenon for the child doing it. A single

category does not well describe it nor do such categories capture understanding as a process rather than as a single acquisition. What is needed is an incisive way of viewing the whole process of gaining understanding.

There have indeed been recent efforts to go beyond a cataloging of kinds of understanding or thinking of mathematical understanding as a singular acquisition. Ohlsson (1988) performed a detailed mathematical and applicational analysis of fraction-related concepts. From this elaborated example, he suggests that mathematical understanding entails three things; knowledge of the mathematical construct and related theory; the class of situations to which this theory can be applied; and a referential mapping between the theory and the situations. He does not however, suggest how this mapping is developed or grows. He infers but does not give a process model.

Herscovics and Bergeron (1988) give a two tiered model of understanding and illustrate it using the understanding of number and pre-number in young children. The first tier involves three levels of physical understanding: intuitive, (perceptual awareness), procedural (e.g. 1-1 correspondence) and logico-physical abstraction (e.g. physical invariance). The second tier is non-levelled and entails as components of understanding the use of mathematical procedures (e.g. counting) to make mathematical abstractions reflected through the use of a notational system.

Both of these models of understanding above involve levels or components which appear to have predicate quality - they define complexes of components in unique categorical terms. In that sense they give a picture of the components which might be involved in the process of understanding. Von Glasersfeld (1987), however sees

understanding as a CONTINUING PROBLEM-SOLVING PROCESS of consistently organizing one's mathematical structures.

Let us consider the following example drawn from a study of 7-9 year old working in groups doing fraction comparison tasks (Wales,1984; Kieren & Pirie, forthcoming). In the task children were asked to compare the amount of pizza a person A would get if sharing 3 pizzas among 7 persons with the amount person B would get sharing 1 pizza among 3 persons. Here is commentary by Hanne working with 2 friends (all aged 7).

Hanne		**A is hard - let's skip it.**
Hanne		**B is easy, you 'Y' it** 
		(Draws 'Y' and explains her process to her friends).
Hanne	**(I)**	**Let's use 'Ys' on A.** 
		(Action 1, draws:
		i.e., Hanne cuts the three pizzas into "fair shares" in order to give one third each).
	(II)	**(Action 2,** 
		i.e., she cuts the remaining two thirds into seven smaller pieces).
Hanne	**(III)**	**Oh, I see! A gets a third and a bite.**
		A gets more.

What has happened here? Hanne starts by not understanding how to divide 3 among 7. It is clear from the complete tape that she can divide 1 among n for n small and in particular has <u>formalized</u> this act for 1 ÷ 3 ('Y it'). In I and II we see her now successfully divide 3 among 7 using the result of 3 replicates of dividing 1 among 3. At 'III' she marks the fact that she realizes that she has a successful new organization of sharing or division.

This leads us to ask what does Hanne's understanding entail? How is it a growing process? Our answer can be summarized as follows:

> **Mathematical understanding can be characterized as a levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur as thinking moves between levels of sophistication (as with Hanne above). Indeed each level of understanding is contained with in succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without.**

While it is beyond the scope of this paper to completely delineate this theory of mathematical understanding which we call TRANSCENDENT RECURSION, or to fully connect it to data on children's mathematical behaviour gathered in England and Canada, some major tenets of the theory are highlighted below. Of course it should be understood that we are not saying that the observed action sequence above exhibits these tenets per se. It is the underlying consistent organization or personal mathematical understanding, which we are trying to typify.

In saying that mathematical understanding is levelled and recursive we are trying to <u>observe</u> it as a complex levelled phenomenon defined by Vitale (1988) which is recursive if each level is in some way defined in terms of itself (self referenced, self similar), yet each level is not the same as the previous level (level-stepping). To this definition we have added an idea taken from Margenau's (1987) notion of growth of scientific constructs. New constructs transcend but are compatible with old ones (they are not simple extensions).

In trying to use recursion to describe understanding we also use the concept of thinking drawn from Maturana and Tomm (1986). Thinking is seen to be a recursive phenomenon - a distinction among distinctions of languaging and languaging is itself recursive. It entails the consensual coordination of consensual coordination of actions. Thus thinking, means having a consistent levelled structure leading back to, or calling, processes from lower levels potentially all the way back to action. Growth of this structure, however, can occur in a non-linear fashion. This view of recursion is useful in considering the personal 'transfer' of understanding. 'Transfer' to a new situation means using ones current understanding to reconstruct or reformulate ones knowledge to accommodate the new situation. Thus, recursion can be used as a tool to "see" the organization underlying this reconstruction. This recursive reconstruction, organization or understanding is seen as Hanne above 'calls' the form of her previous level of dividing knowledge as a basis for her new understanding of dividing up.

What are inter-relationships among levels? If one focuses on any one particular level then understanding at this level depends upon the level below to provide necessary initiating conditions, and on the level above to provide the environmental constraints which "call out" forms or processes at the focus level [Salthe, 1985]. For example, Hanne's '1/3 understanding' level provides an initiating condition for her '3/7 dividing process.' As argued below a recursive, dynamic notion of understanding can provide a description of personal mathematical knowledge building. While it is beyond the scope of this brief paper, this theory can also provide an account of mathematical problem solving.

To illustrate one aspect of our theory in less general terms, we turn to the example of fractions, and focus on fractional knowledge about and built through symbolic manipulation. Such knowledge can play several roles in the recursive structure of understanding fractions or rational numbers. It can be called as a particular example while building, validating or reconstructing knowledge at the "higher" level of quotient groups or fields. On the other hand work on a symbolic task should be able to "call" intuitive knowledge of fractions as quotients or even the action of dividing up equally as a basis to reconstruct or to validate symbolic level activity. Rational number understanding is seen as a dynamic growing whole consisting of sub-levels which are self similar in that they are about fractional knowledge. These levels are not reducible to one another however: knowledge of quotient fields is distinct from computational knowledge of fractions which it organizes, which in turn is distinct from the intuitive knowledge below it.

Nonetheless, these levels of knowing are inter-dependent. One can look at the fractions as quotients and the act of dividing up equally as providing <u>initiating conditions</u> for knowledge gained through symbolically multiplying or dividing fractions. The level above, 'rationals as a multiplicative group' constrains such symbolic knowing. This constraint is environmental; the <u>mathematics itself</u> "calls out" certain symbolic acts as correct. Thus a person's symbolic understanding is environmentally constrained by the normal structure of that domain of mathematical knowledge, in this case that of the rational numbers.

No mention has yet been made as to the relevance of this model of the process of understanding to the teaching of mathematics. We offer here a brief glimpse of how

schooling can affect the environmental levels surrounding symbolic fractional understanding. In this case knowing may be constrained by what the teacher considers to be mathematics. The teacher may see computations as a set of procedures to be learned, which would call forth certain behaviors in a child. The level below subsumed by symbolic knowing under such circumstances, might then provide as initiating conditions all strategies the student has found successful for survival in school or with that teacher, such as the blind memorization of algorithms. Thus a child's understanding of school mathematics can be environmentally constrained by the teacher's (or text's) mathematics and might call as initiating conditions non-mathematical structures or behaviours.

Summary and Concluding Remarks

This viewing of mathematical understanding as a dynamic process allows us to see a person's current state as containing other levels which are different, but compatible, ways of understanding the mathematics, which allow the person to validate upper level knowledge or provide a basis for facing unknown but related mathematics.

Considering mathematical understanding as a recursive phenomenon is not meant to replace the contemporary views of understanding suggested by Ohlsson or Herscovics and Bergeron. It is meant to provide insight into how such understanding grows and how the elements these authors describe are integrated into a whole. As such the theory sketched above should allow for the dynamic levelled analysis of mathematical understanding. In particular, it should allow one to see the self similarity but transcendence in the levels, to see the process of validation of personal knowledge and to comprehend

transfer as recursive reconstruction. It enables one to identify the roles of form (language, symbolism) and process both at any level and in the growth between levels.

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COGNITIVE ASPECTS OF THE LEARNING OF MATHEMATICS IN A MULTICULTURAL SCHOOL

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Research results are reported which indicate that when language-related learning difficulties are discounted, cognitive differences between pupils from three different cultural groups learning mathematics together in the same school are far less evident than are differences between pupils in different achievement groups or in different years. This research suggests that it is viable to use a common curriculum when pupils from different cultural groups learn mathematics together in the same classrooms.

The full title of the project on which this paper is based is as follows: "An investigation of the role of culturally conditioned thinking in the learning of mathematics by pupils in multicultural and in culturally segregated schools: a longitudinal study". The research carried out in 1988 at Uthongathi, a multicultural school in Kwazulu/Natal, addressed the first part of this title. The project is ongoing and in 1989 it will be possible to compare the Uthongathi data with data collected similarly in schools in three culturally segregated school systems. It is possible, in fact likely, that the findings in these schools will be different. This paper deals only with the cognitive aspects of the interview and test data collected at Uthongathi. A second paper, by Manjul Beharie and Yanum Naidoo, reports on the affective aspects of the Uthongathi research. In both papers illustrations are drawn from the data of all four researchers.

Theoretical framework and rationale for the methodology

Evidence of the cultural basis of mathematics which has traditionally been considered culture-free has been drawn from countries as diverse as China, U.S.A., Jordan, Mozambique and Australia (Bishop, 1988). Uthongathi, the first of the New Era Schools Trust (NEST) schools in South Africa, with its policy of nonracial education and balance of numbers, is a natural laboratory in which to study, firstly, the effects (if any) of family cultural background and home language on the learning of mathematics, and secondly, any modification to these effects which may result from prolonged multicultural schooling. Hence the study is longitudinal, and a qualitative, hermeneutic research methodology involving audiotaped interviews was considered appropriate to the exploratory nature of the research.

The pupils

The three mathematics teachers at Uthongathi were given the task of selecting pupils who were paradigm cases inasmuch as they represented the following categories in a "3-dimensional" model: in each of standards 5, 6 and 7 (i.e., grades 7, 8 and 9), pupils of the Indian , Black and White race groups were chosen such that each of three achievement levels was represented, viz., high, medium and low. In view of the complex elements involved in the choice of suitable pupils, it was considered that their mathematics teachers, who had known them for periods ranging from three to fifteen months, would be more competent than the researchers to make the selection. In practice 25 pupils were selected rather than the 27 required in the model, because there were only 30 std 5 pupils at the

school and no pupils could be found to fill the categories high achievement Black and low achievement White in this year. All other categories were filled without difficulty. For interviews, these pupils were allocated to researchers as follows:

Norma	:	std 5 pupils	(7 pupils)
Manjul	:	high achievers in std 6 and 7	(6 pupils)
Anita	:	medium achievers, std 6 and 7	(6 pupils)
Yanum	:	low achievers in std 6 and 7	(6 pupils).

The "cognitive" interviews

Three of the six interviews with each pupil were concerned largely with cognitive aspects of the pupil's learning of mathematics. These interviews were based on the following tasks.

(1) "Matchsticks". Three series of mathematical problems involving matchsticks were solved by all pupils in the project. The understanding of these problems required minimal verbal input, and all solutions were obtainable using spatial ability and logic.

(2) "Verbal problems". Section A (6 problems) from Presmeg's (1985) test for mathematical visuality was given to pupils to solve aloud. Pupils had the choice of reading the problems in Zulu or English or both languages.

(3) "School problems". Pupils solved aloud problems from their school mathematics textbooks.

Language

The data from these three interviews revealed that, where differences in the problem solving performances of pupils from the three cultural groups were evident in a particular standard and achievement group, these differences were largely attributed to language. The 17 Indian and White pupils all named

English, the medium of instruction at Uthongathi, as their home language (even if Hindi, Tamil or Gujarati were also spoken at home). In contrast, of the 8 Black pupils, only two named English as their home language, although a further two indicated that English was a second language (after Zulu) spoken at home. Even when problems were understood and solved, some pupils (and especially those new to the school) could not explain their thought processes in English. Some evidence was found that there are two types of language-related learning difficulties in school mathematics (Berry's 1985 types A and B), but the type A (fluency) difficulties were largely masking the subtler type B (culturally determined) difficulties in the present research. The following protocols are illustrative.

NOMBU (std 7, home language Zulu/English): "We are doing word problems and I'm not enjoying it."
INTERVIEWER: "Why not?"
NOMBU: "I don't understand what the sentence means. Sometimes I mix it up or misunderstand the sentence." (Anita's data.)

NONHLANHLA (std 5, home language Zulu): "Yes but I don't understand this thing" (after reading word problem in English, then its Zulu translation). Only three of the six word problems were attempted, and each of these three required extensive explanation by the researcher.
(Norma's data.)

XOLANI (std 5, home language Zulu), read word problem A-1 for four minutes in English and in Zulu, then pointed out that the Zulu wording did not mean exactly the same as the English: "John miss one day, then go, misses one, then go. Peter misses two days, and then go, two days then go. After four days .... no." Then he speaks in Zulu.
(Norma's data.)

The problem, in English, reads, "One day John and Peter visit a [library] together. After that, John visits the library regularly [every] two days, at noon. Peter visits the library every three

days, also at noon. If the library is open every day, how many days after the first visit will it be before they are, once again, in the library together?"

Xolani shows here the possible type B difficulties which underlie even the translation of mathematical problems into Zulu. Phyllis Zungu (lecturer in the Zulu Department, University of Durban-Westville), who did the translation, confirmed this difficulty, pointing out that it was necessary sometimes to "talk around" English mathematical terms when translating them into Zulu, either because a direct translation was not possible or because the Zulu terms were not well known even to Zulu speakers.

One encouraging finding in the Uthongathi research was that prolonged schooling at Uthongathi tended to reduce the differences between English home language pupils and those for whom English is not a mother tongue. Manjul, whose interviewees were of above average achievement, reported as follows:
"All participants were fluent in English, and could understand the language and terms of mathematics (e...essed through English medium). The Black students who have problems with English attend extra English tutorial classes. The two Black pupils in my group sometimes had problems in expressing themselves but they basically understood the various concepts and terminology in mathematics.

THAMI (std 7) found that learning was more 'enjoyable, because last year I learned my English background' (meaning the language), 'and this year I understand the teachers better.'

TABO (std 6) also pointed out that 'Mr B__, especially, tends to emphasise on Black students reading books every day so that they become fluent with the English language, but I

think this is also good.'"

Manjul pointed out that "these pupils are socialised in part with a western culture (i.e., they are exposed to the fruits of technology - calculators, computers, T.V., scientific and mathematical toys and puzzles, chemistry sets and so on). In addition their parents take a positive interest in their children's learning, especially in mathematics."

By way of contrast, Yanum also found little difference between cultural groups amongst her below average pupils (std 6 and 7) because these pupils <u>all</u> experienced difficulty at times in understanding the language and concepts of mathematics.

CINDY (std 6, White) made the following comment about her textbook: "It's okay if you want to learn from it; it's a bit difficult to understand. They got the writing saying how to do it, how to explain it ... but I can't ... I can't properly understand it."

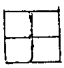
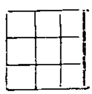
She indicated that it was the language that she "can't properly understand".

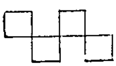
<u>Matchsticks</u>

The problems in the "matchsticks" interview were as follows.

1. Make the following numbers of identical squares, using all 24 matchsticks each time: 1; 2; 3; 6; 7; 8; 9.

2. How many squares?

(a)  (b)  (c)  (d) Predict for 4 X 4.
(e) Predict for 5 X 5.

3. 
(a) Move 2 matches to make five squares.
(b) Move 3 matches to make five squares.
(c) Move 4 matches to make five squares.

After analysing pupils' protocols, all four researchers reported that there were far greater differences in the performances of pupils from different achievement levels and year groups than

there were between pupils from the different cultural groups. In fact we had to conclude that no culturally determined differences were evident for these tasks. The second series of problems ("How many squares?") was of particular interest in this regard because it allowed for a possible generalisation to the nxn case. In std 5 no interviewee was able to generalise and only the two high achievers correctly predicted the 5x5 case (Norma's data). Manjul found that only one std 6 pupil (Tebo, Black) in her high-achieving group correctly predicted the solution for the 100 X 100 case, in which task all of her std 7 interviewees succeeded. Only Marc (std 7, Indian) gave an intuitive generalisation to the nxn case by pointing out the pattern involved (i.e., $1^2+2^2+3^2+....+n^2$). None of Anita's pupils went beyond the 5x5 case, which was solved only by Natasha (std 6, Indian), Nombu (std 7, Black) and Zarina (std 7, Indian). Even the 4x4 case was solved by only one of Yanum's low achievers (Lisa, std 7, White), and no pupil in her group attempted the 5x5 case.

Visualisation

Section B (12 problems) of Presmeg's (1' ", test for mathematical visuality was administered in group mode to all mathematics pupils in standards 5, 6 and 7 at Uthongathi. The reasons for examining mathematical visualisation were twofold:

(1) visualisation may provide a possible bridge to understanding when language difficulties exist;

(2) it was possible that cultural differences would be found in the need for visualisation in mathematics.

Analysis of the test scores of the 136 pupils who wrote section B revealed no significant differences between the three race s in terms of their median scores or frequency distribution

graphs. (All pupils were given the choice of reading the problems in Zulu or in English.) It was concluded that there were no cultural differences in need for visualisation in mathematics among these Uthongathi pupils.

Conclusion

Lawton (1975, p.5) wrote, "One view is that a common curriculum must be derived from a common culture. But this in turn raises other difficult issues. What is meant by a common culture? Is it meaningful to talk of a common culture in a pluralistic society?" The Uthongathi research suggests that a shared school experience provides sufficient elements of a common culture to make it possible to use a common mathematics curriculum under these circumstances.

Acknowledgement

The researchers wish to thank all pupils and members of staff at Uthongathi who made this research possible.

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QUALITATIVE AND QUANTITATIVE PREDICTIONS AS DETERMINANTS OF SYSTEM CONTROL

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Summary: We investigated the influence of qualitative and quantitative predictions for the effectiveness of system control using a simulation of a biological system, the fishing pond. In an experimental paradigm one group of subjects had to give numerical predictions for the optimum of the propagation function, another group had to apply qualitative reasoning by answering a qualitative question. The control group had none of these tasks, but simply played the fishing conflict game. The qualitative group did not perform better than the control group, but the quantitative group was more successful than the other two groups. Explanations are given regarding the function concept for each of the three groups.

1 The role of qualitative reasoning in problem solving

It is obvious that quantitative reasoning seems to be helpful in solving numerical problems. But qualitative predictions can be useful for the solution of quantitative tasks, too (DEKLEER & BROWN, 1984; BOBROW, 1984; HRON, 1988). The authors use the term qualitative reasoning in tasks where one is asked to predict the direction of a quantitative change rather than its absolute value. In such tasks one has to indicate whether the predicted value is going to stay constant, to become greater, or less than the current value.

In our study we investigated the influence of qualitative and quantitative predictions on the effectiveness of control of a biolo   a system. Inferences which merely predict the direction of change shall be called qualitative reasoning; those which result in a numerical value shall be called quantitative reasoning. In some studies qualitative reasoning has been particularly efficient. The authors investigated experts and novices in order to distinguish efficient and non-efficient cognitive processes. A number of studies about processes of qualitative reasoning while solving mathematics or science problems (BEHR, REISS, HARREL, POST & LESH, 1986; BRIARS & LARKIN, 1984; GREENO, 1983; LARKIN, 1983; REISS, BEHR, POST & LESH, 1987; SIMON & PAIGE, 1979) deal with the question of how experts (persons with a developed schema for a given task) and novices (persons with a

schema lacking components and relations between components) can be distinguished. CHI & GLASER (1982) report that experts classified problems according to structural relations within the text. They first dealt with the problem components and its relations in a qualitative manner and then tried to describe the components and their relations in quantitative terms. Their knowledge about structural relations enabled them to choose adequate procedures for a given task. In contrast, novices seemed to associate the solution with surface properties of the problem. Often they concentrated on irrelevant properties which took additional time or did not lead to the solution. Experts started on a top level and worked down to the procedures (top-down approach). Novices started at a low level of the problem and worked to the top by using known procedures until the solution was found (bottom-up approach).

Thus experts and novices could be distinguished according to their starting point and according to the direction of search for a solution. Experts as well as novices used some kind of qualitative reasoning but experts used structural components as a basis for qualitative reasoning, whereas novices used surface properties which were chosen randomly. All the studies emphasize that experts have better problem representation because they make intensive use of qualitative reasoning about problem components and their relations (CHI, FELTOVICH & GLASER, 1981; CHI & GLASER, 1982; CHI, GLASER & REES, 1983). The problem representation enables the expert to determine when qualitative reasoning is adequate and when quantitative reasoning is necessary. Novices, on the other hand, seem to search for formulas, procedures, and equations in an algorithmic manner. They do not take a long time to consider when certain formal structures are useful and which results can be anticipated by using them.

Most of the studies cited deal with physics problems. We want to study the effectiveness of qualitative reasoning within the context of a biological system. SPADA, OPWIS & DONNEN (1985) have developed the fishing conflict game (SPADA, OPWIS, DONNEN & ERNST, 1985; ERNST, 1988). This is a simulation of a

fishing pond. When a fisherman harvests fish from the pool the number of fish decreases. But at the same time the number of fish increases again because of natural propagation. The fisherman is faced with a dilemma: He wants to catch as many fish as possible but in order to be able to have fish on a long term basis he has to refrain from fishing too much.

The propagation can be described by a nonlinear function (cf. the theoretical function in table 2). If the fisherman does not extract too many the increase of fish by propagation is greater than the amount harvested. Therefore, the result is an increase of fish in the pool. If the fisherman catches more than the natural increase of fish within a given period of time then the number of fish in the pool decreases. There is also a point of equilibrium where harvest and propagation are the same. In this case the number of fish in the pool stays constant. This propagation function has to be recognized by the subjects in order to reach the ecological equilibrium. Qualitative reasoning should help in understanding the propagation function.

The fishing conflict game has been used for a number of different questions: The social psychological influence in a game with a number of participants (KNAPP, 1987a, 1988; REISS, 1988), the effect of a time lag of propagation in reaction to harvesting (KNAPP, 1987b). In the current study we also used this experimental paradigm and focussed on the effect of experiment , induced quantitative and qualitative reasoning on the effectiveness of system control (control of the harvest condition in the pool). It was hypothesized that the qualitative experimental group would be most successful because it had to store only a limited number of values in memory and could concentrate on the direction of change (ELLIS & ASHBROOK, 1987; NEUMANN, 1985). It was expected that this group had the clearest understanding for the propagation function (the increase of fish depending on the number of fish in the pool). To a smaller extent we should also find this kind of function concept in the quantitative experimental group and less so in the control group.

2 Method used in the study

The fishing conflict game was given to 79 university students. They took part in the experiment to earn some money, performed the experiment as single persons, started with 120 tons of fish in the pool, and had to indicate how many tons of fish they wanted to withdraw from the pool. The experimenter then reported how many tons of fish was in the pool the next season and the subjects again had to indicate how many tons of fish they would fish. This went on for 25 trials.

In order to investigate our hypotheses experimentally we had to induce the two kinds of reasoning. One experimental group had to focus its attention on qualitative reasoning: On three different occasions during the game we presented the following sentence:

> "The less I take from the pool (1) the less the number of fish in the pool decreases (2) the more the number of fish in the pool increases (3) neither of the two."

Our subjects had to indicate which of the three choices was correct. It was not important for us which of the three choices was preferred by the subjects, but the fact that qualitative reasoning was induced by this question. In fact, none of the three answers alone is correct. Depending on the subject's behavior in previous trials both of the first two answers are simultaneously correct. It was our aim to initiate qualitative reasoning by presenting this question.

The other experimental group received a text during the same three occasions in the game, too:

> With how many tons of fish in the pools does the biggest increase occur?
> With ___ tons of fish in the pool.
> How many tons of increase are there?
> ___ tons of increase.

Our procedure was guided by the following theoretical considerations (REISS, 1988): Quantitative predictions lead to the storage of previous predictions in memory. Memory capacity is used to its limits. The memory load prevents am

optimal performance in the game. There was a third group of subjects, the control group, which took part in the game without any additional questions.
The three groups can be distinguished in respect to their performance in the fishing conflict dilemma. More specifically, one could say that the group under the qualitative condition is going to keep a greater amount of fish in the pool and to harvest more. And the group under the quantitative condition is going to have greater success than the control group.

3 Results

One can draw a graph for the two experimental and for the control group indicating how successful the subjects were in the game. Success was measured by the sum of harvest and resource (tons of fish in the pool). In contrast to our hypothesis the qualitative experimental group did not perform much different from the other two groups. On the contrary, the quantitative experimental group performed much better, there is a significant effect of this kind of induction.

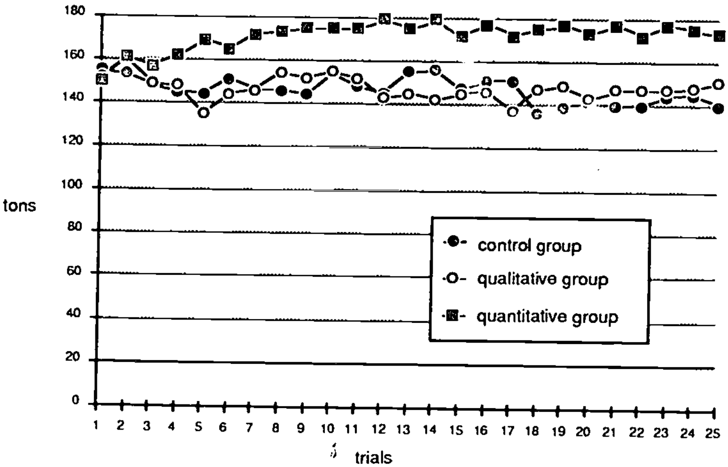

Table 1: Success in the fishing game (sum of harvest and resource) per trial

There was definitely a difference in the function concept between the quantitative and the other two groups. In order to analyze more clearly where this difference comes from after the game we asked the subjects about the expected increase for a given resource (propagation without interference by a fisherman). The results can be seen in table 2.

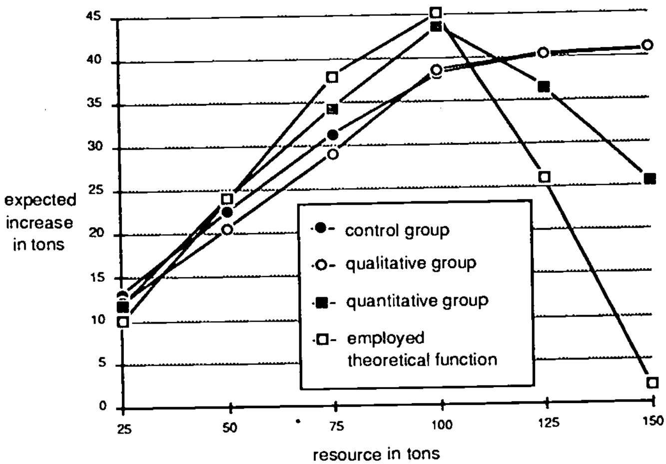

Table 2: Propagation function as used in the game (theoretical function) and as estimated by the three groups

It is evident that the control group and the qualitative experimental group considered the propagation function as a monotonically increasing function, i.e. they thought the number of fish would not stop increasing the more fish there were in the pool. In contrast, the quantitative experimental group estimated a propagation function similar to the one underlying the game (the theoretical function). The highest increase was at 100 tons but the increase did not reach 0 tons as soon as expected.

4 Discussion

Our hypothesis that the qualitative experimental group would perform most successfully in the fishing conflict game could not be verified. The qualitative experimental condition had no effect. The studies cited deal with qualitative reasoning in physics or mathematics problems. GREENO (1973) emphasizes the domain specificity of problem solving. This may have been one factor influencing the results of our study. One could also argue that the experimental condition was too weak; the subjects had to respond to one simple sentence. But this argument would also apply to the quantitative experimental condition. These subjects had to react to one simple sentence, too. But this sentence had its effect. The qualitative condition might also not have been successful because the sentence did not induce this kind of reasoning, whereas the quantitative condition fostered the understanding of the propagation function. It may have been that the task was too easy to overload memory by storing numbers, so that quantitative reasoning in this case was superior but not in general. Another experiment is planned to test the influence of qualitative reasoning with another task.

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TRANSFER BETWEEN FUNCTION REPRESENTATIONS: A COMPUTATIONAL MODEL

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This paper describes a framework within which it is possible to build computational models for problem solving processes in a function curriculum. One such model is described in detail. It serves to measure transfer of information between function representations during a problem solving process. The model has been used in a study with ninth graders who were taught a function curriculum specifically designed to encourage the use of methods from different representations in an integrated manner. While it was found that the computational model does reflect the cognitive aspects of transfer of information, it carries the risk of isolating transfer from other, parallel cognitive processes.

Are computational models an appropriate research tool for investigating cognitive processes in mathematics education? Although this methodological question is legitimate, not enough experience has been accumulated to date in order to discuss it in full generality. Relevant work has been done by Anderson and his group (Anderson, Boyle & Yost, 1985). They combined computer modelling and cognitive psychology in order to design and construct intelligent tutoring systems. The aim of this paper is to further contribute to the discussion of the above question by reporting on the use of a computational model for studying transfer between f .ıon representations. In line with this aim, the relationship between the methodological and the cognitive aspects has been stressed rather than the actual cognitive results of the study.

Function representations

The concept of function is usually introduced in several settings, either simultaneously or in short succession. The same function is represented by different means in each of these settings. The question then naturally arises whether students establish appropriate links between the different representations of the same function.

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In a study with average ability ninth graders Markovits, Eylon and Bruckheimer (1986) found that after studying the relevant part of the curriculum, students had difficulties to find the algebraic rule for a functions given in Cartesian graph form and vice versa. (Algebra-to-graph was easier when the function was familiar, both directions were equally difficult when it was unfamiliar.) Typically, less than a third of the students in the study were able to find the algebraic form of a linear function given in Cartesian graph form.

Smith (1972) studied ninth graders with high aptitude in mathematics and high mental ability, who were taught functions in arrow diagram and in algebraic rule setting. He investigated whether they were able to solve standard exercises, on which they had shown competence in these settings, also in ordered pairs and Cartesian graph settings. He found that they performed well, better in the ordered pairs than in the Cartesian graph setting.

Transfer

Both of these studies use the term "transfer" for students' passage between representations. According to Gagné (1970), horizontal transfer is the process of taking a concept from one setting and applying the same concept in a different setting. Although both discussed studies are concerned with the link between function representations and both provide relevant and valuable results on students' learning of functions, neither study clearly defines what the transferred concepts are. (Similar claims can be made for other studies of transfer between function representations.) Smith checks what is usually called transfer of learning: Does performance in one setting imply performance in the other setting? In order for this to happen, something must be transferred; it could be a mechanism, copied by analogy. Markovits *et al.* look at the ability to translate, rather than transfer. In translation, it is even less clear what exactly is transferred. The method of observation used in the two studies, was to compare student performances; this method is too coarse to allow a refined study of what is transferred between function representations.

We propose to give a more restrictive, but precise definition of transfer. This definition applies specifically to transfer between function representations during a problem solving process. Suppose that while solving a given problem, a student works successively in the representations R_1, R_2, ..., R_k, R_{k+1}, ... The work of the student in R_k will be called stage k of his solution process and we will say that this student used transfer at the transition from stage k to stage k+1 if his work throughout stage k+1 takes into account all the information gathered during his work in stages 1 through k. We stress that this definition is extremely restrictive since the student will only be considered to have used transfer into representation R_k if he takes into account *all* the information gathered during *all* his previous work on the problem and if he does this at *all* times throughout his work in R_k. We also stress that transfer is here considered during a problem solving process. This is a significant type

of transfer because it occurs during a natural process. It is also content related: The student who uses transfer knows how to interpret information gleaned from previous representations in the present representation.

We will now turn to the description of a framework which makes it possible to measure in which representation a student works at any time, what information is available to him from previous stages of his work, and whether he uses this information.

The triple representation model

The triple representation model (TRM) is a computer environment which has been designed as the core of a problem based functions curriculum (Schwarz & Bruckheimer, 1988). Work with TRM is possible in one of three modes: T(able), G(raph) or A(lgebra); each mode corresponds to a functional representation. The link between the representations is realized by operations named Read — Read A(lgebra), Read G(raph), and Read T(able) — which allow the student to consult results previously obtained in one mode while working in another.

The work within any mode is operational; that is, it is organized in operations that the student has to perform. The most important operations are Search, Compute, Draw, Plot, and Findimage. For the purpose of this paper, we will assume that a function has been defined algebraically. The Search operation (Algebraic mode, AS) then enables the student to check algebraic conditions for a large number of equidistant values such as in

From a to b step δ: If $f(x)>C$ then print (1)

where the student has to fix the lower bound a and the upper bound b of the search, the step δ, the type of comparison ($>$, $<$, $=$ or $\neq$), and the goal value C. The Search operation prints on the screen the values of x for which (1) is satisfied; the values of f(x) can be printed as an option. The Compute operation (Algebraic mode, AC) enables the student to compute automatically the value of a function for any given element of the domain. The Draw operation (Graphical mode, GD) enables the student to draw, magnify, stretch or shrink the graph of a function defined algebraically. The Plot operation (Graphical mode, GP) allows the student to put the cursor on any given point of the graph and read its coordinates. The Findimage operation (Tabular mode, TF) enables the student to obtain the value of f(x) in a table by specifying the value of x. These operations will be denoted by their abbreviations AS, AC, GD, GP, and TF.

The TRM has been designed with the intent to stress parallels between the operations in the three representations, e.g. AC, GP, and TF. It enables the student to use methods from different representations in conjunction during problem solving, because they have been integrated to a large extent. A problem typically solved with TRM is the Open Box problem:

An open box is constructed by removing a small square from each corner of a square tin sheet (20cm x 20cm) and folding up the sides. What is the largest possible volume of such a box (to an accuracy of 10^{-4})?

For the solution of this problem the student is forced to use the algebraic representation since the maximal accuracy in the tabular representation is 10^{-2} and the maximal accuracy in the graphical representation is 10^{-3}. Therefore, if he ever uses another representation, he has to carry out at least one passage between representations during the solution process.

The computational model for transfer

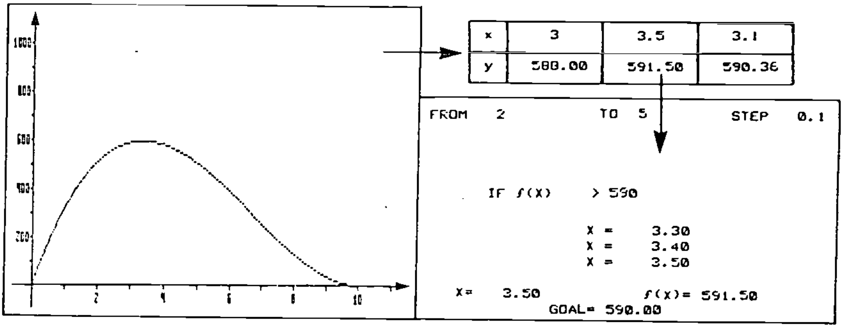
Although the computational model is conceivable within a much wider framework, it will be described here within the solution process of the Open Box problem with TRM. The aim of the computational model is to formalize the analysis needed to decide whether a student has used transfer at the passage from stage k to stage k+1.

This analysis will be based on the notion of *solution domain*. The solution domain of a student at a given moment of the solution process is the interval within which an expert would locate the solution, given that all information collected by the student previously was available to the expert. The information available will be in form of a set of number pairs (x,f(x)) which are known to belong to the function. These number pairs may have been obtained directly through use of the AC, GP or TF operation or they may have been printed by running an AS operation or they may have been read from a graph obtained via GD. In the latter case, only pairs whose x-value is marked on the scale of the x-axis will be taken into account. The solution domain can now be computed formally as follows: Assume that n such number pairs are available, and that after ordering them according to increasing value of x they are $(x_1,f(x_1))$, $(x_2,f(x_2))$, ..., $(x_k,f(x_k))$, ..., $(x_n,f(x_n))$. Assume further that the index m gives the maximum among these points, i.e. $f(x_k) \leq f(x_m)$ for all k, $1 \leq k \leq n$. Then the solution domain will be the interval $P=(x_{m-1},x_{m+1})$. Slight and other obvious modifications have to be made to this definition in some special cases: E. g., if m=1 or m=n, one of the end-points of P will be infinite. Since the set of ordered pairs available after a given operation includes the set available before that operation the solution domain will never increase during a solution process.

For determining whether transfer occurred at the passage from stage k to stage k+1 the solution domain after the last operation of stage k is relevant, because it contains all the information about the solution collected up to and including stage k. Assume that this solution domain is the interval P=(a,b). Each operation at stage k+1 will now be assigned a *transfer index* + or – as follows: The operations AC, GP, and TF use a single value of x; the index will be + if this value of x is in P, – otherwise. For the operations AS and GD, the student needs to specify an entire interval (the interval to be searched for AS and the domain to be graphed for GD); if, for a given operation, this interval is part of P, the transfer index of this operation

will be +, otherwise −. We will say that a student has used transfer from stage k to stage k+1 if the transfer indices of *all* his operations at stage k+1 are positive.

As an example consider a student who first defined a function algebraically, in the domain $0 \leq x \leq 10$, then drew its graph in the same domain, built a table of values for some values of x, and finally used the Search operation to find a sufficiently precise value of the maximum. His work is illustrated in the following figure:



The details of this student's operations are given in the following table:

Operation	Solution Domain	Transfer Index
1 AD $f(x)=x(20-2x)^2$, $0 \leq x \leq 10$	(0,10)	*
2 GD $0 \leq x \leq 10$, $0 \leq y \leq 1000$	(2,4)	+
3 TF f(3)=588	(2,4)	+
4 TF f(3.5)=591.50	(3,4)	+
5 TF f(3.1)=590.36	(3.1,4)	+
6 AS From 2 to 5 step 0.1, if f(x)>590	(3.2,3.4)	−
7 AS From 3.3 to 3.4 step 0.001, if f(x)>592	(3.332,3.334)	+
8 AS From 3.332 to 3.334 step 0.0001, if f(x)>592.5925	(3.3332,3.3334)	+

The student's work thus comprised four stages:

Stage 0 (operation 1): Algebraic mode: Definition of the function.

Stage 1 (operation 2): Graphical mode: Graphing the function.

Stage 2 (operations 3, 4, 5): Tabular mode: Tabulating some values.

Stage 3 (operations 6, 7, 8): Algebraic mode: Searching.

The solution domains in the table were computed, according to the above rule, from the information the student received as a result of his actions. For example after operation 5 (TF at 3.1) the following values were available: The values at f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7), f(8), f(9), and f(10) from the graph and f(3), f(3.5), and f(3.1) from the table of the function. Among these, f(3.5) is the largest. The solution domain is now (3.1,4) because an expert can conclude that the maximum of the function lies between the two known neighbors of 3.5: 3.1 and 4. (Note that when drawing this conclusion, the expert makes some assumptions about the shape of the curve; these assumptions may be justified on the basis of the geometry of the problem or on the basis of the algebraic form of the function). Similarly, operation 7 prints on the screen all those among the 101 pairs (x,f(x)), x=3.300, 3.301, 3.302, ..., 3.399, 3.400, for which f(x)>592. The maximal one among these is f(3.333)=592.5925.. . Thus the solution domain after operation 7 is (3.332,3.334), the interval between the two closest known neighbors of x_m=3.333.

All but one of the student's transfer indices are positive. For example, in operation 5, which is part of stage 2, the student asks for tabulation of x=3.5, which is in the interval P=(2,4), the final solution domain of stage 1. Similarly, the interval $3.332 \leq x \leq 3.334$ of operation 7 (stage 3) is a sub-interval of P=(3.1,4), the final solution domain of stage 2. The interval $2 \leq x \leq 5$ of operation 6, however, is not contained in P=(3.1,4); therefore the transfer index of operation 6 is negative. According to the given definition of transfer, the example student used transfer at the transition from stage 0 to stage 1 and at the transition from stage 1 to stage 2. He did not use transfer at the transition form stage 2 to stage 3.

Remark: The model which was actually used in the research is somewhat more complicated than the one described here. The main reason for this is that a student using AS or GD in an interval that is slightly larger than P may well be using transfer, because the choice of the interval is determined by cognitive style as well as knowledge; for instance, if our example student had [illegible] the interval $3 \leq x \leq 4$ in operation 6, this operation would have been assigned a positive transfer index.

Experiment

The research reported here is part of a larger project for which three ninth grade classes have been taught the TRM curriculum for about four months. The computer environment was an integral part of the classroom activity; there was no separation between work with or without computer. Activity with the environment was p[illegible]minantly problem solving. Usually such activity was followed towards the end of the class period by a teacher-led discussion.

At the end of the instructional period all students (N=55) were given the box problem, and their solution path was recorded in dribble files. Fifteen of the students solved the problem in an interview situation with an experimenter present. Their

activity was also recorded by the computer; in addition they were asked questions which assessed why they used a particular operation, especially when this operation was used just after a passage to a new representation.

Results and discussion

All of the interviewed students solved the problem in either a single stage (algebraic mode) or in four or five stages. We classified them into four categories:

1. Single stage students.
2. Students who used transfer at all transitions.
3. Students who used transfer at all but one transitions.
4. Other students.

The model thus enabled us to classify students into those who used transfer of information, and those who did not. This does not, however, imply anything about the cognitive validity of the model. This cognitive validity was checked in two independent ways. First, three experts were asked to classify several students on the basis of a summary version of the dribble files, into those who use transfer always, often or not often. Second, students' cognitive behavior was further investigated with respect to their problem solving tactics. This was done by another index, the *quality index*. This index was also based on the solution domain and expressed the rate at which the solution domain decreased. Strong correlations were found between students who used transfer and those who had a high quality index, even if in some cases the rapid convergence did not occur at the passage between representations but during work within one representation. The interpretation of the results showed that transfer of information *alone* is not very valuable in a problem solving situation. The combination of both indices, however, was very useful in the assessments of students' cognitive behavior. For instance, students who transferred well but showed moderate convergence behavior were interpreted as using representations at a level of significants and not at a level of signifiers. More specifically, these students, when solving a problem, do not see or use the representation of a function as a link to the unified meaning of a function (in the sense of unifying the meaning its different representations). In addition, interesting observations were obtained from the interaction between the two indices: there were students whose quality index improved as a result of making successful transfer between representations. The cognitive interpretation adopted in this case is that such students see the unified meaning of function through the lens of its different representations.

The detailed cognitive results obtained from the computational model will be presented elsewhere. Briefly stated, it was possible to study students' dynamic understanding of the function concept by examining the relationship between the successive indices during the problem solving sessions. Also, mastery of the software found its expression in the consistency of the indices. Accordingly, students who had not mastered TRM as a tool had unstable indices and their data could therefore not

be interpreted. In summary, the discriminating power of the computational model was prominent in the analysis of the interviews and led to insight about students' cognitive processes when solving problems about functions.

Conclusion

The computational model was found to reflect the cognitive aspects of transfer of information. This may partly be due to the fact that building the model forced us to think through the cognitive processes at a level of detail usually not attended to. There are also disadvantages to computational models for cognitive processes. They are connected to the same reasons as the advantages, namely to the level of detail necessary. This level of detail leads to a separation of component cognitive processes which in reality are closely connected. Models that describe such interconnections have to be very sophisticated. Building computational models for cognitive processes may be one way to make mathematics education a more scientific discipline, but we are still at the very beginning of this undertaking.

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TRANSITION FROM OPERATIONAL TO STRUCTURAL CONCEPTION: THE NOTION OF FUNCTION REVISITED

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The study reported in this paper is a continuation of our research on the role of algorithms in formation of mathematical concepts ([5], [6]). In [5] we suggested that many mathematical notions can be conceived both operationally (as processes) and structurally (as abstract objects), and that in most cases the operational conception is the first to develop. In the present paper we take a closer look at the phenomenon of reification -- converting a process into abstract object. Our theoretical claims are illustrated by experimental findings regarding secondary-school students' understanding of the concept of function. The most important conclusion from the case study is that reification is an intricate and difficult process which, at certain levels, can be practically out of reach for some students.

When analyzing the process of learning mathematics, one should be aware of the crucial role played by such epistemological issues as students' implicit beliefs about the nature of mathematics on the whole, and of mathematical entities in particular.

In [5] and [6] it was suggested that the majority of mathematical notions can be conceived in two fundamentally different ways: as static constructs (structural conception) or as processes (operational conception). For example, functions can be regarded structurally as aggregates of ordered pairs, or operationally -- as certain computational procedures. These two approaches, ostensibly incompatible (how can anything be a process and an object at the same time?) are in fact complementary. The idea of complementarity is not new: in physics, entities at subatomic level must be regarded both as particles and as waves to enable a full description and explanation of observed phenomena (see also [8]). Similarly, the ability of seeing a function or a number both as a process and as an object seems to be indispensable for solving advanced mathematical problems.

In the process of concept formation, the operational conception is often the first to develop. Out of it, the structural approach would gradually evolve. In [5] we argued that these claims apply to historical development as well as to individual learning. Indeed, certain parts of mathematics can be regarded as a kind of hierarchy, in which what is conceived purely operationally at one level should be conceived structurally at a higher level. In another words, processes have to be converted into compact static wholes, or reified, to become basic units of a higher-level theory.

Two important didactic principles can be inferred from the above claims.

PRINCIPLE I: The proposed model of concept formation implies that it would be of little or no avail to introduce a new mathematical notion by means of its structural description. The structural approach is much more abstract than the operational: in order to speak about mathematical objects one must be able to focus on input-output relations ignoring the intervening transformations. Thus, to expect that the student would understand a structural definition without some previous experience with underlying processes seems as unreasonable as hoping that he or she would comprehend the two-dimensional scheme of cube without being acquainted with its "real life" 3-D model. In the classroom, therefore, the operational approach should precede the structural. Some well-known difficulties observed in secondary-schools may be due to the common practice of reversing this order.

PRINCIPLE II: Structural approach should not be assumed untill an actual step was made toward a higher-level theory, for which this approach is indispensable. Indeed. to put up with the "existence" of a new kind of intangible mathematical objects, the student must be highly motivated. The required effort of mind would probably not be made until the operational approach proves insufficient and reification of the given process becomes a necessary condition for further learning. Such a situation arises only when some higher-level processes are performed on the concept in question. For example, as long as the notion of function appears nowhere but in the context of basic calculus, the student can do quite well with operational conception of function alone. Converting computational processes regarded as functions into objects becomes necessary only when the person comes across problems in which several functions have to be manipulated simultaneously, so that each one of them must be treated as self-contained static whole. Such treatment of functions is peculiar to many branches of modern mathematics, functional analysis, topology, and for[illegible] logic among them.

The above two requirements should be understood as necessary conditions for reification (which means that if they are not fulfilled, the reification is rather unlikely). Whether they are also sufficient, namely, whether they actually help in transition from operational to structural conception, was the main question addressed in the study which will be reported now. In this research we revisited the concept of function, our first investigation of which was presented in [5].

The present study was carried out in the Centre for Pre-academic Studies (Hebrew University), among 22-25 years old participants of a regular course on elementary mathematics (secondary-school level). Our first step was to collect as much information as possible about the conceptions which develop in students when principles I and II are not observed. In this

report, our own findings, coming from classroom observations and from a special questionnaire on functions, will be combined with the results obtained by several other researchers. At the second stage of the study, the concept of function was taught to a group of students by an experimental method based on our two principles. The results obtained in the experimental group were then compared to the background findings.

WHAT HAPPENS WHEN THE CONCEPT OF FUNCTION IS TAUGHT STRUCTURALLY

Being one of the central ideas of modern mathematics, the concept of function is given much attention all across the secondary-school curricula. In most cases, however, the way it is taught contradicts our model of concept acquisition. Indeed, to put it into Malik's words ([4], p.189), "function course [is usually] laced with set-theoretic notations" (which almost always means that our first principle is not observed), while "the necessity of teaching the modern definition of function at school level is not at all obvious" (so the second principle is violated either). It is in line with our former claims, therefore, that the general agreement about the importance of the concept of function is accompanied by another consensus ([1], [3], [4], [5], [6], [7], [9]): in a class, the exact meaning of this ostensibly innocent notion invariably turns out to be surprisingly elusive and problematic.

1. Our former studies ([5], [6]) showed that in spite of the "object-oriented" way of teaching, the fully fledged structural conception of function is rather rare in high-school students. In our present investigation some new findings reinforced this conclusion. Firstly, in response to the first item in the questionnaire presented in the box below, only 19% of the pupils (see "control group") agreed that function is a static construct composed of (infinitely many) parts. Secondly, the student's inability to consolidate multitude of ordered pairs into one entity could be responsible for the difficulties observed in the classroom when problems involving sets of functions were dealt with. For example, when faced with functional equations (such as $f(x+y) = f(x)+f(y)$), the students usually were confused as to the nature and the number of the solutions. It is also worth mentioning that the pupils had some serious difficulty with the set-theoretic notions underling the structural version of the concept of function. The student's conception of abstract entities such as domain, range, image and pre-image was usually so fuzzy, that general confusion was the most common reaction to problems requiring identification of the different components of a given function. Several phenomena presented in

other papers (e.g. students' inattention to domain when comparing two functions, [4], or some persistent mistakes in symbolic representation of sets, [7]) indicate the same problem: they show that very often the learner can not distinguish between sets and their members. It is probably the student's inability of "seeing" even these basic entities as fully fledged objects, which makes such distinctions quite meaningless.

2. We shall argue now that the main difficulty with the structural definition of function stems not so much from what is actually included in it, as from what is missing. Indeed, in spite of the fact that in the definition no mathematical operations are mentioned, the responses to our first question (see box) indicate that overwhelming majority of pupils (81%) associate functions with computational processes. We can conclude, therefore, that contrary to the curricula designers' intentions <u>the student's conception of function is closer to operational than to structural</u>. Other studies abound in additional evidence. Vinner and Dreyfus, [9], emphasize the importance of the operational aspect by saying that according to the student: "one has to do something to x in order to obtain the corresponding y". In our own former investigation ([6]), some students refused to apply the adjectives "equal" and "the same" to a couple of functions which assumed identical values but were defined by different computational processes. "General difficulty ... with the constant function" ([4], p.24; [7]) may be interpreted as an evidence for the pupil's implicit belief that in order to speak about function, a change in the independent variable must be followed by a change in the dependent variable. It is interesting to note that the dynamical dimension of the concept was emphasized in a similar way by Euler: according to him, "a quantity" should be called function only if it depends on other quantity "in such a way that if the latter is changed the former undergoes changes itself" (1755, [2]).

The historical analogy will go even further if we analyze students' beliefs about the nature of the computational processes falling into the category of functions. In the response to our second question (see box), 94% of the pupils evaluated as true at least one of the following statements: "Every function expresses certain regularity", "Every function can be expressed by a certain computational formula". For all their fuzziness, these descriptions come strikingly close to the (equally inaccurate) "definitions" of function used by mathematicians for nearly a century (since Euler and his "analytical expression" (1748) until Dirichlet's rebellion against the "algorithmic" approach (1837); see [2]). Moreover, the student's responses to the item A in question 3 (see also [4], [7], [9]) show that, like many mathematicians before them, the today's students can not put up

THE QUESTIONNAIRE

1. Which one of the following sentences is, in vour opinion, a better description of the concept of function?
   A. Function is a computational process which produces some value of one variable (y) from any given value of another variable (x).
   B. Function is a kind of (possibly infinite) table in which to each value of one variable corresponds a certain value of another var.
2. True or false?
   A. Every function expresses a certain regularity (the values of x and y can not be matched in a completely arbitrary manner).
   B. Every function can be expressed by a certain computational formula (e.g. $y=2x+1$ or $y=3\sin(\pi+x)$).
3. Which of the following propositions describe functions? (x and y are natural numbers)
   A. If x is an even number then $y = 2x+5$;
      Otherwise (x is an odd number) $y = 1-3x$.
   B. If $x=0$ then $y=3$.
      If $x>0$ then to find the corresponding value of y we add 2 to the the value of y corresponding to $x-1$.
   C. For every value of x we choose the corresponding value of y in an arbitrary way (e.g. by throwing a dice).

RESULTS		control N=48	exper. N=28
1. perc. of s's who chose item A		81	50
perc. of s's who chose item B		19	50
2. perc. of s's whose ans. to A & B were			
	yes. yes	46	36
	yes, no	26	18
	no. yes	21	3
	no. no	6	43
3. perc. of s's who said it was function			
	item A	50	93
	item B	73	93
	item C	17	50

with the so called "split domain" functions. This attitude is reminiscent of the opinion expressed by d'Alembert in his response to the Euler's idea of "discontinuous" function (by "discontinuous" Euler meant a function given by different analytic expression in various parts of its domain). Finally, our respondents' almost univocal rejection of the "arbitrarily" defined function (item C) brings to mind the long and heated historical dispute over the Dirichlet's definition ([2], [3]).

3. In the light of our own findings combined with those of other researchers, the pupil's tendency to associate functions with algebraic formulae seems to be strong and common enough to deserve special attention. Although this tendency can be indicative of operational conception (the student may perceive a formula as a short description of a computational algorithm) as well as of a structural (the formula may be interpreted as a static relation between ordered pairs), sometimes it is probably neither this nor that. Such tendency may signalize <u>a "mutilated", quasi-structural conception</u>, the deficiency of which would come to light in many different contexts. Indeed, unlike Euler, for whom the "analytic expression" was one of two possible manifestations of an independent abstract entity (a curve was the other one), the today's student often seems to regard a formula as a thing in itself, not standing for anything else. This appears to be the most

plausible explanation for such well-known phenomena as the students' inability to build a reasonable bridge between algebraic and graphic representations of functions ([1]), or the common tendency to interpret functional equalities as nothing but a product of symbol manipulations ([6]).

Both operational and quasi-structural conceptions are deviations from the official "structural" approach. But while the former is a healthy, natural stage in concept development, the latter should be regarded as unsatisfactory and potentially harmful. It seems, however, that the quasi-structural conceptions can hardly be avoided within the usual structural way of teaching. The data summarized above confirm that the idea of the set of ordered pairs, when introduced too early, is doomed to remain beyond the comprehension of many students. In such case, the object-oriented language used by the teacher forces the pupil to look for a more tangible entity which may serve as a reasonable substitute. Being the most natural choice, an algebraic expression turns into the thing it was only meant to symbolize (in a different context the same would happen to a graph, [9]).

... AND WHAT HAPPENS WHEN THE OPERATIONAL APPROACH IS APPLIED

In the experiment performed at the second stage of our study, the concept of function was taught to a group of students as a part of a course on algorithms and computability. This time the approach was operational, namely the principles I and II were faithfully observed. The space limitations prevent us from giving the full description of the teaching material, so we shall confine ourselves to some general remarks.

<u>The course</u> (60 teaching hours) was devoted to the idea of algorithm and the concept of function was introduced as a [illegible]s for dealing with the semantics of alghorithmic languages. At that time the notion was almost completely new to all our students.

According to principle I, the operational approach was the first to be applied. Initially, the term "function" was used almost synonymously with algorithm, and then explained as being a name for "the product" of an algorithm. Although it was described also in structural terms (as "the set of all input-output pairs"), our first structural definition only emphasized the connection between functions and computational processes.

Principle II was implemented as well: the structural approach had not been given much attention until it became truly necessary. The first attempt at separating functions from algorithms was made only after the set of the already known algorithms and the resulting set of functions were broadened

several times, to include recursive and "split domain" calculations, among others. Different methods of constructing functions from other functions (by composition, by recursion or by minimization) were discussed, thus the view of function as a self-contained entity which can serve as a building block for other entities was gradually promoted. For representing functions, the usual algebraic notation was used and the students exercised translating explicit and recursive expressions into computer programs, and vice versa.

The "input-output" description of function was replaced by the abstract Bourbaki's definition only after a long period during which the student's attention was focused on the static "products" of different algorithms rather than on the algorithms themselves. This final generalization led to the question of existence of a noncomputable (not "algorithmic") function. This last problem was expected to be the ultimate trigger for reification. Indeed, without the fully fledged structural conception, the problem was doomed to remain meaningless (to a person who identifies functions with algorithmic processes, the idea of noncomputable function must be as absurd, as the notion of a circle which is not round).

Classroom observations were carried out during the entire course. Initially, almost all the phenomena described in the former section as indicative of operational conception could be witnessed again (not surprisingly so, since at the early stages of learning the operational conception was deliberately fostered). The first attempts at transition to the structural approach were met with resistance and lack of understanding: many students could not cope either with sets of functions or with general definitions of operations performed on functions. The difficulty diminished with time but it did not disappeared completely. When the students were asked to describe the set of the recursive functions (the definition of which had been taught and discussed before), almost half of the group gave faulty answers, indicating a difficulty with treating functions as building blocks for other functions. Not surprisingly, the idea of noncomputable function, when mentioned explicitly, evoked astonishment and opposition.

Our questionnaire on function was administered to the participants at the end of the course. Although many answers still indicated operational rather then structural conception, the results (see box, "experimental group") did show a substantial progress toward the latter, at least in comparison to the control group. Moreover, even though the structural approach was not fully adopted by the students, we have good reasons to believe that the danger of "mutilated" conceptions considerably diminished. Indeed, judging from the answers to the third question, there were only few students left who would still regard a term "function" as synonymous with "formula" or "equation".

DISCUSSION AND CONCLUSIONS

Judging from our results, operational approach does stimulate reification, at least to some degree. Special attention should be given, however, to the fact that for all the progress made by the students, our attempt to promote the structural conception can not be regarded as fully successful. This result may be much more significant than all the others.

One may claim, of course, that it was some deficiency of the teaching method which interfered with our objectives, thus limiting our success. Even if partially true, this explanation does not seem to tell all the story. The gap between the efforts invested and the progress made is so big, it prompts us to risk another conjecture, according to which reification is inherently so difficult that there may be students for whom the structural conception will remain practically out of reach whatever the teaching method.

Apart from the ample empirical evidence, there are some theoretical considerations pointing in this direction. Closer look at the process of reification reveals that it may lead to a psychological vicious circle -- an obstacle which seems almost unavoidable, and for many people would remain insurmountable. Indeed, according to our former argumentation (principle II), reification of a concept would not occur until some higher-level operations to be performed at this concept are introduced; on the other hand, conceiving a concept as an object seems to be prerequisite for dealing (meaningfully) with such higher-level operations. To cope with these apparently incompatible requirements, one must be able to orchestrate the lower-level reification with the higher-level manipulations in a subtle, painless manner. Judging from our results, this ability seems to be rather rare. Such historical examples as the turbulent story of the concept of function and the three century long dispute over the elusive notions of negative and complex numbers show that breaking the vicious circle of reification can be quite difficult even for mathematicians.

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Supercalculators and Research on Learning

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The new technology of hand-held graphics computers with symbol manipulation capabilities (supercalculators) can have a significant impact on the learning research in mathematics education. Technological impacts on curricula often take 10 years (Burkhardt, 1986). However, mathematics learning researchers can respond more quickly to the capabilities of supercalculators. The purpose is to a) describe the capabilities of hand-held graphics computers; b) argue for the expansion, initiation, and elimination of various research areas; and, c) suggest directions for possible future efforts. The remarks, although founded in research on learning, are inductive, speculative, and invite comment and debate.

Capabilities. Reading reports regarding machines such as the Hewlett Packard HP–28S (e.g., Hewlett Packard, 1988; Michel, 1987, 1988; Nievergeldt, 1987; Tucker, 1987, 1988; Wicks, 1988), or *supercalculators* (à la Tucker, 1988), lead one to conclude enormous changes in both fundamental school mathematics topics and fundamental ways of doing mathematics are upon us.

Before examining the impact such changes may have on learning research, consider a few examples of the capabilities of the supercalculators. Some single–key–stroke capabilities of a "personalized" machine include: a) two dimensional graphs and zooms; b) vector and matrix computations; c) numerical equation solving; d) symbolic manipulation commands and tests; and e) structured programs in *RPL* (*Reverse Polish Lisp*) or *FORTH–*, *PASCAL–*, *BASIC–*, or *LISP*–like languages.

With the help of a few references, single–key–stroke capabilities of a "personalized" machine also include: f) symbolic differentiation and simplification (Wicks, 1988); g) keys labeled, say *SARX*, that, given a function and an interval, return the proper integral for surface area of the solid of revolution about the *x*–axis set up in symbolic form, together with the numerical value for the surface area to the nearest 0.01; h) keys labeled *PDIV* or *PROOT* that give, respectively, the symbolic result of the division of two polynomials or the algebraically computed real and complex roots of up to a fourth degree polynomial (Hewlett Packard, 1988); i) *CEQN* gives the characteristic equation of a matrix (Wicks, 1988); or j) symbolic and numeric solutions to classes of differential

equations and curve fitting routines (Hewlett Packard, 1988).

Mathematics texts that place major emphasis on numeric or symbolic computations with, say, numbers (including complex numbers), graphs, polynomials, vectors, matrices, derivatives, integrals, Taylor series, trigonometric identities, or zeros of functions, are essentially measures of supercalulator capabilities, not student learning. Random samples of exercises from textbooks used for K–14 mathematics in the United States reveal few exercises that should remain with supercalculators in the hands of teachers and students. Perhaps as much as 90% of the exercises and explanations should be removed.

Since the early 1950s, numerical computations, structured programming, and symbolic manipulations have been available on computers (Hamming, 1980). Mathematicians have called for mathematical programming (Kemeny, 1966) and computer mathematics systems (Birkhoff, 1972) in mathematics courses for some time. Today, the addition of user–friendly, graphics capabilities and the psychological impact of a hand–held, personal carrier of mathematical ideas (a supercalculator) make curricular changes mandatory (Steen, 1988).

Changes in Learning Research. Past learning research has made progress in many areas. Research based in concepts and problem solving may be the most robust with regard to technological advances such a supercalculators. One may be tempted to conclude research on learning with computers would be some of the most useful research for drawing inferences about learning mathematics with supercalculators. However, there is an order of magnitude difference between former uses of computers and the new supercalculators.

Supercalculators are designed and ready to carry out computations with a single keystroke, whereas, former uses of computers required significant exchanges of data and coding for similar computations. Supercalculators are designed to be personal tools to be used regularly and in almost any setting, whereas, former uses of computers required infrequent, shared use in special settings. Supercalculators are designed to allow special tailoring of key commands for personal mathematical needs, whereas, former uses of computers involved general procedures and programs designed for general use by many users. Supercalculators are designed to be symbolic, personal carriers of mathematics, whereas former uses of computers, while capable of dramatic symbol manipulation (e.g., *MACSYMA*), were, nevertheless, designed for general mathematics users, and not as individual, personal carriers of mathematics. The supercalculator represents a substantial extension

of human capabilities in mathematics.

Few mathematics learning researchers would consider conducting research with subjects without devices for recording mathematical communications. In the past, these devices have involved written symbolic communication, verbal (and nonverbal) communication, spatial communication, and the manipulation of devices in which mathematical ideas have been embedded. The supercalculator enhances all of these forms of communication.

Pollak, in prophetic articles about calculators and computers (Pollak, 1977, 1982), noted substantial changes were needed in two partial orderings of the curriculum (i.e., those based on mathematical prerequisites and those based on social importance) and that fundamental changes were needed in the curriculum. Learning research must face equally dramatic: a) expansion of certain research areas, b) initiation of new research areas, and c) the elimination or deemphasis of other research areas.

Expansion. Learning research which can be expanded and modified to reflect supercalculators centers about the use and meaning of variables, computer coding to define and relate mathematical concepts and principles, representational systems, and cognitive development.

Significant work has been done on the meaning of variable (e.g., Chomsky, 1988; Clement, Lockhead, & Soloway, 1982; Dubinsky, Elterman, & Gong, 1988; Küchemann, 1981; Krutetskii, 1976; Oprea, 1988; Shumway, 1989; Wagner, 1981) with children ranging from age 5 to age 20. However promising this work has been, we need to extend the universe of the *concept of variable* to include variables defined over objects such as: real numbers, complex numbers, strings, vectors, real arrays, complex arrays, lists, global names, local names, programs, algebraic objects, and binary integer numbers. Supercalculators take a unified approach to these objects; calculator operations apply whenever meaningful, and all such objects can be inputs to programs, including programs themselves. Consequently there are dramatic, mathematical generalizations of the meaning of variable available on supercalculators. Systematically exploration of the development of such generalized concepts of variable is needed.

Computer coding and its impact on mathematics learning has been studied and seems to be most related to concept development and problem solving (e.g., Blume & Schoen, 1988; Suydam, 1986). Arguments regarding relative merits of computer languages are often made on the basis of structured programming, recursion, global and local variables, graphics, and the ease of naming and writing procedures. Supercalculators offering flexibility of programming such as graphics, procedures,

lists, symbolic manipulation and recursion can put to rest many arguments. Computer coding on a supercalculator becomes much more procedure–oriented and seems to encourage those programming habits most admired by computer scientists and mathematicians. Again, the personalization of the supercalculator seems to be an important psychological factor. Computer programs are coded and then executed by a single keystroke. They become a part of the supercalculator capabilities and are always available. Algorithm design becomes highly personal but also very important for repeated application by the author. Systematic study of the impact of such availability of authored programs for use, modification, and refinement is needed.

Representational systems have gained deserved attention (e.g., Janvier, 1986) and many interests of this line of research are directly applicable to supercalculators as supercalculators provide access to many of the representational systems being studied. One can only endorse continued efforts in representational systems and encourage their investigation on supercalculators.

Cognitive development research needs to direct some long–term efforts towards study of fundamental concepts of mathematics, their representations, and their development in children in the context of supercalculators as a regular tool for exploring mathematics. The advantages for the supercalculator for such efforts are cost, size, personalization, and generalized mathematical power provided for subjects.

Initiation. Teaching experiments and clinical studies exploring supercalculator representations of many important concepts of mathematics rarely studied with young subjects (ages 3–20) are needed. Research has begun with efforts such as Dick's project to revise and test calculus materials designed for students using supercalculators building on prior experiences with younger subjects (Dick & Shaughnessy, 1988) and Michel's year–long teaching experiment with 15 year–olds studying mathematics, physics, and science for 13 hours per week using supercalculators (Michel, 1988). Significant study of generalized variables, complex variables, matrix representations, differentiation, integration, probability distributions, zeros of functions, Taylor series, computer arithmetic, and theorems such as those of De Moivre, Bolzano, Galois, Euler, Gauss, Cauchy, and Gödel are called for by some and the concept of proof is consider basic mathematics (Shumway, in press). Estimation concepts must be developed for algebraic computations as well as numeric computations. Further identification and exploration of fundamental mathematics is needed.

Elimination. Most analyses of the impact of calculators and computers call for a deemphasis of many traditional computational skills (e.g., Pollak, 1982). Learning research that involves skill

development associated with graphing, solving simultaneous equations, finding roots of functions (e.g., factoring or simplifying), polynomial arithmetic, differentiation, integration, matrix arithmetic, differential equations, characteristic equations of matrices, and hypothesis testing without the use or knowledge of supercalculators should be terminated. Substantial collections of research efforts have become moot because of supercalculator capabilities.

Directions for Future Work. Researchers themselves must use supercalculators to do mathematics. High priorities are the required use of supercalculators for all mathematics, the treatment of concepts and proofs as basic mathematics, the earlier, deeper treatment of fundamental conceptual learning, and the deemphasis of many forms of skill learning. Philosophical analyses leads one to such conclusions. Researchers must raise questions, study the associated implications, study feasibilities, identify limitations, and agitate for change based on research and best wisdom.

Discussion. Require supercalculators for *all* mathematics? One could argue there must certainly be times when one might not want to require the use a supercalculator. Perhaps, but the more likely error is to use the "when appropriate phrase" to fail to explore less obvious but appropriate uses. In fact, it may be impossible to find a mathematical situation for which no supercalculator activity would be appropriate.

Concepts and proofs are basic mathematics because, relieved of the computational burdens, conceptual understanding and proofs of the correctness of results are the remaining essential elements of doing mathematics.

Deeper treatment of fundamental conceptual learning is necessary for effective use of supercalculators. History suggests, when computational power is increased, mathematical understandings are ultimately increased as well.

Deemphasis of many forms of skill learning once thought to be essential for mathematical development seem important and likely. Researchers must test the premise that supercalculator computations will produce the number sense and symbolic intuitions thought to develop from computations.

Finally, researchers are obligated to lead, offer evidence, and help make best-evidence decisions.

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HOW & WHEN ATTITUDES TOWARDS MATHEMATICS & INFINITY BECOME CONSTITUTED INTO OBSTACLES IN STUDENTS?

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Summary. It was conjectured in Sierpinska (1987) that sources of epistemological obstacles related to limits can be found in students' attitudes towards mathematics and infinity. The aim of the present research is to understand conditions in which some children's conceptions of infinity and (implicit) philosophies of mathematics become constituted into obstacles as the children develop from the concrete to the formal operational stages. The research has only just started. In this paper we exhibit behavioral & conceptual differences between two girls (Agnès, 10 & Martha, 14) & we formulate some hypotheses as to the above mentioned conditions.

In the sequel we shall use the following abbreviations: "M" - mathematics, "INF" - infinity, "EO" - epistemological obstacle.

I.- INTRODUCTION

1.- Genesis of research. It was conjectured in Sierpinska (1987) that sources of EO related to limits in 16-17 years old students may lie in their attitudes towards M and INF.

2.- Aims of research. We were interested to know when & how, in the course of their development, students come to construct these obstacles, i.e. in what conditions students' conceptions of INF and attitudes twrds M start functioning as EO (cf. Sierpinska, 1989).

3.- General assumptions. We have assumed that this happens sometime in the transition period from the concrete to the formal operational stages and taken children between 10 and 14 years of age. We assumed it highly improbable that any attitude twrds M as scientific knowledge develops in younger children. As far as conceptions of INF are concerned, Piaget & Inhelder (1948) & Fischbein et al. (1979) conclude that at concrete operational stage children are unable to understand, e.g. the infinite nature of continuous divisions of a geometrical figure. This does not mean, however, that they do not develop some less sophisticated conceptions of INF such as "very big, undetermined number" (which may well occur in much older students,too; cf. Sierpinska, 1988).

4.- Methods of research. The mathematical context we used in our interviews is that of equipotent sets: we have studied children's reactions

to our attempts to make them accept the condition of existance of a 1-1- correspondence between elements of two infinite sets as a criterion of their having "as many" elements. We did not use the term "equipotent sets". The Polish term for it refers to "counting" and "number". We wanted to avoid the suggestion that "there are as many elements in set A as in set B if numbers of their elements are equal". The Polish expression for "as many" (which is not distinguished from "as much") - "tyle samo" - can better be translated with the French expression "autant que".

Why this context of equipotent sets? The 1-1 correspondence criterion is known for its being used by Galileo to solve the paradox of natural & even numbers. It was proposed by Bolzano (1851) to deal with infinite sets, and Dedekind based on it his definition of an infinite set (1888). It further became the corner stone of Cantor's Mengenlehre. The notions of equipotent sets and cardinal number have shown to be elegant solutions to many paradoxes and problems in M. But the choice to extend the notion of number in this way testifies for the change in mathematicians' attitudes twrds M the XIXth century was witness of. INF has a rich meaning outside M: it is a part of our culture, of our beliefs concerning the structure of matter, size of Universe, time ... (cf. Sierpinska, 1989). Now, one cannot accept this notion be reduced to the 1-1 - correspondence criterion without coming to think that, maybe, M is not a discipline describing some kind of reality (be it the reality of our thoughts). To use this criterion with consequence one has to be able to reason against one's intuitions, discoursively and formally, and to accept it.

Certain befiefs about INF and certain attitudes twrds M can, therefore, function as obstacles against a ready and unproblematic acceptance of the 1-1 - correspondence criterion for comparison of infinite sets.

If the criterion is accepted without difficulty and used consistently in comparing sets then, of course, it is possible that these obstacles are overcome. However, this is little probable in 10 or even 14 years old children: Rather, this may mean that those obstacles have not been constructed yet and children perform deductive reasoning in much the same way they observe rules while playing games.

And this is how we come to our working hypothesis: let us observe in what conditions the 1-1 - correspondence criterion ceases to be acceptable for children.

We shall study in detail individual histories of children during the experience, looking for reasons of changes or stagnation in their conceptions and trying to pick up behavioral as well as conceptual differences between

children of different age.

The research has only just started and all we are able to give here is presentation of some behavioral and conceptual differences between two girls (Agnès, 10 & Martha, 14) and formulation of some hypotheses.

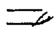
II.- ORGANISATION OF THE EXPERIENCE

There were 4 sessions, each with 4 children of one age group: 10, 11, 12, 14. The 4 children in each group were divided into 2 subgroups of 2 and there were 2 subsessions in each session.

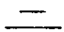
In the first subsession, one subgroup of 2 children was interviewed by us; in the second, these two were asked to interview the other two children.

Here are more details on the first subsession. There were several steps in it:

Step 1. Interviewers suggest the following definition: two sets have as many elements one as the other if their elements can be paired off, that is, if every element from the first of these sets has a pair in the second, and every element from the second has a pair in the first. The suggestion is made by using collections of green and yellow counters and asking: " how can we check that there are as many green as yellow counters?". We start with small collections and go on to larger and larger stopping whenever the children propose to pair off (cf. Brousseau, 1977). Then we negotiate the definition.

Step 2. Children are shown a drawing like:  , and asked: "would you agree with us that if two segments are of equal length then there are as many points in one of them as there are in the other?".

Step 3. "Are there as many points in one as there are in the other of such two lines?": 

Step 4. The same question with: 

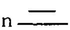
Step 5. The same question with: 

Step 6. "Are there as many natural numbers as there are even numbers?"

Step 7. "What do you think INF is? How do you imagine it?"

III.- COMPARISON OF AGNES (10) & MARTHA (14) BEHAVIOUR & CONCEPTIONS

1.- Behavioral differences

First answers in steps 2 thru 6: Agnès: "Yes" in  & 
Martha: a conditional "Yes" in one step only: 

General reaction to Interviewers' arguments: Agnès: positive; Martha: negative.

Final answers: Agnès: "Yes" in all 5 steps from 2 thru 6. Martha: "Yes" in one step () - rather forced by an Interviewer, non-commital in three steps, negative in one step ().

Below are some excerpts from the detailed analysis of differences between Agnès' and Martha's behaviour.

Step 2. The first reaction to the question (═══), for both girls, consists in saying that there is no univocal answer. The reason given by Agnès is that the number of points on a segment depends upon its width. Martha's reason is that it is impossible to "really" check it, one can only do it "theoretically" by agreeing upon some unit of measure as being the size of a point. Final answers: Agnès explicitely formulates a positive answer and method of proof. Martha formulates explicitely a method of pairing off but declares impossibility of actually performing the procedure (because of the infinity of points on the segments). She gives an answer in a conditional form: if, by joining points that correspond one gets lines that are all parallel then there are as many points on one as there are on the other segment.

Step 4. First reaction to the question (══) is "Yes" in Agnès who starts looking for a pairing off procedure. Martha says "No" and starts showing that there exists an assignment of points which is not one-to-one. Interviewers intervene with criticism. Martha attempts to refute the arguments but fails. Finally states that if we admit that there is an INF of points in a segment and that the notion of next point doesn't make sense then it is impossible to compare, to assign points to points. She tries to give another proof refuting the theorem. After Interviewers' criticism of the proof Martha says: "I give up, because I cannot imagine ...". She criticises proof given by Interviewers: "but no one can ever draw all the lines, either".

Step 6 (natural and even numbers). Agnès first answer is: "No, because there are the odd numbers, still". Martha: in theory it is assumed that there is an infinity of natural as well as even numbers, but, as we imagine these sets, then we see that there should be more natural numbers. Agnès accepts Interviewers' arguments. Martha repeats her argument with force, looks for proofs of the negation of the theorem, refutes arguments given by I., says <u>she doubts whether existence of a 1-1 correspondence indeed proves that there as many natural as there are even numbers</u>. After further interventions, becomes aware of the assumption she has been taking all the time:"I'm considering only bounded sets of numbers, I see", gives up, says: "all this is a matter of convention", and doesn't seem pleased with it.

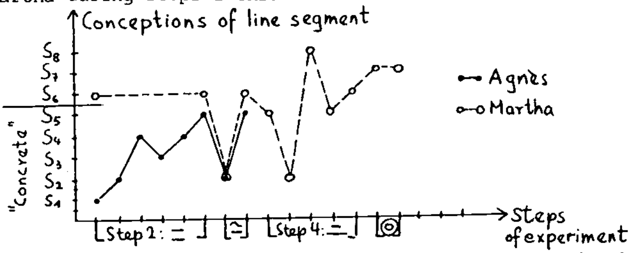
2.- Conceptual differences

Main problems in steps 2 thru 5 were the notions of segment and point. Conceptions that we observed in the girls can be described as follows:

S_1: a line segment is a pencil stroke; a point is a dot; number of points in a segment depends upon its width, length and size of points.

S_2: line segments have no width (or are of the same tiny width that can be ignored); points are small, it may be convened how big they are, say, 1mm.
S_3: segment is a line bounded by two points; there are only two points in a line segment, namely its ends; the line is composed of little <u>segments</u>
S_4: [same as above except for:] the line is composed of small <u>points</u>
S_5: line segment is a line composed od very tiny points which are ordered in a row [drawing illustrating this idea by Agnès: ----------].
S_6: line segment is a mental object composed of an infinity of consecutive points which is represented by a line drawn with the help of a ruler; points have no dimensions but are represented with dots or little segments that have dimensions .
S_7: [same as above without: "which is represented ... "]
S_8: [same as S_7 without "consecutive"]

Diagram below shows the evolution of these conceptions in Agnès and Martha during steps 2 thru 5.



Martha's conception of a mathematical object is characterized by a certain duality: on the one hand, there is the ideal mathematical object, abstract, existing only in one's mind, and on the other - there is its more or less concrete representation: sure, we "assume" that there is an INF of natural numbers, but "as we imagine this set" we think but of a finite, be it very large,set of numbers. Agnès does not seem to have problems of this kind At the beginning she displayed a very "concrete" conception of line segment. Later she started to make abstraction of the width and points became "inimaginably" tiny dots. But she never started thinking of there being something like the "idea of line segment".

The 1-1 correspondence criterion was conceived of operationally (i.e. in terms of operations to be actually performed, cf. Bridgman, 1934) by Martha all along the experience; by Agnès - in steps 2 & 3 only. In further steps Agnès found it sufficient to give verbally and/or iconically the rule for pairing off in a proof. In Martha, her operational conception of the pairing off procedure together with her "dual" conceptions of mathematical objects were constantly sources of mental conflict: it is impossible to actually per

form the procedure on ideal, mathematical line segments (S_6), these being infinite sets of points (and then it is impossible to compare two sets in this sense); it can only be performed on their concrete representations - drawings (where a point is identified with a unit of measure). But then - there is no 1-1 correspondence between points of two line segments of unequal length, so there are not as many points on two such line segments.

This kind of operationalism and "dual" character of mathematical concepts may be characteristic of conceptions in the transition period from concrete to formal operational stages.

From step 4 on Agnès is very keen on precise formulations and proofs. She seems to have now understood what is expected of her and this is how she interprets the rules of the game. She is quite happy with it.

Martha, in step 3 ($\frown$), says she hates formalism as being completely arbitrary, unnecessarily pedantic & contrary to her intuitions. But, in her attempts to refute Interviewers' arguments, she makes big efforts to use "mathematical proofs" (although her logic is sometimes rather strange; e.g. she uses something like $\exists R \subset A \times B$ (R is not 1-1) as a sufficient condition for: A is not equipotent with B).

Agnès had accepted the 1-1 correspondence criterion as soon as she understood that it defines the term "as many as". Martha had understood it so at the very beginning but in step 6 she refused this theoretical choice as being absolutely against "what should be".

IV.- SOME CONJECTURES

1. "Concrete" conceptions of mathematical objects do not prevent one from being able to perform precise deductive reasonings based on assumptions not necessarily conform to one's intuitions.
2. One reason for this may be that, at the concrete operational stage, these intuitions are very superficial: they need not touch the deep conceptual difficulties inherent in a mathematical concept (e.g. problems of density or continuity had to be discussed with Martha but not with Agnès). Younger children may not even "see" the difficulty.
3. Another reason can be that these intuitions are not linked with emotions. A 10 years old child is emotionally open to change of conceptions: she has only started organising her knowledge and she accepts learning from adults. In the course of maturation of personality, the child may start identifying herself with her knowledge. This may become part of her worldview, and an intuition may turn into a conviction or belief. Now linked with strong emotions, it starts to function as an obstacle. This might explain Martha's resistance against the proposed theoretical solutions.

Piaget's theory of intellectual development is not particularly interested in maturation. "Our thinking doesn't become more intellectual just because we are getting more mature (Donaldson, 1978)". But there may be a link between maturation of personality & constitution of conceptions into obstacles.

4. Both Agnès & Martha accepted reasonings in M as being hypothetico-deductive. The difference between them lies in their attitudes towards the relevance of theories thus obtained. Agnès does not care if these theories are absurd: M is a game and it is fun to obtain surprising results. Martha's views are quite opposite: if one of the statements we obtain by deduction is "false", i.e. contrary to what we think there should be, then Axioms, definitions, criteria assumed beforehand must be changed or the whole theory blown up. This difference is analoguous to that between points of view of Russel (formalism) and Lakatos (discoursive empiricism, 1978). However, there seems to be still another difference (perhaps more serious, even) between Agnès' and Martha's attitudes: Martha's attitude may be a result of a conscious reflection on what scientific knowledge is there for, what is scientific and what is not (& "why should I learn it???"). Agnès may just be trying to parrot her Math teacher or the Interviewers. Therefore it may be easy to make her change her attitude. It has shown to be an impossible task with Martha. This difference again seems to be linked with maturation of personality.

5. The difficulty to overcome obstacles in Martha can be linked, also, with her "dual" conceptions of mathematical objects ("ideas" - representations) and operational attitude towards mathematics (Bridgman, 1934) which may be characteristic of the transition period between concrete and formal operational stages.

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Learning Y-Intercept: Assembling the pieces of an "atomic" concept

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In this paper, we report data indicating that some commonly-held assumptions about teaching and learning may be inaccurate. For example, concepts such as y-intercept that are taken to be the unproblematic building blocks of higher-order knowledge of linear functions may be much more complex than they appear. Our analysis emphasizes that "pieces" of y-intercept can be acquired without full conceptual understanding and that concept acquisition is a gradual process, which extends context-bound knowledge to more general fields of application.

Introduction

There are many core concepts in the secondary school curriculum that are generally assumed to require very little instructional development. Concepts such as slope, variable, equality, parabola, and y-intercept are understood to have a simple internal structure and they are taken to be the "atoms" out of which more complex concepts such as function are built. We present some compelling evidence from one student's (named IN) efforts to learn one concept (y-intercept) that questions the simple, all or nothing, "atomic" nature of these concepts. Our results suggest that students acquire pieces of the concept (in the case of y-intercept, graphical and algebraic pieces) before their knowledge becomes atomic, and that learning even these individual pieces can be a highly contextualized, gradual process.

Our analysis focuses specifically on 'N's gradual acquisition of the concept of y-intercept across five distinct graphica, contexts in a computer-based graphing environment. If a significant proportion of our st.·de s learn "atomic" concepts in a similar way (and we will argue that they do), then ınstruction must do much more to support the assembling of atomic concepts than is typically the case.

Background to the Analysis

The data for this case study were the result of the pilot-testing of a computer graphing environment called GRAPHER. IN was a 16 year old high school student enrolled in a summer calculus class for high school students on the Berkeley campus who volunteered to experiment with the software. Our initial background questions revealed that IN was highly motivated and articulate, that she had some deficits in her basic algebra instruction, and yet had overcome these deficits successfully enough to succeed in advanced high school mathematics classes. Her companion in her explorations of GRAPHER was a graduate student, JS. His function was to provide a loose structure for her activity by explaining the basic functionality of the software, suggesting tasks, and posing clinical-style questions

when appropriate. IN liked the system, attending 4 separate sessions averaging 1.5 hours in length. These sessions were videotaped and the entire corpus of 7 hours of interactions was available for analysis. The case study of IN's learning of y-intercept is part of a larger study of her learning in this context (Schoenfeld, Arcavi, & Smith, in preparation).

GRAPHER consists of three separate microworlds, and was designed to assist students in learning polynomial functions. One of these microworlds, "Black Blobs", is a game patterned after "Green Globs" (Dugdale, 1984), in which students choose equations to "shoot at" randomly located sets of squares on a Cartesian graph. The data presented below consists of various situations that IN confronted in Black Blobs and some of the discussions with JS that resulted. (See Schoenfeld (in press) for a more detailed description of GRAPHER.)

A Sketch of IN's Initial Knowledge of Linear Functions

From IN's answers to our preliminary tasks and questions, we concluded that she knew that linear functions can take the form, "y= mx + b", and that "m" was called the slope and "b" the y-intercept. She could construct a table of values to generate the graph of a line. She could also compute the slope from the coordinates of two points, although her understanding of the concept and its properties was faulty. Her knowledge of y-intercept was also a mixture of strengths and weaknesses. On the one hand, she knew that the "b" slot was where the y-intercept was represented in the equation, and she indicated indirectly that intercepts were locations where the graph crossed axes. On the other hand, the graphical and the algebraic pieces of y-intercept were not connected. She did not show any understanding that a "b" value of "1" meant that the line crossed the axis at (0,1). Our subsequent analysis indicates that this missing Cartesian connection between the graphical and algebraic meanings of y-intercept (i.e. that "b" is the y-intercept because (0,b) is a solution of y=mx + b and (0,b) lies on the y-axis) was a fundamental part of IN's struggles to learn y-intercept as a single conceptual atom.

Game situation #1: "Recalling the Graphical Meaning"

Working on her first "Black Blobs" screen (Figure 1 below), IN sought a linear function that would hit the 2 blobs (P_2 and P_3) centered at (-0.5, 2.5) and (0.5, 3.5) respectively. After calculating the slope to be "1", she turned her attention to the intercepts.

IN: OK, let me see. Do we know what the y-intercept is? The y-intercept is...
JS: What is the y-intercept?
IN: The y-intercept is where the point touches y...oh. OK, so then, but then it could ...er 3.5 or 2.5.

JS: Well, 3.5 is where the line is on one side of the y-axis, and 2.5 is where it is on the other.
IN: So which one should I use?
JS: So, if the line passes through this point and this point [points to the two selected blobs], then where is..., does that give you an idea where the y-intercept 's gonna be?
IN: [no response]
JS: Ok, is going to be going like this [shows the line with a pen on the screen] so just knowing that's gonna go like this, can you tell where the y-intercept is
IN: Oh, Oh, Oh!! [gestures with both hands], it's going to be at ...3?

Although she indicated before playing the game that she knew the graphical meaning of y-intercept, that knowledge was not stable enough to apply in a new setting. Instead, she suggested to herself the association of the nearby y-coordinates and y-intercept. This temporary alternative meaning of y-intercept was strong enough to withstand JS's mild prompts to focus on the line through P_2 and P_3 not the blobs themselves. When he was driven to the stronger intervention of representing the line with a pen, she immediately saw the light and determined the correct value "3". This game episode affirmed what our earlier assessment of IN's knowledge had indicated. IN knew pieces of the concept of y-intercept, but these pieces were highly unstable. From the next episode, it is clear that this interchange helped IN to stabilize her graphical meaning, but only in a local sense.

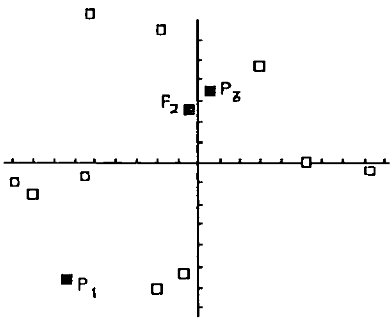

Figure 1

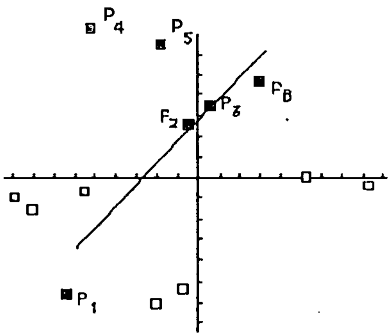

Figure 2

Game situation #2: "Local Competence"

Bolstered by her success at hitting P_2 and P_3 with "y = x + 3", IN turned her attention to 3 other nearby blobs, P_4, P_5, and P_6 centered at (-4,8), (-1,6.5) and (3,5) respectively (see Figure 2 above). IN correctly calculated the value of the

slope to be -0.5 and turned to find the y-intercept. Her first estimate of the y-intercept ("6") was exactly correct. Unfortunately for IN, JS offered her a folded paper as a means to represent the line and check her estimate. She then changed her estimate to "5.5", perhaps as a result of the parallax of the computer screen, and entered the equation "y = -0.5x + 5.5". She was perturbed by the miss that resulted but proceeded to adjust the value in two steps ("5.75" then "6" again) to hit all three targets.

Her performance in this episode would tend to indicate that she had consolidated her graphical meaning of y-intercept and would have no more problems with that issue. In short, it looks as if she "has the concept". As will be apparent from latter episodes, this assessment was clearly incorrect. The competence that she had gained was limited to a narrow graphical context: those situations in which blobs bracketed and were close to the positive y-axis. Game situations that did not fit those conditions presented new and substantial difficulties. As the data from Situation #3 and 4 below will show, the extension of the graphical meaning of y-intercept from the limited context of application of Situations #1 and 2, to other more general contexts was anything but automatic and effortless.

Game situation #3: "What Should Have Been Easy Was Not"

Six days after what had been an enjoyable first round with GRAPHER, IN returned for a second round of work. On her first game screen (illustrated below in Figure 3) she selected 3 blobs, P_1, P_2, and P_3, centered at (0,7), (-1,5) and (-2,3) respectively. She spontaneously asserted that they looked like the ones that she had shot at in her previous session. She then miscalculated the slope to be "-1.5" (the correct value was "1.5") and declared the y-intercept to be "zero", typing in the equation "y = -1.5x + 0". When the resulting line was off-target in slope and location on the y-axis, IN was quite taken aback. In response, JS put both the slope and the y-intercept values up for discussion.

JS: OK, did either of them come out the way that you wanted, or is, are they both wrong?

IN: Well, the y-intercept should be zero, shouldn't it, because that third dot on the top is zero, isn't it?

JS: This one? [pointing to P_1]

IN: Yes.

JS: Ah, well, let's see, it has two coordinates, right, an x and a y?

IN: Yes.

JS: And one of the coordinates is zero.

IN: Yes, the x, and the y is 1,2, 3, 4, 5, 6, 7, zero 7.

JS: So if we need to include the y-intercept...
IN: Aha.
JS: Which one of those numbers, zero or 7, is the y-intercept?
IN: Oh, 7! Oh, I didn't know that.

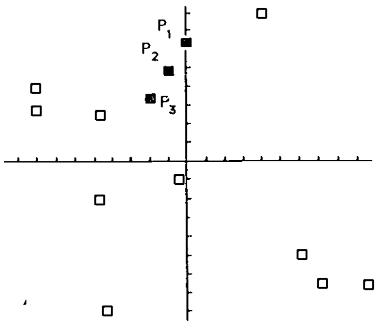

Figure 3

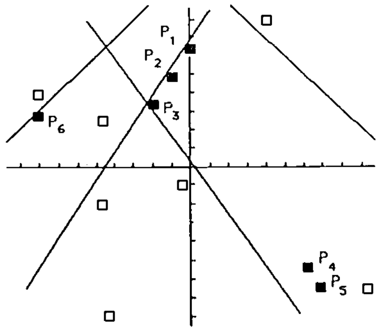

Figure 4

If there was any doubt before, her final comment indicated quite clearly that her difficulty in this situation was rooted in a matter of substance and was not just in a slip of the tongue. This collection of blobs would be the easiest of all possible game situations for someone who understood y-intercept. With a blob located on the y-axis, the y-intercept of the desired line is the y-coordinate of that blob. But this "easy" situation was anything but straightforward for IN. She knew the coordinates of P_1 but did not know that the y-coordinate of P_1 since it was located on the y-axis was the y-intercept. Instead, she used the salient x-coordinate. We take this as strong evidence that the graphical and the algebraic meanings of y-intercept were isolated from each other. She lacked the knowledge that "b" is the y-intercept value, precisely and necessarily because the ordered pair (0,b) was a solution to the equation "y= mx + b" and (0,b) lies on the y-axis. In the absence of this unifying "Cartesian" connection the situation of a blob on the y-axis presented a new context for IN's limited notion of y-intercept, one that required an extension of what she had just learned.

Game situation #4: "Success in a New Context"

About 15 minutes later in Session 2, IN decided to shoot at two blobs that were distant from the y-axis -- P_4 and P_5 centered at (6,-5.5) and (7,-6.5) (see Figure 4 above). After incorrectly calculating the slope to be "1" (the correct value was "-1"), she turned to JS with a question.

IN: [writing "y = 1x +   " on her scratch paper] How do I find the y-intercept?

JS: What is the y-intercept?

IN: Yeah, or how do I find it?

JS: I know, ...what can we, what can we remember from just the word, y-intercept?

IN: Where it touches the y-axis.

JS: OK, um...

IN: But it would be far away, see, and so I'd have, I'd probably make a mistake if I just guessed at it.

JS: Well, let's be a little experimental. Let's see if we can guess.

IN: Oh. It'd probably be here [tracing a line with the mouse out of the graph window near (0,10)] if we can guess.

This episode indicated that IN's knowledge of y-intercept had advanced in a number of ways. First, she was able to successful handle a new situation -- two blobs which did not straddle the y-axis and were quite distant from it. Secondly, her difficulties were due to the limitations of her ability to visualize the line and therefore its y-intercept, not to any difficulty with the graphical meaning itself. In fact, she was able to give a reasonable graphical definition, despite the use of language ("touches") that was involved in her previous confusions with y-coordinate. Finally, her empirical success in estimating the y-intercept had pushed her to seek a more direct and deterministic method for finding the y-intercept values for given blobs. (She asked JS for a "shortcut" after bemoaning the vagaries of "guessing".) If we measure along these micro-dimensions, IN learned a great deal in her interactions with GRAPHER. But, as the final episode we present indicates, there were still definite limitations to her understanding.

Game situation #5: "A New Kind of Context-Dependence"

On a new game screen later on in Session 2, IN experimented with different slope values to get lines of different inclination through the same y-intercept, "-2". JS suggested that she find a linear equation that would hit the blob P_5 centered at (-8.5,-4.5) and went through the same y-intercept she had been using (see Figure 5 below).

IN: This one [pointing to P_5] and what other point? Shouldn't I have two points before I can solve it?

JS: ... what I meant was whether you could draw a point through this blob that went through the y-intercept, -2.

IN: Ok, ah, but what other point would it reach?.... there is no point that I can reach that would make me go through that ax, through that intercept.

JS: Right, OK, so the problem there is what? We only have one point?

IN: Ah, yes.

JS: OK, is there anything else on the screen that we could treat as another point?

IN: That would make this, that would be straight here [tracing the line]? No. See. There's nothing else here.

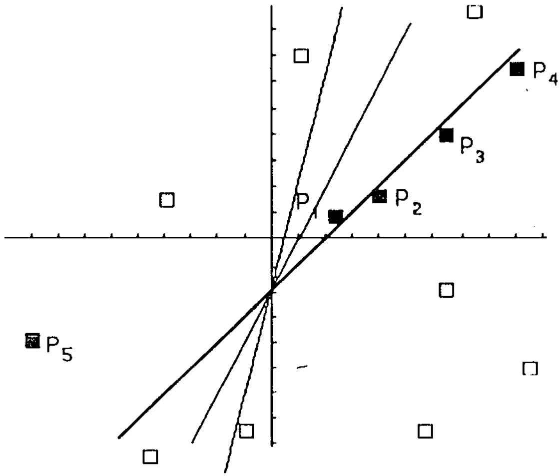

Figure 5

At this point, JS gave up and told her to use the y-intercept as if it was a blob. She needed 3 attempts to guess the slope, "-.5", "-.1", and "-.3". and then typed in the equation, "y = -.3x - 2". Then watching attentively as the line approached the y-axis, she declared with pride and exhilaration, "Exactly!" as the line crossed the y-axis. We take this final episode to indicate that IN's knowledge of y-intercept was very dependent on the particular constraints of the game and was not situated in the ideal Cartesian plane. Her reiteration of the need to start with 2 or more blobs and her expression of exultation when the line crossed at (0,-2) indicates that she had failed to abstract that blobs were sloppy approximations of points in the plane. As a result she failed to see that a y-intercept and a blob were sufficient to determine a line. When JS asked her to simply treat the y-intercept as a blob, she accepted his suggestion as simply an amendment to the rules of the game. (We here note that her requests to find another point were not driven by the need to compute the slope, as the careful reader might think. In this sequence she was estimating the values of the slope.) IN had learned as much as she could about y-intercept from shooting at collections of 2 or 3 blobs in the game. She knew that every collinear set had an associated y-intercept and became quite skilled at estimating its value.

Her learning led her toward the deep connection between the value of "b" in the equation and its graphical meaning, but she never grasped it. Perhaps for this reason, she failed to abstract her knowledge from the game to the more general context of the Cartesian plane.

Discussion

We have presented results from a single case study that indicate that learning simple "atomic" concepts is a much more contextualized and gradual process than is commonly understood. Despite having learned enough mathematics to get herself placed in the calculus class for advanced high school students, IN still had to learn (or relearn) the concept of y-intercept one graphical context at a time. Local competence (as demonstrated in Situation #2 and then again in Situation #4) did not at all imply a general and robust understanding.

This research indicates that some ostensibly simple notions are quite complex and subtle for students and that the appearance of mastery may hide, in fact, only the barest understanding. These results have implications for mathematics curricula whose goal is that students build deep and meaningful understandings of mathematical concepts rather than the superficial and fragile ability to repeat only the procedures they have been taught.

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COMPUTERS, VIDEO, BOTH OR NEITHER. WHICH IS BETTER FOR TEACHING GEOMETRY?

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Nitsa Movshovitz-Hadar, Technion-Israel Institue of Technology

This study was designed to assess the impact of instruction via computers, video, both or neither on learning processes and achievement in geometry. A 2X2 (TV by Computer) factorial design was employed. Participants were 268 fifth grade students who studied the same contents for the same duration of time. Results showed that more media does not necessarily imply better math learning. In fact, the no-media and multi-media yielded similar low learning outcomes, while the video by itself exerted the highest achievement scores, even higher than those obtained in the computer environment. A similar pattern of differences were found on Time On Task.

Five years after Bloom (1984) phrased "the two sigma problem", the solution is still far away. Bloom argued that "optimal learning conditions" can promote cognitive outcomes by approximately two standard deviations (sigma) above what can be achieved under "conventional conditions". The problem, of course, is how to define "optimal learning conditions". Bloom (1976, 1984) maintained that to be effective, educational environments must adequately provide for the four elements of the quality of instruction: appropriate cues, reinforcement, participation, and feedback-correctives. In this view, the four elements are additively related to achievement; if one is missing, achievement will be lower. Therefore, Bloom (1984) suggested to combine instructional methods that emphasize different elements of the quality of instruction.

In the last decade, several attempts have been made to explore the impact of combined methods on achievement. For example, Mevarech (1980) found that combining programs emphasizing cues with programs emphasizing

feedback-correctives enhanced achievement more than each program by itself. Bloom (1984) reviewed a number of studies showing effect-sizes higher than one standard deviation of combined methods compared to control groups that did not employ any specific method. Finally, Tenenbaum (1986) reported strong impact of an instructional method consisted of the four elements of the quality of instruction.

What are the implications of Bloom's theory (1984) to technology assisted instruction? Does the exposure to two instructional media imply better math learning than the exposure to one or to none? Moreover, does the use of one medium improve achievement more than "conventional" instruction with no technology? Undoubtedly, video programs have the potential to enhance different types of cues including verbal, visual, and vocal. As a result, the exposure to new video programs has tended to facilitate learning (Clark, 1983). On the other hand, considerable research has indicated that computer assisted instruction (CAI) that provides immediate feedback-correctives yielded significant better academically oriented achievement and affective outcomes than learning with no computers (Kulik and Kulik, 1987; Mevarech et. al., 1985a, 1985b, 1987,1988). Based upon Bloom's theory (1984) and these findings it was hypothesized that instruction aided by multi-media consisted of computers and TV would promote academically oriented outcomes more than would be expected on the basis of each medium separately or instruction which is not aided by any technology.

Although the cognitive outcomes arising from CAI and instructional video programs have been the focus of systematic research, relatively little research has been directed to other outcomes. In particular, very little is known on learning processes that take place in computer or TV environments. Research has indicated that both TV and computers have been viewed as

having important roles in increasing attention and time-on-task (TOT) (Clark and Salomon, 1986). Thus, it was hypothesized that instruction via multi-media (TV and computers) would increase TOT more than each medium by itself.

The purpose of this study was to investigate the roles of instruction via computers, video, both, or neither on learning processes and achievement in geometry. To examine the study hypotheses, a 2X2 (TV by Computer) factorial design was employed. The four resulting treatments were: multi-media (TV and computers), TV, Computer, and absence of technological devices. The research design holds constant quality and content of instruction, as well as allocated time; the only differences between the treatments were related to the different media as will be described below.

METHOD

Participants

Participants were 268 fifth grade students, 126 boys and 142 girls, who studied in eight classrooms in four Israeli elementary schools. Classrooms were comparable in terms of students' SES, previous exposure to computers and TV educational programs, and teachers' years of experience. Intact classes in the four schools were assigned randomly to the four treatments. As a result, the size of the groups were: multi-media (N=74), TV (N=69), Computer (N=55), and neither TV nor Computer (N=59).

Treatment Groups

All classrooms studied the same unit in geometry: **Tiles and Corners**, for equal amount of time (four weeks) with the same geometry book (Shacham, Snir, and Movshovitz-Hadar, 1987). This unit is a new part of the

Israeli elementary school curriculum in geometry and thus none of the classes had been exposed to its contents prior to the beginning of the study. Indeed, knowledge pre-assessment showed an average grade of less than 20% correct answers. All teachers were told that they were experimenting a new unit. They received the same training and used the same instructional method: introducing a new concept or skill to the whole class followed by individualized practice and application sessions based on the same student workbook.

The difference among the treatment groups was in the exposure to the media. The "TV group" learned the prerequisites concepts related to the unit in the first four sessions as described above and then watched a 25-minute video program called: "Tiles and Corners" of the Dra-Math series produced by the Israeli Instructional Television (Reiner and Movshovitz-Hadar, 1986). Watching was followed by augmented activities designed by the film designers. At the end of the study, the video program was played once again to ensure that understanding had been attained. (For more information about the Dra-Math series, see Movshovitz-Hadar and Reiner, 1983).

The "computer group" used LOGO or BASIC to practice skills and to apply the concepts introduced during the whole class instruction. At each session, after the teacher introduced the new concepts, groups of six students practiced individually at the computers. The rest of the class continued to practice with the student workbook. Thus, similarly to the TV group, also in the Computer group, about 25% of the time was spent at the media.

The multi-media group watched the program as did the "TV group" and they practiced the skills and applied the concepts using LOGO or BASIC as did the "computer group". Finally, as was mentioned above, the "no technology" group spent an equivallent amount of time learning with the workbook only.

Instruments

Mathematics achievement were assessed by two instruments: Standardized Mathematics Achievement Test developed by the Israeli Ministry of Education (Kudar-Richardson reliability coefficient = .86) and geometry achievement test developed by the unit designers (Alpha-Cronbach reliability coefficient= .79). The geometry test was administered prior to- and at the end of the study; the Mathematics Achievement Test was administered prior to the beginning of the study and its results were used as a covariate in all analyses. In addition, students Time On Task (TOT) was assessed at the beginning, in the middle, and at the end of the study by a short questionnaire designed by us for the purposes of the present study (Kudar-Richardson reliability coefficient=. 80).

RESULTS AND DISCUSSION

To assess the effects of the different media on achievement in geometry, a one way analysis of covariance (ANCOVA) was employed with the pre measures used as a covariate. A test of the homogeneity of the slopes indicated that the regression slopes were equal for all four cells and thus the usual analysis of covariance model could be applied.

Significant differences were found between treatment groups on the post geometry test controlling for initial differences in mathematics and pre-geometry achievement tests ($F(3, 262)=35.81$, $p<.001$). However, in contrast to our hypothesis, the multi-media group did not attain the highest mean scores. Evidently, Duncan comparisons indicated that the "TV group" attained the highest achievement mean score. Their mean score was approximately one standard deviation higher than the "Computer group" which in turn was approximately half standard deviation higher than the multi-

media and the "no technology" groups; no significant differences were found between the last two groups.

Analyses of students' TOT indicated that although no significant differences were found among the four treatment groups at the beginning of the study, significant differences among groups were manifested at the middle and at the end of the study. Generally speaking, TOT data supported the results reported above. Duncan comparisons showed that TOT of the "no technology" group remained stable during the time of the study. The "TV group" consistently increased TOT and so did the "Computer group". In contrast, however, the multi-media group increased TOT between the first and the second measures, but than a sharp decrease was manifested.

The results will be discussed at the conference from three perspectives. First, Bloom's model (1976, 1984) will be applied to illustrate the roles of the elements of the quality of instruction. Second, theories in metacognition will be used to explain the small impact of the multi-media on achievement. Finally, the implications of the findings to actual classroom teaching will be presented: what to do and what not to do.

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VOCATIONAL MATHEMATICS

Teachers' Cognition of Mathematical and Vocational Knowledge

The paper reports on an empirical study of the way, teachers in vocational colleges perceive the relation of mathematical and vocational knowledge. A content analysis of 40 interviews shows that the majority of the teachers think of the relation in terms of examples from both domains. Only few of them come up with descriptions relating these domains as a whole. When asked about the purpose of mathematics in their teaching, half of them call mathematics a helpful tool (and nothing else), whereas a third also mentions the conceptual help of mathematics for understanding professional situations.

1 Research Question

Research on teacher cognition analyses the concepts and decision processes of teachers (cf. BROMME & BROPHY 1986, CLARK & PETERSON 1986, HOFER 1986), but did not pay too much attention to the 'professional knowledge' of teachers - taken as the mixture of pedagogical, didactical and matter knowledge, routines and experience including the emotions related to the teaching practice. Teachers' professional knowledge is based on preservice teacher-training and develops during the teacher's actual teaching.

Only recently, empirical research in the professional knowledge of teachers started (cf. SHULMAN 1986) and has to specify the *teachers'* professional knowledge against the knowledge of different professions - e.g. managers of a company or lawyers. The research reported below starts from the assumption that the professional knowledge of teachers is characterized by an integration of knowledge from two domains, namely curricular, subject-matter knowledge and pedagogical knowledge (cf. BROMME 1987, BROMME to appear).

The professional knowledge of teachers is analysed in a specific setting - namely technical and vocational colleges in the FR Germany. This is a part-time classroom-type education of normally two days per week complemented by three days vocational training in companies (the west-German 'dual system' of vocational training, for a detailed description cf. STRÄSSER 1985). This type [illegible]l vocational training would last three years and offers a vocational

certificate as qualified worker ('Facharbeiter') to the successful student. In the college part of the training, the 'Berufsschule', mathematics would be taught for two or three hours per week with the aim of a numerical foundation and interpretation of vocational phenomena, underpinning vocational knowledge by numerical analysis ("zahlenmäßige Deutung und Durchdring ... von beruflichen Erscheinungen", "Untermauerung der Fachkunde durch rechnerische Durchdringung", cf. GRÜNER 1955, p. 477, and WOLFF 1958). The use of metaphors in the widely accepted descriptions of goals of the mathematics teaching may be taken as a hint that there is no explicitly consented didactics for this teaching and only little research in vocational mathematics education.

Analysing the professional knowledge of teachers in vocational colleges, these teachers' curricular knowledge should be additionally subdivided into mathematical knowledge and knowledge related to the vocation they train the students for ('vocational knowledge'). From the point of view of the related disciplines (mathematics, pedagogy and e.g. engineering), the three domains of kowledge differ widely and may be even taken as different cultures (cf. SNOW 1959). The research reported below concentrates on a pair of the triple pedagogical-vocational-mathematical knowledge and analyses the way teachers think of the relation between mathematical and vocational knowledge (for an empirical analysis of the relation of curricular and pedagogical knowledge in general education cf. BROMME & JUHL 1988). The relation of mathematical and vocational knowledge here is a particular revealing case of an integration of different domains of knowledge because of the sp :fic task these teachers have to fulfil.

2 Methodology

Professional knowledge is not directly accessible and differences occur between the knowledge used and the knowledge which is talked about - even in professions which need use of speech when being practiced (cf. ARGYRIS & SCHÖN 1974). For an empirical analysis, one should generate occasions to observe the use of professional knowledge - e.g. ask for a description of lessons in mathematics recently taught.

In 1981 and 1983, 40 teachers of vocational colleges were interviewed to learn about the teachers' concepts on the relation of mathematics and the correspon- ; vocational domain. The teachers were trained in the vocational domain

(e.g. metalwork or business) and trained teachers, not trained mathematicians, who had to teach vocational mathematics in "Berufsschule". At the beginning of the interview. They were asked to relate their answer to a specific course they had taught last year. The interview started with a detailed description of this course (number of students/distribution of sexes/school leaving certificates etc.) in order to secure a relatively narrow relation to the teaching reality of the interviewee. They were also asked to describe difficulties the students had with mathematics, the topics taught to the course, their teaching methodology and the manuals used. At the end of the interview, data on the biography of the interviewee (age, type of vocational training, academic training, teacher training, years of active teaching etc.) were gathered.

In the 1981 and 1983, in total 40 teachers from a whole range of vocations (from business-administration to electricity and tailoring) were interviewed. The interviews lasted from 1 to 2 hours. A comparison of data on the courses and teachers interviewed with data available on vocational teachers and classes in the FR Germany shows that average full-time vocational teachers were in the sample, teaching courses with relatively good school leaving certificates and the usual competencies (for details cf. STRÄSSER 1982, p. 60ff).

The teachers' perception of the relation of mathematical and vocational knowledge was reconstructed by a content analysis relying on categories modelling the two-domain-approach (mathematical vs. vocational knowledge). Two independent raters had to search the interview-parts on the topics and methodology of teaching in order to identify those sentences directly speaking of the function mathematics has in the vocational training of the students. These passages were classified into mathematics (1) for communication purposes, (2) as operative help for vocational problems, as tool, (3) as description of a vocational situation and (4) mathematics in other functions. In a second step, the raters identified passages directly speaking on the relation of mathematical and vocational knowledge, analysing these passages as binary relations (mathematical/vocational topic) by classifying their degree of abstractness (example vs. whole discipline/subject for both carriers of the relation) and the way, mathematical and vocational knowledge are described to act upon each other (e.g.: related mathematics taught before vs. taught after the related vocational knowledge).

3 Results

The 6 female and 34 male teachers (with average age of 42 when interviewed) had all (except two) gone through a preservice teacher training. 16 of them mentioned special studies in mathematics during their teacher training. As a mean, they had seven years of active teaching. They had been teaching courses in business/administration (14 teachers), technical domains (22 teachers) and others (4 teachers, e.g. courses for future florists) with higher school leaving certificates and more female students in the business/administration courses than in the other courses (business/administration: 66.5 % average female students against 21.8 % average female students in the technical courses).

Only half of the teachers remembered teaching mathematics more than four consecutive lessons in isolation from vocational contexts. Less than half of the topics of mathematics lessons can clearly be labelled mathematics. The teachers gave examples like "rule of three" and "equations" as well as "torque and power" and "calculating investment" as topics of their lessons. Most of the teachers described their teaching method in the following way: A lesson would begin with a description of a situation or a technical or scientific experiment specifically prepared for teaching vocational mathematics. Having developed a solution in terms of a formula or a calculation rule, the teachers seem to underestimate the re-interpretation of the mathematical solution in terms of the vocational contexts (for details cf. STRÄSSER 1985a, for a description of this teaching method cf. BLUM 19°

To learn about the three domains in the professional knowledge of the teachers, they were asked to distribute 100 points to three descriptions of themselves: "educator"/"specialist in mathematics"/"specialist in the vocational domain they trained for". 37 teachers answered with the distribution shown in the graph (see next page). As can be seen from the answers, the teachers' self-concept was rather being an educator or a specialist in the vocational domain, not in mathematics.

As for the function(s) of mathematics, half of the teachers call mathematics a helpful tool for vocational contexts (and nothing else). A third also mentions the help of mathematics for understanding vocational situations (see table 1 xt page; both raters had only a 50-%-agreement on both the identification

Graph: Teachers' Self-Description

The graph is the orthogonal, parallel projection of the self-descriptions onto the plane defined by 0 points for 'educator'.

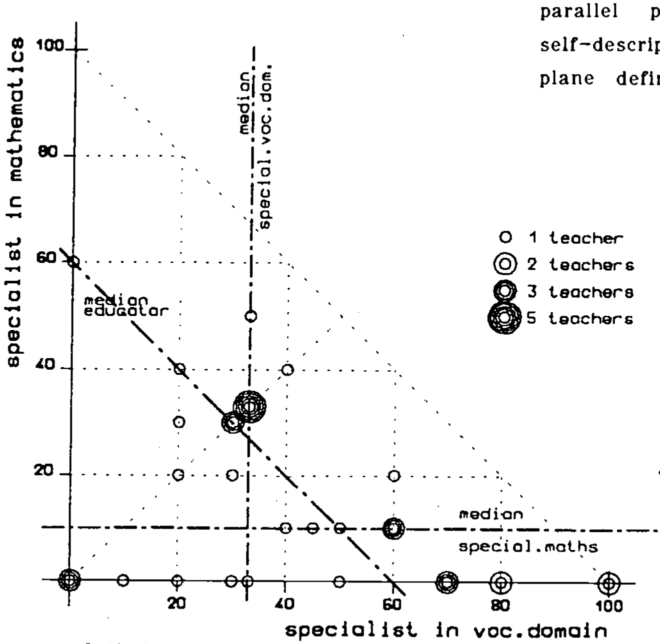

Table 1: Functions of Mathematics for Vocational Contexts

	none	ONLY tool	ONLY understanding	tool AND underst.	other
number of teachers	8	16	7	3	6

and classification of interview-parts). "Other" functions of teaching mathematics were e.g. 'fostering logical thinking', and 'general education'.

The detailed analysis of passages relating mathematics and vocational contexts is summarized in table 2 (see next page; both raters show an accordance of .78 for both the identification and classification of the relation). Obviously, the majority of the teachers think of the relation between these two domains in terms of examples from both domains, only few of them come up with descriptions relating these domains as a whole.

Table 2: Relations of Mathematics and Vocational Contexts

Degree of Abstractness	Contents of Relation: maths helps vocational knowledge	voc. knowl. helpful for maths	mutual help	other	
ONLY examples	18	0	1	1	20
maths ONLY as discipline	5	0	1	3	9
other	4	2	1	3	10
	27	2	3	7	39

Statistical tests show no connection between the self-description of the teachers and the function of mathematics for vocational contexts or the relation between mathematical and vocational knowledge. Teachers with mathematical studies tend to give significantly more points for the "specialist in mathematics" than the others. Teachers having more years of active teaching seem to put more stress on the 'understanding'-function than those with less teaching experience. There is no correlation between age, mathematical studies, characteristics of the courses and their cognition of the function of mathematics or the relation mathematical/vocational knowledge (for details see BROMME & STRÄSSER in press).

4 Interpretation

An interpretation of the results presented should start with the fact that the 'communication'-functions of mathematics for vocational contexts is not at all mentioned by the teachers. The teachers only perceive the 'tool'- and the 'understanding'-function, which may be taken as a redefinition of the descriptive function. This more or less pragmatic view on mathematics comes up also in the widespread use of examples rather than whole disciplines/subjects when the relation of mathematical and vocational knowledge is mentioned in the interviews. Mathematics has its fundamental purpose in helping with vocational problems, serving as an operative tool. It is taught integrated with vocational contexts and has nearly no role in itself (cf. the few teachers mentioning the underpinning of mathematics by vocational knowledge). The most ·iking result is the overall agreement on the relation of mathematical and

vocational knowledge in the interviews. A majority of interviewees only mentions the 'tool'-function of mathematics. An 'objective' interpretation could take this perception as so dominant and widespread that different views can only rarely be found in the vocational colleges. A 'measurement' interpretation could take this as an indication for the inappropriateness of interviews to empirically analyse such cognitive structures. Interviews and content analysis might be not sensitive enough to really mirror delicate differences. A 'subjective' interpretation could mention the difficulties to speak about the relation of mathematical and vocational knowledge. Even within mathematics education as an emerging discipline, there is an obvious conceptual deficit on the way in which mathematics and domains of its application relate to each other. A decision which interpretation should be favoured has to wait for additional empirical and conceptual work in the field.

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TRAINING ELEMENTARY TEACHERS IN PROBLEM SOLVING STRATEGIES: IMPACT ON THEIR STUDENTS' PERFORMANCE

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This paper descibes a course in mathematical problem solving strategies for elementary school teachers and the results of this training on the teachers' students' performance on select problem solving items from the fourth National Assessment of Educational Progress in mathematics. Overall, students of teachers participating in the course outperformed the students whose teachers did not take the course.

Results from the Fourth National Assessment of Educational Progress document that the mathematical performance of elementary and middle school students in this country is alarmingly poor (Dossey, et al, 1988). For example, NAEP results indicate that over seventy percent of third graders cannot correctly solve a problem involving two or more steps, or that over one-half of the seventh grade students have difficulty with problems involving logical reasoning based on simple syllogisms. Other research in teacher knowledge of mathematics makes a strong case that efforts to improve children's mathematics learning might first begin with enhancing teachers' knowledge about mathematics. Ball (1988) points out, for example, that only twenty percent of a group of pre-serivce teachers could definitely say that the statement, "as the perimeter of a closed figure increases, the area also increases," was incorrect. Others have found that a group of pre-service teachers could answer correctly only slightly more than one-half of a set of problems that could be solved using a variety of strategies (Oprea and Stonewater, 1988).

Funding for this research is sponsored by Title II of the Education for Economic Security Act and administered by the Ohio Board of Regents.

Yet the question of whether or not direct inservice training of teachers is an effective means of improving their students' learning is still open. Wheately (1983) found substantial student gains in subtest scores on the Iowa Test of Basic Skills after their teachers received training. Szetela and Super (1987) found gains in student performance due to teacher training on only two of five problem solving tests. It appears that the effects of inservice teacher training on subsequent student performance are unclear, at least with respect to mathematical problem solving ability.

Partly in response to these research findings, The Ohio Problem Solving Consortium has received funding to form a cooperative venture between public school teachers and university personnel. (Stonewater and Kullman, 1985; Stonewater and Oprea, 1988). The Consortium trains elementary and middle school teachers in problem solving strategies and assists the teachers in using their newly-learned knowledge of problem solving as a basis for redesigning their own instruction to improve children's problem solving abilities. The purpose of this article is to describe a problem solving course and to report the results of improvements in the problem solving abilities of the teachers' students as measured by select problem solving items from the fourth National Assessment for Educational Progress in mathematics.

<u>Methodology</u> To assess the effectiveness of the course on the participating teachers' students, Consortium teachers and six teachers not involved in the project administered select items from the NAEP in October and again six months later.

Eight items were chosen from the fourth mathematics assessment and were selected to represent problems which could be solved using at least one of the strategies learned by the teachers. A total of 516 experimental students and 122 control students completed all testing. Data were analyzed using a multiple analysis of covariance, with pre-test scores as the covariate. A Wilks' Criterion F-value was computed, as well as adjusted post-test scores, providing a post-test statistic that could be used to compare groups with possible pre-test differences controlled. For a statistically significant F-value, post-hoc analyses of variance were computed for each of the eight test items separately.

<u>Problem Solving Course</u> Teachers were expected to learn and be able to use seven different problem solving strategies: Guess and Check; Patterns; Simpler Problem; Elimination; Working Backwards; and Simulation. Project data indicate that elementary school students do fairly poorly applying these problem solving strategies (Stonewater, 1988). Teachers were also exprected to reorient their own teaching to include units on each of the problem solving strategies. A typical class began with a review of assigned problems and a discussion of the teachers' experiences applying the strategy to their own teaching, followed by a short lecture introducing the next strategy. Once an example problem was worked by the professor, the class worked a problem in groups of four to five teachers. After solutions and solution methods were discussed with the entire class, homework was assigned for the next week. The problem solving course was designed on the basis of the Instructional Model for Problem Solving

(Stonewater, Stonewater, & Perry, 1988), a model grounded in cognitive developmental theory and intended to key in to the teachers' cognitive developmental levels as a means of ehnhancing understanding. The model includes three categories of instructional approaches which, on the one hand provide support for the learner to engage in complex and difficult learning tasks, and, on the other, create what Piaget (1952) termed disequilibrium, or an upsetting of how teachers traditionally think about problem solving so that new and more sophisticated ways can be accommodated. These catagories are structure, direct experience, and diversity.

Structure -- The IMPS model suggests that in order to enable students to attend to difficult and complex problem solving, the course should provide a high degree of structure as a support for engaging in difficult learning tasks. One method used to provide structure was to develop a task analysis or a list of heuristics and guidelines that described how to carry out the strategy. For example, a task analysis was written for the Working Backwards strategy and includes a sequence of steps that the teacher or student could use in applying the strategy.

Direct Experience -- The IMPS model also suggests that activities which engage teachers in direct application of what they are learning will enhance learning. A number of methods were used to do this. First, in conjunction with the local public broadcasting television station, a series of four video tapes, entitled Problem Solving in the Middle School, were developed as examples of what "master teachers" do when

teaching problem solving. These were viewed by the class. One particular useful portion of the tapes shows middle school teachers actually using various problem solving strategies in their classes and teachers could often relate their own students' reactions and problems to what they saw on the tape.

As another direct experience method, teachers were asked to apply each of the strategies in their own classes and to keep a journal of their experiences. While this activity did not engage the teachers directly in actual problem solving, it helped them build confidence in their abilities to teach problem solving. As one junior high teacher commented, "I can give a class a problem now without making sure I 'know' the answer first. What I do know is that I'll figure out the problem by day's end!" Teachers also saw in their own students' problem solving many of their own difficulties with mathematical problem solving.

Diversity -- Another approach used in the IMPS model is that students must realistically engage in the complexities of what is to be learned in order for them to experience disequilibrium. Presenting diverse situation for the teachers to engage with is a method of doing this. Thus, problems that required the teachers to generalize beyond their current levels of mathematical knowledge and thinking ability were presented. For example, teachers rarely had difficulty with pattern problems like predicting the next term in the sequence 1/(1*2), 1/(2*3), 1/(3*4),.... But in order to challenge them and create the required disequilibrium, we introduced a second level of pattern problems that required generalization beyond the mathematics they knew. Such a

problem extension would be to predict the sum of the series, 1/(1*2) + 1/(2*3) + ... + 1/(n*[n+1]). This variation was much more difficult, but after some struggling, most teachers began the process of making generalizations. Thus, in addition to learning the basics of each strategy, the teachers went beyond what they might need to teach elementary or middle school into important mathematical thinking skills.

Another method of introducing diversity was to show the teachers that more than one strategy or approach to a problem was often appropriate. By using diverse problem solving approaches, teachers had to confront the often-held belief that there is "only one solution to a mathematics problem".

<u>Results</u> Results of this study indicate that the students of particiating teachers performed significantly better on the post-test than did the students of control teachers ($F(8,621) = 3.82$; $p < .0002$). Adjusted post-test scores for the experimental group were 5.20 items out of 8 correct (65%), compared to 4.56 items out of 8 correct for the control group (57%). In addition, the experimental group's adjusted post-test means for each of the eight items separately were higher than the control group's adjusted post-test means. Of these differences, five of the eight items were statistically significant in favor of the experimental group. These data are reported in Table 1.

The five items in which the experimental students outperformed the control students were catagorized as problems that could be solved using silulation, elimination, patterns, or guess and check. Note that in every case, the experimental

students gains are substantially greater than those of the control group. For example, the increase from pre- to post-test in the percent of students in the control group who answered one of the guess and check items correctly was only 2.5%, while it was almost 15% for the experimental students.

Discussion The results of the NAEP study utilized in conjunction with the problem solving course as well as the feedback received from teachers indicate that the major objective of enhancing children's problem solving abilities by way of training their teachers was accomplished. Teachers' self-reports indicate they felt more confident at doing as well as teaching mathematical problem solving. In general, it appears that efforts by universities to offer subject matter coursework to teachers can be an effective catalyst for bringing about changes in curricular content in the schools and for positively influencing children's learning.

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Table 1

Percent Correct and F–Values on NAEP Items

ITEM		PRE	POST	Percent Gain Pre to Post	Adjusted Post	F	p ≤
1. Guess & Check	E	32.4	40.7	8.3	40.8		
	C	33.6	35.2	1.6	35.0	1.45	.23
2. Simulation	E	54.7	75.4	20.7	74.8		
	C	45.9	59.8	13.9	62.3	9.19	.01
3. Elimination	E	41.3	60.5	19.2	59.4		
	C	26.2	41.0	14.8	45.6	8.93	.01
4. Patterns	E	41.9	55.8	13.9	55.7		
	C	37.7	40.2	2.5	40.7	9.20	.01
5. Elimination	E	86.6	95.0	8.4	94.8		
	C	82.0	85.2	3.2	85.8	13.23	.01
6. Elimination	E	66.3	79.8	13.5	79.4		
	C	50.8	72.1	21.3	73.9	1.83	.18
7. Guess & Check	E	58.9	69.2	10.3	68.8		
	C	41.8	61.5	19.7	62.9	1.56	.21
8. Guess & Check	E	35.5	50.0	14.5	49.6		
	C	27.0	29.5	2.5	31.1	14.57	.01
TOTAL TEST	E	52.2	65.8	13.6	64.9	3.82	.01
	C	43.1	53.1	10.0	56.8		

DEVELOPING ALGEBRAIC UNDERSTANDING: THE POTENTIAL OF A COMPUTER BASED ENVIRONMENT

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This paper will discuss the potential of a Logo environment for developing pupils' algebraic understanding. Results from a three year longitudinal study of pupils (aged 11-14) programming in Logo indicate that Logo experience does enhance pupils' understanding of variable in an algebra context, but the links which pupils make between variable in Logo and variable in algebra depend very much on the nature and extent of their Logo experience. The algebraic understandings which pupils are likely to develop are described and related to categories of variable use outlined in the paper. Although the focus of the paper is predominantly on Logo there will be discussion within the presentation of preliminary results from pupils' work with a spreadsheet (Excel). Studies with Logo and with spreadsheets indicate that for some pupils interaction with the computer plays a crucial role in their developing understanding of a general method.

BACKGROUND

This paper will discuss the potential of a computer-based environment for developing pupils' algebraic understanding. The focus in the paper will be predominantly on Logo programming, although there will be some discussion in the presentation of preliminary results from pupils' work with a spreadsheet (Excel). The paper derives mainly from a "Longitudinal Study of Pupils' Algebraic Thinking in a Logo Environment" (Appendix 1). The ideas and results presented are also informed by an ongoing study "The Role of Peer Group Discussion in a Computer Environment" (Appendix 2).

Algebra as a mathematical language has developed over the centuries from its first introduction as a tool to solve equations in which a letter or symbol represented a particular but unknown number, to classical generalised arithmetic in which symbols were used to represent relationships between variables to what we now know as modern algebra. Modern algebra can be thought of as a language which enables the similarities in structure between different mathematical systems to be made explicit. Algebra has played a central role in school mathematics for many years and although more recently the teaching of algebra has been given less emphasis Byers and Erlwanger stress that "we can no more dispense with teaching algebraic symbolism than teaching place-value notation. Symbolic expressions are transformed more easily than their verbal conterparts so that they not only save time and labour but they also aid the understanding of content" (Byers & Erlwanger, 1984, p.265). Vygotsky believed that "the new higher concepts in turn transform their meaning of the lower. The adolescent who has mastered algebraic concepts has gained a vantage from which he sees arithmetic concepts in a broader perspective" (Vygotsky, 1934, p.115).

We must also recognise that "school algbera" is not a uniform practice. In Britain there exists a wide range of mathematics curriculae, all reflecting differing emphases on "school algebra". Pupils are now introduced to algebraic ideas with more caution and in some curricula (for example SMILE[1]) there are many pupils who are no longer introduced to algebra within school mathematics. We are now approaching a new era in Britain with the introduction of a national curriculum for mathematics and here again the emphasis on "school algebra" is likely to change.

Despite these differences one general trend is that pupils' first introduction to algebra is now more likely to be in the context of generalising mathematical relationships resulting from practical or psuedo practical activity. Previously pupils' first introduction to algebra was more likely to be in the context of manipulating algebraic symbols derived from generalised arithmetic.

PREVIOUS RESEARCH ON PUPILS" UNDERSTANDING OF ALGEBRA

Before considering the computer context, it is important to take into account previous research related to pupils' understanding of algebra. One important research finding is that there is a gap between arithmetical and algebraic thinking which relates to pupils' use of informal methods in arithmetic (Booth, 1984). This means for example that pupils might find it difficult to express the area of a rectangle in the form A = W x L (where A, L and W are the respective area, length and width of the rectangle) because their informal method for solving area of rectangle problems in arithmetic is counting the number of squares in a rectangle.

There has also been considerable research identifying pupils' misconceptions when dealing with algebraic objects, focussing mainly on pupils' use and understanding of variable (Collis, 1974; Booth, 1984; Küchemann, 1981; Wagner, 1981). This research suggests that many pupils lack understanding that a letter can represent a range of values (Collis, 1974; Booth, 1984; Küchemann,1981) and lack understanding that different letters can represent the same value (Wagner, 1981). They find it difficult to accept an "unclosed" expression in algebra (for example a + 6) which relates to their difficulty in operating on these expressions (Booth, 1984; Collis, 1974). They also find it difficult to understand that a systematic relationship exists between two variable dependent expressions (Küchemann, 1981).

There is also considerable literature related to pupils' difficulties with the manipulation of algebraic objects, but our research has not yet addressed these issues within a computer context.

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[1] condary Mathematics Individualised Learning Experiment

FRAMEWORK FOR CATEGORISING PUPILS' USE OF VARIABLE IN LOGO

By carrying out an ongoing analysis of the situations in which pupils use variable to define a general procedure in Logo categories of variable use have been identified (Sutherland, 1988a). We use them to provide a framework for analysing pupils' understanding of algebra related ideas.

(I) One variable input to a procedure. When pupils use one variable input they are using variable as a place holder for a range of numbers.

(S) Variable as scale factor. In this situation the variable input is used to scale all the distance commands in a turtle graphics procedure. This type of variable input can be used by pupils as a way of generalising a fixed procedure (Fig 1a) without making explicit the geometrical relationships within the procedure.

(N) More than one variable input to a procedure. This category is concerned with situations in which pupils use more than one variable input to their procedure often as a means of avoiding expressing a general relationship between variables within a procedure (Fig. 1.b).

(O) Variable input operated on within a procedure. In this category any general relationship between variables within a procedure is made explicit by operating on one or more variable inputs within the procedure (Fig 1c).

```
TO TOM :SCALE
LT 90
PU
BK :SCALE * 60
PD
FD :SCALE * 60
LT 45
FD :SCALE * 20
RT 90
FD :SCALE * 20
RT 90
FD :SCALE * 20
RT 90
FD :SCALE * 20
END
```

```
TO KITE :YT :HT
RT 45
FD :YT
RT 90
FD :YT
RT 90
FD :YT
RT 90
FD :YT
BK :YT
RT 90
FD :YT
RT 45
FD :HT
END
```

```
TO SQUAN :NUM
LT 135
REPEAT 4 [FD :NUM RT 90]
LT 135
FD :NUM * 3
END
```

Fig.1 a) Variable as Scale Factor b) More than One Input c) Variable Operated On

(F) Variable input to define a mathematical function in Logo. In this category variable is input to a procedure, which acts like a mathematical function, that is it is operated on within the procedure and the result is output from the procedure to be used by another Logo function or command

```
TO FUNC :X              equivalent to    F(x) = X + 4
OUTPUT :X + 4
END
```

FRAMEWORK FOR ANALYSING ALGEBRAIC UNDERSTANDING

Our work has primarily been concerned with investigating pupils'

1) Use of a formal language to represent a generalisation
2) Understandings associated with the use of variable

These understandings have been categorised as follows, with reference to the previously discussed literature:

- Understanding that a variable name can represent a range of numbers
- Understanding that any variable name can be used
- Understanding that different variable names can represent the same value
- Acceptance of "lack of closure" in a variable dependent expression
- Understanding the nature of the second order relationship between two variable dependent expressions

Using this framework we will now discuss the potential of a computer based environment on pupils' developing algebraic understanding.

1) Use of a Formal Language to Represent a Generalisation

The review of literature suggests that pupils often use informal methods which cannot easily be generalised and formalised. "If children do not have that structure available in the arithmetic case, they are unlikely to produce (or understand) it in the algebra case" (Booth, 1984, p.102). In the Logo environment pupils are able to interact with the computer and negotiate with their peers so that their intuitive understanding of pattern and structure is developed to the point where they can make a generalisation and formalise this generalisation in Logo. There is evidence that in many cases pupils could not do this without both "hands on" interaction with the computer and discussion with their peers (Sutherland, 1988b).

Our studies indicate that pupils' ability to use Logo to represent a general method is linked to their experience of using variable in the category of "(O) Variable operated on". It is suggested that it is only when pupils are able to use variable in this category that they have made the break from arithmetical to algebraic thought (Filloy and Rojano, 1987). Work with pupils who have had no previous experience of "paper and pencil" algebra suggests that these pupils can use variable in category (O) but that they are unlikely to do so without specific teaching

sequences directed at this idea. In our more recent work with 12-13 year olds (Appendix 2) we have presented pupils with problems which specifically need variable in the category (O) as part of their first experience of variable in Logo. These pupils are more confidently able to use variable in this category than pupils of a similar age group who were part of our previous study (The Logo Maths Project) and who were not subject to such extensive direction.

Our more recent work with spreadsheets (Healy & Sutherland, 1988) suggests that this computer environment provides another context for expressing a generalisation, but one which appears to be substantially different from the Logo programming context. Naming and declaring the variable is no longer a focus and within a "mouse driven" spreadsheet environment the generalisation can be encapsulated without reference to a formal language.

Expressing a generalisation in either a spreadsheet or Logo language helps to convince pupils of the validity of their generalisation. We now need to study more carefully what would constitute a proof for pupils that this computer generated generalisation is valid and the related implications for the learning of Mathematics.

2) Understandings Associated With the Use of Variable

Perhaps the most important result from our studies is that the algebraic understandings which pupils develop are closely related to the particular computer environment and the types of problem situations with which the pupils have been engaging. This means that the role of the teacher is crucial in both provoking pupils to work on problems for which the use of variable is an essential problem solving tool and in providing pupils with information about the constraints on using variable within the relevant programming context. Within this section we will discuss the variable related understandings which pupils derive from working with Logo.

Understanding that a variable name can be used to represent a range of numbers. Pupils who have used variable in the category of "(I) One variable input to a procedure" are likely to have developed an understanding that a variable can represent a range of numbers. However pupils understanding of "range of numbers" is likely to be restricted to positive whole numbers unless they have worked on problems in which it is necessary to use both decimal and negative numbers. We have found that when pupils use variable in the category of "(S) Variable as scale factor" they are provoked to use decimal numbers as input.

Understanding that any variable name can be used. In a computer programming context pupils are often introduced to variables with meaningful variable names (e.g. SIDE or SCALE). Our Logo studies indicate that when pupils are first introduced to variables they attach too much significance to these meaningful names and think for example that the name SIDE for the length of a square conveys some meaning to the computer. We have found that if pupils are encouraged to use a range of variable names, including "nonsense" names (which they know have no meaning) and abstract and single letter names (which they will use in their algebra work) they come to understand that any name can be used.

Understanding that Different Variable Names Can Represent the Same Value. Pupils overinterpret the constraints on the variable name itself. Algebra research has shown that pupils do not understand that different variable names can represent the same value (Küchemann, 1981). Our studies indicate that if pupils have within their Logo programming experience, defined a procedure with at least two variables (using variable in the category of "(N) More than one variable input") and then in the context of using this procedure assigned both inputs the same value they are likely to develop an understanding that different variable names can represent the same value in Logo.

Acceptance of "Lack of closure" in a Variable Dependent Expression. Our studies indicate that pupils who have used "unclosed" expressions in Logo either within the context of defining simple functions (see Fig. 2), or within the context of operaing on a variable have no difficiulty in accepting "lack of closure" in variable dependent expressions.

Understanding the nature of the second Or ·; Relationship Between two variable dependent expressions. None of the eight 13-14 year old pupils who were part of the Logo Maths Project (Hoyles & Sutherland, 1989) developed an understanding of the nature of the second order relationship between two variable dependent expressions. Analysis of their Logo experience indicates that they had never used this idea in Logo. Subsequently a task was developed for a group of five 10-11 year old pre-algebra pupils (Appendix 1) in which they were specifically confronted with this idea. Three of these pupils showed, by their response to Logo structured interview questions (identical to those given to the 13-14 year olds) that they had developed an understanding of this idea. This suggests that it is possible for pupils, if they use this idea during their "hands on" Logo programming sessions, to develop a related understanding.

This provides evidence that a crucial factor in learning is first the use of an idea within a problem solving situation.

LINKS WITH "PAPER AND PENCIL" ALGEBRA

As part of the "Longitudinal Study of the Development of Pupils' Algebraic Thinking in a Logo Environment" (Appendix 1) eight case study pupils worked on materials which were aimed at helping them make links between their Logo work and "paper and pencil" algebra. These materials were based on the similarity between using variable to define a function in Logo and on using variable to define a function in algebra. These pupils were presented with items from the C.S.M.S[2] study in the form of a structured interview in order to probe whether or not they had made any links to a "paper and pencil" algebra context. The results of these interviews indicate that pupils can make links betwen the two contexts, but the links which they make are as much related to their particular experiences in Logo as to the specifically designed linking materials. More research needs to be carried out in this area with specific attempts made to integrate the computer based and the "paper and pencil" algebra curriculum.

CONCLUSION

Our studies indicate that within a computer-based environment there does not have to be a gap between pupils' informal methods and the formal representation of this method. Pupils, through interacting with the computer and discussion with their peers are able to develop their intuitive understanding of pattern and structure to the point where they can make a generalisation and formalise this generalisation in Logo. For some pupils in particular the interaction with the computer appears to play a crucial role in their developing understanding of a general method.

We have found that Logo experience does enhance pupils' understanding of variable in an algebra context, but the links which pupils make between variable in Logo and variable in algebra depend very much on the nature and extent of their Logo experience.This suggests that it is the using of an idea which is the crucial factor influencing understanding. We need to carry out more research both to understand more about the mathematical processes in which the pupils are engaged when working in a computer-based environment and to discover how best to integrate pupils' computer based experiences with a developing mathematics curriculum.

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[2] As part of the research programme "Concepts in Secondary mathematics and Science" just under s aged 14+ were tested on their understanding of algebra (küchemann, 1981).

Appendix 1: A Longitudinal Study of Pupils' Algebraic Thinking in a Logo Environment

This research was carried out by the author for her Phd thesis (Sutherland, 1988). It was both part of and an extension of the Logo Maths Project (Hoyles & Sutherland, 1989). The research consisted predominantly of a three year longitudinal case study of four pairs of pupils programming in Logo during their "normal" mathematics lessons. The data collected consisted of video recordings of all their Logo sessions. In addition pupils were individually presented with structured Logo programming tasks and individually interviewed to probe their developing understanding in both a Logo and a "paper and pencil" algebra context. A subsidiary one year study was carried out with a group of eight pre-algebra 10-11 year old primary school pupils.

Appendix 2: The Role of Peer Group Discussion in a Computer Environment (1988-1989)

This is an ongoing project funded by the Leverhulme Trust and carried out by the author in conjunction with Lulu Healy and Celia Hoyles. One of the aims of the project is to investige the relationship between pupils' negotiation of a generalisation in natural language and their formal representation of this generalisation. Pupils (aged 12-13) work in Logo, a spreadsheet and a "paper and pencil" mathematics environment. The data collected consists of video tapes of four pairs of pupils working in all three environments.

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Verbal Evidence for Versatile Understanding of Variables in a Computer Environment

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We have previously reported (Thomas and Tall, 1986, 1988) on experiments demonstrating the value of a computer-based pre-algebra module of work in aiding 11 and 12 year-old pupils to reach a higher level of understanding of the use of letters in algebra than that found in a more traditional approach. We have also put forward the hypothesis that one reason for this success is the way "cognitive integration" (Thomas 1988) of the child's global/holistic and serialist/analytic cognitive abilities leads to versatile thinking. Further, this may be actively promoted using the "enhanced Socratic mode" of teaching (Tall 1986) using the computer as a resource for teacher demonstration, pupil exploration and discussion to develop appropriate concept imagery. This paper considers evidence in support of the theory from interviews with the students involved, taken six months after the computer treatment.

Some Theoretical Considerations

When algebra is perceived, and hence taught, as an essentially logical, serialist activity with little or no recourse to either its inherent structure or its underlying concepts - such as the use of letters as generalized numbers or variables - one would expect this view of algebra to prevail among pupils. A substantial body of research points to just such a lack of understanding as contributing to poor performance in algebra throughout secondary school and beyond (e.g.Rosnick and Clement 1980, Matz 1980, Küchemann 1981, Wagner, Rachlin and Jensen 1984). The results of our work have suggested differential effects between the computer-based approach to algebra, with its emphasis on letters as generalized numbers and the traditional skill-based type of module with its emphasis on acquiring manipulative skills. It seems that the computer work promoted a deep conceptual understanding better, while the other work, as expected, initially facilitated better surface skills. However, when the computer module was combined with the skill-based one then it led to a superior overall performance without detrimental effect on skills. It is our view that the computer is providing an environment in which pupils acquire a global/holistic view of algebraic concepts - relating the symbols on paper to meaningful ideas such as the mental picture of a letter representing a variable number - in

contrast to the more serialist/analytic view nurtured by emphasizing the operation on symbols. An illustration of this is the following type of question, with which many will be familiar:

Factorize $(2x + 1)^2 - 3x(2x + 1)$.

Many pupils faced with this type of question seem locked into a sequential/operational mode of working where they "multiply out the brackets","collect together like terms" and factorize the resulting quadratic function. Few are able to apply the versatility of thought to switch from an analytical approach to a global/holistic one which "chunks" together the symbols 2x+1 as a single conceptual entity, allowing them to move more directly to the answer. We believe that the activities carried out in the computer context encourages flexible mental constructs more likely to lead to this global/holistic view.

Evidence For Versatility and Conceptual Understanding

Conceptual understanding in algebra is not evidenced by test performance alone. Correct answers to routine problems may be produced by incorrect understanding and incorrect responses to non-routine problems may have a sensible foundation. In order to examine pupil's understanding of algebra beyond the test performances indicated in Tall and Thomas (1988), we conducted a number of interviews with selected students and administered a broadly based questionnaire to see if certain phenomena which occurred in the interviews were replicated on a wider scale.

The Interviews

The teaching experiment (Tall & Thomas 1988) had comprised two groups of 13 year-old secondary-school children taken from six mixed abilit) forn .ranged into 57 matched pairs. The experimental group used the computers for three weeks, following a module of investigational activities, while the controls followed their traditional algebra course. Six months later, all the pupils were given the same traditional module for a two week period, the controls as revision, the others for the first time.

After the post-test in this experiment a cross-section of 11 experimental and 7 control pupils (of comparable performance on the post-test), were given a semi-structured interview lasting about twenty minutes. During the interviews, which were recorded, the pupils were required to attempt certain key questions and to explain their thinking and strategies. The following examples taken from the transcripts of the interviews show a marked difference between the experimental pupils, who often attempted to give a relational explanation for their reasoning, and control pupils, who were more likely to be concerned simply with carrying out routine algebraic processes.

Question : Solve 2p - 1 = 5.

The following response from a control pupil illustrates the confusion that may arise from mechanically carrying out routine processes:

Pupil 11 : 2p minus 1 equals 5. If you add the 1 to the 5 that's 6 so, because there's no other minus p, I forget the p and do the 2p minus 1 equals. If you add the 1 to the 5 which is 6 and then you take 1 from the 6... No, I don't get that. I know I've done it but...

Interviewer : What would the value of p be did you say?

pupil 11 : Six.

Here the explanation is solely in terms of the operations with no reasons for their use being cited. This may be compared with the following reasoning from one of the experimental group pupils :

Pupil 2 : Well find out what minus 1 so you would add 1 to that so you get rid of the 1, so that would be 6 and then its obvious that 2 times 3 equals 6, so p would be 3.

The pupils in the interviews were also asked to compare the above equation with

2s - 1 = 5.

This was in order to see if they were able to conserve equation (Wagner 1977) under a change of variable. A distinct difference in the type of comment between the two groups shows the superior understanding in this area of those pupils who had used the computer.

Control group :

Those unsure of the relationship :

Pupil 10 : s could be 3 as well.

Pupil 12 : So s could be 3 as well.

Pupil 13 : They could both equal 4.

Those who needed to solve both equations:

Pupil 11 : Well what I have put is 2p equals 6 and 2s equals 6.

Pupil 14 : 2s...add the 1 and 5, 6 er 2 and 2, 6, 3 times, so s is 3 as well.

Experimental (computer) group :

Pupil 1 : I can say that p and s have the same value...it's the same sum.

Pupil 2 : Well they are both the same...Yes, because they are both the same but different letters.

Pupil 3 : They are both...p and s both equal 3.

Pupil 4 : It's just a different letter but it would have to be 2 times 3 minus 1 equal to 5.

Pupil 5 : The same. Just using a different letter.

Pupil 8 : It is 3 the p and s...because they are basically the same sum, but are different letters.

Pupil 9 : They are both the same. It's the same apart from the letters,exactly the same except the letters.

These pupils offer verbal evidence of a global/holistic view of the equations enabling them to develop the understanding of conservation of equation by seeing the common structure of the equations. This concept of conservation of equation under a change of variable was further tested with several of the children by the use of an extension to the first question above to :

Solve $2(p + 1) - 1 = 5$.

The insight of the computer group pupils is shown by their comments:

Pupil 1 : Yes, p equals 2.

Interviewer : How did you work that out then?

Pupil 1 : Well its the same, but its plus 1, so minus 1 add 3.

Pupil 2 : Oh it would be 2.

Interviewer : Can you tell me why?

Pupil 2 : Because p plus 1 if that's 3 its the same as the last one only the p is less because you've got to add 1 to the sum.

Deep and powerful insights such as these, which are facilitated bv a global/holistic view leading to the structure of the equations was not matched by the con..ols. Instead we have:

Pupil 15 : Say p plus 1, there is already 1 plus p plus another one, I'd say that was 2p, and then outside plus another 2 that is 4 minus 1 is 3 I would say.

Interviewer : So what is the answer?

Pupil 15 : p equals 1 I would say.

Extension of algebraic ideas

Research has indicated that the type of algebraic equation where there are variables on both sides of the equation is considerably more difficult, since it involves algebraic manipulation (of variables) rather than arithmetic (e.g. Herscovics and Kieran 1980). Neither experimental or control pupils in the the experiment had been taught to solve this type of equation. It was hypothesized that the relational understanding of the experimental pupils would lead to their

greater ability in handling such equations. Several interviewees were asked to tackle the question:

Solve 3x - 5 = 2x + 1.

The replies again gave evidence of superior understanding on the part of those who had used the computer.

Controls :

Pupil 15 : I'd say it was minus 2x and here you've got 3x, 2x plus 1x,so I'd put that as 1x

[Writes 3x - 5 = 2x + 1 = 1x]

Interviewer : And is that the answer?

Pupil 15 : Yes

Hence, although the surface operation of subtracting 2x is carried out it does not seem to be in the context of any understanding of an overall purpose in the question, and no reasons for the operation are given. One of the pupils in this group had lost sight of the objective altogether:

Pupil 12 : I'm trying to work out how you could take 5 from that to leave that.

Interviewer : Can you see any way of doing it?

Pupil 12 : You would have to find the value of x before you could start.

In contrast, the experimental group pupils given this question responded more purposefully :

Pupil 1 : Well the value of x must be the same because it's in the same sum... I'm thinking that maybe take x some number away from both sides. That wouldn't leave anything in there to go on. You'd have nothing there if you take 2x away and 1x minus 5 equals plus 1.

[Writes x - 5 = +1]

Interviewer : So how might you do it now?

Pupil 1 : I was thinking maybe get rid of this and forget about that 4 by putting, adding 5 to both sides - that should do it - so it would be 3x equals 2x plus 6...try to take x away.

[Writes 3x = 2x + 6]

Shortly after this he solved the equation.

Pupil 2 : You would add 5 to that to get rid of the minus 5 and then that plus 6 so it would be 3x equals 2x plus 6....Well that plus 6 has got a bigger x because 2x plus 6 equals 3x, that means another 6 would be equal to x, so make that 3x as well...Well x equals 6.

We can see that this pupil starts off with a serialist/analytical approach, but accompanied by clear reasons for the steps taken. However, in the middle of the question the pupil is versatile enough to change viewpoint to a global/holistic one and see the equation in terms of its balancing structure, enabling the equating of an extra x with 6.

The Questionnaire

A questionnaire given to 147 pupils, whilst not giving the opportunity to follow up answers as in an interview, gave evidence of a wider dispersal of the phenomena found in the interviews. It included three types of questions; one where they were required to explain, with reasons, whether two algebraic expressions were equal or not; one where they had to explain to an imaginary visitor from Mars the meaning of some algebraic notation and the third where harder algebraic questions, beyond the level they had studied, were to be attempted.

Question	Experimental Proportion Correct	Control Proportion Correct	z	p
Is $\frac{6}{7}$ the same as 6+7 ?	0.76	0.44	3.38	<0.0005
Is 2+3c the same as 5c ?	0.41	0.31	1.24	n.s.
Is 2(a+b) the same as 2a+2b ?	0.57	0.31	2.69	<0.0005
Solve 13-y=2y+7	0.43	0.27	1.83	<0.05
Simplify 5h-(3g+2h)	0.24	0.08	2.16	<0.025
Solve 17-3e>2	0.31	0.13	2.37	<0.01

Table 1 - A comparison of some questionnaire facilities

Error	Experimental Proportion Making Error	Control Proportion Making Error	z	p
$3 + m = 3m$	0.09	0.27	2.54	<0.01
$ab = a + b$	0.06	0.13	1.77	<0.05
$b - 2 \times c = (b - 2) c$	0.09	0.23	1.77	<0.05
$3 + 2m = 5m$	0.04	0.13	1.57	n.s.

Table 2 - A comparison of some questionnaire errors

The results in tables 1 and 2 from selected, and the fact that *the controls did not perform significantly better than the experimental group on any question*, support the hypothesis that the experimental students have a better understanding of algebraic notation. Moreover, it also seems that one of the main failings of the controls is that the traditional skill-based module has encouraged a predominantly left-to-right sequential method of processing algebraic notation. In contrast to this, the computer group, seem to have a better, more global, view of the notation which in turn has reduced the occurrence of some of the more common notational errors such as conjoining in addition and the wrong use of brackets. An interesting example of this, although arithmetic rather than algebraic, is the first question in table 1, where many of the controls did not consider the two notations as the equivalent because

"$\frac{6}{7}$ is a fraction, 6+7 is a sum".

This is a good example of a response which is based on sound conceptual reasoning, but one that is limited because it implies the inability to encapsulate the *process* 6+7, as a single

conceptual entity. The encapsulation occurred far more often amongst the computer group, again underlying what we believe is a more flexible global view.

The difficulties that pupils had with the question

> Is 2(a+b) the same as 2a+2b ?

again revealed the difference between the symbols representing *a process* and the *result* of that process as a conceptual entity. So firmly had it been ingrained in them that "calculations inside brackets must be done first" that the symbol 2(a+b) is read as "first add a and b, then multiply by 2" whilst 2a+2b requires both multiplications to be carried out before the addition, that they saw the *processes* as being different rather than the *results* being the same. Even so, the experimental group were once again more likely to attempt to surmount this conceptual obstacle, one student proposing an interesting way out of his dilemma:

> Pupil 1 : Well its brackets, so you've got to add these two numbers before you times it [...]
>
> Interviewer : You can't see any way round that problem?
>
> Pupil 1 : I know there is one, but I can't find it. [...] Unless you went along and put a+b equals c and then put 2 times c, but that's a long way round.

Conclusions and further research

Through interviews it is manifestly clear that the students involved in the enhanced Socratic approach had developed a more versatile understanding of the concept of variable, in which they were able to encapsulate the algebraic processes as objects and to chunk information in expressions in a way which enabled them to take a more versatile approach to solving algebraic problems. However, it should be noted that it has not proved possible to follow up the initial three week algebra module with further algebraic experiences using the computer and, subsequently, the classes have been reorganized in a way which has led to a variety of different experiences for pupils matched in pairs during the experiment. Some eighteen months after the delayed post-test, a similar test has revealed that the difference between the experimental and control groups is no longer statistically significant. We have still to administer interviews to see if there remain differences detectable by these means. This suggests that, although computer experiences may be able provoke different kinds of understanding in the short and medium term, if these experiences are not continued then their effect may wane in the face of the overwhelming influence of more recent experiences.

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CONCEPTUAL ADJUSTMENTS IN PROGRESSING FROM REAL TO COMPLEX NUMBERS

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This study assesses the difficulties that high school students experience when progressing from real to complex numbers. It was found that students are reluctant to accept complex numbers as numbers, and that students incorrectly attribute to complex numbers the ordering relation which holds for real numbers. The paper presents some sources of these difficulties and suggests ways to help students overcome them.

One of the main concepts in mathematics is that of number. Students learn various number systems starting with the natural numbers, and progressing through integers, rational, real and complex numbers.

The transition from one realm of numbers to an extended one requires a major adjustment in each student's concept of number. One major difficulty for students is realizing that the new elements in the extended domains are numbers even though these numbers often differ in appearance and properties from those in the less-extended domains. Another problem for students is their tendency to incorrectly attribute properties of the less-extended domains to the more general ones.

The difficulties that children and adolescents encounter when progressing from natural to rational numbers have been extensively investigated. Researchers found that many students do not accept rational numbers as numbers (Kerslake, 1986); also, students tend to incorrectly attribute properties of operations with natural numbers (such as that multiplication never makes smaller) to all rational numbers (Bell, 1982; Hart, 1981; Fischbein, Deri, Nello, & Marino, 1985). These attributions influence the students' beliefs about numbers and arithmetic operations, and thereby limit their ability to solve certain kinds of word problems involving rational numbers.

However, the difficulties that students encounter in other extensions of the number system are rarely discussed in the research literature. We found only one study (Vinner, 1988) that deals with the extension from real to complex numbers. Vinner's study shows that many

students find it extremely difficult to accept complex numbers, such as the non-digit number i, as numbers. The present study explores this issue. The two principal questions addressed are:

1. Do students accept complex numbers as numbers?
2. Do they incorrectly attribute properties of the real number system to the complex one?

METHOD

Subjects: Seventy-eight eleventh-grade students from three high schools in Israel participated in this study. The students had just finished eight lessons on complex numbers, and completed a summative test which included calculation examples and equations involving complex variables. Ninety-six percent of them passed, getting at least 60% of the answers correct.

The three mathematics teachers introduced complex numbers as an extension of the field of real numbers. The solution of the equation $x^2=-1$ was denoted by the imaginary number i, and other imaginary numbers were obtained by multiplying i by real numbers. Complex numbers were shown in the general form a+bi, where a and b are real numbers. Equality of two complex numbers and arithmetical operations with complex numbers were defined. The geometrical representation of complex numbers in the Gauss plane was introduced. Most of the class time was spent on practicing the operations and solving equations involving complex variables.

Instruments: Post-test and delayed post-test questionnaires were developed to examine the students' perceptions of complex numbers. The post-test questionnaire included the following four items:

1. *Circle the numbers in the following list:*

   $2/0$ 3 0 0.25 $\sqrt{-3}$ -0.434334333

   $-\sqrt{-4}$ $a+b$ $3/4$ $3+2i$ $0.2\dot{3}$ $5\sqrt{3}$.

2. *Solve the following equations:*

   a. $x^2+x+2=0$ b. $x^2+9=0$

3. *Answer "true" or "false" and explain your choice of answer:*

   For every two given numbers p and q, one of the following relationships holds:

   $p < q$ $p > q$ $p = q$

4. *For each of the following pairs of numbers, write >, <, or = (only if possible):*

   $2-3i$ 2 0.3 0.333 $-0.1\dot{6}$ -0.166 i $4+i$

The delayed post-test included similar items. For example, the equation $x^2+2x+6=0$ in the delayed post-test is similar to equation 2a above.

Interviews: Semi-structured individual interviews of about an hour each were conducted with 14 of the students in order to obtain more information about their concepts of complex numbers. Students were encouraged to explain their answers to the questionnaire, and to answer other related questions such as: "When you key $\sqrt{-3}$ into a calculator, you get an error message. Why?" The interviewees were asked for their criteria for determining whether a given entity is a number, and for their opinions on the existence of ordering relation among numbers.

Procedure: The post-test questionnaire was administered to the students at their regular classes immediately after they had finished their studies of complex numbers. A delayed post-test was given two months later. The individual interviews were conducted a few days after the students had responded to the delayed post-test. The interviews were tape recorded and transcribed. Systematic data on the taped interviews are not presented here; excerpts illustrating the students' reasoning are included.

RESULTS

1. Identifying complex numbers as numbers

Table 1 shows that immediately after instruction, most students recognized complex numbers of the form a+bi ($a\neq0$, $b\neq0$) as numbers. They were less willing to accept pure imaginary numbers as numbers. Two months later, there was a significant decrease in the number of students who responded that complex numbers and pure imaginary numbers are numbers.

Table 1: Recognition of complex numbers as numbers (%)

Item	*Post-test*		*Delayed post-test*	
	Yes	*No*	*Yes*	*No*
Is $\sqrt{-3}$ a number?	69	31	50	50
Is $-\sqrt{-4}$ a number?	69	31	49	51
Is 3+2i a number?	87	13	65	35

The students' solutions to quadratic equations with negative discriminants (item 2) also showed a significant decrease in correct responses from post-test to delayed post-test; Immediately after instruction, 84% of the students solved the quadratic equations correctly, 4%

of them correctly claimed that there is no real solution, 6% argued that these equations have no solution, and 6% did not respond. Two months later, however, only 17% of the students solved these equations correctly. Most of them (83%) claimed that quadratic equations with negative discriminants have no solutions.

Students' decisions on whether complex numbers are numbers stem from their concepts of what numbers are. Those who perceived complex numbers as numbers described a number as an entity that one can do mathematics with (calculate, solve equations, etc.) or as an entity that is represented by a point on a real line or on a plane.

Students who did not perceive complex numbers as numbers claimed that numbers are entities which are written with numerical digits, entities which are represented by points on the real line, or entities which describe positive or negative quantities. Some of these students viewed complex numbers as operations rather than as numbers. These students argued that the expressions 3+2i and $\sqrt{-3}$ describe operations that still need to be executed.

Other arguments used by students to counteract the statement that complex numbers are numbers reflect their uneasiness about the non-digit number i, and their confusion over the terms real, imaginary and complex numbers. Some of them claimed that i was a variable and not a number. Others volunteered that the term "imaginary numbers" implies something that does not exist, is not real, is something strange, is not a number. They reasoned that complex numbers are composed from a real part, which is a number, and an imaginary part which is not a number -- hence complex numbers are not numbers.

The error sign displayed by a calculator when numbers such as $\sqrt{-3}$ are keyed in was a source of support to the claim that imaginary numbers are not numbers. Students argued that since the error was the same as for 2/0, which is not a number, it followed that $\sqrt{-3}$ is not a number either.

2. Understanding that the ordering relation "less than" does not hold for the complex number system.

The students were taught that the ordering relation "less than," which holds for the real number system, does not hold for the complex number system. However, only a small percentage of the students realized that some unequal complex numbers are incomparable according to the ordering relation "less than" (see Table 2).

Some students explained that the geometrical representation of complex numbers as points on a plane illustrates that it is impossible to determine which of two given unequal complex numbers is greater than the other. Others claimed that complex numbers do not describe quantities and therefore are incomparable.

Table 2: Responses to the statement: "For every two given numbers p and q, one of the following relationships exists: $p = q$; $p > q$; $p < q$" (%)

	Post-test	*Delayed post-test*
True	88	95
False	12	5

Most students argued that the ordering relation "less than" holds for all numbers. Common justifications were that the three conditions $p > q$, $p < q$, and $p=q$ described the entire range of possibilities; and that numbers describe quantities, so any given number must be equal to, greater than, or smaller than any other given number.

The information in Table 3 is consistent with that in Table 2. Few students understood that the given numbers are incomparable by "less than". Students who claimed that $i<i+4$ perceived the symbol "+" as signifying addition in its usual sense, and argued that when a positive number is added to another number, the sum is greater than the first number. Those who claimed that $2-3i>2$ said that i is a negative number because it is related to -1, so -3i is a positive number.

Table 3: Responses to the Item: Write <, >, or = whenever possible (%)

	i ___ 4+i			*2-3i ___ 2*	
	Post-test	*Delayed post-test*		*Post-test*	*Delayed post-test*
i and i+4 are incomparable	4	4	2-3i and 2 are incomparable	8	5
$i > 4+i$	0	0	$2-3i > 2$	12	12
$i < 4+i$	95	96	$2-3i < 2$	78	82
$i = 4+i$	0	0	$2-3i = 2$	1	1
$i \neq 4+i$	1	0	$2-3i \neq 2$	1	0

CONCLUSIONS AND IMPLICATIONS

This study pointed out two major difficulties that students encounter when progressing from real to complex numbers: reluctance to accept complex numbers as numbers, and a tendency to incorrectly attribute to complex numbers the ordering relation "less than" which holds for real numbers.

These are due largely to the students' perceptions of numbers as (1) entities which are written with numerical digits, (2) entities which are represented as points on the real line, or (3) entities which describe quantities. These perceptions are anchored in the students' relatively long experience with numbers; therefore, the students find it difficult to assimilate imaginary and complex numbers into their scheme of number.

Another major cause for these problems is that some students view complex numbers as operations that need to be executed. The fact that a calculator does not differentiate between complex numbers and expressions which are not numbers contributes to the students' reluctance to integrate these numbers.

Mathematics educators should be aware that complex numbers do not fit readily into their students' notion of what a number is. They should attempt to help the students overcome this obstacle. Some ways of increasing the students' acquaintance with complex numbers are:

(1) Relate the extension of the real number system to previous extensions of the concept of number, starting with the natural numbers and progressing through integers, rational and real numbers.

(2) Debate the gains and losses which accompany each of these extensions (e.g., gaining closure under subtraction and losing the existence of the smallest number when progressing from natural numbers to integers; gaining the ability to solve every polynomial equation and losing the ordering relation when progressing from real to complex numbers).

(3) Encourage students to reflect on the development of their own concepts of numbers.

(4) Represent other views of complex numbers (e.g., as ordered pairs of real numbers).

(5) Demonstrate practical uses of complex numbers in mathematics and in other domains such as electronics.

The difficulties that students face when progressing from real to complex numbers are similar to those found during extensions of other number systems. Therefore, beyond the issue of complex numbers, we suggest that teachers use the concept of extended systems as a formal mathematical tool at the middle and high-school levels. This concept may help students grasp the idea of complex number by taking entities which look different and grouping them under a single handle: "number".

When discussing this concept with students, it can also be beneficial to describe the difficulties that mathematicians had experienced when extending number systems. Such enlightment may help students develop a perception of mathematics as a man-made domain (see Kleiners' paper, 1988).

A final comment, not directly connected to the main theme of this paper: some students interpret a calculator's error sign, which appears when imaginary numbers are keyed in, as an indicator that these entities are not numbers. It is important to discuss the limitations of calculators, and to explain that there are numbers that calculators cannot represent. It should be stressed that a calculator does not determine whether a certain entity is a number; that decision is a theoretical, purely mathematical one.

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DOES THE SEMANTIC STRUCTURE OF WORD PROBLEMS AFFECT SECOND GRADERS' EYE-MOVEMENTS?

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In the present study eye-movement registration was used to examine the influence of the semantic structure of one-step addition and subtraction word problems (simple versus complex) on the eye-fixation patterns of high-ability and low-ability second graders. Semantic complexity had a significant effect on the partition of the total fixation time over the words and the numbers in the problem: the proportion of time spent on the words was higher for complex problems than for simple ones. This result provides additional support for the hypothesis that semantic processing is a crucial component in a skilled solution process. On the other hand, the effect of the pupils' ability level was not significant. Those findings are interpreted taking into account the available theory on word problem solving.

INTRODUCTION

During the past decade children's solution processes for one-step addition and subtraction word problems have been extensively investigated using techniques such as paper-and-pencil tests, individual interviews, and computer simulation. Recently, we started to apply eye-movement registration as a new data-gathering technique. In a first exploratory eye-movement study we analyzed the eye-movement behavior of nine high-ability and eleven low-ability first graders while reading and solving a series of eleven elementary addition and subtraction word problems (De Corte & Verschaffel, 1987). While the main goal of that pioneering study was to explore the usefulness and the limitations of eye-movement data as access to young children's solutions of word problems, it yielded already some remarkable empirical findings. First, it was found that the high-ability children looked more and longer at the non-numerical elements in the problem text than the low-ability children. Second, our data supported the frequently heard statement that errors on word problems are due to

inattentively reading the problem; in fact, pupils sometimes answered without even casting a glance at some crucial parts of the problem text. Due to several technical and methodological problems encountered during the gathering and the analysis of the eye-movement data, the results of that pioneering study could not be considered as strong evidence in favor of those conclusions; however, these findings suggested hypotheses for further study. Therefore, the main goal of the present investigation was to test several hypotheses concerning the processes underlying skilled and unskilled word problem solving in a more controlled and systematic way.

THEORETICAL FRAMEWORK

Solving one-step arithmetic word problems

In the late seventies, Greeno and his associates introduced a theoretical model of skill in solving elementary arithmetic word problems (Riley, Greeno & Heller, 1983). Two basic assumptions underlied their approach: (1) word problems that require the same formal arithmetic operation can be described in terms of different semantic structures underlying the problem, and (2) the construction of an appropriate representation of that semantic structure is a crucial aspect of a skilled solution process.

Concerning the first assumption, Greeno c.s. constructed a classification scheme for elementary addition and subtraction word problems based on their underlying semantic relations. They distinguished three main categories of problems (Change, Combine, and Compare), and within each of the three problem types, further distinctions are made depending on the identity of the unknown quantity. Furthermore, Change and Compare problems are also subdivided depending on the direction of the event (increase or decrease) or tionship (more or less) respectively.

Referring to the second assumption, Greeno c.s. developed a theoretical model in which semantic processing is considered to be the most important component of a skilled solution process. According to that model, one first constructs a global, internal representation of the problem in terms of sets and set relations using semantic problem schemata. On the basis of this internal representation, the problem solver then selects and executes an arithmetic operation to find the unknown quantity in the problem.

Furthermore, Greeno c.s. (1983) identified three different levels of problem-solving skill, each associated with a distinct pattern of correct answers and errors on the problem types within the three main categories. They also developed computer models that simulate these levels of performance. The main difference between those levels relates to the way in which problem information is represented. Models with more detailed semantic knowledge refer to more advanced levels of problem-solving skill, and therefore, they can solve more problems of a certain categorie.

It is important to remark that according to Greeno c.s., the main difference between good and poor problems solvers does not lie in the presence or the absence of semantic processing respectively; poor problem solvers try to construct a semantic problem representation too, but due to their less-developed schemata, they do not succeed in building an appropriate one. This view contrasts with another possible explanation for the errors of poor problem solvers, namely that they are mainly due to the absence of a semantic processing stage. According to this latter view those children apply a rash and impulsive style of responding, in which the selection of the arithmetic operation is not based on a careful reading and a thorough analysis of the semantic relations between the known and the unknown elements of the problem, but on superficial strategies such as always adding the two given

numbers or looking for keywords in the problem text (see e.g. Goodstein, Cawley, Gordon & Helfgott, 1971).

Eye-movements and cognitive processes

The use of eye-movement registration to unravel children's internal processes when solving math problems, is a recent development. In our research we took as a starting point the two fundamental assumptions formulated by Just and Carpenter (1987) as a result of their work in the area of reading, namely the immediacy and the eye-mind hypothesis. In terms of children's word problem solving, the immediacy hypothesis implies that a pupil does not postpone the interpretation of a word or a sentence until he has read the whole problem, but instead tends to process each element from the first time when the cognitive system has access to it. The eye-mind assumption implies for example that when a pupil is fixating words we assume that he is mentally processing them, and that when he is fixating the numbers, he is 'doing' something with those numbers (e.g. calculating).

METHOD AND HYPOTHESES

Subjects, tasks and procedure

Twenty second graders (10 high and 10 low-ability pupils) participated in our study. These children were selected among the whole sample of second graders of a local school, on the basis of their scores on a paper-and-pencil test consisting of a series of one-step addition and subtraction word problems. They were also administered test for technical reading and computational skills.

During the eye-movement session each child had to solve 16 one-step addition and subtraction word problems: half of the problems had a simple semantic structure; the other half a complex one. The simple problems were Change 2 or Combine 1 problems; the complex tasks had a

Change 5 or a Compare 6 structure (Riley e.a., 1983). These 16 items were formulated and presented in a way that allowed us to control for all possible task variables that were not central to the present investigation, such as the amount of sentences, words and characters in the problem, the complexity of the grammatical structures, the technical reading difficulty of the names of the persons and the objects in the problem, and the size of the given numbers.

The word problems were presented on a tv-screen. While the pupils read and solved the problems, their eye-movements were registered with DEBIC 80, a system that uses the "pupil center-corneal reflection" method as its measurement principle. Every 20 milliseconds the system registrates the X- and Y-coordinates of the subject's point of regard. This raw material was subjected to a reduction program, that transforms these data into a series of consecutive fixations with a particular duration and location. These fixation data were the basis for calculating the dependent variables, the most important ones being the proportion of the total fixation time spent on the words and on the numbers in the problem, and on those parts of the visual field that did not contain any problem information. However, as the fixation time spent on those "empty" fields was less than 5% of the total fixation time, we will neglect those fixations. Consequently, we will describe our hypotheses and our results as if the total fixation time was the sum of the fixations on the words and on the numbers, the latter being the complement of the former.

Hypotheses

The first hypothesis was that <u>problems with a complex semantic structure will elicit a larger proportion of the total fixation time on words than simple problems</u>. The argumentation underlying this hypothesis can be summarized as follows: problems with a more complex

semantic structure elicit more complex understanding and reasoning processes before the computational activities with the given numbers; this is reflected in more and longer fixations on the non-numerical elements of the problem text.

Second, we expected that high-ability children will spend a larger proportion of their total fixation time on words than their low-ability peers. The basic assumption underlying this hypothesis is that constructing and manipulating a global problem representation is a major characteristic of a skillful solution process; low-ability children, on the other hand, will immediatly jump into calculations without trying to really understand the problem situation, or even without reading the whole problem. This latter assumption is based on the available literature on children's use of superficial solution strategies on the one hand (see e.g. Goodstein et al., 1971) and on the results of our own exploratory study on the other (De Corte & Verschaffel, 1987). As said before, this hypothesis is incongruent with Riley et al.'s (1983) theoretical analysis of skilled and unskilled word-problem solving.

Finally, we also expected an interaction between problem complexity and problem-solving ability. More precisely, it was predicted that the difference between the simple and the complex problems in the proportion of the total fixation time on words, will be greater in the high-ability than in the low-ability group. This interaction hypothesis is argued as follows. As high-ability children's solutions are assumed to be "semantic" in nature, the complexity of the semantic structure will strongly determine the speed at which they succeed in building an appropriate problem representation. If low-ability children's choices of an arithmetic operation are based on stereotyped, superficial strategies that do not take into account the semantic structure of the problem, their eye-movement patterns for

simple and complex problems will be much more alike.

To test the hypotheses mentioned above, the dependent variable was subjected to an analysis of variance with the semantic complexity of the problem and the pupil's problem-solving ability as independent variables (2*2 split-plot design). Because significant correlations were found between the pupils' scores on the word-problem-solving pretest on the one hand and their reading ability ($r=0.46$, $p>0.05$) and computational ability ($r=0.67$, $p>0.01$) on the other, we also carried out analyses of covariance with reading or computational scores as covariates.

RESULTS

First, analysis of variance revealed a significant effect of problem structure on the proportion of fixation time on words ($F(1,18)=5.35$, $p<0.05$). While 56 % of the fixation time was spent on the words in simple problems, this percentage increased till 61% for complex ones. This is in accordance with our hypothesis.

Second, high-ability pupils tended to look proportionally more at the words (61%) in the problems than their low-ability peers (56%). Although these percentages were in line with our second hypothesis, the effect of problem-solving ability did not reach the 5% significance level, neither with reading scores, nor with computational scores as covariate. Finally, we did not find an interaction effect.

CONCLUSIONS

The hypotheses of our study were only partially confirmed. Problem complexity had a significant effect on the proportion of fixation time on words. Since the simple and the complex problems in our study differed only with respect to their semantic structure this finding provides additional support for the hypothesis that the construction of

an appropriate problem representation is a crucial component of skilled word problem solving. On the other hand, we did not find a significant effect of problem-solving ability. This finding is incongruent with the frequently heard statement that low-ability children's bad performances on word problems are mainly due to the application of superficial strategies, such as always adding the given numbers or looking for particular keywords. An alternative explanation that fits better with the present eye-movement data, is that the low-ability children's failures are not the result of the absence of a semantic processing stage, but of their faulty semantic analysis, which in turn can probably be attributed to a lack of sophisticated conceptual knowledge such as semantic problem schemata. This latter explanation is in accordance with the theoretical analysis of skill in word problem solving by Riley et al. (1983).

NOTE

(1) L. Verschaffel is a Research Associate of the National Fund for Scientific Research, Belgium

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The lesson - A preconceptional Stage

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The notions of preconceptional and conceptional stages are discussed. It is claimed that many students while learning mathematics are in a preconceptional stage. Namely, it is not that they have wrong ideas about the mathematical notions. They have no idea at all. In spite of that, they have to perform on mathematical tasks and to react to their teachers' questions. Thus, they are involved in a meaningless communication. This common behavior gets almost no attention in the mathematical education research which focuses mainly on misconceptions. The preconceptional stage deserves research efforts. Before clearing the misconceptions, which are part of the conceptional stage, we should clarify to ourselves what makes the transition from the preconceptional stage to the conceptional stage possible.

On February 20th, 1951, a lesson was given to a young girl by a middle aged professor. The lesson ended with a homocide. The teacher assassinated the student. This was unavoidable. The quality of communication between the teacher and the student was unbearable. The only way to save the profession of teaching was to kill the student.

Fortunately enough, this happened only on the stage of the pocket theater in a play by Eugene Ionesco. However, the phenomena of that lesson occur every day, in every school in almost every mathematics class. Teachers and students are engaged in a meaningless communication.

There are several ways to explain why communication is so bad. Of course, there is Ionesco's view that meaningful communication between human beings is impossible. Mathematics educators, as such, cannot accept this. They believe that meaningful communicaton, at least at the domain of mathematics, is possible and if it does not occur then there are reasons for it. The reasons that mathematics educators point at in order to explain communication failures are of two kinds:

1. Misconceptions. 2. Unappropriate mathematical level of the student.

The last approach can be considered as the level theory (See for instance van Hiele, 1987). As to misconceptions, the assumption is that the student handles meaningfully the mathematical tasks imposed on him. By "meaningful" we mean that the student associates certain meaning to the mathematical notions involved in the task. This meaning is not necessarily the correct meaning but can be considered as reasonable if you are tolerant and sensitive enough. This is contrary to the situation where the student does not associate any meaning to the notions involved. On the other hand, he does not refrain himself from reacting to the task. This we call a meaningless behavior. When misconceptions are involved, the student associates to the mathematical notions a meaning which is different from the meaning associated to them by the mathematical community. Thus, the task of the mathematics educator is to explore the misconceptions, to understand why they were formed and to suggest ways to overcome them. Comparing now the misconception theory to the level theory, it is not clear how they are related. Assume a student at the k-th level of a certain mathematical domain performing on a k+j-th level task ($j>0$). The level theory predicts that a success at such a situation can be only incidental. But how should we interpret the students' behavior? Is it a meaningful behaviour resulting from misconceptions or is it a meaningless behaviour determined by unknown factors which should be investigated? We are not sure that level theory has made itself clear about this point. Our impression is that both meaningful and meaningless behaviors can occur when you are not at the appropriate mathematical level of the task. For instance, assume that a student is at the first level in geometry according to van Hiele theory (van Hiele, 1987) and he has to deal with rectangles. For him the concept of the rectangle is a collection of pictures that he saw in the past.

These are usually pictures of quadrilaterals that have four right angles and their adjacent sides are not congruent. At the second or third level of van Hiele theory, a rectangle is by definition a parallelogram with a right angle. From the first level student's point of view, a square is not a rectangle. From the teachers point of view, a square is a rectangle. The meaning the teacher assigns to "rectangle" is different from the meaning the student assigns to it. Hence, this is a misconception. On the other hand, if a student is in the first van Hiele level in geometry and he has to prove a certain geometric claim then his behavior will probably be meaningless.

The general impression is that the main focus of the psychological research in mathematics education is on misconceptions and not on meaningless behavior. This is quite natural. First, misconceptions explaim many of the students' mistakes and difficulties. Second, the stage of misconceptions is a stage where there is a good chance of learning. The fact that you understand your student's behavior, that you can discuss it with him and that you know what modifications in his thought are needed in order to reach the correct concept, all this is a good starting point for learning. On the other hand, when somebody is in the meaningless stage, the situation is much harder. You see somebody who acts in a meaningless way, but you cannot tell what makes him act the way he acts. In addition to that, you do not know what to do in order to make him understand the notions involved. Usually, you repeat almost the same words you uttered to him earlier, perhaps more slowly. Nevertheless, it is impossible to ignore the meaningless stage. Let us consider as one the stage of misconceptions and the correct conceptions and call it the conceptional stage (or the meaningful stage). The other stage will be called the preconceptional stage (or the meaningless stage). Our question is the following: at a given moment of a common mathematics lesson, what percentage of the students is in the conceptional stage and what percentage of them is in the preconceptional stage? For a mathematics teacher the answer to this question is critical. It is an invaluable information. Unfortunatelly, there is no satisfactory method to answer this question. Of course, one can use quizzes. But quizzes show knowledge or lack of knowledge about a certain restricted topic. From common quizzes it is very hard to tell whether a student is or is not at the conceptional stage.

Interviews are a very effective means, but you cannot intrview the entire class. Thus, the above information is usually non-available, especially in big classes where the teacher talks most of the time and after that the students are asked to solve problems similar to those which were solved on the blackboard. It seems that many teachers believe that a good percentage of their students is in the conceptional stage. Otherwise, how can they teach?

However, the moment you start interviewing the students you realize how many of them are not in the conceptional stage. In this paper we would like to illustrate this. As we explained above, this cannot be proved statistically. Our claim is that the quality of communication we have in the following interview is typical to many mathematics lessons and many teacher-student interactions.

We have documented and analysed over twenty long interviews with college students but finally we decided that fiction is more convincing than reality. Fiction has all the elements of reality but in a clear concentrated form. Thus, we have chosen Ionesco's lesson mentioned above and we will use it in order to characterize the teacher-student communication the way we see it. We are using Watson's translation (Ionesco, 1958) where we replaced "pupil" by "student". The teacher is a middle aged professor and the student is an eighteen year old girl. After posing some addition exercises to the student which were solved correctly, the teacher assumes that she is ready for the subtraction exercises. (p.11 - p.12).

Professor: Let's try subtraction. Just tell me, that if you are not too tired, what is left when you take three from four? Student: Three from four?...three from four? Professor:Yes, that's it. I mean to say, what is four minus three? Student: That makes...Seven? Professor: I am extremely sorry to have to contradict you, but three from four does not make seven. You're muddling it up. Three plus four makes seven, take three away from four and that makes?...It's not a question of adding up, now you have to subtract. Student: (struggling to understand) Yes...I see... Professor: Three from four, that makes...How many...how many? Student: Four? Professor: No, Mademoiselle. That's not the answer. Student: Three then? Professor: That is not right either, Mademoiselle...I really do beg your pardon...It does not make three...I am terribly sorry...Student: Four minus three...three away from four...four minus three? I suppose it wouldn't make ten?

Professor: Oh, dear me, no Mademoiselle. But you mustn't rely on guesswork. You must reason it out...

The above dialogue might sound absurd to the common ear but a mathematics educator can see here some typical elements. The student has no idea about subtraction. For her, "to take three from four" is a meaningless expression. However, she must react to the question. A common reaction is an attempt to gain time by repeating the question. By this, she might gain also some hints. The professor realizes this and he is ready to give such a hint. His method is rephrasing the question. The result is quite typical: a phrase which is harder to understand than the original question. This is because it uses a new notion ("minus") which is unfamiliar to the student. At this stage, although the question is still meaningless for her, the student has no alternative but answering the question. A common way of doing it in such a situation is regressing to a previous familiar situation in which she was successful, to ignore the differences and to act as if the present situation were the previous situation. This can be, undoubtedly, considered as a preconceptional stage. The teacher is quite sensible to the student's behavior. He explains to her what caused her mistake and hopes that this will help. But of course, it doesn't because the question is still meaningless for the student. At the same time, the pressure to answer does not stop and therefore the only alternative now is guessing (note that to this student it never occurs that she could have said "I do not know". She is not the only one it never occurs to her). Guessing is very common practice in mathematics learning and it is typical to the preconceptional stage. Guessing has its own rules and it deserves a special study in mathematics education research.

Here, for instance, the student is trying first to repeat one of the numbers mentioned in the question. When this fails she tries the second one. Only after that she tries a wild guess ("ten") and then she is stopped by the teacher. Note that the strategy of repeating the numbers mentioned in the question could have been successful if the question were: which number is the greater? three or four? (a question which is posed to the student a little bit later, on p. 12). Our professor, being aware of the student's guessing, tries to construct in her some meaning for subtraction. In order to do that, he invites her to perform some mental acts. Elsewhere these were called imagination acts (see Vinner & Tall, 1982). Here another major problem is involved.

The teacher believes that the student is capable of performing these imagination acts, but this is not necessarily the case. In geometry, for instance, we very often say "continue this segment infinitely to both sides in your mind" or "think of a point which has no width and no length". How do we check whether our students can do it? There is a good chance that our student won't be able to do it as illustrated by the following dialogue
(p.14 - p.15).

Professor: ...If you had two noses and I'd plucked one off, how many would you have left? Student: None. Professor: What do you mean, none? Student: Well, it's just because you haven't plucked me off that I've still got one now. If you had plucked it off, it wouldn't be there any more. Professor: You did not quite understand my example. Suppose you had only one ear. Student: Yes, and then? Professor: I stick on another one, how many would you have? Student: two. Professor: Good. I stick yet another one on. How many would you have? Student: Three ears. Professor: I take one of them away...how many ears...do you have left? Student: two. Professor: Good. I take another one away. How many do you have left? Student: Two. Professor: No. You have two ears. I take away one. I nibble one off. How many do you have left? Student: Two. Professor: I nibble one of them off. One of them...Student: Two. Professor: One! Student: Two! Professor: One!! Student: Two!! Professor: One!! Student: Two!! Professor: One!! Student: Two!!

As we said above, the student is asked here to perform some imagination acts. In some of them she succeeds and in some of them she fails. It is even hard to characterize those in which she fails versus those in which she succeeds.

It seems that she specially fails to imagine strong counter reality situations. She cannot imagine herself with two noses. Therefore, she fails to answer the question about the two noses and the one being plucked off. On the other hand, she is able to imagine herself with one ear, sticking on another one and another one. But she is uncapable of performing in her mind the inverse procedure. This looks strange but mathematics teachers are familiar with the phenomena. The inability to perform imagination acts is another characteritic of the preconceptional stage. This is related somehow to hypothetical thinking required very often from mathematics students, an impossible mission in many cases (Professor:...You have ten fingers. Student: Yes, Sir. Professor: How many would you have if you had five of them?

Student: Ten, Sir. (p.15))

All the above examples can be considered as examples from the preconceptional stage. Ionesco's lesson does not lack interesting examples that can be considered as examples from the misconceptional stage. Because of space problem, we will not discuss them here. We would only like to note that not always there is a clear distinction between the preconceptional stage and the misconceptional stage (which is part of the conceptional stage). This fact does not have to be a reason to reject the distinction. There are many distinctions that do not have clear cuts, like the distinction between good and bad, clever and stupid etc., yet they are very useful distinctions in most of the cases.

The fact that a student is in a preconceptional or misconceptional stage does not enable him or her a meaningful learning. The only alternative left to him is rote learning. This is illustrated by the following (p.17 - p.18): Professor:...How much is three billion, seven hundred and fifty five million, nine hundred and ninety-eight thousand two hundred and fifty one, multiplied by five billion, one hundred and sixty-two million, three hundred thousand, five hundred and eight? Student: (very rapidly) That makes nineteen quintillion, three hundred and ninety quadrillion, two billion, eight hundred and forty-four billion, two hundred and nineteen million, a hundred and sixty-four thousand, five hundred and eight...

Professor: (astonished) No. I don't think so. That must make nineteen quintillion, three hundred and ninety quadrillion, two trillion, eight hundred and forty-four billion, two hundred and nineteen million, a hundred and sixty-four thousand, five hundred and nine...Student: No...five hundred and eight...Professor: (growing more and more astonished and calculating in the head) Yes...you are right, by Jove...Yours is the correct product...(Muttering unintelligibly)...quintillion, quadrillion, trillion, billion, million...(distinctly)...a hundrad and sixty-four thousand five hundred and eight...(stupefied) but how did you arrive at that, if you don't understand the principles of arithmetical calculation? Student: Oh, it is quite easy, really. As I can't depend on reasoning out, I learnt of by heart all the possible combinations in multiplication. Professor: But the conbinations are infinite.

Student: I managed to do it, anyway.

As we claimed above, Ionesco's fictious lesson is almost an accurate mirror image of a great deal of the practice in mathematics education. We know very little about the preconceptional stage. Therefore, we also know very little about the methods of carrying forward our students from the preconceptional stage to the conceptional stage. On the other hand, we do not believe in miracles. The current situations in mathematics teaching is not only a result of bad pedagogy. It is also a result of posing too many mathematical topics which are beyond the mathematical abilities of great percentage of the students. Being more aware of the preconceptional stage and of the fact that so many students are stuck there might have some influence on the curriculum as well.

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AN ANALYSIS OF THE EMOTIONAL ACTS OF YOUNG CHILDREN WHILE LEARNING MATHEMATICS

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Emotion acts of children as they engage in mathematical activity are analyzed in terms of the children's cognitive appraisals of situations which, in turn, are based on the classroom social norms. In the classroom we observed, the teacher and children mutually constructed social norms that fostered generally favorable emotional acts which, in turn, sustained and perpetuated the operative social norms. Examples from the classroom illustrate the relationship between the social norms and the children's emotion acts.

Introduction

During a teaching experiment in a second grade mathematics classroom we observed an unusually positive emotional tone which seemed to contribute substantially to the children's learning of mathematics. Since doing mathematics is thought by many, including many mathematics educators, to be associated with negative emotion (McLeod, 1985), we set out to analyze our observations. The discrepancy between our observations and the commonly expressed view is heightened since negative emotion is associated especially with those mathematical activities that involve problem solving (McLeod, 1985) and in the project classroom our approach, which was based on the constructivist theory of learning, was that all mathematics, including the so-called basics such as arithmetical computation, was taught through problem solving. The primary instructional strategies used in the project classroom were small group problem solving and whole class discussion. (For a clarification of what we mean by problem solving see Cobb, Wood, and Yackel, in press, and Cobb, Yackel, and Wood, in press.)

Our analysis focuses on that aspect of emotional experience which involves cognitive appraisal of a situation, emotional act, as opposed to the physiological arousal, emotional state. Since they involve cognitive appraisal, emotional acts have an underlying rationale which, in turn, is based on the social order within which the mathematical activity takes place. Accordingly, our analysis necessarily includes an analysis of the social norms that were operative in the classroom and how they were mutually constructed by the children and the teacher. We argue that it was because the teacher and children established social norms that constrast sharply with those of typical classrooms that we observed generally desirable emotional acts.

Theoretical Framework

The theoretical framework that forms the basis for our analysis is that of the constructivist approach to emotion. According to this approach emotions are viewed as "socioculturally constituted" (Armon-Jones, 1986a) and involve cognitive appraisal or interpretation (Bedford, 1986; Armon-Jones, 1986a). In this approach attention is not focused on physiological states of the individual(s) involved but on the interpretation the individual gives to the situation that causes him/her to judge it as desirable or undesirable. In this sense emotion acts involve cognitive appraisal "in that they depend upon the agent's knowledge and his capacity to judge and compare" (Armon-Jones, 1986a, p. 42). The cognitive appraisal, in turn, is based on what is and is not acceptable or appropriate in the given culture. From this perspective "our capacity to experience certain emotions is contingent

upon learning to make certain kinds of appraisals and evaluations ... [I]t is learning to interpret and appraise matters in terms of norms, standards, principles and ends or goals judged desirable or undesirable" (Pritchard, 1976, p.219). For example, for an individual to feel embarrassment he/she must interpret the situation as one in which he/she has failed to act in accordance with the expectations of the local social order. Specifically, if, in a classroom it is expected that when a child responds to a question the only acceptable responce is a correct answer to the question posed, then it is appropriate for a child to feel embarrassed when he/she gives an incorrect answer. In contrast, if incorrect answers are routinely given and discussed along with correct answers, it is not appropriate for a child to feel embarrassed when he/she given an incorrect answer. As this example illustrates, the emotion acts of children while engaged in learning mathematics are influenced by the social norms that are operative in the classroom.

This is not to say that children enter a classroom that has a ready-made, pre-existing set of established social norms. Social norms are not static prescriptions or rules to be followed but are instead regularities in the process of social interaction (Voigt, 1985). These regularities are mutually constructed by the participants in the course of their interaction. In this view, meaning is negotiated by the teacher and the students in the course of their social interactions. In this regard we follow Blumer (1969) when he said that " ... human beings act toward things on the basis of the meanings that the things have for them. ... the meaning of

such things is derived from, or arises out of, the social interactions that one has with one's fellows" (Blumer, 1969, p.5). The norms are, from the observer's perspective, continually reconstructed in concrete situations and do not exist apart from the interactions that give rise to them. As in any collective body "there is one group or individual who is empowered to assess the operating situation and map out a line of action" (Blumer, 1969, p.56). In our case it was the classroom teacher who guided and directed the construction of the norms.

From the psychological perspective, the norms come into being through the expectations that the teacher and children have for each other and the largely implicit obligations that they have for themselves in specific situations (Voigt, 1985). Emotion acts, because they involve interpretations based on the social norms, serve the function, therefore, of sustaining and endorsing the norms from which they derive (Armon-Jones, 1986b). Conversely, socially inappropriate emotional acts indicate either that the student has misconstrued the situation or that the student's beliefs are incompatible with social norms that are acceptable to the teacher and other students. Because emotion acts are cognition-based these acts are open to criticism by reference to the norms. Further, there does not have to be evidence of a specific emotion before that emotion can be ascribed (Armon-Jones, 1986b). For example, a student can be told that he "ought to" feel a certain way in a given situation, such as, that he ought to feel pleased when he has persisted in solving a challenging problem. In this way interpretations that are deemed

appropriate in light of the social norms can be brought to the attention of participants in the local society and can serve to sustain the norms as well as to endorse certain emotion acts as appropriate.

Children's beliefs about the nature of mathematics, and their own and the teacher's role also influence their interpretations of situations and hence their emotion acts. These are, in turn, influenced by the social norms. For a detailed discussion of the relationship of beliefs to emotion acts and social norms see Cobb, Yackel, and Wood, in press.

Examples of the Relationship of Children's Emotional Acts to Social Norms

In the project classroom social norms included that students cooperate to solve problems, that meaningful activity was valued over correct answers, that persistence on a personally challenging problem was more important than completing a large number of activities, and that while working in pairs students should reach consensus as they completed the instructional activities. The mutual construction of the meaning of each of these through interaction of the children with each other and of the children with the teacher served at the same time to indicate whether or not an emotion act was appropriate. As an illustration consider the following example.

The following episode occurred at the beginning of a class discussion that followed small group work. One pair of children volunteered that they had spent the entire twenty minutes allocated to group wrok on a single problem.

Kara and Julie: Because at first we didn't understand.

Teacher: How did you feel when you finally got your solution?

Kara and Julie: Good.

Kara and Julie's excitement at having solved the problem was indicated by the way in which they jumped up and down as they talked with the teacher. By calling the attention of the entire class to this incident the teacher endorsed the girls' construal of the situation as one warranting excitement and simultaneously perpetuated the social norm that persistence on a personally challenging problem is more important than completing a large number of activities.

In the same way, when one child displayed anger because a child from a neighboring group told him the answer to a problem he was trying to figure out for himself, the teacher affirmed the rationale for his anger. In this way she indicated that his interpretation of the situation was warranted and in doing so simultaneously reaffirmed that in this classroom meaningful activity was valued over correct answers.

In the first few weeks of the school year children often interpreted situations in ways that were consistent with their prior school experience but were not compatible with the teacher's expectations for the children's activity in this classroom. She then initiated a conversation in which she talked with the children about her expectations and how she as a specially empowered member of the group (Blumer, 1969) assessed the situation. For example, during one lesson at the beginning of the school year Peter went to the front of the class to explain his solution to a problem. In the course of

his explanation he realized that his answer was wrong, looked down at the floor and then quickly returned to his seat. The teacher, realizing that peter construed this as a situation that warranted embarrassment said, "That's okay Peter. It's all right. Boys and girls even if your answer is not correct, I am most interested in having you think. That's the important part. We are not always going to get answers right, but we want to try." By telling the children how she interpreted the situation the teacher expressed her expectations for the children. Simultaneously she expressed her belief that it was more important in this class to think about mathematics than to get correct answers.

We have presented examples which illustrate how emotions are socially constituted through interpretations of situations and how they function to sustain and perpetuate the local social order, in particular the social norms that operated in the project classroom. The social norms that were established in the project classroom differ sharply from those of typical classrooms. It was for this reason that children were able to interpret situations when they were engaged in mathematical activity in ways that made positive emotion acts the standard rather than the exception.

The implication of this work is that teachers can promote positive emotional experiences for children when they engage in mathematical activity by guiding the construction of classroom social norms which are conducive to mathematical problems solving.

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THE USE OF GRAPHS AS VISUAL INTERACTIVE FEEDBACK WHILE CARRYING OUT ALGEBRAIC TRANSFORMATIONS

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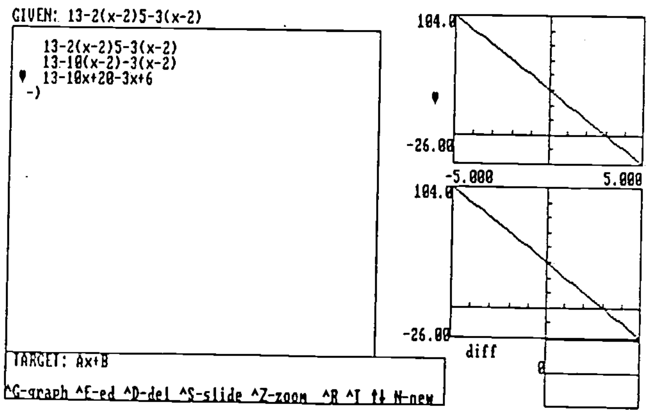
Errors in performance of algebraic transformations by algebra beginners are a common phenomenon. Among many reasons for the difficulties there is one which had been investigated and described in this work: the absence of meaningful feedback mechanism that could not only immediately identify a mistaken process but could also reflect the algebraic situation and the student's action. The RESOLVER, a computer environment had been used with algbera beginners of different ability to identify the impact of visual linked multiple representations while procedurally performing algebraic simplifications and transformations. The article concentrates on the effect of a feedback mechanism, on the effect of the visualization of algebraic expressions on the performance and on the ways in which students analyze their own mistakes.

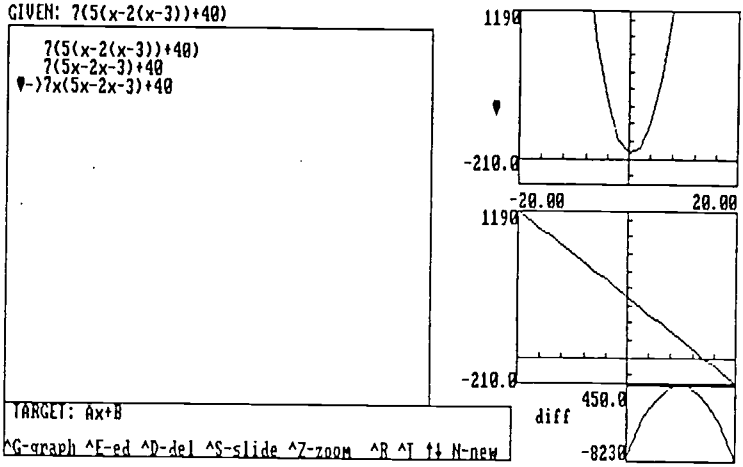
A major part of the learning of algebra for beginners is devoted to the learning of techniques to transform expressions. The literature reports on difficulties in carrying out algebraic transformation; difficulties that are rooted in the misinterpretation of the major essences of algebra (Booth, Davis et. al 1978, Matz 1982, Thompson 1987). An obstacle blocking the way to carry transformations is the lack of checking mechanism to use as a feedback while simplifying. The only available mechanism is the numerical checking; students may substitute numbers and compare the values of transformed expressions (Lee & Wheeler 1985). Most students even do not bother to do that. The introduction of computers into the secondary school algebra curriculum could affect the learning to transform expressions in several ways. A Computer's uses range from a tutor which helps students to carry the right simplification (Brown 1985), through computerized tools which direct students to understand the deep structure of algebraic expressions (Thompson 1987) to the use of programs that could carry symbolic transformations for the user such as MuMath (Fey 1984, Heid 1988). In parallel, several studies have been carried out to observe the impact of computers on another topic in the algebra curriculum: the investigation of functions. As part of a recent study (Yerushalmy in prep.), we studied the effect of linked multiple representation software on students' performance including their technical performance within the traditional Algebra 1 curriculum. One of the results suggests that students presented a rich repertoire of visual arguments, but they did not link them to parallel results reached by symbolic and numerical procedures. The picture one can draw from the studies mentioned above and others is that

multiple representation tools could adequately and successfully serve algebra students and help them understand major concepts; however, methods of using such tools to enforce procedural algebraic understanding and performance are yet to be established. The questions investigated in this work were (1) How do learners handle immediate identification of a false or correct step by the computer, while transforming algebraic expressions? Do they need the judgmental feedback to verify their action or do they use the computerized feedback to try and understand why they got the truth or the false action? (2) What are possible roles of graphs serving as an automatic display of qualitative feedback while transforming algebraic expressions?

USING THE RESOLVER: HOW DOES THE SOFTWARE WORK?

The RESOLVER, the software used in the experiment is an environment that promotes students' experimentation and verification while transforming algebraic expressions. The program allows the user to carry out a process of transformations between two expressions and allows the user to indicate the effect of each transformation on the expression. The RESOLVER (designed by Schwartz and Yerushalmy) is mainly an algebraic notepad which allows the input of any expression whose syntax is acceptable in algebra. It provides in parallel three graphs for each transformation: one graph displays the original expression, one displays the current transformed expression and one presents the difference between the two expressions. Since any legitimate operation of transformed expressions does not affect the graph of the expression, the graph of differences provides qualitative and quantitative information about the correctness of each step. Here is an example of two transformations: one is correct and the other is incorrect.





There are multiple goals to the difference graph. First it allows an easier diagnosis of the comparison between the two graphs; second, and more important, the graph of the difference is an indicator of possible terms that are not correct. For example: a parabola in the difference window points to a mistake in at least the X^2 term.

SAMPLE AND METHOD

Seven students participated in an experiment, each received about five hours of work. The participants were: A seventh grader of average ability in mathematics who had recently first learned to transform and simplify expressions; four eighth graders, average and above average ability, who had concluded the topic of functions and graphs using the Function Analyzer (Schwartz & Yerushalmy 1988); two ninth graders from the very low track in their school. All students volunteered to participate and to stay after school hours for the experiment. Each of the three different populations worked separately with a researcher. During each of the meetings students were asked to transform a file of expressions, either on paper or using the RESOLVER. Each file included expressions from various levels of complexity. The level of difficulty had been matched to the previous knowledge of each group and each file included transformations which had been learned in the classrooms (such as computations and grouping) and others which were assumed be new (such as multiplication of binomials and factoring). The experiment did not include teaching intervention of any kind. Each session was audiotaped and all the algebraic actions were recorded on paper.

THE IMPACT OF QUALITATIVE FEEDBACK

The data gathered from the experiment were analyzed with three aspects: the use of feedback of any kind while transforming expressions; the visualization of the expression using the graph, and the linkage between the analysis of the differences between graphs and between the expressions. Here are a few descriptions from this work.

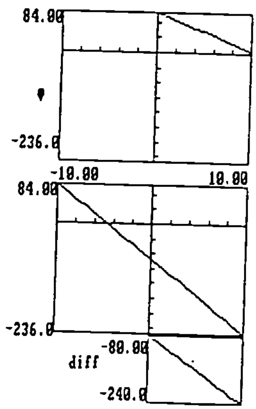
The case of S: an experiment with a seventh grader.

S had learned graphic representation of numerical information in an introductory chapter to statistics. With the RESOLVER, there are two options to check the correctness of simplified form. First, by comparing the result to the target expression and second by using the interactive feedback to each step of the transformation process. S did not use the target feedback to check his answers. In all cases he was very careful to try and correct each error immediately and was not bothered at all to reach the certain format specified as a target. However, the existence of an automatic constant feedback that informs of mistakes affected S, both positively and negatively. Since he was so anxious to get the "good" feedback at each step he frequently gave up on solving while he could not get rid of mistakes. On the other hand, the existence of an interactive immediate feedback encouraged S to conjecture and experiment while simplifying. Facing a problem he often said:"I'll write what I think is true and then we will see". On another occasion S evaluated the dimension of his mistake by an evaluation of the numerical values.

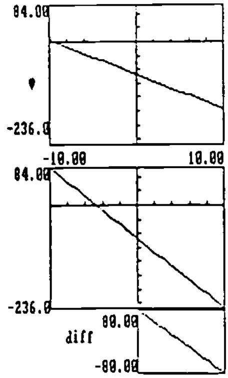
EXPRESSION | DIFFERENCE GRAPH

4-8(x+5)2

4-8x+80



S: I have done an awful mistake....the numbers on the difference graph are so large!



He then tried another transformation:

4-8x-30

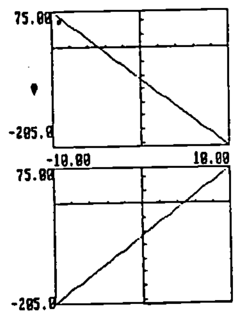
S: That is better, the numbers are smaller now.

At other times he used the graph to analyze his next step and here are two examples. In the first instance S was able to correct a mistake, caused by an incorrect multiplication, by the analysis of the direction of the graph.

EXPRESSION | CURRENT EXPRESSION'S GRAPH

3-7(9-2x)-5

3-63-14x-5



S: I watched the graphs and they looked as in the <u>opposite</u> directions. It reminds me that when you open parenthesis with a negative sign the expression gets the <u>oppposite</u> signs so I changed the expression to: 3-63+14x-5.

In the following case, S made a technical error again and was able to locate it by observing the difference graph. In all cases, S evaluated the difference window as an entity and not as a product of difference between the two graphs. For him, it was an independent entity and, despite all his enthusiasm to understand the graphs, he never asked how exactly the difference graph is connected to the two main graphs.

(1) 3(2x-8) - (3x-1)4 + 5(x+2)

(2) 6x - 24 - 12x + 1 +5x + 10

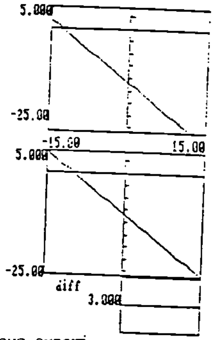
(3) (-x) + (-13)

S simplified the expression on paper and then checked his answer using the RESOLVER:

EXPRESSION

$(-x) + 3$

DIFFERENCE GRAPH



S: I know that I have to add 3 but where does it come from?

An experiment with experienced eighth graders.

The special purpose of the work with that group was to study if and how the previous experience with functions and graphs affected the performance in algebraic transformation. Also it was necessary to explore if above-average students have any need to get feedback on their technical performance or whether such feedback is a waste for advanced students. The main difference in the work between this group and the other three students was that this group made extensive use of the graphs of all kinds; most of the excerpts include both: diagnosis of the property of the graph itself and diagnosis of differences. They analyzed the difference graph geometrically (as opposed to S who mainly paid attention to the numerical data). Here is a description of their attempts to simplify an expression (organized chronologically):

(1) The given is a cubic expression.

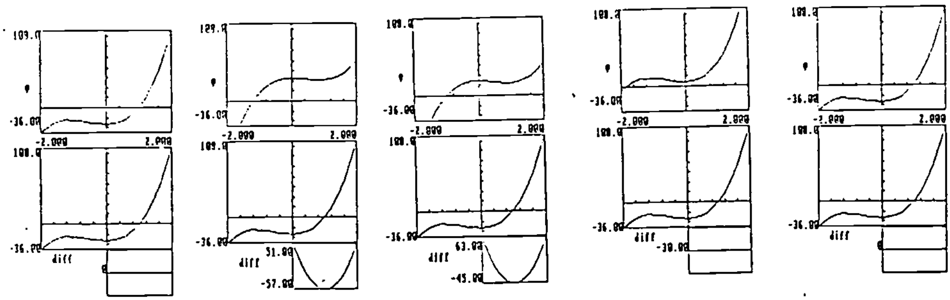
They expanded the expression while performing two mistakes

(2) The difference graph is a paravola. Should we try (-6)?

They got rid of one mistake; the difference graph shows a constant difference.

(4) It looks better now, but the difference graph is -30. The inteception is above the x-axis and it has to be below the axis.

1 2 3 4 5



9xx(x+3)–4(xx+2)3 9xxx+27–12xx+6 9xxx+27–12xx–6 9xxx+27xx–12xx+6 9xxx+27xx–12xx–24

They observed the complexity of the given task by the shape of the graph. They assumed that the more complex the graph, the more difficult the transformation. A similar reaction had been already observed while working with S. However, the graphic display did not help them to identify a mistake caused by a repeated false strategy. In such cases, students made many attempts that brought them closer to the answer, but they could not reach the requested target expression. The feedback motivated them to define criteria of the quality of estimation of an algebraic result.

CONCLUSIONS

The results confirm the hypothesis that students do not usually develop and use any strategies to evaluate their algebraic transformations and that the computer could be of help. Errors of different types appeared in the work of all participants, at all levels of ability, but they did not expect any feedback while simplifying expressions. The work with the RESOLVER developed a need for feedback; this need, however, varied with ability and knowledge of graphs and functions. Weaker students were looking more for judgmental feedback; they tended to use the difference graph as an indicator of right or wrong answers.

We had watched an increasing tendency of all participants to spend time in conjecturing about

the process instead of aiming for "the single right answer". Several occurrences led us to this conclusion; they did not hesitate to write a transformation even if they suspected that something might be wrong. Students usually did not use the Target answer, they were concerned more about the process and less about the exact format of the answer. We are able to show that there are advantages that even beginners could benefit with the integration of visual understanding and analysis of graphs with the procedural action of transforming. Beginners have developed estimation processes to evaluate the complexity of the task by the shape of the graph, used the graph interceptions to evaluate factors at the given expressions and they were able to identify the type of their error, even if they could not find the reason.

During the last few months we carried a continuous study of a group of seventh graders, using graph feedback as well as other types of feedback. The full data had not been analyzed yet, however, we find clear evidence about the willingness of students to correct their mistakes once they find that they are able to reflect on their own actions.

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IMAGES OF GEOMETRICAL TRANSFORMATIONS : FROM EUCLID TO THE TURTLE AND BACK

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Abstract. The relationship between turtle geometry and Euclidean geometry is investigated through their groups of transformations. In spite of a strong intuitive kinship between the two geometries, they are still different enough for each to illuminate non-trivial aspects of the other. From a psychological and educational perspectives, the comparison between the two geometries allows an examination of the mental images associated with each and, in particular, a comparison of different 'levels' of thinking about transformations: as moving a physical object, a point, a particular shape in the plane, or the whole plane.

Introduction. Ever since Klein's Erlanger Program, which described the various geometries through their transformation groups, there have been attempts to use transformations in the teaching of geometry (e.g. Coxford 73). Formally, geometrical transformations are defined as maps of the whole plane. This, however, is hard for novices to visualize, especially when composition of transformations is involved. Hence the effect of transformations is sometime introduced by its effect on a single point. But a single point is not enough to determine the transformation: By a well-known theorem, three non-collinear points (that is, a triangle) will be required. We can thus visualize transformations by considering their effect on a fixed triangle. Furthermore, we can represent the group operations of composition and inverse of transformations by their composite effect on the triangle.

The device of representing a transformation by its effect on a triangle, helps in bridging the gap between the rigorous mathematical

definition of transformations as maps of the whole plane, and their more intuitive representation as operations on a physical (or mental) object.

The motions of the triangle in the plane under the various transformations bring to mind the motions of the Logo turtle on the computer screen (e.g. Abelson and diSessa 81). Most of the people we have asked in an informal survey to relate turtle operations to Euclidean plane transformations, tended to identify FORWARD with translations and RIGHT with rotations. While these intuitions are quite natural, they are not, as we shall see, entirely correct.

Even though the precise nature of the relationship between turtle geometry and Euclidean geometry has not been hitherto articulated, there are quite a few projects and 'microworlds' in transformation geometry, based on this relation (e.g. Thompson 85, Goldstein 86, Edwards 88). The research reported in this article attempts to put the relations between these two geometries on sound foundations by comparing their groups of transformations. In particular, it will be shown that the group of turtle operations is isomorphic to the group of *direct* isometries (i.e. translations and rotations but no reflections). The intriguing question of what in the turtle world corresponds to *reflections* will also be discussed. Some of the psychological and educational implications of these results will be considered.

The Group of the Turtle. Intuitively, the elements of the turtle group are the turtle operations FORWARD (FD) and RIGHT (RT) with all possible inputs, and sequences thereof. For example, the sequence [FD 50 RT 90 FD 36 RT 14 FD -70 RT -56] is such an element. The group operation is composition of functions. Note that the turtle operations LEFT and BACK are also included via FD and RT with negative inputs. In this intuitive view, turtle operations are just that - physical (or computational, or mental) actions on a physical (computational, mental) object - the turtle, and their mathematical nature is unspecified. To make this intuitive approach more rigorous, we need several changes in the way we view turtle operations. First, we need to view FORWARD and RIGHT as operating on the turtle *state* rather than the turtle itself. Second, we need to view these

operations as acting on the whole (infinite) *set* of turtle states rather than on a single state.

The *turtle state*, then, consists of the turtle's position and heading. Analytically, we define the turtle state to be the triple (x, y, h), where (x, y) are the coordinates of the turtle's position in a Cartesian system, and h is its heading, measured in degrees clockwise from the north. We denote by S the set of all turtle states and call it *the turtle plane*. Given real numbers a and b, RT a and FD b are transformations of the turtle plane, defined for all (x, y, h) in S as follows.

RT a: $(x, y, h) \to (x, y, h+a)$

FD b: $(x, y, h) \to (x + b \sin h,\ y + b \cos h)$

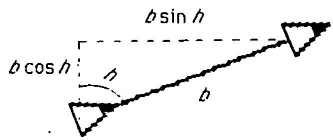

Fig. 1: FD b as a state-change operator

As mentioned above, the turtle group consists of all finite sequences of FDs and RTs. Having formally defined FD and RT as transformations of the turtle plane, we now define the *turtle group* to be the group of transformations of the turtle plane generated by the set $\{\text{FD } a, \text{RT } b \mid a, b \text{ real numbers}\}$. We shall denote the turtle group by G.

From the definition of the turtle group, it follows that the group operation is the composition of maps, the unit element is the identity transformation, the inverse of FD a is BK a (which is the same as FD $-a$) and the inverse of RT b is LT b (which is the same as RT $-b$). Two elements f and g of G are considered equal if they are equal as functions, i.e. if $f(s) = g(s)$ for all turtle states s in S.

The Two Geometries Compared. We now proceed to establish the fundamental correspondence between the two

geometries; namely, we construct an isomorphism (a one-to-one structure-preserving map) from the turtle group onto the group of direct isometries of the Euclidean plane.

The intuitive idea is simple: We correspond to the turtle a particular isosceles triangle in the Euclidean plane, and to turtle motions triangle motions. However, to make this idea precise we need to move from the correspondence between motions to a correspondence between transformations. This we do with the aid of the following two theorems:

(a) Given two turtle states, there is a unique element in the turtle group carrying one to the other.

(b) given two congruent triangles with corresponding vertices, there is a unique plane isometry carrying one to the other (Coxeter 61). (What we shall actually need is a variant of this last theorem, namely that given two congruent *isosceles* triangles, there is a unique *direct* isometry that does the job.)

We can now describe the correspondence as consisting of three steps, the middle of which is the intuitive idea mentioned above. First, we fix an arbitrary turtle state, say the HOME state (0,0,0), and view elements of the turtle group as acting on a *single* turtle (in its HOME state) via theorem (a). Second, we view turtle motions as motions of the corresponding isosceles triangle as described above. Third, we view motions of the isosceles triangle as plane isometries by theorem (b). Inverting this three-step process, we can find turtle operations corresponding to each translation and rotation. Thus our map is one-to-one and *onto*.

Note: For a more formal definition of this map and a proof that it is indeed an isomorphism between the two groups, see (Zazkis 89).

A Turtle View on Plane Isometries. We now apply the above scheme to find explicit interpretations of plane translations and rotations in turtle terms, and vice versa. For a start, we work out the plane isometry corresponding to the element FD 50 of the turtle group. First we view the effect of FD 50 on the turtle in its HOME position. Second, we view the same picture as a motion of a triangle in the Euclidean plane.

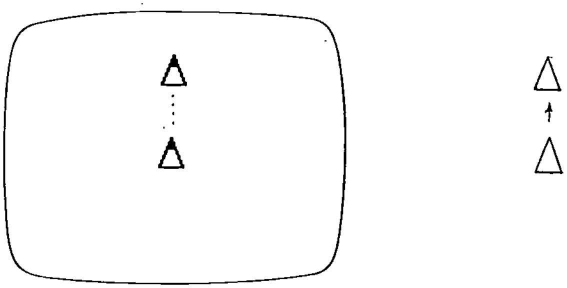

Fig. 2: Viewing FD 50 as a translation of the plane.

Finally, we determine the plane isometry that performs this motion. In this case, the resulting isometry is the translation of 50 units along the (positive) y-axis. Thus, the above isomorphism, carries FD 50 to this plane translation. Similarly, we see that the isomorphism maps all turtle operations of the form FD a onto the translations along the y-axis.

Applying next the same scheme to RT 90, we find that the isomorphism carries it to a 90-degree rotation about the origin. Similarly, the isomorphism carries all the operations of the form RT a onto the rotations of the plane about the origin.

Since the elements of the form [FD a] and RT b generate the turtle group, we are now in the position to easily calculate the plane isometry corresponding to each element of the turtle group. However, it is not yet clear what the *reverse* correspondence is. In particular, what turtle operation corresponds to an arbitrary translation? To answer, we look at a particular translation, say the 50-unit translation in the direction of 45 degrees clockwise from the positive y-axis, and consider its effect on our chosen triangle.

As can be seen from fig. 3 below, the turtle operation that accomplishes the same effect on the turtle is [RT 45 FD 50 LT 45]. Since translation shifts the triangle parallel to itself, we can expect the same from the corresponding turtle operation. In turtle terms this means that the transformation should be *heading preserving*, i.e. the initial and final headings should be the same.

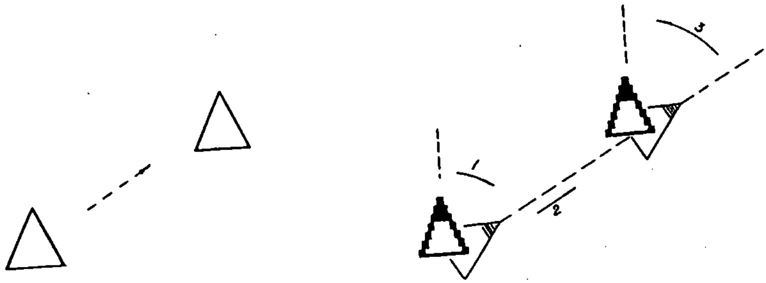

Fig. 3: (a) A plane translation (b) Translation as turtle operation

In general, there is a one-to-one correspondence between translations and heading-preserving turtle transformations. Since the turtle can only move in the direction it is facing, in order to execute a heading preserving transformations, it needs to first *turn* towards its destination, then *move* there and, finally, *turn back* the same amount (to keep the heading invariant). Thus, these transformations are characterized by their special form [RT *a* FD *b* LT *a*].

While the equivalence between these two definitions of heading preserving transformations is obvious in turtle terms, interpreting it back in the group of isometries yields an interesting insight, namely, that every translation can be obtained by conjugating a translation along the y-axis by an appropriate rotation. (Recall that the *conjugate* of B by A is the transformation ABA^{-1}. As can be seen from this example, this is a formal way to express the intuitive notion of "doing the same thing in a different place" (Leron 86).)

By a similar line of reasoning, one can show that the turtle analog of a *general* rotation (not necessarily about the origin), is a cojugate of a RIGHT by a suitable heading preserving operation. Interpreting this back in the Euclidean plane yields a decomposition of a general rotation as conjugate of a rotation about the origin by a translation.

This is a typical demonstration of how such isomorphism can be useful: Properties which are quite obvious in one system, can yield

interesting insights when interpreted through the isomorphism in the other system.

Turtle Reflections. We have now found (through the isomorphism) turtle interpretations for two of the fundamental types of plane isometries - translations and rotations. A natural question at this point is, what in the turtle world corresponds to *reflections*? More formally we ask, how can we extend the turtle group to a group isomorphic to the *entire* group of plane isometries?

In search for an answer, we turn back to our isosceles triangle - the Euclidean object analogous of the turtle - and consider the effect of a reflection on the triangle. A natural mental image of this operation is that of physically lifting the triangle out of the plane, inverting it, then putting it back into the plane. Since this operation interchanges left and right, it is called an *indirect* isometry. But the above physical description of the triangle under reflection, lends itself easily to formulation in turtle terms. We call the corresponding new turtle operation FLIP. Intuitively, FLIP can be described as "turning the turtle on its back" or, equivalently, switching its right and left. Formally, we extend the turtle state to include a fourth component - the flip-state - which can take on two values: face-down and face-up. The FLIP operation switches the values of the flip-state, leaving all other components of the state invariant. A closer look shows that FLIP actually corresponds to reflection in the y-axis, and all other reflections can be obtained from it by appropriate conjugations. We conclude that the *extended* turtle group, the one generated by FDs, RTs and FLIP, is isomorphic to the entire group of plane isometries.

Conclusion. This article gives a fresh outlook on Euclidean geometry in two ways. Mathematically, turtle geometry can be considered to give an *intrinsic* view of Euclidean geometry (Abelson and diSessa 81, p. 13). Psychologically, turtle geometry gives us new mental images with which to view plane isometries. Viewing isometries as turtle operations (through the isomorphism) brings back and legitimates our original intuitions of acting on a physical object, intuitions that are all but lost when working with transformations of th[illegible] plane. In the Logo literature turtle geometry is often

considered as alternative to Euclidean geometry. Our discussion of the isomorphism between the two groups establishes a different relationship between the two geometries: Turtle geometry as adding another, perhaps more intuitive view of Euclidean geometry, rather than replacing it. The words of Abelson and diSessa (81, p.185) are appropriate here:

"... whenever we have two different representations of the same thing we can learn a great deal by comparing representations and translating descriptions from one representation into the other. Shifting descriptions back and forth between representations can often lead to insights that are not inherent in either of the representations alone."

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A knowledge-base of student reasoning about characteristics of functions

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The present paper describes an incorporation of experience in mathematics teaching, cognitive research, and logic programming techniques. It involves abstract high level activities in secondary mathematics in relation to representation systems of functions. The paper presents an approach to the development of a computerized component which will be the core of the "knowledge model" and the "student model" of an intelligent tutoring system. Based on data gathered from nearly "perfect students" and "actual students", a two part expert system was constructed. The algebraic expert can derive the properties of a function and exhibit the inference chains of perfect students. It attempts to try to analyze student reasoning by matching the actual and perfect answers. The graphical expert takes into account student visual thinking in graphical presentation of functions by software.

The general shift in the area of curriculum development towards the emphasis of cognitive issues in the process of developing instructional systems, is the general background for the dream of intelligent tutoring systems (ITS) which have diagnostic/predictive possibilities. ITS should in principle, enable a better interaction between student and system and lead to better instruction. The first attempts in this direction are described in Sleeman and Brown (1982) and discussed in many papers and reviews. Among the strengths of these systems are the well-articulated curriculum embodied in the domain expertise and an explicit theory of instruction represented by its tutoring strategies. The weaknesses against these strengths, are inadequate models of what the student knows and how the student learns new knowledge (Wenger, 1987; Lawler and Yazdani, 1987). It seems that for several years researchers and developers have been reflecting on the first attempts. At the same time advanced techniques and theories (Kearsley, 1987; Holland, 1987) were developed, and more recently a second generation of systems is being designed and investigated with focus on the student model.

Methodology

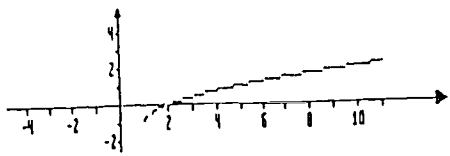
A starting point of research and development of an intelligent computer tutor is the choice of a pedagogical problem which is suited for a computational diagnostic model. Based on experience in mathematics teaching, educational studies, and research and development of instructional software, we identified three main criteria for choosing such an appropriate problem:

(a) The solution of the problem requires processes of an inferential nature rather than associations. The cognitive behavior can be hypothesized to be a knowledge-based process that is built of simple inferential processes.

(b) The required cognitive processes are not dealt explicitly by the curriculum. They involve aspects of reflective abstraction (Piaget's notion) such as generalization, interiorization, encapsulation and coordination (Ayers et al., 1988).

(c) The problem should be interesting and comprehensive, but at the same time its manageability must be ensured.

These criteria are found in the basic tutorial activities that we investigate. The tutorial activity we started with reflects our belief of effective pedagogy, and is as follows. The student is presented with a certain algebraic expression of a function and the graph of another function of the same type. For example, the algebraic expression is $y = 1 - \sqrt{x-1}$ and the graph is:



The student has to propose and justify a function characteristic which proves the non-equivalence of the graph and the algebraically presented function. The characteristics which we introduced are:

- intersection points with the axes;
- quadrants through which the graph passes;
- maximum possible domain of the function;
- range (image-set) of the function.

In the example both algebraic rule and graph have the same intersection point with the axes (2,0), same domain $x \geq 1$, and pass through quadrants 1 and 4. They differ in their range. (There may be more than one distinguishing characteristic.)

Dealing with such problems invokes some "mathematical maturity" and we believe that the development of transfer skills between the two representations can enhance the learning process. In order to develop a prototype of a computerized component which will be the core of the "knowledge model" and the "student model" we decided to gather data from nearly "perfect students" and from "actual students". A questionnaire was applied to 12th grade students (n=26) at the top level of 5 credit points in mathematics, and to 10th grade students (n=32) at the level of 4 credit points.

Knowledge Representation: Perfect Student

Introduction

The questionnaire was designed to examine how students handle problems of finding characteristics of functions given in each of the two representations, algebraic and graphical. The specific choice of items was intended to provide information on the intrinsic difficulties of students within each representation and on the role of the type of function and the complexity of a particular function relative to its type. As expected we found, for example, that the concept *image-set* causes difficulties across representation and type of function, and that the *greatest-integer* function which is constant piecewise is "pathologic" regarding all four characteristics.

The analysis of students' responses to the questionnaire gave much more than that. It stressed some salient conclusions:

- Students' reasoning is generally *logic*, that is, students can generally explain their actions by a succession of rules.
- Students' knowledge tends to be *consistent*, that is, it is possible to carry out a cross-examination of the various answers and to understand student behavior.
- Students' answers reveal a partly hierarchical system of levels in relation to characteristics of functions.

These conclusions will be illustrated by means of one example taken from the questionnaire. In the algebraic part of the questionnaire, the students have to justify their answers and we will see how some of their explanations (which were clarified during interviews) constituted the basis of the *algebraic perfect-student-expert*. For example when

asked about the range of the function $y = -\sqrt{4-x} - 2$, student A (a high achiever in Grade 12) gave the following argument:

1. There is a square root, and *I know* that the square root function of x has always positive values.
2. *So* the square root of everything is always positive,
3. There is a sign "-" before the square root, *so* the term $-\sqrt{\square}$ is negative,
4. $-\sqrt{\square}$ is negative, *so* the term $-\sqrt{\square} - 2$ is less than -2.

Student B (another "perfect" student) answered the same problem as follows:

1. as above,
2. as above,
3. *So* $\pm\sqrt{\square} \pm 2$ is greater or lesser than ± 2,
4. *So* $-\sqrt{\square} - 2$ is less than -2.

The results of the analysis of the questionnaire led us to construct an *expert system as a first approximation of the knowledge-base of students* in the realm of the four characteristics of functions. This expert system will be referred to as "The Perfect Student Expert". The currently developed expert is divided into two parts: an algebraic expert and a graphical expert. Each part can be considered as a separate student modeling implementation.

The Algebraic Expert

The algebraic expert deals with families of functions $\pm f(\pm x \pm a) \pm b$ where $f(x)$ is $1/x, \sqrt{x}$, or $[x]$. Our main concern in the construction of the algebraic expert, was to build a system preserving the cognitive structures revealed by the questionnaire. To achieve this, we represented the functions in symbolic list structures which reflect the syntactics of the expression as observed by the student:

$[E1, OP, L, E2, B]$ (or some permutation of this list).

OP is one of the operators: division_sign, square_sign, integer_sign.

L is a list $[E3, x, E4, A]$ (or some permutation of this list).

Ei are variables, whose values are ± 1, related to unary or binary plus and minus.

A, B take the values of the parameters of the function.

For the previous example, $y = -\sqrt{4-x} - 2$, the algebraic expert records the function in its list form: $[-1, sqr, [1,4,-1,x], -1, 2]$. The rules formulated for deciding the characteristics translate easily into Prolog clauses. The interpreter can be used directly for deriving the properties of a function. A user-interface presents the output of the expert as illustrated in the following for finding the image-set of $y = -\sqrt{4-x} - 2$.

Please enter a function F $\gg [-1, sqr, [1,4,-1,x], -1, 2]$.

Enter a characteristic (d(omain), r(ange), i(ntersection), q(uadrants), h(elp))$\gg$ *r*.

Expert answer:

(1) $[\_, sqr, \ldots x, \_, \_]$ *applies range* $\sqrt{x}$ *is* $[0, inf(inity)]$

(2) $[\_, sqr, [\_,\_,\_,x], \_, \_]$ *implies range* $\sqrt{\pm x \pm A}$ *is* $[0, inf]$

(3) $[-1, sqr, [\_,\_,\_,x], \_, \_]$ *implies range* $(-\sqrt{\pm x \pm A})$ *is* $[inf, 0]$

(4) $[-1, sqr, [\_,\_,\_,x], E_4, B]$ *implies range* $(-\sqrt{\pm x \pm A} + E_4 * B)$ *is*$[inf, E_4 * B]$

or

(1) $[\_, sqr, \ldots x, \_, \_]$ *applies range* $\sqrt{x}$ *is* $[0, inf(inity)]$

(2) $[\_, sqr[\_,\_,\_,x], \_, \_]$ *implies range* $\sqrt{\pm x \pm A}$ *is* $[0, inf]$

(3) $[\_, sqr, [\_,\_,\_,x], \_, B]$ *implies range* $(\pm\sqrt{\pm x \pm A} \pm B)$ *is* $[\pm B, inf]$ *or* $[inf, \pm B]$

(4) $[-1, sqr, [\_,\_,\_,x], E_4, B]$ *implies range* $(-\sqrt{\pm x \pm A} + E_4 * B)$ *is* $[inf, E_4 * B]$.

This output exhibits precisely the inference chains of the "perfect" students A and B. *Applies* stands for *"I know"* and *implies* stands for *"so"*. After instanciation of the variables the expert's response is $[inf, -2]$.

The expert can do more than that: it locates students' answers relative to the perfect student by *matching* the actual and perfect answers. For example, if a student answers that the range of the function is $y \leq 0$, the expert will try to match the answer and present the following output for the student's answer:

(1) $[\_, sqr, \ldots x, \_, \_]$ *applies range* $\sqrt{x}$ *is* $[0, inf]$

(2) $[\_, sqr, [\_,\_,\_,x], \_, \_]$ *implies range* $\sqrt{\pm x \pm A}$ *is* $[0, inf]$

(3) $[-1, sqr, [\_,\_,\_,x], \_, \_]$ *implies range* $(-\sqrt{\pm x \pm A})$ *is* $[inf, 0]$

or

(3) $[-1, sqr[\_,\_,\_,x], -1, \_]$ *implies range* $(-\sqrt{\pm x \pm A} - B)$ *is* $[inf, 0]$

The expert provides two alternative explanations: Rule 4 is missing or an incorrect rule ("The range of the term $-\sqrt{\square} - B$ is $y \leq 0$ because of the two "_" signs) was applied. How could the expert produce such a rule? A system of meta-rules which reflects some

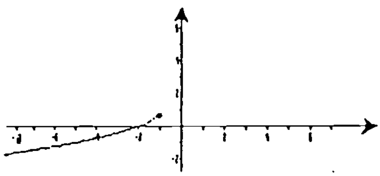
most common errors of students has been inserted to the expert. The analysis of student reasoning, such as searching for missing or incorrect rules, can be done using techniques of meta-programming. Prolog is especially suited for meta-programming since Prolog clauses are themselves terms in Prolog.

The Graphical Expert

The development process of this expert was based on the application of the questionnaire to the 10th grade class. The aim of the graphical expert is to restore the four characteristics of a function given in its graphical representation.

The nature of the processes needed for this task is very different from the processes involved in the algebraic representation: the students "see" the "graphing" and have only to translate it in a formal way. Thus, in the questionnaire, we could not ask for explanations of the visual answers. This fact influenced the construction of the graphical expert; while the algebraic expert has been based on the explanations of perfect students, the graphical expert has been based on classification of actual students' answers.

Let us clarify this point by means of an example taken from the questionnaire. The student is presented with the following graph (without being told that the graph belongs to a square-root function):



A typical answer for the domain and the range of the function is: $-8 \leq x \leq -1, \quad -2 \leq y \leq 1$. Students who gave this answer did not process the visual information they received; they just answered what they saw. It is difficult here to say that the students were wrong, but of course we hope that they will implicitly translate the visual information to: $x \leq -1, \quad y \leq 1$. The answer $y \leq 1$ is associated with some familiarity with the graph of a square-root function.

Another source of problems is connected with the limitations of the resolution of the computer screen. Some students gave the following result to the intersection with the x-axis: $(x, 0)$ where $-2.2 \leq x \leq -1.8$. Here again, there exists a gap between the

implicit information and the visible; the student grasps the visible. It is also found that many students do not cope successfully with reading points when non-integer coordinates occur. Based on the actual students' answers to the graphical tasks the following aspects in the solving process were identified:

(a) Objective mathematical skills for interpreting graphs (e.g., linking "range" with the y-axis).

(b) Association with familiar graphs.

(c) Linking the visible "graphing" with the implicit graph.

(d) Concern with the accuracy of the graphics, and reading points on a graph.

We illustrate how the graphical expert acts in the last example. The "graphing" presented to the student is symbolized by the list:

[features $([-8,-1],[-2,1])$, inter $([-2.2,-1.8], \text{none})$, quads $([\text{none}, 2, 3, \text{none}])$]

Legend;

features $([-8,-1],[-2,1])$: the visible part of the graph is constituted by the x part $-8 \leq x \leq -1$ and by the y part $-2 \leq y \leq 1$.

inter $([-2.2,-1.8], \text{none})$: The intersection point with the x-axis is "spread" over the interval $[-2.2,-1.8]$; there is no intersection of the graph with the y-axis.

quads $([\text{none}, 2, 3, \text{none}])$: The graph passes through quadrants 2 and 3.

Note: while the "inter" and "quads" functors have the same format for all these types of functions in the system, the "features" format, features $([Lx, Ly])$, is different for the three types. The length of the sublists Lx, Ly is determined by the "graphing" of $\sqrt{x}, 1/x$ and $[x]$.

The expert finds the characteristics by applying hierarchic..l rules expressed as logic clauses. For example, the rules for deciding the range of the current example are:

1. length $Ly = 2$ *implies that the range is* $[\_,\_]$

2. length $Ly = 2$ *applies the range is* $[\_, inf]$ *or* $[inf, \_]$

3. $Lx = [-8,-1]$ *and* $Ly = [-2,1]$ *implies that the range is* $[inf, 1]$.

The user interface exhibits the inference chain of the expert and the matching of correct rules when an actual student answer is entered. As for the algebraic expert the inference chains enable to capture actual student knowledge (i.e., failure in rules 1, 2, 3 reflect difficulties related to aspect (a), (b), (c) respectively).

Next Steps

We started the construction of the "student model" by gathering information from perfect and actual students. Empirical data and clinical observations contributed researcher-defined rules that simulate student reasoning. In the present paper, we have not described the tutorial component. Roughly, it chooses a triplet: algebraic representation, graph, and characteristics, and asks whether the the given characteristic discriminates between the two presented functions. The Perfect Student Expert (the domain expertise) simulates students' correct responses and determines the level required for solution. The process can be reversed by logical programming techniques, i.e. a certain level can 'determine' possible tasks. Data collected by the questionnaire and analyzed by the student model provide diagnostic information about students' knowledge. The tutorial needs to select tasks at the level of the student in order to generate learning. Experiments with the pilot version of the tutorial showed that the tasks may provide sufficient challenge for learning and progress. In case it is not sufficient, the expert may present the "rules" invoked for the situation. The tutorial is designed to be used not only for treatment and evaluation, but also to sharpen the diagnosis.

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