DOCUMENT RESUME

ED 411 139 SE 058 667

AUTHOR	Booker, George, Ed.; Cobb, Paul, Ed.; de Mendicuti, Teresa N.
TITLE	Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education with the North American Chapter 12th PME-NA Conference (14th, Mexico, July 15-20, 1990), Volume 3.
INSTITUTION	International Group for the Psychology of Mathematics Education.
PUB DATE	1990-07-00
NOTE	357p.; For volumes 1 and 2, see SE 058 665-666.
PUB TYPE	Collected Works - Proceedings (021)
LANGUAGE	French, Spanish, English
EDRS PRICE	MF01/PC15 Plus Postage.
DESCRIPTORS	Educational Research; Elementary Secondary Education; Foreign Countries; Higher Education; *Mathematics Education

ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) contains the following research papers: "The Construct Theory of Rational Numbers: Toward a Semantic Analysis" (M. Behr & G. Harel); "Reflections on Dealing: An Analysis of One Child's Interpretations" (G. Davis); "About Intuitional Knowledge of Density in Elementary School" (J. Gimenez); "Understanding the Multiplicative Structure: Concepts at the Undergraduate Level" (G. Harel & M. Behr); "A Contextual Approach to the Teaching and Learning of Mathematics: Outlining a Teaching Strategy that Makes Use of Pupil's Real World Experiences and Strategies, and the Results of the First Teaching Experiment of Project" (T.O. Keranto); "On Children's Mathematics Informal Method" (F.L. Lin & L.R. Booth); "A Case Study of the Role of Unitizing Operations with Natural Numbers in the Conceptualization of Fractions" (A. Ludlow); "Constructing Fractions in Computer Microworlds" (J. Olive & L.P. Steffe); "Proportional Reasoning: From Shopping to Kitchens, Laboratories, and Hopefully, Schools" (A.D. Schliemann & V.P. Magalhaes); "The Fraction Concept in Comprehensive School at Grade Levels 3-6 in Finland" (T. Strang); "Critical Decisions in the Generalization Process: A Methodology for Researching Pupil Collaboration in Computer and Non Computer Environments" (L. Healy, C. Hoyles, & R.J. Sutherland); "'Scaffolding' a Crutch or a Support for Pupils' Sense-Making in Learning Mathematics" (B. Jaworski); "The Role of Mathematical Knowledge in Children's Understanding of Geographical Concepts" (R.G. Kaplan); "Speaking Mathematically in Bilingual Classrooms: An Exploratory Study of Teacher Discourse" (L.L. Khisty, D.B. McLeod, & K. Bertilson); "The Emergence of Mathematical Argumentation in the Small Group Interaction of Second Graders" (G. Krummheuer & E. Yackel); "Potential Mathematics Learning Opportunities in Grade Three Class Discussions" (J.J. Lo, G.H. Wheatley, & A.C. Smith); "Certain Metonymic Aspects of Mathematical Discourse" (D.J. Pimm); "Inverse Relations: The Case of the Quantity of Matter" (R. Stavy & T. Rager); "The Development of Mathematical Discussion" (T. Wood); "Estrategias y Argumentos en el Estudio Descriptivo de la Asociacion Usando Microordenadores" (J.D. Godino, C. Batanero, & A.E. Castro); "Computerized Tools and the Process of Modeling" (C. Hancock & J. Kaput); "Examples of Incorrect Use of Analogy in Word Problems" (L. Bazzini); "Children's Pre-concept of Multiplication: Procedural

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Understanding" (C. Beattys, N. Herscovics, & N. Nantais); "The Kindergartners' Knowledge of Numerals" (J.C. Bergeron); "An Analysis of the Value and Limitations of Mathematical Representations Used by Teachers and Young Children" (G.M. Bouton-Lewis & G.S. Halford); "A Study on the Development of Second-Graders' Ability in Solving Two-Step Problems" (F. Cao); "Understanding the Division Algorithm from New Perspectives" (D.W. Carraher); "Negative Numbers Without the Minus Sign" (T.N. Carraher); "Learning Difficulties Behind the Notion of Absolute Value" (I. Chiarugi, G. Fracassina, & F. Furinghetti); "Le Role de la Representation dans la Resolution des Problemes Additifs" (R.F. Damm); "Using a Computerized Tool to Promote Students' 'Number Sense' and Problem Solving Strategies" (N. Hativa, V. Bill, S. Hershkovitz, & I. Machmandrow); "Children's Understanding of Compensation, Addition and Subtraction in Part/Whole Relationships" (K.C. Irwin); "Factors Affecting Children's Strategies and Success in Estimation" (C. Morgan); "Assessment in Primary Mathematics: The Effects of Item Readability" (J.A. Mousley); "Social Sense-Making in Mathematics: Children's Ideas of Negative Numbers" (S. Mukhopadhyay, L. B. Resnick, & L. Schauble); "Children's Pre-Concept of Multiplication: Logico Physical Abstraction" (N. Nantais & N. Herscovics); "Building on Young Children's Informal Arithmetical Knowledge" (A.I. Oliver, A. Murray, & P. Human); "From Protoquantities to Number Sense" (L.B. Resnick, S. Lesgold, & V. Bill); "Prospective Elementary Teachers' Knowledge of Division" (M.A. Simon); "Relative and Absolute Error in Computational Estimation" (J. Sowder, L. Sowder, & Z. Markovits); "A Child Generated Multiplying Scheme" (L. Steffe); and "Salient Aspects of Experience with Concrete Manipulatives" (P. Thompson & A.G. Thompson). Includes a listing of author addresses. (MKR)

International Group For the Psychology of Mathematics Education

PROCEEDINGS

Fourteenth PME Conference

With the North American Chapter
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Published by the Program Committee
of the 14th PME Conference, Mexico.

Sponsored by:
Consejo Nacional de Ciencia y Tecnología (CONACYT)
Gobierno del Estado de Morelos
IBM de México, S.A.
Sección de Matemática Educativa del CINVESTAV

Editors:

George Booker
Paul Cobb
Teresa N. de Mendicuti

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Printed in Mexico.

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Rational Number

THE CONSTRUCT THEORY OF RATIONAL NUMBERS: TOWARD A SEMANTIC ANALYSIS

Merlyn J. Behr, Northern Illinois Univ. Guershon Harel, Purdue Univ.

A semantic analysis which is in progress will attempt to demonstrate the numerous interconnections among concepts in the multiplicative conceptual field. The analysis is based on two theoretical frameworks: mathematics of quantity and formation and reformation of units of quantity. The analysis employs two representational systems: noncontextualized drawings and the symbolism of the mathematics of quantity. The noncontextualized drawings are given in a notational system which might be a bridge between contextualized drawings or manipulative materials and mathematical symbolism. This paper presents, very briefly, new insights which have been gained through this type of analysis about subconstructs of rational numbers.

Numerous issues remain about how to facilitate children's construction of rational number knowledge. One of these issues is gaining deeper understanding of rational number subconstructs. We are gaining new insights into these subconstructs through an analysis based on the two perspectives of composition and recomposition of units (Steffe, Cobb and von Glasersfeld, 1988; Steffe, 1986, 1988) and mathematics of quantity (Kaput, 1985; Schwartz, 1988). The objective of our analysis is to provide a more firm theoretical foundation for research and development in this domain. An overview of our progress on this analysis is given in Figure 1.

A brief explanation of Figure 1 follows: The chart indicates that a rational number (3/4 is used to illustrate) has different interpretations depending on the rational number subconstruct considered. From the perspective of part-whole, a rational number is an extensive quantity and has a units formulation of either one composite 3/4-unit or three separate 1/4-units, independent of whether the analysis is based on a discrete or continuous quantity. Considering the quotient construct, a rational number is an extensive or intensive quantity according to whether the division of the numerator by the denominator is partitive or quotitive. For partitive division, 3/4 is the extensive quantity 3(1/4-unit)s per [1-unit] when the numerator, 3, has the units formulation of three singleton units of discrete quantity or one singleton unit of continuous quantity. When the units conceptualization of the numerator is one composite (3-unit) of discrete

quantity it is 1/4(3-unit) per [1-unit]. For quotitive division, 3/4 is one composite [3/4-unit] when the numerator is conceptualized as 3(1-unit)s of discrete quantity (the analysis is incomplete for the numerator as one (3-unit) of discrete quantity and for continuous quantity). For the operator construct, a rational number is a mapping for which the domain and range varies according to the affect of several variables considered in the analysis: the interpretation given to the numerator and denominator in terms of type of operator, the order in which the numerator and denominator operators are applied, whether the operand of these operators is a discrete or continuous quantity, and according to the unitization of this quantity. We illustrate the operator interpretation of 3/4 in Figure 1 for the duplicator and partition-reducer interpretation of the numerator and denominator. When the operand is a discrete quantity, 3/4 is a function which maps quantities of the form 4n(1-unit)s to quantities of the form 3n(1-unit)s independent of the operator order; for continuous quantity, 3/4 is a function which maps a quantity of the form 1(1-unit): (a) onto a quantity of the form 1/4(3-unit) when the numerator operator (duplicator) is applied first and (b) onto a quantity of the form 3(1/4-unit)s when the denominator operator (partition-reducer) is applied first.

In the remainder of this communication we will briefly illustrate the procedures of the analysis. We employ two forms of analyses: diagrams to represent the physical manipulation of objects and the notation of mathematics of quantity. The diagrams provide a semantic analysis and the mathematics of quantity model a mathematical analysis; a very "close" step-by-step correspondence between the two representations suggests the mathematical accuracy embodied by the diagrams.

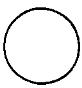
<u>The Part-Whole Subconstruct</u>. Our analysis of this subconstruct leads to two interpretations of rational number which we illustrate with 3/4. One interpretation is that 3/4 is three separate one-fourth parts of a whole and the second is that it is one composite three-fourths unit. Three-fourths as

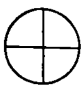
separate parts of a whole is 3(1/4-unit)s for a continuous quantity and is 3(1/4(4n-unit)-unit)s (i.e., three one-fourth-unit of units where each one-fourth-unit of units is a unitized one-fourth of a (4n-unit)) for a discrete quantity. We illustrate how our analysis leads to these interpretations in Figures 2 and 3. Three-fourths as a composite part of a whole is 1(3/4-unit); illustrations are omitted due to lack of space.

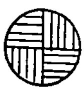
Figure 2. Three-fourths as <u>parts</u> of a discrete unit-whole leads to the interpretation that three-fourths is three one-fourth units.

1.	(0 0 0 0 0 0 0 0)	One (8-unit). We could have started with any (4n-unit).
2.	(0 0 / 0 0 / 0 0/ 0 0)	The (8-unit) is partitioned into 4 parts.
3.	((00) (00) (00) (00))	Each part is unitized as 1(2-unit), and a unit-of-units is formed, (4(2-unit)s-unit).
4.	((0 0) (0 0) (0 0) (0 0))	Each (2-unit) is reconceptualized as (1/4(4(2-unit)s)-unit).
5.	(~~(0 0)~~ ~~(0 0)~~ ~~(0 0)~~ (0 0))	Three (1/4(4(2-unit)s)-unit)s are distinguished.

Figure 3. Three-fourths as <u>parts</u> of a continuous unit-whole leads to the interpretation that three-fourths is three one-fourth units.

1.  1(1-unit)

2.  The (1-unit) is partitioned into four parts.

3.  Each part is conceptualized as 1/4(1-unit).

4.  Three of the 4 parts are singled out to give 3 (1/4-unit)s.

<u>The Quotient Construct</u>. In this analysis we consider several variables: quotitive and partitive division, discrete and continuous quantity, and different unitizations of the numerator and denominator.

Partitive division gives two interpretations of a rational number which can be illustrated by 3/4 as the two intensive quantities: 3(1/4-unit)s per [1-unit] and 1/4(3-unit) per [1-unit]. We illustrate the first interpretation using discrete quantities.

Figure 4. Partitive division of 3 ÷ 4, based on discrete quantities leads to three-fourths as three one-fourth units in one measure space per one unit of quantity from another measure space.

```
1. (0 0 0 0) (0 0 0 0) (0 0 0 0)         3(4-unit)s
     [*]   [*]   [*]   [*]               4[1-unit]s

2. (0/0/0/0) (0/0/0/0) (0/0/0/0)         Each (4-unit) is partitioned into 4
     [*]   [*]   [*]   [*]               parts.

3.  (Q Q Q Q) (Ø Ø Ø Ø) (@ @ @ @)        Each of the 4 parts of each (4-unit)
     [*]   [*]   [*]   [*]               is reunitized as 1/4(4-unit).

4.   Q     Q     Q     Q                 Four 1/4(4-unit)s are distributed
    [*]   [*]   [*]   [*]                equally among the 4 [1-unit]s to
                                         give 1/4(4-unit)/[1-unit].

5.   Ø     Ø     Ø     Ø                 Four more 1/4(4-unit)s are dis-
     Q     Q     Q     Q                 tributed among the 4 [1-unit]s,
    [*]   [*]   [*]   [*]                another 1/4(4-unit)/[1-unit].

6.   @     @     @     @                 The third four 1/4(4-unit)s are
     Ø     Ø     Ø     Ø                 distributed equally among the
     Q     Q     Q     Q                 4 [1-unit]s, this gives a third
    [*]   [*]   [*]   [*]                1/4(4-unit)/[1-unit].

7.   @     @     @     @                 The three (1/4(4-unit))s are
     Ø     Ø     Ø     Ø                 accumulated to
     Q     Q     Q     Q                 3(1/4(4-unit)-unit)s per [1-unit].
    [*]   [*]   [*]   [*]
```

In each step of the mathematics of quantity model below, the step number from the preceding pictorial model is given in parentheses to demonstrate the closeness between the two models.

Figure 5. The partitive division of 3 ÷ 4 which leads to the interpretation that three-fourths is three one-fourth units in one measure space per one unit of quantity in another measure space.

1. 3/4 = 3(1-unit)s ÷ 4[1-unit]s (1)

2. = 3 x (4(1/4-unit)s) ÷ 4[1-unit]s (2)

3. = (4(1/4-u)s + 4(1/4-u)s + 4(1/4-u)s) ÷ 4[1-unit]s (3)

4. = 4(1/4-u)s ÷ 4[1/4-u]s + (4(1/4-u)s + 4(1/4-u)s) ÷ 4[1-unit]s (4)

5. = $\frac{\text{1(1/4-u)}}{\text{4[1-u]s}}$ + (4(1/4-u)s + 4(1/4-u)s) ÷ 4[1/4-u]s (4)

$$6.\quad = \frac{1(1/4\text{-u})}{[1\text{-unit}]} + 4(1/4\text{-u})s + 4[1/4\text{-u}]s + 4(1/4\text{-u})s) + 4[1/4\text{-u}]s \qquad (5)$$

$$7.\quad = \frac{1(1/4\text{-u})}{[1\text{-unit}]} + \frac{1(1/4\text{-u})}{[1\text{-unit}]} + 4(1/4\text{-u})s) \div 4[1/4\text{-u}]s \qquad (5)$$

$$8.\quad = \frac{1(1/4\text{-unit})}{[1\text{-unit}]} + \frac{1(1/4\text{-unit})}{[1\text{-unit}]} + \frac{1(1/4\text{-unit})}{[1\text{-unit}]} \qquad (6)$$

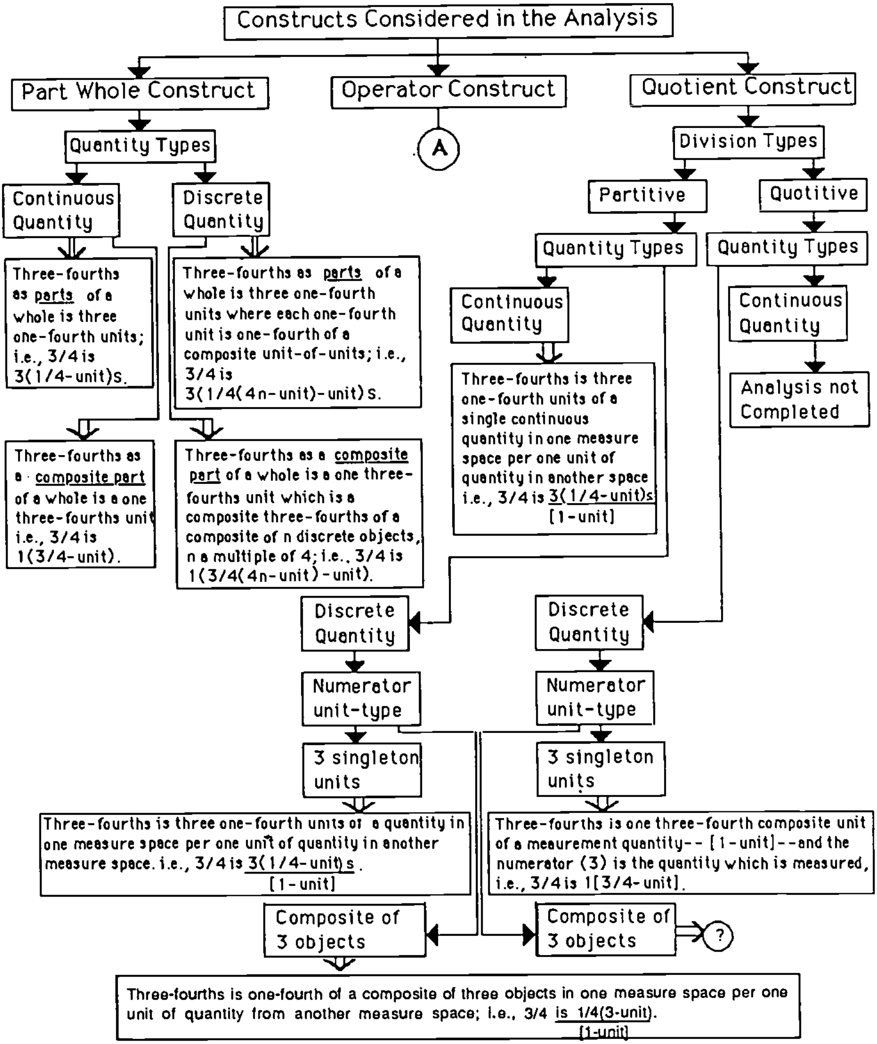
$$9.\quad = \frac{3(1/4\text{-unit})s}{[1\text{-unit}]} \qquad (7)$$

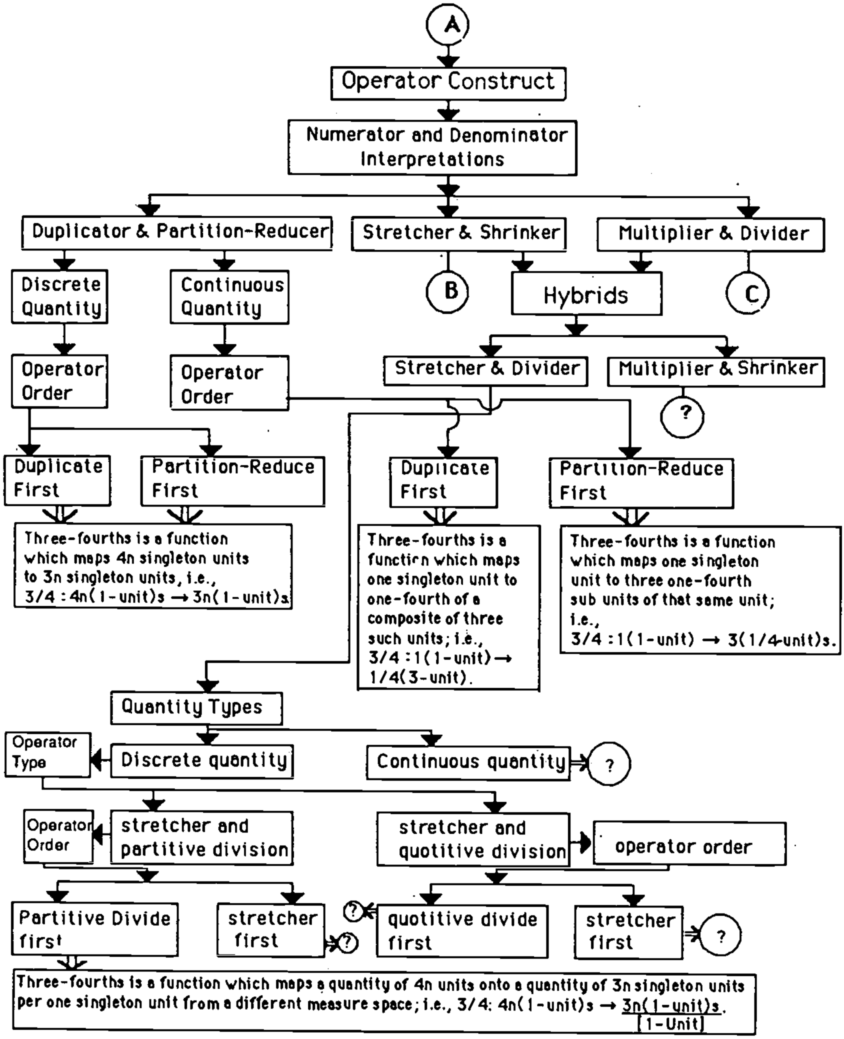
Quotitive division, given the extent of our analysis, leads to the interpretation that 3/4 is the extensive quantity 1[3/4-unit], where 4{1-unit}s/[1-unit] is a measurement unit which is used to measure the numerator, 3, expressed as 3(1-unit)s.

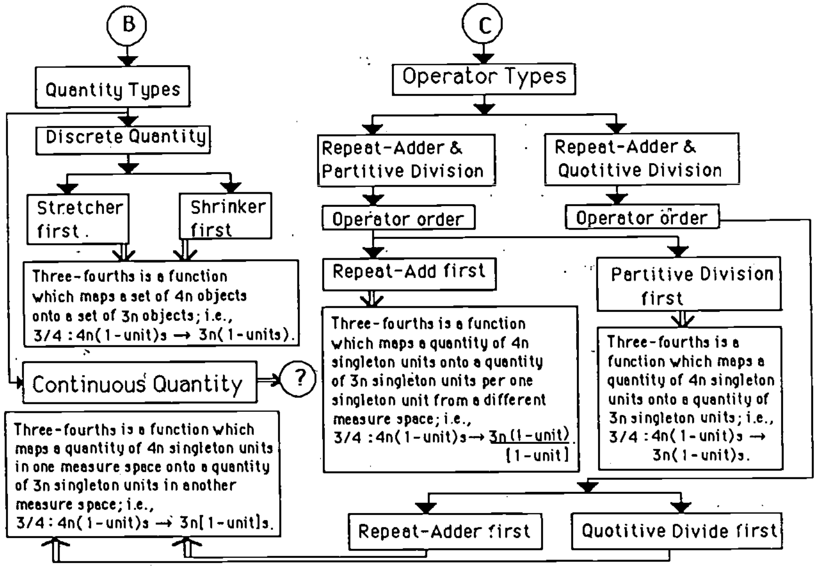
Figure 6. Quotitive division of 3(1-units) by 4{1-units}/[1-unit] which leads to the interpretation that 3/4 is 1[3/4-unit]

(0) (0) (0) [{*} {*} {*} {*}]	a. 3(1-unit)s 4{1-unit}s/[1-unit]
(0) (0) (0) [{*} {*} {*} {*}]	b. The measurement quantity is subunitized as 4[1/4-units].
(0) (0) (0) [{*} {*} {*} {*}]	c. The "first" (1-unit) of the to-be-measured quantity is matched with the "first" sub measurement-unit (i.e. iteration of the measurement unit is begun).
(0) (0) (0) [{*} {*} {*} {*}]	d. The "second" (1-unit) is matched with the "second sub measurement-unit. Meanwhile the first matched pair is dropped from attention.
(0) (0) (0) [{*} {*} {*} {*}]	e. The matchings are accumulated; i.e. both matched pairs are brought into cognitive attention.
(0) (0) (0) [{*} {*} {*} {*}]	f. The "third" (1-unit) is matched with the "third" sub measurement unit. Meanwhile, the first two matched pairs are dropped from attention.
(0) (0) (0) [{*} {*} {*} {*}]	g. The matchings are accumulated.
((0) (0) (0)) [((*) (*) (*)) (*)]	h. The 3(1-unit)s of the measured quantity are unitized to a 1(3-unit) and the 3 matched [1/4-unit] of the measurement quantity are unitized as 1[3/4-unit]s. The measure of the quantity that was to be measured is 1[3/4-unit].

Figure 1 The Construct Theory of Rational Numbers Semantic Analysis
Overview--Interpretations of Three-fourths




A
Operator Construct
Numerator and Denominator Interpretations
Duplicator & Partition-Reducer
Stretcher & Shrinker
Multiplier & Divider
Discrete Quantity
Continuous Quantity
B
Hybrids
C
Operator Order
Operator Order
Stretcher & Divider
Multiplier & Shrinker
?
Duplicate First
Partition-Reduce First
Duplicate First
Partition-Reduce First
Three-fourths is a function which maps 4n singleton units to 3n singleton units, i.e., 3/4 : 4n(1-unit)s → 3n(1-unit)s
Three-fourths is a function which maps one singleton unit to one-fourth of a composite of three such units; i.e., 3/4 : 1(1-unit) → 1/4(3-unit).
Three-fourths is a function which maps one singleton unit to three one-fourth sub units of that same unit; i.e., 3/4 : 1(1-unit) → 3(1/4-unit)s.
Quantity Types
Operator Type
Discrete quantity
Continuous quantity
?
Operator Order
stretcher and partitive division
stretcher and quotitive division
operator order
Partitive Divide first
stretcher first
?
?
quotitive divide first
stretcher first
?
Three-fourths is a function which maps a quantity of 4n units onto a quantity of 3n singleton units per one singleton unit from a different measure space; i.e., 3/4: 4n(1-unit)s → 3n(1-unit)s / [1-Unit].



References

Kaput, J. (1985). Multiplicative word problems and intensive quantities: An integrated software response (Tech. Report). Harvard Graduate School of Education, Educational Technology Center.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithemetic operations. In J. Heibert & M. Behr (Eds.), Research agenda for mathematics education: Number concepts and operations in the middle grades (pp. 41 - 52). Reston, VA: The National Council of Teachers of Mathematics.

Steffe, L. (1986). Composite units and their constitutive operations. Paper presented at the Research Presession to the Annual Meeting of the National Council of Teachers of Mathematics, Washington, D.C.

Steffe, L. (1988). Children's construction of number sequences and multiplying schemes. In J. Heibert & M. Behr (Eds.), Research agenda for mathematics education: Number concepts and operations in the middle grades (pp. 119 - 140). Reston, VA: The National Council of Teachers of Mathematics.

Steffe, L., Cobb, P., & von Glasersfeld, E. (1988). Construction of arithmetical meanings and strategies. New York, NY: Springer-Verlag.

REFLECTIONS ON DEALING: AN ANALYSIS OF ONE CHILD'S INTERPRETATIONS

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We detail the responses of a grade 2 child who was shown video episodes of preschool children engaged in sharing activities, and was asked why the preschoolers counted or made height checks after dealing. The grade 2 child's responses indicate a keen awareness of the logical completeness of dealing to establish fair shares, and an acute sense of the need to check. We discuss this child's responses in the context of language as a vehicle for heightened awareness of mathematical procedures.

The procedure of dealing out a collection of discrete objects in order to share them evenly is interesting, because although a majority of pre-school and primary school children can, and do, use it in response to a sharing question (Clements and Lean (1988), Hunting and Sharpley (1988), Miller (1984)) it appears that few of them express a belief that dealing is actually sufficient to establish a fair share. Instead, young children of this age mainly refer to counting or height or length checking of piles of discrete items to check for equality of shares (Davis and Pitkethly, 1990). Children between the ages of 5 and 8 who can express the sufficiency of dealing to establish fair shares seem to be relatively rare. In a study of preschoolers, Hunting and Davis (1989) found only one child in 85 who expressed the sufficiency of dealing to establish fair shares, and in a study of 17 grade 2 children, davis and Pitkethly (1990), observed only one such child.

In the Davis and Pitkethly (1990) study, edited highlights of videotapes of preschoolers involved in sharing activities were compiled. The edited tape consisted of three five-year old children, one boy and two girls, performing sharing tasks with 12 biscuits. Each child was presented with two dolls and asked to share all the biscuits evenly between the two dolls. Then the interviewer presented a third doll just before the other dolls were supposed to eat the biscuits, and the child was asked to share all the biscuits evenly between the three dolls. These video-taped episodes were shown to 17 grade 2 children in small groups, and the grade 2 children were questioned for their views why the preschoolers counted or measured after dealing. There was a specific list of questions that were asked of the children in each group. The questions were asked of the group as a whole and then of each child who had not already answered. The major conclusion of that study was that the grade 2 children interviewed, with a notable exception, expressed almost no awareness that dealing by itself is sufficient to establish fair shares. Rather they saw it as necessary to count, or to do height or length comparisons of shared items.

This article is about the responses of a particular eight year old child, Tom, who gave a clear statement of the sufficiency of dealing in the davis and Pikethly (1990) study. I have selected Tom and analysed his responses to questions in detail because, in the context of the interviews, he demonstrated an unusual depth of access to procedural matters and to the applicability of different procedures. The purpose of the analysis of Tom's responses is to indicate the depth to which a particular eight year old child was capable of expressing the logic inherent in the dealing strategy, and to examine the reasons why he could attain such depth of expression.

Tom thought about and reported on the procedures used by the younger children. He was reporting not only what they did but what he thought they were thinking when they did it. In this sense he was reflecting on the cognitive activity of other children, but at the same time providing evidence of his cognitive activity.

Tom's responses are important not simply because he appears to be an exceptional child. They are important because he expresses a clear awareness of his and others procedures without the use of explicit symbolism. Steffe (1988) says that "Children's operations seem to be primarily outside their awareness, and, without the use of symbols, they have little chance of becoming aware of them nor can they elaborate those operations beyond their primitive forms." I think that Tom, and the methodology we used with him, provides us with another way in which awareness of operations and procedures can be made explicit, and can potentially lead to an elaboration of those operations and procedures.

Tom's answers seemed to reflect his thought processes: it is as if one could see things clicking over in his head as he digested what was asked of him. His hesitations and restatements of the questions indicate an attempt on his part to understand fully what it was the interviewer was asking. It seems that he was able to put himself in another person's position and to imagine what they were thinking; indeed what they were capable of thinking. Furthermore Tom himself had a predominantly pictorial way of thinking; it seemed most times that he was simply reporting on something that he could literally see in his mind. He communicated in a very vivid way and he constantly used his hands to help communicate his thoughts.

Tom's responses were of a different character, both qualitatively and quantitatively, to those of the other children in the study. There were 28 situations in which statements or questions were put to Tom. They elicited 57 responses from him that were judged to be related in some way to aspects of the dealing procedure or more general procedures. This number was more than

twice as many responses per question as that given by the next most responsive child, Davis and Pitkethly (1990).

Tom seemed to have a clear understanding of the sufficiency of dealing to establish fair shares: his opening statement below that uses a card analogy establishes this, and he did not express a contrary point of view throughout the interview. He is aware of human error and the need to check because he himself sometimes makes mistakes: the 9 statements he made concerning possible or actual errors in the application of procedures supports this conclusion. What is also interesting is the number of statements of Tom's that are inferences about thought processes, or logical aspects of procedures.

These responses indicate a high level of intellectual activity on Tom's part. He appears to be observant, aware of human error, and has conceptual models of how people think. Some of Tom's statements about logical aspects of procedures actually refer to definitions. It is remarkable that a child who was just 8 years of age was capable of expressing a definition succinctly so that it would be accepted by most reflective adults. Other statements of Tom's that related to logical aspects of procedures dealt clearly with the underlying reason that the procedure was carried out as it was. The other remarkable point was the number of inferences about the thought processes of the video-taped preschoolers: Tom made 11 statements of this nature.

More importantly however, Tom expressed an understanding of the logical aspects of the dealing procedure without using symbolism. Of course it is a debatable point whether pictures that one sees in one's head are symbols in this sense, but I will take it that they are not: a symbol indicates a written mark that is used consistently as a sign. Tom, and children like him, are counter examples to the notion that without the use of symbols they have little chance of becoming aware of their procedures and operations. We see in Tom's statements a clear understanding of the nature of dealing, of its different applications, its sufficiency to establish fair shares, and the desirability - but not necessity - of a check because of the possibility of error.

Whether, without symbolism, Tom could elaborate the dealing procedure beyond a primitive form into the concept of fractions, for example, is a moot point: there is no evidence either way.

What appeared to allow Tom to be so clear in his statements about dealing was a primarily vivid pictorial manner of thinking. The video tape record shows, and the transcript below indicates, a child whose speech was hesitant and who often backtracked, because he appeared to be scanning images in his mind. Tom used his hands in a vivid and suggestive way as he talked and appeared to be relating pictures of imagined events. This, of course, does not explain his considerable and unusual awareness of the facets of the dealing procedure, but it does allow

that something other than symbols allowed him to talk about these facets with considerable accuracy. In fact Tom used spoken language to express his awareness of the dealing procedure. He was an articulate child who was capable of searching for and successfully finding words to describe his thoughts, which appeared to have a large pictorial component.

In my interpretation of Tom's ability to have, and to express, an awareness of logical aspects of a particular procedure, there are implications for learning via communication in primary mathematics classrooms. One way to allow students to progress to an awareness of operations and procedures in mathematics is to enhance their ability to use written symbols. Another approach that is suggested by Tom's responses is to engage the children in dialogue in order to give them an opportunity to develop an awareness of the logic inherent in a given mathematics episode. This awareness will not normally develop in the course of a single interview episode (Davis and Pitkethly, 1990). On-going discussion and reflection would appear to be required to allow children in the early grades of school to obtain the clarity of vision and expression shown by Tom. What is important is that if Tom can use words to express an awareness of logical aspects of a procedure there seems to be no reason why other children, through extended and directed dialogue, could not attain a similar awareness of this and other procedures and operations.

It is easy in mathematics classes to overestimate the power of symbolism and to underestimate the power of visual images that are normally dealt with by oral or written language. It may be that for many children in their later years of mathematical experiences these two aspects of mathematical awareness and thought are in opposition, but there is no reason that they should be. Pre-school and the early grades of school are where the process of dialogue and discussion in mathematics can and should begin.

COMMENTARY ON A SELECTION OF TOM'S ANSWERS

The following is a transcription of a selection of Tom's remarks and the questions and comments which prompted them, together with commentary on the answers. Anne and Gary are the interviewers and Alexis is another grade 2 child - one of the three children interviewed at this time.

Excerpt 1

The preschool boy in the video episodes gave one biscuit to each doll and stopped. He then continued to give out one to each, until all biscuits were used, after the interviewer indicated that all the biscuits should be shared.

Anne : " Do the dolls have the same? Tell me why."

This question was asked of the three children present. Tom's answer to the question is revealing. It indicated that he understood the dealing procedure to be

sufficient to ensure equality of shares, and then related it to another familiar example. Alexis then suggested that one can deal by twos and Tom explained that there is no essential difference, just that it is faster.

Tom: " The child put out the biscuits. They give one to one doll and then another to the other doll and they kept on doing it until there was no biscuits left So that the dolls both had the same. Like when you're playing cards and you want to get the equal number of cards . Sometimes. So you say, one for me and one for you and so on and until all the cards have filled up. And ya sometimes put some of the cards out in the middle. "

Alexis: " You could do it by twos."

Tom: " Like you pick up two biscuits. You go two, two, two. Its just a bit faster way of doing it."

Tom's replies to the next three question show that he did not: think it was necessary to count after dealing to establish fair shares, but that he was aware of human error, including his own.

Excerpt 2

The preschool boy counted the number of biscuits and correctly stated that there were 6 in each stack.

Anne: " Do you have to count, to know if they are fair shares?"

Tom: " Well if you know what you are doing you probably don't; but if ah, ya sh... I always count them after, just to make sure, before I put the answer."

Anne : " Did he know the dolls had a fair share even before he counted?"

Tom : " Yeh, I think so. I thought he was going to count them in the next thing ."

Anne : " You thought it was logical that he would have actually counted them at some stage. Did he know they had fair shares before he even counted them, do you think?"

Tom : " Well, if he'd done it quite a lot he would probably know it was alright , but since he was only in Kindergarten he'd probably count them, just to make sure."

Excerpt 3

The preschool boy in the video episodes picked up three biscuits at a time from each pile and stacked them in front of the corresponding doll. He then picked up two biscuits from Joey's stack and placed one on each of the other dolls' stacks. He correctly counted the number of biscuits in Joey's stack. He moved to another stack and continued counting on from 4 until he reached 6. He then started again with Joey's stack and counted all stacks correctly to conclude that each doll had 4 biscuits.

Tom is then asked why the boy counted. His answer indicates, as before, that counting is not a part of the dealing process but it is a sensible check:

Anne : " So he did count, didn't he, and you watched how he shared them out and then he counted them. Why did he count?

Tom : " Ah, because he probably counted and ah, the same as he..the same reason he did last time, and also because it was a little bit harder. So he counted just to make sure that he'd put the right amount of biscuits in each pile."

Anne : " Did he do it the way you suggested?" (i.e. deal from one pile completely then the other).

Tom : " He did it kind of, he did it... he did it the same except he didn't put all the piles together. He kind of shared them. He kind of looked at this...looked at both the piles and he ah...he saw how many was on that pile and how many was on that pile and he shared them around ah... to Joey and then... then he looked at the piles again. If they weren't fair he kept sharing them until they were fair and then he checked them with a knife."

Excerpt 4

The first preschool girl in the video episodes gave one biscuit to each doll and asked if she had to give out all the biscuits. She did so, one biscuit at a time to each doll in turn, and ended with a stack in front of each doll. In this excerpt Tom has seen the girl deal out the 12 biscuits.

Anne : " Do the dolls have the same?"

Tom : " Well probably ... since she used the right method of counting, ah, sharing them out, she'd have probably got them right."

Tom's answer was somewhat curious. He had seen the girl deal the biscuits, and he had previously stated that dealing was sufficient to ensure even shares. Yet he said "she'd have probably got them right". My interpretation of Tom's remarks here is that he has placed himself in the position of the girl in the video episode, and he is reporting what he thought she was feeling. This interpretation is strengthened by Tom's answer to the next question:

Anne :" How do you think she knows the dolls have got the same? She hasn't counted them, has she? How could she tell?"

Tom :" Well she doesn't... probably she doesn't really know that, she doesn't, she's not absolutely sure yet. And she'll probably count them."

Excerpt 5

The first girl in the video episodes then checked the height of the two stacks of biscuits she had made. Tom was asked if that is reasonable, and he gave a wonderful image for his answer. In so doing he tells us in precise detail that a height check is indeed one way to check for fairness:

Anne :" She looked to see if they were the same height. Now, does that seem reasonable to you Tom?"

Tom :" Yes. It seems alright because since they were the same size biscuits and the same length, if she puts them together and they're, they're, one's higher and

one's lower, it wouldn't be fair and if one, if one's lower and one's higher then it wouldn't be fair. But if they're flat across the top,if you can ah... if you can walk... if you were very little you can walk across the top without walking up a step or anything then it will be fair."

Tom was then asked about the relative merits of height checks and counting, and whether a height check is something he himself would have done:

Anne :" So that's another, is that another way to tell if you've got fair shares?"

Tom :" Yes."

Anne :" Is it perhaps as reliable as counting?"

Tom :" Well probably about the same ."

Anne :" About the same."

Tom :" It's.. it..., counting would be a better idea if the things were..., counting would be a better idea if the things weren't the same shape and size but if they're all the same like the biscuits were, it's alright to put them together."

Anne : "But it wasn't an idea that came to you until you saw the little girl do it, was it? Was it something you would do yourself? Have you got a preference?"

Tom :" I ... if things were different shapes I would normally count them but if they're the same I would normally count them, or I would put them together "

This is a revealing answer. If we take Tom at his word then he would count the biscuits after dealing or else do a height comparison. This is despite the fact that he has stated that dealing is sufficient to establish fair shares. His previous statements about possible errors give us a clue that he is not satisfied until he has checked by counting or a height comparison. This may well be the same for children other than Tom: the difference that appeared in the Davis and Pitkethly (1990) study is that most children did not express an awareness that dealing is by itself sufficient to establish fair shares. To the contrary the other children in that study were adamant that counting or some other check was essential for the establishment of fair shares.

Excerpt 6

The second girl in the video episodes gave out biscuits to the two dolls, in turn: 2 biscuits, 2 biscuits, 1 biscuit, and 1 biscuit. She did not place them in a stack and she did not overtly/count the biscuits, and insisted both dolls had "the same".

Tom was asked why the girl did not stack the biscuits like the other two children. His answer indicated that she could count the biscuits, as did the others, but with the biscuits in a line rather than stacked. He was not referring to counting out the biscuits but rather to counting them after they had been shared:

Anne : " Why didn't she stack them? The others stacked them, she didn't. Is there some reason why she didn't stack them?

Tom :" Maybe because she's using a different method of sharing. She puts them on the side and she can just count one, two, three, instead of one, two three, four, five, six,...... if they're way in a line they won't get mixed up and she can just count them, one, two, three, four, five ."

Tom was asked again, in the context of this episode, how it is that the preschool interviewer, Maggie, seemed to know that the girl had shared evenly. He once again asserted that it is sensible, even important, to check, unless one knows what one is doing. This latter remark again indicated that he was aware that dealing is sufficient by itself to ensure equality of shares:

Anne :" Maggie asks Amanda how she knows they've got the same; how can you tell? Amanda says they just have. Do you think she was confident she had got it right? Do you think she is going to count the biscuits to find out? Will she count them do you think?"

Tom :" She may count them or do what the second girl did ... just pile them and see if she got it right."

Anne :" Is it important that she does either of those, to know if they are the same?"

Tom :" I think it is important to do, unless you are really good and she knows exactly what she is doing."

References.

Clements, M.A. and Lean, G.A. (1988). Discrete fraction concepts and cognitive structure. In A. Borbas (Ed.), Proceedings of the Twelfth Annual Conference of the International Study Group for the Psychology of Mathematics Education vol 1, 215-222. Veszprem, Hungary: Hungarian National Centre for Education Technology.

Davis, G.E. and Pitkethly, A. (1990) Cognitive aspects of sharing. Journal for Research in Mathematics Education, 21 (2), 145 - 153.

Hunting, R.P. and Davis G.E. (1987). Dimensions of young children's knowledge of the fraction one-half. Submitted for publication.

Hunting, R.P. and Sharpley, C. F. (1988) Preschoolers' cognitions of fractional units. British Journal of Educational Psychology, 58, 172-183.

Miller, K. (1984). Child as the measurer of all things: measurement procedures and the development of quantitative concepts. In C. Sophian (ed.), Origins of cognitive skills. Hillsdale, NJ: Erlbaum.

Steffe, L.P. (1988) Children's construction of number sequences and multiplying schemes. In J. Hiebert and M. Behr (Eds), Number Concepts in the Middle Grades Research Agenda for Mathematics Education, National Council of Teachers of Mathematics, vol 2, 119-140. Reston, Virginia: Lawrence Erlbaum Associates.

ABOUT INTUITIONAL KNOWLEDGE OF DENSITY IN ELEMENTARY SCHOOL

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We found some characteristics of density conceptions with 1900 students (11 and 14 years old). We describe a sequential hierarchy assisting this pre-notions that obtained good results with a set of didactical situations bearing in mind this scheme .

Introduction and reasons for the Study .

The study about the first conception that pupils have regarding the notion of density is a matter of great concern for researchers today, and many aspects have been introduced :

(1) On the basis of the studies carried out by K.Hart (Hart, CSMS 1980) , the analysis concentrated on the excessive application of processes and intuition relating to natural numbers to very different structures ,i.e. rationals ,in their expression as decimals or as fractions. Find numbers between 0,41 and 0,42 (Brown 1981) , find a fraction between 1/2 and 2/3 ,etc. Until now, the problem has been analysed in very specific situations .

Kieren (1976, 1981, 1988) , Behr et al. (1983,1985,1986) emphasized the transfer of structures in the methodical arrangement and application of algorithms which are known for natural numbers and which are deviated towards fractions.

(2)Brousseau' epistemological thought (1977-1981) made him see the problem in a wider context : the analysis of didactic situations with decimal filters . The data collected from his study are already an indication to the way in which the problem could be dealt with reasonable success (67 % of the sample are able to give decimal numbers between 1,2 y 1,3 or indeed , give 3 fractions between two specific rationals). Here Brousseau uses fractions greater than the unit .

M.J.Perrin (1986) when introducing the problem of different expressions of a rational (greater than the unit), finds much lower parcentages of correcte replies. His analysis , along the lines of R.Douady (1984) integrates Brousseau's didactic position and poses the problem of boundaries , as the basis for the notion of a real number.

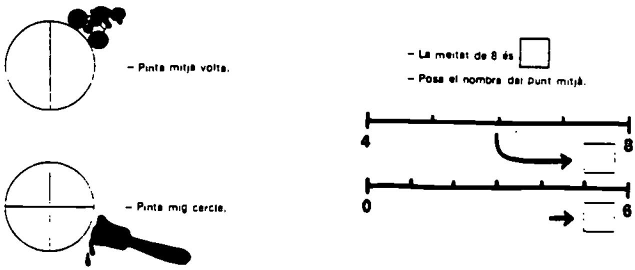
(3) Kidron - Vinner (1983) added density to a more generic problem : how many fractions are there between two rationals ? This fell into the vision of infinity and Fischbein's corresponding characterisations (Fischbein 1979) , the role of irrationals and the strength of the number line in the students' conception.

(4) Much more recently , Chevallard (1989) situates the problem in the framework

of didactical situationsin which the didactic contract is broken. His research uses the CSMS item.

The application of the archimedian property of the natural numbers to Q (Kieren 1981a), is added to the difficulty of the lexical order of decimals (Davis 1982) .

Chevallard informs us (1989) of the fact that whilst having algorithmic ways of looking for an intermediate fraction between two others , they are not used in practice. According to him , all open and productive research without a unique solution is outside of the contract (Chevallard 1989, pg. 71) .If indeed it is correct that no systematic training exists , it seems to us that in order to have at least an intuitive knowledge of density (in accordance with Kieren 's (1981) more generic proposition), there must be a conception of the algorithmic calculation of an intermediate situation. This is not surprising considering the fact that the problem of finding the mid-point between two natural numbers and his difference from half of the greater one is posed very little (from an early age).



Justification and Aims of our Study .

- In the context of evaluating the knowledge of an idea of order in rational numbers (Giménez 1987) we ask ourselves particulary about the utility of analysing the problem of preconceptions which we have about the notion of density in relation to the knowledge of comparison in diverse subconstructs . The work of Kidron-Vinner (1983) already cited , situated the problem of decimal comparison in a specific context of symbolic lexicographical order which is not the only one to be taken into account.

- We consider that an important topological conception underlies Brousseau's propositions. However, this conception remains to be analysed outside (and even within) the context of the didactic situation and epistemological point of view. What influence does the use of the "limit-patterns" and aproximation have on the idea of density ? We consider that the representation of rationals in the number line may be important and significant . M.J. Perrin (whose work in 1985 we did not know about) analyse the same aspect.

(c) Estimation has its own characteristics, and variables which afect it (Reys et al. 1982), some of a cognitive type (Siegel 1982) . We ask ourselves ,therefore, if the use of estimation in adding fractions (used for the RNP in the observation of comparison , Behr et al. 1986) carried out in the number line, is related to a correct intuitive observation of the idea of density .

Hypothesis and design .

Thus , it is thought that the conception of density in elementary schools may be influenced by :

(a) The representational aspect which appears to be the key to the comprehension of the rational number (number line).

(b) The idea of approximation which we have in the Q set .

(c) Master of the notion of order in diverse contexts.

We propose to observe the extent of these influences and their possible interrelation .

On the basis of a former brief analysis , having held interviews with 27 pupils aged from 10 to 14 years , a test on 47 items is devised and given to 100 pupils in two groups - 10 year olds and 14 year olds - in Barcelona province. These age groups were chosen because the first is when they begin systematically to learn fractions and the second when they complete compulsory "Formación Básica" (General Education) in Spain, known as EGB. As a result , it was hoped that it would be possible to find out what conceptions they have of the concepts related to comparison and we wanted to relate this to the cognitive level (Giménez 1989) .

Results .

The analysis "Item response theory" and the factorial analysis makes it clear that it is not a viable proposition to take the said test as a scale of recognition of the factors involved, but we consider that the observations deduced from the categorisation of the replies is important . We found perceptual variability and different factors in comparison tasks :

- Contextualised situation (adding and estimation in a football match).
- Ordering with fraction -equation situations (inequalities)
- Use of number line
- Find a fraction between two others
- Algorithmical situations with symbolic fractions
- Direct approximation
- Find an addition close to a point in number line

- Find a fraction belonging two neighbords.
- Equivalence situations

Amongst the wide range of aspects noted, we will observe the four following :

(1) In the limit -pattern (use of boundaries) there is a low return in the algorithmic level , altough we observe the expected strategies of : conversion to common denominator, numerator-denominator division , looking for a known fraction by visual approximation . There are no general strategies : each case is solved in a particular way.

In the first group (10 year olds) , a considerable percentage (56,17% appear not to understand the term "between" (linguistic difficulty) . Three years later, the percentage drops to 20 %. The following table indicates the strategies adopted in relation to the different fractions .

TABLE OF STRATEGIES ADOPTED IN RESPONSE TO THE QUESTION "FIND THE FRACTION BETWEEN ... AND ..."

Between.....	% error	convert to c.d. take "next" numerator that lower	convert c.d. take before last numerator	convert c.d., take some middle one	divide num./ denom. express by decimal	divide num. by denom. & take decimal correspond	Add numerators and denominators	use some approximation	1 more num. 1 more denom.
13/16 1/2	20,5	9,5	7,78	12,7	10,87	—	—	—	—
5/8 7/16	12,2	14,87	8,46	—	9,83	—	2,29	1/2 5,37	—
9/4 15/8	16,1	19,5	8,24	—	4,80	—	0,65	2 4,64	—
2/3 4/5	13,6	20,59	—	—	2,29	1,37	0,8	—	17,05
1/4 3/12	diverse	8,58	They are equivalent , then, "there isn't one " 43,13%						

(2) In situations where two conditions are given to find the intermediate fraction which fulfills both , we observe an inability to operate and lack of intuitive knowledge of the limit-pattern as far as "superimposed neighbords" are concerned

In the next table, we can see how, apart from leaving the reply blank, some 15% do not understand the double condition . The increase from 1,5% to 7,55% is significant in the reply saying that a condition is missing and resolves one of the patterns .

percentage error Strategies / (question: a,b) →	10 year olds (a)	(b)	14 year olds (a)	(b)
Gives two solutions	8,4	1 9,3	7,8	7,3
Leaves it without reply	62,3	45,3	66,5	52,5
Puts down one of the fractions given	4,8	3,9	0,6	3,2
Says " there isn't one "	0,11	0,11	0,11	0,11
Gives a wrong fraction	23,3	26,4	21,8	23,8
Condition missing	0,7	1,5	2,8	7 ,55

When noting the strategies used , we observe (next table) how the only one that does not change is the addition of numerators and denominators (only 1 pupil in the 14 year old age group out of the whole sample !)Even the use of decimals is not stable in the percentage of correct replies given to other items of the test.

percentage error Strategies /	question a 10 y old	14 y olds	question b 10 y old	14 y olds
Reduces a c.d. considering the boundaries	0,11	7,21	0	2,17
Gives the answer in decimals and/or converts it into a decimal fraction .. .	0,11	4,12	0,11	2,17
Add the numerator and denominator taking into account the boundary	0	0,11	0	0,11
Correctly calculated by heart	0,11	1,71	0	0,80
Total	0,33	13,15	0,11	5,25

The said problem ,very important in order to understand the notion of the limit of a series , underlines the value of not only the idea of approximation but also the unassumed topological value leading to the compactness of R and the density of Q in R. On the basis of the global results , we dare to suggest the independent influence which the topological structure of Q exercises over the idea of density , in addition to the visin of infinity (Kidron-Vinner 1983) The factorial analysis also fitted in to the said hypothesis.

All this serves to demonstrate that the algorithmic process (in some cases known) is not able of resolving a case with superimposed boundaries. The fact that "two steps" are required in the common denominator algorithm (Chevallard 1989)

further contributes to the difficulty of the problem . Moreover, it appears possible to suggest a lack of clear transfer of the equivalence representation in the number line (Streefland 1983) ,which must be completed through didactic situations with neighbords intersection analysis. This would explain the success that Brousseau (1986) had with 10-11 year old pupils !!!

(3) We confirm the expected coincidence with the Kidron-Vinner study where pupils tend to give replies in concrete numbers (concrete versus abstract thinking)

Summarising our replies we note the categorisation of the following strategies :

Type of strategy	percentage (%)	11 y	14 y.
--> Intuitional thought	(yes, they are infinite)	7,2	12,8
	(yes, forced argument)	11,36	12,7
--> Algorithmic thought	(it exist if I can manage to find it)	8,28	12,45
--> Recalling a studied property *.		0,11	13,16
--> Denial(by finding particular case where it does not work)			
	(when they are not equivalent)	3,93	6,52
	(no,when they are not consecutive)	4,59	2,97
--> Other errors or incongruencies		6,55	3,2
--> No reply		49,29	30,6

* To what point can the idea of density become explicit without a complete vision ?

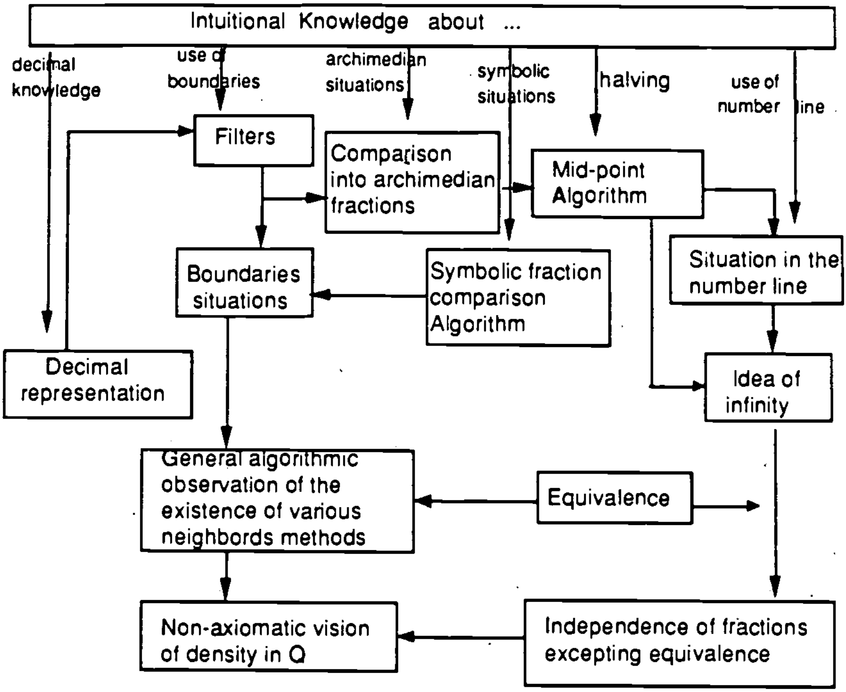
(4) The idea of situation in the number line bears relatively little correlation to the replies given about density. However, with the replies to the questions about approximation of additions in the number line, there is a greater correlation It's thought, therefore, that decimal filters should not be the only problem to be considered, but that the topological configuration of rational should also be taken into account. One observation about the phaenomenological difficulties encountered in these problems is found in the vision of more or less narrow division (Lemay 1978) in relation to the difficulty in comprehension of equality of parts (see Fredudenthal 1983) .

We also note that the uniform property (which is applied due to the fact that addition is an increasing function of two variables) , is an implicit reasoning which we expected from the pupils, but does not happen in practice. Also it appears to contribute to the notion of density in parallel to situations of boundaries-pattern (Giménez 1990b).

Brief conclusion .

We cosider that the diagram given below is an accurate representation of the observations we have noted and described (and others which, for lack of space, we cannot included here) :

Sequential Hierarchy Assisting the Pre-notion of Density



This diagram should be able to guide learning situations. The backward feed of the diagram through the observation of filters (decimals in Brousseau's sense, or archimedian based on the intuition of capacity in Giménez 1987) will make a way for a topological vision of the rational number which will complete the diagram of Kieren's intuitional knowledge (1988) . Effectively , a concept has a strength when it is situated in relation to others (Vergnaud 1981) .

Bearing in mind the above diagram , we have carried out experiments on 14 year old pupils (belonging to a socio-cultural group with greater difficulties than those of the cited studies) with a set a didactic situations and have obtained good results

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References .

BEHR ,M. et al.(1983) Rational number concepts in Lesh-Landau (eds)*Acquisition of mathematical concepts and processes* . Academic Press . New York.

BEHR,M et al. (1985) Main results about Rational Number Project in L.Streefland (ed.) *Proceedings of the IX PME.* Utrecht. Holland.

BEHR (1985) Order and equivalence of rational numbers : A clinical teaching experiment in Journal for *Research in Mathematics Education vol. 15* NCTM Reston VA pp. 323-341.

BEHR ,WACHSMUTH , POST (1986) Children's concept and estimation of rational numbers in H. Schoen (ed) *Estimation 1986 Yearbook of the NCTM* . Reston VA.

BROUSSEAU (1980) Problèmes de l'enseignement des décimaux en *Recherches en didactique des mathématiques nº 1, 1.* La Pensé Sauvage. Grenoble.

BROUSSEAU (1981) Problèmes de l'enseignement des décimaux en *Recherches en didactique des mathématiques nº 2, 1.* La Pensé Sauvage. Grenoble.

BROUSSEAU (1986) *Théorisation de phenomènes d'enseignement des mathématiques.* Thèse d'état . Univ. Bordeaux . Annexes .

BROWN (1981) *Children's understandig of number operations . Place value and decimals in Secondary school children.* ph D thesis.Chelsea College. Univ. of London

CHEVALLARD (1989) *L'enseignement des fractions au college* . Irem d' Aix -Marseille.

DAVIS ,R et al. (1982) *The roles of "understanding" in the learning of mathematics* . Report Univ. of Illinois Urbana.

DOUADY, R (1984) Jeux de cadres et dialectique outil-objet dans l'enseignement des mathématiques *Comptes rendus III Ecole d'été Didactique des Mathématiques* IMAG Grenoble.

FISCHBEIN,E,TIROSH,D and HESS,P. (1979) The intuition of infinity , *Ed Studies in Mathematics 10* , 1979 , pp3-40.

FREUDENTHAL,H (1983) *Didactical phaenomenology of mathematical structures.* Reidel. Dordrecht.

GIMENEZ,J (1987) Estudis exploratoris sobre concepcions de l'ordenació de fraccions . Dept. Educació i Psicologia Unpublished paper . Div. VII. Univ.Barcelona.

GIMENEZ ,J (1987) The use of fractions on capacity measures by young children in elementary school. (Exploratory study) Paper presented at the ICME V . Budapest .

GIMENEZ ,J (1988 a) Propuesta metodológica para la ensenanza de las fracciones en la educación básica . Paper presented to the *Reunion Centroamericana de ensenanza matematicas* S.Carlos de Guatemala.

GIMENEZ,J (1989) About continuous operator subconstruct in rational numbers. *Proceedings of PME 13* Paris pp. 10-14.

GIMENEZ,J (1990) *Realidad y juegos para un conocimiento intuitivo del órden en las fracciones.* Comunicación aceptada en el 1er congreso Iberoamericano de educación matemática. Sevilla 1990.

GIMENEZ,J. (1990) *Reflexiones sobre las propiedades de las operaciones y su necesaria interpretación funcional*. Suma . por aparecer.

HART (1980) CSMS Report. London .

HASEMANN (1989) Children's indibiduality in solving fraction problems in *Proceedings PME 13* . Paris pp 67-74.

KIDRON - VINNER (1983) Rational numbers and decimals at the senior high level . Density and comparison. *Proceedings of the PME 7*. Israel , pp.301-306.

KIEREN ,T (1976) On the mathematical , cognitive and instructional foundations of rational numbers in Lesh (ed) , *Number and measurement* ERI SMEAC . Ohio.

KIEREN (1981) *Five faces of mathematical knowledge building.* Occasional paper n. 20 Dept. of Secondary Education Univ. of Alberta.

KIEREN,T(1988) Personal knowledge of rational numbers Its intuitive and formal development in Hiebert-Behr (eds) *Number Concepts and operating in the middle grades* Reston. NCTM , LEA

LEMAY ,F (1978) Une conversation sur la division . Instantanés Mathématiques. Québec

PERRIN,M.J (1985) Réprésentation des fractions et des nombres décimaux chez les élèves de CM_2 et du college . Publ. IREM Univ. Paris VII . núm. 24.

PERRIN,M.J. (1986) A propos de l'enseignement des nombres décimaux et rationnels au collège a *Petit X* n.10 Irem de Grenoble .

STREEFLAND ,L(1983) *Aanzet tot een nieuwe breukendidactiek volgens Wiskobas* . Utrecht. OW&OC .

UNDERSTANDING THE MULTIPLICATIVE STRUCTURE

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Problem situations in the multiplicative conceptual field (MCF) are analyzed with respect to a wide range of task variables from five structures: numeric, semantic, propositional, contextual, and mathematical. This paper focuses on the Invariance Structure--a substructure of the Semantic Structure--and the mathematical structure which consists of principles underlying the Invariance Structure. These principles are organized into two main classes--order determinability principles class and order determination principles class--corresponding to two solution actions--the action of declaring whether an order relation between two quantities is determinable and the action of specifying what that order relation is. The analysis is believed to (a) reflect the structure of the MCF; (b) contributes to our understanding of what constitutes a conceptual knowledge of the MCF and solutions of multiplicative problems; (c) guide researchers to design activities for children to construct MCF knowledge; and (d) aid to investigation of the effect of various multiplicative variables on children's performance.

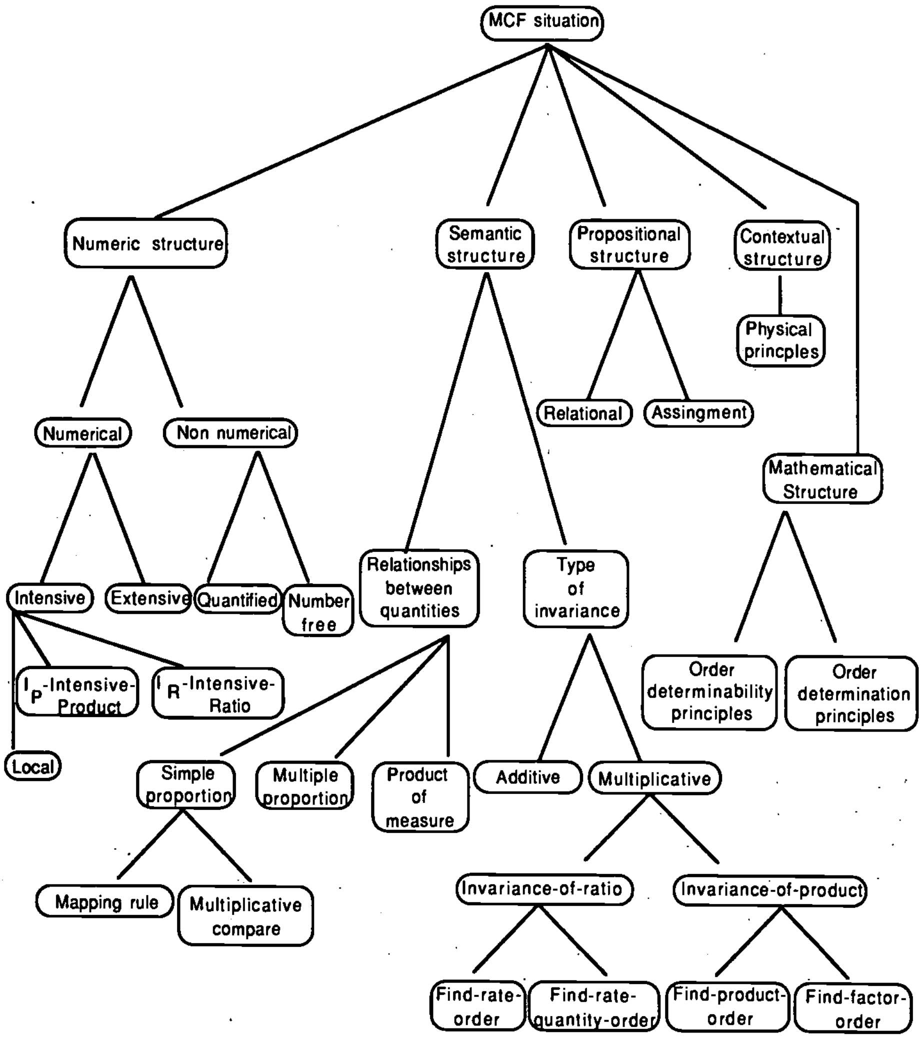
In the last few years, we have focused our research on theoretical analyses of the multiplicative conceptual field (MCF), based on our own and others' previous work on the acquisition of multiplicative concepts--such as, multiplication, division, fraction, ratio, and proportion--and relationships among them. The goal of this resesarch is to better understand the mathematical, cognitive, and instructional aspects of the MCF structure. Results from this research indicate that the mathematical structure of the MCF is very complex and cognitively very demanding (see Harel and Behr, 1989; Harel, Behr, Post, and Lesh, in press; Behr, Harel, Post, and Lesh, in press; Behr and Harel, this volume). Thus, because of the importance and ubiquity of the MCF in mathematics, it is an important challenge for mathematics educators, instructional psychologists, and curriculum developer, to study this field and understand children's acquisition of the knowledge it involves, so that instruction that facilitates children's construction of this knowledge can be devised.

Vergnaud (1988) introduced the notion of conceptual field "as a set of situations, the mastery of which requires mastery of several concepts of different natures" (p. 141). In particular, Vergnaud characterized the multiplicative conceptual field (MCF) as consisting of all problem situations whose solutions involve multiplication or division, and classefied these situations into three categories: simple proportion, product of measure, and multiple proportion (Vergnaud, 1983, 1988). In our recent work on the structure of the MCF, we analyzed multiplicative situations with respect to a wide range of task variables, some of which are known to the research on multiplicative concepts, others are new. This analysis, integrates Vergnaud's analysis and others' analyses (e.g., Nesher, 1988; Thompson, 1990), and results in a system of structures which are believed to reflect the nature of the the MCF, both mathematically and cognitively.

The analysis is organized around five structures: Numeric, Semantic, Propositional, Contextual, and Mathematical, with several subtructures (Figure 1). Because of space restrictions, we limit our discussion to two strongly interrelated structures: the Invariance Structure (one of the

substructure of the Semantic Structure) and the Mathematical Structure. The latter consists of mathematical principles classes underlying situations in the Invariance Structure.

Figure 1



Multiplicative Invariance: Problem Situations and Solution Principles

Variation is one of the most important and ubiquitous actions in mathematical reasoning. In

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many problem situations in mathematics, quantities are varied by applying to them a certain transformation. The goal in these problems is to identify the type of transformation applied in order to determine whether the relation between the initial quantities is invariant under the transformation applied, and, if not, to determine the new relation between the quantities that resulted from the transformation, and/or to search for an appropriate compensation for the change that results. For example, in problems such as "divide 163 by 0.3," children are taught to change the decimal divisor 0.3 to the whole-number divisor 3 by multiplying by 10, divide 163 by 3, and, multiply the result by 10. What constitutes the understanding of this procedure is the awareness that the *equality* relation between the dividend 163, the divisor 0.3, and their quotient is not invariant under the change of the divisor 0.3 to 3, and that the "multiply by 10" transformation--applied to the quotient of 163÷3--is an appropriate compensation for this change. Moreover, many of children's inventions of computations can be explained in terms of reasoning on variation and invariance. We bring two examples of such inventions. The first is form the domain of basic addition. Cobb and Merkel (1989) reported that children solve the problem 8+9=? by changing the 9 into 8+1, to get the well-remembered basic fact 8+8=16, and then add 1. We hypothesize that when children use this strategy, they are aware (at least implicitly) of the fact that the equality relation in this problem is invariant under the change of 9 to (8+1). The second example is from the domain of division. We asked an eight-year-old child to solve the problem 156÷2.75, before he was taught the standard algorithm for dividing decimals. In responding to this problem, he multiplied 2.75 by 2 and got 6.5, then he multiplied the 6.5 again by 2 and got 13. He then divided 156 by 13, using the standard algorithm, and multiplied the result, 12, by 4. We believe that the conceptual base for this solution is the awareness that *changing the problem quantities is an allowable action provided that appropriate compensations are applied* . This child applied consecutively the "multiply by 2 transformation" to change the decimal divisor 2.75 into the whole number divisor 13; he did so to change the given problem, 156÷2.75, into a familiar one, 156÷13. He knew that this quantity change is allowable but it must be followed by an appropriate compensation with respect to the equality relation between the dividend and divisor on the one hand and the their quotient on the other. He chose a multiplicative compensation by applying the "multiply by 4" transformation to the quotient of 165÷13. This solution as well as many other solution procedures invented by or taught to children are based on basic*invariance principles* which will be specified below.

Problem situations. Additive reasoning and multiplicative or proportional reasoning can be characterized in terms of the type of compensation, additive or multiplicative, children employ for the transformation(s) applied to the problem quantities. Children who are additive reasoners interpret changes to the values of the quantities as additive transformations, and, therefore, employ additive compensations even in situations in which multiplicative compensations are required. Proportional reasoners, on the other hand, differentiate between situations in which quantity changes must be interpreted as additive transformations and those which must be interpreted as

multiplicative transformations, and, accordingly, employ the appropriate compensations. When the transformation "add k," for example, is applied to the quantity a in the inequality a < b, children have little difficulties finding an appropriate compensation that keeps the direction of the order relation unchanged (e.g., add k to the quantity b). On the other hand, children have great difficulty solving problems in which multiplicative compensations are involved; for example they do not easily understand that the "smaller than" relation in the inequality $a / b < c / d$ is invariant under the transformation "add k" (where k is positive) to the quantity b, and offer the transformation "add k" to the quantity d as a compensation for this transformation. It is the awareness of the type of transformation (additive or multiplicative) applied and the ability to find the appropriate compensation for that transformation that constitutes proportional reasoning.

An important goal for mathematics education is *to identify the knowledge that constitutes adequate reasoning for additive invariance and multiplicative invariance tasks, and to offer learning activities that can help children to develop such knowledge* . In the rest of this paper, we focus on categories of multiplicative situations, and present classes of mathematical principles that constitute their solutions. From a pure mathematical point of view, these principles are easily derived from the axioms of the ordered field, and thus, they are self-evidence, at least for mathematically sophisticated people. From a cognitive point of view, however, as research on the concept of proportion imply, these principle are not obvious.

Multiplicative and proportion tasks are instances of two general categories: **Invariance-of-ratio** and**Invariance-of-product** (see Figure 1). The classification of tasks into hese categories was made according to the type of solutions--whether are based on ratio comparison or on product comparison--commonly used by subjects. For example, the orange concentrate task used by Noelting (1980) belongs to the invariance-of-ratio category, because the correct solutions used by children to solve this task always involve comparison of two ratios. On the other hand, the balance scale task used by Siegler (1976) belongs to the invariance-of-product category, because the correct solutions used by children in Siegler's study involve comparison of the products of the distance-weight values for each side of the fulcrum to determine which side goes down.

Quite different mathematical principles, and thus different reasoning patterns, are involved in the solution of these types of tasks. To describe these principles, a refinement of each category is needed (see Figure 1). There are two subcategories of the invariance-of-product category: The **Find-product-order** Subcategory consists of problems which give two order relations between corresponding factors in two products and asks about the order relation between the two products (e.g., given that a > b and c = d and the question is about the order relation between a X c and b X d); and the **Find-factor-order** Subcategory consists of problems which give an order relation between two products and an order relation between two corresponding factors in the two products, and asks about the order relation between the other two factors (e.g., given that a X c < b X d, and a>b, and the question is about the order relation between c and d). Likewise, there are two subcategories of the invariance-of-ratio category: The **Find-rate-order** Subcategory which

consists of problems which give two order relations between corresponding quantities in the two rate pairs and asks about the order relation between the two rates (e.g., given that a > b and c = d and the question is about the order relation between a / c and b / d); and the **Find-rate-quantity-order** Subcategory which consists of problems which give an order relation between two rates and an order relation between two corresponding quantities in the two rate pairs and asks about the order relation between the other two quantities in the two rate pairs (e.g., given that a / c = b / d, and a > b, and the question is about the order relation between c and d).

Solution principles. Each task from either one of these four subcategories ivolves three number pairs--a and b, c and d, and either a pair of product a X c and b X d or a pair of ratios a / c and b / d--and the order relation between quantities within two of the three pairs are given and the problem is to determine, if possible, the order relation between the quantities within the third pair. A solution of such a task includes two actions: finding out whether the third order relation is determinable, and, if so, then determining what that order relation is. Accordingly, the knowledge involved in solving these types of tasks relies on two categories of classes: the **order determinability principles** class and the **order determination principles** class. The order determinability principles specify the conditions under which order relations between quantities within two pairs can lead to the action of declaring whether the order relation between quantities within the third pair is determinate or indeterminate (e.g., if a and b are equal but c and d are unequal, then the order relation between the products a X c and b X d, or between the ratios a / c and b / d, is determinate). The order determination principles specify the conditions under which order relations between two quantities within two pairs can lead to the action of declaring that the order relation between the two quantities within the third pair to be less than, greater than, or equals (e.g., if a and b are equal but c is greater than d, then a / c < b / d).

We have identified two pairs of classes of multiplicative principles: one pair concerns the order determinability principles, the other the order determination principles (Figure 2). Two classes, one from each pair, correspond to the invariance-of-product category--one is called the **product order determinability principles class**, the other the **product order determination principles class**. The other two classes correspond to the invariance-of-ratio category--one is called the **ratio order determinability principles class**, the other the **ratio order determination principles class**. Each of the two product order (determinability or determination) principles classes is further divided into two subclasses--**product composition (PC) subclass** and**product decomposition (PD) subclass**--depending on whether the principles solve tasks from the find-product-order subcategory or the find-factor-order subcategory, respectively. Similarly, each of the two ratio order (determinability or determination) principles classes is further divided into two subclasses--**ratio composition (RC) subclass** and **ratio decomposition (RD) subclass**--depending on whether the principles solve tasks from the find-rate-order subcategory or find-rate-quantity-order subcategory, respectively.

Figure 2

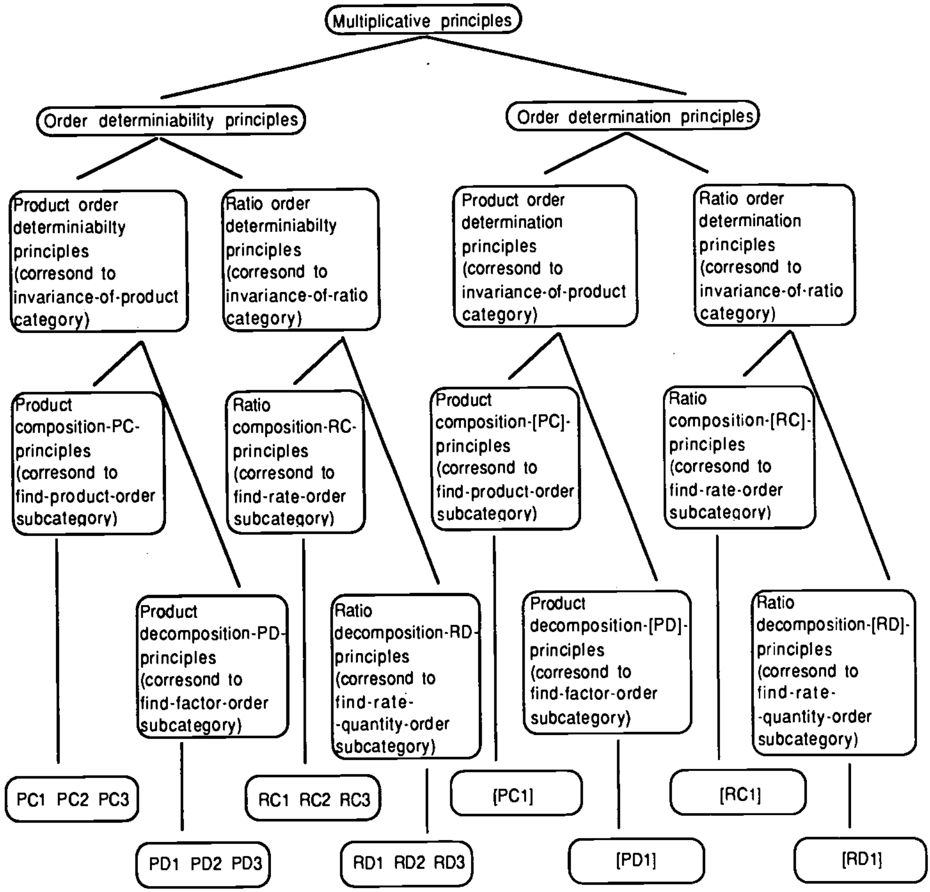

Figure 2 further presents the specific principles in each subclass. The product composition subclass consists of: **PC1**--the order relation between the products a X c and b X d is determinate if the order relations between a and b is the same as the order relation between c and d or if one of them is the equals relation; **PC2**--the order relation between the products a X c and b X d is indeterminate if the order relations between a and b conflicts with (i.e., is in the opposite direction) the order relation between c and d; and **PC3**--the order relation between the products a X c and b X d is indeterminate if one of the order relations, either between a and b or between c and d, is unknown.

The second subclass of order determinability principles is the product decomposition (PD)

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subclass; it consists of: **PD1**--the order relation between the factors a and b in the products a X c and b X d is determinate if the order relation between the factors c and d conflicts with the order relation between the products a X c and b X d or if one is the equals relation; **PD2**--the order relation between the factors a and b in the products a X c and b X d is indeterminate if the order relations between the other two quantities, c and d, is the same as the order relation between the products a X c and b X d and neither is the equals relation; and **PD3**--the order relation between the factors a and b in the products a X c and b X d is indeterminate if one of the order relations, either between c and d or between a X c and b X d, is unknown.

The class of ratio order determinability principles also consists of two subclasses. The first is the ratio composition (RC) subclass; it consists of the following principles: **RC1**--the order relation between the ratios a / c and b / d is determinate if the order relations between a and b conflicts with the order relation between c and d or one is the equals relation; **RC2**--the order relation between the ratios a / c and b / d is indeterminate if the order relations between a and b is the same as the order relation between c and d and neither one is the equals relation; and **RC3**--the order relation between the ratios a / c and b / d is indeterminate if one of the order relations, either between a and b or between c and d, is unknown.

The second subclass of the ratio order determinability class is the ratio decomposition (RD) subclass, which consists of the following principles: **RD1**--the order relation between the rate quantities a and b in the ratios a / c and b / d is determinate if the order relation between the rate-quantities c and d conflicts with the order relation between the ratios a / c and b / d or if one is the equals relation; **RD2**--the order relation between the rate quantities a and b in the ratios a / c and b / d is indeterminate if the order relations between the other two rate-quantities, c and d, is the same as the order relation between the ratios a / c and b / d and neither is the equals relation; and **RD3**--the order relation between the rate quantities a and b in the ratios a / c and b / d is indeterminate if one of the order relations, either between c and d or between a / c and b / d, is unknown.

Once a determinability principle is applied and the requested order relation is found to be determinable, then a determination principle can be applied to ascertain whether that relation is the less than, equals, or greater than relation. There is a determination principle which corresponds to each of the determinability principles which ascertains that the required order relation is determinate (i.e., the first principle in each of the above subclasses). We denote them by [PC1], [PD1], [RC1], [RD1], respectively. For example, an instantiation of [PD1] is: if c > d and a X c < b X d, then a < b.

Summary

The MCF analysis, partly presented in this paper, can contribute to research on learning, curriculum development, and teaching, in several ways. First, an important objective for mathematics education is to find instructional ways that enable children to construct conceptual knowledge--knowledge which is rich in relationships among concepts (Hiebert and Lefevre, 1986).

Research on knowledge development has widely dealt with the characteristics of this knowledge, especially in mathematics, but has done little to identify the means by which such a knowledge is constructed. In our view, what constitutes a relationship among mathematical objects in a student's mind is a set of mathematical principles which explicitly or implicitly employed by the student in the process of constructing that relationship. The analysis presented in this paper can guide researchers to design activities that help children construct mathematical principles which would facilitate the understanding of relationships among different concepts in the MCF. Second, these mathematical principles are believed to be the foundation for multiplicative and proportional reasoning; that is, these principles are the basic theorems in actions (ala Vergnaud, 1988), from which more complex theorem in actions can be constructed by children in solving advanced multiplicative and proportional reasoning problems. Finally, this analysis identifies several variables with subvariables which researchers could manipulate in multiplicative and proportion tasks and in experimental instruction to investigate the effect on children's performance. Finally, this analysis draws attention to the important issue of mathematical invariance, in investigating children's reasoning and in designing instructional activities. Some related research questions are: (1) does an awareness of the invariance principles described here help children to solve multiplicative and proportion problems? (2) what instructional activities can help children construct these principles? (3) what constitutes children's ability to distinguish between additive invariance and multiplicative or proportional invariance? (4) how could additive and multiplicative concepts and relationships and distinction among them be introduced through the mathematical invariance idea?

References

Behr, M., Harel, G., Post, T., & Lesh, R. (In press). Rational number, ratio, and proportion. In D. Grouws (Ed) Handbook for Research on Mathematics Teaching and Learning.

Harel, G., & Behr, M. (1989). Structure and hierarchy of missing value proportion problems and their representations. *The Journal of Mathematical Behavior*, *8*, 77-119.

Harel, G., Behr, M., Post, T., & Lesh, R. (in press). The blocks task; comparative analysis of the task with other proportion task, and qualitative reasoning skills of 7-th grade children in solving the task. *Cognition and Instruction*.

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades* (pp. 19-40). Reston: National Council of Teachers of Mathematics.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part I, Differentiation of stages. *Educational Studies in Mathematics*, *II*, 217-253.

Siegler, R. S. (1976). Three aspects of cognitive development. *Cognitive Psychology*, *8*, 481-520.

Thompson, P. (1989). Cognitive model of quantity base reasoning in algebra, Paper read at the AERA Annual meeting, San Francisco, California.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh &M. Landau (Eds.) *Acquisition of Mathematics Concepts and Processes* (pp. 127-124). New York: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades* (pp. 141-161). Reston: NCTM.

Cobb, P. & Merkel, G. (1989). Thinking strategies: Teaching arithmetic thourgh problem solving. In P. Trafton (Ed.), *New Direction for Elementary School Mathematics*. Reston: NCTM.

A CONTEXTUAL APPROACH TO THE TEACHING AND LEARNING OF MATHEMATICS: OUTLINING A TEACHING STRATEGY THAT MAKES USE OF PUPILS' REAL WORLD EXPERIENCES AND STRATEGIES, AND THE RESULTS OF THE FIRST TEACHING EXPERIMENT OF PROJECT

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Firstly this paper outlines an alternative teaching strategy to prevailing methods of teaching new mathematical concepts and techniques, one in which the pupils' related real world experiences and intuitive models of thought and action are taken into account and made use of from the very first stages of the teaching of mathematical contents. Secondly the main results of the first teaching experiment of the project are examined. This teaching experiment focused on developing the pupils' proportional reasoning in a continual and discrete mixing context.

1. Theoretical backgroud and outlines for the teaching of mathematics in accordance with the contextual approach

There are various alternatives regarding how to organize the teaching of purely traditional mathematics and mathematical models and applications. According to the prevailing "formalistic" method fof teaching mathematical concepts and techniques, the teaching of mathematics may be arranged as follows: first, a symbolized definition of mathematical idea to be learned is given, then algorithmic exercises based on certain rules are performed and finally, an effort is made to "apply" the techniques learned to certain spesific exercises. In short, we teach mathematical abstractions by concretizing them (Freudenthal 1983).

The teaching strategy described above involves the risk that the idea of mathematics conveyd to the pupils may remain too narrow. Mathematics is experienced as a symbolic calculation system bearing few connections with real world phenomena and events, and mathematics as a model forming science that analyzes phenomena and events of the physical, social and intllectual world is afforded little attention. Indeed, the pupils are deprived of the opportunity to invent and develop their own the tools for analyzing and resolving quantitative real world phenomena

and events. Another risk is that the solutions reached do not take into account or make use of the experiences and cognitive schemes brought to the learning situation by the pupils. It can be proved that this is exactly what has happened in the case of the teaching of ratio and proportion in the comprehensive school in Finland (Keranto 1989).

Thus, good grounds exist for searching and experimenting with an alternative teaching strategy, one that takes into consideration and makes use of the pupils' real world experiences and available means for solving problems. Such a teaching strategy should also enable the formation of general mathematical models, giving the pupils an opportunity to formulate and develop the mathematics needed to solve exercises belonging to certain categories.

The aim was therefore to develop a strategy which is the inverse of "formalistic" approach, which sets out from real world quantitative phenomena and problems that really require the kind of pure mathematics that is intended to be learned. In this way, the pupil is in a position to discover gradually the true meaning of the mathematical tools, like ratio and proportion, presented in a symbolic form, and the driving force force crystallized in them through the work of the mathematicians of previous generations. This teaching strategy based on both real world connections and the pupils' experiences and models for solutions will be referred to later contextual approach (cf. the development of scientific concepts, Davidov 1982, Leontyev 1977, Vygotsky 1962, "mixing approach, Carss (ed.), 1986, 202, "didactical solution", Freudenthal 1983, "modelling approach", Heller et al 1989).

2. The contextual approach from the viewpoint of the pupil's learning acts

In the following, I will try to provide a brief outline of the contextual strategy from the viewpoint of a pupil's learning acts. My purpose here is also to prepare the reader for the first teaching experiment of the project. The following model has been elaborated on the basis of the ideas presented by Davidov (see Davidov 1982). Ratio, proportion and proportionality were chosen as the mathematical example in figure 1 below, as their teaching/learning process is the most important object for the research and elaboration in the project so far as content is concerned.

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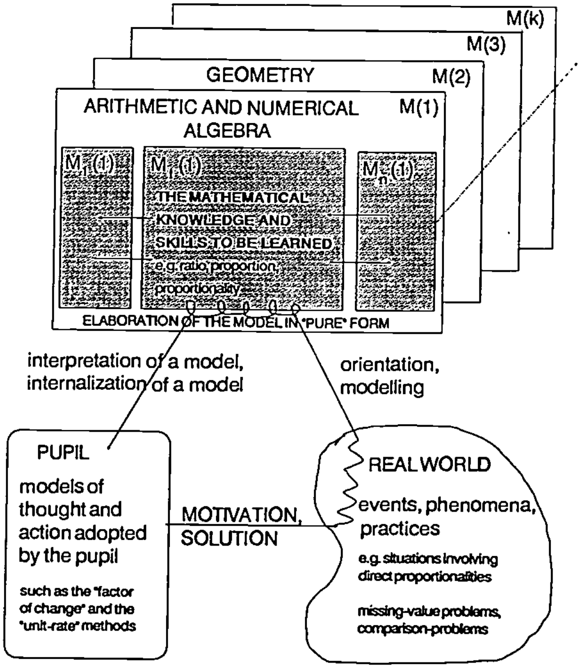

Figure 1. The whole of learning acts in the contextual approach to the teaching of mathematics

In the first stage, the attention of the pupils is derected to the examination of such real world situations, a valid analysis of which calls for mastery of the mathematical knowledge and skills that are meant to be learned. When the situations are analyzed, a cognitive conflict that may be established spontaneously and explicitly is generated between the pupils' output and the objective requirements how the problems should be solved. The main purpose of this stage is to generate an internal tension that motivates the pupils to find a valid model for the solution of the problems that were presented. In the teaching of proportional reasoning, this conflict may be produced for example in a mixing context. The (erranous) output by pupils obtained through mental reasoning is compared to the strengths of taste and colour in mixed liquids that were actually produced.

In the second stage, an attempt is made to direct the pupils' attention to features relevant for the solution of the entiry category of problems. By modifying systematically the way in which the problems are put, the pupils are helped to find the basic principle. In the context of liquid mixtures, for instance, the basic principle for producing mixtures of the same strenght is the invariance of the proportional shares of the liquides to be mixed. The within and between ratios remain constant.

In the third stage, an attempt is made to model the observed basic relations. The discovered mathematical model is developed in a "pure" form to such an extent that the entire category of problems that was originally under discussion can be solved in a valid way. For instance, mental identification and production of equally strong mixtures of liquid can be based on the observation of within and between ratios and on the formation of equivalent ratios (see within and between ratios, Karplus et al 1983, cf. within and between strategies, Noelting 1980 the factor of change method, Post et al 1988, scalar and function operator, Vergnaud 1983).

In the fourth stage, one reverts to solve the problems that proved too difficult at first with the aid of the mathematical tools that were learned. A new cycle starts when the pupils are taught to observe that the mathematical method that was developed is still defective of difficult to make good use in a certain problems. For instance, the pupils can be first taught to use the factor of change method to solve comparison problems involving integral ratios. At the end of the first cycle, one can present problems involving such difficult numbers and ratios that other methods are needed or worth using to solve these problems, such as the unit-rate method or cross-multiplication method (for a summary of these methods, see Post et al 1988).

3. Attempts to implement teaching in accordance with the contextual approach: the first teaching experiment

What should be done to enable mathematics teachers to teach mathematics as described above? It is obvious that an emphatic supply of new experiences and alternative teaching programs is needed. A basic task is to construct and experiment contextual teaching programs like in the project initiated by me in the autumn term of 1988. The most important practical goal of the project is to construct a school course of ratio and proportionality that could be used as a model to teach also other appropriate types of demanding mathematical knowledge and skills in the way described.

The first teaching experiment of this project was put into effect in the autumn term of 1988 in one of the lower classes of Oulu Teacher Training Shool, class 5-6a (n=20). The writer composed a 12-lesson syllabus and acted as teacher, with Viljo Vänskä (the teacher of the class) assisting as necessary.

The main aim of this experiment was find out how the syllabus designed to develop proportional reasoning affects the pupils' performance level and the strategies used by them in the comparison and missing value problems. At the same time, it was the intention to determine and validity of purpose-made multiple-choice tests for evaluating the pupils' levels of performance and the strategies used by them.

The teaching experiment had fore phases:
(1) assesment of the subjects' starting level in propornal reasoning by means of classroom tests developed for the purpose (juice I, 22 comparison problems, continual mixing con- text; lottery I, 23 comparison problems, discrete mixing context; lottery II, 14 missing value problems, discrete mixing context);
(2) the actual nine lessons: the purpose was to learn how to solve mentally missing-value and comparison problems involving integral pairs by observing the within and between multiplicity of these pairs and by forming equivalent ratios if necessary (cf. the factor of change method, Post et al 1988);
(3) Final testing to assess the development of proportional reasoning in the subjects (lottery I, discrete mixing context);
(4) delayed final testing to assess permanency and transfer of the level of performance and the solution strategies (juice I, continual mixing context, in february 1989).

It may be mentioned that the number structures of the tests juice I and lottery I were identical with the expection of two tasks (E, IE, WB, W, B, N, WBX, WX, BX, NX, see problem structures W, B, N and X, Karplus et al 1983). The numbers in the pairs were positive integrals. The problems in lottery II experiment were similar in number structure to those problems in lottery I experiment in which the ratios examined were equivalent.

The starting point for the construction of these tests were the individual tests developed by Noelting and the research group formed by Karplus, Pulos and Stage, as weel as the classroom tests used by the author in a previous study (see orange juice experiment, Noelting 1980, the lemon juice experiment, Karplus et al 1983, the juice experiments I & II, Keranto 1986).

For a summarirized general idea of what kind of problems these tests include, which developmental level of proportional reasoning is presented by perfect performance in each problem and how the used invalid strategies are separated from each other and valid multiplicative strategies, let us take a look at table 1.

It is thus possible, in principle, to try to infer by this test the level of proportional reasoning in each subject in terms of strategies used and performance levels attained. In short, the strategies used are inferred as follows: the performance pattern of the pupil's choices is compared to an solution pattern resulting from the consistent use of each strategy presented above (see columns 4 to 8 in the table 1). The procedure is the same both for items involving equivalent ratios (WB, W, B, N) and for ones involving inequivalent ratios (WBX, WX, BX, NX).

TABLE 1. The items of lottery test I, the developmental levels of proportional reasoning corresponding to valid performance in each item, the answer patterns derived from the consistent use of each strategy, and the order in which the problems were presented (question: "From which can, A or B, are you more likely to draw the winning ticket, or are the chances equal?"

	items cans A (w e) B (w e)	stage Noelting	problem structure Karplus et al	strategies & patterns 1-dim (w)	1-dim (e)	Am	Add	Pro (oikea rivi)	order of presentation
0.	3:2 vs. 2:3	IA		A	A	E	A	A	instruction
1.	4:2 vs. 5:2	IB		B	E	B	B	B	15
2.	2:3 vs. 2:5			E	A	B	A	A	1
3.	4:2 vs. 3:3	IC	IE	A	A	E	A	A	6
4.	4:4 vs. 3:2			A	B	A	B	B	10
5.	2:2 vs. 3:3	IIA	E	B	A	B	E	E	14
6.	5:5 vs. 2:2			A	B	A	E	E	2
7.	1:2 vs. 2:4	IIB	WB	B	A	B	A	E	11
8.	4:2 vs. 2:1			A	B	A	A	E	18
9.	2:4 vs. 3:6		W	B	A	B	A	E	16
10.	6:3 vs. 2:1			A	B	A	A	E	3
11.	3:2 vs. 6:4		B	B	A	B	B	E	21
12.	5:2 vs. 10:4			B	A	B	B	E	7
13.	4:6 vs. 10:15		N	B	A	B	A	E	12
14.	4:10 vs. 2:5			B	A	B	A	E	23
15.	2:1 vs. 4:3	IIIA	WBX	B	A	B	E	A	4
16.	1:2 vs. 2:5			B	A	B	A	A	20
17.	4:2 vs. 5:3		WX	B	A	B	E	A	13
18.	3:6 vs. 2:5			A	B	A	E	A	17
19.	3:2 vs. 6:5		BX	B	A	B	E	A	8
20.	4:5 vs. 2:3			A	B	A	E	A	19
21.	2:3 vs. 3:4	IIIB	NX	B	A	B	E	B	5
22.	4:3 vs. 3:2			A	B	A	E	B	9
23.	5:2 vs. 7:3			B	A	B	B	A	22

A, B, E: A, if the chance to draw winning ticket from can A is bigger; B, if the chance to draw winning ticket from can B is bigger; E, if the chances are equal.

w = the number of winning tickets
e = the number of empty tickets

Stages (Noelting 1980a, b):
IA (lower intuitive); IB (middle intuitive); IC (high intuitive); IIA (lower concrete operations); IIB (higher concrete operations); IIIA (lower formal operations); IIIB (hugher formal operations).

Problems structure (cf. Karplus et al 1983, 52-59):
IE (n:n vs. k:p), E (equivalency category 1:1), W (within combination ratio integral); B (between combination ratio integral); X (unequal ratios).

Strategies: 1-dim(w), 1-dim(e) (unidimensional comparison of winning/empty tickets); Am (additive comparison of the amounts of winning and empty tickets); Add (comparison of the differences of winning and empty tickets); Pro (proportional reasoning; multiplicative strategies, observation of within and/or between ratios, formation of equivalent ratios).

4. The principal results of the teaching experiment

It is only possible in this article to summarize some of the most important results. The pricipal results of this first teaching experiment were as follows:

(1) The test items constructed on the basis of earlier empirical research and rational task analysis formed hierarchically increasingly difficult séries which corresponded for the most part to the hypothesized order of difficulty for the items in question and to the empirical results of previous interviews.

(2) For the most part, the pupils used certain strategies very consistently over the combinations IIB:= (WB, W, B, N) and III:= (WBX, WX, BX, NX). These strategies could be identified with relative ease and reliability by means of the developed technique.

(3) The syllabus proved to be most beneficial to those pupils who had done well or ver well in their previous mathematics studies. Most of them advanced from using additive strategies to using multiplicative strategies. As far as those whose previous performance had been bad or fair concerned, the syllabus consentrating to the factor of change method had little effect on their level of performance and the additive strategies they had using at the beginning.

(4) The results of the delayed final test juice I showed that the use of strategies was relatively permanent and generalized also to the solution of liquid mixture problems.

The experiences and the results from this first teaching experiment of the project suggest that it is possible and gainful to develop proportional reasoning with the contextual teaching strategies in some cases. However, to construct a really good contextual syllabus for the teaching and learning ratio and proportionality, a series of teaching experiments need to be performed which is also the intention of this project.

References

Carss, M. (ed.) 1986. Proceedings of the fifth International Congress on Mathematical Education. Boston: Birkhauser.

Davidov, V.V. 1982. The psychological structure and contents of the learning activity in school children. In R. Glaser & J. Lompscher (eds.) Cognitive and motivational aspects of instruction. Amsterdam, Holland: North Holland Publishing Company, 37-44.

Freudenthal, H. 1983. Didactical phenomenology of mathematical structures. Dordrecht, Holland: Reidel.

Heller, P.M., Ahlgren, A., Post, T., Behr, M. & Lesh, R. 1989. Proportional reasoning: the effect of two context variables, rate type, and problem setting. Journal of research in science teaching 3, 205-220.

Karplus, R., Pulos, S. & Stage, E.K. 1983. Proportional reasoning of early adolescents. In R.Lesh & M.Landau (eds.) Acquisition of mathematics concepts and processes. Orlando, FL: Academic Press, 45-90.

Keranto, T. 1986. Rational and empirical analysis for the measurement and instruction of proportional reasoning in school. In P.Kupari (ed.) Mathematics Education in Finland: Yearbook 1985. University of Jyväskylä. Institute for Educational Research. Publication Series B. Theory into practice 3, 1-33.

Keranto, T. 1989. The significance of cognitive psychology on mathematics teaching and didactic research. In P. Kupari (ed.) Mathematics Education in Finland: Yearbook 1987-1988. University of Jyväskylä. Institute for Educational research. Publication Series B. Theory into practice 39, 25-33.

Keranto, T. 1989. A contextual approach to the teaching and learning of mathematics. In P.Heikkinen & L.Rautio (eds.) Academic Lectures from Oulu. University of Oulu. Acta Universitatis Ouluensis, F3, 15-19.

Leontyev, A.N. 1977. Toiminta, tietoisuus, persoonallisuus. Helsinki: Kansankulttuuri.

Noelting, G. 1980. The development of proportional reasoning and ratio concept. Part I: differentiation of stages. Educational Studies in Mathematics 11, 217-253.

Post, T., Behr, M. & Lesh, R. 1988. Proportionality and the development of prealgebra understanding. In the yearbook 1988: The ideas of algebra, K-12. NCTM, 78-89.

Vergnaud, G. 1983. Multiplicative structures. In R.Lesh & M.Landau (eds.) Acquisition of mathematics concepts and processes. Orlando, FL: Academic Press, 127-174.

Vygotsky, L.S. 1962. Thought and language. Cambridge, Mass: M.I.T. Press.

ON CHILDREN'S MATHEMATICS INFORMAL METHOD

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ABSTRACT

Unlike to their counterparts in some western countries, Taiwan students do not use informal methods prevalently within school mathematics context. A small sample of Junior high school students are interviewed to examine whether they do or do not have informal methods available to them. The nature of children's informal methods and its implications in teaching are discussed.

INFORMAL METHOD

There have been many observations made in different societies that children often do not use the formally taught procedures to solve mathematics problems, but rather use informal procedures of their own (e.g. Carpenter et al, 1983; Booth, 1981; Hart, 1981; Carraher et al, 1985). Some of this work (e.g. Hart, 1981) has suggested that children's

use of such procedures enables them to solve 'simple' problems in mathematics (such as those involving small integers) successfully, but limits their progress in mathematics since these methods are generally not readily applicable to 'harder' problems (such as problems involving large numbers or non-integers). Linked to this use of informal procedures, is almost certainly holding by the child of intuitive models which are also integer models (e.g. Fischbein et al, 1985).

The development of these informal procedures may well relate in part to the child's experiences either in the elementary mathematics classroom or in the living. Formal procedures may require recall of certain concepts and facts. Children who do not have these concepts and facts readily available may be 'forced' to use counting or addition-based procedures which result in the continued use of the informal rather than formal procedures (Hart, ibid). This might suggest that in countries where less emphasis is placed in the classroom upon informal mathematics, and where considerable stress is placed upon the learning of rules, algorithms and number facts, the development of such informal procedures may be less apparent.

In Taiwan , the considerable educational premium is placed upon academic achievement. Mathematics teaching methods tend to be traditional instruction in emphasis. Taiwan students are more prepared to learn facts and algorithms than perhaps are their western counterparts. It was confirmed that items (such as fractions computation) which call primarily for recall of standard rules and procedures have a higher facility in Taiwan than in, say, England (e.g. Yang, 1988; Lin, 1988). However, items which

can be solved by informal procedures have higher facility in England than in Taiwan (e.g. Lin, 1988). For example, on a recipe question in which the amount for eight people were given, English children tend to use 'halving' to find the amount for four people, and 'building up using halving' to find the amount for six people; while Taiwan children use prevalently taught multiplicative algorithms to solve the question in both situations. Since the lower level Taiwan students use taught algorithms badly, the facilities of items in the recipe question in Taiwan are roughly 20% lower than in England. In Taiwan, no particular tendency for children to use informal procedures appears to have been noted by teachers, and preliminary research in this area has so far reported only very low incidences of such usage (e.g. Lin, 1988).

The fact that children in Taiwan do not spontaneously report the use of informal procedures within a test context may not, of course, be a sound indicator that they do not have such methods available to them. Carraher et al (1985) have shown that Brazil children who are selling coconut in the street out of school can use informal counting procedures to make exchange within their business context, but can not use them in school mathematics problem context. So, it will be interesting to examine Taiwan children whether they do have informal methods available to them but do not perform it in school test context, or they do not develop such informal methods?

INVESTIGATION

In order to make an initial examination of this question, a small investigation was conducted within a 'conversation' setting during which students were given the task of suggesting how to solve certain problems without recourse to 'school mathematics'. For this purpose, a small sample of six grade 7 (age 13 years) students from a junior high school in Taipei was selected to represent the top, middle and lower ranges of mathematical ability within their grade, as judged by the teacher of the mixed-ability class from which they came.

The students were 'interviewed' as a group so that discussion could take place among them, but the tasks were worked on, and solutions presented individually. Subsequent checks were made between written and verbally presented methods of solution to ensure that the presentation given verbally had not been changed to coincide with ideas already presented by another student.

The discussion which follows focuses upon only one of the tasks presented, namely, the recipe question.

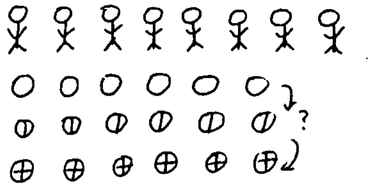
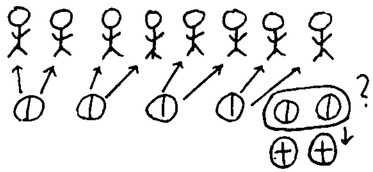
> " To make soup, we need 6 potatos for 8 people.
> How many potatos do we need for 6 people? "

The children were asked first to solve the problem. All six students applied an multiplicative algorithm of some kind. Two of these attempts were incorrect, in that the order of the ratio was wrongly recorded (e.g. $8 \div 6 = \frac{6}{8}$; $6 \div 8 = \frac{6}{8}$, $6 \div \frac{6}{8}$)
The students were then asked the following Key Question.

> " Could you think of any way that might solve the question if you had never been to school, and so had never learned the methods you had just used"

This appeared to be a novel idea to the students, but all responded quite readily (see table 1).

Table 1. Incidence of informal methods N=6

Strategy	number of students used strategy
	2
	2
	1
find the weight of the 6 potatos, shared the weight by 8, and then take 6 of them.	1

As a result, all six students have shown an informal strategies of some kind. It seems that Taiwan students may also be able to 'invent' certain informal procedures, but typically may not use within the school context. Of particular note here is that the two students who made an error in their algorithmic approach solved the problem correctly by means of their second method.

DISCUSSION

The performance of these two students would seem to imply that, had they typically used the informal approach instead of the algorithmic method, they would scored sucessfully on this recipe question. The algorithmic errors made by these two students are typical among the poorer performance of students at the lower end of a national study scale of understanding in ratio of Taiwan junior high school students (Lin, 1988). This might explain the 20% difference of facility on the recipe question between Taiwan and English students.

One note of caution needs making here. It is not clear to what extent the 'informal' methods produced by the students in this sample were 'primitive' in the sense of having some existence prior to the taught procedure, or to what extent they were in fact concrete representations of an algorithm already internalised.

It is interesting to note that all the informal methods reported here reflected a unitary approach, which is in fact the basis of their two steps multiplicative algorithm. This is in contrast to the 'child-method: building up using halving' most commonly found in the English sample (Hart, 1981, 1984), which is based on matching the recipe for 6 people with that for 8 by havling the quantity needed for 8 people (thereby obtaining the amount needed for 4), halving again (to obtain the amount needed for 2), and finally adding these two amounts together to find the requirement for 6 people. Whilst in some senses the English children's approach can be regarded as a version of the unitary

approach, it has the disadvantage of being not generalisable to problems in which the numerical relationships are different nor does it directly mirror the algorithm which is usually thaught.

REFLECTION

Firstly, the findings, if substantiated, raise once more the question of what is meant by 'understanding' and what 'tests of understanding' may be measuring.

Secondly, they also draw attention again to the need to arrive the procedures which students use, as well as those which they can use, since the two may not necessarily be the same.

Thirdly, the question of the informal methods students use is shown again to require examination. To what extent may the use of informal methods help students in their mathematical progress, and to what extent may these prove to be a disadvantage? The answer to this last question must require attention to the kinds of informal procedures developed, and their relation to the more formal procedures.

REFERENCE

Booth, L.R. (1981). Child-methods in secondary mathematics. Educational Studies in Mathematics, 12, 29-41.

Carpenter, T.P., Hiebert, J., & Moser, J.M. (1983). The effect of instruction on children's solutions of addition and subtraction word problems. Educational Studies in Mathematics, 14, 55-72.

Carraher, T.N., Carraher, D.W., & Schliemann, A.D. (1985). Mathematics in the street and in schools. British Journal of Developmental Psychology, 3, 21-29.

Fischbein, E., Deri, M., Nello, M.S., & Marino, M.S. (1985). The role of implicit models in solving verbal problems in multiplication and division. Journal of Research in Mathematics Education, 16(1), 3-17.

Hart, K.M. (1981). Children's Understanding of Mathematics, 11-16, John Murray, London.

Hart, K.M. (1984). Ratio: Children's Strategies and Errors, NFER-Nelson Publishing Company Ltd., Slough.

Lin, F.L. (1988). Societal differences and their influences on children's mathematics understanding. Educational Studies in Mathematics, 19, 471-497.

Yang, J.S. (1988). Primary and Junior Secondary School Children's Understanding of Fraction, A report of mathematics understanding of Taiwan students programme, Dept. of Mathematics, Taiwan Normal University (in Chinese).

Michael: A case study of the role of unitizing operations with natural numbers in the conceptualization of fractions

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This article provides an interpretation of the behavior of a third grader, Michael, in his effort to solve problems with integers and the use of this knowledge to generate the concept of the unit fraction. Michael's mental strategies provide a window on his capacity to simultaneously manage different types of units and his ability to infer multiplication as a repetitive addition by means of iteration of different types of units. Michael uses these unitizing operations to construct the concept of a unit fraction, and the inverse relationship between the number of parts and the size of each part when partitioning a given quantity of money. Underlying the study of Michael's solutions is the assumption that children construct more sophisticated schemes out of more elementary schemes of operating. It seems that fraction schemes spring out of iterating schemes that lead to partitioning schemes.

Theoretical Rational

The study is based on the constructivist principle that children generate their own strategies to solve problematic situations based on what they know and how they are taught. If children are taught in a formal symbolic way, the strategies they devise will focus on the symbols and will be largely devoid of meaning. If, on the other hand, children are encouraged to operate in experiential contexts where natural language is stressed, they will construct meaning using natural language as the first source of symbolization. To this effect, Ginsburg (1977) has observed that

> Just as children can speak their native language before being able to write it, so they understand a good deal of arithmetic before they can produce its written symbolism. Before his arrival in school, the five or six-year-old already knows something about adding and subtracting and equivalence. Indeed much of the symbolism children have to learn refers to ideas they already possess in at least a crude form. (p. 178)

Such a process that uses natural language to develop mathematical concepts before conventional symbolization is introduced might facilitate the natural evolution of the individual's mathematical ways of operating. Finally, if mathematical concepts are built upon concepts that are already available to the children and are natural extensions of those concepts, then they will have a built-in context for constructing new ideas. This basic Piagetian principle has been documented by researchers in the

field (e.g. Ginsburg, 1977; Steffe, Cobb and Richards, 1983; Steffe and Cobb, 1988). In the case of fractions, it seems necessary to identify contexts from which fractions can spring and which are intuitive and natural to children, not just to the teacher. As Vygotsky (1986) has observed, both the children and the adult may "refer to the same object but each will think of it in a fundamentally different framework; the child's framework is purely situational with the word tied to something concrete, whereas the adult's framework is conceptual" (p. 133).

Method

Michael, age 9 years, is one of six third graders who has participated in the study since October, 1989. The students in the study were drawn from the same third grade class in a Clarke County, Georgia, elementary school, and are supposed to have above average, average, and low ability in mathematics.

The research being conducted uses the teaching experiment method as described by Cobb and Steffe (1983). The teaching experiment is a methodology based on active interaction between the teacher and the students, in a one-to-one basis, by means of teaching episodes. Each child has participated in eleven teaching episodes in the course of three and a half months. Each teaching episode lasts approximately 60 minutes and is videotaped and subsequently analized to diagnose the actual operational level of the child (i.e., retrospective development) in order to plan prospective development. That is, the teacher acts in function of the child's zone of proximal development. This zone of potential generative mental activity has been characterized by Vygotsky (1978) as the zone that "defines those functions that have not yet matured but are in the process of maturation, functions that will mature tomorrow but are currently in an embrionic state" (p. 86).

Every task or problem posed to the students is a goal directed activity whose primary objective is not only to elicit the solution of the specific problem but to encourage reorganization of the schemes involved. Such reorganization of the schemes leads to interiorization of them by means of abstraction, which in turn induces more operative mathematical concepts. Each problem was conceived within the context of money which was already familiar, in a certain way, to the child in order to facilitate the perception of regularities. Money (e.g. dollar system) is in itself a well structured system of different types of units (coins and bills) that also facilitates the formation of new units. Every problem was solved verbally and no written symbols were used. The children always solved the problems mentally using their own conceptual ways of operating and verbally trying to make explanations of their answers.

Michael's knowledge of Unitizing, Iteration and Partitioning

Michael is able to exchange a dollar bill for four quarters and each quarter for twenty-five cents, or a dollar bill for ten dimes and each dime for ten cents, or fourteen dimes for one

hundred and forty cents. Inversely he can establish four quarters as one dollar and fourteen dimes as four quarters and one more quarter leaving a left over that he characterizes as "not enough for another quarter to fit in". The following dialogue gives evidence of his ability to deal with different types of units simultaneously.

Michael puts twelve quarters into one cup and five one dollar bills into another cup. The cups are covered and the teacher starts questioning Michael about the number of dollars and the number of quarters.

T : How many quarters altogether?
M : (He touches the cup with the quarters and extends four fingers in his left hand and then one finger in his right hand, and thinks for some time....) Thirty-two.
T : Why?
M : Because there are twelve quarters here (he touches the cup with the quarters) and added 12 plus 4, 16. 16 plus 4 is 20 (showing two fingers in his right hand), 20 plus 4 is 24. That's three dollars (showing three fingers in the same hand). 24 plus 4 is 28 quarters (showing 4 fingers) and 28 quarters plus 4 quarters is 32 quarters (showing five fingers).

Michael is then asked to put 14 dimes in a new cup and the new cup is placed with the others and covered.

T : How many quarters altogether?
M : (Michael shows the two initial cups) Thirty-two quarters plus ... thirty-seven.
T : How did you find that out?
M : Thirty-two quarters, and for a hundred and forty (refers to the fourteen dimes) and four quarters in a dollar, and 32 plus 4 is 36, and 36 plus 40 quarters (meaning cents) is 37 and there is not enough cents for another quarter to fit in, so there is 37 quarters.
T : Very good. How many dollars do you have?
M : Five (he touches the cup with the one dollar bills) and, ... twelve quarters? ... nine.
T : How did you find that out?
M : Because five dollars here (he touches the cup with the dollar bills) and three dollars (he touches the cups with quarters) and one dollar here (he touches the cup with dimes) is nine dollars.
T : How many dimes do you have altogether?
M : (He touches the cup with five dollars and puts up five fingers of his right hand and silently vocalizes something then puts up three fingers of his left hand and then one more, he repeats the process again and says ...) Ninety, no wait, ninety four.
T : How did you find that out?
M : 10 - 20 - 30 - 40 - 50 and that is five dollars and here is three dollars, 60 - 70 - 80 plus one dollar is

90 plus 40 cents left over and I added 4 dimes. That is 94.

The explanation for the total number of quarters shows how Michael was able to exchange each dollar for four quarters and then iterate 4 five times to add to the twelve given quarters. A similar process of exchange and iteration of the units generated by the exchange is given in his explanation of the 94 dimes. In addition, he included the unitizing of twelve quarters as three dollars each of which he inmediately represents as ten dimes in the middle of his iteration of units of ten. Moreover, the unitization of four quarters as one dollar and ten dimes as one dollar leads him to find nine dollars out of five one dollar bills, twelve quarters and fourteen dimes.

At this point, there is indication that Michael is acting at the level of empirical abstraction since he carries out the operation in thought with re-presented sensory material (four quarters re-presented as one dollar bill, one dollar bill as ten dimes, or fourteen dimes as one hundred and forty cents re-presented as four quarters and one quarter and some left over that "is not enough for another quarter to fit in").

The following episode shows how Michael is able to anticipate the structure of his strategy, in order to find the number of quadruplets out of four different sets of cards. Moreover, he unitizes and iterates the results of the prior steps of his strategy in order to arrive at his final answer. After making pairs and triplets, Michael is asked to find the number of quadruplets out of four cards (A, B, C, D), three cards (1, 2, 3), two cards (●, Õ), and two cards (*, +) that were vertically displayed over four pieces of construction paper.

T : How many quadruplets can you make?
M : Let's see how many triplets I have (he tries to make them but loses track of the combinations and the number of them). This is hard! (thinks). Let's see how many pairs I have right there (he takes the third group of cards in his hands and after looking at the first two group of cards and nodding ...) O.K. (he puts the cards down, and touches both of them) Twenty-four (he touches the last card in the last group) forty-eight.

After looking at all the cards and having clear what the task was, he starts working on the problem and he tries to make triplets so then he could find the number of quadruplets. Here, he is applying a similar method of operating to the one he used to find triplets (first find the number pairs and then iterate this result to find the number of triplets). Very soon he feels the need to go even backwards and find the number of pairs before the number of triplets. Then he iterates the number of triplets twice in order to find the number of quadruplets.

Michael's behavior seems to indicate that he is beginning to operate at an abstract level since he was able to reflect on his prior way of finding triplets and used the same strategy for finding quadruplets. Michael uses repetitive addition as a

consequence of his systematic way of forming pairs, triplets, and quadruplets.

In summary, it seems that Michael presents sophisticated ways of operating within the context of the natural numbers.

Michael's knowledge of fractions

Money permits making change of known quantities into different types of units that foster the creation of other types of units that are not the given by the actual coins and bills. Furthermore, partitions seem to be facilitated as generalizations of change. This variety of partitions give a concrete and familiar context to introduce equivalent fractions, improper fractions and to compare fractions without the use of algorithms and based only on children's mental ways of operating.

The following episode shows how Michael's construction of the unit fraction is a direct consequence of the re-presentation of one thousand dollars as a multiple of a five dollar bill.

T : A five dollar bill is what part of one thousand dollar bill?
M : (Michael uses twice two full hands, extends his arms in front of him and then uses once, the two hands and says quickly) One two hundredth.
T : Why?
M : Because there are twenty of this (refering to a five dollar bill) in a hundred, and there is ten one hundred in a thousand, so I take those ten one hundreds plus twenty, take twenty, ... it is two hundred.
T : How did you find out that ten twenties is two hundred?
M : Because five twenties is one hundred and ten twenties is two hundred.

It is worthwhile to make explicit here that neither Michael nor any other child in the study had learned the "times tables" so any concept of multiplication that they have is the result of their own understanding. Here as in many other occasions, Michael uses the word "plus" to mean "times" and so he says "... I take those ten one hundred plus twenty, take twenty, ... it is two hundred."

In the following three dialogues Michael starts working within the concrete task of splitting one hundred pennies in two, four, five, and ten cups, and suddenly he seems to be able to operate without the need of knowing the exact amount of pennies in each cup but again in a similar task finds the number of pennies in each cup (part). However, he also displays an indirect way of comparing fractions using the complementary fraction. Michael's efforts seem to lead him to more stable conceptualization of the inverse relationships between the number of parts and the size of each part and the importance of the complementary fraction.

T : What is bigger, eight tenths or four fifths?
M : They are the same.

T : Yes. How did you find that out?
M : Because there are twenty cents left here (showing two of ten cups) and here (showing four of five cups) there are 20 - 40 - 60 - 80 so there is twenty left in here too. So they are the same.

T : What is bigger twenty-nine thirtieths or ninety-nine hundredths?
M : They are the same, wait, ninety-nine hundreths.
T : Why?
M : Because all you need is one more. You have to have how much is left in a cup to reach a hundred, and one thirtieth is bigger than one hundredth.

T : What is bigger nineteen twenthieths or forty-nine fiftieths?
M : Forty-nine fiftieths.
T : Why?
M : Because there are two cents in every fiftieth so you are only two cents away from the dollar. And then if you have nineteen twentieths there is five cents in each so you are five cents away from the dollar.

Michael's inverse relationship between the number of parts and the size of each part and also the inverse relationship between the size of the complement and the complementary part is shown in the context of an unknown but fixed amount of money. The teacher puts some dollar bills in Michael's shirt-pocket and asks him to give to her part of his money. Michael gives her half. Upon the request of even less money he gives her one fourth, one eighth, one tenth, one hundredth, one five hundredth, one thousandth, and one millionth. Upon her request of more money he gives her one thousand and fiftieth, one thousandth, a half, "one third, wait, two thirds", eight nineths, nine tenths, ninety-nine one hundredths, nine hundred ninety-nine thousandths. Finally the teacher asks for explanations of the last answer.

T : Well, why do you think that nine hundred ninety-nine thousandths is bigger than ninety-nine hundredths?
M : Because ninety-nine hundredths is only about 21 dollars to reach one thousand and nine hundred ninety-nine thousandths you only need one more.
T : Why?
M : Because you have one little part away and in the other one more bigger part to reach what I have in here (showing his shirt-pocket)
T : What is that little part that you are talking about?
M : One dollar.
T : So you are thinking about one thousand dollars?
M : Yes.
T : But if you do not think about one thousand dollars, can you compare one hundredth and one thousandth?
M : One hundredth is bigger because one hundredth is this big (showing a big space between his thumb and index

finger of his left hand) and one thousandth is this big (showing a little space between the same two fingers of his right hand).

Michael's initial notions of equivalent fractions and inverse relationships leads him to add fractions in a natural way by means of establishing relationships between fractions and with no need for an algorithm.

T : Suppose that I give you first one half of my money and then one fourth of my money. What part of my money have I given to you?
M : (Without hesitation) Three fourths.
T : Why?
M : Because half is two fourths and then you give me one fourth.
T : Suppose I give you one fifth of my money and then one tenth of my money. What part of my money have I given to you?
M : One fifteenth.
T : Why?
M : Because five and ten equal fifteen so you gave me one fifteenth.
T : But before you did not add four and two. Can you see a relationship between one fifth and one tenth?
M : You gave me three fifths, wait, you gave me one and a half fifths.
T : What is bigger one fifth or one tenth?
M : One fifth.
T : One fifth is how many tenths?
M : One fifth and one tenth is three tenths.
T : How much is two sevenths and one fourteenth?
M : (Without hesitation) Five fourteenths.
T : Why?
M : Because one seventh is the same as two fourteenths and that's three and the other seventh is two, and two and two and one is five.

Discussion

Michael has considerable potential for solving fraction problems. He is able to work with fractions of a definite number of discrete units and he seems to abstract his concept of fraction by solving fraction problems of an indefinite number of discrete units. To start with, he had well developed schemes for dealing with units and unit relationships that permit him to construct multiples of units by means of repetitive addition. His strong unitizing operations facilitate his development of the concept of fractions. In sum, Michael's capacity to perform on a high level of abstraction for his age seems to be due to his great capacity to unitize and iterate.

Bibliography

Cobb, P. & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. Journal for Reserach in Mathematics Education, Vol. 14, No. 2, 83-94.

Ginsburg, H. (1977). Children's arithmetic: the learning process. New York: D. Van Nostrand Company.

Gunderson, A. G., Gunderson, E. (1957). Fraction concepts held by young children. The Arithmetic Teacher, 3-4, 1956-57 (pp. 168, 173).

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert, M. Hehr (Ed.) Number concepts and operations in the middle grades. Reston, VA: National Council of Teachers of Mathematics.

Kieren, T., & Nelson, L.D. (1981). Partitioning and unit recognition in performances on rational number tasks. In Proceedings of the Third Annual Meeting of the North American Chapter of I.G.P.M.E., 91-102.

Piaget, J. (1972). The principles of genetic epistemology. London: Routledge & Kegan Paul.

Piaget, J. (1973). Introduction a l'epistemologie genetique: la pensee mathematique. Paris: Presses Universitaires de France.

Steffe, L.P., von Glasersfeld, E., Richard, J., & Cobb, P. (1983). Children's counting types: Philosophy, theory and application. New York: Praeger.

Steffe, L. P., & Cobb, P. (1988). Construction of arithmetical meanings and strategies. New York: Springer-Verlag.

von Glasersfeld, E. (1980). The concept of equilibration in a constructivist theory of knowledge. In F. Benseler, P.M. Hejl, & W.K. Kock (Eds.) Autopoiesis, Communication and Society. New York: Campus.

Vygotsky, L. S. (1978). Mind in Society: The development of higher psychological processes. Cambridge, Mass.: The MIT Press.

Vygotsky, L. S. (1986). Thought and Language. Cambridge, Mass.: The MIT Press.

Constructing Fractions in Computer Microworlds

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The purpose of this paper is to illustrate by example how the specific structure and constraints built into a computer microworld, together with the interactive communication between child and teacher within the context of a constructivist teaching experiment, promoted a functional accomodation in one child's operations on fractions; specifically, the operations required to form quantitative equivalences among an improper fraction of a single unit, a mixed number, and a proper fraction of a double unit. The construction of these operations is an important step in establishing a comparison scheme for fractions as quantities.

Learning Environments

We view mathematical knowledge as being constructed by the child as a result of dynamic goal-directed mathematical activity in an environment. If an environment is an invention of the child, it follows that an environment is established as the result of an assimilation, which is "the integration of new objects or situations and events into previous schemes" (Piaget, 1980). The result of an assimiliation of a particular situation is an experience of the situation and this experience constitutes a learning environment in the immediate here and now. When a child takes the learning environments of others into account, the child's mathematical learning environment becomes a variable experiential field. Possible mathematical learning environments for a child are designed by a teacher with respect to the child's current schemes.

In "goal-directed mathematical activity" we include the accomodations that a child makes as a result of his or her experiences. These accomodations account for mathematical learning, as they consist of modifications of current mathematical concepts to neutralize perturbations that arise as a result of experience. Interactive communication with a child provides an opportunity for us to modify problematic situations so that the perturbations which drive mathematics learning can be engendered. If a child is successful in neutralizing these perturbations by modifying his or her current knowledge, the child can establish a modified environment and can see the old environment in a new way.

A Computer Microworld

In this study the possible learning environment for Karia, a fifth grade child in a rural public school, was constructed in the context of a LogoWriter microworld. The notion of a microworld was advanced by Seymor Paper (1980) and has been elaborated by many people. For us, a microworld is specifically structured to engage the user in activities directly related to some learning objective we have. In order to encourage children's accomodations, the microworld needs to have built-in flexibility of use and an easily adaptable and extensible command structure.

The particular microworld used in this study (called CANDYBAR) uses rectangular regions of a fixed size to represent candy bars. An early partitioning activity experienced by most children is the sharing of a candy bar with one or more friends. This microworld provides the user with opportunities to model such partitioning activity. A candy bar can be drawn on the screen by the child by simply typing the letter C and pressing the "return" (or "enter") key. An on-screen menu of commands is available to the child. These commands allow the child to perform the following operations on a candy bar:

- partition a bar into a specified number of parts (e.g. P 12);

- move a small arrow marker immediately underneath the candy bar a specified number of pieces to the right (R) or to the left (L) of its current position;

- fill a specified fraction of the bar with a different color (e.g. F [3/4] will shade in three-fourths of the bar (starting from the left edge of the bar) and stamp the fraction "3/4" underneath the bar at the limit of the shaded part);

- draw and iterate a specified fraction or amount of a candy bar (e.g. D [1/4] D [1/4] D [1/4] will draw three 1/4 pieces of a candy bar, creating a rectangle which is 3/4 of a unit bar, partitioned into three equal pieces. D can also be used to model mixed numerals: e.g. D 1 D [1/2] will draw one whole candy bar followed by half of a candy bar.)

Both F and D will accept improper fractions as input: e.g. F [5/4] will shade in 5/4 of the current candy bar by shading in the whole bar and 1/4 more past the end of the bar. If two bars have been placed end to end then F [5/4] will shade all of the first bar and 1/4 of the second bar.

There are also commands to move the marker around the screen, clear the screen, change the color of the outline of the candy bar and the color to be used for shading a fraction of the bar.

The current design of this microworld has evolved over the past 12 months through children's interactions with it. The intent is to provide a model for generating fractional quantities which, although continuous in the geometric sense (a continuous rectangular region), allows the child to use whole number knowledge and counting strategies to partition units. Because the computer creates the equidivisions, practical geometric measuring problems for the child are avoided. Thus the child can apply whole number relationships meaningfully in the context of continuous quantities -- this might help the child in the transition from fractions as parts of things to fractions as amounts or measures of quantities. The microworld also provides operations which use fraction symbols as quantities (F) and as measurement units (D). These operations can help the child make use of and compare fractions as quantities which is necessary for the construction of the rational numbers of arithmetic.

The Teaching Episodes

Karla worked with one of the researchers once a week for approximately 45 minutes over a period of six weeks. She worked primarily with the CANDYBAR microworld; however, the situations of learning included our verbal and nonverbal interactions; they were not simply supplied by the microworld.

All our communicative interactions with Karla were video taped using a camcorder, and her interactions with the computer were recorded directly from the video output of the computer on a separate video recorder. In this way, both her interaction with us as teacher and the complete record of her computer interactions have been preserved. Analyses of these video records were performed by viewing both tapes simultaneously, synchronized through their common audio recording.

It was clear from Karla's first session with the CANDYBAR microworld that she had already constructed some notion of equivalence of fractions. Given a portion of a candy bar on the screen partitioned into six pieces and being told that this portion was three-fourths of the whole bar, she successfully created the whole bar using P 8, explaining her action in the following way: "If there are six pieces in three-fourths then there must be two more [in the whole bar]." She also gave similar explanations when presented with two-fifths of a bar partitioned into four pieces, three-fourths partitioned into nine pieces, and two-thirds partitioned into eight pieces. In each

case she used the numerator of the given fraction to calculate the number of pieces in a unit fraction and used this information to calculate the total number of pieces in the whole bar.

During the fourth session Karla experienced a constraint built into the F command (it always shades from the left edge of the current candy bar and its fraction input always refers to a fraction of one unit candy bar) which was in conflict with her goal of shading in part of two candy bars. She had drawn two candy bars end to end using the command D [2/2] twice. She partitioned each bar into six pieces using the P command. She now had a double candy bar consisting of 12 equal pieces. She issued the command F [6/12] and was surprised when only half of the first bar was filled (three of the 12 pieces). She explained that she expected the computer to fill in 6 of the 12 pieces in the double bar. The teacher explained that the computer only knows that a candy bar is so big (outlining one of the two bars): "It doesn't understand, even though you've drawn a candy bar twice that size, it thinks there are two candy bars there; not one."

As Karla had (on her own initiative) introduced the idea of forming fractions of more than one candy bar, the teacher posed the following challenge to Karla:

T: See if you can draw on the screen a fraction of a candy bar that is greater than one bar; that's more than one bar.

K: *You mean like six over two, or something?*

T: Ah-hah! ... But less than two. Greater than one but less than two. You've drawn two candy bars and we know that they fit.

Karla used D [2/2] to draw a whole candy bar (without partitions) and then added a half candy bar to the end of it using D [1/2], stating that she had one and a half candy bars. She again encountered a constraint in trying to fill the one and a half candy bars using the F command. She did not know how to express one and a half as a fraction greater than one. She tried to use the F command in the same way that she had used the D command, but because of the constraint that F always fills from the left edge of the bar this only succeeded in shading one whole bar, half in blue and half in white. The teacher again explained that the F command always starts from the left edge of the whole candy bar and thus must be told what the total amount is in order to fill the whole amount.

K: *Ok, so that's one and a half candy bars.*

T: Yeah, but unfortunately Fill doesn't know mixed numbers...So is there a fraction that means one and a half? Just a single fraction that means one and a half?...It's called an improper fraction. Have you ever heard of those?

K: *Yeah.*

T: Ok. What do you know about improper fractions?

K: *Well, the number on top is larger than the one on the bottom.*

Karla immediately typed in the command F [2/1] but before pressing the "return" key to execute the command the following dialogue concerning the meaning of "2/1" ensued:

T: How much of a candy bar is two over one?

K: *That's one whole candy bar.*

T: Is it?

K: *Two over one: two pieces of one candy bar.*

T: What's two over two?

K: *It's one whole candy bar.*

T: So what do you think two over one is?

K: *Uh, its....one half a candy bar.*

T: What's one over two?

K: *That's one half of a candy bar.*

T: So are they the same? Is two over one the same as one over two?

Karla nods her head in the affirmative. At this point in the teaching episode Karla has <u>assimilated</u> the phrase "two over one" and the numeral "2/1" using her part-whole concept of one-half. This part-whole interpretation of fraction breaks down when applied to improper fractions.

In order to confirm this assimilation she executes the command F [2/1] and the whole screen ended up being filled, as the fill went beyond the closed region of the one and a half candy bars. The symbol "2/1" was, however, stamped on the screen at the appropriate position which indicated the end point of two candy bars (see figure 1).

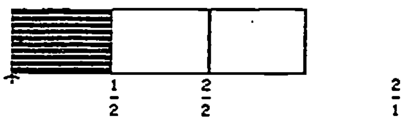

Figure 1

T: You see where the two over one is? What is the two over one marking?

K: *It's marking, uhm, two pieces...and it's different from two over one.*

Karla's attempt to confirm her assimilation using the operations of the microworld produced an unexpected result which Karla took as being (for her) different from "two over one"--which for her meant one-half:

T: Yeah. You've drawn that before. What is the two over one?

K: *Two pieces of one candy bar. It's, uh...*

T: What's another way of saying that?

K: *Uh, one-half.* (A classic "part-whole" interpretation.)

A critical teaching move took place at this point which helped Karla confront the inconsistency of her assimilation and forced her to reinterpret her notion of a fractional unit and restructure her operations on those units. As the restructuring of these operations takes place while in the act of using them, we call this a <u>functional accomodation</u> (Steffe, 1990):

T: (responding to Karla's assertion that 2/1 is one-half) Is it? Here's one-half...right here (pointing to the numeral "1/2" stamped at the point half way along one candy bar)...Here's two over two (pointing to that numeral at the end of the first bar). How much of a candy bar is that?

K: *One candy bar.*

T: Ok. Here's two over one. The whole thing from here to here is two over one (pointing at the left edge of the first bar and then at the numeral "2/1" at the end of where the second bar would have been). How <u>much</u> of a candy bar is that?

K: *That's going to be....two candy bars.*

The teacher spontaneously introduces a critical change of language at this point, giving the phrase "two

over one" meaning in terms of a multiple quantity--"two of one"

T: Right...Two of one. Two of one candy bar. If I give you two of one how much have I given you?

K: *You've given me two whole candy bars.*

T: Right. If I give you "one of two" how much have I given you?

K: *Uhm...half of a candy bar.*

T: That's right. Good! If I give you "two of two" how much have I given you?

K: *Uh....the whole thing.*

The accomodation that Karla has made at this point is in what counts as a unit for her and how she operates with those units. The teacher now tests his hypothesis that Karla can now take, for example, one-half as a unit quantity (rather than "one of two parts") and iterate it to form an improper fraction. The **D** command provides the analog for this operation within the computer microworld:

T: Now use **D** with a half to draw one and a half candy bars. (Karla enters **D [1/2]** and then pauses.)

T: That's half.

K: *That's half.*

T: Ok, I want you to keep using **D** with a half to create one and a half candy bars. (Karla enters **D [1/2]** two more times.)

T: Do you have one and a half candy bars?

K: *Yeah.*

T: How many halves do you have there?

K: *Three halves...so that'd be three over one? Three of one?* (Indicating that the "half" had indeed been taken as one unit!)

The teacher now attempted to help Karla construct the relation between this unit "half" and the unit "one candy bar":

T: Three of one is how many?

K: *It's uh, three whole candy bars.*

T: Right. So this isn't three of one, it's three...three what?

K: *Three...three of these* (pointing to the pieces on the screen).

T: What are each of those?

K: *A half.* (A "half" is now a unit quantity for Karla rather than "one of two parts")

T: So it's three-halves. (An unfortunate intrusion on the teacher's part!)

K: *Three-halves.*

T: Do you know how to write that as an improper fraction?

K: *Uh-huh. No.*

T: How would you write one-half?

K: *I'd say it's one-half; it's one over two.*

T: So how do you think you would write three-halves?

K: *Uhm...three over two?* (This conjecture suggests an extrapolation of Karla's current numeral system for fractions. The denominator now names the unit quantity (a "2-piece") rather than indicating the number of parts in a unit. The operations of the microworld provide Karla with the opportunity to test her conjecture.)

T: Try it...with **D**. (Karla types in **D [3/2]** and an unpartitioned rectangle is drawn the same size as the three half

candy bars.)

K: *Oh! So it's three-halves.* (Verification of her conjecture.)

T: Ok. Try it with F. (Karla enters F [3/2] and it fills the whole rectangle.)

T: Did it fill one and a half candy bars? (Karla nods yes.)

T: Sure did! Ok?

K: *Ok.*

The above episode illustrates how the constraints built into the F command prevented Karla from accomplishing her task using the part-whole operations of her current fraction scheme and the language of mixed numbers. The part-whole operations which had been effective for her when dealing with proper fractions did not apply to the situations involving improper fractions. In order to construct new operations which would work in this new situation she had to modify her concept of a "fractional unit."

The structure of the **D** command allowed Karla to use a half as an iterable unit, and by doing so she was able to eventually construct a composite unit consisting of three halves, which was perceptually equivalent to one and a half candy bars. A numeral was eventually constructed to signify three halves. We describe this process of restructuring as a <u>functional accomodation</u> made in the restricted situations of learning, that is, using iterable halves. This was not a complete reorganization of her fraction scheme as will be evident in the following episode which took place a week later.

<u>Fifth teaching episode with Karla</u>

The teacher reviewed with Karla what they had done the week before. Karla stated that she learned how to make a mixed number into a fraction. The teacher then asked Karla to create two candy bars end to end, each a different color, and to represent with those two candy bars as many fractions that she could think of that were greater than one and less than two.

K: *Ok. Comparing or just making?*

T: Making them and, yeah, you can compare them as well, if you'd like.

Karla drew two bars using the following sequence of commands:

D [8/8] (draws one whole, <u>unpartitioned</u> candy bar)

D [8/8] (draws a second unpartitioned candy bar on the end of the first)

S, P 1. (colors the outline of the first bar orange).

F [8/9] (shades 8/9 of the first bar – see figure 2).

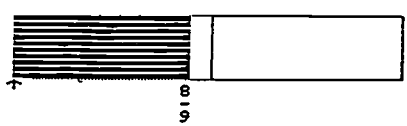

Figure 2

T: Ok, that's eight-ninths. Is that greater than one?

K: *Uh...yes.*

T: It is?

K: *Eight over eight equals one, and eight into nine...?* (We interpret this as meaning nine-eighths were intended to be filled rather than eight-ninths. This would explain Karla's use of **D** [8/8] to create her two bars.)

T: Show me...Is more than one candy bar shaded?

K: *Uh, no. I need two candy bars...two for one...so...?*

T: What does eight-ninths mean?

K: *It means that, uhm...getting eight-ninths of the two candy bars (I think), and ...I have two candy bars and...*

T: Tell me what you see.

K: *Uh, I get less than one because it might be more than eight but I want eight but it's less than, uhm, its less than nine over...*

T: I'm sorry I can't hear you. You have to speak louder.

K: *Ok. Uhm, it's eight-ninths and its 8 over 8 but less than 9 over 9 so that means it has to be improper. It has to have the larger number on top.*

T: To be...?

K: *To be, uhm, larger than one.*

T: Ok. But what does eight-ninths, what is this eight-ninths?

K: *Uhm...it's part of one candy bar.*

Although Karla eventually realized her error, this sequence illustrated the bounds of her accomodation -- she did not yet have a general comparison scheme for fractions and conflated the results of two different partitions of the same unit (eighths and ninths). The following episode illustrates, however, that the functional accomodation which occurred in the previous session was available in contexts more general than the occasion of construction.

Karla was eventually successful in creating a fraction greater than one. She used the **D** command twice with an input of **[1/1]** to put two candy bars end to end, partitioning each into quarters using **P 4** and then filled 5/4 of a candy bar using **F [5/4]**. The teacher then asked her to create another fraction greater than one in the bottom half of the screen. Karla immediately created two candy bars end to end, each partitioned into five pieces and then successfully filled 6/5 of a candy bar using **F [6/5]**. The screen now looked like figure 3.

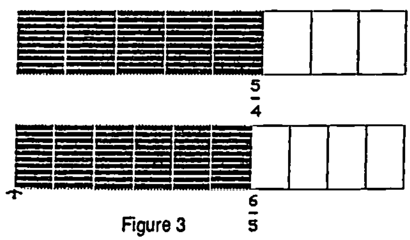

Figure 3

The teacher asked Karla the following: "Which is greater, five-fourths or six-fifths?" Karla's response to the question led to the following activity:

K: *Uhm...you can't exactly see but they look kind of equal.*

T: They look equal?

K: *Kind of...but five-fourths just looks a little bigger.*

T: Uhm...What are those as mixed numbers?

K: *Uhm...One and one fifth* (pointing to the bottom bars) *and...one and one fourth* (pointing to the top bars).

T: Good! Good! Uhm, do a D followed by a 1. I think **D** will take a whole number if you want it to. (Karla complies) Yeah, ok. That marks the one so you can see you've got the one and one fifth there. In fact, if you do a D

followed by a fifth, see what happens. (Karla enters D [1/5]) Ok? Now you have a way of comparing mixed numbers and improper fractions, right?

K: *Yeah.*

T: Ok...What's left over...unshaded?

K: *Unshaded...that's...four-tenths* (pointing to the bottom bars), *and that's three-twelths* (top bars)*...No, that's not quite right...three-eighths.*

T: Sorry...In the top one it's?

K: *It's three-eighths.*

T: Uhm, you said unshaded on the top was three-eighths. Why did you say that?

K: *Cause there's three unshaded...well, not exactly, if you count those as two candy bars...But if you counted those as one big candy bar you'd have three-eighths.*

T: Good...Uhm, if we count them as two candy bars, how much of one candy bar is unshaded?

K: *That's three-quarters.*

T: Ok...and the bottom one?

K: *That's...that's gonna be four-fifths.*

T: Ok...uhm...I like what you did though in terms of...if we regard this as one big candy bar, how much is shaded?

K: *That would be...*

T: In the top one.

K: *Five-fourths, which would be five-eighths counting it as one.*

Karla appears to demonstrate an ability to express the same quantity as an amount of two different units at this point. She is able to take a proper fraction of two units and transform it into an improper fraction of one unit. This would appear to indicate an ability to analyze her old operations on proper fractions in terms of her new operations on improper fractions. This is an important step in establishing a comparison scheme for fractions. It also indicates a reorganization of her fraction scheme, within the context of this microworld situation, from fractions as "parts of things" to fractions as "amounts or measures of quantities." Karla has assigned to the denominator of a fraction the role of naming a unit quantity (rather than the number of parts in a unit), and, further more, she appears to realize that the numerical value of the denominator is inversely related to the size of that unit quantity, as well as being relative to a referent super-unit. This is a tremendous change for a child to make in her fraction concept!

T: Yeah, you know, we actually say that that's five-eighths of 2 candy bars...Ok, five-eighths of 2 is the same as how much of one?

K: *Uhm...five-fourths.*

References

Papert, S. (1980). Mindstorms. New York: Basic Books.

Piaget, J. (1980). Shemes of action and language learning. In M. Piattelli-Palmarini, (Ed.), Language and Learning: The debate between Jean Piaget and Noam Chomsky (pp. 164-167). Cambridge: Harvard University Press.

Steffe, L. P. (1990). The learning paradox: A plausible counter example. In L. P. Steffe (Ed.), Epistemological foundations of mathematical experience. New York: Springer Verlag.

PROPORTIONAL REASONING: FROM SHOPPING TO KITCHENS, LABORATORIES, AND, HOPEFULLY, SCHOOLS

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We analyze how the understanding of proportional relations by illiterate subjects, in the context of commercial transactions, may encourage the adoption of the same model to solve problems in situations that are otherwise not seen by them as requiring proportional reasoning. Results show a strong transfer effect, especially to situations that are familiar to the subjects. Implications for mathematics education are discussed.

In everyday life we apply proportional reasoning whenever we determine the cost of multiple items of the same unit price. Even at a very young age children may solve proportionality problems when buying candies or toys. However, at school children often have difficulty solving problems requiring the same mathematical model. Moreover, even though they are successful in solving proportionality problems in school, via algorithms such as the "rule of three", students may not recognize the appropriateness of school procedures to situations outside the classroom. These difficulties are even more puzzling if we consider that workers with little schooling, who deal with proportional relationships at work, spontaneously transfer their problem solving strategies to problems different from those usually encountered at work (Carraher, 1986; Schliemann & Carraher, 1988).

Why is it that a mathematical model such as the proportional model is used in the solution of some problems but not in others? How does experience with the problem context affect the choice of the model? How does the use of the model in one type of context facilitate its adoption for other contexts?

Experimental research tends to show that, except when an explicit hint is given to subjects (see Gick & Holyoak, 1980, 1983), people tend not to use prior relevant knowledge to solve new problems. Under certain conditions, however, some kind of transfer seems possible. Bassok & Holyoak (in press), for instance, showed that students who had studied arithmetic progressions as a general model, that could be applied to any context, spontaneously recognized that the same equations could be used to solve physics problems, while those who dealt with progressions in the

[1] The data presented in this study are part of the Master's Dissertation of the second author, under the supervision of the first.

specific context of physics problems about constant-acceleration hardly ever used the same type of solution when faced with isomorphic algebra problems. Schliemann (1988) and Schliemann & Acioly (1989) showed that schooled lottery bookies who deal with permutations in the context of a lottery game transfer their strategies to other contexts, but this does not occur among illiterate bookies. Schliemann & Carraher (1988) showed transfer of proportional solutions among fishermen with little schooling, from a context of price of goods to other contexts. But the new contexts (rate of processed to unprocessed seafood) was known to be proportionaly related, although they did not have to solve problems about them. It is possible that no tranfer would occur if the new contexts were completely unknown by the subjects.

Brazilian cooks constitute an interesting group for studying transfer of mathematical models. First, it is possible to study their knowledge idependently of schooling, since most are unschooled. Second, as part of their everyday life they deal with two domains where proportionality plays a role: (a) Cooking, since they have to decide on the amount of ingredients to be put in a recipe, and (b) Purchase of items in the marketplace, where they have to compute the price of items they buy. Prices have to be computed exactly and the proportional model has to be used. A problem like "To buy 3 candies you need 5 cents; how much do you need to buy 9 candies?" has an exact proportional answer, namely, 15. The equivalent problem in the kitchen, "To make a cake with 3 cups of flour, you need 5 spoonfuls of water; how many spoonfuls do you need if you put 9 cups of flour?" can be satisfatorily solved by increasing the amount of water until the dough attains the right consistency. Although the proportional model is not used in the kitchen, some qualitative understanding of proportions may develop as a result of cooking activities. In this case, it is possible that transfer of the exact proportional model from the market could be easily achieved. What would happen, however, if the context of the problems were completely unknown to the subject? Would transfer also occur? Under which conditions? If it occurs, is it just analogical transfer, where, through a hint, the same set of procedures is transfered to a new situation, or does it involve understanding of the mathematical relations in both situations and recognition that the same mathematical model applies to both?

In this paper we experimentally analyze how problem solving strategies regarding the price of goods relate to the solution of problems about contexts that are part of the subjects experience (cooking recipes), but to which they don't usually apply the exact proportional mathematical

model, and to contexts that are completely unknown to the subject (mixtures of medicinal ingredients).

METHOD

Subjects were 28 women, aged from 16 to 40 years old, with no formal mathematical instruction on proportions and who had from three months to one year of schooling in an adult literacy program. They all worked as cooks in domestic employment in Recife, Brazil.

Problems involving proportional reasoning, with similar relations and quantities, were given orally to each subject in three different versions: (a) as part of prices of things to buy, a context where exact answers are called for in everyday situations, (b) as part of cooking recipes, a context to which approximate answers are usually accepted in everyday activities, and (c) as part of medicine formula, a context unknown to the subjects. Three different orders for problem presentation were used.

Ten subjects (Group 1) were first given eight proportional problems as part of cooking recipes and immediatly after four problems about prices. This was followed by a second presentation of the recipe problems that were not initially solved. Finally they were given eight medicine problems. This sequence (recipes, prices, recipes, medicine) was designed to evaluate subjects' ability to solve proportional problems about quantities of ingredients in a recipe, before and after solving proportional problems about prices of goods. It also allowed analysis of a possible transfer of the proportional model used for prices to medicine problems, after the subject was given the oportunity to transfer the model from prices to recipes.

Ten cooks (Group 2) started with the four price problems and were then given the eight recipe problems, followed by the eight medicine problems. This condition (prices, recipes, medicine) allows for control of possible effects of practice due to the repeated presentation of recipe problems in Order 1 condition.

Finally, eight subjects (Group 3) started the eight medicine problems, followed by the four price problems, then the medicine problems that were not solved in the first trial and, finally, the 10 recipe problems. This sequence (medicine, prices, medicine, recipes) was presented to evaluate subjects' ability to solve proportional problems about unknown contexts, before and after solving proportional problems about prices. It also permits analysis of possible transfer effects of use of the proportional model in price problems to problems with unknown contexts, without the mediation of recipe problems.

RESULTS

Results are shown in Table 1. When recipe problems were given first (Group 1), only one quarter of the answers (23.7%) were precisely correct and only two subjects found correct answers for more than half of the problems. When, after solving money problems, subjects were presented again with recipe problems they had not solved, there was a dramatic increase in the percentage of correct answers: Four subjects gave 100% correct answers and in the worst cases subjects correctly answered half of the problems. An identical overall percentage was found for medicine problems but, in this case a larger range occured with two subjects presenting only 37.5% of correct answers.

Table 1

Percentage of correct answers for each group in each context

Group 1	Recipes	Prices	Recipes*	Medicine
	23.7	95.0	76.2	76.2
Group 2		Prices	Recipes	Medicine
	--	100	68.7	76.2
Group 3	Medicine	Prices	Medicine*	Recipes
	14.06	100	40.6	75.0

Including problems solved before price problems were given; these were not presented once more after price problems.

In Group 2, after solving the four money problems, subjects were able to solve 68.7% of the recipe problems they were seeing for the first time, and 76.2% of the medicine problems. The slightly lower results in recipe problems were due to two subjects who presented less than half of the answers correct.

In Group 3 results were very low when medicine problems were first given (14.6%) and, after price problems, did not increase as much as recipe problems, reaching a percentage of only 40.6% correct. Recipe problems, however, were correct in 68.5% of the cases.

When first given Recipe problems subjects tended to use rough estimates of quantities and to give justifications that appealed to their customary practice: "I think that's enough" or "That's how I do it".

Sometimes the conflict between the mathematical model and everyday practice determined the choice of approximate answers as in the case of a subject who was asked how many passion fruits she would need to make 6 glasses of juice, knowing that for 4 glasses 5 passion fruits were needed, and gave the following answer: "To make 6, if you put 8, the taste will be stronger. I would put 7. The difference would be very little. One cannot do it exactly."

When given price problems, the answers were precise and the strategies revealed use of both functional or scalar relations. The functional solutions consisted in finding the unit price for the product mentioned in the problem, and then finding the total price to be paid. For instance, for the problem "If 2 kilos of rice cost 5 cruzados, how much do you have to pay for 3 kilos?", the answer given by one subject was: "Each kilo costs 2 cruzados and 50 cents; then 3 kilos cost 7 cruzados and 50 cents". Scalar solutions consisted in finding the relationship between the number of items for which a price was stated in the problem and the number of items to which the total price was to be found. This ratio would then be applied to the price, either directly or through successive additions. For instance, in a problem like "To make a certain cake, for 2 cups of flour one has to put 3 eggs. How many eggs do you need if you want to make the cake with 6 cups of flour?", a direct application of the ratio was found in the following answer: "In 2 cups one puts 3; then if you put 6, you have to put 9... Because it would make 3 times 3, it would be the same thing. For the same problem, the correct proportional answer was found by another subject through successive additions, as follows: "Because for 2 (cups of flour) there were 3 eggs. For 2 you put 3. Then for 2 one puts 3, for 2 more one puts 3 more, makes 6, 2 more makes 6 and takes 9".

There was an interesting difference for errors in medicine problems as opposed to recipe problems. While in recipe problems errors tended to be reasonable approximations of the precise answers, in medicine problems subjects tended to apply any kind of arithmetical transformation to the given quantities, obtaining nonsensical answers such as a smaller amount of one substance when the other ingredient was increased. Such responses were never given to recipe problems.

DISCUSSION

In the present study we obtained results similar to those of Schliemann & Carraher (1988) showing that transfer of the proportional

model used in one specific everyday context (prices) is easily achieved when the situations are known by' the subjects (recipes). We have now shown that this does not occur when the contexts of the new situation are completely unknown to the subject (medicine). However, when the known situation, to which transfer had occurred and the proportional model was applied, was presented before the unknown situation, subjects recognized the adequacy of the model even though they did not know the context. It is important to notice that both recipe problems and medicine problems deal with quantities of ingredients to be mixed, allowing, therefore, a perfect analogy between the variables in the two contexts. This is not true for price problems. Such results suggest that different mechanisms may be responsible for transfer to each of the two, known and unknown contexts. In the first case, knowledge about how the variables in the problem relate helps the subject to immediately identify that the same model applies to the relations. To use Vergnaud's (1982) terms, the process may be described as a recognition by the subject that the invariants of the two situations are the same and, therefore, the same procedure for solution could be applied. With completely unknown contexts there is nothing to help the subjects in identifying the invariants and only by analogy to a known context that deals with the same sort of activity (mixture of ingredients) the subject may recognize that the same procedure could be applied.

If the cognitive abilities of the illiterate cooks participating in this study were to be judged from their results while solving proportional problems in the area of cooking, the conclusion might be drawn that they were unable to understand proportional relations. However, these same subjects showed that they were able to solve problems involving the same logical structure when the situation was one that, in everyday life, required its use, as is the case of prices. Moreover, despite the differences between the real situation of prices and recipes, they were able to identify that, at a more abstract level, the problems were mathematically similar. This recognition, however, seems to require knowledge of how the variables relate in both contexts.

These results give support to the suggestion that, to ensure understanding and transfer of mathematical models, educators should provide students with experiences to familiarize them with the problem context to which the models are relevant.

REFERENCES

Bassok, M. & Holyoak, K. (in press). Interdomain transfer between isomorphic topics in Algebra and Physics. Journal of Experimental Psychology: Learning, Memory and Cognition.

Carraher, T. (1986). From drawings to buildings: working with mathematical scales. International Journal of Behavioral Development, 9, 527-544.

Gick, M.L. & Holyoak, K.J. (1980). Analogical problem solving. Cognitive Psychology, 12, 306-355

Gick, M.L. & Holyoak, K.J. (1983). Schema induction and analogical transfer. Cognitive Psychology, 15, 1-38.

Schliemann, A.D. (1988). Understanding the combinatorial syatem: Developement, school learning and everyday experience. The Quarterly Newsletter of the Laboratory of Comparative Human Cognition, 10(1), 3-7.

Schliemann, A.D. & Acioly, N.M. (1989). Mathematical knowledge developed at work: The contribution of practice versua the contribution of schooling. Cognition and Instruction, 6(3), 185-221.

Schliemann, A.D. & Carraher, T.N. (1988, April) Everyday experience aa a source of mathematical learning: Knowledge complexity and tranafer. Paper presented at the Meeting of the American Educational Research Association. New Orleans.

Vergnaud, G. (1982). Cognitive and developmental paychology and reaearch in mathematics education: Some theoretical and methodological isauea. For the Learning of Mathematics, 3(2).

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THE FRACTION-CONCEPT IN COMPREHENSIVE SCHOOL AT GRADE-LEVELS 3-6 IN FINLAND

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SUMMARY: The difficulties of learning the fraction-concept are analysed and compared with the results of other researchers, who have investigated fraction-learning. The most difficult sectors showed to be the number-line exercises, compareing the magnitudes of several fractions and constructing equivalent fractions. This research consisted of two phases, analysing the textbooks and testing 3000 pupils at grade-levels 3-6 in comprehensive school. The test was made of 34 items, which measured the understanding of the fraction-concept, not any algoritms.

Both teachers and many children find the learning of fractions quite difficult in comprehensive school. In Finland the pupils are at grade-level 3 (about 9 years old) when they face fraction for the first time. But what makes the fraction-learning so difficult?

The fraction-concept has many different aspects. The part-whole aspect is usually the first side of fraction, which textbooks present to children. They often use pictures of circles divided into 1-8 equal parts, some of the parts coloured. The task of the pupils, is to give the fraction-symbol of that picture or on the contrary colour a part of a picture, when the fraction-symbol is given.

Another aspect of fraction is proportion. This is connected with proportional reasoning and probability. This aspect has also many kinds of applications in real life.

Fraction can also mean a measure. This aspect is connected with number-line. And fraction means also the quotient. We need that concept in connection with solving linear equations, because px=q and x=q/p. Fraction can also be an operator, which is connected with transformations and different scales. We use this aspect for instance when we draw maps or calculate some distances on a map. Fraction-concept has also near connections with procent and decimal fractions. And these concepts are very common in everyday life, specially when the calculators have become so common.

So we use many different picture-models of the different aspects of the fractions. Children must find out the meaning of different aspects and models and how to use them. They must construct quite a large net of facts about fraction-concept to be able to use the right strategies in solving fraction-problems.

Many wellknown researches have studied the learning of fraction-concept in different countrys. T.Kieren has writed a detailed model of the rational number learning. M.Behr, T.Post and R.Lesh have tested 650 pupils' fraction-learning very carefully in connection with the Rational Number Project (1979-1983). L.Streefland has done an important work in testing the influence of N-distractor in fraction-learning. He has also constructed teaching-methods, which make the learning more meaningsfull. Other wellknown fraction-researchers are for instance K.Hart, C.Novillis-Larson and K.Hasemann.

Which are the most common difficulties with fraction? First, the children must understand, that a whole must be divided into equal parts to get fraction. The devision is not always so easy task. It is easier to deal a whole into two, four or eight parts, but much harder to get three or five equal parts of a whole. It is also difficult to see, that when a whole is devided into eigth parts, one must take two of them to get one quarter. Or to see which part of a whole the given part is, if all the devision marks are not in the picture.

When the children face the fractions for the first time they have constructed schemes for natural numbers and operations with them. The fraction-concept is in conflict with some of these schemes so, that children must construct new, different schemes for fractions. They must understand, that 2/3 doesn't mean two different numbers but a whole in itself. It is something different from numbers 2 and 3 as natural numbers. It is also difficult to realize, that $1/3 < 1/2$, althought $3 > 2$.

Children learn, that fraction means a piece of pizza or a coloured part of a circle. After that it is very difficult for them to understand, that it can also be a point on the number-line. Most difficult it is, if the number-line is longer than one unit.

Quite a big difficulty with fractions is in compareing the magnitudes of different fractions. It is easier, if the denominators are the same. Children use to have more difficulties in compareing fractions with the same numerators than in compereing fractions with the same denominators.

One difficult point more with fractions is the concept of equivalence. The children don't realize, that equivalent fractions have the same magnitude, though they are composed of different number-symbols and they are often made by converting.

My research consisted of two phases. First I analysed three series of mathematic-textbooks at grade-levels 3-6 in our comprehensive school. I classified all the exercises of fractions in these textbooks to see which aspects of fraction-learning each textbook-serie emphasized at different grade-levels.

The textbooks learn the fractions quite mechanically. They give a rule and lots of exercises, which must be solved useing that rule. The textbooks include lots of algoritm-exercises. They don't make the children to think, just follow the ready rules. They include very little problem solving and exercises, where children must find out themselves which operations to use.

I classified all the exercises in each textbook into four main-groups, which are:

1. transformations between the different presentations of fraction concept (verbal, picture, symbol, real-life)
2. transformations between different symbol-forms
3. algoritm-exercises
4. contextual exercises.

I give here first the number of all the fraction-exercises which the three textbook-series included at the different grade-levels. Let's call the textbooks just numbers 1, 2 and 3.

Grade-level textbook-serie	3.	4.	5.	6.
1	585	645	1150	931
2	235	626	1273	824
3	327	387	921	908

And next the procent of algoritm-exercises of all the fraction-exercises at each grade-level.

Grade-level textbook-serie	3.	4.	5.	6.
1	33%	28%	40%	47%
2	5%	26%	65%	73%
3	29%	26%	61%	61%

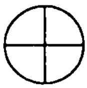
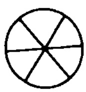
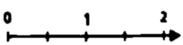
And the procent of contextual exercises of all the fraction-exercises at each grade-level.

Grade-level textbook-serie	3.	4.	5.	6.
1	24%	26%	26%	30%
2	7%	17%	13%	12%
3	10%	18%	16%	22%

As we see the different textbook-series emphasizes the learning quite differently.

In the second phase I tested nearly 3000 pupils at grade-levels 3-6 in our comprehensive school. The test which I had constructed for this purpose was composed of 34 items, which measured understanding of the fraction-concept, not any algoritms. As I expected, the children had difficulties with number-line items, with compareing the magnitudes of fractions, which had the same numerators and with making equivalent fractions.

I give here the results of some items. The given procent means those pupils, who have solved the item.

Grade-level	3.	4.	5.	6.
Colour 1/4 	99.8%	99.5%	99.6%	100.0%
Colour 2/3 	46.0%	57.5%	70.1%	80.9%
Point the 1/4 	32.4%	55.1%	58.7%	80.3%
Point the 1 1/4 	13.7%	29.4%	51.3%	70.8%
Mark the biggest fraction				
$\frac{2}{6}$ $\frac{3}{6}$ $\frac{1}{6}$ $\frac{4}{6}$	73.1%	93.8%	87.6%	90.7%
$\frac{1}{3}$ $\frac{1}{7}$ $\frac{1}{6}$ $\frac{1}{10}$	50.6%	40.5%	67.3%	83.3%
Fill the number				
$\frac{1}{3} = \frac{2}{\square}$	32.1%	52.4%	74.5%	85.8%
$\frac{2}{4} = \frac{\square}{2}$	33.2%	54.3%	80.0%	89.8%

The first item represents those exercises, which are very common in the textbooks. Nearly everybody can solve it. The other items are very rare or never seen in those textbooks, which I analysed.

Researcher in Finland have warned already for some years, that the children learn mathematics too mechanically in our comprehensive school. They learn rules and tricks, but not thinking mathematically. It is rote-learning without meaning (E.Lehtinen 1989). I think this is what happens often with the fraction-concept. The children learn to do the different operations just mechanically. They don't learn to understand the different properties of the fraction-concept, not the connections between different aspects and models. They learn to follow rules, which they don't understand and which they can forget in some days or weeks.

SOME REFERENCES

Hart K.M, Kerslake D, Brown M.L, Ruddock G, Küchemann D.E, McCartney M (1981) Children's Understanding of Mathematics: 11-16. John Murray.

Hasemann K (1986) Mathematische Lernprozesse, Analysen mit Kognitionstheoretischen Modellen. Germany.

Hasemann K (1987) Pupils' Individual Concepts of Fractions and the Role of Conceptual Conflict in Conceptual Change. In Articles on Mathematic Education.

Keranto T (1985) Rational and Empirical Analysis for the Measurement and Instruction of Proportional Reasoning in School Mathematics. Education Research in Finland Yearbook 1985, University of Jyväskylä (ed. Pekka Kupari)

Kerslake D (1986) Fractions, Children's Strategies and Errors. A Report of the Strategies and Errors in Secondary Mathematics Project. NFER-NELSON. London.

Kieren T (1980) The Rational Number Construct - Its Elements and Mechanisms. Recent Research on Number Learning. Columbus. ERIC.

Lehtinen E (1989) Matematiikka on ajattelun ja tunteen asia . Teoksessa: Ekebom U-M, Helin M, Tulusto R (toim.) Koulu apua. Weilin+Göös. Espoo.

Lesh R (1980) Directions for Research Concerning Number and Measurement, Concepts in Number and Measurement. Papers from Research Workshop.

Lesh R, Landau M (1983) Acquision of Mathematics Concepts and Processes. Academic Press. New York.

Lesh R, Behr M, Post T (1987) Rational Number Relations and Proportions. In Janvier (ed.) Problems of Representation in the Teaching and Learning of Mathematics.

Novillis C (1976) An Analysis of the Fraction Concept into a Hierarchy of Selected Subconcepts and the Testing of the Hierarchical Dependencies. In Journal for Research in Mathematics Education 7, 1976 (131-144).

Novillis-Larson C (1980) Sevent-grade Student's Ability to Associate Proper Fractions with Points on the Number Line. Recent Research on Number Learning. Columbus. ERIC.

Post T, Behr M, Lesh R, Wachsmuth I (1985) Selected Results from the Rational Number Project. In Psychology of Mathematics Education Ninth International Conference Papers Noordwijkerhout (342-351).

Steffe L, Parr R (1968) The Development of the Concept of Ratio and Fraction in the Fourth, Fifth and Sixth Years of the Elementary School. Report from the Mathematics Concept Learning Project. Technical Report no 49.

Streefland L (1984) Unmasking N-Distractors as a Source of Failures in Learning Fractions. In Proceedings of Psychology of Mathematics Education 8, Sydney.

Streefland L (1986) Rational Analysis of Realistic Mathematics Education as a Theoretical Source for Psychology, Fractions as a Paradigm. European Journal of Psychology of Education 1986 vol 1, nro 2.

Social Interactions, Communication and Language

CRITICAL DECISIONS IN THE GENERALISATION PROCESS: A METHODOLOGY FOR RESEARCHING PUPIL COLLABORATION IN COMPUTER AND NON COMPUTER ENVIRONMENTS.

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This paper presents a methodology for researching pupil collaboration in a computer and non computer environment. The research takes place within the three mathematical environments: Logo; a spreadsheet environment; and a paper and pencil environment. It is concerned with an analysis of the discourse processes whereby pairs of pupils come to make generalisations and formalise them. Critical decisions in the generalisation process for all three environments have been identified in order to provide a framework for analysing pupil collaboration. In particular, we present our analysis of the critical decisions across environments; the incidence of collaborative decisions; and the pupil roles in identifying relationships and their formalisation in computer environments.

Background

It has been argued that computer based activities "invite" collaboration (Shiengold 1987, p204). In our view, however, there is, a need for microanalysis of the interactions in a computer context — that is inter-pupil and pupil-computer — in order to investigate this assertion. One would hope that such an analysis would uncover significant incidents and critical decision points whereby pupils come to experience their mathematics in a different way as a result of collaborative exchanges.

In the Logo Maths Project (Hoyles & Sutherland, 1989) we started to address peer collaboration through the analysis of the transcripts of pairs of pupils working with Logo over a period of 3 years. The Logo project also highlighted the complexity of the role of peer interaction in a computer environment (Hoyles and Sutherland 1986). It was found, for example, that "collaborative work or discussion does not necessarily lead to individual learning gains in tightly specified circumstances" . The research reported here has attempted to investigate more closely the origins and process of collaboration.

Description of the Research

This paper will discuss some of the findings of the research project "The Role of Peer Group Discussion in a Computer Environment" Project (1988-1989)[1] . This research was concerned with an analysis of the way pupils working in pairs make generalisations and come to formalise them in a mathematical environment. The research took place within computer and non computer environments, namely Logo , a spreadsheet package[2] and a paper and pencil environment. The aim was to investigate the inter-relationship between the negotiation of a generalisation by

[1] Funded by the Leverhulme Trust
[2] A mouse driven spreadsheet package, EXCEL was chosen for the research.

pupil pairs and its formal representation, and any effects on pupil response of the problem solving tools made available by the different environments. Here we address the second of these issues, and will examine any effects of the environment on the patterns of collaboration of the pupil pairs.

We have been working with a class of 2nd year secondary school pupils (aged 12-13), from which we selected four case study pairs. The study involved two phases: a preliminary phase to introduce Logo and spreadsheets; and a research phase.

Preliminary Phase In the Logo environment, preliminary activities involved giving pupils experience of writing and editing procedures, and experience of using and operating on variables. We had planned that where possible these processes should be introduced within the context of the pupils' own projects. However we found that pupils usually needed to consolidate their understanding of using and operating on variables through an exploration within specially designed tasks before they had sufficient confidence to take up teacher interventions suggesting the use of variables within their own projects. In the spreadsheet environment, preliminary activities involved entering data and formulae, and replicating formulae containing relative references, again through exploration within specially designed tasks.

Research Phase Three research tasks (one for each environment) were designed (see appendix 1). Each task required the construction and formalisation of mathematical relationships. The research tasks were carried out by the four case study pairs, over a period of one lesson (70 mins). Each session was video recorded[3]. The video recordings were transcribed and used along with researcher's observations and pupil's written work as a basis for the analysis.

Analysis of Transcripts Analysis of the transcripts has been concerned with four inter-related aspects of the role of discussion in learning, that is distancing, conflict, scaffolding and monitoring (see Hoyles et al 1990 for discussion). In order to provide a framework for this analysis we decided to focus on critical decision points in the process of generalisation (see Table 1). These were identified from the pupil transcripts and supported by task analysis. They also were seen to have theoretical justification (see for example Polya 1957, Mason 1982).

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[3] Data was collected by pointing a video camera at the computer screen (or at the paper in the non computer environment), which obtained all computer input and output, discussion between the pupils and gestures to computer screen.

ORIENTATE	Decide to Construct a General Case	
SPECIALISE	Generate Specific Cases	(a) By constructing specific cases (b) By applying unformalised rule to specific cases
	Tabulate Specific Cases	
GENERALISE	Name Parameter	
	Choose Parameter	(a) With no reference to relationship (b) With implicit reference to relationship[4]
	Identify Relationships	(a) After choosing parameters (b) Incorporating implicit choice of parameter[5] (c) Using figurative relationships
	Formalise Relationship	
CONVINCE	Check Relationship	(a) By constructing specific case (b) By applying unformalised rule to specific cases (c) By calculating formalisation for specific cases

Table 1: Critical Decisions in the Generalisation Process

A sample transcript and our identification of the critical decisions made is given below.

Richard and Sunil were working on the Logo research task (Appendix 1). They drew out the head in direct drive, and then wrote a fixed procedure, using the dimensions shown on the task sheet. They then started to construct a general procedure:

S: what we gonna do now?	}	**Decide to construct**
R: we're gonna make it variable........	}	**a General Case**

Having decided to write a general procedure, the pair go on to discuss the name of the parameter.

R: move to FACE1	}	
S: FACE1 dot dot something, isn't it FACE1 dot dot	}	
R: something like that	}	
S: dot dot S, is that what we're gonna call it	}	
R: hold it we can have any variable	}	**Name Parameter**
S: lets say S	}	
R: lets just have, we had S last time, lets just have S all the time to make it easier	}	
S: ok	}	
R: S...dot dot	}	

---

4 By this we mean occasions when pupils choose a parameter in a way which suggests that they have identified a mathematical relationship, for example the selection of a common factor.

5 By this we mean occasions when pupils make no attempt to first identify parameters but nonetheless use them as a basis for defining a relationship; that is the way the generalisation is articulated implies a specification of the parameters.

They added the parameter name to the title line of their procedure (TO FACE1 :S), and started to discuss the choice of a particular value to be represented. They adopt a strategy for identifying the relationship between the dimensions within the head, which involves the recognition of a common factor of 5 (classified as Choose Parameter (b) , see Table 1).

R: FD 45, what are we gonna have as the	}	
S: 5 would be quite small	}	**Choose Parameter**
R: well what's...	}	**(b) with implicit**
S: times 9	}	**reference to**
R: they all go into 5 don't they	}	**relationship**
S: they all go...	}	
R: yeah they all go into 5, go on lets just have it ok why not ...so 9	}	

Having chosen the parameter in a way which also identified the general relationship between the different parts of the head, the pair were ready to formalise this relationship. They do this using a substitution method, that is, for example, replacing FD 45 in their procedure by FD :S*9:

S: delete	}	
R: dot dot	}	
S: dot dot S isn't it	}	
R: dot dot S, S	}	**Formalise**
R: where's that star....times	}	**Relationship**
S: 9	}	
(types :S * 9)	}	
R: ok umm	}	

Discussion of Results

Having identified these critical decisions, as illustrated above, we then approached our research questions by analysing:

- the incidence of critical decisions across environments
- the incidence of collaborative decisions
- pupil roles in identifying relationships and their formalisation in computer environments

Analysis of the results is still ongoing. We present here our preliminary findings, and will further elaborate these in the presentation.

Incidence of critical decisions across environments. For each transcript we calculated the frequency of occurrence of each type of decision and found that the pattern of these frequencies varied across environments. This was particularly evident with regard to the decisions as to parameter choice and decisions concerning the identification of the mathematical relationships embedded in the tasks(see Table 2).

	LOGO frequency	SPREADSHEET frequency	PAPER AND PENCIL frequency
Choose Parameter			
(a) With no reference to relationship	16	2	2
(b) With implicit reference to relationship	8	0	1
Identify Relationships			
(a) After choice of parameter	22	2	9
(b) Incorporating choice of parameter	5	50	16
(c) Using figurative relationships	0	3	9

Table 2:Occurrence of Critical Decisions: Choose Parameter and Identify Relationships

In the Logo environment, the generalisation tends to be constructed in one of two ways: a) choosing a parameter with no reference to a relationship, e.g.

A: FORWARD,FORWARD, do the FORWARD the highest, which is 45, no its not...
J: no it...
A: it's 70, shall we use the highest
J: OK, use 70 as dot dot S then

Choose Parameter (a) With no reference to relationship

and then using this as a basis to identify the numerical relationships, e.g.

J: if 70 is dot dot s then it's, it must be divided by, can't we say...
A: 70 divided by what equals......70 divided by 30
J: equals what?

Identify Relationship (a) After choice of parameter

b) choosing a parameter simultaneously with the identification of a relationship (Choose Parameter (b) in Table 1). In the earlier Richard and Sunil extract this approach was adopted.

In contrast, in the spreadsheet environment, pupils almost exclusively adopt a strategy whereby they identify relationships incorporating implicit choice of parameter e.g:

J2: look at the numbers going down 1, 5, 12, aaah
J1: 4, 7, 10, 13, oh that's right, it's easy.........this one....the gap between the two is always different, so it's 4 then the gap's 7, then the gap's 10, then the gap's 12, 1, I mean, then the gap's...
J2: 16
J1: 16, this should be 19

Identify Relationship (b) Incorporating implicit choice of parameter

This strategy was also very evident in the paper and pencil environment e.g:

R: 2, 3, 4, 5, 6 , that's it that's how you do it
S: what?
R: look you add 2 to that one then 3 then 4 then 5

Identify Relationship (b) Incorporating implicit choice of parameter

However additionally, in the paper and pencil environment, unlike the spreadsheet environment, pupils were likely to use their figures to assist in the identification of relationships[6] :

S: keep to side, keep to the side.....from each point there's 4, so you're gonna need it from each point right there's 4 going away right 1, 2, 3 ,4, 1, 2, 3, 4, right see if there's	} **Identify Relationship** } **(c) Using figurative** } **relationships**

Incidence of Collaborative Decisions For each pair, the percentage of collaborative decisions is shown in Table 3. A decision was judged to be collaborative where there was evidence in the transcript that both pupils made some contribution. Looking across all the pairs, the data suggests that the Logo programming environment provided a setting most fertile for collaboration. However, when we analyse the decision making process more closely by identifying the suggestions within the critical decisions that formed the *basis of subsequent action*, we find large between pair differences related to issues such as task involvement and task difficulty. For example, when the task is too easy, there is no need for a high level of collaboration, or indeed task involvement. Consequently one pupil, the "driver", can dominate the solution process, either by making the majority of suggestions which are subsequently acted upon within collaborative decisions, or by making the decisions on their own (for example, Pair 2 in the Logo environment). In these cases, we have difficulty at present in defining the role of the "driver's" partner in any generalisable way. When a task is too hard, there again tends to be individual dominance as above and collaboration from the perspective of both partners is poor – from the "driver's" perspective little appropriate help is received, and from the "passenger's" perspective the solution process is opaque (for example pair 4 in the paper and pencil environment).

	LOGO	SPREADSHEET	PAPER AND PENCIL
	%	%	%
Pair 1: Alice & Joanna	80	44	65
Pair 2:Jamie & Jake	52	67	71
Pair 3: Richard & Sunil	82	48	25
Pair 4: Joku & Simone	44	50	38

Table 3: Percentage of Collaborative Decisions in each Environment

6 Given the similarity between the paper and pencil and spreadsheet task it would seem that the spreadsheet environment provokes pupils to focus primarily on the given numbers rather than the figures.

Pupil Roles in Identifying Relationships and their Formalisation in Computer Environments For successful task completion, the mathematical relationships have to be both identified and formalised. In the computer environments this formalisation takes place on the computer. If both identification and formalisation of relationships are exclusively determined in a pupil pair by one pupil the collaboration is unlikely to be effective. For effective collaboration, these processes must be shared in some way. We have found that in the computer based research tasks characterised by a particularly high level of collaboration, pupils have tended implicitly to decide to separate these two process, one making the majority of suggestions as to the mathematical relationships (the "pattern spotter") and the other formalising them. Additionally, we found that the pupils taking on this latter role also tended to dominate the keyboard (the "programmer"). Moreover, in three out of four pairs (pairs 1, 2 and 3) this same individual took on the role of "programmer" in *both* the computer environments.

REFERENCES

Hoyles, C. & Sutherland, R.(1986) *Peer Interaction in a Programming Environment* Proceedings of the tenth PME , London p. 254-260

Hoyles, C. & Sutherland, R. (1989) Logo Mathematics in the Classroom Routledge

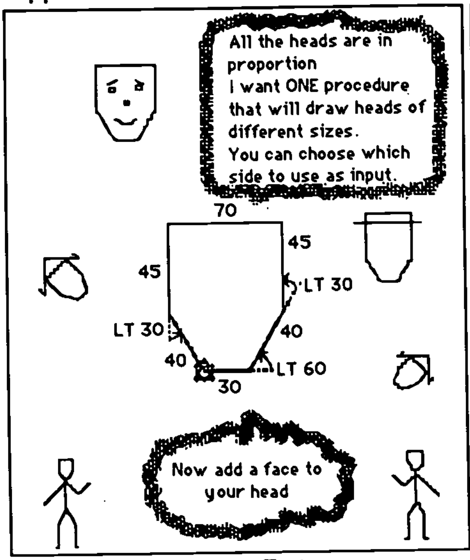
Hoyles, C., Sutherland, R.,& Healy, L. (in press) *Children Talking in Computer Environments: New Insights on the Role of Discussion in Mathematics Learning* . In Durkin, K. and Shire, B., Language and Mathematical Education. OUP

Mason, J with Burton, L. & Stacey, K (1982) Thinking Mathematically Addison-Wesley Publishers Ltd

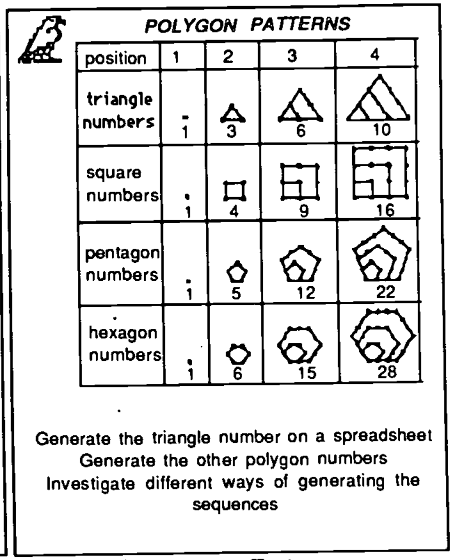
Polya, G. (1957), How to Solve It Oxford University Press

Shiengold, K., (1987), The Microcomputer as a Symbolic Medium, in Pea, R.D., & Shiengold, K., (Eds.), Mirrors of Minds: Patterns of experience in Educational Computing, Ablex Publishing Corporation, New Jersey, USA, 198-208.

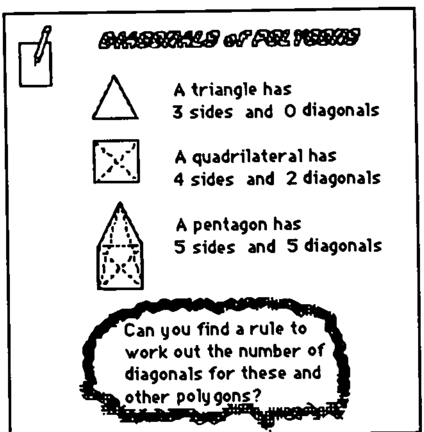
Appendix 1



The Logo Task



The Spreadsheet Task



The Paper and Pencil Task

The criteria for devising the research tasks were that they should: be closed in terms of goal, but open in terms of approach; involve abstract rather than "real life" mathematics; involve a search for relationships defined by specific cases, including both visual images and numbers; involve relationships which are not immediately obvious (e.g. more complex than "times by 2"); involve some element of choice in the variables to be used for defining the relationship.

"SCAFFOLDING"

- a crutch or a support for pupils' sense-making in learning mathematics?

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In this paper I explore the relationship between teacher and pupil as the pupil strives to make sense of the mathematics which she encounters in the classroom and the teacher endeavours to provide an appropriate level of support. I have taken the word "scaffolding" from the work of Bruner and his colleagues Wood and Ross They use the term to describe a role for a teacher working within Vygotsky's "zone of proximal development", providing a form of "vicarious consciousness" for the learner, and undertaking "contingent teaching". I contrast two possible views of this role; one as encouraging a dependency in the pupil from which it may be hard to break free, and the other as providing a supportive environment from within which the pupil is encouraged to begin growth towards independence. I examine issues which these raise for the teacher and provide cases from my own research which highlight decisions which teachers are called upon to make.

Introduction

Any study of teaching would be sterile without considerations of the learner, and should take account of the immense literature on child development, thinking and learning. However, much of this literature considers the learner alone without emphasising the role which can be played by a teacher or teacher figure. In constrast Jerome Bruner, with particular reference to the work of Vygotsky, offers a view of the teaching role. Bruner, (1985 p25) contrasts a view of the learner as seen by Vygostsky with other prevalent views, in particular interpretations of those of Piaget:

> "Too often, human learning has been depicted in the paradigm of a lone organism pitted against nature - whether in the model of the behaviourists' organism ... , or in the Piagetian model where a lone child struggles single-handed to strike some equilibrium between assimilating the world to himself or himself to the world."

The Piagetian model encompasses some concept of 'readiness', in which a child is unlikely to develop conceptually until such readiness is manifested. In this model there seems little that a teacher can do to help the child progress, and may actually hinder progress by interupting the child's natural development. Vygotsky however, promoted the view that progress could be enhanced by instruction, and his "zone of proximal development" (Vygotsky 1978 p86) provided a measure of potential enhancement:

> "It [the zone of proximal development] is the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers."

Vygotsky uses as an example the idea of two children entering a school, both of whom are aged ten years, but who are eight years old in terms of mental development.

> "Can I say that they are the same age mentally? Of course. What does this mean? It means that they can independently deal with tasks up to the degree of difficulty that has been standardised for the eight year old level. If I stop at this point, people would imagine that the subsequent course of

The task for the pupils involved the building of a tower from a series of interlocking blocks of varying size. In the beginning, according to Bruner (ibid) the tutor is "consciousness for two". Having persuaded the child to engage in the activity, her general task

> "is that of scaffolding - reducing the number of degrees of freedom that the child must manage in the task. ..., she sees to it that the child does only what he can do and then she fills in the rest - as in slipping the pegs of certain blocks into the holes of others to which they are mated, the child having brought them next to each other. She limits the complexity of the task to the level that the child can just manage, *even to the point of shielding his limited attention from distractors.*" (My italics)

Once the child had achieved, with help, one level of mastery, the tutor, with some prudence, could then invite the child to undertake a higher level assembly - in Vygotskian terms, "leading the child on ahead of his development" This not only keeps the child within the ZPD, but "keeps him from getting bored".(Bruner ibid) The teacher is thus seen to provide support for the pupil, but, as my italics above stress, this can encourage the pupil to depend on the teacher for support and thus inhibit her own initiative.

Implications for the classroom

Wood points out that although the 'formula' for contingent teaching sounds simple, even trite, the tutor often violated the agreed rules, sometimes repeating an instruction at the same level when she should have given more help, on other occasions giving help when none was called for. He points out that, "Understanding the 'rules' of contingency and *teaching* according to the rules are two quite different things." Contingent teaching, however perceived, is no straightforward matter for the teacher.

Although on the face of it notions of 'readiness' and those of 'contingent teaching' might appear to be at odds, it seems that the teacher, about to interact with a pupil in the classroom, must, perhaps implicitly, bring considerations of both to the situation. She must be sensitive to the pupil's needs, but prepared to enter some teaching act relevant to these needs. A crucial consideration seems to be of what such teaching acts consist if they are to maximise the potential of the ZPD, and how the teacher knows in any instant what act is appropriate.

Valerie Walkerdine (1988) examines the concept of 'readiness', looking closely at transcripts of classroom discourse involving a teacher and very young children. In one case where the teacher has judged the children 'ready' to consider concepts of 'bigger' and 'smaller' they make a mistake which appears to contradict this readiness, and the teacher has to make an instant decision as to how to respond. Walkerdine analyses the contexts which might have contributed to the mistake and it becomes clear what complexity of thinking and decision making is required of the teacher in providing appropriately for the children's development.

Wood, concluding his remarks on contingent teaching (ibid), made the following observations with regard to classroom teaching, and to mathematics teaching particularly:

> "In the 'real world' of the classroom of course, the problem of achieving contingent instruction is far more difficult. ... many lessons taught in school often involve tasks that do not have a clear, obvious structure ... .Even mathematics, which seems well structured, does not have a single clear cut structure ... most classroom teaching takes place with *groups* of children. ...Does it make sense to talk about contingent teaching in a situation where many children are being taught simultaneously?

Despite such reservations, when I discussed the principles of contingent teaching, of scaffolding and the ZPD with one progressive reflective teacher whom I have studied in depth

mental development and of school learning for these children will be the same, because it depends on their intellect."

He goes on to suppose, however, that he does not stop here but encourages the children to solve problems with his assistance, acknowledging that this might involve any of a variety of methods:

"...some [experimenters] might run through an entire demonstration and ask the children to repeat it, others might initiate the solution and ask the child to finish it, or offer leading questions."

Ultimately he hypothesises that, given this assistance, the first child can deal with problems up to a twelve-year-old's level, whereas the second achieves success only at the level of a nine-year old, and asks the question, "Now are these children mentally the same?"

Vygotsky, while admitting to the naivety of claiming actual development levels, nevertheless claims that these children are not mentally the same age, and that from here onwards their progress would be different. He suggests moreover, that , "what is the zone of proximal development today, will be the actual development tomorrow - that is, what a child can do with assistance today she will be able to do by herself tomorrow." He further claims:

"The zone of proximal development can become a powerful concept in developmental research, one that can markedly enhance the effectiveness and ultility of the application of diagnostics of mental development to educational problems."

Bruner (ibid) paraphrases as follows:

"If the child is enabled to advance by being under the tutelage of an adult or a more competent peer, then the tutor or the aiding peer serves the learner as *a vicarious form of consciousness* until such a time as the learner is able to master his own action through his own consciousness and control. When the child achieves that conscious control over a new function or conceptual sustem, it is then that he is able to use it as a tool. Up to that point, the tutor in effect performs the critical function of "*scaffolding*" the learning task to make it possible for the child, in Vygotsky's word, to internalise external knowledge and convert it into a tool for conscious control." (My italics)

Thus the tutor has to make judgements about the degree of control which a child is capable of assuming at any stage, and the *handover* of control is a crucial part of the scaffolding process.

These notions of 'zone of proximal development' (ZPD), 'vicarious consciousness', 'scaffolding' and 'handover' are attractive metaphors through which we can begin to examine the teaching-learning interface. However, they raise a number of issues which it is my purpose to address.

Contingent teaching

Whatever the attraction of such theoretical notions, their translation into practical situations involving teacher and learner is far from obvious. I quoted above some of Vygotsky's own examples of strategies which a teacher figure might use. Bruner and some colleagues undertook a teaching experiment (See Wood, Bruner and Ross 1976) , using what they called 'contingent teaching', in which they attempted to implement a form of 'scaffolding'. In this a tutor was trained to work *contingently* with pupils. According to Wood (1988):

"Contingent teaching, as defined here, involves the pacing of the amount of help children are given on the basis of their moment-to-moment understanding. If they do not understand an instruction given at one level, then more help is forthcoming. When they do understand, the teacher steps back and gives the child more room for initiative."

(See Jaworski 1989) he felt that many of the principles were ones which he would wish to espouse and that they were not inconsistent with an awareness of 'readiness'. I reviewed data which I gathered in his lessons in this context.

Mathematical Challenge

In my study of a number of teachers, one of my focuses has been that of 'mathematical challenge' (See for example Jaworski 1988). I have been interested in what acts a teacher undertakes to facilitate a pupil's mathematical progress. The teacher above, Ben, claimed on a number of occasions that he made judgements about how to respond to pupils, or what to offer them, according to his perception of their *readiness* for it. His intervention with particular pupils or groups of pupils varied considerably according to this. Sometimes it seemed particularly fierce, as in the following example involving a pupil Rachel who had been working on a problem which she had selected to tackle as part of her GCSE course work.

The problem itself is immaterial to the discussion which follows. She had been working on some particular cases from which she was trying to find a general pattern. Ben had been working with other pupils at Rachel's table, and before leaving he looked over at what she was writing. She looked up questioningly, and he said, "Yes, Rachel, what's next?". She replied, "I'm just doing some more of these", referring to her examples which she had discussed with him earlier. He replied,

B Do you know you actually haven't proved it? You've just shown that it could possibly be true. Can you think, - what *is* your conjecture? Can you give it me in words?

R Erm, you add up the perimeter, and add four on - add four on, and that gives you the number of -

B Hang on, you've got a proof there in the making. (A distinct change in the tone and pace of his words occurs here) You've nearly said why it's true, haven't you?

R Have I?

B Why you add four on, why do you add four on?

R There's four corners. It's got to be an extension.

B Could you write that as a proof to show your conjecture is actually true, yes?

One level of my analysis[1] of this excerpt can be summarised as follows. The teacher's focus here was on *proof*. He judged Rachel to be at a position to think about proof and challenged her accordingly. In responding to his instruction to express her conjecture in words, she convinced Ben that in fact she was close to proving her assertion. His tone of voice conveyed his excitement and pleasure when he realised that she was so close to a proof.

At first glance there seems to be very little of scaffolding taking place here. Almost immediately the teacher seems to hand control to the pupil in terms of proving her conjecture. Yet looking at the four sentences of the teacher's first words, which are uttered in a fairly

[1]The analysis of a piece of transcript such as this is very complex and relies heavily on contextual considerations and perceptions of the observer and participants. Analysis would be at a number of levels - higher levels including consideration and justification of analysis at lower levels. These are major considerations of my PhD Thesis on which I am currently working and from which the above extract is taken. I offer analysis rather superficially here without the weight of such considerations, partly as they could take up the rest of my allotted space, but also because they would detract from my main argument in this paper.

slow, low key, measured manner, it is possible to see a rapidly developing structure of support:

1. *"Do you know you actually haven't proved it?"* - drawing her attention to proof;

2. *"You've just shown that it could possibly be true."* - qualifying his rather bald assertion in terms of her work on examples.

3. *"Can you think, - what is your conjecture?"* - hesitatingly beginning to provide support. How can he help her to see what he means?

4. *"Can you give it me in words?"* - a specific request with which she can comply. He has brought the focus on proof into a task which is within her grasp.

The pupil's response is exciting for the teacher. His tone of voice when he replies to her is quite different from when he uttered the above sentences. In her short, poorly articulated response there is enough for the teacher to gain insight into her thinking and observe that she is close to what he wants in terms of proof. However, his talk of proof perhaps does not accord with what she sees herself doing, because she asks in surprise, *"Have I?"* He responds by further narrowing the focus, asking, *"Why do you add four on?"*

He leaves her to *"write that as a proof to show your conjecture is actually true"*. We do not know what she actually wrote, or what further help was given. It is possible to envisage, with different judgements by the teacher, what further help might have been given at the stage describe above. For example, the teacher might have been drawn to explain in more detail what he meant by proof.

What seems interesting is where the pupil was left. It is impossible to judge whether this was appropriate or inappropriate for *her*. What it offered was a chance for her to use her own thinking to decide what sort of explanation to provide and begin to be conscious of the nature of this as a proof. Further explanation from the teacher about the nature of proof may have made it clearer to her what he particularly required, but could simultaneously have taken away her own necessity to judge for herself and develop her own concept of proof.

The above scenario could be construed in terms of Rachel's ZPD. Without the teacher's remarks she may have gone no further than vague expressions of generality from a number of particular cases. With the introduction of the idea of *proof* she could begin to think in those terms, starting to become aware of what a proof might mean and opening up the opportunity for looking for proof when tackling further problems. The scaffolding which the teacher's remarks provided could thus have an enabling effect.

Making Judgements

The teacher was well aware of his making of judgements and their potential implications, and we discussed this overtly on many occasions. For example, at the end of one lesson I asked him about a particular response which he had made to one pupil in the lesson. In expressing his reasons for it, he indicated that he had been aware of a number of possible ways to respond,, and had made an instant decision. He said, "it's this business of *judgement* again, isn't it?" We contrasted, on another occasion, the potential dichotomy between planning and sponteneity in a lesson, and I remarked how often the lesson seemed to go very much the way he had suggested to me that it would, although he appeared to be spontaneously responding to pupils remarks and questions. He replied,

"Now, that could be interpreted in two ways. It could have been interpreted that Ben has a plan in his mind and he's going to get there irrespective of any obstacles placed in his way. Ben sticks slavishly to his planning and won't even be pushed off. Or is the other way of looking at it that Ben knows his group fairly well and can fairly predict their reactions? I think we're getting nearer the second."

I asked if he planned in a way that left flexibility. *Flexibility* had been his word originally, but he paused and then said,

"I think you'll have to expand on flexibility. If you mean outcome, *no*. Because at some point I have got to get round the idea of surface area - at some point in the future. If it's today or tomorrow, it's flexibility.

If it's a worthwhile mathematical excursion - why not go. It's got to be worthwhile. Then it's back to judgements, making decisions, letting things go. We keep coming back to that today."

And on yet another occasion he said,

"Yeah, teaching's a lot of judgement as I call 'on the hoof'. You're making a lot of judgements as you go along.

I think a lot of the time you don't have the time to sit back and have the luxury of saying, 'yes, I'll make a decision'. I think they're there. The judgements are there

On what basis? On *need* I suppose. On *perceived* needs - I would say that."

Interpretation of 'needs'

As a counterpoint to 'Mathematical Challenge', another focus of my research has been that of 'Sensitivity to Students' (See Jaworski 1988). The relationship between these focuses has proved a central feature in my study of various teachers. Ben's reference to 'perceived needs' is part of what I came to see as his knowledge of particular pupils and his consequent degree or style of challenge. However, identification of a pupil's needs is highly interpretive, and it is important to examine the basis of an interpretation.

Walkerdine (1988) implied that a concept such as 'readiness' could be used as a panacea to avoid searching out particular reasons for children's mistakes. Suggesting that the children were after all not 'ready' for something could avoid having to look for deeper reasons. A teacher's perception of the 'needs' of a pupil can similarly be used as a panacea for justifying whatever action was taken. For example, "She needed an explanation, so I gave it." This could hide a severe case of teacher-lust of the form, "This was an opportunity for me to give an explanation about which I could feel good afterwards, and so I launched in without further thought." It might have been, in this highly hypothetical (and provocatively described) situation, that what the child would most benefit from was some encouragement to produce such an explanation herself.

Another example from Ben's classroom illustrates this point. Pupils were seeking triangles which had numerically the same area as perimeter, and their approach had been left to them. One girl had drawn a number of triangles from which she was roughly calculating area and perimeter. One of these triangles had its sides labelled 3, 6 and 9 cm. Ben came and looked at this and said to her:

T If I've got a length of line nine, it must lie straight over the top, so I can't actually make a triangle like that.

It seemed as if this logic should immediately alert her to a mistake. However, the conversation continued,

P Yeah, but its a triangle

T That's not accurate is it?

P Yeah, There's one side and the other and the other

T That's three and that's nine?

P Yeah

T No, it's eight and a bit

P Well!

T Now hang on. If that becomes nine, it's got to go a bit longer.

P Yeah, but that means that's got to be longer

T So that must bring it down to make it longer doesn't it?

P Yeah, well that's what I was doing

T Now, the only way you can make it surely, if that's six and that's three, that's the only way you can do it to make it nine is to make it into a line isn't it?

P Well, that's nearly right isn't it?

With some noncommital sound he moved on at this point, and I was quite surprised that he had not tried some other form of explanation. In discussion later, Ben said, "there's no more I could say in that moment that could convince her". He had made a conscious judgement *not* to be drawn into further explanation. He knew that at some stage he would have to reintroduce the concept of an x, y, $x+y$ triangle with her, but judged that further discussion at this point was inappropriate.

It is possible to view the teaching of this concept as one in which a *contingent* approach might usefully be employed, - perhaps in introduction by the teacher of a number of triangles which the pupil might try to draw, in progressing from the teacher explaining to gradually prompting the pupil to explain, and finally in asking the pupil to suggest some 'impossible' triangles herself. It might be believed that leaving the pupil in her misconception was irresponsible of the teacher, and that there was some *duty* to take the matter further at that point.

Such moral issues have to remain within the province of the teacher concerned, as no one else is knowledgeable enough to make judgements. However, we can speculate on the outcome of such a contingent approach. It is likely that the pupil would gain a demonstrable perception of the triangle concept which would satisfy the teacher in that instant. Thus the contingent teaching could be seen to have been successful. It is also likely that the pupil's dependence on the teacher would have increased as a result. It is a tension of the teaching/learning interface that what is seen as the most helpful input in a teaching moment might ultimately be the most inhibiting (See for example Mason 1989). If the pupil learns to expect that the teacher will cushion the thinking, making concepts 'easy' to perceive, then the pupil is likely to grow into an inability to struggle to make sense alone.

A crutch or a support?

Crucial to the notion of *scaffolding* is that of *handover*. In the hypothetical case above, handover could be achieved *locally* as the pupil demonstrated her ability to produce her own 'impossible' triangles. In the Wood, Bruner, Ross experiment, handover involved the child locally demonstrating ability to construct a tower more complex than had been built with the tutor's help. My reading of scaffolding in the literature has led an interpretation of handover in this *local* sense. The issue as I see it is that scaffolding, perceived locally, may encourage learning of a particular concept, but is likely also to inhibit independent thinking and development by encouraging a dependency on the teacher's support. Thus it becomes a crutch on which the pupil develops increasing dependence.

Contrary to perceptions of 'readiness' which presume that development precedes learning, Vygotsky's theory is that development *follows* learning via the ZPD. The ZPD *can* be perceived *locally* according to a particular concept or task. However, there seems to be no need

for this, nor does Vygotsky seem to suggest it. Vygotsky rejects theories of the Gestalt school which suggest that learning *causes* development; that "the child, while learning a particular operation, acquires the ability to create structures of a certain type regardless of the diverse materials with which she is working and regardless of the particular elements involved." (1978 p83) One objection is that research has shown that learners do not automatically generalise experiences in this way. Rather, according to Vygotsky, what is learned will be shared, through peer interaction and conversations with peers and adults, allowing an assimilation and modification of experience from which development proceeds. He claims that *schooling* is a vital element in the movement from learning to development, as an environment can be created which supports and facilitates this process. However, he recognises that the formal discipline in which learning of school subjects leads to development requires, "extensive and highly diverse concrete research based on the concept of the ZPD" (p 91)

To return to *scaffolding*; perhaps the scaffolding process can be extended to approach this formal discipline of which Vygotsky speaks. Two essentials of this discipline seem to be, firstly that the pupil is not encouraged to become dependent on the teacher to instigate and regulate thinking; secondly that the pupil is able to generalise particular learning experiences to more general thinking strategies. This requires a lifting of the scaffolding process to a level at which the learned act, for example the building of a tower or the recognition of an impossible triangle, is just one of the experiences that the teacher uses in order to promote learning initiative. The teacher needs also to be able to point to occurrences of the pupil having made intuitive leaps, or having struggled with apparent contradictions to reach order and consistency as instances of successful achievement. Implicit in this is that the teacher needs to be as ready to withhold comfort as to provide it. In either case it seems important to draw the pupil to an awareness of what has taken place - "What was the result of my giving you that help?" or "What was the result of my *not* giving you that help?"

It could be that the scaffolding metaphor is inadequate for the process which I suggest, in that it carries too much of the sense of dependency or of a crutch. However, often a crutch is something which a person casts aside when a limb has become strong. Without any doubt pupils depend on the help of their teachers. Ultimately they have to leave this behind. In my paper 'To inculcate versus to Elicit knowledge' (Jaworski 1989) I gave examples of a teaching approach which I feel goes some way to 'fit' (See von Glasersfeld 1984) a potentially extended view of scaffolding. I invite those who are interested to explore this extension.

References:

Bruner J.S (1985) 'Vygotsky: a historical and conceptual perspective'. In J.V.Wertsch (ed), *Culture Communication and Cognition: Vygotskian Perspectives.* Cambridge: Cambridge University Press.

Jaworski B. (1988) 'One Mathematics Teacher' *Proceedings of PME 12, Hungary*

Jaworski B. (1989b) 'To inculcate versus to elicit knowledge.' *Proceedings of PME 13, Paris*

Mason J. (1989) 'Tensions' ..In D. Pimm (ed) *Mathematics Teachers and Children.* London: Hodder & Stoughton

von Glasersfeld E. 'An introduction to radical constructivism'. In P. Watzlawik, *The Invented Reality.*London W.W. Naughton & Co.

Vygotsky L.S. (1978) *Mind in Society. The development of the higher psychological processes.* London: Harvard University Press

Walkerdine V. (1988) *The Mastery of Reason. Cognitive development and the production of rationality.* London: Routledge.

Wood D. (1988) *How Children Think and Learn* Oxford: Basil Blackwell

Wood D.J. Bruner J.S. & Ross G. (1976) 'The Role of Tutoring in Problem Solving' *Journal of Child Psychology and Psychiatry*, 17, 2, 89-100

THE ROLE OF MATHEMATICAL KNOWLEDGE IN CHILDREN'S UNDERSTANDING OF GEOGRAPHICAL CONCEPTS

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This study examines the relationship between children's procedural and conceptual understanding of mathematics and their accuracy in interpreting geography text material containing mathematical information. It is expected that inexperience with specific mathematical concepts and/or children's mathematical misconceptions will be associated with inaccurate interpretations of geography content. It is also anticipated that mathematical competence will not necessarily be applied to reasoning about mathematically related geographical concepts. Sixty-four children, 16 in each of grades 3 - 6 are being interviewed about related topics in mathematics and geography to test these hypotheses.

Recently there has been a great deal of concern expressed regarding Americans' geographical illiteracy. Studies indicate that serious gaps and misconceptions in geographical knowledge are evident in students' performance at all ages (Daniels, 1988; National Geographic Society, 1988; Solorzano, 1985). One factor contributing to this problem may be the failure to view the acquisition of geography concepts in the context of students' understanding and application of knowledge from other academic content areas (Adler, 1989; Blaut & Stea, 1971; Downs, Liben, & Daggs, 1988).

In particular, students' knowledge of mathematical concepts and procedures seems to be a critical variable in developing an appreciation of many geographical ideas. A survey of geography curricular materials supports this contention and indicates that many geographical concepts, indeed, presuppose knowledge of particular mathematics concepts. For example, in third grade (Silver Burdett & Gin, 1988) children are instructed about how maps are "drawn to scale" such as "1 inch to 20 feet," yet the concept of ratio does not appear in most mathematics curricula until later on in sixth grade. In

addition, even if the mathematics has been "taught" it does not necessarily imply that children have attained an accurate knowledge of given concepts and procedures (Baroody, 1987; Ginsburg, 1989). Further, it has yet to be demonstrated that even when children have attained competence in a particular mathematical content area that they can apply this competence to another context.

This research examines the relationship between students' knowledge of mathematics concepts and procedures and their ability to understand and interpret geography content in which these mathematical concepts are embedded. It is hypothesized that a)children's misconceptions or lack of experience with particular mathematical content areas will be associated with inaccuracy and misunderstanding of geographical concepts and that b)children who demonstrate competence in particular mathematical concepts and procedures will not necessarily apply this knowledge to related geographical contexts.

The subjects of this study are 64 students, 8 boys and 8 girls from each of grades 3 through 6 attending a middle class public suburban school district in New Jersey. Within grade and sex, the students were randomly selected from over 200 volunteers from two schools in the district. Each student is being individually interviewed outside the classroom for approximately 45 minutes. All interviews are being videotaped.

In the first part of the procedure students are asked to read two short excerpts from a grade-level geography text. One contains content dealing with knowledge of maps and the other is related to information about the population, climate, or industry of a given geographical area. For each excerpt students are asked a)factual information questions based on the

content, b)interpretive questions that go beyond the given information in the text and that require the application of some mathematical knowledge, c)definition questions about some mathematically loaded terms used in the text, and d)concept extension questions in which the same map and graphing concept(s) described in the text need to be applied to an analogous situation.

In the second part of the procedure students are asked to work out computational examples for each mathematical concept embedded in the geography text, to demonstrate their understanding of the computational procedures, to solve word problems involving the application of the same computational procedures, to construct the solution to a geometry or measurement problem, and to interpret a graph problem in a non-geographical context.

All students in each grade are asked the same core questions. However, clinical interviewing procedures are utilized to clarify students' responses and identify misconceptions in their reasoning. Within both the geography and mathematics tasks, subjects are evaluated on accuracy of answers and/or procedures used, appropriateness of reasoning and solution strategies used regardless of accuracy of execution, and on the type and frequency of misconceptions expressed. An accuracy/appropriateness score of 0, 1, or 2 is obtained for each item and total scores are obtained for all items in each domain and for subsections of items within each domain. Based on total scores within domains, subjects will also be divided into high and low accuracy groups.

Data analysis will focus on the correlational relationship between knowledge of particular mathematical contents and the

attainment and application of specific geographical concepts at each grade level. It also will focus on a comparison of the types of misconceptions held by students in high and low accuracy groups. Trends across grade levels will be examined.

This study is currently in progress and data have not yet been analyzed. Complete results will be available in time for reporting at the scheduled meeting in July, 1990. These should indicate that mathematically inaccurate children have lower accuracy scores in factual geographical information than children who are accurate in their mathematical concepts and procedures. They should also demonstrate that children's mathematical misconceptions interfere with the application and interpretation of geographical concepts. Further, it is expected that children who demonstrate accurate knowledge of mathematical concepts and procedures will not necessarily be able to apply this knowledge to geographical contexts.

The results of this study should have significance in several areas. First it will add to our general understanding of how children's existing knowledge base interacts with school curricular content. Second it will expand our knowledge about how children are or are not able to apply concepts from one domain to another (in this case mathematics to geography). Third, the videotaped illustrations obtained during the data collection can be used in teacher education programs to demonstrate to practitioners that there are a variety of ways in which students interpret "objective" content and that effective instruction must take into account the fact that academic subject areas often overlap with one another.

References

Adler, T. (1989). Site visit to LRDC educates Hill staff. APA Monitor, January, 7.

Baroody, A. (1987). Children's mathematical thinking: A developmental framwork for preschool, primary, and special education teachers. New York: Teachers College Press.

Blaut, J. & Stea, D. (1971). Studies in geographic learning. Annals of the Association of American Geographers, 61, 387-393.

Daniels, L. (1988). Finding new ways to make geography exciting. The New York Times, March 22.

Downs, R., Liben, L. & Daggs, D. (1988). On education and geographers: The role of cognitive developmental theory in geographic education. Annals of the Association of American Geographers, 78, 680-700.

Ginsburg, H. (1989). Children's arithmetic, second edition. Austin, TX: Pro-Ed.

Loftin, R. & Ainsley, W. (1988). Our country's communities. Morristown, NJ: Silver Burdett & Ginn.

National Geographic Society. (1988). An international Gallup survey. Washington, DC: National Geographic Society.

Solorzano, L. (1985). Why Johnny can't read maps, either. U.S. News & World Report, March 25.

SPEAKING MATHEMATICALLY IN BILINGUAL CLASSROOMS: AN EXPLORATORY STUDY OF TEACHER DISCOURSE

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The education of Spanish-speaking students in the United States is often inadequate, particularly in mathematics. This project investigated linguistic factors which might help or hinder the acquisition of mathematical knowledge by students with limited English proficiency (LEP). Data from a qualitative study of four elementary classrooms suggest that Spanish is seldom used to develop mathematical understanding, even by bilingual teachers of LEP students. When Spanish was used, linguistic errors often hampered student learning.

Recent data suggest that Hispanic students in the USA continue to perform below national norms in mathematics. The purpose of this study was to investigate some of the language factors that might contribute to this pattern of underachievement. The focus of the study was on the language used by the teacher, with particular emphasis on his/her mathematical explanations.

The role of teacher discourse has played an important part in research on teaching (Cazden, 1986). In studies in mathematics education (e.g., Good, Grouws, & Ebmeier, 1983) the nature of teacher explanations, especially in the development portion of the lesson, has been a major focus. This study provides additional information on critical aspects of the teacher's use of language in bilingual classrooms.

Conceptual Framework

The analysis of teacher discourse has become an important part of research on teaching. This kind of research began with the specification of categories of teacher speech acts and their frequencies, but the paradigm

has since shifted to an interactionist perspective in an effort to unite the cognitive and social dimensions of learning. Mehan's (1979) study of lessons provides an important example of the analyses of actual classroom dialogues and how language use structures lessons. In research on mathematics teaching, Leinhardt (1988) emphasizes the role of routines, scripts, and agendas in structuring teacher discourse and classroom interaction.

Along with the structure of classroom discourse, there are linguistic factors that have implications for research on mathematics teaching. Pimm (1987), for example, notes the complex relationships between mathematics and language, and uses linguistic analyses to illuminate how communication takes place in mathematics classrooms. One of the tools in his analysis is the notion of "register", the use of natural language in a way which is particular to a role or function (Halliday, 1978). Mathematics has its own register (Cuevas, 1984). This register is not just special terminology but also a set of unique meanings and structures applied to everyday language. The development of a mathematics register is accomplished in many ways, including reinterpretation of existing words (e.g., carry, borrow, reduce) as well as the introduction of new terminology (e.g., common denominator) (Halliday, 1978). In addition, certain syntactic and semantic structures are characteristic of mathematics. For example, there are four semantically different verbal subtraction problems, each of which could be solved with the same symbolic sentence (Moser, 1988). Furthermore, a common mathematical question is: How many are left? When a Spanish-speaker interprets this, "left" can become linguistically redundant and can easily be confused with its directional meaning.

Lastly, effective instruction for non-English or limited English

proficient (NEP/LEP) students requires the use of the students' native language particularly for concept development, the integration of English language development with academic skills (Tikunoff, 1985), and most importantly, teaching strategies utilizing interaction and context (Cummins, 1986). In teaching mathematics, attention must be given to clarification of terms since there are differences in ways of expressing mathematical concepts in Spanish and English which can cause confusion in comprehension (Cuevas, 1984).

The importance of these points is that language issues in the teaching and learning of mathematics may be more crucial than previous research would suggest. Further study of these factors is particularly important in the context of bilingual classrooms.

Methods

Four classrooms were selected from elementary schools that have significant numbers of Hispanic non-English and limited English proficient (NEP/LEP) students. Two classrooms were chosen from the primary level and two from upper level elementary classrooms.

Each classroom was video taped for seven to ten hours on days when the teacher indicated that new concepts such as place value and rational numbers would be explained. Formal interviews were conducted with each teacher regarding personal and teaching background, academic and language backgrounds of his/her students, and perceptions of teaching mathematics. Informal interviews were conducted with some students to assess their grasp of the mathematical meanings presented in the lesson and to enhance our observations.

The analyses of video tapes, the primary source of data, were guided by the following constructs: a) the nature and use of the mathematical

register; b) the nature and use of L1 (Spanish) and L2 (English) and the comprehensible nature of L2; c) the use of language for the negotiation of meaning or to emphasize rote learning; and d) the clarity or ambiguity of language in concept development. Triangulation among three independent observers was used to provide validation of the items deemed to be linguistically troublesome.

One teacher did not speak Spanish, but was known as a highly effective teacher of mathematics even with LEP students. The other teachers were bilingual instructors. One of them was not a native speaker of Spanish but did have extensive academic training in the language and had lived in a Spanish-speaking country. Another teacher had all of her schooling including higher education in Mexico. Both of these teachers taught at the same primary grade level. The third teacher was Hispanic but had had most of his schooling in the United States and had not maintained a high level of fluency in Spanish.

Results

Analyses of the data present three striking patterns of teacher discourse. The first relates to the differences among teachers in their efforts to develop the mathematics register. Effective techniques included emphasizing meanings by variations in voice tone and volume, pointing to written words as they were used orally in order to highlight differences in meaning, and frequent "recasting" of mathematical ideas and terms. Less effective techniques were characterized by missed opportunities to establish and clarify the mathematics register. For example, during an explanation of place value and regrouping, the Spanish word "decena" (meaning a group of ten) was used. This is a specialized word unfamiliar to young students and is very similar to "docena" which means dozen. The two spoken words can

easily be confused, particularly if the teacher's accent is difficult to understand. Such a misunderstanding can make the discussion incomprehensible. In another example, also having to do with regrouping, the teacher intended to convey to the students that once they had counted ten items to a group, they had to start again with a new group. However, what was said in Spanish was "no se cuenta mas el diez" which means "don't count the ten anymore," or eliminate ten as a counting number.

Second, very little Spanish was actually used by two of the bilingual teachers. What they used fell into two categories which we are calling "instrumental use" and "markers of solidarity." One teacher used Spanish in a perfunctory manner as an "instrument" to discipline and call students' attention to the subject of the lesson, or randomly, to "punctuate" a statement. The other teacher used Spanish to give encouragement and to motivate the class; it was also used when the teacher worked individually with a student as a private but shared mode of expression. In both of these classrooms, very few whole thoughts were conveyed in Spanish, the only language of proficiency for some students. Overall, Spanish was not used when mathematical meaning was being developed.

Thirdly, with the exception of one classroom, very little contextualized instruction and verbal interaction occurred during the mathematics lesson. Consequently, students were expected to learn by listening, contrary to recommendations by Cummins (1986).

Conclusions

The application of communication and interaction constructs to the teaching and learning processes in mathematics is relatively new theoretical ground, particularly as it relates to bilingual students. As such, a study of this nature generates new insights and variables for further research.

As we have seen from the foregoing discussion, the characteristics of language use which emerged suggest that bilingual teachers need help in strategies for developing the mathematics register in both languages and in making appropriate use of the students' first language. Furthermore, the general nature of mathematics instruction with bilingual students is consistent with what is generally known from observations of mainstream classrooms, i.e., students work individually and in silence. This is a critical dimension since effective instruction for bilingual students requires a verbally rich environment.

These patterns of language use also raise some salient questions. On what basis do teachers choose to use Spanish in instruction? How do they check whether their English is comprehensible and adjust accordingly? Finally, how does the teachers' expertise in mathematics affect the use of the register in the development of mathematical meaning? It is hoped that answers to these and other questions will enhance our understanding of the effects of language and communication factors on the development of mathematical ability.

Preparation of this paper was supported in part by National Science Foundation Grant No. MDR-8850535. Any opinions, conclusions, or recommendations are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Cazden, C. (1986). Classroom discourse. In M.C. Wittrock (Ed.) Handbook of research on teaching, 3rd Edition (pp.432-463). New York: Macmillan.

Cuevas, G. (1984). Mathematics learning in English as a second language. Journal for Research in Mathematics Education, 15, 134-144.

Cummins, J. (1986). Empowering minority students: A framework for intervention. Harvard Educational Review, 56, 18-36.

Good, T., Grouws, D, & Ebmeier, H. (1983). Active mathematics teaching. New York: Longman, Inc.

Halliday, M.A.K. (1978). Language as social semiotic. London: Edward Arnold.

Leinhardt, G. (1988). Expertise in instructional lessons: An example from fractions. In D.A. Grouws & T.J. Cooney (Eds.), Effective mathematics teaching (pp. 47-66). Hillsdale, NJ: Erlbaum.

Mehan, H. (1979). Learning lessons. Cambridge, MA: Harvard University Press.

Moser, J. (1988). Arithmetic operations on whole numbers: Addition and subtraction. In T.R. Post (Ed.) Teaching mathematics in grades K-8, (pp. 111-145). Newton, MA: Allyn and Bacon, Inc.

Pimm, D. (1987). Speaking mathematically. New York: Routledge & Kegan Paul.

Tikunoff, W. (1985). Applying significant bilingual instruction in the classroom. Rosslyn, VA: National Clearinghouse for Bilingual Education.

THE EMERGENCE OF MATHEMATICAL ARGUMENTATION IN THE SMALL GROUP INTERACTION OF SECOND GRADERS

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An analysis of a videotaped small group situation of second graders taken from the Purdue Problem Centered Mathematics Project of COBB, WOOD, and YACKEL will be presented. The main focus is on the interactively constituted process of argumentation. The emergence of mathematical argumentation will be seen both as a result of social interaction and as an individual process of understanding the argument in terms of one's own cognitive structure. Theoretically this analysis is based on the attempted coordination of a psychological perspective with a sociological one.

In recent research the importance of small group activities in education and especially in mathematics education has been emphasized (Slavin, Sharan, Kagan, Hertz-Lazarowitz, R., Webb, C. & Schmuck, R., 1985). However, the reality of everyday mathematics classroom situations shows many difficulties in effectively stabilizing small group interaction as a usual form of learning in school. In West Germany, as in the United States, for example, the main form of classroom interaction is still organized as so-called "frontal-teaching" (Hoetker & Ahlbrandt, 1969; Hopf, 1980; Mehan, 1979; National Research Council, 1989). Small group activities in mathematics classes are very often used for routinized practice in the teaching-learning process and not for peer-activities about learning new mathematical concepts

This research was supported in part by the National Science Foundation under grants No. MDR-874-0400 and No. MDR-885-0560. All opinions expressed are solely those of the authors.

(Slavin, 1986). This is not the place to discuss all the problems which are possible obstacles in the institutionalization of small group activities as a regular learning setting. Here the focus is on the process of argumentation in small group interaction while doing mathematical problems.

In the Purdue Problem Centered Mathematics Project of Cobb, Wood, and Yackel, the classroom is organized so the fundamental learning processes are to take place in the process of small group interaction, while consolidating the new learned content is placed in the whole class discussion (Yackel, Cobb, & Wood, in press, p.5). In this paper we illustrate the nature of mathematical argumentation and justification in small group collaborative work by analyzing an episode in which two children are solving a sequence of multiplication sentences. Thereafter we will discuss the relation between argumentation and learning mathematics. The analysis coordinates a psychological and a sociological perspective. The nature of the children's individual conceptualizations of the tasks will be discussed and will be seen to be influenced by the social interaction between the children. The presence of an adult observer further influences the social interaction. While the observer's intent is simply to ask for clarification so that he understands each child's solution method, his comments serve the function of eliciting explicit mathematical arguments and justifications from each child which they may not have otherwise given. As the analysis will show, these justifications give insights into the children's evolving conceptualizations of the tasks. Over the course of the episode we see that the children use several different types of mathematical argumentation. We will discuss two of these to show how the children's mathematical conceptualizations and the social interaction between the children influence which form of argumentation

each child uses on a specific task as well as to show the evolution of one child's understanding of one of the forms of argumentation.

An illustration of mathematical argumentation in small group work

The episode is taken from a small group problem solving activity in which the children John and Andy are working on the following problems: 2 x 4 = _; 4 x 4 = _; 5 x 4 = _; 10 x 4 = _; 9 x 4 = _; 8 x 4 = _; 8 x 5 = _; 7 x 5 = _; _ x 5 = 30. The problems are written one under the other on a single worksheet.

The children generate a number of explanations of these problems. From the stance of mathematical argumentation the types of explanations are not all different from each other. We can construct several types of mathematically different argumentations in this interaction. Only two will be discussed here. These are the arguments of: (A1) counting up in steps or in the rhythm of the multiplier and (A2) relating the problem to one of the previously solved problems. Two examples are given to clarify this.

Argumentation A1: For the problem "8 x 5 = __", John counts in steps of five, "5, 10, 15, 20, 25, 30, 35, 40."

Argumentation A2: For the problem "8 x 4 = __" John suggests taking away 4 from the result of the problem "9 x 4= __":

By way of background it is necessary to say that this episode is taken from the first day these children had encountered the multiplication symbol. The teacher explained the symbol in terms of sets. For example, 5 x 4 means five sets of four.

John and Andy solve the first task by saying "Four and four is eight." This is a direct use of the meaning the teacher gave for the "x" symbol and is an example of argumentation A1. For the next two problems John uses argumentation A2 while Andy uses argumentation A1. These two forms of argumentation are almost indistinguishable because of

the numbers involved. The difference in these approaches is much more apparent on the problem 10 x 4 = __ and it is here where it is especially evident that the social interaction becomes significant. At the beginning of the following episode John's approach is a form of argumentation A2.

John: Oooh! Just 5 more than that [5 x 4].

Andy: No. No way!

John: No, look. It's five more sets [of 4]. Look.

Andy: Yeah.

John: Five more sets than 20.

Andy: Oh! 20 plus 20 is 40. So its gotta be 40.

The initial misunderstanding about what John means by "five more" is evidence that Andy is not thinking in terms of argumentation A2. In fact, even after producing the result "40" Andy generates an (incomplete) argumentation of the first type. For Andy, "40" is simply the answer to 20 plus 20. He still must generate a solution to 8 x 5 = _ for himself, which he does by counting by fours. He pauses at 28 and coincidentally John repeats the answer of "40". Then Andy repeats "40" and the observer intervenes to clarify for himself what solution method each child used. The intervention of the observer has the effect of requesting an explanation before Andy has completed his argument. Thus, instead of describing his incomplete solution, Andy describes the first solution method which was suggested by John but computed by Andy. His halting explanation suggests that his understanding of it as a valid justification is tentative.

Andy: 5 plus 4 is -- 5 times 4 is 20, so just 5, I mean 20 more than that makes 40.

On the next problem, 9 x 4 = __, Andy again suggests thinking in terms of sets of four, the first type of argumentation, but John suggests the second type.

Andy: 9 times 4. 9 sets of 4.

John: Just take away 4 from that [10 x 4].

Andy: Thirty -- six....I get it.

We see that John's repeated use of the second type of argumentation has some effect on Andy. To solve 8 x 4 = __ both children use the second form of argumentation, but they differ in its application.

John: Look! Look! Just take away 4 from that [9 x 4]. Just take 4 away from that [9 x 4] to get that [8 x 4].

Andy: Just take away 2 from there [10 x 4]. Take 8 away from there [10 x 4].

...

Observer: And how did you do it John? Did you do it the same way?

John: Yeah, same way.

Observer: Okay.

John: But I used that one [9 x 4]. Take 4 away. It makes 32.

Again, the observer's intervention provides clarification. John's reply to the observer shows that he realizes that they both used the same type of argumentation.

At this point we might conclude that through the interaction with John, Andy has developed an understanding of the second type of argument. The next task 8 x 5 = __ illustrates the tentative nature of his understanding. In this task the multiplicand changes from 4 to 5.

Andy attempts to solve the task by adding 5 to 8 x 4 and gets 37. John, on the other hand says "Eight sets of 4. Eight sets of 5." clearly placing these tasks in juxtaposition but then proceeds to use argumentation A1 to solve the problem. Andy repeats John's argument verbatim, as if to think it through for himself. In doing so he abandons his attempt to use an argument of the second type.

On the final task in this episode 7 x 5 = __ Andy uses the first form of argumentation, counting up seven sets of 5 and John uses the second form.

John: Oh, It's just 5 lower than that [8 x 5].

The greater flexibility shown by John in being able to use both types of argumentation, while Andy is quicker at completing the calculations, is consistent with our knowledge of John's and Andy's individual mathematical conceptual development. Over the course of the episode we see Andy's emerging but still tentative understanding of the second form of argumentation. After hearing John use it Andy begins to use it. He succeeds in using it but after later using it incorrectly, in attempting to solve 8 x 5, he reverts back to the first form of argumentation which he understands completely. Episodes like this one form the basis for children's learning of mathematics through social interaction.

Argumentation and learning mathematics

This is only a single example and it should not be misunderstood as a paradigmatic example for "good small group activity". The intention of presenting this example was that it demonstrates in a very illustrative way the theoretical problems which have to do with the relationship between the social interaction process that occurs when partners give different forms of arguments and the individual process of learning.

As the remarks at the beginning concerning the normal use of small group activities in a classroom situation emphasized, a good working collaboration is not necessarily a good social condition for facilitating mathematical learning. Often the smooth functioning of such interactions is instead an indicator of a situation where everything that needs to be learned for participation in this interaction is already learned. Smooth functioning then means nothing other than the collaborative mechanical execution of a routinized mathematical method. Thus learning is based on conflicts which demands resolution by the participants (Bauersfeld, Krummheuer, & Voigt, 1988; Krummheuer, 1989; Miller, 1986; Perret-Clermont, 1980). These conflicts can be social in origin and include trying to take the perspective of a partner, recognizing the fact that a different individual has developed a different interpretation of the situation, or attempting to understand a different form of argument presented by a partner. In terms of Piaget's theory this has to do with the development of a more flexible cognitive structure which is less egocentric (Piaget, 1975). In terms of Mead's theoretical approach, one can characterize this phenomenon as a more differentiated process of perspective taking (Mead, 1934). In this sense the learning of an individual has a social genesis.

References

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), Foundations and methodology of the discipline of mathematics education (pp. 174-188). Proceedings of the TME Conference, Antwerp.

Hoetker, J., & Ahlbrandt, W.(1969). The persistence of recitation. American Educational Research Journal, 6, 145-167.

Hopf, D. (1980). Mathematikunterricht. Eine emprische Untersuchung zur Didaktik und Unterrichtsmethode in der 7. Klasse des Gymnasiums. Stuttgart: Klett.

Krummheuer, G. (1989). Die menschliche Seite am Computer. Weinheim: Deutscher Studien Verlag.

Mead, G. H. (1934). Mind, self, and society. Chicago: University of Chicago Press.

Mehan, H. (1979). Learning lessons. Cambridge, MA: Harvard University Press.

Miller, M. (1986). Kollektive Lernprozesse. Studien zur Grundlegung einer soziologischen Lerntheorie. Frankfurt: Suhrkamp.

National Research Council. (1989). Everybody counts: A report to the nation on the future of mathematics education. Washington, D.C.: National Academy Press.

Perret-Clermont, A. N. (1980). Social interaction and cognitive development in children. New York: Academic Press.

Piaget, J. (1975). Die Entwicklung des rumlichen Denkens. Stuttgart: Klett (original title: La representation de le'espace chez l'enfant).

Slavin, R. (1986, April). Developmental and motivational perspectives on cooperative learning: A reconciliation. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Slavin, R., Sharan, S., Kagan, S., Hertz-Lazarowitz, R., Webb, C., Schmuck, R. (1985, Eds.) Learning to cooperate, cooperating to learn. New York: Plenum Press.

Voigt, J. (1985). Patterns and routines in classroom interaction. Recherches en Didactique des Mathematiques, 6, 69-118.

Yackel, E., Cobb, P., & Wood, T. (in press). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education Monograph.

Potential Mathematics Learning Opportunities in Grade Three Class Discussion

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The purpose of this paper is to report findings of an on going research which tries to understand the potential learning opportunities during mathematics whole class discussion which is compatible with constructivist's view of learning. One third grade class was video taped during mathematics whole class activities for one semester. Individual interviews were conducted with students and the teacher in order to reflect on certain events occurring during class discussion. Analysis of one transcript of class discussion showed that mathematics meanings were negotiated through interactions, and the potential learning opportunities were constructed during these interactions.

From a constructivist's perspective learning occurs when a child tries to adapt her functioning schemes to neutralize perturbations that arise through interactions with our world (Steffe, 1988). Two important aspects, constructions and interactions, are important in the above statement. Although construction of knowledge is a personal act, it is by no means an isolated activity as many people's interpretations of constructivism imply. Constructivists recognize the important of social interaction as " the most frequent source of perturbations" (von Glaserfeld, 1989. p.136).

Recently an instructional model which is compatible with constructivsm's view of learning has been developed and implemented in twenty-four second grade classrooms (Cobb, et. al. 1989). Two quantitative analyses showed that the students in the project group scored as well as the students in the control group on a computation subtest and significantly better on the concepts and application sections of the standardized test. (Nicholls, et. al. in press a). Also, the students were found to be more task oriented, less ego-oriented and showed less work avoidance than students in the control group (Nicholls, et al. in press b). However, quantitative analysis alone can tell little about what really happened in the classrooms and what factors contributed to the success of this model. The present research, which focuses on class discussion is an attempt to develop "a coherent framework within which to talk about both teaching and learning" (Cobb, et. al., 1988). The goal of this

research is to describe the social dynamics and identify potential learning opportunities in class discussions.

In this instructional model, Problem Centered Learning, a typical mathematics lesson has three parts, a short introduction of carefully selected tasks by the teacher, 30-40 minutes of small group work on matheamtics tasks, followed by 15 minutes of whole class discussion (Wheatley, in press) . Unlike research in understanding social interaction in a small group and its role in learning (Yackel et al., in press), little research has been done on whole class interaction (Gilah, 1987). In contrast to small group interaction which only involves a small number of students with each group working along different paths (sometimes even on different tasks) with or without the assistance from the teacher, in whole class discussion the students and teacher form an intellectual community whose goal is to construct meaning. The interaction in class discussion is more complicated than those in small groups. Norms and interactive patterns are more easily negotiated in whole class discussion since the teacher is a member of the group.

Unlike the commonly seen teacher-lead didactic discussions which are an adjunct to exposition, the purpose of class discussion in Problem Centered Learning is for students to share their methods and to negotiate mathematics meanings. In other words, students' opinions are the focus of the discussion , not the teacher's. The role of the teacher is to facilitate the verbal interactions among her students and not "lead" students to previously determined statements. She frames situations as paradigm cases which she uses to negotiate social norms. In no case does she evaluate student answers on correct choices of the solution methods. Because she does not want the discussion becomes a guessing game (Voigt,1985), neither does she want to damper students' enthusiasm by saying "You are wrong!". Her goal is to facilitate student-to-student commnication. In most cases, students can restructure their solutions by simply verbalizing it, or by using suggestions from other students.

The Research Plan

All the whole class mathematics discussions of one third grade classroom which used Problem Centered Learning were video taped and field notes of daily events were taken everyday for a whole semester. Individual student's perceptions of mathematics learning in

general and whole class discussion in specific were gathered through individual interviews and a mathematics belief questionnaire (Nicholls, et. al. in press a), which was administered at the beginning of the semester. Also, formal and informal interviews were conducted with the teacher which focused on her interpretations of certain events and her rationale for certain interventions which she made during class discussion. The multiple perspectives model (Cobb, in press) are used in analyzing class discussion. Because page limitation, this paper will discuss only how mathematics meaning can be negotiated through interactions and how potential learning opportunities are constructed during these interactions.

The following is a transcript of one instance of a whole class discussion which occurred in early September. The task was " Find different ways to make 50."
Brandon and Travis wrote down 9 9 on the board.

9 9

8 5

(A was the teacher and all the other letters indicate different students.)

B: **See, I was doing one problem** [B was making 50 another way. B is explaining T's method even though he did not initially participate in the solution. The solution being explained was developed by T.] **Travis, he was using those unifix cubes. He took...** [took 50 cubes. The fifty cubes were arranged in five stacks of ten. T took one from each of the stack in an attempt to make a different '50', different from adding five tens.] **First he thought all these were 9** [pointing to the five nines written on the board] . **But then we counted them all up.** [It is interesting to note that T began with 50 cubes, partitioned them into six sets and then T and B counted them to find how many. Apparently, once the partitioning was begun, the 50 no longer existed for T. They had obtained 51 when they determined how many 9, 9, 9, 9, 9, and 5 made. They were in the action of counting and did not relate the result to the 50 cubes with which T had begun. This action suggests T was not at the level of part-whole]. **So we took 1 away from one of the 9.** [leaving 8] **He knew it was not all 9's** [it did not add to 50], **and he had a 5. This** (B pointed to the '8') **was a 9 once, but he took one away** [because their count had yielded 51

which was one too many] **then we counted it up, we got 50** [It is likely they inferred it would be 50 since there had been 51 and they took one away].

S: **I don't understand what you did. You told us you counted all the 9's this and that, but how did they make 50? Because that's,.. that's a higher number** [referring to the 8 5 and thinking of it as eighty-five].

[S is intent on making sense of B and T's explanation. A review of the video recording indicated that during this time she rarely looked at the board and seemed to be disinterested. This is an excellent example of negotiating meaning of terms.]

B: **See...**

T: **I broke it down into 9's. See...**

S: **Of 50?** [meaning "Did you break down 50?"]

T: **I counted up to 50** [started with 50 cubes], then we broke down into 9's.

B: **Right.**

T: **Then I have 5, then I took off 1 from another 9.** [because when we counted there were 51 and that was one too many. Even though he had started with fifty because that is the sum he was attempting to obtain, the fifty no longer existed for T when he began the partitioning.]

B: **See. We had 51. So we took 1 off . It was a 9, so we took 1 off. That's where the 8 came from.**

A: **Most people said they could understand Travis broke down into 9's, and took one away, but what is that 8?**

S: **Why is 85? It was supposed to make 50.** [Why did you write eighty-five? 85 is greater that 50!]

B: **No, they were 8 and 5.**

S: **Oh, you mean 9, 9, 9, 9, 8 and 5.**

B: **Right.**

S: **Oh, I understand now.**

A: **I don't understand.** [This facilitative intervention by the teacher in no way interrupted the rhythm and flow of the discussion. The students understood it as a further

attempt to help them make sense of the explanation.]

B: **All right. Let me talk it over for you. It was four 9's, an 8 and a 5.**

J: **I know what they were trying to say.** [J is attempting to clarify B's explanation. In order to do this she had to take B's perspective, to think like he did.] **You added all the 9's and them 8, and 5. You come up with 50.**

(H raised her hand indicating she wanted to contribute to the discussion. E shook his head which suggesting he had made sense of the explanation and disagreed with some aspect of the work, most likely the sum of 50.)

A: **Does four 9's, an 8 and a 5 makes 50?** [Another facilitative intervention by the teacher.]

(K raised her hand.)

B: **See. We counted it up. 9 and 9 is 18. 18 and 18 is....**

(Some students tried to figure out 18 + 18, while others went beyond that and tried to figure out what four 9's, an 8 and a 5 would make.)

C: **It was 49.**

B: **You told me it was 51 .** [Apparently B was surprised at this result. We infer he had not participated in obtaining the 51 originally.]

A: **Did they have a creative way of showing 50?**

(Unison: **Yeah!**)

A: **Five 9's and a 5.**

S: **O.K. I agree.**

A: **We really have very different ways. Thank you for sharing it. And thank you for you people to pay enough patience. They had a really good idea but needed a little help.**

(Class in unison: **Yeah.**)

In this statement, the teacher was negotiating social norms and attempting to communicate her goals for student learning. She was saying, "Our goal is making sense and as long as people are doing something meaningful it is a valuable contribution to the intellectual community (the class). It is important to explain your reasoning. Class members should be

trying to make sense of the explanations. I aliked the way they explained their solution even though it was not perfect. Our goal is not correct answers but good thinking." In effect the teacher was framing this episode as a pardigm case of doing mathematics, a move which has been identified as effective in negotiating social norms of the class (Cobb et. al. 1988).

The above example illustrates the dynamics of a class discussion in a constructivism type classroom and how it constrasts with a typical teacher-lead discussion. Students are freely expressing their ideas, they are trying to make sense of each other's method and they carry on a conservation among themselves. There is a sense of owenership of what they are doing, and they make an effort to explain their methods. B and T helped each other in explaining their method by trying to figure out what might be left out by the other person, and by interpreting the comments and questions (verbal and nonverbal) of the students in the class. The potential learning opportunities for B and T were apparent as they were trying to reconstruct and verbalize their solutions (Levina, 1981) and as they attempted to distance themselves from the action of explaining to coordinate the other person's views (Sigel, 1981). Simlilar opportunities existed for all students who verbally participated in these discussions as they jointly resolved this conflict, for example S and J. However, for the students who did not participate verbally, they could still carry out conversations within themselves in trying to make sense of the whole situation since the task was at a cognitive level at which all could engage. By analyzing the tape and student interviews, we now question the use of the term "off task behavior". For example, even the girl S who participated verbally in the whole discussion did not appear to be listening much of the time. We also found some students who played with their pencils throughout the entire discussion (faced the board only once in a while) and yet still gave a detailed description of what happened during that period without any prompting. These findings certainly make us cautious in judging student engagement by any overt "engaging behaviors".

We use the phrase potential learning opportunities instead of learning opportunities in this paper because we recognize the complex relationship between the context (students perceived learning environment and their intentions) which students construct and operate in (can only be inferred by observers) and the setting (learning environment) which can be

directly observed. The teacher can select tasks which are likely to create conflicts, thus serving as an opportunity for students to restructure their thinking. However, students may not benefit from these opportunities simply because they have different intentions. That is why negotiating social norms becomes an important aspect of the teacher's role in class discussion (Bishop, 1985).

As we indicated above, the results reported in this paper are only a partial aspect of this research project. Analyses are continuing as we try to explain why some discussions were more effective than the others, and the influence of the teacher's decisions, the particular tasks used and how the roles played by different students effect the potential learning opportunities. As Pirie and Schwarzenberger (1988) pointed out although people believe and accept that mathematics discussion can help learning, little research had been conducted to determine its effectiveness. A study of the complex interactions during whole class discussion may help us determine the contribution of this instructional component to mathematics learning.

References

Bishop, A. (1985). The social construction of meaning-a significant development for mathematics education? For the Learning of Mathematics, 5 (1), 24-28.

Cobb, P. (in press). Multiple perspectives. In L. P. Steffe & T. Wook (Eds.), Transforming early childhood mathematics education. Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Yackel, E. and Wood, T. (1988). Curriculum and teacher development as the coordination of psychological and anthropological perspectives. Paper presented at the meeting of the Instruction/Learning working Group of the National Center for Research in Mathematical Sciences Education, Madison, Wisconsin, May.

Cobb, P. Yackel, E. and Wood, T. (1989). Young children's emotional acts while doing matheamtical problem solving. In D. B. McLeod &V.M. Adams (Eds.), Affect and matheamtics problem solving: A new perspective (pp.117-148). New York: Springer-Verlag.

Gilah C. L. (1987). Teacher student interaction: a case study. Educational studies in Mathematics, 18, 255-271.

Levina, R. E. (1981). L. S. Vygotsky's ideas about the planning function of speech in children. In J. V. Wertch (Ed.), The concept of activity in Soviet psychology (pp. 279-299). Armont, NY: Sharpe.

Nicholls, J., Cobb, P., Yackel, E., Wood, T., Yackel, E. & Patashnick, M. (In press a). Dimensions of success in matheamtics: Individual and classroom differences. Journal for Research in Mathematics Ecuation.

Nicholls, J., Cobb, P., Wood, T. Yackel, E. & Wheatley, G. (in pressb). Assessing young children's mathematical learning. In G. Kulm (Ed.), Assessing higher order thinking in Mathematics. washington, D. C. : Association for the Advancement of Science.

Pirie S. E. B. and Schwarzenberger R. L. E. (1988). Mathematical discussion and mathematical understanding. Educational Studies in Mathematics, 19, 450-470.

Sigel, I. E. (1981). Social experience in the development of representational thought: Distance theory. In I. E. Sigel, D. M. Brodzinsky, & R. M. Golinkoff (Eds.), New directions in Piagetian theory and practice (pp. 203-218). Hillsdale, NJ: Erlbaum.

Steffe, L. (1988). Overview of the Action Group A1: Early childhood years. Paper presented at the Sixth International Congress on Mathematics Education, Budapest, Hungary, July 27-August 3.

von Glaserfeld, E. (1989). Cognition, construction of knowledge, and teaching. Syntheses. 80. 121-140.

Voigt, J. (1985). Patterns and routines in classroom interaction. Researches en Didactique des Mathamtitiques, 6 (1), 69-118.

Wheatley, G. (in press). Constructivist perspectives on science and matheamtics learning. Science Education.

Yackel, E., Cobb, P. & Wood, T. (in press). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education Monograph.

CERTAIN METONYMIC ASPECTS OF MATHEMATICAL DISCOURSE

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This paper outlines certain aspects of language use in mathematics which have not been frequently attended to, yet which are important to understand if the teaching of mathematics is to draw draw on the creative strengths of language with which pupils are already familiar. Following a theoretical illustration of the linguistic distinction between *metaphoric* and *metonymic*, a class lesson is analysed indicating places where these notions appear.

The Aboriginal Creation myths tell of the legendary totemic beings who had wandered over the continent in the Dreamtime, singing out the name of everything that crossed their path - birds, animals, plants, rocks, waterholes - and so singing the world into existence. (*The Songlines*, Bruce Chatwin, p.2)

In this paper, I want to start to examine how mathematical language can be and is used to 'sing' mathematical objects into existence, and how the 'songs' change over time and the objects with them. Mathematics has much in common with other ritual activities: the mystique, the incantations, the initiates and the masters. Certain old songs are no longer sung (the ones about coordinates in differential geometry, or calculations with infinitesimals, for instance) and the awarenesses they encourage diminish and die back (or go back into the Dreamtime if you prefer).

As with songs (and with novels and pictures, poems and plays, and other 'art' forms), so also with mathematical theorems, there is always the question: 'What is the song about?' Asking such an apparently straightforward question immediately imposes a time order that can misdirect. It pulls attention in the direction of songs coming *after* the 'things' that they are about, that they are *merely* representations of something else, rather than being the primary thing in themselves. If the Aboriginal songs bring the referents into existence, what then of traditional accounts of reference and meaning?

The same is true of the current mathematics education preoccupation with 'recording' and 'representation' (see Janvier, 1987, for example), where the assumption seems to be that there is always something being represented that predates it, and hence relegates language to a far less powerful position in the hierarchy of mathematical creation and existence: in particular, it emphasises the passive role often attributed to language in merely describing or representing experience, rather than being either a constituent component of the experience or even the experience itself.

Corran and Walkerdine (1981) comment: "Their [the teachers in their sample schools] framework did not provide a way of understanding language to have other functions than recording." This point is echoed in Douglas Barnes' remark (1976) that "communication is not the only function of language" With respect to children's acquisition of their first language, it would be a palpable nonsense to assert that pupils only acquire a particular aspect of language (e.g. use of the passive mood or formation of negatives or learning how and when to swear and which words to use according to context) *after* they have understood it. Ritual elements such as repetition or rehearsing rhyming sequences of sounds or words independent of apparent relevance to their immediate context are commonplace, self-imposed techniques of acquisition for young children.

Rehearsal of a system independent of its application seems to be a notion with which children are very familiar. Intransitive counting provides but one instance of exploring and getting a system right in production predominantly before approaching the harder task of figuring out what it might have to do with the material world. This suggests that 'teaching

for understanding' includes a manipulative facility which need not necessarily *follow* acquisition of 'meaning'.Here is a brief example of such counting which may help to show up some differences. Herbert Ginsburg (1977) reports a conversation between two twin sisters, Deborah and Rebecca (almost five years old).

D(eborah): 1,2,1,2,1,2. [The spoken words are recorded as numerals in the original.]
R(ebecca): 1,2,3.
D: No, not like that. I said 1,2,1,2,1,2.
R: 1,2,3,4,5,6,7,8.
R: 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19 ...(to her mother) What's after 19?
M(other): 20.
R: 20,21,22,23,24,25,26,277,28,29. What's after?
M: 30.
R: 31,32,33,34,35,36,37,38,39, now 40.

Later on she starts recounting - from 22 to 29; at this point she indicated by a glance that she wanted her mother to supply the next number.
M: 30.
R: 40,50,60,70,80,90,tenny.

This is linguistic experience of mathematics - there is no transitive counting here: it is 'about' learning the song, one of the ones which sing the numbers into existence.

Many writers have attempted to explore aspects of the concepts of 'meaning' and 'understanding' in the context of teaching and learning mathematics and have frequently adopted the natural history approach of offering a taxonomy of types, as well as judgements about preferred variants. This work (among others) has influenced school practice and resulted in current beliefs about the importance of always understanding what you are doing and the devaluing of certain traditional practices (such as chanting multiplication tables or working at 'pages of sums') in search of 'real understanding' (a concept which is pounced on by Walkerdine and Lucey in their insightful analysis (1989) of differing attitudes to and interpretations of girls' and boys' mathematical performance). This paper is also concerned with the notion of mathematical meaning, but from a different perspective.

Tahta (1988) has written of a current reluctance among certain teachers to assist pupils to share with them or textbooks, and this is a part of a reaction to and rejection of certain traditional methods in the teaching of mathematics I mentioned above. However, a reassessment is under way, and from a somewhat unlikely direction. It is not primarily a backlash, a rejection of the new ways and a nervous return to the old (though there are examples of such appeals). It is coming from certain unashamedly modern influences such as semiotics being applied to the teaching of mathematics and allowing value and direction to be perceived in certain traditional teaching practices that had been condemned wholesale under the heading of rote-learning methods. (These include chanting, repetitive apparently ritualistic aspects of mathematical activity, whether spoken (e.g. multiplication tables) or written (arithmetic or algebraic manipulations and algorithms) - indeed the rejection was frequently so strong as to deny that these practices were mathematical in nature.

The second was an over-narrow conception of meaning in mathematics in terms of reference rather than nets of connections in both form and content, and meaning in this restricted sense being claimed to be the most important, indeed only goal of mathematics teaching. In England, at least, an increasingly common dogma is if in doubt at any stage in anything mathematical, then go back to the 'meaning' (often the concrete) from which everything is presumed to stem. Valerie Walkerdine (1988) has recently drawn attention to the implausibility of such an account in the case of the teaching of place value. She offers a much more telling if complex account, one that intimately implicates the teacher's language and positioning within classroom activity.

In all these examples of the children beginning place value, the teacher, in a sense, *tells*, the children what they are supposed to be learning and discovering .. she is providing the children with cues which reveal the properties of place value which the objects they are manipulating are supposed to supply.

The teachers intention in giving the matchsticks to count and group - and represent the numerals in a place-value system - is that the relations of value between the numerals in the system will be apparent to the children because they will be presented concretely as relations between bundles and single matchsticks. On the contrary, it is the properties of the place-value system which are used to make the matchsticks, and the grouping of them into bundles of ten, signify in particular ways. The children's activities (grouping in tens) are being determined by a convention not vice versa.

Metaphor and metonymy are concepts partaking of an ancient study - that of rhetoric. Mathematics is often contrasted with rhetoric (see Nash,1990, for example) as being a discourse composed solely of logical (and hence non-rhetorical) statements. In addition, I hope to show (both here and in my talk) how both the classical and reformulated concepts (by Jakobson, and subsequently by Lacan) of metaphor and metonymy can highlight aspects of the nature and principles of mathematical discourse which is at the heart of the activity known as 'doing mathematics'.

One mathematics lesson

Below I have described a mathematics lesson and later transcribed and analyse both the nature of the activity and some of the dialogue from a videotape of the lesson.(In my proposed talk, I will show some of the videotape.) The entire class is seated in a circle (including the teacher) - a mixed-ability class of eleven-year-olds. In some sense, the lesson was 'about ' modular arithmetic (base five). The teacher responded to a pupil who initially asked what they are going to do today, with "we're going to play a game", but felt the need to add a rider "You need to concentrate though, It's a concentrating game", as if she wanted to block some readings that the term *game*" might entail. She then announced to the whole class, "We're going to play a game with numbers and colours." William, you're one; , what number are you? [to an adjacent pupil in the circle] and so on round [including the teacher who says "twenty-seven"]. Then, "William, will you go and take a white rod. To the next person "a red" and then green, pink and yellow in turn. (These rods are the first five Cuisenaire rods, cuboids whose lengths ar whole number multiples of the smallest one.)

Once everyone has a number and some have taken their rods, the teacher starts questioning pupils about what colour certain pupils further round the circle will take, and then attempts to encourage pupils to be able to move between any number and the corresponding colour. When she is satisfied they can do this quite fluently, she moves on to look at the possibility of adding colours.

Discussion

A great deal of what occurred relies on the fact that there are now three different naming systems at work. Firstly, there are the individual pupil names that the teacher uses a lot to direct her questions and to focus attention (e.g. "Sally, what colour are you?", "What colour is Debbie going to take?") Then there are the ordinal names relatively arbitrarily started (William is quite close to the teacher), but then systematically reflecting the order of the circle.(Early on the teacher asks Can you all remember what number you are? and when someone shakes their head responds with "Well, how could you find out?) Thus, knowing one or a few number name assignations enables anyone to work out the others (whereas that is not true of personal names: the ordinals assign a unique name whereas there are a number of pairs of pupils with the same Christian name, and the teacher on occasions makes mistakes). Finally, there are the colour names of the individual Cuisenaire rods

picked up in a way that respects the ordinal naming (William, who was "one", takes the first rod, the shortest in length) as well as imposing the structure of the colours, and as with personal names, the colour names are not unique.

Note how these associations, indeed identifications are made The focus of the activity is on the individuals present, but all the relations are attended to by name. The activity identifies first a number name and then a colour name with a person name, and then the teacher-led activity (in which the teacher is very strongly positioned) works on relations and associations among the different sets of names.

The metonymy in the classical sense is apparent in the use of colour or number names for pupil names. (T is Teacher, P any pupil.)

T: Who's a pink? Who else is yellow? Can I have a different red number?

At one point she asks: Who's going to be the same colour? and then reformulates it to: What numbers are going to be the same colour?

T: Fiona's twelve and I'm twenty-seven. What other numbers are going to be red?

P: Seventeen's going to be red.
P: Miss, it's all the numbers that have the end in 2 or 7 ... are going to be red.

This last statement is a lovely metonymic creation, linking the form of the numeral (numbers after all don't have 2s or 7s in them) with the colour name (neither numerals nor numbers have colours), and is entirely typical of the knowledge claims that this activity gives rise to. What is this song about?

The teacher then asks: Can we work out any numbers? Suppose we had two first year classes in here. What about forty-four? What colour will they be?

T: Suppose we had all the first year here what colour would number one hundred be?

(She is careful here to extend the basis of reference that she has set up, namely that it is people which signify most, in order to get past the actual and into the hypothetical, and slowly towards the infinite.)

She then shifts focus by saying: Well, let's have a think about adding them, because you are quite good at telling me what colours they are going to be.

T: What colour will we get if we add a pink and a yellow?

There is a subsequent shift from talk of "a pink" or "a yellow" (where it might be plausible to see this as a contraction of "a pink rod" and so have a relatively concrete referent) to "pink" and "yellow" as in a pupil remark "yellow and pink is pink".

The teacher shifts once again and asks, "Can anyone give me a sum and tell me what the answer is going to be using colours?"

P: Red and pink will be white.
T: (echoes but reformulates) A red and a pink will be a white. Can you explain how you know that?

P: Twenty-two and forty-nine equals a white. Forty-nine ends in a 9 and pinks are 9" Twenty-two ends in a 2 and reds are 2. So two and four make 6 and so five and one equal six. I nearly got it. It equals white anyway, because if you go from white which equals one four is a pink, five is yellow and six is white again, so it must be a white, twenty-two and forty-nine.

Later on, P: 104 and 76 will be yellow - if you added them together the colour will be a yellow.

P: Twenty-four add three equals red

T: Twenty-four add three equals red - is that right? I'm 27 and I'm red. (The boy has apparently counted on from his number - 24 to get that of the teacher and formed his sum that way.)

P: One and five is white, two and seven is red, three and eight is green, four and nine is pink, five and zero is yellow.

P: Two thousand one hundred and ninety-four will be pink - but I'm not going to work it out!
(No need of people here.)

Then faced with sums such as R + P = they encounter commutativity (you don't need to do red plus white when I've done white plus red) and whether you can add two the same, By this stage, Red is standing for red and not a red, these objects have no more articles, but they are what the symbols R and P stand for (linked by alliterative first-letter naming).

Final comment from a boy who was having difficulty with his mathematics in general, "It's quite easy, this."

In her book, *The Mastery of Reason,* Walkerdine writes about home and school practices and the importance for analysing what happens in terms of the practices set up and the various positions held by the participants. In discussing the occurrences of 'more' and 'less' in her corpus, she remarks:

> In every case initiated by the child, she either wants more precious commodities, of which the mother sees it her duty to limit consumption, or the child does not want to finish food which the mother sees it as her duty to make the child eat. The differential position of the mother and child is made salient by the mother-initiated episodes in which the mother can stop the child giving her more, and can announce that she is going to do more of something. It is important that the child is not in that position. This, therefore, presents a relational dynamic or a position within the practices relating to power, rather than a simple view of turn-taking in those practices. (1988, p. 26)

In this videotape, we see the teacher position herself very carefully in the practice that is to provide the setting for the introduction of the signification of 'adding colours', the ostensible focus of the activity. She is part of the circle and has a number like every one else (including the visiting teacher as well). Amongst other things, this allows her to ask, "I'm 27. Who's like me? Who's got a red one?" And the identifying metonymy works when the pupils give her numbers rather than the names of pupils back. In addition is the assumption that being like the teacher is an important and desirable status-marked thing to be, though indeed it is not one of the pupils who is 'like the teacher' who offers conjectures in response to her question. And he falters interestingly when coming to name the teacher's number in a pattern he has seen in the signifiers (a reflexive question of is the teacher like herself), as if the metaphoric content/referent (in this case the teacher herself) gets in the way of the metonymic pattern (those ending in a 2 or a 7).

Walkerdine adds: If material phenomena are only encountered within their insertion into, and signified within, a practice, this articulation is not fixed and immutable, but slippery and mobile. That is, signifiers do not cover fixed 'meanings' any more than objects have one set of physical properties or function. It is the very multiplicity which allows us to speak of a 'play' of signifier and signified, and of the production of different dynamic

relations within different practices. It is for this reason that I used the terms 'signify'* and 'produce' rather than represent. If social practices are points of creation of specific signs then semiotic activity is productive, not a distortion or reflection of a material reality elsewhere." (1988, p. 30)

(*When typing this document I originally rendered this word as 'singify', a delightful metonymic creation which made unexpected links with the opening quotation.)

In this activity we have got away from the confusion of clock arithmetic of what 0-4 actually are - the problem of finding a name for the class. The traditional solution (of use to experts or adepts but potentially confusing to neophytes) is that of giving the name of one of the objects to the class (see Mason and Pimm, 1984) What you lose in marking this distinction is the ability to relate [1] + [2] = [3] to 1 + 2 = 3.(or **1** + **2** = **3** as some texts have it.) - that is to use the system you are familiar with. There is also he difficulty of having 1 (itself a symbol) standing for another symbol e.g. 6. A complex general issue of making, marking and ignoring distinctions.(It is there with fractions, where the 'lowest form' is the name for the rational number, but is not the best symbol to work with when trying to add two fractions. Part of the attitude evident in the New Mathematics of the 1960s was the willingness to change the symbolism and frequently to make it more elaborate, ostensibly to help learners (Other instances include marking vowels in Hebrew or Arabic the differentiated by position notation for positive and negative numbers (SMP books) or the box notation of the Madison project.)

The activity offers the possibility of gaining experience with symbolising. The language is set up and then it can drive the activity. The activity is self-checking to the extent that numbers and colours are on public display to check conjectures. But what are the objects and how do the names relate to them?

Meaning?

Being aware of structure is one part of being a mathematician. Algebraic manipulation can allow some new property to be apprehended that was not 'visible' before - the transformation was not made on the meaning, but only on the symbols - and that can be very powerful. Where are we to look for meaning? Mathematics is at least as much in the relationships as in the objects, but we tend to see (and look for) the objects. Relationships are invisible objects to visualise. Caleb Gattegno, writing in his book The *Generation of Wealth* (p. 139), claimed:

> My studies indicate that "mathematization" is a special awareness, *an awareness of the dynamics of relationships*. To act as a mathematician, in other words, is always to be aware of certain dynamics present in the relationships being contemplated. (It is precisely because the essence of mathematics is relationships that mathematics is suitable to express many sciences.) Thus, it is the task of education in mathematics to help students reach the awareness that they can be aware of relationships and their dynamics. In geometry, the focus is on the relationships and dynamics of images; in algebra, on dynamics per se.

Mathematics has a problem with reference so it tends to reify its discourse in order to meet the naive desire for reference. "The questions 'What is length?', 'What is meaning?', 'What is the number one?', etc. produce in us a mental cramp. We feel that we can't point to anything in reply to them and yet ought to point to something. (We are up against one of the great sources of philosophical bewilderment: a substantive makes us look for a thing that corresponds to it.)" Ludwig Wittgenstein, *The Blue and Brown Books*.

Partly this rejection stems from a two-fold misunderstanding. The first is an over-narrow conception of understanding and meaning, as well as the teacher's responsibility for providing it. Reversing Lewis Carroll's punning suggestion from

Alice's Adventures in Wonderland so that it now reads "take care of the sounds and the sense will take care of itself" offers an exciting possibility for teachers.

The second arises from a presumed necessary temporal dependence between the development of symbolic fluency and understanding, with the latter always and necessarily preceding the former. Part of what I am arguing for is a far broader concept of mathematical meaning, one that embraces both of these aspects (the metaphoric and metonymic foci for mathematical activity) and the relative independence of these two aspects of mathematical meaning with respect to acquisition.

Dick Tahta has claimed (1985, p. 49) that:

> We do not pay enough attention to the actual techniques involved in helping people gain facility in the handling of mathematical symbols. ... in some contexts, what is required - eventually - is a fluency with mathematical symbols that is independent of any awareness of current 'external' meaning. In linguistic jargon, 'signifiers' can sometimes gain more meaning from their connection with other signifiers than from what is being signified.
>
> Linguists have called the movement 'along the chain of signifiers' *metonymic* whereas 'the descent to the signified' is *metaphoric*. ... The important point is that there are two sharply distinguished aspects (metonymic relations along the chain of signifiers and metaphoric ones which descend into meaning) which may be stressed at different times and for different purposes.

There is a fundamental tension between what is currently called 'mathematical understanding' and symbolic fluency and automaticity. One increasingly important question for school mathematics at all levels is deciding what value is to be placed on the later, given the increasing sophistication and decreasing cost of symbolic manipulation devices on calculators and computers. The relative independence of these two aspects is hidden in a presumed temporal priority arising from a presumed conceptual priority of metaphoric understanding.

Note: I am most grateful to Eric Love for conversations about the mathematical activity described in this paper.

Bibliography

D. Bames, *From Communication to Curriculum*, Penguin, Harmondsworth, 1976.

B. Chatwin, *The Songlines*, Picador, London, 1987.

G. Corran and V. Walkerdine, *The Practice of Reason*, Institute of Education, London, 1981.

H. Ginsburg, Children's Arithmetic, Van Nostrand, New York, 1977.

C. Janvier (ed.), *Problems of Representation in the Teaching and Learning of Mathematics*, Lawrence Erlbaum Associates, Brighton, 1987.

J. Mason and D.Pimm, 'Generic examples: seeing the general in the particular', *Educational Studies in Mathematics*, **15**(3), pp. 277-289.

G. Nash, *Rhetoric*, Oxford University Press, Oxford, 1990.

D. Tahta, 'On notation', *Mathematics Teaching*, **112**, pp. 48-51.

D. Tahta, 'Lucas turns in his grave', in D. Pimm (ed.), *Mathematics, Teachers and Children*, Hodder and Stoughton, Sevenoaks, 1988.

V. Walkerdine, *The Mastery of Reason*, Routledge, London, 1988.

V. Walkerdine and H. Lucey, *Counting Girls Out*, Virago, London, 1989.

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Inverse Relations: The Case of the Quantity of Matter

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Paper prepared for the Conference of The International Group for the Psychology of Mathematics Education, Mexico, July 15-20.

January 1990

Inverse Relations: The Case of the Quantity of Matter

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Quantity of matter is measured by mass (m), volume (v) and number of particles (n). These three dimensions are related to each other (m/v=density, m/n=molar mass, v/n=molar volume). The purpose of this paper was to test the ability of students to differentiate between these dimensions and their ability to qualitatively convert from one to the other. This was done by presenting 9th and 10th grade students with several substances of equal quantity in one dimension, and asking them to decide whether or not they were equal in the two other dimensions. The results indicate that: a) inverse ratio tasks were more difficult than direct ratio tasks; b) the conversion between n and v was more difficult than other conversions, and problems related to the gaseous state were more difficult than problems related to other states of matter, c) girls had more difficulties than boys and d) teaching had almost no effect on students' performance. These results are discussed.

The chemist *measures* the quantity of substance by *weighing* it or by determining its *volume*, but he or she thinks about it in terms of *particles*. Thus, the quantitative aspect of chemistry deals with three dimensions: Two of these - *mass* and *volume* - are physical quantities which can be sensed and directly measured by instruments, and therefore can be considered *concrete*. The concepts of mass and volume are acquired either by experience and intuition or by the learning of physics in school prior to the learning of chemistry. The *third* dimension, which according to IUPAC is named *amount* (of substance) cannot be measured directly by an instrument and cannot be sensed. Thus, in contrast to mass and volume, it is not a concrete concept, but may be regarded as *theoretical*. Students become acquainted with this concept through chemistry lessons rather than by experience or intuition.

This idea may be summarized as follows: The "quantity" of a given substance can be expressed by three different physical quantities which represent three aspects of matter or three properties of all materials: mass, volume and amount ("particulate").

Each of these expressions of physical quantity has a name, a symbol and a unit which emerge from the different property it represents (see Table 1).

Table 1: Basic physicial quantities related to matter and their units.

Physical quantity	Symbol	Unit	Property underlying the physical quantiy
Mass	m	kg	the inertia of an object
Volume	v	liter (meter^3)	the space filling of an object
Amount	N n	particle (1ee) mole	the particulate nature of matter

The three dimensions of the quantity of matter for a given substance are related to each other: a) m/v=d(density). density depends on temperature and in gases also on pressure; b) m/n-Mm(molar mass). molar mass is independent of external conditions and derives only from the mass of the particles; c) v/n=Vm(molar volume). molar volume is affected by both the volume of the particles and the distances between them. These distances change with temperature. pressure and state of matter. In gases at standard conditions molar volume is dependent only on the distances between particles and is equal for all gases.

Many studies (e.g., Ingle and Shayer, 1971; Duncan, 1973; MacDonald, 1975; Novick, 1976; Dierks, 1981; Umland, 1984; Lazonby, 1985; Lybeck et al., 1988) reveal that students encounter difficulties in solving problems related to the quantity of matter. Others report on difficulties students have with inverse ratio reasoning (e.g.. Stavy, 1981, Surber & Gzesh, 1984). Therefore, the purpose of this study was to test the ability of students to differentiate between these three dimensions and their ability to qualitatively convert from one to another. This was done by presenting students with several substances (solids, liquids and gases) of equal quantity in one dimension, and asking them to decide whether or not they were equal in the two other dimensions.

Several variables were tested: a) the nature of the reasoning processes required in solving the different tasks. For example, judging the inequality of mass when presented with equal volumes of different substances (v→m) requires **direct ratio** reasoning. For the substances1 and 2, $m_1/v_1=d_1$ and $m_2/v_2=d_2$. If $v_1=v_2$ and $d_1>d_2$ then $m_1>m_2$. The same type of reasoning is required for judging he inequality of mass of equal number of particles of different substances (n→m) and for judging the inequality of volume (of solids and liquids) of equal number of particles of different substances (n→v). Judging the inequality of volume of equal masses of different substances (m→v) requires **inverse ratio** reasoning. For the substances 1 and 2 $m_1/v_1=d_1$ and $m_2/v_2=d_2$, when $m1=m2$ and $d1>d2$ then $v2>v1$. The same type of reasoning is required for judging the inequality of number of particles of equal masses of different substances (m→n) and for judging the inequality of the number of particles (of solids and liquid) of equal volumes of different substances (v→n); b) the nature of the domain specific knowledge required in solving the different tasks: mass, volume and the particulate nature of matter and the three states of matter, solid, liquid and gas; c) grade, ninth and tenth grade students were tested - ninth grade students have not studied yet about particulate amount while tenth grade students have studied it; d) gender.

A. Subjects and design

The sample included 66 middle class students from grades nine and ten. The distribution of students according to grade and gender is presented in Table 2. Each student was individually interviewed while being shown the materials. Eighteen problems were presented to each of the students. The problems related to all three states of matter (solid, liquid and gas). Israeli students do not deal with the particulate amount of matter and its unit, the mole, until the 10th grade. They are taught about the particulate nature of matter, about elements, compounds and ions, and about basic chemical reactions. We therefore asked about "number of particles" rather than "amount". The 10th grade subjects had studied particulate amount and its unit and had solved stoichiometric problems. Teaching was initiated from the gas laws (Gay-Lussac and Avogadro).

Table 2: Distribution of the research population by grade and gender.

Grade	Boys	Girls	Total
Ninth	20	17	37
Tenth	14	15	29

Tasks

I. Three different known solid objects of equal volume and shape were presented to the student (wood, aluminum and "plastic"). The student was asked to judge: 1) whether they had the same or different mass; 2) whether they had the same or different number of particles. The student was then presented with three different known liquids (water, alcohol and oil) of equal volume (in containers of the same size and shape) and asked: 3) whether they had the same or different mass; and 4) whether they had the same or different number of particles. Finally, the student was presented with three different known gases (O_2, CO_2 and air) of equal volume (in syringes of the same size and shape) and asked: 5) whether they had the same or different mass; and 6) whether they had the same or different number of particles. In each case, the student was asked to explain his/her answer.

II. The following questions were posed to each of the students: If we take equal masses of the three solid objects 7) will they have the same or different volume and 8) will they have the same or different number of particles; If we take equal masses of the three liquids 9) will they have the same or different volume and 10) will they have the same or different number of particles; If we take equal masses of the three gases 11) will they have the same or different volume and 12) will they have the same or different number of particles. In each case the student was asked to explain his/her answer.

III. The following questions were posed to each of the students: If we take an equal number of particles of the three solid objects 13) will they have the same or different

volume and 14) will they have the same or different mass; If we take an equal number of particles of the three liquids 15) will they have the same or different volume and 16) will they have the same or different mass; If we take an equal number of particles of the three gases 17) will they have the same or different volume and 18) will they have the same or different mass. In each case the student was asked to explain his/her answer.

Results and Discussion

Table 3 presents students' success in solving the direct and inverse ratio tasks in the different pairs of dimensions. As can be seen from Table 3, 9th and 10th grade students are able to differentiate and qualitatively convert between volume and mass and between mass and number of particles in the direct ratio tasks presented to them. However, they have difficulty in performing the same operations with regard to other, essentially similar and logically related, tasks. The level of success in judging the inequality of mass of equal volumes of different substances (direct ratio) was 83%, while the level of success in judging the inequality of volume of equal masses of different substances (inverse ratio) was significantly lower - 66%. Similarly, the level of success in judging the inequality of mass of an equal number of particles of different substances was 82%, while the level of success in judging the inequality of the number of particles of equal masses of different substances was significantly lower - 66%. This finding can be related to the different reasoning processes required in solving the different tasks.

Table 3: Percentage of students who correcly answered the direct and inverse ratio tasks in the different pairs of dimensions

Pairs of dimensions	RATIO			
	Direct		Inverse	
	Task	Percent	Task	Percent
Volume & mass	$v \rightarrow m$	83	$m \rightarrow v$	66
Number of particles & mass	$N \rightarrow m$	82	$m \rightarrow N$	66
Number of particles & volumes	$N \rightarrow v$	59	$v \rightarrow N$	56

Although each pair of tasks ($m \rightarrow v$, $v \rightarrow m$) is logically identical and one could represent the "inverse ratio" problems in terms of direct ratio e.g. v/m=constant (or N/m=constant for the pair $m \rightarrow N$, $N \rightarrow m$), students apparently prefer to use that ratio which has an intuitive meaning. The ratio m/v represents density or "heaviness" of a substance - a property which can be intuitively grasped and visualized, whereas,the ratio v/m has no immediate meaning and cannot be similarly grasped and visualized. The same is true with regard to the conversions between mass and number of particles ($N \rightarrow m$, $m \rightarrow N$). While the

ratio m/N has a meaning - the mass of one particle, and can be easily grasped and mentally visualized, the ratio N/m has no such immediate meaning and is therefore difficult to grasp or to see with the mind's eye. Support for this explanation can be found in the students' explanations of their judgments. In the conversion from volume to mass they used the idea of density or heaviness to a larger extent than in the conversion from mass to volume; in the conversion from number of particles to mass they used more the idea of particle mass more frequently than in the conversion from mass to number of particles. No such difference was found between the two tasks involving conversion between volume and number of particles (see Table 3). These two tasks showed a lower level of success resembling that of the inverse ratio in the other conversions (59% and 56%). It is possible that in this case the direct ratio in itself - v/N is a more difficult quantity to grasp because molar volume is affected by both the volume of the particles and the distances between them. These distances change from substance to substance with the state of matter, with temperature, pressure, etc. In addition, the behavior of gases is different than that of solids and liquids. Indeed the hardest tasks in this conversion were those relating to gases (see Table 4).

Table 4: Percentage of students who correctly answered the direct and inverse ratio tasks in the different states of matter

Pairs of dimensions	RATIO					
	Direct			Inverse		
	Task	State of matter	Percent	Task	State of matter	Percent
Volume & mass	v→m	solid	98	m→v	solid	71
		liquid	80		liquid	67
		gas	71		gas	61
Number of particles & mass	N→m	solid	85	m→N	solid	66
		liquid	83		liquid	73
		gas	78		gas	59
Number of particles & volume	N→v	solid	68	V→N	solid	62
		liquid	65		liquid	58
		gas	45		gas	49

Another interesting finding is that, in general, performance on tasks involving solids was higher than on tasks involving liquids which in turn was higher than on tasks involving gases (see Table 4). Students' difficulties on the gaseous state problems can be attributed also to their difficulty in conceiving of gases as materials (Stavy 1988). If they do not believe a gas to be a substance, it is very difficult for them to think of it in terms of mass, volume or number of particles.

The difference in performance between problems in the liquid and the solid states is not very well understood. It has been found (Stavy, 1985) that the concept of liquid is easier

to understand than that of solid, but it may be that it is easier to think about the density of a solid than about the density of a liquid, or about the particulate nature of a solid rather than that of a liquid. Ben-Zvi (1984) has shown that students visualize particles as small bits of matter and this is probably easier to do with solids than with liquids.

A very significant difference between the performance of boys and girls was observed, with boys consistently outperforming girls (for all tasks: ANOVA - $p<0.005$) (see Table 5). Sex-related differences, favoring boys, have been reported for several Piagetian tasks that are related to concepts and topics in the science curriculum (Howe and Shayer 1981, Linn and Polus 1983, Robert 1989). It has been suggested that this difference is due to an experiential deficit on the part of the girls or to an underlying difference in cognitive structure or style. Although we cannot explain the observed difference between boys and girls in our study, we can suggest that it may be related to girls' inferiority in spatio-visual abilities (Liben 1978). Our tasks, if not worked out on a formal mathematical level alone, require some manipulation of mental images.

Table 5: Percentage of students who correctly answered the direct and inverse tasks in the different sexes

Pairs of dimensions	RATIO					
	Direct	Percent		Inverse	Percent	
	Task	Boys	Girls	Task	Boys	Girls
Volume & mass	$v \rightarrow m$	83	73	$m \rightarrow v$	83	48
Number of particles & mass	$N \rightarrow m$	85	79	$m \rightarrow N$	76	56
Number of particles & volume	$N \rightarrow v$	59	59	$V \rightarrow N$	37	56

Another important point that should be raised is the ineffectiveness of teaching (see Table 6). No significant differences were observed between 9th and 10th grade students (except for the problems demanding conversion between volume and number of particles). It seems that in the course of chemistry teaching, insufficient attention is paid to the serious problems which students (especially girls) have with the basic concept of the quantity of matter.

Table 6: Percentage of students who correctly answered the direct and inverse ratio tasks in the different grades

Pairs of dimensions	RATIO					
	Direct			Inverse		
		Percent			Percent	
	Task	9th grade	10th grade	Task	9th grade	10th grade
Volume & mass	v→m	87	79	m→v	66	66
Number of particles & mass	N→m	84	79	m→N	67	64
Number of particles & volume	N→v	49	65	V→N	53	67

Conclusions

The results reported in this paper indicate that students' performance in solving simple qualitative problems in chemistry is affected by several factors. It was shown that both the logico-mathematical aspect of the content and the specific nature of the subject matter affect students' performance. Inverse ratio reasoning was found to be more difficult than direct ratio reasoning. It would be very interesting to find out what is the source of this difficulty and whether students' difficulty in solving the inverse ratio conversion tasks is related to their difficulty in solving other parallel inverse ratio problems, e.g. in mathematics, or whether it is related only to the specific content of the present tasks.

It was also shown that the conversion between number of particles and volume was more difficult than other conversions (m→v, m→N) and that problems in the gaseous state were more difficult than problems in other states of matter. These two areas of subject matter are indeed more complicated.

Two other important findings were that girls have more difficulties than boys and that teaching had almost no effect on students' performance. These findings imply that while teaching, one should start from the easier tasks (conversion of the direct ratio nature n→m, v→m, in the solid state) and gradually advance to the more complex and more difficult tasks, emphasizing the inverse ratio aspect and the volume and gaseous sate problematics. One should also emphasize the analogy between density (d); Molar mass (Mm) and molar volume (Vm) as characteristic intensive properties of substances and the equivalence between the three dimensions, volume, mass and amount (particulate), as properties on which we can base our measurements of a given quantity of substance.

Bibliography

Ben-Zvi, R., Eylon, B. & Silberstine, J. (1986). Is an atom of copper malleable? *Journal of Chemical Education, 63*, 64-66.

Dierks, W. Stoichiometric calculation: Known problems and proposed solutions at a chemistry-mathematics interface. *Studies in Science Education, 8* (1981) 93-105.

Duncan, I.U. and Johnstone, A.H. (1973). The mole concept. *Education in Chemistry, 10*, 213-4.

Howe, A. & Shayer, M. (1981). Sex-related differences on a task of volume and density. *Journal of Research in Science Teaching 18*, 169-175.

Ingle, R.B. and Shayer, M. (1971). Conceptual demands in Nuffield O-level chemistry. *Education in Chemistry 8*, 182-3.

Lazonby, J.N., et al (1985). The mole: Questioning format can make a difference. *Journal of Chemical Education 1*, 60-61.

Liben, L.S. (1978). Performance on Piagetian spatial tasks as a function of sex, field dependence and training. *Merrill-Palmer Quarterly, 24*, 97-109.

Linn, M.C. and Polus, S. (1983). Male-female differences in predicting displaced volume: Strategy usage, aptitude relationships and experience. *Journal of Educational Psychology 75*, 86-96.

Lybeck, L., et al (1988). *Empirical research in science and mathematics education.* The International Council of Associations for Science Education (Univ. Dortmund), 39-55.

MacDonald, J.J. (1975). The mole: What's all the fuss about. *School Science Review, 96*, 635-636.

Michell, I. and Gunstone, R. (1984). Some student conceptions brought to the study of stoichiometry. *Research in Science Education, 4*, 78-88.

Novick, S. and Menis, J. (1976). A study of student perceptions of the mole concept. *Journal of Chemical Education, 53*, 720-722.

Robert, M. (1989). Robustness of cognitive gender differences: The case of horizontality representation in the water-level task. Paper presented at the Tenth Biennial Meeting of the ISSBD, Jyvaskyla, Finland.

Stavy, R. (1981). Teaching inverse functions via the concentration of salt water solution. *Archives de Psychologie, 49*, 267-287.

Stavy, R. (1988). Children's conceptions of gas. *International Journal of Science Education, 10*, 553-560.

Stavy, R. (1985). Children's ideas about solid and liquid. *International Journal of Science Education, 7*, 407-421.

Surber, C.F. and Gzesh, S.M. (1984). Reversible operations in the balance scale task. *Journal of Experimental Psychology, 38*, 254-274.

Umland, J.B. (1984). A recipe for teaching stoichometry. *Journal of Chemical Education*, 1036-1037.

THE DEVELOPMENT OF MATHEMATICAL DISCUSSION

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The development of mathematical discussion in a second-grade classroom is presented. In this classroom, the students and teacher mutually construct a form of discourse in which mathematical meaning is negotiated. Two sample episodes are analyzed to illustrate the evolving nature of the interaction patterns as the students and teacher interactively constitute a basis for activity that creates opportunities for learning. The tension created between the teacher and students' meanings as they engage in the process of negotiation of mathematical meaning will be considered in light of their personally constructed basis for activity.

The importance of whole class discussion in mathematics for primary-aged children can be considered both from the point of view of individual mathematical construction and the development of the taken-to-be-shared meaning. The nature of the routines and pattems of interaction that are mutually established between the teacher and students create learning opportunities that ultimately influence what is learned. The purpose of this paper is to illustrate the evolving nature of the pattems of interaction that occur between the students and teacher as they interactively constitute a basis for activity that encourages the development of mathematical meaning. Specifically, the discussion will focus on the negotiation of meaning through a process involving a genuine commitment to communicate.

The interactively constituted discourse that influences the child's evolving sense of number has as a crucial feature the child's involvement in problem solving activity. As children, participate in discussion which emphasizes their ideas and solutions to problems, they have an opportunity to provide explanations and justifications which form the basis of mathematical argumentation. Students' who engage in dialogue in which accounting for their ideas to others is anticipated are provided opportunities in which to negotiate mathematical meanings. It is these taken-to-be-shared concepts that come to be accepted as "mutually congruent mental representations" and become "real objects whose existence is just as 'objective' as mother love..." (Davis & Hersh, 1981). Patterns of interaction in which these discussions occur are

characterized by a genuine commitment to communicate in which the students' as well as the teacher are active participants in the dialogue. From this perspective, communication is viewed as an activity, that requires two or more autonomous partners, in which the students' are responsible for describing their worlds to their teachers. For their part, teachers are obligated to convey a genuine interest in and willingness to learn about what their students are discussing (Cazden, 1988).

Mathematical Discourse in Elementary School

Whole-class discussions in which children talk over their solutions to mathematical problems provides opportunities for learning not available in most traditional school classrooms. The typical discourse pattern found in these classrooms has been extensively described by Hoetker & Ahlbrandt (1969), Mehan (1979), and Sinclair and Coulthard (1975), as one in which the teacher controls, directs and dominates the talk. The patterns of interaction become routinized in such a way that the students do not need to think about mathematical meaning, but instead focus their attention on making sense of the teacher's directives (Bauersfeld, 1980; Voigt, 1985).

Our ongoing research and development project in second-grade (7-year olds) is an attempt to analyze children's construction of mathematical concepts and operations within the complexity of the social setting of the classroom. A constructivist's perspective of children's learning forms the basis of the project in which the cognitive models of early number learning (Steffe, Cobb, von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards & Cobb, 1983) are used to develop instructional activities. The general instructional strategies are small group collaboration followed by whole class discussion of children's solutions. Moreover, we believe that mathematics learning is an interactive as well as a problem solving activity that involves the negotiation of taken-to-be-shared mathematical meanings by members, of the classroom communities (Bauersfeld, 1988). We have previously described and discussed the nature of these classrooms with regard to the mutual construction of the social norms that constitute the obligations and expectations crucial for the development of a classroom setting in which learning

Cobb, Yackel, & Wheatley, 1990; Yackel, Cobb, & Wood, in press). Although the project has been expanded to include additional classrooms, this paper will be drawn from video-recorded data collected during the initial year-long classroom teaching experiment in one second-grade classroom.

Interactively Constituted Basis for Mathematical Discourse

Class discussion following the collaborative work of the small groups provides an opportunity for children to monitor their activity during a period of retroactive thematization (Steffe, 1990). As children explain their solutions to instructional activities, opportunities arise for them to reiterate their earlier activity in order to bring it to a level of conscious awareness and evaluate it in terms of the ongoing discussion. It is through this monitoring activity that children build confidence in their actions.

The nature of the social interaction that is interactivelly constituted in the class is crucial to the development of negotiation that creates opportunities for learning. Discourse that is characterized by a genuine commitment to communicate provides opportunities for children to reflect on and evaluate their prior activity. As such, class discussions in which the emphasis is on children's explanations and justifications of their mathematical solutions provides occasions for children to engage in negotation of mathematical activity taken-to-be-shared.

The following episodes have been selected to illustrate the evolving nature of the whole class discussion in which the teacher and children attempted to communicate and negotiate mathematical meaning. The first example occurs at the beginning of the year as the class has been discussing the following problem which is on the chalkboard. A picture is shown of a pencil underneath which are 5 paper clips followed by the statement of the problem:

Each paper clip is 3 centimeters long.

How long is the pencil?

The episode begins:

Teacher: How long is the pencil, Lisa?

Lisa: 15 centimeters long.

Teacher: 15 centimeters. How did you get that answer?

Lisa: There were all 3's and we added them up by 3's and got 15.

Teacher: Did anyone do it a different way? Chuck?

Chuck: I got 5 centimeters.

Teacher: You got 5 centimeters. How long is each one of these [clips]? Let's take a look at this. If each one of these is 3 centimeters long, we have a 3, 3, 3, 3, and a 3. How much is that altogether? Karen?

Karen: 15.

In this early dialogue, the children are obligated to provide an explanation for their answers. In this case, the teacher assumes that Chuck does not understand the problem and attempts to direct him to the answer.

Teacher: Do you agree or disagree?

Chuck: Disagree.

Teacher: Disagree? That's alright, but there are 3 centimeters in each clip.

Chuck: There are 5 clips and that's 5 centimeters.

Teacher: He is right. There are 5 clips, but how long is each clip? A 3 and a 3 and a 3 and a 3 (measuring the distance on each clip with her hands). How long is that?

Chuck: 5 centimeters.

Teacher Okay. (pause) Did anybody else do it a different way?

In this class, the teacher and children have mutually constituted the social norm that their opinions are respected, thus it is appropriate for Chuck to disagree with the answer given. Moreover, in this exchange, the teacher attempts to engage him more directly in finding the solution. Chuck's contribution to the interaction is influenced by his current level of conceptual understanding in which he is limited by the fact that he has not constructed a scheme for coordinating units of different ranks. The teacher is also constrained in the interaction by her current conceptions and her limited understanding of the possible constructions children might possess. Thus, she simply indicates to Chuck that she accepts that his answer is meaningful to

him and directs the discussion to the other students. As yet, the students and the teacher while attempting to interactively constitute a basis for activity are unable to negotiate mathematical meaning.

The second episode that occurred later in the year centered around the following problem:

> Daisy Duck invited 50 children to her birthday party. Nineteen of them were girls. How many were boys.?

The children as they were discussing the problem, have been offering solutions which involve subtracting 19 from 50. The episode begins:

Teacher: Okay Alex what do you say?

Alex: It's 31.

Teacher: You think it's 31.

Alex: Because 30 plus 20 is 50, 30 plus 20 is 50.

Unfortunately, at this moment, the teacher is distracted by the comments of another pair of children and stops to carry on a conversation with them. Returning to Alex, she attempts to restate his explanation.

Teacher: Alright, look he said... 3 plus 20?

Alex: 30 plus 20.

Teacher: 30 plus 20. Where did you get the 30?

Alex's explanation of the problem as *"30 plus 20"* indicates that he has interpreted the problem as a missing addend. The teacher, attempting to understand his method, asks for further elaboration. In so doing, Alex tries to adjust his explanation to fit with the previous subtraction solutions that have been given.

Alex: [It] equals 50, and that takes up the 50 children that were at the party.

Teacher: But this is 19, right? (point to the number).

Alex: I know.

Teacher: Alright.

Alex: And so 50 minus 20 would be 30.

Teacher: Okay. What he is saying is instead of taking the 19, I [Alex] made it 20.

Alex: No.

Teacher: No, you didn't?

The teacher who has been trying to make sense of Alex's explanation offers her interpretation of his method to the rest of the class. Alex challenges her comments as incorrect, and continues to alter his initial solution from addition to subtraction. The teacher accepts this, and encourages his further explanation.

Alex: 50 minus 30 is 20.

Teacher: 50 minus 30.

Alex: 50... Well, I don't know what I did. 50 minus 20 is 30...

Teacher: Right.

Alex: But its a 19 instead of a 20 so it has to be one higher than it, because that number is one less than 20, so it's 31.

Teacher: Alright.

As Alex rethinks his explanation, he realizes that just changing his initial interpretation of the task to subtraction creates a situation that does not fit with the problem as written. In so doing, he offers a final explanation that provides a rationale and justification for his solution. The teacher's comment *"Alright"* indicates her agreement. The episode closes with both Alex and the teacher coming to accordance through a process of negotiation which has provided opportunities for Alex to provide a justification for his explanation.

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In T. Cooney & D. Grouws (Eds.) Effective mathematics teaching (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. Educational Studies in Mathematics. 11, 23-41.

Cazden, C. (1988). Classroom discourse: The language of teaching and learning. New York: Heineman.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D.B. McLeod & V.M. Adams (Eds.), Affect and mathematical problem solving: A new perspective (pp. 117-148). New York: Springer-Verlag.

Cobb, P. (1990b). Multiple perspectives. In L.P. Steffe, & T. Wood (Eds.), Transforming childhood mathematics education: An international perspective (pp. 200-215). Hillsdale, NJ: Lawrence Erlbaum Associates.

Davis, P., and Hersch, R. (1981). The mathematical experience. Boston, MA: Houghton Mifflin.

Hoetker, J., & Ahlbrandt, W. (1969). The persistence of recitation. American Educational Research Journal, 6, 145-167.

Mehan, H. (1979). Learning lessons. Cambridge, MA: Harvard University Press.

Sinclair, J., & Coulthard, R. (1975). Towards an analysis of discourse: The English used by teachers and pupils. London: Oxford University Press.

Steffe, L.P. (1990). Personal communique.

Steffe, L.P., & Cobb, P., & von Glasersfeld, E. (1988). Construction of mathematical meanings and strategies. New York: Springer-Verlag.

Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). Children's counting types: Philosophy, theory, and application. New York: Praeger Scientific.

Wood, T., Cobb, P., & Yackel, E., & Wheatley, G. (1990). Reflections on Learning and Teaching Mathematics in Elementary School. Paper presented at the seminar series "Constructivism in Education." Athens, University of Georgia.

Voigt, J. (1985). Patterns and routines in classroom interaction. Recherches en Didactique des Mathematiques, 6, 69-118.

Yackel, E., Cobb, P., & Wood, T. (in press). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education Monograph.

Note

The research reported in this paper was supported by the National Science Foundation (NSF Grant MDR 874-0400 and MDR 885-0560) and all opinions expressed are those of the author.

Statistical Reasoning

ESTRATEGIAS Y ARGUMENTOS EN EL ESTUDIO DESCRIPTIVO DE LA ASOCIACION USANDO MICROORDENADORES

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SUMMARY

In this work problem solving procedures about statistical association using microcomputers are analysed. The effect of several didactical variables on these procedures is also studied. The sample was formed by 18 training teachers that received a previous teaching during a seven week period. The study of the arguments expressed by the students allows us to know the scope and meaning given by them to the concept of association and to infer criteria to desing new didactical situations.

INTRODUCCION

El uso de ordenadores como recurso en los procesos de enseñanza - aprendizaje de las matemáticas plantea problemas de investigación de reconocido interés en didáctica de las matemáticas, como se pone de manifiesto en Howson y Kahane (1986). Para el caso de la enseñanza de la Estadística, el hecho de que los alumnos dispongan de la herramienta informática para el proceso de datos hace posible ponerles ante situaciones - problemas relativos al análisis de ficheros de datos de aplicaciones reales. De este modo, los programas de cálculo y representación gráfica pueden actuar como amplificadores conceptuales (R. Lesh, 1987) de las nociones y procedimientos estadísticos, al permitir el planteamiento de cuestiones más abiertas e inducir una actividad matemática exploratoria sobre los conjuntos de datos.

En este contexto teórico situamos este trabajo, que analiza los procedimientos seguidos por los estudiantes en la resolución de problemas sobre el estudio descriptivo de la asociación entre variables estadísticas y el efecto sobre

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Este trabajo forma parte del Proyecto PS88-0104 subvencionado por la Dirección General de Investigación Científica y Técnica (M.E.C. Madrid)

dichos procedimientos de distintas variables didácticas. Este análisis permite deducir el alcance y sentido dado por losalumnos al concepto de asociación y determinar criterios para la construcción de nuevas situaciones didácticas.

Desde el punto de vista metodológico usamos la técnica del registro de la interacción alumno- ordenador, lo que nos permite identificar los procedimientos de resolución de los problemas. Sin embargo, se muestra la necesidad de recurrir también a las argumentaciones escritas en que basan sus respuestas para poder apreciar la correción y pertinencia de los procedimientos empleados y para controlar el sentido de los conocimientos estadísticos adquiridos.

METODOLOGIA

Experiencia de enseñanza y logicial utilizado

Este trabajo es parte de una investigación sobre enseñanza de la Estadística Descriptiva que se ha desarrollado con un grupo de 18 futuros profesores (estudiantes de edades entre 19 y 20 años) durante un periodo de 7 semanas. En este tiempo se han alternado dos horas semanales de exposición teórica y de resolución de ejercicios con papel y lápiz, con otra de práctica en el aula de informática. Los contenidos abarcados en las 21 horas de enseñanza han sido los de estadística descriptiva uni y bivariante y una introducción a la filosofía y algunos gráficos del análisis exploratorio de datos.

Los programas empleados en la experiencia, forman parte del paquete PRODEST que se describe en Batanero y cols. (1988). Los procedimientos empleados, además de los de grabación y depuración de ficheros, son los siguientes:

<u>Práctica 1 (9 actividades): programa CONTAJE</u>

Realización de tablas de frecuencias de variables estadísticas cualitativas o discretas, diagramas de barras, diagramas acumulativos y gráficos de sectores.

<u>Práctica 2 (9 actividades): programa HISTO</u>

Cálculo de tablas de frecuencias de variables estadísticas continuas, agrupando en intervalos, histogramas, polígonos de frecuencias y polígonos acumulativos.

Práctica 3 (7 actividades): programa TRONCO

Realización del gráfico del tronco (stem and leaf).

Práctica 4 (6 actividades): programa MEDIANA:

Cálculo de los estadísticos de orden y gráfico de la caja (box and whiskers).

Práctica 5 (7 actividades): programa ESTADIS

Estadísticos elementales de valor central, dispersión y forma.

Práctica 6 (9 actividades): programa TABLAS:

Tablas de contingencia, con posibilidad de agrupación de datos.

Práctica 7 (10 actividades): programa REGRESIO

Cálculos y gráficos de regresión y correlación bivariante.

Evaluación del aprendizaje

El proceso de aprendizaje se ha observado mediante la grabación en ficheros de las interacciones alumno - ordenador en todas las actividades, así como la respuesta escrita de los alumnos a las cuestiones planteadas y sus argumentaciones. Además se ha realizado una evaluación final de cada alumno, también grabada, que consistió en la respuesta a una serie de cuestiones, usando el ordenador, sobre el análisis de un fichero relativo a datos deportivos reales de alumnos de un centro de Educación General Básica. Las variables incluídas en el fichero se muestran en la Tabla 1.

TABLA 1: TIPOS DE VARIABLES INCLUIDAS EN EL FICHERO DE DATOS

Tipo:	Variables:
Dicotómicas:	Sexo Practica o no deporte
Discretas, (con necesidad de agrupación):	Pulsaciones en reposo Pulsaciones después de 30 flexiones
Continuas:	Tiempo en recorrer 100 m. en Septiembre Tiempo en recorrer 100m. en Diciembre

Las cuestiones sobre asociación planteadas en la evaluación final fueron las siguientes:

1.- ¿Hay diferencia en la práctica de deporte entre chicos y chicas?

2.-¿Hay relación entre las variables NUMERO DE PULSACIONES EN REPOSO y PRACTICA ALGUN DEPORTE?

3.-¿Hay relación entre las varibles PULSACIONES EN REPOSO y PULSACIONES DESPUES DE 30 FLEXIONES? ¿De qué tipo?

4.- ¿Ha disminuido el tiempo que tardan los alumnos en recorrer 30 metros de Septiembre a Diciembre?

Análisis de los datos

Con objeto de obtener una codificación de la información recogida para cada alumno se ha aplicado la técnica de análisis de contenido. Al tratar de responder a las cuestiones planteadas, el alumno utiliza uno de los programas disponibles y puede ocurrir que en la primera ejecución resuelva el problema o que se produzca un error y deba intentarlo de nuevo. De este modo, se ha tomado el intento de solución como unidad de análisis, obteniéndose en total 120 unidades de análisis para las 4 cuestiones propuestas y los 18 alumnos, lo que proporciona un número medio de 1.66 intentos por alumno y pregunta. En la Tabla 2 se muestran los diferentes procedimientos empleados en la solución de las cuestiones planteadas.

RESULTADOS Y DISCUSION

Procedimientos de análisis de datos.

Un primer hecho que puede observarse en la Tabla 2 es que el total de intentos para una misma cuestión es, en general, mayor que el número de alumnos (18). Aparte de los errores de ejecución (no incluídos en la Tabla 2) que han supuesto el 6.1 por ciento, en algunos problemas el alumno debe elegir una solución entre otras posibles (por ejemplo, la amplitud del intervalo), por lo que hace varios ensayos con un mismo programa. En otros casos no es así, pero emplea más de un programa para dar su respuesta.

Observamos que el procedimiento empleado para estudiar la asociación depende de la forma en que se plantea la pregunta, influencia ya señalada por diversos autores. Así, los ejercicios 3 y 4, referidos a la diferencia entre muestras relacionadas son resueltos en forma distinta. En el primer

caso, al plantear la pregunta en términos de **relación** entre las dos variables, induce el empleo casi exclusivo de la regresión. En el segundo, la referencia a una **diferencia** provoca el estudio separado de cada variable por diversos métodos, principalmente por medio de las tablas de frecuencias (HISTO) o de los estadísticos de orden (MEDIANA).

TABLA 2
FRECUENCIA DE INTENTOS SEGUN CUESTION Y PROGRAMA USADO

	Cuestión				
Procedimiento	1	2	3	4	Total
CONTAJE	19			2	21
HISTO	2			7	9
TRONCO				1	1
MEDIANA				6	6
ESTADIS				4	4
TABLAS	9	23	2	3	37
REGRESIO		20	20	2	42
Total	30	43	22	25	120

Este no es, sin embargo, el único factor observado. Dentro de una pregunta planteada en los mismos términos, es esencial el tipo de variable estadística presente en la tarea. Por ejemplo, en el problema 1, la pregunta se plantea en términos de **diferencias**, pero en este caso, aunque todos los alumnos lo resuelven estudiando separadamente las tablas de frecuencias de las variables (CONTAJE), un número importante de ellos, además, confirma sus resultados empleando el estudio conjunto (TABLAS). Asimismo, la pregunta 2 que se plantea en términos de relación es resuelta, en primer lugar, con el de la regresión y, además, mediante la tabla de contingencia.

Argumentaciones

En los párrafos anteriores hemos analizado los procedimientos de resolución seguidos por los alumnos. Este análisis no es, sin embargo, suficiente para asegurar la corrección de las respuestas: por un lado, hemos visto que cada pregunta podría resolverse con varios procedimientos, pero, lo esencial es la información utilizada de los mismos y la interpretación dada a esta información. Por este motivo, se han analizado las **argumentaciones** ofrecidas por los

estudiantes, que se relacionan en la Tabla 3. Los tipos de argumentos hallados en las preguntas de asociación se han clasificado en la forma siguiente:

Frecuencias absolutas: Razonamientos basados en la frecuencia absoluta de las distribuciones univariantes o de las condicionales o marginales de una tabla de contingencia.

Frecuencias relativas. proporciones o porcentajes en distribuciones univariantes o en las condicionales o marginales de una tabla de contingencia.

Percentiles o rangos de percentiles, frecuencias acumuladas ascendentes o descendentes y argumentos basados en los estadísticos de orden, a excepción de la mediana.

Valores centrales: comparación de medias, modas o medianas de las distribuciones.

Coeficiente de correlación (valor absoluto y signo).

Nube de puntos: Forma creciente o decreciente de la nube, forma aproximada de línea recta, mayor o menor dispersión de los puntos de la nube.

Dispersión: Comparación de varianzas, desviaciones típicas u otros estadísticos de dispersión.

Otros argumentos: forma de los gráficos, presencia de varias modas, realización de cálculos adicionales, etc.

El número de argumentos es mayor que el número de estudiantes, porque a veces utilizan dos o más argumentos para el mismo problema. En las dos preguntas en que interviene una variable dicotómica (1 y 2), los argumentos más utilizados correctamente son los relacionados con frecuencias: relativas, absolutas y rangos de percentiles, a pesar de que en la segunda cuestión casi todos los alumnos han empleado el programa REGRESIO, además de la tabla de contingencia.

Independientemente de que la pregunta haya inducido a estudiar las variables en forma conjunta (segunda cuestión) o no (primera), parece deducirse de la argumentación que la asociación de una variable con otra dicotómica se interpreta preferentemente como diferencia en las distribuciones de frecuencias condicionales. También concluímos que el uso del procedimiento REGRESIO en la cuestión segunda se ha utilizado a modo de confirmación visual de la respuesta, ya que sólo unos pocos alumnos han empleado un argumento sobre el valor

del coeficiente de correlación.

La tendencia a basar sus argumentos en valores numéricos, se presenta de nuevo en la tercera cuestión, en la que las razones basadas en el coeficiente de correlación superan a las basadas en la forma de la nube de puntos. Ambos tipos de argumentos ponen de relieve que la idea de relación entre variables cuantitativas es sinónima de la de dependencia, aunque aleatoria.

Al analizar los argumentos empleados en la última cuestión, observamos de nuevo la influencia de la forma en que está hecha la pregunta, ya que pocos alumnos identifican la asociación existente entre las dos variables cuantitativas (tiempo en septiembre y tiempo en diciembre). Muy pocos argumentos correctos o incorrectos se refieren a ello. La mayor parte de argumentos correctos se refieren a la comparación de los valores centrales de dihas variables. Aunque la mitad de alumnos han empleado el programa HISTO, pocos han comparado las frecuencias relativas de las dos distribuciones para los mismos intervalos de valores de los tiempos. Por ello pensamos que tampoco han llegado a identificar el problema como uno de estudio de asociación de una variable cuantitativa (tiempo en recorrer 100 m.) respecto a una dicotómica (antes/después del entrenamiento), sino más bien como de diferencias de dos variables cualesquiera no relacionadas entre sí.

CONCLUSIONES

El análisis de los resultados ha puesto de manifiesto que un proceso de enseñanza de nociones estadística basado en la resolución de problemas realistas de análisis de datos, hecho posible al disponer del recurso informático, es insuficiente para que los estudiantes adquieran por completo todo el alcance y sentido de las nociones estadísticas de asociación e independencia. El diseño de situaciones didácticas, basadas en el uso de logiciales, precisa tener en cuenta la naturaleza de las distintas variables didácticas, y atribuir un papel relevante al profesor como gestor de las situaciones, especialmente para explicitar los matices del saber matemático incorporado en los problemas.

TABLA 3

FRECUENCIAS DE ARGUMENTOS EN LAS CUESTIONES SOBRE ASOCIACION

Cuestión	Argumentos correctos		Argumentos incorrectos		Total
1	Frec. relativa	15	Frec. relativa	2	
	Frec. absoluta	7			
	Otros argumentos	5			
	Total cuestion 1:	27		2	29
2	Frec. relativa	7	Correlación	1	
	Percentil/rango	6	No justifica	1	
	Frec. absoluta	4			
	Dispersión	3			
	Correlación	3			
	Otros argumentos	4			
	Total cuestión 2:	27		2	29
3	Correlación	13	Frec. relativa	2	
	Nube de puntos	6	Otras	3	
	Otras	3			
	Total cuestión 3:	22		5	27
4	Valor central	11	Frec. absoluta	3	
	Otras	7			
	Total cuestión 4:	18		3	21

REFERENCIAS

Batanero Bernabeu, M. C.; Díaz Godino y Estepa Castro, A. (1987). Un paquete didáctico de programas para el laboratorio de estadística. En I. Brincones y cols. (Ed.). **Actas del I Simposio Internacional de Educación e Informática**. I.C.E. de la Universidad Autónoma de Madrid.

Howson, A. B. y Kahane, J. P. (1986) **The influence of computers and informatics on mathematics and its teaching.** Cambridge: Cambridge University Press.

Lesh, R. (1987). The evolution of problem representatios in the presence of powerful conceptual amplifiers. En C. Janvier (Ed). **Problems of representation in the teaching and learning of mathematics.** London: Lawrence Erlabaum Assoc.

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COMPUTERIZED TOOLS AND THE PROCESS OF DATA MODELING*

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We are attempting to identify conceptual challenges that students encounter as they design, collect and analyze data about a real situation. We propose the term "data modeling" to describe this process, and present a new computerized tool for working with data called the Tabletop. While the Tabletop is a tool for analyzing data, we conjecture that it can help students become better designers of data. Examples from clinical research in progress help to show how closely intertwined are the phases of data modeling, and thus begin to resolve the apparent paradox of how a technological tool for one phase can benefit others.

In the last few years there has been a renewed interest among mathematics educators in the curricular value of working with real data, as a way to learn to construct and evaluate mathematically supported arguments, and as a meaningful context for learning mathematical concepts and techniques. The Hands On Data project, in which the authors are currently engaged, is one of many projects exploring this general theme. Two terms around which interest has crystallized are "data analysis" and "statistics." Under the heading "data analysis", for example, the Mathematical Sciences Education Board (MSEB, 1990, p. 42) lists "collection, organization, representation, and interpretation of data; construction of statistical tables and diagrams; and the use of data for analytic and predictive purposes." Under "statistics", the recent NCTM standards (NCTM, 1989, p.105) stipulate that students in grades 5-8 should learn to:

- systematically collect, organize and describe data;
- construct, read, and interpret tables, charts and graphs;
- make inferences and convincing arguments that are based on data analysis;
- evaluate arguments that are based on data analysis;
- develop an appreciation for statistical methods as powerful means for decision making.

These are good lists: they mark off a coherent set of interdependent skills and concepts. To collect and organize data systematically, after all, one needs to understand how the data will be graphed, charted, analyzed and used in arguments; to evaluate an argument based on data analysis one needs to consider whether the data was collected and organized appropriately to support the argument. But do the headings do justice to the lists?

We think that it is important to emphasize, as terms such as "data analysis" and "statistics" do not, that designing and collecting data is a creative act. The word "data" (from the Latin "givens") is itself something of a misnomer. Data are not given to us by the world; rather, we create data to model aspects of our experienced world that interest us (Goodman, 1978). We often use the word "data" as a mass noun, like "hay" or "cheese." But data does not come by the bale or by the pound, or even by the kilobyte; data is built in structures. These structures are in turn amenable to

* This paper has been prepared with the support of the National Science Foundation, grant # MDR-8855617. We also gratefully acknowledge the donation of equipment by Apple Computer, Inc. The views expressed are not necessarily those of the NSF or of Apple Computer. The ideas in this paper are being developed in collaboration with the other members of the Hands On Data research team: Phyllis Kalowski, Anne Lopoulos, Alana Parkes and Linda Waxler.

manipulations which yield results to which we may assign meaning. In inquiry and in pedagogy, we can begin to make sense of data when we consider the data, the structure and the manipulations as coherent whole — a *data model*. We therefore use the term *data modeling* to describe the highly intertwined sets of skills and activities by which data models are created, manipulated and interpreted.

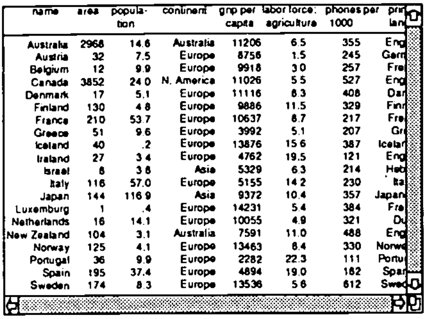
The process by which one creates a data model of a real situation ("systematically collecting and organizing data") is complicated and conceptually rich. If a teacher is not expecting these complexities, their intrusion in a classroom data modeling activity is likely to be perceived as a failure of the activity. For example:

> One mixed-grade class wanted to order special T-shirts for a group to which they belonged. They decided to make a database of peoples' names, sizes and color preferences, to help assemble the order. However, the students did not think of establishing data conventions, and it turned out at the end that the sizes had been entered in many different formats — as "L," "Large," "Children's Large," "Size 12," "Women's Large," et cetera. No time had been allotted to deal with this issue, and in the end the order was assembled by hand.
>
> Students in another class tried to answer the question, "How do students in our school spend their time?" A questionnaire yielded a bewildering variety of descriptions of how time was spent, including "playing outside," "playing baseball," "watching TV," "doing chores," "doing homework," "mowing the lawn," "transportation," "at home," "outside" and so on. Neither the students nor their teacher had anticipated the challenging task that remained, which was to organize these responses, some at different hierarchical levels, others overlapping, into a structured categorical system that they could use for further data collection and analysis.

Of course, we are not the first to recognized the complexity and importance of data creation. Both Taba (1967) and the USMES project (USMES, 1976) designed curricula with careful attention to the process of creating data, and the MSEB urges that students deal with "the messy reality of worldly data" and comments, "The inevitable dialog that emerges between the reality of measurement and the reality of calculations -- between the experimental and the theoretical -- captures the whole science of mathematics" (1990, p.43). The contribution which we hope to make here, beyond a plea for better terminology, is to discuss the interrelationship of data creation and data analysis in the light of some clinical examples and a software environment being developed as part of our project.

The Tabletop

The *Tablemaker* is an integrated environment for creating, organizing, exploring and analyzing record-oriented data (i.e. data about a set of objects with the same attributes, or "fields," recorded for each object). It is currently implemented in prototype form. It includes a conventional record-oriented database view which allows one to define fields and enter and edit data (illustrated in figure 1, with sample data about 24 countries of the world). It also includes the *Tabletop*, a radically different interactive representation for exploring and analyzing the data. The Tabletop's representational system proceeds from two principles: first, the screen shows one icon for each item in the database (or in the subset of the database currently being examined); and, second, the icons can be moved about the screen in a variety of ways to reveal properties of the data.



name	area	population	continent	gnp per capita	labor force: agriculture	phones per 1000	pri lan
Australia	2968	14.6	Australia	11206	6.5	355	Eng
Austria	32	7.5	Europe	8756	1.5	245	Gern
Belgium	12	9.9	Europe	9918	3 0	257	Fre
Canada	3852	24 0	N. America	11026	5 5	527	Eng
Denmark	17	5.1	Europe	11116	6.3	408	Dar
Finland	130	4 8	Europe	9886	11.5	329	Finr
France	210	53.7	Europe	10637	8.7	217	Fre
Greece	51	9.6	Europe	3992	5.1	207	Gr
Iceland	40	.2	Europe	13876	15 6	387	Icelar
Ireland	27	3 4	Europe	4762	19.5	121	Eng
Israel	6	3 6	Asia	5329	6.3	214	Heb
Italy	116	57.0	Europe	5155	14 2	230	Ita
Japan	144	116 9	Asia	9372	10.4	357	Japan
Luxemburg	1	.4	Europe	14231	5.4	384	Fre
Netherlands	16	14.1	Europe	10055	4.9	321	Du
New Zealand	104	3.1	Australia	7591	11.0	488	Eng
Norway	125	4.1	Europe	13463	6.4	330	Norw
Portugal	36	9.9	Europe	2282	22.3	111	Portu
Spain	195	37.4	Europe	4894	19.0	182	Spar
Sweden	174	8.3	Europe	13536	5 6	612	Swed

Fig. 1: In addition to the Tabletop (figures 2-12), the Tablemaker provides this conventional row-and-column view for entering and editing data.

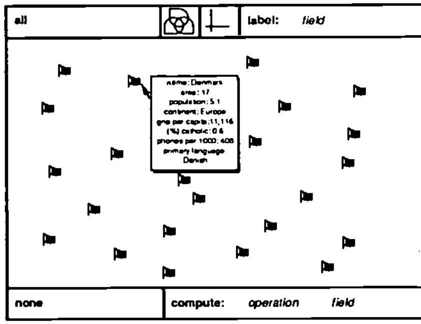


Fig. 2: The Tabletop shows one icon for each item in the database. The icons are initially scattered randomly. Any icon's detailed information can be examined at any time by pointing and clicking the mouse.

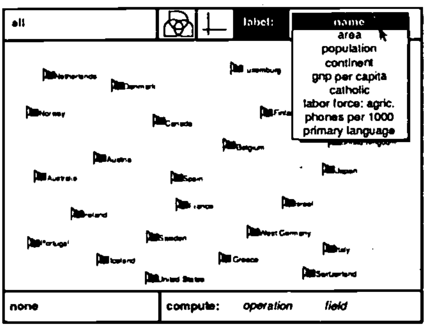


Fig. 3.: Icons can be labelled with their values for any field in the database. When an icon moves, its label moves with it.

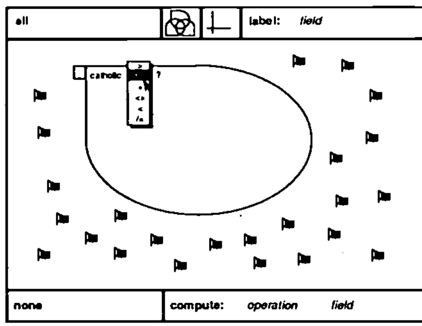


Fig. 4: A set constraint can be set up using pop-up menus.

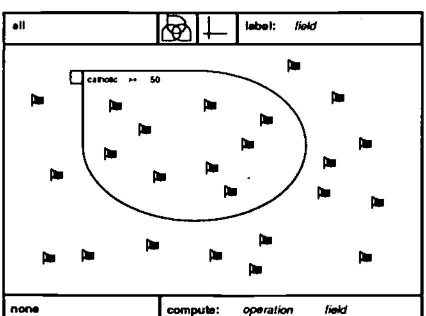


Fig. 5: Icons satisfying the constraint move into the circle. They move quickly and simultaneously, but smoothly so that any individual can be tracked.

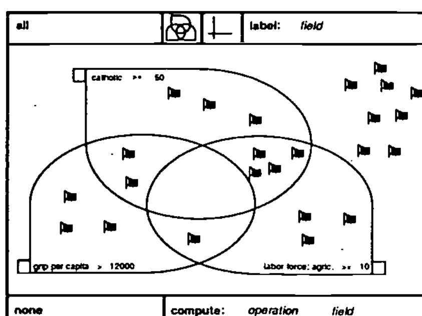


Fig. 6: Up to three circles can be active at once. Any part of any constraint can be modified directly, and the affected icons will immediately move to new positions.

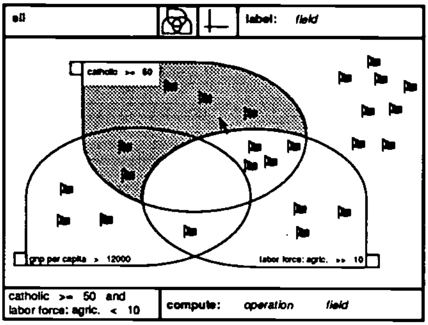


Fig. 7: Once identified, subsets of the database can be operated on in a variety of ways. For example, the selected icons can be marked so they can be tracked during later analysis. They can also be printed out, deleted, etc.

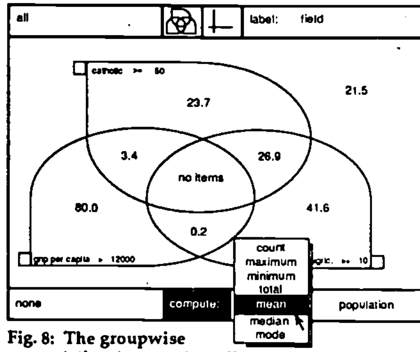


Fig. 8: The groupwise computation feature. The illustration shows mean population for each of the eight subgroups created by the Venn diagram.

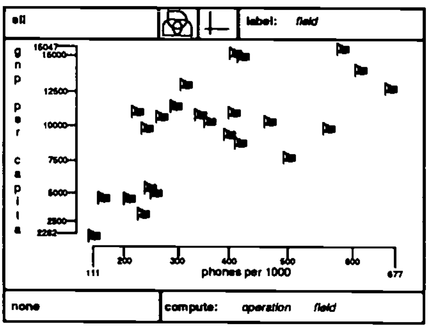


Fig. 9: To set up an axis, choose a field from the pop-up menu attached to the axis label. An appropriate scale appears automatically and the icons move to align themselves with it. Each axis constrains the icons in only one direction.

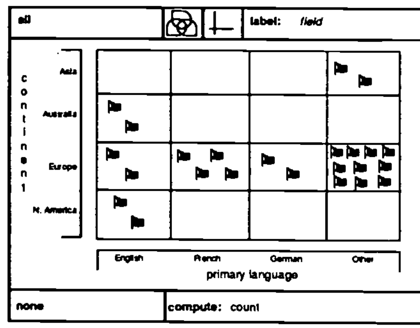


Fig. 10: If the fields associated with an axis are discrete, a cellular graph results. Hybrids with one discrete and one continuous axis are also possible.

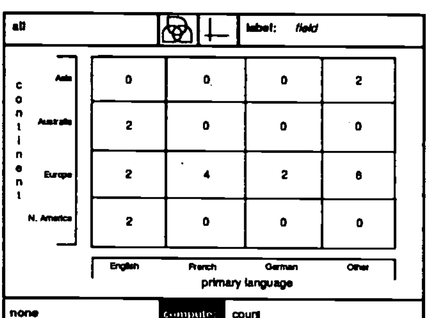


Fig. 11: In this case, performing groupwise computations produces a crosstabulation.

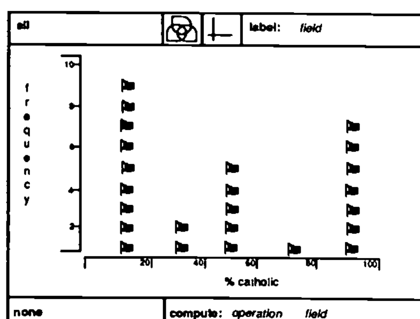


Fig. 12: Putting "frequency" on one axis causes the icons to pile up, producing a frequency distribution graph.

Figures 2-12 are snapshots of the screen's appearance during a sample session with the Tabletop. It is left to the reader to imagine what happens between snapshots. The general flow of interaction is that the user imposes structure on the screen space -- by setting up a circle with a constraint, or by associating the vertical or horizontal axis with one of the fields -- after which the icons move to take up positions dictated by the new structure. The icons move simultaneously and quickly, but smoothly, so that each individual's movement can be tracked. If one "drags" an icon to another part of the screen, it will hasten back to its proper place (a favorite feature of younger users). The icons can be made to carry labels based on any field in the database, and summary computations can be generated over subsets of the data. From these simple, pseudo-physical principles emerge scatter plots, frequency histograms, crosstabulations, Venn diagrams and a number of other "plots," less familiar but equally informative. All of these plots are built by the user in a way that helps to make their meaning clear. They can all be incrementally modified (changing "less than" to "less than or equal" in a Venn circle, for example) with immediate feedback. And they are all open to further querying: one can always change the labels on the icons, or examine an interesting one in detail by double-clicking (as in figure 2).

This software is new in many ways; there is much to discuss and more to investigate concerning its potential educational uses. Many questions relate to the intelligibility of the representations. In our piloting we have found that students as young as eight can interpret many of the representations. In fact, the enthusiasm and engagement with which many youngsters explore the system's behavior seems to hint that the Tabletop's representations resonate somehow with issues (categorization, inclusion and exclusion, ranking by order) which are of special concern at this age. The success of these children in interpreting and constructing scatter plots (as well as their initial misinterpretations and misconstructions) suggest, among other things, that the animated, manipulable scatterplot of the Tabletop may be an ideal context in which to learn the basic principles of two-dimensional graphing, well before encountering the extra complications of graphs of continuous functions.

As a data analysis tool, the Tabletop offers a number of important advances over the software currently in use in schools. It provides an unprecedented range of analytical operations together with representations that help to make them intelligible. Its flexibility and rich feedback stimulate exploration rather than holding students to a predetermined path of analysis. It invites students to look for unanticipated patterns, and to cross-check their first results. Time and effort previously devoted to simply generating graphs, charts and summary data can now be shifted to thinking about what they mean. But to predict the Tabletop's role in the complete data modeling process we must first consider the nature of that process.

Data Modeling

We begin with a truly naive and preliminary breakdown of stages of one basic kind of computer-aided data modeling. Of special interest to us is how the "phases" act, not as sequential components of a temporal process (which they can be on a trivial level), but rather as sets of "fingers" which grow into one another and mutually influence one another. Indeed, an important

measure of overall data modeling competence might be the degree to which the mutual influence of these phases is internalized in a way that structures planning and execution of each stage.

I. **Early Phases of Data modeling: Designing the Data Model**
   A. Problem Specification
      1. Finding or accepting the problem (determining a goal structure or understanding and accepting a given goal structure)
      2. Refining the problem's goal structure by refining the problem's elements:
         a) Defining data structures (choosing one or many variables, establishing the unit of analysis. More advanced students might consider hierarchical or relational data structures).
         b) Defining categories
         c) Defining relations among categories
   B. Solution Specification
      1. Determining a solution method (e.g. a survey, observations, a data search, etc)
      2. Defining solution instruments or resources; establishing conventions for measurement and coding
      3. Designing sampling and control strategies
      4. Testing or piloting the solution

II. **Middle Phases of Data Modeling: Data Gathering and/or Measurement**
   A. Administering data gathering instruments or executing data search, etc.
   B. Coding data for computer entry
   C. Entering data into a computer database

III. **Late Phases of Data Modeling: Data Analysis**
   A. Exploring and describing the data (with the help of the computer) through charts, graphs, summary statistics, etc.
   B. Analyzing and interpreting relationships in the data
   C. Drawing conclusions, e.g. about causal relationships or the best course of action
   D. Presenting arguments for the conclusions, supported by data displays

To help give the flavor of how these issues actually arise during student inquiry projects, and to illustrate the significant challenges that come up in the design phases of data modeling, we present examples drawn from clinical work in progress.

One group, which we have seen for four one-hour sessions at the time of writing, consists of three fifth-graders of normal ability, experienced with computers but not with our software. They have been investigating student practices and preferences at lunchtime. In an initial brainstorming session the students (with some help from us) drew up a questionnaire which they would administer as interviewers. The first question asked whether the interviewee bought the school lunch, brought his or her lunch from home, or had no lunch. Additional questions established the contents of the meal, which parts were preferred and which parts, if any, were thrown away. The students planned to interview about ten students each on a day when the school cafeteria offered a popular lunch, and again on a day when the lunch was unpopular. The students' initial grasp of the logic of this kind of study was tentative, however:

> In the first session they felt that it was important *not* to interview the same students on two days (they had collected questionnaire data once before, but not in a varied-condition context). When they came to the second session with data for one day's lunch, Kamal and Glenn remarked that they had spoken to students who ate no lunch and had decided, contrary to the now-forgotten original plan, not to record them, since most questions on the questionnaire did not apply. Tyesha, however, had recorded one such case, marking all the other questions with a line. Of course this data would have made it possible to compare proportions of non-lunch-eaters on different days. But after two session devoted to entering the first day's data and beginning to learn how to manipulate it with the Tabletop, the

students seemed on their own initiative to have developed a renewed interest in and a somewhat sturdier comprehension of how aggregating the questionnaire data might help to reveal the difference between the two days.

The initial batch of completed questionnaires included many incomplete or idiosyncratic notations, from which the students (sometimes consciously, more often unconsciously) reconstructed the facts they needed. The data entry phase helped bring many of these issues to light:

> The first question on the lunch questionnaire, for example, asked "bought lunch, brought lunch or no lunch?" Glen tried to make sense of a questionnaire that appeared to say "no" lunch but went on to list favorite parts, etc. It turned out that Kamal had coded answers to this question as "yes" or "no," meaning bought or brought, respectively (he had disregarded those with no lunch). In another questionnaire, rather than enumerate the parts of the meal, another student had recorded simply "school lunch." The students reconstructed by recalling what had been in the standard school lunch that day. Reconstruction was less successful in the case of a lunch brought from home, the contents of which had been recorded as "lots of things." The students also found that they had written "bread" in several places, sometimes meaning the sub roll of a meatball sub and at other times meaning the slice of bread that came with meal. They resolved this ambiguity by coding the latter as "bread and butter." The researchers offered to enter the next round of data for the students, but warned that they would type exactly what they saw on the questionnaires. The second set of questionnaires had far fewer ambiguities.

Some of these issues in the recording and coding of data seem clearly allied with, if not identical to, what writers on literacy have called decontextualization. This is not surprising: once students begin to work with data in situations of purposeful activity with coordinated action by many participants, the appropriate use of the symbols and processes involved must clearly constitute a kind of literacy (Gee, 1989).

Our fifth-graders have not yet had to grapple with category design issues. They did encounter some questionnaires that showed evidence of students having brought some lunch from home and bought more at school. These prompted discussion, but no one has yet suggested modifying the bought/brought/none classification scheme. An older group of students, however, has proved more able to anticipate problems of categorization before collecting data. These three eighth-grade students of above-average ability began by defining the problem: to study the relations between their peers' musical tastes and other personal and academic characteristics.

> As they began to think about the design of a questionnaire a number of category design issues arose, mostly relating to music (categories of students -- "preppy," "burnout," etc. -- seemed less subject to debate). Is rhythm and blues a form of rock? Is there a difference between light rock and classical rock? Does a song need to be old in order to be a "classic?" How do you determine whether a song is hard rock or heavy metal? Is heavy metal a subcategory of hard rock? Interestingly, the word "subcategory" did not appear in the first session, so the last question was actually formulated as "is heavy metal hard rock?" However, that they were thinking in terms of hierarchy is clearly reflected in their use of indentation when writing notes.

More data design issues were forced as they actually began to draft the questionnaire:

> It occurred to them to gather additional data by asking about favorite songs and favorite artists. An important question arose when they needed to decide whether they would request a rank-order of music preferences or use a Likert-like preference scale. After initially deciding to have students list their preferences, they began to worry about how long the form would take to administer. This forced reconsideration of their decision. Their

first revision of the idea was to ask whether the respondent listened to a given category and then only request rank order of those catogories and their subcategories. The messiness of this approach led them to switch to a preference scale which would be used only if the student first responded "yes" that they listened to that type of music. The phrasing of the preference question was briefly problematic, turning on the question of whether to ask how much the student listened to this type of music, or how much they liked it. They decided on the amount of listening. This stimulated a further decision to request the respondent's favorite radio station.

This data is extremely preliminary, given especially that the activities described are still in progress. However, it is already possible to see the importance of junctures in the data modelling process where reflection is provoked. The eighth-graders began to think more carefully about categories when they actually tried to write the questionnaire questions. The fifth-graders began to reflect on the problems in their coding as they tried to enter it into the computer, and they seemed to develop a clearer idea of their multiple-day data collection strategy during a period of "messing about" with their first day's data on the computer. We are beginning to see the iterative process of data modeling, not as a grand loop from data design to data collection to data analysis and back to the beginning, but as a tangle of loops small and large, connected at many such junctures. For experienced data modelers, the issues of each phase are implicitly present at every other phase, with the net effect that looping is minimized (but not eliminated, of course). The less experienced the student, however, the more these loops need to be actually traversed. One challenge to curriculum development is to insert junctures that trigger the right data modeling questions authentically and quickly, so as to conserve students' time and enthusiasm.

The Tabletop's improved support of data analysis should help students become better data designers by helping them better understand what they would like to do with the data, perhaps using the Tabletop's visual representations as guiding images of the desired end result. Procedurally, the Tabletop eliminates previous bottlenecks in the data analysis phase and thus facilitates iteration throughout the entire data modeling process (Data entry is a remaining bottleneck which we hope to widen with future extensions to the software). But perhaps its role is best understood by thinking of the Tabletop not as a tool for doing data analysis, but as a medium for embodiment of the data model itself. It provides a near-transparent interface to the formal system constituted by the data together with the formally allowable operations on the data, which is precisely what we have defined the data model to be. Since all phases of data modeling are concerned with the data model, the central role of the software is thus no surprise. This perspective on the role of technology in inquiry, namely as a medium for the embodiment of models and theories, has guided the design of our software. We hope to test and refine it as our clinical work continues.

REFERENCES

Gee, J.P. (1989). What is literacy? The Literacies Institute tech. rep. #2. Newton, MA: Education Development Center.

Goodman, N. (1978). *Ways of worldmaking*. Indianapolis: Hackett.

Mathematical Sciences Education Board of the National Research Council (1990). *Reshaping School Mathematics: a philosophy and framework for curriculum*. Washington: National Academy Press.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics* . Reston, VA: NCTM.

Taba, H. (1967). Teacher's handbook to elementary social studies. Reading, MA: Addison-Wesley.

Unified Sciences and Mathematics for Elementary Schools (1976). *The USMES guide*. Fourth edition. Newton, MA: Education Development Center

ORIENTATE	Decide to Construct a General Case	
SPECIALISE	Generate Specific Cases	(a) By constructing specific cases (b) By applying unformalised rule to specific cases
	Tabulate Specific Cases	
GENERALISE	Name Parameter	
	Choose Parameter	(a) With no reference to relationship (b) With implicit reference to relationship[4]
	Identify Relationships	(a) After choosing parameters (b) Incorporating implicit choice of parameter[5] (c) Using figurative relationships
	Formalise Relationship	
CONVINCE	Check Relationship	(a) By constructing specific case (b) By applying unformalised rule to specific cases (c) By calculating formalisation for specific cases

Table 1: Critical Decisions in the Generalisation Process

A sample transcript and our identification of the critical decisions made is given below.

Richard and Sunil were working on the Logo research task (Appendix 1). They drew out the head in direct drive, and then wrote a fixed procedure, using the dimensions shown on the task sheet. They then started to construct a general procedure:

S: what we gonna do now? R: we're gonna make it variable........	} **Decide to construct** } **a General Case**

Having decided to write a general procedure, the pair go on to discuss the name of the parameter.

R: move to FACE1 S: FACE1 dot dot something, isn't it FACE1 dot dot R: something like that S: dot dot S, is that what we're gonna call it R: hold it we can have any variable S: lets say S R: lets just have, we had S last time, lets just have S all the time to make it easier S: ok R: S...dot dot	} **Name Parameter**

[4] By this we mean occasions when pupils choose a parameter in a way which suggests that they have identified a mathematical relationship, for example the selection of a common factor.
[5] By this we mean occasions when pupils make no attempt to first identify parameters but nonetheless use them as a basis for defining a relationship; that is the way the generalisation is articulated implies a specification of the parameters.

They added the parameter name to the title line of their procedure (TO FACE1 :S), and started to discuss the choice of a particular value to be represented. They adopt a strategy for identifying the relationship between the dimensions within the head, which involves the recognition of a common factor of 5 (classified as Choose Parameter (b) , see Table 1).

R: FD 45, what are we gonna have as the	}	
S: 5 would be quite small	}	**Choose Parameter**
R: well what's...	}	**(b) with implicit**
S: times 9	}	**reference to**
R: they all go into 5 don't they	}	**relationship**
S: they all go...	}	
R: yeah they all go into 5, go on lets just have it ok why not ...so 9	}	

Having chosen the parameter in a way which also identified the general relationship between the different parts of the head, the pair were ready to formalise this relationship. They do this using a substitution method, that is, for example, replacing FD 45 in their procedure by FD :S*9:

S: delete	}	
R: dot dot	}	
S: dot dot S isn't it	}	
R: dot dot S, S	}	**Formalise**
R: where's that star....times	}	**Relationship**
S: 9	}	
(types :S * 9)	}	
R: ok umm	}	

Discussion of Results

Having identified these critical decisions, as illustrated above, we then approached our research questions by analysing:

- the incidence of critical decisions across environments
- the incidence of collaborative decisions
- pupil roles in identifying relationships and their formalisation in computer environments

Analysis of the results is still ongoing. We present here our preliminary findings, and will further elaborate these in the presentation.

Incidence of critical decisions across environments. For each transcript we calculated the frequency of occurrence of each type of decision and found that the pattern of these frequencies varied across environments. This was particularly evident with regard to the decisions as to parameter choice and decisions concerning the identification of the mathematical relationships embedded in the tasks(see Table 2).

	LOGO frequency	SPREADSHEET frequency	PAPER AND PENCIL frequency
Choose Parameter			
(a) With no reference to relationship	16	2	2
(b) With implicit reference to relationship	8	0	1
Identify Relationships			
(a) After choice of parameter	22	2	9
(b) Incorporating choice of parameter	5	50	16
(c) Using figurative relationships	0	3	9

Table 2:Occurrence of Critical Decisions: Choose Parameter and Identify Relationships

In the Logo environment, the generalisation tends to be constructed in one of two ways: a) choosing a parameter with no reference to a relationship, e.g.

A: FORWARD,FORWARD, do the FORWARD the highest, which is 45, no its not... J: no it... A: it's 70, shall we use the highest J: OK, use 70 as dot dot S then	**Choose Parameter** **(a) With no reference to relationship**

and then using this as a basis to identify the numerical relationships, e.g.

J: if 70 is dot dot s then it's, it must be divided by, can't we say... A: 70 divided by what equals......70 divided by 30 J: equals what?	**Identify Relationship** **(a) After choice of parameter**

b) choosing a parameter simultaneously with the identification of a relationship (Choose Parameter (b) in Table 1). In the earlier Richard and Sunil extract this approach was adopted.

In contrast, in the spreadsheet environment, pupils almost exclusively adopt a strategy whereby they identify relationships incorporating implicit choice of parameter e.g:

J2: look at the numbers going down 1, 5, 12, aaah J1: 4, 7, 10, 13, oh that's right, it's easy.........this one....the gap between the two is always different, so it's 4 then the gap's 7, then the gap's 10, then the gap's 12, 1, I mean, then the gap's... J2: 16 J1: 16, this should be 19	**Identify Relationship** **(b) Incorporating implicit choice of parameter**

This strategy was also very evident in the paper and pencil environment e.g:

R: 2, 3, 4, 5, 6 , that's it that's how you do it S: what? R: look you add 2 to that one then 3 then 4 then 5	**Identify Relationship** **(b) Incorporating implicit choice of parameter**

However additionally, in the paper and pencil environment, unlike the spreadsheet environment, pupils were likely to use their figures to assist in the identification of relationships[6] :

S: keep to side, keep to the side.....from each point there's 4, so you're gonna need it from each point right there's 4 going away right 1, 2, 3 ,4, 1, 2, 3, 4, right see if there's	} **Identify Relationship** } **(c) Using figurative** } **relationships**

Incidence of Collaborative Decisions For each pair, the percentage of collaborative decisions is shown in Table 3. A decision was judged to be collaborative where there was evidence in the transcript that both pupils made some contribution. Looking across all the pairs, the data suggests that the Logo programming environment provided a setting most fertile for collaboration. However, when we analyse the decision making process more closely by identifying the suggestions within the critical decisions that formed the *basis of subsequent action*, we find large between pair differences related to issues such as task involvement and task difficulty. For example, when the task is too easy, there is no need for a high level of collaboration, or indeed task involvement. Consequently one pupil, the "driver", can dominate the solution process, either by making the majority of suggestions which are subsequently acted upon within collaborative decisions, or by making the decisions on their own (for example, Pair 2 in the Logo environment). In these cases, we have difficulty at present in defining the role of the "driver's" partner in any generalisable way. When a task is too hard, there again tends to be individual dominance as above and collaboration from the perspective of both partners is poor – from the "driver's" perspective little appropriate help is received, and from the "passenger's" perspective the solution process is opaque (for example pair 4 in the paper and pencil environment).

	LOGO	SPREADSHEET	PAPER AND PENCIL
	%	%	%
Pair 1: Alice & Joanna	80	44	65
Pair 2:Jamie & Jake	52	67	71
Pair 3: Richard & Sunil	82	48	25
Pair 4: Joku & Simone	44	50	38

Table 3: Percentage of Collaborative Decisions in each Environment

[6] Given the similarity between the paper and pencil and spreadsheet task it would seem that the spreadsheet environment provokes pupils to focus primarily on the given numbers rather than the figures.

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Pupil Roles in Identifying Relationships and their Formalisation in Computer Environments For successful task completion, the mathematical relationships have to be both identified and formalised. In the computer environments this formalisation takes place on the computer. If both identification and formalisation of relationships are exclusively determined in a pupil pair by one pupil the collaboration is unlikely to be effective. For effective collaboration, these processes must be shared in some way. We have found that in the computer based research tasks characterised by a particularly high level of collaboration, pupils have tended implicitly to decide to separate these two process, one making the majority of suggestions as to the mathematical relationships (the "pattern spotter") and the other formalising them. Additionally, we found that the pupils taking on this latter role also tended to dominate the keyboard (the "programmer"). Moreover, in three out of four pairs (pairs 1, 2 and 3) this same individual took on the role of "programmer" in *both* the computer environments.

REFERENCES

Hoyles, C. & Sutherland, R.(1986) *Peer Interaction in a Programming Environment* Proceedings of the tenth PME , London p. 254-260

Hoyles, C. & Sutherland, R. (1989) Logo Mathematics in the Classroom Routledge

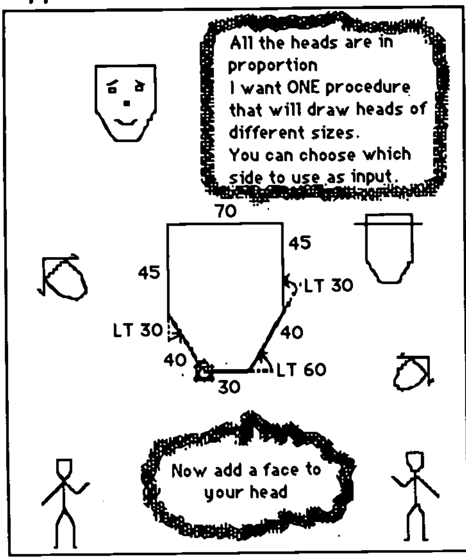
Hoyles, C., Sutherland, R.,& Healy, L. (in press) *Children Talking in Computer Environments: New Insights on the Role of Discussion in Mathematics Learning* . In Durkin, K. and Shire, B., Language and Mathematical Education, OUP

Mason, J with Burton, L. & Stacey, K (1982) Thinking Mathematically Addison-Wesley Publishers Ltd

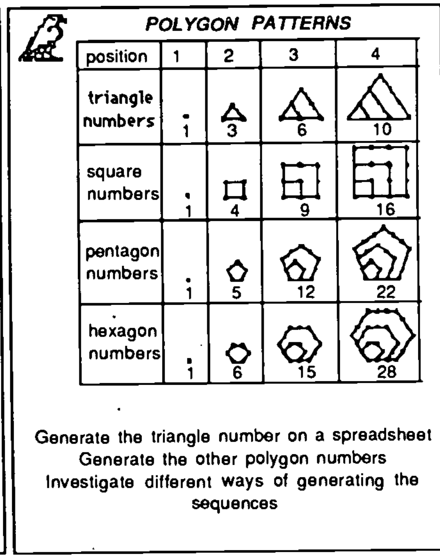
Polya, G. (1957), How to Solve It Oxford University Press

Shiengold, K., (1987), The Microcomputer as a Symbolic Medium, in Pea, R.D., & Shiengold, K., (Eds.), Mirrors of Minds: Patterns of experience in Educational Computing, Ablex Publishing Corporation, New Jersey, USA, 198-208.

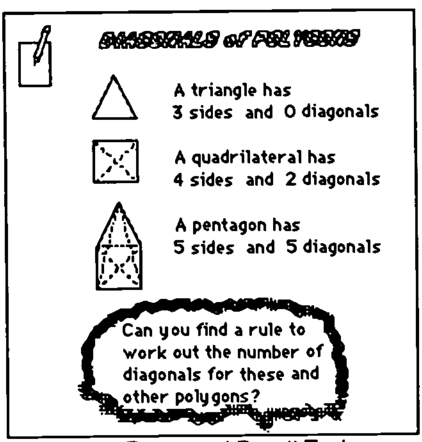
Appendix 1



The Logo Task



The Spreadsheet Task



The Paper and Pencil Task

The criteria for devising the research tasks were that they should: be closed in terms of goal, but open in terms of approach; involve abstract rather than "real life" mathematics; involve a search for relationships defined by specific cases, including both visual images and numbers; involve relationships which are not immediately obvious (e.g. more complex than "times by 2"); involve some element of choice in the variables to be used for defining the relationship.

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"SCAFFOLDING"

- a crutch or a support for pupils' sense-making in learning mathematics?

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In this paper I explore the relationship between teacher and pupil as the pupil strives to make sense of the mathematics which she encounters in the classroom and the teacher endeavours to provide an appropriate level of support. I have taken the word "scaffolding" from the work of Bruner and his colleagues Wood and Ross They use the term to describe a role for a teacher working within Vygotsky's "zone of proximal development", providing a form of "vicarious consciousness" for the learner, and undertaking "contingent teaching". I contrast two possible views of this role; one as encouraging a dependency in the pupil from which it may be hard to break free, and the other as providing a supportive environment from within which the pupil is encouraged to begin growth towards independence. I examine issues which these raise for the teacher and provide cases from my own research which highlight decisions which teachers are called upon to make.

Introduction

Any study of teaching would be sterile without considerations of the learner, and should take account of the immense literature on child development, thinking and learning. However, much of this literature considers the learner alone without emphasising the role which can be played by a teacher or teacher figure. In constrast Jerome Bruner, with particular reference to the work of Vygotsky, offers a view of the teaching role. Bruner, (1985 p25) contrasts a view of the learner as seen by Vygostsky with other prevalent views, in particular interpretations of those of Piaget:

> "Too often, human learning has been depicted in the paradigm of a lone organism pitted against nature - whether in the model of the behaviourists' organism ... , or in the Piagetian model where a lone child struggles single-handed to strike some equilibrium between assimilating the world to himself or himself to the world."

The Piagetian model encompasses some concept of 'readiness', in which a child is unlikely to develop conceptually until such readiness is manifested. In this model there seems little that a teacher can do to help the child progress, and may actually hinder progress by interupting the child's natural development. Vygotsky however, promoted the view that progress could be enhanced by instruction, and his "zone of proximal development" (Vygotsky 1978 p86) provided a measure of potential enhancement:

> "It [the zone of proximal development] is the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers."

Vygotsky uses as an example the idea of two children entering a school, both of whom are aged ten years, but who are eight years old in terms of mental development.

> "Can I say that they are the same age mentally? Of course. What does this mean? It means that they can independently deal with tasks up to the degree of difficulty that has been standardised for the eight year old level. If I stop at this point, people would imagine that the subsequent course of

The task for the pupils involved the building of a tower from a series of interlocking blocks of varying size. In the beginning, according to Bruner (ibid) the tutor is "consciousness for two". Having persuaded the child to engage in the activity, her general task

> "is that of scaffolding - reducing the number of degrees of freedom that the child must manage in the task. ..., she sees to it that the child does only what he can do and then she fills in the rest - as in slipping the pegs of certain blocks into the holes of others to which they are mated, the child having brought them next to each other. She limits the complexity of the task to the level that the child can just manage, *even to the point of shielding his limited attention from distractors.*" (My italics)

Once the child had achieved, with help, one level of mastery, the tutor, with some prudence, could then invite the child to undertake a higher level assembly - in Vygotskian terms, "leading the child on ahead of his development" This not only keeps the child within the ZPD, but "keeps him from getting bored".(Bruner ibid) The teacher is thus seen to provide support for the pupil, but, as my italics above stress, this can encourage the pupil to depend on the teacher for support and thus inhibit her own initiative.

Implications for the classroom

Wood points out that although the 'formula' for contingent teaching sounds simple, even trite, the tutor often violated the agreed rules, sometimes repeating an instruction at the same level when she should have given more help, on other occasions giving help when none was called for. He points out that, "Understanding the 'rules' of contingency and *teaching* according to the rules are two quite different things." Contingent teaching, however perceived, is no straightforward matter for the teacher.

Although on the face of it notions of 'readiness' and those of 'contingent teaching' might appear to be at odds, it seems that the teacher, about to interact with a pupil in the classroom, must, perhaps implicitly, bring considerations of both to the situation. She must be sensitive to the pupil's needs, but prepared to enter some teaching act relevant to these needs. A crucial consideration seems to be of what such teaching acts consist if they are to maximise the potential of the ZPD, and how the teacher knows in any instant what act is appropriate.

Valerie Walkerdine (1988) examines the concept of 'readiness', looking closely at transcripts of classroom discourse involving a teacher and very young children. In one case where the teacher has judged the children 'ready' to consider concepts of 'bigger' and 'smaller' they make a mistake which appears to contradict this readiness, and the teacher has to make an instant decision as to how to respond. Walkerdine analyses the contexts which might have contributed to the mistake and it becomes clear what complexity of thinking and decision making is required of the teacher in providing appropriately for the children's development.

Wood, concluding his remarks on contingent teaching (ibid), made the following observations with regard to classroom teaching, and to mathematics teaching particularly:

> "In the 'real world' of the classroom of course, the problem of achieving contingent instruction is far more difficult. ... many lessons taught in school often involve tasks that do not have a clear, obvious structure ... .Even mathematics, which seems well structured, does not have a single clear cut structure ... most classroom teaching takes place with *groups* of children. ...Does it make sense to talk about contingent teaching in a situation where many children are being taught simultaneously?

Despite such reservations, when I discussed the principles of contingent teaching, of scaffolding and the ZPD with one progressive reflective teacher whom I have studied in depth

mental development and of school learning for these children will be the same, because it depends on their intellect."

He goes on to suppose, however, that he does not stop here but encourages the children to solve problems with his assistance, acknowledging that this might involve any of a variety of methods:

"...some [experimenters] might run through an entire demonstration and ask the children to repeat it, others might initiate the solution and ask the child to finish it, or offer leading questions."

Ultimately he hypothesises that, given this assistance, the first child can deal with problems up to a twelve-year-old's level, whereas the second achieves success only at the level of a nine-year old, and asks the question, "Now are these children mentally the same?"

Vygotsky, while admitting to the naivety of claiming actual development levels, nevertheless claims that these children are not mentally the same age, and that from here onwards their progress would be different. He suggests moreover, that , "what is the zone of proximal development today, will be the actual development tomorrow - that is, what a child can do with assistance today she will be able to do by herself tomorrow." He further claims:

"The zone of proximal development can become a powerful concept in developmental research, one that can markedly enhance the effectiveness and ultility of the application of diagnostics of mental development to educational problems."

Bruner (ibid) paraphrases as follows:

"If the child is enabled to advance by being under the tutelage of an adult or a more competent peer, then the tutor or the aiding peer serves the learner as *a vicarious form of consciousness* until such a time as the learner is able to master his own action through his own consciousness and control. When the child achieves that conscious control over a new function or conceptual sustem, it is then that he is able to use it as a tool. Up to that point, the tutor in effect performs the critical function of "*scaffolding*" the learning task to make it possible for the child, in Vygotsky's word, to internalise external knowledge and convert it into a tool for conscious control." (My italics)

Thus the tutor has to make judgements about the degree of control which a child is capable of assuming at any stage, and the *handover* of control is a crucial part of the scaffolding process.

These notions of 'zone of proximal development' (ZPD), 'vicarious consciousness', 'scaffolding' and 'handover' are attractive metaphors through which we can begin to examine the teaching-learning interface. However, they raise a number of issues which it is my purpose to address.

Contingent teaching

Whatever the attraction of such theoretical notions, their translation into practical situations involving teacher and learner is far from obvious. I quoted above some of Vygotsky's own examples of strategies which a teacher figure might use. Bruner and some colleagues undertook a teaching experiment (See Wood, Bruner and Ross 1976) , using what they called 'contingent teaching', in which they attempted to implement a form of 'scaffolding'. In this a tutor was trained to work *contingently* with pupils. According to Wood (1988):

"Contingent teaching, as defined here, involves the pacing of the amount of help children are given on the basis of their moment-to-moment understanding. If they do not understand an instruction given at one level, then more help is forthcoming. When they do understand, the teacher steps back and gives the child more room for initiative."

(See Jaworski 1989) he felt that many of the principles were ones which he would wish to espouse and that they were not inconsistent with an awareness of 'readiness'. I reviewed data which I gathered in his lessons in this context.

Mathematical Challenge

In my study of a number of teachers, one of my focuses has been that of 'mathematical challenge' (See for example Jaworski 1988). I have been interested in what acts a teacher undertakes to facilitate a pupil's mathematical progress. The teacher above, Ben, claimed on a number of occasions that he made judgements about how to respond to pupils, or what to offer them, according to his perception of their *readiness* for it. His intervention with particular pupils or groups of pupils varied considerably according to this. Sometimes it seemed particularly fierce, as in the following example involving a pupil Rachel who had been working on a problem which she had selected to tackle as part of her GCSE course work.

The problem itself is immaterial to the discussion which follows. She had been working on some particular cases from which she was trying to find a general pattern. Ben had been working with other pupils at Rachel's table, and before leaving he looked over at what she was writing. She looked up questioningly, and he said, "Yes, Rachel, what's next?". She replied, "I'm just doing some more of these", referring to her examples which she had discussed with him earlier. He replied,

B Do you know you actually haven't proved it? You've just shown that it could possibly be true. Can you think, - what *is* your conjecture? Can you give it me in words?

R Erm, you add up the perimeter, and add four on - add four on, and that gives you the number of -

B Hang on, you've got a proof there in the making. (A distinct change in the tone and pace of his words occurs here) You've nearly said why it's true, haven't you?

R Have I?

B Why you add four on, why do you add four on?

R There's four corners. It's got to be an extension.

B Could you write that as a proof to show your conjecture is actually true, yes?

One level of my analysis[1] of this excerpt can be summarised as follows. The teacher's focus here was on *proof*. He judged Rachel to be at a position to think about proof and challenged her accordingly. In responding to his instruction to express her conjecture in words, she convinced Ben that in fact she was close to proving her assertion. His tone of voice conveyed his excitement and pleasure when he realised that she was so close to a proof.

At first glance there seems to be very little of scaffolding taking place here. Almost immediately the teacher seems to hand control to the pupil in terms of proving her conjecture. Yet looking at the four sentences of the teacher's first words, which are uttered in a fairly

[1]The analysis of a piece of transcript such as this is very complex and relies heavily on contextual considerations and perceptions of the observer and participants. Analysis would be at a number of levels - higher levels including consideration and justification of analysis at lower levels. These are major considerations of my PhD Thesis on which I am currently working and from which the above extract is taken. I offer analysis rather superficially here without the weight of such considerations, partly as they could take up the rest of my allotted space, but also because they would detract from my main argument in this paper.

slow, low key, measured manner, it is possible to see a rapidly developing structure of support:

1. *"Do you know you actually haven't proved it?"* - drawing her attention to proof;

2. *"You've just shown that it could possibly be true."* - qualifying his rather bald assertion in terms of her work on examples.

3. *"Can you think, - what is your conjecture?"* - hesitatingly beginning to provide support. How can he help her to see what he means?

4. *"Can you give it me in words?"* - a specific request with which she can comply. He has brought the focus on proof into a task which is within her grasp.

The pupil's response is exciting for the teacher. His tone of voice when he replies to her is quite different from when he uttered the above sentences. In her short, poorly articulated response there is enough for the teacher to gain insight into her thinking and observe that she is close to what he wants in terms of proof. However, his talk of proof perhaps does not accord with what she sees herself doing, because she asks in surprise, *"Have I?"* He responds by further narrowing the focus, asking, *"Why do you add four on?"*

He leaves her to *"write that as a proof to show your conjecture is actually true"*. We do not know what she actually wrote, or what further help was given. It is possible to envisage, with different judgements by the teacher, what further help might have been given at the stage describe above. For example, the teacher might have been drawn to explain in more detail what he meant by proof.

What seems interesting is where the pupil was left. It is impossible to judge whether this was appropriate or inappropriate for *her*. What it offered was a chance for her to use her own thinking to decide what sort of explanation to provide and begin to be conscious of the nature of this as a proof. Further explanation from the teacher about the nature of proof may have made it clearer to her what he particularly required, but could simultaneously have taken away her own necessity to judge for herself and develop her own concept of proof.

The above scenario could be construed in terms of Rachel's ZPD. Without the teacher's remarks she may have gone no further than vague expressions of generality from a number of particular cases. With the introduction of the idea of *proof* she could begin to think in those terms, starting to become aware of what a proof might mean and opening up the opportunity for looking for proof when tackling further problems. The scaffolding which the teacher's remarks provided could thus have an enabling effect.

Making Judgements

The teacher was well aware of his making of judgements and their potential implications, and we discussed this overtly on many occasions. For example, at the end of one lesson I asked him about a particular response which he had made to one pupil in the lesson. In expressing his reasons for it, he indicated that he had been aware of a number of possible ways to respond,, and had made an instant decision. He said, "it's this business of *judgement* again, isn't it?" We contrasted, on another occasion, the potential dichotomy between planning and sponteneity in a lesson, and I remarked how often the lesson seemed to go very much the way he had suggested to me that it would, although he appeared to be spontaneously responding to pupils remarks and questions. He replied,

"Now, that could be interpreted in two ways. It could have been interpreted that Ben has a plan in his mind and he's going to get there irrespective of any obstacles placed in his way. Ben sticks slavishly to his planning and won't even be pushed off. Or is the other way of looking at it that Ben knows his group fairly well and can fairly predict their reactions? I think we're getting nearer the second."

I asked if he planned in a way that left flexibility. *Flexibility* had been his word originally, but he paused and then said,

"I think you'll have to expand on flexibility. If you mean outcome, *no*. Because at some point I have got to get round the idea of surface area - at some point in the future. If it's today or tomorrow, it's flexibility.

If it's a worthwhile mathematical excursion - why not go. It's got to be worthwhile. Then it's back to judgements, making decisions, letting things go. We keep coming back to that today."

And on yet another occasion he said,

"Yeah, teaching's a lot of judgement as I call 'on the hoof'. You're making a lot of judgements as you go along.

I think a lot of the time you don't have the time to sit back and have the luxury of saying, 'yes, I'll make a decision'. I think they're there. The judgements are there

On what basis? On *need* I suppose. On *perceived* needs - I would say that."

Interpretation of 'needs'

As a counterpoint to 'Mathematical Challenge', another focus of my research has been that of 'Sensitivity to Students' (See Jaworski 1988). The relationship between these focuses has proved a central feature in my study of various teachers. Ben's reference to 'perceived needs' is part of what I came to see as his knowledge of particular pupils and his consequent degree or style of challenge. However, identification of a pupil's needs is highly interpretive, and it is important to examine the basis of an interpretation.

Walkerdine (1988) implied that a concept such as 'readiness' could be used as a panacea to avoid searching out particular reasons for children's mistakes. Suggesting that the children were after all not 'ready' for something could avoid having to look for deeper reasons. A teacher's perception of the 'needs' of a pupil can similarly be used as a panacea for justifying whatever action was taken. For example, "She needed an explanation, so I gave it." This could hide a severe case of teacher-lust of the form, "This was an opportunity for me to give an explanation about which I could feel good afterwards, and so I launched in without further thought." It might have been, in this highly hypothetical (and provocatively described) situation, that what the child would most benefit from was some encouragement to produce such an explanation herself.

Another example from Ben's classroom illustrates this point. Pupils were seeking triangles which had numerically the same area as perimeter, and their approach had been left to them. One girl had drawn a number of triangles from which she was roughly calculating area and perimeter. One of these triangles had its sides labelled 3, 6 and 9 cm. Ben came and looked at this and said to her:

T If I've got a length of line nine, it must lie straight over the top, so I can't actually make a triangle like that.

It seemed as if this logic should immediately alert her to a mistake. However, the conversation continued,

P Yeah, but its a triangle

T That's not accurate is it?

P Yeah, There's one side and the other and the other

T That's three and that's nine?

P Yeah

T No, it's eight and a bit

P Well!

T Now hang on. If that becomes nine, it's got to go a bit longer.

P Yeah, but that means that's got to be longer

T So that must bring it down to make it longer doesn't it?

P Yeah, well that's what I was doing

T Now, the only way you can make it surely, if that's six and that's three, that's the only way you can do it to make it nine is to make it into a line isn't it?

P Well, that's nearly right isn't it?

With some noncommital sound he moved on at this point, and I was quite surprised that he had not tried some other form of explanation. In discussion later, Ben said, "there's no more I could say in that moment that could convince her". He had made a conscious judgement *not* to be drawn into further explanation. He knew that at some stage he would have to reintroduce the concept of an x, y, $x+y$ triangle with her, but judged that further discussion at this point was inappropriate.

It is possible to view the teaching of this concept as one in which a *contingent* approach might usefully be employed, - perhaps in introduction by the teacher of a number of triangles which the pupil might try to draw, in progressing from the teacher explaining to gradually prompting the pupil to explain, and finally in asking the pupil to suggest some 'impossible' triangles herself. It might be believed that leaving the pupil in her misconception was irresponsible of the teacher, and that there was some *duty* to take the matter further at that point.

Such moral issues have to remain within the province of the teacher concerned, as no one else is knowledgeable enough to make judgements. However, we can speculate on the outcome of such a contingent approach. It is likely that the pupil would gain a demonstrable perception of the triangle concept which would satisfy the teacher in that instant. Thus the contingent teaching could be seen to have been successful. It is also likely that the pupil's dependence on the teacher would have increased as a result. It is a tension of the teaching/learning interface that what is seen as the most helpful input in a teaching moment might ultimately be the most inhibiting (See for example Mason 1989). If the pupil learns to expect that the teacher will cushion the thinking, making concepts 'easy' to perceive, then the pupil is likely to grow into an inability to struggle to make sense alone.

A crutch or a support?

Crucial to the notion of *scaffolding* is that of *handover*. In the hypothetical case above, handover could be achieved *locally* as the pupil demonstrated her ability to produce her own 'impossible' triangles. In the Wood, Bruner, Ross experiment, handover involved the child locally demonstrating ability to construct a tower more complex than had been built with the tutor's help. My reading of scaffolding in the literature has led an interpretation of handover in this *local* sense. The issue as I see it is that scaffolding, perceived locally, may encourage learning of a particular concept, but is likely also to inhibit independent thinking and development by encouraging a dependency on the teacher's support. Thus it becomes a crutch on which the pupil develops increasing dependence.

Contrary to perceptions of 'readiness' which presume that development precedes learning, Vygotsky's theory is that development *follows* learning via the ZPD. The ZPD *can* be perceived *locally* according to a particular concept or task. However, there seems to be no need

for this, nor does Vygotsky seem to suggest it. Vygotsky rejects theories of the Gestalt school which suggest that learning *causes* development; that "the child, while learning a particular operation, acquires the ability to create structures of a certain type regardless of the diverse materials with which she is working and regardless of the particular elements involved." (1978 p83) One objection is that research has shown that learners do not automatically generalise experiences in this way. Rather, according to Vygotsky, what is learned will be shared, through peer interaction and conversations with peers and adults, allowing an assimilation and modification of experience from which development proceeds. He claims that *schooling* is a vital element in the movement from learning to development, as an environment can be created which supports and facilitates this process. However, he recognises that the formal discipline in which learning of school subjects leads to development requires, "extensive and highly diverse concrete research based on the concept of the ZPD" (p 91)

To return to *scaffolding*; perhaps the scaffolding process can be extended to approach this formal discipline of which Vygotsky speaks. Two essentials of this discipline seem to be, firstly that the pupil is not encouraged to become dependent on the teacher to instigate and regulate thinking; secondly that the pupil is able to generalise particular learning experiences to more general thinking strategies. This requires a lifting of the scaffolding process to a level at which the learned act, for example the building of a tower or the recognition of an impossible triangle, is just one of the experiences that the teacher uses in order to promote learning initiative. The teacher needs also to be able to point to occurrences of the pupil having made intuitive leaps, or having struggled with apparent contradictions to reach order and consistency as instances of successful achievement. Implicit in this is that the teacher needs to be as ready to withhold comfort as to provide it. In either case it seems important to draw the pupil to an awareness of what has taken place - "What was the result of my giving you that help?" or "What was the result of my *not* giving you that help?"

It could be that the scaffolding metaphor is inadequate for the process which I suggest, in that it carries too much of the sense of dependency or of a crutch. However, often a crutch is something which a person casts aside when a limb has become strong. Without any doubt pupils depend on the help of their teachers. Ultimately they have to leave this behind. In my paper 'To inculcate versus to Elicit knowledge' (Jaworski 1989) I gave examples of a teaching approach which I feel goes some way to 'fit' (See von Glasersfeld 1984) a potentially extended view of scaffolding. I invite those who are interested to explore this extension.

References:

Bruner J.S (1985) 'Vygotsky: a historical and conceptual perspective'. In J.V.Wertsch (ed), *Culture Communication and Cognition: Vygotskian Perspectives.* Cambridge: Cambridge University Press.

Jaworski B. (1988) 'One Mathematics Teacher' *Proceedings of PME 12, Hungary*

Jaworski B. (1989b) 'To inculcate versus to elicit knowledge.' *Proceedings of PME 13, Paris*

Mason J. (1989) 'Tensions' ..In D. Pimm (ed) *Mathematics Teachers and Children.* London: Hodder & Stoughton

von Glasersfeld E. 'An introduction to radical constructivism'. In P. Watzlawik, *The Invented Reality.*London W.W. Naughton & Co.

Vygotsky L.S. (1978) *Mind in Society. The development of the higher psychological processes.* London: Harvard University Press

Walkerdine V. (1988) *The Mastery of Reason. Cognitive development and the production of rationality.* London: Routledge.

Wood D. (1988) *How Children Think and Learn* Oxford: Basil Blackwell

Wood D.J. Bruner J.S. & Ross G. (1976) 'The Role of Tutoring in Problem Solving' *Journal of Child Psychology and Psychiatry*, 17, 2, 89-100

THE ROLE OF MATHEMATICAL KNOWLEDGE IN CHILDREN'S UNDERSTANDING OF GEOGRAPHICAL CONCEPTS

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This study examines the relationship between children's procedural and conceptual understanding of mathematics and their accuracy in interpreting geography text material containing mathematical information. It is expected that inexperience with specific mathematical concepts and/or children's mathematical misconceptions will be associated with inaccurate interpretations of geography content. It is also anticipated that mathematical competence will not necessarily be applied to reasoning about mathematically related geographical concepts. Sixty-four children, 16 in each of grades 3 - 6 are being interviewed about related topics in mathematics and geography to test these hypotheses.

Recently there has been a great deal of concern expressed regarding Americans' geographical illiteracy. Studies indicate that serious gaps and misconceptions in geographical knowledge are evident in students' performance at all ages (Daniels, 1988; National Geographic Society, 1988; Solorzano, 1985). One factor contributing to this problem may be the failure to view the acquisition of geography concepts in the context of students' understanding and application of knowledge from other academic content areas (Adler, 1989; Blaut & Stea, 1971; Downs, Liben, & Daggs, 1988).

In particular, students' knowledge of mathematical concepts and procedures seems to be a critical variable in developing an appreciation of many geographical ideas. A survey of geography curricular materials supports this contention and indicates that many geographical concepts, indeed, presuppose knowledge of particular mathematics concepts. For example, in third grade (Silver Burdett & Gin, 1988) children are instructed about how maps are "drawn to scale" such as "1 inch to 20 feet," yet the concept of ratio does not appear in most mathematics curricula until later on in sixth grade. In

addition, even if the mathematics has been "taught" it does not necessarily imply that children have attained an accurate knowledge of given concepts and procedures (Baroody, 1987; Ginsburg, 1989). Further, it has yet to be demonstrated that even when children have attained competence in a particular mathematical content area that they can apply this competence to another context.

This research examines the relationship between students' knowledge of mathematics concepts and procedures and their ability to understand and interpret geography content in which these mathematical concepts are embedded. It is hypothesized that a)children's misconceptions or lack of experience with particular mathematical content areas will be associated with inaccuracy and misunderstanding of geographical concepts and that b)children who demonstrate competence in particular mathematical concepts and procedures will not necessarily apply this knowledge to related geographical contexts.

The subjects of this study are 64 students, 8 boys and 8 girls from each of grades 3 through 6 attending a middle class public suburban school district in New Jersey. Within grade and sex, the students were randomly selected from over 200 volunteers from two schools in the district. Each student is being individually interviewed outside the classroom for approximately 45 minutes. All interviews are being videotaped.

In the first part of the procedure students are asked to read two short excerpts from a grade-level geography text. One contains content dealing with knowledge of maps and the other is related to information about the population, climate, or industry of a given geographical area. For each excerpt students are asked a)factual information questions based on the

content, b)interpretive questions that go beyond the given information in the text and that require the application of some mathematical knowledge, c)definition questions about some mathematically loaded terms used in the text, and d)concept extension questions in which the same map and graphing concept(s) described in the text need to be applied to an analogous situation.

In the second part of the procedure students are asked to work out computational examples for each mathematical concept embedded in the geography text, to demonstrate their understanding of the computational procedures, to solve word problems involving the application of the same computational procedures, to construct the solution to a geometry or measurement problem, and to interpret a graph problem in a non-geographical context.

All students in each grade are asked the same core questions. However, clinical interviewing procedures are utilized to clarify students' responses and identify misconceptions in their reasoning. Within both the geography and mathematics tasks, subjects are evaluated on accuracy of answers and/or procedures used, appropriateness of reasoning and solution strategies used regardless of accuracy of execution, and on the type and frequency of misconceptions expressed. An accuracy/appropriateness score of 0, 1, or 2 is obtained for each item and total scores are obtained for all items in each domain and for subsections of items within each domain. Based on total scores within domains, subjects will also be divided into high and low accuracy groups.

Data analysis will focus on the correlational relationship between knowledge of particular mathematical contents and the

attainment and application of specific geographical concepts at each grade level. It also will focus on a comparison of the types of misconceptions held by students in high and low accuracy groups. Trends across grade levels will be examined.

This study is currently in progress and data have not yet been analyzed. Complete results will be available in time for reporting at the scheduled meeting in July, 1990. These should indicate that mathematically inaccurate children have lower accuracy scores in factual geographical information than children who are accurate in their mathematical concepts and procedures. They should also demonstrate that children's mathematical misconceptions interfere with the application and interpretation of geographical concepts. Further, it is expected that children who demonstrate accurate knowledge of mathematical concepts and procedures will not necessarily be able to apply this knowledge to geographical contexts.

The results of this study should have significance in several areas. First it will add to our general understanding of how children's existing knowledge base interacts with school curricular content. Second it will expand our knowledge about how children are or are not able to apply concepts from one domain to another (in this case mathematics to geography). Third, the videotaped illustrations obtained during the data collection can be used in teacher education programs to demonstrate to practitioners that there are a variety of ways in which students interpret "objective" content and that effective instruction must take into account the fact that academic subject areas often overlap with one another.

References

Adler, T. (1989). Site visit to LRDC educates Hill staff. APA Monitor, January, 7.

Baroody, A. (1987). Children's mathematical thinking: A developmental framwork for preschool, primary, and special education teachers. New York: Teachers College Press.

Blaut, J. & Stea, D. (1971). Studies in geographic learning. Annals of the Association of American Geographers, 61, 387-393.

Daniels, L. (1988). Finding new ways to make geography exciting. The New York Times, March 22.

Downs, R., Liben, L. & Daggs, D. (1988). On education and geographers: The role of cognitive developmental theory in geographic education. Annals of the Association of American Geographers, 78, 680-700.

Ginsburg, H. (1989). Children's arithmetic, second edition. Austin, TX: Pro-Ed.

Loftin, R. & Ainsley, W. (1988). Our country's communities. Morristown, NJ: Silver Burdett & Ginn.

National Geographic Society. (1988). An international Gallup survey. Washington, DC: National Geographic Society.

Solorzano, L. (1985). Why Johnny can't read maps, either. U.S. News & World Report, March 25.

SPEAKING MATHEMATICALLY IN BILINGUAL CLASSROOMS:

AN EXPLORATORY STUDY OF TEACHER DISCOURSE

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The education of Spanish-speaking students in the United States is often inadequate, particularly in mathematics. This project investigated linguistic factors which might help or hinder the acquisition of mathematical knowledge by students with limited English proficiency (LEP). Data from a qualitative study of four elementary classrooms suggest that Spanish is seldom used to develop mathematical understanding, even by bilingual teachers of LEP students. When Spanish was used, linguistic errors often hampered student learning.

Recent data suggest that Hispanic students in the USA continue to perform below national norms in mathematics. The purpose of this study was to investigate some of the language factors that might contribute to this pattern of underachievement. The focus of the study was on the language used by the teacher, with particular emphasis on his/her mathematical explanations.

The role of teacher discourse has played an important part in research on teaching (Cazden, 1986). In studies in mathematics education (e.g., Good, Grouws, & Ebmeier, 1983) the nature of teacher explanations, especially in the development portion of the lesson, has been a major focus. This study provides additional information on critical aspects of the teacher's use of language in bilingual classrooms.

Conceptual Framework

The analysis of teacher discourse has become an important part of research on teaching. This kind of research began with the specification of categories of teacher speech acts and their frequencies, but the paradigm

has since shifted to an interactionist perspective in an effort to unite the cognitive and social dimensions of learning. Mehan's (1979) study of lessons provides an important example of the analyses of actual classroom dialogues and how language use structures lessons. In research on mathematics teaching, Leinhardt (1988) emphasizes the role of routines, scripts, and agendas in structuring teacher discourse and classroom interaction.

Along with the structure of classroom discourse, there are linguistic factors that have implications for research on mathematics teaching. Pimm (1987), for example, notes the complex relationships between mathematics and language, and uses linguistic analyses to illuminate how communication takes place in mathematics classrooms. One of the tools in his analysis is the notion of "register", the use of natural language in a way which is particular to a role or function (Halliday, 1978). Mathematics has its own register (Cuevas, 1984). This register is not just special terminology but also a set of unique meanings and structures applied to everyday language. The development of a mathematics register is accomplished in many ways, including reinterpretation of existing words (e.g., carry, borrow, reduce) as well as the introduction of new terminology (e.g., common denominator) (Halliday, 1978). In addition, certain syntactic and semantic structures are characteristic of mathematics. For example, there are four semantically different verbal subtraction problems, each of which could be solved with the same symbolic sentence (Moser, 1988). Furthermore, a common mathematical question is: How many are left? When a Spanish-speaker interprets this, "left" can become linguistically redundant and can easily be confused with its directional meaning.

Lastly, effective instruction for non-English or limited English

proficient (NEP/LEP) students requires the use of the students' native language particularly for concept development, the integration of English language development with academic skills (Tikunoff, 1985), and most importantly, teaching strategies utilizing interaction and context (Cummins, 1986). In teaching mathematics, attention must be given to clarification of terms since there are differences in ways of expressing mathematical concepts in Spanish and English which can cause confusion in comprehension (Cuevas, 1984).

The importance of these points is that language issues in the teaching and learning of mathematics may be more crucial than previous research would suggest. Further study of these factors is particularly important in the context of bilingual classrooms.

Methods

Four classrooms were selected from elementary schools that have significant numbers of Hispanic non-English and limited English proficient (NEP/LEP) students. Two classrooms were chosen from the primary level and two from upper level elementary classrooms.

Each classroom was video taped for seven to ten hours on days when the teacher indicated that new concepts such as place value and rational numbers would be explained. Formal interviews were conducted with each teacher regarding personal and teaching background, academic and language backgrounds of his/her students, and perceptions of teaching mathematics. Informal interviews were conducted with some students to assess their grasp of the mathematical meanings presented in the lesson and to enhance our observations.

The analyses of video tapes, the primary source of data, were guided by the following constructs: a) the nature and use of the mathematical

register; b) the nature and use of L1 (Spanish) and L2 (English) and the comprehensible nature of L2; c) the use of language for the negotiation of meaning or to emphasize rote learning; and d) the clarity or ambiguity of language in concept development. Triangulation among three independent observers was used to provide validation of the items deemed to be linguistically troublesome.

One teacher did not speak Spanish, but was known as a highly effective teacher of mathematics even with LEP students. The other teachers were bilingual instructors. One of them was not a native speaker of Spanish but did have extensive academic training in the language and had lived in a Spanish-speaking country. Another teacher had all of her schooling including higher education in Mexico. Both of these teachers taught at the same primary grade level. The third teacher was Hispanic but had had most of his schooling in the United States and had not maintained a high level of fluency in Spanish.

Results

Analyses of the data present three striking patterns of teacher discourse. The first relates to the differences among teachers in their efforts to develop the mathematics register. Effective techniques included emphasizing meanings by variations in voice tone and volume, pointing to written words as they were used orally in order to highlight differences in meaning, and frequent "recasting" of mathematical ideas and terms. Less effective techniques were characterized by missed opportunities to establish and clarify the mathematics register. For example, during an explanation of place value and regrouping, the Spanish word "decena" (meaning a group of ten) was used. This is a specialized word unfamiliar to young students and is very similar to "docena" which means dozen. The two spoken words can

easily be confused, particularly if the teacher's accent is difficult to understand. Such a misunderstanding can make the discussion incomprehensible. In another example, also having to do with regrouping, the teacher intended to convey to the students that once they had counted ten items to a group, they had to start again with a new group. However, what was said in Spanish was "no se cuenta mas el diez" which means "don't count the ten anymore," or eliminate ten as a counting number.

Second, very little Spanish was actually used by two of the bilingual teachers. What they used fell into two categories which we are calling "instrumental use" and "markers of solidarity." One teacher used Spanish in a perfunctory manner as an "instrument" to discipline and call students' attention to the subject of the lesson, or randomly, to "punctuate" a statement. The other teacher used Spanish to give encouragement and to motivate the class; it was also used when the teacher worked individually with a student as a private but shared mode of expression. In both of these classrooms, very few whole thoughts were conveyed in Spanish, the only language of proficiency for some students. Overall, Spanish was not used when mathematical meaning was being developed.

Thirdly, with the exception of one classroom, very little contextualized instruction and verbal interaction occurred during the mathematics lesson. Consequently, students were expected to learn by listening, contrary to recommendations by Cummins (1986).

Conclusions

The application of communication and interaction constructs to the teaching and learning processes in mathematics is relatively new theoretical ground, particularly as it relates to bilingual students. As such, a study of this nature generates new insights and variables for further research.

As we have seen from the foregoing discussion, the characteristics of language use which emerged suggest that bilingual teachers need help in strategies for developing the mathematics register in both languages and in making appropriate use of the students' first language. Furthermore, the general nature of mathematics instruction with bilingual students is consistent with what is generally known from observations of mainstream classrooms, i.e., students work individually and in silence. This is a critical dimension since effective instruction for bilingual students requires a verbally rich environment.

These patterns of language use also raise some salient questions. On what basis do teachers choose to use Spanish in instruction? How do they check whether their English is comprehensible and adjust accordingly? Finally, how does the teachers' expertise in mathematics affect the use of the register in the development of mathematical meaning? It is hoped that answers to these and other questions will enhance our understanding of the effects of language and communication factors on the development of mathematical ability.

Preparation of this paper was supported in part by National Science Foundation Grant No. MDR-8850535. Any opinions, conclusions, or recommendations are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Cazden, C. (1986). Classroom discourse. In M.C. Wittrock (Ed.) Handbook of research on teaching, 3rd Edition (pp.432-463). New York: Macmillan.

Cuevas, G. (1984). Mathematics learning in English as a second language. Journal for Research in Mathematics Education, 15, 134-144.

Cummins, J. (1986). Empowering minority students: A framework for intervention. Harvard Educational Review, 56, 18-36.

Good, T., Grouws, D, & Ebmeier, H. (1983). Active mathematics teaching. New York: Longman, Inc.

Halliday, M.A.K. (1978). Language as social semiotic. London: Edward Arnold.

Leinhardt, G. (1988). Expertise in instructional lessons: An example from fractions. In D.A. Grouws & T.J. Cooney (Eds.), Effective mathematics teaching (pp. 47-66). Hillsdale, NJ: Erlbaum.

Mehan, H. (1979). Learning lessons. Cambridge, MA: Harvard University Press.

Moser, J. (1988). Arithmetic operations on whole numbers: Addition and subtraction. In T.R. Post (Ed.) Teaching mathematics in grades K-8, (pp. 111-145). Newton, MA: Allyn and Bacon, Inc.

Pimm, D. (1987). Speaking mathematically. New York: Routledge & Kegan Paul.

Tikunoff, W. (1985). Applying significant bilingual instruction in the classroom. Rosslyn, VA: National Clearinghouse for Bilingual Education.

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THE EMERGENCE OF MATHEMATICAL ARGUMENTATION IN THE SMALL GROUP INTERACTION OF SECOND GRADERS

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An analysis of a videotaped small group situation of second graders taken from the Purdue Problem Centered Mathematics Project of COBB, WOOD, and YACKEL will be presented. The main focus is on the interactively constituted process of argumentation. The emergence of mathematical argumentation will be seen both as a result of social interaction and as an individual process of understanding the argument in terms of one's own cognitive structure. Theoretically this analysis is based on the attempted coordination of a psychological perspective with a sociological one.

In recent research the importance of small group activities in education and especially in mathematics education has been emphasized (Slavin, Sharan, Kagan, Hertz-Lazarowitz, R., Webb, C. & Schmuck, R., 1985). However, the reality of everyday mathematics classroom situations shows many difficulties in effectively stabilizing small group interaction as a usual form of learning in school. In West Germany, as in the United States, for example, the main form of classroom interaction is still organized as so-called "frontal-teaching" (Hoetker & Ahlbrandt, 1969; Hopf, 1980; Mehan, 1979; National Research Council, 1989). Small group activities in mathematics classes are very often used for routinized practice in the teaching-learning process and not for peer-activities about learning new mathematical concepts

This research was supported in part by the National Science Foundation under grants No. MDR-874-0400 and No. MDR-885-0560. All opinions expressed are solely those of the authors.

(Slavin, 1986). This is not the place to discuss all the problems which are possible obstacles in the institutionalization of small group activities as a regular learning setting. Here the focus is on the process of argumentation in small group interaction while doing mathematical problems.

In the Purdue Problem Centered Mathematics Project of Cobb, Wood, and Yackel, the classroom is organized so the fundamental learning processes are to take place in the process of small group interaction, while consolidating the new learned content is placed in the whole class discussion (Yackel, Cobb, & Wood, in press, p.5). In this paper we illustrate the nature of mathematical argumentation and justification in small group collaborative work by analyzing an episode in which two children are solving a sequence of multiplication sentences. Thereafter we will discuss the relation between argumentation and learning mathematics. The analysis coordinates a psychological and a sociological perspective. The nature of the children's individual conceptualizations of the tasks will be discussed and will be seen to be influenced by the social interaction between the children. The presence of an adult observer further influences the social interaction. While the observer's intent is simply to ask for clarification so that he understands each child's solution method, his comments serve the function of eliciting explicit mathematical arguments and justifications from each child which they may not have otherwise given. As the analysis will show, these justifications give insights into the children's evolving conceptualizations of the tasks. Over the course of the episode we see that the children use several different types of mathematical argumentation. We will discuss two of these to show how the children's mathematical conceptualizations and the social interaction between the children influence which form of argumentation

each child uses on a specific task as well as to show the evolution of one child's understanding of one of the forms of argumentation.

An illustration of mathematical argumentation in small group work

The episode is taken from a small group problem solving activity in which the children John and Andy are working on the following problems: 2 x 4 = _; 4 x 4 = _; 5 x 4 = _; 10 x 4 = _; 9 x 4 = _; 8 x 4 = _; 8 x 5 = _; 7 x 5 = _; _ x 5 = 30. The problems are written one under the other on a single worksheet.

The children generate a number of explanations of these problems. From the stance of mathematical argumentation the types of explanations are not all different from each other. We can construct several types of mathematically different argumentations in this interaction. Only two will be discussed here. These are the arguments of: (A1) counting up in steps or in the rhythm of the multiplier and (A2) relating the problem to one of the previously solved problems. Two examples are given to clarify this.

Argumentation A1: For the problem "8 x 5 = __", John counts in steps c five, "5, 10, 15, 20, 25, 30, 35, 40."

Argumentation A2: For the problem "8 x 4 = __" John suggests taking away 4 from the result of the problem "9 x 4= __":

By way of background it is necessary to say that this episode is taken from the first day these children had encountered the multiplication symbol. The teacher explained the symbol in terms of sets. For example, 5 x 4 means five sets of four.

John and Andy solve the first task by saying "Four and four is eight." This is a direct use of the meaning the teacher gave for the "x" symbol and is an example of argumentation A1. For the next two problems John uses argumentation A2 while Andy uses argumentation A1. These two forms of argumentation are almost indistinguishable because of

the numbers involved. The difference in these approaches is much more apparent on the problem 10 x 4 = __ and it is here where it is especially evident that the social interaction becomes significant. At the beginning of the following episode John's approach is a form of argumentation A2.

John: Oooh! Just 5 more than that [5 x 4].

Andy: No. No way!

John: No, look. It's five more sets [of 4]. Look.

Andy: Yeah.

John: Five more sets than 20.

Andy: Oh! 20 plus 20 is 40. So its gotta be 40.

The initial misunderstanding about what John means by "five more" is evidence that Andy is not thinking in terms of argumentation A2. In fact, even after producing the result "40" Andy generates an (incomplete) argumentation of the first type. For Andy, "40" is simply the answer to 20 plus 20. He still must generate a solution to 8 x 5 = _ for himself, which he does by counting by fours. He pauses at 28 and coincidentally John repeats the answer of "40". Then Andy repeats "40" and the observer intervenes to clarify for himself what solution method each child used. The intervention of the observer has the effect of requesting an explanation before Andy has completed his argument. Thus, instead of describing his incomplete solution, Andy describes the first solution method which was suggested by John but computed by Andy. His halting explanation suggests that his understanding of it as a valid justification is tentative.

Andy: 5 plus 4 is -- 5 times 4 is 20, so just 5, I mean 20 more than that makes 40.

On the next problem, 9 x 4 = __, Andy again suggests thinking in terms of sets of four, the first type of argumentation, but John suggests the second type.

Andy: 9 times 4. 9 sets of 4.

John: Just take away 4 from that [10 x 4].

Andy: Thirty -- six....I get it.

We see that John's repeated use of the second type of argumentation has some effect on Andy. To solve 8 x 4 = __ both children use the second form of argumentation, but they differ in its application.

John: Look! Look! Just take away 4 from that [9 x 4]. Just take 4 away from that [9 x 4] to get that [8 x 4].

Andy: Just take away 2 from there [10 x 4]. Take 8 away from there [10 x 4].

...

Observer: And how did you do it John? Did you do it the same way?

John: Yeah, same way.

Observer: Okay.

John: But I used that one [9 x 4]. Take 4 away. It makes 32.

Again, the observer's intervention provides clarification. John's reply to the observer shows that he realizes that they both used the same type of argumentation.

At this point we might conclude that through the interaction with John, Andy has developed an understanding of the second type of argument. The next task 8 x 5 = __ illustrates the tentative nature of his understanding. In this task the multiplicand changes from 4 to 5.

Andy attempts to solve the task by adding 5 to 8 x 4 and gets 37. John, on the other hand says "Eight sets of 4. Eight sets of 5." clearly placing these tasks in juxtaposition but then proceeds to use argumentation A1 to solve the problem. Andy repeats John's argument verbatim, as if to think it through for himself. In doing so he abandons his attempt to use an argument of the second type.

On the final task in this episode 7 x 5 = __ Andy uses the first form of argumentation, counting up seven sets of 5 and John uses the second form.

John: Oh, It's just 5 lower than that [8 x 5].

The greater flexibility shown by John in being able to use both types of argumentation, while Andy is quicker at completing the calculations, is consistent with our knowledge of John's and Andy's individual mathematical conceptual development. Over the course of the episode we see Andy's emerging but still tentative understanding of the second form of argumentation. After hearing John use it Andy begins to use it. He succeeds in using it but after later using it incorrectly, in attempting to solve 8 x 5, he reverts back to the first form of argumentation which he understands completely. Episodes like this one form the basis for children's learning of mathematics through social interaction.

Argumentation and learning mathematics

This is only a single example and it should not be misunderstood as a paradigmatic example for "good small group activity". The intention of presenting this example was that it demonstrates in a very illustrative way the theoretical problems which have to do with the relationship between the social interaction process that occurs when partners give different forms of arguments and the individual process of learning.

As the remarks at the beginning concerning the normal use of small group activities in a classroom situation emphasized, a good working collaboration is not necessarily a good social condition for facilitating mathematical learning. Often the smooth functioning of such interactions is instead an indicator of a situation where everything that needs to be learned for participation in this interaction is already learned. Smooth functioning then means nothing other than the collaborative mechanical execution of a routinized mathematical method. Thus learning is based on conflicts which demands resolution by the participants (Bauersfeld, Krummheuer, & Voigt, 1988; Krummheuer, 1989; Miller, 1986; Perret-Clermont, 1980). These conflicts can be social in origin and include trying to take the perspective of a partner, recognizing the fact that a different individual has developed a different interpretation of the situation, or attempting to understand a different form of argument presented by a partner. In terms of Piaget's theory this has to do with the development of a more flexible cognitive structure which is less egocentric (Piaget, 1975). In terms of Mead's theoretical approach, one can characterize this phenomenon as a more differentiated process of perspective taking (Mead, 1934). In this sense the learning of an individual has a social genesis.

References

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), Foundations and methodology of the discipline of mathematics education (pp. 174-188). Proceedings of the TME Conference, Antwerp.

Hoetker, J., & Ahlbrandt, W.(1969). The persistence of recitation. American Educational Research Journal, 6, 145-167.

Hopf, D. (1980). Mathematikunterricht. Eine emprische Untersuchung zur Didaktik und Unterrichtsmethode in der 7. Klasse des Gymnasiums. Stuttgart: Klett.

Krummheuer, G. (1989). Die menschliche Seite am Computer. Weinheim: Deutscher Studien Verlag.

Mead, G. H. (1934). Mind, self, and society. Chicago: University of Chicago Press.

Mehan, H. (1979). Learning lessons. Cambridge, MA: Harvard University Press.

Miller, M. (1986). Kollektive Lernprozesse. Studien zur Grundlegung einer soziologischen Lerntheorie. Frankfurt: Suhrkamp.

National Research Council. (1989). Everybody counts: A report to the nation on the future of mathematics education. Washington, D.C.: National Academy Press.

Perret-Clermont, A. N. (1980). Social interaction and cognitive development in children. New York: Academic Press.

Piaget, J. (1975). Die Entwicklung des rumlichen Denkens. Stuttgart: Klett (original title: La representation de le'espace chez l'enfant).

Slavin, R. (1986, April). Developmental and motivational perspectives on cooperative learning: A reconciliation. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Slavin, R., Sharan, S., Kagan, S., Hertz-Lazarowitz, R., Webb, C., Schmuck, R. (1985, Eds.) Learning to cooperate, cooperating to learn. New York: Plenum Press.

Voigt, J. (1985). Patterns and routines in classroom interaction. Recherches en Didactique des Mathematiques, 6, 69-118.

Yackel, E., Cobb, P., & Wood, T. (in press). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education Monograph.

Potential Mathematics Learning Opportunities in Grade Three Class Discussion

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The purpose of this paper is to report findings of an on going research which tries to understand the potential learning opportunities during mathematics whole class discussion which is compatible with constructivist's view of learning. One third grade class was video taped during mathematics whole class activities for one semester. Individual interviews were conducted with students and the teacher in order to reflect on certain events occurring during class discussion. Analysis of one transcript of class discussion showed that mathematics meanings were negotiated through interactions, and the potential learning opportunities were constructed during these interactions.

From a constructivist's perspective learning occurs when a child tries to adapt her functioning schemes to neutralize perturbations that arise through interactions with our world (Steffe, 1988). Two important aspects, constructions and interactions, are important in the above statement. Although construction of knowledge is a personal act, it is by no means an isolated activity as many people's interpretations of constructivism imply. Constructivists recognize the important of social interaction as " the most frequent source of perturbations" (von Glaserfeld, 1989. p.136).

Recently an instructional model which is compatible with constructivsm's view of learning has been developed and implemented in twenty-four second grade classrooms (Cobb, et. al. 1989). Two quantitative analyses showed that the students in the project group scored as well as the students in the control group on a computation subtest and significantly better on the concepts and application sections of the standardized test. (Nicholls, et. al. in press a). Also, the students were found to be more task oriented, less ego-oriented and showed less work avoidance than students in the control group (Nicholls, et al. in press b). However, quantitative analysis alone can tell little about what really happened in the classrooms and what factors contributed to the success of this model. The present research, which focuses on class discussion is an attempt to develop "a coherent framework within which to talk about both teaching and learning" (Cobb, et. al., 1988). The goal of this

research is to describe the social dynamics and identify potential learning opportunities in class discussions.

In this instructional model, Problem Centered Learning, a typical mathematics lesson has three parts, a short introduction of carefully selected tasks by the teacher, 30-40 minutes of small group work on matheamtics tasks, followed by 15 minutes of whole class discussion (Wheatley, in press) . Unlike research in understanding social interaction in a small group and its role in learning (Yackel et al., in press), little research has been done on whole class interaction (Gilah, 1987). In contrast to small group interaction which only involves a small number of students with each group working along different paths (sometimes even on different tasks) with or without the assistance from the teacher, in whole class discussion the students and teacher form an intellectual community whose goal is to construct meaning. The interaction in class discussion is more complicated than those in small groups. Norms and interactive patterns are more easily negotiated in whole class discussion since the teacher is a member of the group.

Unlike the commonly seen teacher-lead didactic discussions which are an adjunct to exposition, the purpose of class discussion in Problem Centered Learning is for students to share their methods and to negotiate mathematics meanings. In other words, students' opinions are the focus of the discussion , not the teacher's. The role of the teacher is to facilitate the verbal interactions among her students and not "lead" students to previously determined statements. She frames situations as paradigm cases which she uses to negotiate social norms. In no case does she evaluate student answers on correct choices of the solution methods. Because she does not want the discussion becomes a guessing game (Voigt,1985), neither does she want to damper students' enthusiasm by saying "You are wrong!". Her goal is to facilitate student-to-student commnication. In most cases, students can restructure their solutions by simply verbalizing it, or by using suggestions from other students.

The Research Plan

All the whole class mathematics discussions of one third grade classroom which used Problem Centered Learning were video taped and field notes of daily events were taken everyday for a whole semester. Individual student's perceptions of mathematics learning in

general and whole class discussion in specific were gathered through individual interviews and a mathematics belief questionnaire (Nicholls, et. al. in press a), which was administered at the beginning of the semester. Also, formal and informal interviews were conducted with the teacher which focused on her interpretations of certain events and her rationale for certain interventions which she made during class discussion. The multiple perspectives model (Cobb, in press) are used in analyzing class discussion. Because page limitation, this paper will discuss only how mathematics meaning can be negotiated through interactions and how potential learning opportunities are constructed during these interactions.

The following is a transcript of one instance of a whole class discussion which occurred in early September. The task was " Find different ways to make 50."

Brandon and Travis wrote down 9 9 on the board.

9 9

8 5

(A was the teacher and all the other letters indicate different students.)

B: **See, I was doing one problem** [B was making 50 another way. B is explaining T's method even though he did not initially participate in the solution. The solution being explained was developed by T.] **Travis, he was using those unifix cubes. He took...** [took 50 cubes. The fifty cubes were arranged in five stacks of ten. T took one from each of the stack in an attempt to make a different '50', different from adding five tens.] **First he thought all these were 9** [pointing to the five nines written on the board] . **But then we counted them all up.** [It is interesting to note that T began with 50 cubes, partitioned them into six sets and then T and B counted them to find how many. Apparently, once the partitioning was begun, the 50 no longer existed for T. They had obtained 51 when they determined how many 9, 9, 9, 9, 9, and 5 made. They were in the action of counting and did not relate the result to the 50 cubes with which T had begun. This action suggests T was not at the level of part-whole]. **So we took 1 away from one of the 9.** [leaving 8] **He knew it was not all 9's** [it did not add to 50], **and he had a 5. This** (B pointed to the '8') **was a 9 once, but he took one away** [because their count had yielded 51

which was one too many] **then we counted it up, we got 50** [It is likely they inferred it would be 50 since there had been 51 and they took one away].

S: **I don't understand what you did. You told us you counted all the 9's this and that, but how did they make 50? Because that's,.. that's a higher number** [referring to the 8 5 and thinking of it as eighty-five].

[S is intent on making sense of B and T's explanation. A review of the video recording indicated that during this time she rarely looked at the board and seemed to be disinterested. This is an excellent example of negotiating meaning of terms.]

B: **See...**

T: **I broke it down into 9's. See...**

S: **Of 50?** [meaning "Did you break down 50?"]

T: **I counted up to 50** [started with 50 cubes], then we broke down into 9's.

B: **Right.**

T: **Then I have 5, then I took off 1 from another 9.** [because when we counted there were 51 and that was one too many. Even though he had started with fifty because that is the sum he was attempting to obtain, the fifty no longer existed for T when he began the partitioning.]

B: **See. We had 51. So we took 1 off . It was a 9, so we took 1 off. That's where the 8 came from.**

A: **Most people said they could understand Travis broke down into 9's, and took one away, but what is that 8?**

S: **Why is 85? It was supposed to make 50.** [Why did you write eighty-five? 85 is greater that 50!]

B: **No, they were 8 and 5.**

S: **Oh, you mean 9, 9, 9, 9, 8 and 5.**

B: **Right.**

S: **Oh, I understand now.**

A: **I don't understand.** [This facilitative intervention by the teacher in no way interrupted the rhythm and flow of the discussion. The students understood it as a further

attempt to help them make sense of the explanation.]

B: **All right. Let me talk it over for you. It was four 9's, an 8 and a 5.**

J: **I know what they were trying to say.** [J is attempting to clarify B's explanation. In order to do this she had to take B's perspective, to think like he did.] **You added all the 9's and them 8, and 5. You come up with 50.**

(H raised her hand indicating she wanted to contribute to the discussion. E shook his head which suggesting he had made sense of the explanation and disagreed with some aspect of the work, most likely the sum of 50.)

A: **Does four 9's, an 8 and a 5 makes 50?** [Another facilitative intervention by the teacher.]

(K raised her hand.)

B: **See. We counted it up. 9 and 9 is 18. 18 and 18 is....**

(Some students tried to figure out 18 + 18, while others went beyond that and tried to figure out what four 9's, an 8 and a 5 would make.)

C: **It was 49.**

B: **You told me it was 51 .** [Apparently B was surprised at this result. We infer he had not participated in obtaining the 51 originally.]

A: **Did they have a creative way of showing 50?**

(Unison: **Yeah!**)

A: **Five 9's and a 5.**

S: **O.K. I agree.**

A: **We really have very different ways. Thank you for sharing it. And thank you for you people to pay enough patience. They had a really good idea but needed a little help.**

(Class in unison: **Yeah.**)

In this statement, the teacher was negotiating social norms and attempting to communicate her goals for student learning. She was saying, "Our goal is making sense and as long as people are doing something meaningful it is a valuable contribution to the intellectual community (the class). It is important to explain your reasoning. Class members should be

trying to make sense of the explanations. I aliked the way they explained their solution even though it was not perfect. Our goal is not correct answers but good thinking." In effect the teacher was framing this episode as a pardigm case of doing mathematics, a move which has been identified as effective in negotiating social norms of the class (Cobb et. al. 1988).

The above example illustrates the dynamics of a class discussion in a constructivism type classroom and how it constrasts with a typical teacher-lead discussion. Students are freely expressing their ideas, they are trying to make sense of each other's method and they carry on a conservation among themselves. There is a sense of owenership of what they are doing, and they make an effort to explain their methods. B and T helped each other in explaining their method by trying to figure out what might be left out by the other person, and by interpreting the comments and questions (verbal and nonverbal) of the students in the class. The potential learning opportunities for B and T were apparent as they were trying to reconstruct and verbalize their solutions (Levina, 1981) and as they attempted to distance themselves from the action of explaining to coordinate the other person's views (Sigel, 1981). Simlilar opportunities existed for all students who verbally participated in these discussions as they jointly resolved this conflict, for example S and J. However, for the students who did not participate verbally, they could still carry out conversations within themselves in trying to make sense of the whole situation since the task was at a cognitive level at which all could engage. By analyzing the tape and student interviews, we now question the use of the term "off task behavior". For example, even the girl S who participated verbally in the whole discussion did not appear to be listening much of the time. We also found some students who played with their pencils throughout the entire discussion (faced the board only once in a while) and yet still gave a detailed description of what happened during that period without any prompting. These findings certainly make us cautious in judging student engagement by any overt "engaging behaviors".

We use the phrase potential learning opportunities instead of learning opportunities in this paper because we recognize the complex relationship between the context (students perceived learning environment and their intentions) which students construct and operate in (can only be inferred by observers) and the setting (learning environment) which can be

directly observed. The teacher can select tasks which are likely to create conflicts, thus serving as an opportunity for students to restructure their thinking. However, students may not benefit from these opportunities simply because they have different intentions. That is why negotiating social norms becomes an important aspect of the teacher's role in class discussion (Bishop, 1985).

As we indicated above, the results reported in this paper are only a partial aspect of this research project. Analyses are continuing as we try to explain why some discussions were more effective than the others, and the influence of the teacher's decisions, the particular tasks used and how the roles played by different students effect the potential learning opportunities. As Pirie and Schwarzenberger (1988) pointed out although people believe and accept that mathematics discussion can help learning, little research had been conducted to determine its effectiveness. A study of the complex interactions during whole class discussion may help us determine the contribution of this instructional component to mathematics learning.

References

Bishop, A. (1985). The social construction of meaning-a significant development for mathematics education? For the Learning of Mathematics, 5 (1), 24-28.

Cobb, P. (in press). Multiple perspectives. In L. P. Steffe & T. Wook (Eds.), Transforming early childhood mathematics education. Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Yackel, E. and Wood, T. (1988). Curriculum and teacher development as the coordination of psychological and anthropological perspectives. Paper presented at the meeting of the Instruction/Learning working Group of the National Center for Research in Mathematical Sciences Education, Madison, Wisconsin, May.

Cobb, P. Yackel, E. and Wood, T. (1989). Young children's emotional acts while doing matheamtical problem solving. In D. B. McLeod &V.M. Adams (Eds.), Affect and matheamtics problem solving: A new perspective (pp.117-148). New York: Springer-Verlag.

Gilah C. L. (1987). Teacher student interaction: a case study. Educational studies in Mathematics, 18, 255-271.

Levina, R. E. (1981). L. S. Vygotsky's ideas about the planning function of speech in children. In J. V. Wertch (Ed.), The concept of activity in Soviet psychology (pp. 279-299). Armont, NY: Sharpe.

Nicholls, J., Cobb, P., Yackel, E., Wood, T., Yackel, E. & Patashnick, M. (In press a). Dimensions of success in matheamtics: Individual and classroom differences. Journal for Research in Mathematics Ecuation.

Nicholls, J., Cobb, P., Wood, T. Yackel, E. & Wheatley, G. (in pressb). Assessing young children's mathematical learning. In G. Kulm (Ed.), Assessing higher order thinking in Mathematics, washington, D. C. : Association for the Advancement of Science.

Pirie S. E. B. and Schwarzenberger R. L. E. (1988). Mathematical discussion and mathematical understanding. Educational Studies in Mathematics, 19, 450-470.

Sigel, I. E. (1981). Social experience in the development of representational thought: Distance theory. In I. E. Sigel, D. M. Brodzinsky, & R. M. Golinkoff (Eds.), New directions in Piagetian theory and practice (pp. 203-218). Hillsdale, NJ: Erlbaum.

Steffe, L. (1988). Overview of the Action Group A1: Early childhood years. Paper presented at the Sixth International Congress on Mathematics Education, Budapest, Hungary, July 27-August 3.

von Glaserfeld, E. (1989). Cognition, construction of knowledge, and teaching. Syntheses. 80. 121-140.

Voigt, J. (1985). Patterns and routines in classroom interaction. Researches en Didactique des Mathamtitiques, 6 (1), 69-118.

Wheatley, G. (in press). Constructivist perspectives on science and matheamtics learning. Science Education.

Yackel, E., Cobb, P. & Wood, T. (in press). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education Monograph.

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CERTAIN METONYMIC ASPECTS OF MATHEMATICAL DISCOURSE

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This paper outlines certain aspects of language use in mathematics which have not been frequently attended to, yet which are important to understand if the teaching of mathematics is to draw draw on the creative strengths of language with which pupils are already familiar. Following a theoretical illustration of the linguistic distinction between *metaphoric* and *metonymic*, a class lesson is analysed indicating places where these notions appear.

The Aboriginal Creation myths tell of the legendary totemic beings who had wandered over the continent in the Dreamtime, singing out the name of everything that crossed their path - birds, animals, plants, rocks, waterholes - and so singing the world into existence. (*The Songlines*, Bruce Chatwin, p.2)

In this paper, I want to start to examine how mathematical language can be and is used to 'sing' mathematical objects into existence, and how the 'songs' change over time and the objects with them. Mathematics has much in common with other ritual activities: the mystique, the incantations, the initiates and the masters. Certain old songs are no longer sung (the ones about coordinates in differential geometry, or calculations with infinitesimals, for instance) and the awarenesses they encourage diminish and die back (or go back into the Dreamtime if you prefer).

As with songs (and with novels and pictures, poems and plays, and other 'art' forms), so also with mathematical theorems, there is always the question: 'What is the song about?' Asking such an apparently straightforward question immediately imposes a time order that can misdirect. It pulls attention in the direction of songs coming *after* the 'things' that they are about, that they are *merely* representations of something else, rather than being the primary thing in themselves. If the Aboriginal songs bring the referents into existence, what then of traditional accounts of reference and meaning?

The same is true of the current mathematics education preoccupation with 'recording' and 'representation' (see Janvier, 1987, for example), where the assumption seems to be that there is always something being represented that predates it, and hence relegates language to a far less powerful position in the hierarchy of mathematical creation and existence: in particular, it emphasises the passive role often attributed to language in merely describing or representing experience, rather than being either a constituent component of the experience or even the experience itself.

Corran and Walkerdine (1981) comment: "Their [the teachers in their sample schools] framework did not provide a way of understanding language to have other functions than recording." This point is echoed in Douglas Barnes' remark (1976) that "communication is not the only function of language" With respect to children's acquisition of their first language, it would be a palpable nonsense to assert that pupils only acquire a particular aspect of language (e.g. use of the passive mood or formation of negatives or learning how and when to swear and which words to use according to context) *after* they have understood it. Ritual elements such as repetition or rehearsing rhyming sequences of sounds or words independent of apparent relevance to their immediate context are commonplace, self-imposed techniques of acquisition for young children.

Rehearsal of a system independent of its application seems to be a notion with which children are very familiar. Intransitive counting provides but one instance of exploring and getting a system right in production predominantly before approaching the harder task of figuring out what it might have to do with the material world. This suggests that 'teaching

for understanding' includes a manipulative facility which need not necessarily *follow* acquisition of 'meaning'.Here is a brief example of such counting which may help to show up some differences. Herbert Ginsburg (1977) reports a conversation between two twin sisters, Deborah and Rebecca (almost five years old).

D(eborah): 1,2,1,2,1,2. [The spoken words are recorded as numerals in the original.]
R(ebecca): 1,2,3.
D: No, not like that. I said 1,2,1,2,1,2.
R: 1,2,3,4,5,6,7,8.
R: 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19 ...(to her mother) What's after 19?
M(other): 20.
R: 20,21,22,23,24,25,26,277,28,29. What's after?
M: 30.
R: 31,32,33,34,35,36,37,38,39, now 40.

Later on she starts recounting - from 22 to 29; at this point she indicated by a glance that she wanted her mother to supply the next number.
M: 30.
R: 40,50,60,70,80,90,tenny.

This is linguistic experience of mathematics - there is no transitive counting here: it is 'about' learning the song, one of the ones which sing the numbers into existence.

Many writers have attempted to explore aspects of the concepts of 'meaning' and 'understanding' in the context of teaching and learning mathematics and have frequently adopted the natural history approach of offering a taxonomy of types, as well as judgements about preferred variants. This work (among others) has influenced school practice and resulted in current beliefs about the importance of always understanding what you are doing and the devaluing of certain traditional practices (such as chanting multiplication tables or working at 'pages of sums') in search of 'real understanding' (a concept which is pounced on by Walkerdine and Lucey in their insightful analysis (1989) of differing attitudes to and interpretations of girls' and boys' mathematical performance). This paper is also concerned with the notion of mathematical meaning, but from a different perspective.

Tahta (1988) has written of a current reluctance among certain teachers to assist pupils to share with them or textbooks, and this is a part of a reaction to and rejection of certain traditional methods in the teaching of mathematics I mentioned above. However, a reassessment is under way, and from a somewhat unlikely direction. It is not primarily a backlash, a rejection of the new ways and a nervous return to the old (though there are examples of such appeals). It is coming from certain unashamedly modern influences such as semiotics being applied to the teaching of mathematics and allowing value and direction to be perceived in certain traditional teaching practices that had been condemned wholesale under the heading of rote-learning methods. (These include chanting, repetitive apparently ritualistic aspects of mathematical activity, whether spoken (e.g. multiplication tables) or written (arithmetic or algebraic manipulations and algorithms) - indeed the rejection was frequently so strong as to deny that these practices were mathematical in nature.

The second was an over-narrow conception of meaning in mathematics in terms of reference rather than nets of connections in both form and content, and meaning in this restricted sense being claimed to be the most important, indeed only goal of mathematics teaching. In England, at least, an increasingly common dogma is if in doubt at any stage in anything mathematical, then go back to the 'meaning' (often the concrete) from which everything is presumed to stem. Valerie Walkerdine (1988) has recently drawn attention to the implausibility of such an account in the case of the teaching of place value. She offers a much more telling if complex account, one that intimately implicates the teacher's language and positioning within classroom activity.

> In all these examples of the children beginning place value, the teacher, in a sense, *tells*, the children what they are supposed to be learning and discovering .. she is providing the children with cues which reveal the properties of place value which the objects they are manipulating are supposed to supply.
>
> The teachers intention in giving the matchsticks to count and group - and represent the numerals in a place-value system - is that the relations of value between the numerals in the system will be apparent to the children because they will be presented concretely as relations between bundles and single matchsticks. On the contrary, it is the properties of the place-value system which are used to make the matchsticks, and the grouping of them into bundles of ten, signify in particular ways. The children's activities (grouping in tens) are being determined by a convention not vice versa.

Metaphor and metonymy are concepts partaking of an ancient study - that of rhetoric. Mathematics is often contrasted with rhetoric (see Nash,1990, for example) as being a discourse composed solely of logical (and hence non-rhetorical) statements. In addition, I hope to show (both here and in my talk) how both the classical and reformulated concepts (by Jakobson, and subsequently by Lacan) of metaphor and metonymy can highlight aspects of the nature and principles of mathematical discourse which is at the heart of the activity known as 'doing mathematics'.

One mathematics lesson

Below I have described a mathematics lesson and later transcribed and analyse both the nature of the activity and some of the dialogue from a videotape of the lesson.(In my proposed talk, I will show some of the videotape.) The entire class is seated in a circle (including the teacher) - a mixed-ability class of eleven-year-olds. In some sense, the lesson was 'about ' modular arithmetic (base five). The teacher responded to a pupil who initially asked what they are going to do today, with "we're going to play a game", but felt the need to add a rider "You need to concentrate though, It's a concentrating game", as if she wanted to block some readings that the term *game*" might entail. She then announced to the whole class, "We're going to play a game with numbers and colours." William, you're one;, what number are you? [to an adjacent pupil in the circle] and so on round [including the teacher who says "twenty-seven"]. Then, "William, will you go and take a white rod. To the next person "a red" and then green, pink and yellow in turn. (These rods are the first five Cuisenaire rods, cuboids whose lengths ar whole number multiples of the smallest one.)

Once everyone has a number and some have taken their rods, the teacher starts questioning pupils about what colour certain pupils further round the circle will take, and then attempts to encourage pupils to be able to move between any number and the corresponding colour. When she is satisfied they can do this quite fluently, she moves on to look at the possibility of adding colours.

Discussion

A great deal of what occurred relies on the fact that there are now three different naming systems at work. Firstly, there are the individual pupil names that the teacher uses a lot to direct her questions and to focus attention (e.g. "Sally, what colour are you?", "What colour is Debbie going to take?") Then there are the ordinal names relatively arbitrarily started (William is quite close to the teacher), but then systematically reflecting the order of the circle.(Early on the teacher asks Can you all remember what number you are? and when someone shakes their head responds with "Well, how could you find out?) Thus, knowing one or a few number name assignations enables anyone to work out the others (whereas that is not true of personal names: the ordinals assign a unique name whereas there are a number of pairs of pupils with the same Christian name, and the teacher on occasions makes mistakes). Finally, there are the colour names of the individual Cuisenaire rods

picked up in a way that respects the ordinal naming (William, who was "one", takes the first rod, the shortest in length) as well as imposing the structure of the colours, and as with personal names, the colour names are not unique.

Note how these associations, indeed identifications are made The focus of the activity is on the individuals present, but all the relations are attended to by name. The activity identifies first a number name and then a colour name with a person name, and then the teacher-led activity (in which the teacher is very strongly positioned) works on relations and associations among the different sets of names.

The metonymy in the classical sense is apparent in the use of colour or number names for pupil names. (T is Teacher, P any pupil.)

T: Who's a pink? Who else is yellow? Can I have a different red number?

At one point she asks: Who's going to be the same colour? and then reformulates it to: What numbers are going to be the same colour?

T: Fiona's twelve and I'm twenty-seven. What other numbers are going to be red?

P: Seventeen's going to be red.
P: Miss, it's all the numbers that have the end in 2 or 7 ... are going to be red.

This last statement is a lovely metonymic creation, linking the form of the numeral (numbers after all don't have 2s or 7s in them) with the colour name (neither numerals nor numbers have colours), and is entirely typical of the knowledge claims that this activity gives rise to. What is this song about?

The teacher then asks: Can we work out any numbers? Suppose we had two first year classes in here. What about forty-four? What colour will they be?

T: Suppose we had all the first year here what colour would number one hundred be?

(She is careful here to extend the basis of reference that she has set up, namely that it is people which signify most, in order to get past the actual and into the hypothetical, and slowly towards the infinite.)

She then shifts focus by saying: Well, let's have a think about adding them, because you are quite good at telling me what colours they are going to be.

T: What colour will we get if we add a pink and a yellow?

There is a subsequent shift from talk of "a pink" or "a yellow" (where it might be plausible to see this as a contraction of "a pink rod" and so have a relatively concrete referent) to "pink" and "yellow" as in a pupil remark "yellow and pink is pink".

The teacher shifts once again and asks, "Can anyone give me a sum and tell me what the answer is going to be using colours?"

P: Red and pink will be white.
T: (echoes but reformulates) A red and a pink will be a white. Can you explain how you know that?

P: Twenty-two and forty-nine equals a white. Forty-nine ends in a 9 and pinks are 9" Twenty-two ends in a 2 and reds are 2. So two and four make 6 and so five and one equal six. I nearly got it. It equals white anyway, because if you go from white which equals one four is a pink, five is yellow and six is white again, so it must be a white, twenty-two and forty-nine.

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Later on, P: 104 and 76 will be yellow - if you added them together the colour will be a yellow.

P: Twenty-four add three equals red

T: Twenty-four add three equals red - is that right? I'm 27 and I'm red. (The boy has apparently counted on from his number - 24 to get that of the teacher and formed his sum that way.)

P: One and five is white, two and seven is red, three and eight is green, four and nine is pink, five and zero is yellow.

P: Two thousand one hundred and ninety-four will be pink - but I'm not going to work it out!
(No need of people here.)

Then faced with sums such as R + P = they encounter commutativity (you don't need to do red plus white when I've done white plus red) and whether you can add two the same, By this stage, Red is standing for red and not a red, these objects have no more articles, but they are what the symbols R and P stand for (linked by alliterative first-letter naming).

Final comment from a boy who was having difficulty with his mathematics in general, "It's quite easy, this."

In her book, *The Mastery of Reason,* Walkerdine writes about home and school practices and the importance for analysing what happens in terms of the practices set up and the various positions held by the participants. In discussing the occurrences of 'more' and 'less' in her corpus, she remarks:

> In every case initiated by the child, she either wants more precious commodities, of which the mother sees it her duty to limit consumption, or the child does not want to finish food which the mother sees it as her duty to make the child eat. The differential position of the mother and child is made salient by the mother-initiated episodes in which the mother can stop the child giving her more, and can announce that she is going to do more of something. It is important that the child is not in that position. This, therefore, presents a relational dynamic or a position within the practices relating to power, rather than a simple view of turn-taking in those practices. (1988, p. 26)

In this videotape, we see the teacher position herself very carefully in the practice that is to provide the setting for the introduction of the signification of 'adding colours', the ostensible focus of the activity. She is part of the circle and has a number like every one else (including the visiting teacher as well). Amongst other things, this allows her to ask, "I'm 27. Who's like me? Who's got a red one?" And the identifying metonymy works when the pupils give her numbers rather than the names of pupils back. In addition is the assumption that being like the teacher is an important and desirable status-marked thing to be, though indeed it is not one of the pupils who is 'like the teacher' who offers conjectures in response to her question. And he falters interestingly when coming to name the teacher's number in a pattern he has seen in the signifiers (a reflexive question of is the teacher like herself), as if the metaphoric content/referent (in this case the teacher herself) gets in the way of the metonymic pattern (those ending in a 2 or a 7).

Walkerdine adds: If material phenomena are only encountered within their insertion into, and signified within, a practice, this articulation is not fixed and immutable, but slippery and mobile. That is, signifiers do not cover fixed 'meanings' any more than objects have one set of physical properties or function. It is the very multiplicity which allows us to speak of a 'play' of signifier and signified, and of the production of different dynamic

relations within different practices. It is for this reason that I used the terms 'signify'* and 'produce' rather than represent. If social practices are points of creation of specific signs then semiotic activity is productive, not a distortion or reflection of a material reality elsewhere." (1988, p. 30)

(*When typing this document I originally rendered this word as 'singify', a delightful metonymic creation which made unexpected links with the opening quotation.)

In this activity we have got away from the confusion of clock arithmetic of what 0-4 actually are - the problem of finding a name for the class. The traditional solution (of use to experts or adepts but potentially confusing to neophytes) is that of giving the name of one of the objects to the class (see Mason and Pimm, 1984) What you lose in marking this distinction is the ability to relate [1] + [2] = [3] to 1 + 2 = 3.(or **1** + **2** = **3** as some texts have it.) - that is to use the system you are familiar with. There is also he difficulty of having 1 (itself a symbol) standing for another symbol e.g. 6. A complex general issue of making, marking and ignoring distinctions.(It is there with fractions, where the 'lowest form' is the name for the rational number, but is not the best symbol to work with when trying to add two fractions. Part of the attitude evident in the New Mathematics of the 1960s was the willingness to change the symbolism and frequently to make it more elaborate, ostensibly to help learners (Other instances include marking vowels in Hebrew or Arabic the differentiated by position notation for positive and negative numbers (SMP books) or the box notation of the Madison project.)

The activity offers the possibility of gaining experience with symbolising. The language is set up and then it can drive the activity. The activity is self-checking to the extent that numbers and colours are on public display to check conjectures. But what are the objects and how do the names relate to them?

Meaning?

Being aware of structure is one part of being a mathematician. Algebraic manipulation can allow some new property to be apprehended that was not 'visible' before - the transformation was not made on the meaning, but only on the symbols - and that can be very powerful. Where are we to look for meaning? Mathematics is at least as much in the relationships as in the objects, but we tend to see (and look for) the objects. Relationships are invisible objects to visualise. Caleb Gattegno, writing in his book The *Generation of Wealth* (p. 139), claimed:

> My studies indicate that "mathematization" is a special awareness, *an awareness of the dynamics of relationships*. To act as a mathematician, in other words, is always to be aware of certain dynamics present in the relationships being contemplated. (It is precisely because the essence of mathematics is relationships that mathematics is suitable to express many sciences.) Thus, it is the task of education in mathematics to help students reach the awareness that they can be aware of relationships and their dynamics. In geometry, the focus is on the relationships and dynamics of images; in algebra, on dynamics per se.

Mathematics has a problem with reference so it tends to reify its discourse in order to meet the naive desire for reference.."The questions 'What is length?', 'What is meaning?', 'What is the number one?', etc. produce in us a mental cramp. We feel that we can't point to anything in reply to them and yet ought to point to something. (We are up against one of the great sources of philosophical bewilderment: a substantive makes us look for a thing that corresponds to it.)" Ludwig Wittgenstein, *The Blue and Brown Books*.

Partly this rejection stems from a two-fold misunderstanding. The first is an over-narrow conception of understanding and meaning, as well as the teacher's responsibility for providing it. Reversing Lewis Carroll's punning suggestion from

Alice's Adventures in Wonderland so that it now reads "take care of the sounds and the sense will take care of itself" offers an exciting possibility for teachers.

The second arises from a presumed necessary temporal dependence between the development of symbolic fluency and understanding, with the latter always and necessarily preceding the former. Part of what I am arguing for is a far broader concept of mathematical meaning, one that embraces both of these aspects (the metaphoric and metonymic foci for mathematical activity) and the relative independence of these two aspects of mathematical meaning with respect to acquisition.

Dick Tahta has claimed (1985, p. 49) that:

> We do not pay enough attention to the actual techniques involved in helping people gain facility in the handling of mathematical symbols. ... in some contexts, what is required - eventually - is a fluency with mathematical symbols that is independent of any awareness of current 'external' meaning. In linguistic jargon, 'signifiers' can sometimes gain more meaning from their connection with other signifiers than from what is being signified.
>
> Linguists have called the movement 'along the chain of signifiers' *metonymic* whereas 'the descent to the signified' is *metaphoric*. ... The important point is that there are two sharply distinguished aspects (metonymic relations along the chain of signifiers and metaphoric ones which descend into meaning) which may be stressed at different times and for different purposes.

There is a fundamental tension between what is currently called 'mathematical understanding' and symbolic fluency and automaticity. One increasingly important question for school mathematics at all levels is deciding what value is to be placed on the later, given the increasing sophistication and decreasing cost of symbolic manipulation devices on calculators and computers. The relative independence of these two aspects is hidden in a presumed temporal priority arising from a presumed conceptual priority of metaphoric understanding.

Note: I am most grateful to Eric Love for conversations about the mathematical activity described in this paper.

Bibliography

D. Barnes, *From Communication to Curriculum*, Penguin, Harmondsworth, 1976.

B. Chatwin, *The Songlines*, Picador, London, 1987.

G. Corran and V. Walkerdine, *The Practice of Reason*, Institute of Education, London, 1981.

H. Ginsburg, Children's Arithmetic, Van Nostrand, New York, 1977.

C. Janvier (ed.), *Problems of Representation in the Teaching and Learning of Mathematics*, Lawrence Erlbaum Associates, Brighton, 1987.

J. Mason and D.Pimm, 'Generic examples: seeing the general in the particular', *Educational Studies in Mathematics*, **15**(3), pp. 277-289.

G. Nash, *Rhetoric*, Oxford University Press, Oxford, 1990.

D. Tahta, 'On notation', *Mathematics Teaching*, **112**, pp. 48-51.

D. Tahta, 'Lucas turns in his grave', in D. Pimm (ed.), *Mathematics, Teachers and Children*, Hodder and Stoughton, Sevenoaks, 1988.

V. Walkerdine, *The Mastery of Reason*, Routledge, London, 1988.

V. Walkerdine and H. Lucey, *Counting Girls Out*, Virago, London, 1989.

Inverse Relations: The Case of the Quantity of Matter

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Paper prepared for the Conference of The International Group for the Psychology of Mathematics Education, Mexico, July 15-20.

January 1990

Inverse Relations: The Case of the Quantity of Matter

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Quantity of matter is measured by mass (m), volume (v) and number of particles (n). These three dimensions are related to each other (m/v=density, m/n=molar mass, v/n=molar volume). The purpose of this paper was to test the ability of students to differentiate between these dimensions and their ability to qualitatively convert from one to the other. This was done by presenting 9th and 10th grade students with several substances of equal quantity in one dimension, and asking them to decide whether or not they were equal in the two other dimensions. The results indicate that: a) inverse ratio tasks were more difficult than direct ratio tasks; b) the conversion between n and v was more difficult than other conversions, and problems related to the gaseous state were more difficult than problems related to other states of matter, c) girls had more difficulties than boys and d) teaching had almost no effect on students' performance. These results are discussed.

The chemist *measures* the quantity of substance by *weighing* it or by determining its *volume*, but he or she thinks about it in terms of *particles*. Thus, the quantitative aspect of chemistry deals with three dimensions: Two of these - *mass* and *volume* - are physical quantities which can be sensed and directly measured by instruments, and therefore can be considered *concrete*. The concepts of mass and volume are acquired either by experience and intuition or by the learning of physics in school prior to the learning of chemistry. The *third* dimension, which according to IUPAC is named *amount* (of substance) cannot be measured directly by an instrument and cannot be sensed. Thus, in contrast to mass and volume, it is not a concrete concept, but may be regarded as *theoretical*. Students become acquainted with this concept through chemistry lessons rather than by experience or intuition.

This idea may be summarized as follows: The "quantity" of a given substance can be expressed by three different physical quantities which represent three aspects of matter or three properties of all materials: mass, volume and amount ("particulate").

Each of these expressions of physical quantity has a name, a symbol and a unit which emerge from the different property it represents (see Table 1).

Table 1: Basic physicial quantities related to matter and their units.

Physical quantity	Symbol	Unit	Property underlying the physical quantiy
Mass	m	kg	the inertia of an object
Volume	v	liter (meter^3)	the space filling of an object
Amount	N n	particle (lee) mole	the particulate nature of matter

The three dimensions of the quantity of matter for a given substance are related to each other: a) m/v=d(density), density depends on temperature and in gases also on pressure; b) m/n-Mm(molar mass), molar mass is independent of external conditions and derives only from the mass of the particles; c) v/n=Vm(molar volume), molar volume is affected by both the volume of the particles and the distances between them. These distances change with temperature, pressure and state of matter. In gases at standard conditions molar volume is dependent only on the distances between particles and is equal for all gases.

Many studies (e.g., Ingle and Shayer, 1971; Duncan, 1973; MacDonald, 1975; Novick, 1976; Dierks, 1981; Umland, 1984; Lazonby, 1985; Lybeck et al., 1988) reveal that students encounter difficulties in solving problems related to the quantity of matter. Others report on difficulties students have with inverse ratio reasoning (e.g.. Stavy, 1981, Surber & Gzesh, 1984). Therefore, the purpose of this study was to test the ability of students to differentiate between these three dimensions and their ability to qualitatively convert from one to another. This was done by presenting students with several substances (solids, liquids and gases) of equal quantity in one dimension, and asking them to decide whether or not they were equal in the two other dimensions.

Several variables were tested: a) the nature of the reasoning processes required in solving the different tasks. For example, judging the inequality of mass when presented with equal volumes of different substances ($v \rightarrow m$) requires **direct ratio** reasoning. For the substances1 and 2, $m_1/v_1=d_1$ and $m_2/v_2=d_2$. If $v_1=v_2$ and $d_1>d_2$ then $m_1>m_2$. The same type of reasoning is required for judging he inequality of mass of equal number of particles of different substances ($n \rightarrow m$) and for judging the inequality of volume (of solids and liquids) of equal number of particles of different substances ($n \rightarrow v$). **Judging the** inequality of volume of equal masses of different substances ($m \rightarrow v$) requires **inverse ratio** reasoning. For the substances 1 and 2 $m_1/v_1=d_1$ and $m_2/v_2=d_2$, when $m1=m2$ and $d1>d2$ then $v2>v1$. The same type of reasoning is required for judging the inequality of number of particles of equal masses of different substances ($m \rightarrow n$) and for judging the inequality of the number of particles (of solids and liquid) of equal volumes of different substances ($v \rightarrow n$); b) the nature of the domain specific knowledge required in solving the different tasks: mass, volume and the particulate nature of matter and the three states of matter, solid, liquid and gas; c) grade, ninth and tenth grade students were tested - ninth grade students have not studied yet about particulate amount while tenth grade students have studied it; d) gender.

Method

A. Subjects and design

The sample included 66 middle class students from grades nine and ten. The distribution of students according to grade and gender is presented in Table 2. Each student was individually interviewed while being shown the materials. Eighteen problems were presented to each of the students. The problems related to all three states of matter (solid, liquid and gas). Israeli students do not deal with the particulate amount of matter and its unit, the mole, until the 10th grade. They are taught about the particulate nature of matter, about elements, compounds and ions, and about basic chemical reactions. We therefore asked about "number of particles" rather than "amount". The 10th grade subjects had studied particulate amount and its unit and had solved stoichiometric problems. Teaching was initiated from the gas laws (Gay-Lussac and Avogadro).

Table 2: Distribution of the research population by grade and gender.

Grade	Boys	Girls	Total
Ninth	20	17	37
Tenth	14	15	29

Tasks

I. Three different known solid objects of equal volume and shape were presented to the student (wood, aluminum and "plastic"). The student was asked to judge: 1) whether they had the same or different mass; 2) whether they had the same or different number of particles. The student was then presented with three different known liquids (water, alcohol and oil) of equal volume (in containers of the same size and shape) and asked: 3) whether they had the same or different mass; and 4) whether they had the same or different number of particles. Finally, the student was presented with three different known gases (O_2, CO_2 and air) of equal volume (in syringes of the same size and shape) and asked: 5) whether they had the same or different mass; and 6) whether they had the same or different number of particles. In each case, the student was asked to explain his/her answer.

II. The following questions were posed to each of the students: If we take equal masses of the three solid objects 7) will they have the same or different volume and 8) will they have the same or different number of particles; If we take equal masses of the three liquids 9) will they have the same or different volume and 10) will they have the same or different number of particles; If we take equal masses of the three gases 11) will they have the same or different volume and 12) will they have the same or different number of particles. In each case the student was asked to explain his/her answer.

III. The following questions were posed to each of the students: If we take an equal number of particles of the three solid objects 13) will they have the same or different

volume and 14) will they have the same or different mass; If we take an equal number of particles of the three liquids 15) will they have the same or different volume and 16) will they have the same or different mass; If we take an equal number of particles of the three gases 17) will they have the same or different volume and 18) will they have the same or different mass. In each case the student was asked to explain his/her answer.

Results and Discussion

Table 3 presents students' success in solving the direct and inverse ratio tasks in the different pairs of dimensions. As can be seen from Table 3, 9th and 10th grade students are able to differentiate and qualitatively convert between volume and mass and between mass and number of particles in the direct ratio tasks presented to them. However, they have difficulty in performing the same operations with regard to other, essentially similar and logically related, tasks. The level of success in judging the inequality of mass of equal volumes of different substances (direct ratio) was 83%, while the level of success in judging the inequality of volume of equal masses of different substances (inverse ratio) was significantly lower - 66%. Similarly, the level of success in judging the inequality of mass of an equal number of particles of different substances was 82%, while the level of success in judging the inequality of the number of particles of equal masses of different substances was significantly lower - 66%. This finding can be related to the different reasoning processes required in solving the different tasks.

Table 3: Percentage of students who correcly answered the direct and inverse ratio tasks in the different pairs of dimensions

Pairs of dimensions	RATIO			
	Direct		Inverse	
	Task	Percent	Task	Percent
Volume & mass	$v \rightarrow m$	83	$m \rightarrow v$	66
Number of particles & mass	$N \rightarrow m$	82	$m \rightarrow N$	66
Number of particles & volumes	$N \rightarrow v$	59	$v \rightarrow N$	56

Although each pair of tasks ($m \rightarrow v$, $v \rightarrow m$) is logically identical and one could represent the "inverse ratio" problems in terms of direct ratio e.g. v/m=constant (or N/m=constant for the pair $m \rightarrow N$, $N \rightarrow m$). students apparently prefer to use that ratio which has an intuitive meaning. The ratio m/v represents density or "heaviness" of a substance - a property which can be intuitively grasped and visualized, whereas,the ratio v/m has no immediate meaning and cannot be similarly grasped and visualized. The same is true with regard to the conversions between mass and number of particles ($N \rightarrow m$, $m \rightarrow N$). While the

ratio m/N has a meaning - the mass of one particle, and can be easily grasped and mentally visualized, the ratio N/m has no such immediate meaning and is therefore difficult to grasp or to see with the mind's eye. Support for this explanation can be found in the students' explanations of their judgments. In the conversion from volume to mass they used the idea of density or heaviness to a larger extent than in the conversion from mass to volume; in the conversion from number of particles to mass they used more the idea of particle mass more frequently than in the conversion from mass to number of particles. No such difference was found between the two tasks involving conversion between volume and number of particles (see Table 3). These two tasks showed a lower level of success resembling that of the inverse ratio in the other conversions (59% and 56%). It is possible that in this case the direct ratio in itself - v/N is a more difficult quantity to grasp because molar volume is affected by both the volume of the particles and the distances between them. These distances change from substance to substance with the state of matter, with temperature, pressure, etc. In addition, the behavior of gases is different than that of solids and liquids. Indeed the hardest tasks in this conversion were those relating to gases (see Table 4).

Table 4: Percentage of students who correctly answered the direct and inverse ratio tasks in the different states of matter

Pairs of dimensions	RATIO					
	Direct			Inverse		
	Task	State of matter	Percent	Task	State of matter	Percent
Volume & mass	v→m	solid	98	m→v	solid	71
		liquid	80		liquid	67
		gas	71		gas	61
Number of particles & mass	N→m	solid	85	m→N	solid	66
		liquid	83		liquid	73
		gas	78		gas	59
Number of particles & volume	N→v	solid	68	V→N	solid	62
		liquid	65		liquid	58
		gas	45		gas	49

Another interesting finding is that, in general, performance on tasks involving solids was higher than on tasks involving liquids which in turn was higher than on tasks involving gases (see Table 4). Students' difficulties on the gaseous state problems can be attributed also to their difficulty in conceiving of gases as materials (Stavy 1988). If they do not believe a gas to be a substance, it is very difficult for them to think of it in terms of mass, volume or number of particles.

The difference in performance between problems in the liquid and the solid states is not very well understood. It has been found (Stavy, 1985) that the concept of liquid is easier

to understand than that of solid, but it may be that it is easier to think about the density of a solid than about the density of a liquid, or about the particulate nature of a solid rather than that of a liquid. Ben-Zvi (1984) has shown that students visualize particles as small bits of matter and this is probably easier to do with solids than with liquids.

A very significant difference between the performance of boys and girls was observed, with boys consistently outperforming girls (for all tasks: ANOVA - $p<0.005$) (see Table 5). Sex-related differences, favoring boys, have been reported for several Piagetian tasks that are related to concepts and topics in the science curriculum (Howe and Shayer 1981, Linn and Polus 1983, Robert 1989). It has been suggested that this difference is due to an experiential deficit on the part of the girls or to an underlying difference in cognitive structure or style. Although we cannot explain the observed difference between boys and girls in our study, we can suggest that it may be related to girls' inferiority in spatio-visual abilities (Liben 1978). Our tasks, if not worked out on a formal mathematical level alone, require some manipulation of mental images.

Table 5: **Percentage of students who correctly answered the direct and inverse tasks in the different sexes**

Pairs of dimensions	RATIO: Direct Task	Direct Percent Boys	Direct Percent Girls	Inverse Task	Inverse Percent Boys	Inverse Percent Girls
Volume & mass	$v \rightarrow m$	83	73	$m \rightarrow v$	83	48
Number of particles & mass	$N \rightarrow m$	85	79	$m \rightarrow N$	76	56
Number of particles & volume	$N \rightarrow v$	59	59	$V \rightarrow N$	37	56

Another important point that should be raised is the ineffectiveness of teaching (see Table 6). No significant differences were observed between 9th and 10th grade students (except for the problems demanding conversion between volume and number of particles). It seems that in the course of chemistry teaching, insufficient attention is paid to the serious problems which students (especially girls) have with the basic concept of the quantity of matter.

Table 6: Percentage of students who correctly answered the direct and inverse ratio tasks in the different grades

Pairs of dimensions	RATIO Direct Task	Direct Percent 9th grade	Direct Percent 10th grade	Inverse Task	Inverse Percent 9th grade	Inverse Percent 10th grade
Volume & mass	v→m	87	79	m→v	66	66
Number of particles & mass	N→m	84	79	m→N	67	64
Number of particles & volume	N→v	49	65	V→N	53	67

Conclusions

The results reported in this paper indicate that students' performance in solving simple qualitative problems in chemistry is affected by several factors. It was shown that both the logico-mathematical aspect of the content and the specific nature of the subject matter affect students' performance. Inverse ratio reasoning was found to be more difficult than direct ratio reasoning. It would be very interesting to find out what is the source of this difficulty and whether students' difficulty in solving the inverse ratio conversion tasks is related to their difficulty in solving other parallel inverse ratio problems, e.g. in mathematics, or whether it is related only to the specific content of the present tasks.

It was also shown that the conversion between number of particles and volume was more difficult than other conversions (m→v, m→N) and that problems in the gaseous state were more difficult than problems in other states of matter. These two areas of subject matter are indeed more complicated.

Two other important findings were that girls have more difficulties than boys and that teaching had almost no effect on students' performance. These findings imply that while teaching, one should start from the easier tasks (conversion of the direct ratio nature n→m, v→m, in the solid state) and gradually advance to the more complex and more difficult tasks, emphasizing the inverse ratio aspect and the volume and gaseous sate problematics. One should also emphasize the analogy between density (d); Molar mass (Mm) and molar volume (Vm) as characteristic intensive properties of substances and the equivalence between the three dimensions, volume, mass and amount (particulate), as properties on which we can base our measurements of a given quantity of substance.

Bibliography

Ben-Zvi, R., Eylon, B. & Silberstine, J. (1986). Is an atom of copper malleable? *Journal of Chemical Education, 63*, 64-66.

Dierks, W. Stoichiometric calculation: Known problems and proposed solutions at a chemistry-mathematics interface. *Studies in Science Education, 8* (1981) 93-105.

Duncan, I.U. and Johnstone, A.H. (1973). The mole concept. *Education in Chemistry, 10*, 213-4.

Howe, A. & Shayer, M. (1981). Sex-related differences on a task of volume and density. *Journal of Research in Science Teaching 18*, 169-175.

Ingle, R.B. and Shayer, M. (1971). Conceptual demands in Nuffield O-level chemistry. *Education in Chemistry 8*, 182-3.

Lazonby, J.N., et al (1985). The mole: Questioning format can make a difference. *Journal of Chemical Education 1*, 60-61.

Liben, L.S. (1978). Performance on Piagetian spatial tasks as a function of sex, field dependence and training. *Merrill-Palmer Quarterly, 24*, 97-109.

Linn, M.C. and Polus, S. (1983). Male-female differences in predicting displaced volume: Strategy usage, aptitude relationships and experience. *Journal of Educational Psychology 75*, 86-96.

Lybeck, L., et al (1988). *Empirical research in science and mathematics education.* The International Council of Associations for Science Education (Univ. Dortmund). 39-55.

MacDonald, J.J. (1975). The mole: What's all the fuss about. *School Science Review, 96*, 635-636.

Michell, I. and Gunstone, R. (1984). Some student conceptions brought to the study of stoichiometry. *Research in Science Education, 4*, 78-88.

Novick, S. and Menis, J. (1976). A study of student perceptions of the mole concept. *Journal of Chemical Education, 53*, 720-722.

Robert, M. (1989). Robustness of cognitive gender differences: The case of horizontality representation in the water-level task. Paper presented at the Tenth Biennial Meeting of the ISSBD, Jyvaskyla, Finland.

Stavy, R. (1981). Teaching inverse functions via the concentration of salt water solution. *Archives de Psychologie, 49*, 267-287.

Stavy, R. (1988). Children's conceptions of gas. *International Journal of Science Education, 10*, 553-560.

Stavy, R. (1985). Children's ideas about solid and liquid. *International Journal of Science Education, 7*, 407-421.

Surber, C.F. and Gzesh, S.M. (1984). Reversible operations in the balance scale task. *Journal of Experimental Psychology, 38*, 254-274.

Umland, J.B. (1984). A recipe for teaching stoichometry. *Journal of Chemical Education*, 1036-1037.

THE DEVELOPMENT OF MATHEMATICAL DISCUSSION

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The development of mathematical discussion in a second-grade classroom is presented. In this classroom, the students and teacher mutually construct a form of discourse in which mathematical meaning is negotiated. Two sample episodes are analyzed to illustrate the evolving nature of the interaction patterns as the students and teacher interactively constitute a basis for activity that creates opportunities for learning. The tension created between the teacher and students' meanings as they engage in the process of negotiation of mathematical meaning will be considered in light of their personally constructed basis for activity.

The importance of whole class discussion in mathematics for primary-aged children can be considered both from the point of view of individual mathematical construction and the development of the taken-to-be-shared meaning. The nature of the routines and patterns of interaction that are mutually established between the teacher and students create learning opportunities that ultimately influence what is learned. The purpose of this paper is to illustrate the evolving nature of the patterns of interaction that occur between the students and teacher as they interactively constitute a basis for activity that encourages the development of mathematical meaning. Specifically, the discussion will focus on the negotiation of meaning through a process involving a genuine commitment to communicate.

The interactively constituted discourse that influences the child's evolving sense of number has as a crucial feature the child's involvement in problem solving activity. As children, participate in discussion which emphasizes their ideas and solutions to problems, they have an opportunity to provide explanations and justifications which form the basis of mathematical argumentation. Students' who engage in dialogue in which accounting for their ideas to others is anticipated are provided opportunities in which to negotiate mathematical meanings. It is these taken-to-be-shared concepts that come to be accepted as "mutually congruent mental representations" and become "real objects whose existence is just as 'objective' as mother love..." (Davis & Hersh, 1981). Patterns of interaction in which these discussions occur are

characterized by a genuine commitment to communicate in which the students' as well as the teacher are active participants in the dialogue. From this perspective, communication is viewed as an activity, that requires two or more autonomous partners, in which the students' are responsible for describing their worlds to their teachers. For their part, teachers are obligated to convey a genuine interest in and willingness to learn about what their students are discussing (Cazden, 1988).

Mathematical Discourse in Elementary School

Whole-class discussions in which children talk over their solutions to mathematical problems provides opportunities for learning not available in most traditional school classrooms. The typical discourse pattern found in these classrooms has been extensively described by Hoetker & Ahlbrandt (1969), Mehan (1979), and Sinclair and Coulthard (1975), as one in which the teacher controls, directs and dominates the talk. The patterns of interaction become routinized in such a way that the students do not need to think about mathematical meaning, but instead focus their attention on making sense of the teacher's directives (Bauersfeld, 1980; Voigt, 1985).

Our ongoing research and development project in second-grade (7-year olds) is an attempt to analyze children's construction of mathematical concepts and operations within the complexity of the social setting of the classroom. A constructivist's perspective of children's learning forms the basis of the project in which the cognitive models of early number learning (Steffe, Cobb, von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards & Cobb, 1983) are used to develop instructional activities. The general instructional strategies are small group collaboration followed by whole class discussion of children's solutions. Moreover, we believe that mathematics learning is an interactive as well as a problem solving activity that involves the negotiation of taken-to-be-shared mathematical meanings by members, of the classroom communities (Bauersfeld, 1988). We have previously described and discussed the nature of these classrooms with regard to the mutual construction of the social norms that constitute the obligations and expectations crucial for the development of a classroom setting in which learning

Cobb, Yackel, & Wheatley, 1990; Yackel, Cobb, & Wood, in press). Although the project has been expanded to include additional classrooms, this paper will be drawn from video-recorded data collected during the initial year-long classroom teaching experiment in one second-grade classroom.

Interactively Constituted Basis for Mathematical Discourse

Class discussion following the collaborative work of the small groups provides an opportunity for children to monitor their activity during a period of retroactive thematization (Steffe, 1990). As children explain their solutions to instructional activities, opportunities arise for them to reiterate their earlier activity in order to bring it to a level of conscious awareness and evaluate it in terms of the ongoing discussion. It is through this monitoring activity that children build confidence in their actions.

The nature of the social interaction that is interactivelly constituted in the class is crucial to the development of negotiation that creates opportunities for learning. Discourse that is characterized by a genuine commitment to communicate provides opportunities for children to reflect on and evaluate their prior activity. As such, class discussions in which the emphasis is on children's explanations and justifications of their mathematical solutions provides occasions for children to engage in negotation of mathematical activity taken-to-be-shared.

The following episodes have been selected to illustrate the evolving nature of the whole class discussion in which the teacher and children attempted to communicate and negotiate mathematical meaning. The first example occurs at the beginning of the year as the class has been discussing the following problem which is on the chalkboard. A picture is shown of a pencil underneath which are 5 paper clips followed by the statement of the problem:

> Each paper clip is 3 centimeters long.
>
> How long is the pencil?

The episode begins:

Teacher: How long is the pencil, Lisa?

Lisa: 15 centimeters long.

Teacher: 15 centimeters. How did you get that answer?

Lisa: There were all 3's and we added them up by 3's and got 15.

Teacher: Did anyone do it a different way? Chuck?

Chuck: I got 5 centimeters.

Teacher: You got 5 centimeters. How long is each one of these [clips]? Let's take a look at this. If each one of these is 3 centimeters long, we have a 3, 3, 3, 3, and a 3. How much is that altogether? Karen?

Karen: 15.

In this early dialogue, the children are obligated to provide an explanation for their answers. In this case, the teacher assumes that Chuck does not understand the problem and attempts to direct him to the answer.

Teacher: Do you agree or disagree?

Chuck: Disagree.

Teacher: Disagree? That's alright, but there are 3 centimeters in each clip.

Chuck: There are 5 clips and that's 5 centimeters.

Teacher: He is right. There are 5 clips, but how long is each clip? A 3 and a 3 and a 3 and a 3 (measuring the distance on each clip with her hands). How long is that?

Chuck: 5 centimeters.

Teacher Okay. (pause) Did anybody else do it a different way?

In this class, the teacher and children have mutually constituted the social norm that their opinions are respected, thus it is appropriate for Chuck to disagree with the answer given. Moreover, in this exchange, the teacher attempts to engage him more directly in finding the solution. Chuck's contribution to the interaction is influenced by his current level of conceptual understanding in which he is limited by the fact that he has not constructed a scheme for coordinating units of different ranks. The teacher is also constrained in the interaction by her current conceptions and her limited understanding of the possible constructions children might possess. Thus, she simply indicates to Chuck that she accepts that his answer is meaningful to

him and directs the discussion to the other students. As yet, the students and the teacher while attempting to interactively constitute a basis for activity are unable to negotiate mathematical meaning.

The second episode that occurred later in the year centered around the following problem:

> Daisy Duck invited 50 children to her birthday party. Nineteen of them were girls. How many were boys.?

The children as they were discussing the problem, have been offering solutions which involve subtracting 19 from 50. The episode begins:

Teacher: Okay Alex what do you say?

Alex: It's 31.

Teacher: You think it's 31.

Alex: Because 30 plus 20 is 50, 30 plus 20 is 50.

Unfortunately, at this moment, the teacher is distracted by the comments of another pair of children and stops to carry on a conversation with them. Returning to Alex, she attempts to restate his explanation.

Teacher: Alright, look he said... 3 plus 20?

Alex: 30 plus 20.

Teacher: 30 plus 20. Where did you get the 30?

Alex's explanation of the problem as *"30 plus 20"* indicates that he has interpreted the problem as a missing addend. The teacher, attempting to understand his method, asks for further elaboration. In so doing, Alex tries to adjust his explanation to fit with the previous subtraction solutions that have been given.

Alex: [It] equals 50, and that takes up the 50 children that were at the party.

Teacher: But this is 19, right? (point to the number).

Alex: I know.

Teacher: Alright.

Alex: And so 50 minus 20 would be 30.

Teacher: Okay. What he is saying is instead of taking the 19, I [Alex] made it 20.

Alex: No.

Teacher: No, you didn't?

The teacher who has been trying to make sense of Alex's explanation offers her interpretation of his method to the rest of the class. Alex challenges her comments as incorrect, and continues to alter his initial solution from addition to subtraction. The teacher accepts this, and encourages his further explanation.

Alex: 50 minus 30 is 20.

Teacher: 50 minus 30.

Alex: 50... Well, I don't know what I did. 50 minus 20 is 30...

Teacher: Right.

Alex: But its a 19 instead of a 20 so it has to be one higher than it, because that number is one less than 20, so it's 31.

Teacher: Alright.

As Alex rethinks his explanation, he realizes that just changing his initial interpretation of the task to subtraction creates a situation that does not fit with the problem as written. In so doing, he offers a final explanation that provides a rationale and justification for his solution. The teacher's comment *"Alright"* indicates her agreement. The episode closes with both Alex and the teacher coming to accordance through a process of negotiation which has provided opportunities for Alex to provide a justification for his explanation.

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In T. Cooney & D. Grouws (Eds.) Effective mathematics teaching (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. Educational Studies in Mathematics. 11, 23-41.

Cazden, C. (1988). Classroom discourse: The language of teaching and learning. New York: Heineman.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D.B. McLeod & V.M. Adams (Eds.), Affect and mathematical problem solving: A new perspective (pp. 117-148). New York: Springer-Verlag.

Cobb, P. (1990b). Multiple perspectives. In L.P. Steffe, & T. Wood (Eds.), Transforming childhood mathematics education: An international perspective (pp. 200-215). Hillsdale, NJ: Lawrence Erlbaum Associates.

Davis, P., and Hersch, R. (1981). The mathematical experience. Boston, MA: Houghton Mifflin.

Hoetker, J., & Ahlbrandt, W. (1969). The persistence of recitation. American Educational Research Journal, 6, 145-167.

Mehan, H. (1979). Learning lessons. Cambridge, MA: Harvard University Press.

Sinclair, J., & Coulthard, R. (1975). Towards an analysis of discourse: The English used by teachers and pupils. London: Oxford University Press.

Steffe, L.P. (1990). Personal communique.

Steffe, L.P., & Cobb, P., & von Glasersfeld, E. (1988). Construction of mathematical meanings and strategies. New York: Springer-Verlag.

Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). Children's counting types: Philosophy, theory, and application. New York: Praeger Scientific.

Wood, T., Cobb, P., & Yackel, E., & Wheatley, G. (1990). Reflections on Learning and Teaching Mathematics in Elementary School. Paper presented at the seminar series "Constructivism in Education." Athens, University of Georgia.

Voigt, J. (1985). Patterns and routines in classroom interaction. Recherches en Didactique des Mathematiques, 6, 69-118.

Yackel, E., Cobb, P., & Wood, T. (in press). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education Monograph.

Note

The research reported in this paper was supported by the National Science Foundation (NSF Grant MDR 874-0400 and MDR 885-0560) and all opinions expressed are those of the author.

Statistical Reasoning

ESTRATEGIAS Y ARGUMENTOS EN EL ESTUDIO DESCRIPTIVO DE LA ASOCIACION USANDO MICROORDENADORES

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SUMMARY

In this work problem solving procedures about statistical association using microcomputers are analysed. The effect of several didactical variables on these procedures is also studied. The sample was formed by 18 training teachers that received a previous teaching during a seven week period. The study of the arguments expressed by the students allows us to know the scope and meaning given by them to the concept of association and to infer criteria to desing new didactical situations.

INTRODUCCION

El uso de ordenadores como recurso en los procesos de enseñanza - aprendizaje de las matemáticas plantea problemas de investigación de reconocido interés en didáctica de las matemáticas, como se pone de manifiesto en Howson y Kahane (1986). Para el caso de la enseñanza de la Estadística, el hecho de que los alumnos dispongan de la herramienta informática para el proceso de datos hace posible ponerles ante situaciones - problemas relativos al análisis de ficheros de datos de aplicaciones reales. De este modo, los programas de cálculo y representación gráfica pueden actuar como amplificadores conceptuales (R. Lesh, 1987) de las nociones y procedimientos estadísticos, al permitir el planteamiento de cuestiones más abiertas e inducir una actividad matemática exploratoria sobre los conjuntos de datos.

En este contexto teórico situamos este trabajo, que analiza los procedimientos seguidos por los estudiantes en la resolución de problemas sobre el estudio descriptivo de la asociación entre variables estadísticas y el efecto sobre

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Este trabajo forma parte del Proyecto PS88-0104 subvencionado por la Dirección General de Investigación Científica y Técnica (M.E.C. Madrid)

dichos procedimientos de distintas variables didácticas. Este análisis permite deducir el alcance y sentido dado por losalumnos al concepto de asociación y determinar criterios para la construcción de nuevas situaciones didácticas.

Desde el punto de vista metodológico usamos la técnica del registro de la interacción alumno- ordenador, lo que nos permite identificar los procedimientos de resolución de los problemas. Sin embargo, se muestra la necesidad de recurrir también a las argumentaciones escritas en que basan sus respuestas para poder apreciar la correción y pertinencia de los procedimientos empleados y para controlar el sentido de los conocimientos estadísticos adquiridos.

METODOLOGIA

Experiencia de enseñanza y logicial utilizado

Este trabajo es parte de una investigación sobre enseñanza de la Estadística Descriptiva que se ha desarrollado con un grupo de 18 futuros profesores (estudiantes de edades entre 19 y 20 años) durante un periodo de 7 semanas. En este tiempo se han alternado dos horas semanales de exposición teórica y de resolución de ejercicios con papel y lápiz, con otra de práctica en el aula de informática. Los contenidos abarcados en las 21 horas de enseñanza han sido los de estadística descriptiva uni y bivariante y una introducción a la filosofía y algunos gráficos del análisis exploratorio de datos.

Los programas empleados en la experiencia, forman parte del paquete PRODEST que se describe en Batanero y cols. (1988). Los procedimientos empleados, además de los de grabación y depuración de ficheros, son los siguientes:

<u>Práctica 1 (9 actividades): programa CONTAJE</u>

Realización de tablas de frecuencias de variables estadísticas cualitativas o discretas, diagramas de barras, diagramas acumulativos y gráficos de sectores.

<u>Práctica 2 (9 actividades): programa HISTO</u>

Cálculo de tablas de frecuencias de variables estadísticas continuas, agrupando en intervalos, histogramas, polígonos de frecuencias y polígonos acumulativos.

Práctica 3 (7 actividades): programa TRONCO

Realización del gráfico del tronco (stem and leaf).

Práctica 4 (6 actividades): programa MEDIANA:

Cálculo de los estadísticos de orden y gráfico de la caja (box and whiskers).

Práctica 5 (7 actividades): programa ESTADIS

Estadísticos elementales de valor central, dispersión y forma.

Práctica 6 (9 actividades): programa TABLAS:

Tablas de contingencia, con posibilidad de agrupación de datos.

Práctica 7 (10 actividades): programa REGRESIO

Cálculos y gráficos de regresión y correlación bivariante.

Evaluación del aprendizaje

El proceso de aprendizaje se ha observado mediante la grabación en ficheros de las interacciones alumno - ordenador en todas las actividades, así como la respuesta escrita de los alumnos a las cuestiones planteadas y sus argumentaciones. Además se ha realizado una evaluación final de cada alumno, también grabada, que consistió en la respuesta a una serie de cuestiones, usando el ordenador, sobre el análisis de un fichero relativo a datos deportivos reales de alumnos de un centro de Educación General Básica. Las variables incluídas en el fichero se muestran en la Tabla 1.

TABLA 1: TIPOS DE VARIABLES INCLUIDAS EN EL FICHERO DE DATOS

Tipo:	Variables:
Dicotómicas:	Sexo Practica o no deporte
Discretas, (con necesidad de agrupación):	Pulsaciones en reposo Pulsaciones después de 30 flexiones
Continuas:	Tiempo en recorrer 100 m. en Septiembre Tiempo en recorrer 100m. en Diciembre

Las cuestiones sobre asociación planteadas en la evaluación final fueron las siguientes:

1.- ¿Hay diferencia en la práctica de deporte entre chicos y chicas?

2.-¿Hay relación entre las variables NUMERO DE PULSACIONES EN REPOSO y PRACTICA ALGUN DEPORTE?

3.-¿Hay relación entre las varibles PULSACIONES EN REPOSO y PULSACIONES DESPUES DE 30 FLEXIONES? ¿De qué tipo?

4.- ¿Ha disminuido el tiempo que tardan los alumnos en recorrer 30 metros de Septiembre a Diciembre?

Análisis de los datos

Con objeto de obtener una codificación de la información recogida para cada alumno se ha aplicado la técnica de análisis de contenido. Al tratar de responder a las cuestiones planteadas, el alumno utiliza uno de los programas disponibles y puede ocurrir que en la primera ejecución resuelva el problema o que se produzca un error y deba intentarlo de nuevo. De este modo, se ha tomado el intento de solución como unidad de análisis, obteniéndose en total 120 unidades de análisis para las 4 cuestiones propuestas y los 18 alumnos, lo que proporciona un número medio de 1.66 intentos por alumno y pregunta. En la Tabla 2 se muestran los diferentes procedimientos empleados en la solución de las cuestiones planteadas.

RESULTADOS Y DISCUSION

Procedimientos de análisis de datos.

Un primer hecho que puede observarse en la Tabla 2 es que el total de intentos para una misma cuestión es, en general, mayor que el número de alumnos (18). Aparte de los errores de ejecución (no incluídos en la Tabla 2) que han supuesto el 6.1 por ciento, en algunos problemas. el alumno debe elegir una solución entre otras posibles (por ejemplo, la amplitud del intervalo), por lo que hace varios ensayos con un mismo programa. En otros casos no es así, pero emplea más de un programa para dar su respuesta.

Observamos que el procedimiento empleado para estudiar la asociación depende de la forma en que se plantea la pregunta, influencia ya señalada por diversos autores. Así, los ejercicios 3 y 4, referidos a la diferencia entre muestras relacionadas son resueltos en forma distinta. En el primer

caso, al plantear la pregunta en términos de relación entre las dos variables, induce el empleo casi exclusivo de la regresión. En el segundo, la referencia a una diferencia provoca el estudio separado de cada variable por diversos métodos, principalmente por medio de las tablas de frecuencias (HISTO) o de los estadísticos de orden (MEDIANA).

TABLA 2
FRECUENCIA DE INTENTOS SEGUN CUESTION Y PROGRAMA USADO

	Cuestión				
Procedimiento	1	2	3	4	Total
CONTAJE	19			2	21
HISTO	2			7	9
TRONCO				1	1
MEDIANA				6	6
ESTADIS				4	4
TABLAS	9	23	2	3	37
REGRESIO		20	20	2	42
Total	30	43	22	25	120

Este no es, sin embargo, el único factor observado. Dentro de una pregunta planteada en los mismos términos, es esencial el tipo de variable estadística presente en la tarea. Por ejemplo, en el problema 1, la pregunta se plantea en términos de diferencias, pero en este caso, aunque todos los alumnos lo resuelven estudiando separadamente las tablas de frecuencias de las variables (CONTAJE), un número importante de ellos, además, confirma sus resultados empleando el estudio conjunto (TABLAS). Asimismo, la pregunta 2 que se plantea en términos de relación es resuelta, en primer lugar, con el de la regresión y, además, mediante la tabla de contingencia.

Argumentaciones

En los párrafos anteriores hemos analizado los procedimientos de resolución seguidos por los alumnos. Este análisis no es, sin embargo, suficiente para asegurar la corrección de las respuestas: por un lado, hemos visto que cada pregunta podría resolverse con varios procedimientos, pero, lo esencial es la información utilizada de los mismos y la interpretación dada a esta información. Por este motivo, se han analizado las argumentaciones ofrecidas por los

estudiantes, que se relacionan en la Tabla 3. Los tipos de argumentos hallados en las preguntas de asociación se han clasificado en la forma siguiente:

Frecuencias absolutas: Razonamientos basados en la frecuencia absoluta de las distribuciones univariantes o de las condicionales o marginales de una tabla de contingencia.

Frecuencias relativas. proporciones o porcentajes en distribuciones univariantes o en las condicionales o marginales de una tabla de contingencia.

Percentiles o rangos de percentiles, frecuencias acumuladas ascendentes o descendentes y argumentos basados en los estadísticos de orden, a excepción de la mediana.

Valores centrales: comparación de medias, modas o medianas de las distribuciones.

Coeficiente de correlación (valor absoluto y signo).

Nube de puntos: Forma creciente o decreciente de la nube, forma aproximada de línea recta, mayor o menor dispersión de los puntos de la nube.

Dispersión: Comparación de varianzas, desviaciones típicas u otros estadísticos de dispersión.

Otros argumentos: forma de los gráficos, presencia de varias modas, realización de cálculos adicionales, etc.

El número de argumentos es mayor que el número de estudiantes, porque a veces utilizan dos o más argumentos para el mismo problema. En las dos preguntas en que interviene una variable dicotómica (1 y 2), los argumentos más utilizados correctamente son los relacionados con frecuencias: relativas, absolutas y rangos de percentiles, a pesar de que en la segunda cuestión casi todos los alumnos han empleado el programa REGRESIO, además de la tabla de contingencia.

Independientemente de que la pregunta haya inducido a estudiar las variables en forma conjunta (segunda cuestión) o no (primera), parece deducirse de la argumentación que la asociación de una variable con otra dicotómica se interpreta preferentemente como diferencia en las distribuciones de frecuencias condicionales. También concluímos que el uso del procedimiento REGRESIO en la cuestión segunda se ha utilizado a modo de confirmación visual de la respuesta, ya que sólo unos pocos alumnos han empleado un argumento sobre el valor

del coeficiente de correlación.

La tendencia a basar sus argumentos en valores numéricos, se presenta de nuevo en la tercera cuestión, en la que las razones basadas en el coeficiente de correlación superan a las basadas en la forma de la nube de puntos. Ambos tipos de argumentos ponen de relieve que la idea de relación entre variables cuantitativas es sinónima de la de dependencia, aunque aleatoria.

Al analizar los argumentos empleados en la última cuestión, observamos de nuevo la influencia de la forma en que está hecha la pregunta, ya que pocos alumnos identifican la asociación existente entre las dos variables cuantitativas (tiempo en septiembre y tiempo en diciembre). Muy pocos argumentos correctos o incorrectos se refieren a ello. La mayor parte de argumentos correctos se refieren a la comparación de los valores centrales de dihas variables. Aunque la mitad de alumnos han empleado el programa HISTO, pocos han comparado las frecuencias relativas de las dos distribuciones para los mismos intervalos de valores de los tiempos. Por ello pensamos que tampoco han llegado a identificar el problema como uno de estudio de asociación de una variable cuantitativa (tiempo en recorrer 100 m.) respecto a una dicotómica (antes/después del entrenamiento); sino más bien como de diferencias de dos variables cualesquiera no relacionadas entre sí.

CONCLUSIONES

El análisis de los resultados ha puesto de manifiesto que un proceso de enseñanza de nociones estadística basado en la resolución de problemas realistas de análisis de datos, hecho posible al disponer del recurso informático, es insuficiente para que los estudiantes adquieran por completo todo el alcance y sentido de las nociones estadísticas de asociación e independencia. El diseño de situaciones didácticas, basadas en el uso de logiciales, precisa tener en cuenta la naturaleza de las distintas variables didácticas, y atribuir un papel relevante al profesor como gestor de las situaciones, especialmente para explicitar los matices del saber matemático incorporado en los problemas.

TABLA 3

FRECUENCIAS DE ARGUMENTOS EN LAS CUESTIONES SOBRE ASOCIACION

Cuestión	Argumentos correctos		Argumentos incorrectos		Total
1	Frec. relativa	15	Frec. relativa	2	
	Frec. absoluta	7			
	Otros argumentos	5			
	Total cuestion 1:	27		2	29
2	Frec. relativa	7	Correlación	1	
	Percentil/rango	6	No justifica	1	
	Frec. absoluta	4			
	Dispersión	3			
	Correlación	3			
	Otros argumentos	4			
	Total cuestión 2:	27		2	29
3	Correlación	13	Frec. relativa	2	
	Nube de puntos	6	Otras	3	
	Otras	3			
	Total cuestión 3:	22		5	27
4	Valor central	11	Frec. absoluta	3	
	Otras	7			
	Total cuestión 4:	18		3	21

REFERENCIAS

Batanero Bernabeu, M. C.; Díaz Godino y Estepa Castro, A. (1987). Un paquete didáctico de programas para el laboratorio de estadística. En I. Brincones y cols. (Ed.). Actas del I Simposio Internacional de Educación e Informática. I.C.E. de la Universidad Autónoma de Madrid.

Howson, A. B. y Kahane, J. P. (1986) The influence of computers and informatics on mathematics and its teaching. Cambridge: Cambridge University Press.

Lesh, R. (1987). The evolution of problem representatios in the presence of powerful conceptual amplifiers. En C. Janvier (Ed). Problems of representation in the teaching and learning of mathematics. London: Lawrence Erlabaum Assoc.

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COMPUTERIZED TOOLS AND THE PROCESS OF DATA MODELING*

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We are attempting to identify conceptual challenges that students encounter as they design, collect and analyze data about a real situation. We propose the term "data modeling" to describe this process, and present a new computerized tool for working with data called the Tabletop. While the Tabletop is a tool for analyzing data, we conjecture that it can help students become better designers of data. Examples from clinical research in progress help to show how closely intertwined are the phases of data modeling, and thus begin to resolve the apparent paradox of how a technological tool for one phase can benefit others.

In the last few years there has been a renewed interest among mathematics educators in the curricular value of working with real data, as a way to learn to construct and evaluate mathematically supported arguments, and as a meaningful context for learning mathematical concepts and techniques. The Hands On Data project, in which the authors are currently engaged, is one of many projects exploring this general theme. Two terms around which interest has crystallized are "data analysis" and "statistics." Under the heading "data analysis", for example, the Mathematical Sciences Education Board (MSEB, 1990, p. 42) lists "collection, organization, representation, and interpretation of data; construction of statistical tables and diagrams; and the use of data for analytic and predictive purposes." Under "statistics", the recent NCTM standards (NCTM, 1989, p.105) stipulate that students in grades 5-8 should learn to:

- systematically collect, organize and describe data;
- construct, read, and interpret tables, charts and graphs;
- make inferences and convincing arguments that are based on data analysis;
- evaluate arguments that are based on data analysis;
- develop an appreciation for statistical methods as powerful means for decision making.

These are good lists: they mark off a coherent set of interdependent skills and concepts. To collect and organize data systematically, after all, one needs to understand how the data will be graphed, charted, analyzed and used in arguments; to evaluate an argument based on data analysis one needs to consider whether the data was collected and organized appropriately to support the argument. But do the headings do justice to the lists?

We think that it is important to emphasize, as terms such as "data analysis" and "statistics" do not, that designing and collecting data is a creative act. The word "data" (from the Latin "givens") is itself something of a misnomer. Data are not given to us by the world; rather, we create data to model aspects of our experienced world that interest us (Goodman, 1978). We often use the word "data" as a mass noun, like "hay" or "cheese." But data does not come by the bale or by the pound, or even by the kilobyte; data is built in structures. These structures are in turn amenable to

* This paper has been prepared with the support of the National Science Foundation, grant # MDR-8855617. We also gratefully acknowledge the donation of equipment by Apple Computer, Inc. The views expressed are not necessarily those of the NSF or of Apple Computer. The ideas in this paper are being developed in collaboration with the other members of the Hands On Data research team: Phyllis Kalowski, Anne Lopoulos, Alana Parkes and Linda Waxler.

manipulations which yield results to which we may assign meaning. In inquiry and in pedagogy, we can begin to make sense of data when we consider the data, the structure and the manipulations as coherent whole — a *data model*. We therefore use the term *data modeling* to describe the highly intertwined sets of skills and activities by which data models are created, manipulated and interpreted.

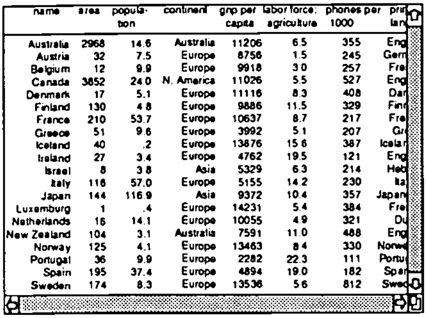
The process by which one creates a data model of a real situation ("systematically collecting and organizing data") is complicated and conceptually rich. If a teacher is not expecting these complexities, their intrusion in a classroom data modeling activity is likely to be perceived as a failure of the activity. For example:

> One mixed-grade class wanted to order special T-shirts for a group to which they belonged. They decided to make a database of peoples' names, sizes and color preferences, to help assemble the order. However, the students did not think of establishing data conventions, and it turned out at the end that the sizes had been entered in many different formats — as "L," "Large," "Children's Large," "Size 12," "Women's Large," et cetera. No time had been allotted to deal with this issue, and in the end the order was assembled by hand.
>
> Students in another class tried to answer the question, "How do students in our school spend their time?" A questionnaire yielded a bewildering variety of descriptions of how time was spent, including "playing outside," "playing baseball," "watching TV," "doing chores," "doing homework," "mowing the lawn," "transportation," "at home," "outside" and so on. Neither the students nor their teacher had anticipated the challenging task that remained, which was to organize these responses, some at different hierarchical levels, others overlapping, into a structured categorical system that they could use for further data collection and analysis.

Of course, we are not the first to recognized the complexity and importance of data creation. Both Taba (1967) and the USMES project (USMES, 1976) designed curricula with careful attention to the process of creating data, and the MSEB urges that students deal with "the messy reality of worldly data" and comments, "The inevitable dialog that emerges between the reality of measurement and the reality of calculations -- between the experimental and the theoretical -- captures the whole science of mathematics" (1990, p.43). The contribution which we hope to make here, beyond a plea for better terminology, is to discuss the interrelationship of data creation and data analysis in the light of some clinical examples and a software environment being developed as part of our project.

The Tabletop

The *Tablemaker* is an integrated environment for creating, organizing, exploring and analyzing record-oriented data (i.e. data about a set of objects with the same attributes, or "fields," recorded for each object). It is currently implemented in prototype form. It includes a conventional record-oriented database view which allows one to define fields and enter and edit data (illustrated in figure 1, with sample data about 24 countries of the world). It also includes the *Tabletop*, a radically different interactive representation for exploring and analyzing the data. The Tabletop's representational system proceeds from two principles: first, the screen shows one icon for each item in the database (or in the subset of the database currently being examined); and, second, the icons can be moved about the screen in a variety of ways to reveal properties of the data.



name	area	popula-tion	continent	gnp per capita	labor force: agriculture	phones per 1000	pri... lan...
Australia	2968	14.6	Australia	11206	6.5	355	Eng
Austria	32	7.5	Europe	8756	1.5	245	Gern
Belgium	12	9.9	Europe	9918	3.0	257	Fre
Canada	3852	24.0	N. America	11026	5.5	527	Eng
Denmark	17	5.1	Europe	11116	8.3	408	Dar
Finland	130	4.8	Europe	9886	11.5	329	Finr
France	210	53.7	Europe	10637	8.7	217	Fre
Greece	51	9.6	Europe	3992	5.1	207	Gr
Iceland	40	.2	Europe	13876	15.6	387	Icelar
Ireland	27	3.4	Europe	4762	19.5	121	Eng
Israel	8	3.8	Asia	5329	6.3	214	Heb
Italy	116	57.0	Europe	5155	14.2	230	Ita
Japan	144	116.9	Asia	9372	10.4	357	Japan
Luxemburg	1	.4	Europe	14231	5.4	384	Fre
Netherlands	16	14.1	Europe	10055	4.9	321	Du
New Zealand	104	3.1	Australia	7591	11.0	488	Eng
Norway	125	4.1	Europe	13463	8.4	330	Norwe
Portugal	36	9.9	Europe	2282	22.3	111	Portu
Spain	195	37.4	Europe	4894	19.0	182	Spar
Sweden	174	8.3	Europe	13536	5.6	812	Swed

Fig. 1: In addition to the Tabletop (figures 2-12), the Tablemaker provides this conventional row-and-column view for entering and editing data.

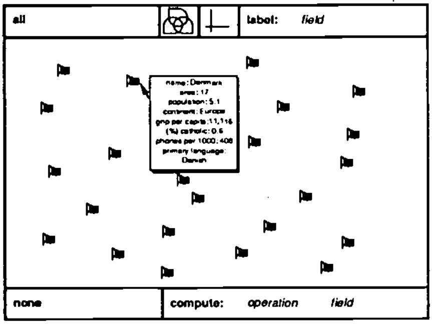


Fig. 2: The Tabletop shows one icon for each item in the database. The icons are initially scattered randomly. Any icon's detailed information can be examined at any time by pointing and clicking the mouse.

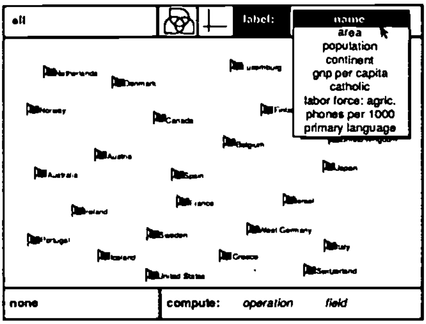


Fig. 3.: Icons can be labelled with their values for any field in the database. When an icon moves, its label moves with it.

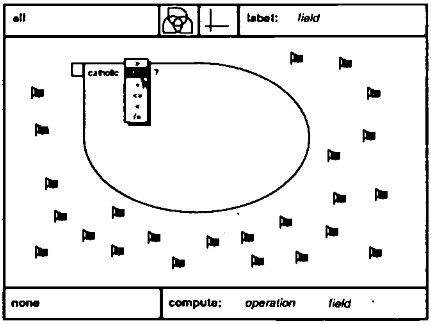


Fig. 4: A set constraint can be set up using pop-up menus.

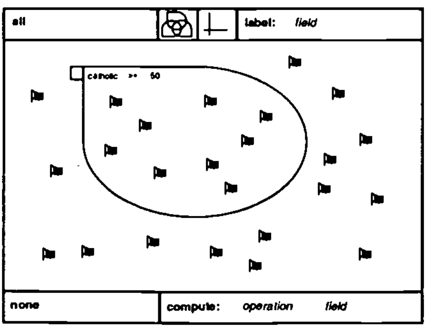


Fig. 5: Icons satisfying the constraint move into the circle. They move quickly and simultaneously, but smoothly so that any individual can be tracked.

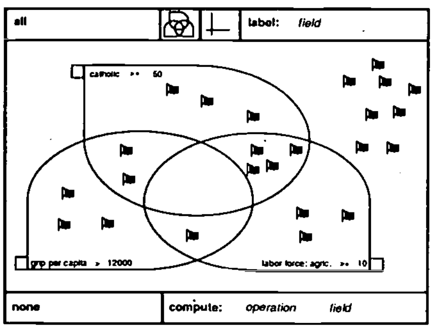


Fig. 6: Up to three circles can be active at once. Any part of any constraint can be modified directly, and the affected icons will immediately move to new positions.

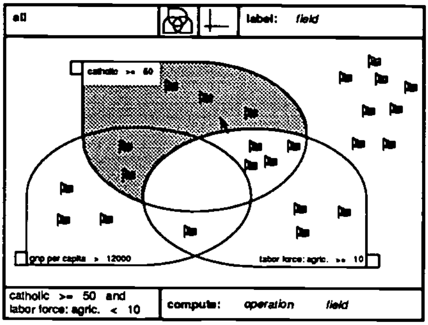


Fig. 7: Once identified, subsets of the database can be operated on in a variety of ways. For example, the selected icons can be marked so they can be tracked during later analysis. They can also be printed out, deleted, etc.

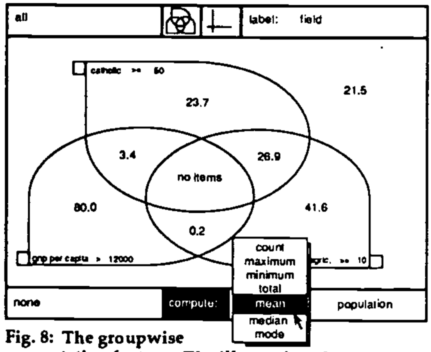


Fig. 8: The groupwise computation feature. The illustration shows mean population for each of the eight subgroups created by the Venn diagram.

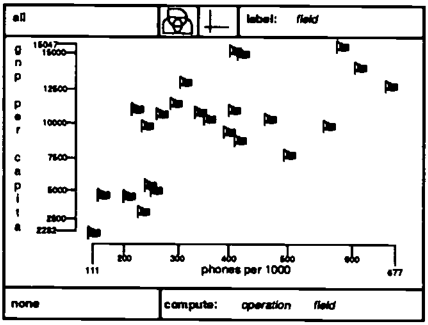


Fig. 9: To set up an axis, choose a field from the pop-up menu attached to the axis label. An appropriate scale appears automatically and the icons move to align themselves with it. Each axis constrains the icons in only one direction.

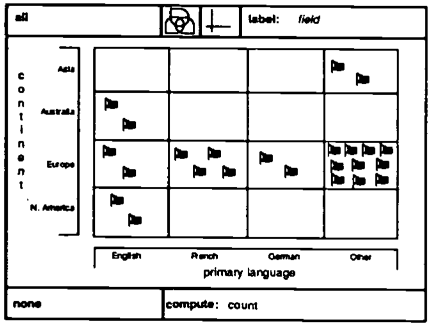


Fig. 10: If the fields associated with an axis are discrete, a cellular graph results. Hybrids with one discrete and one continuous axis are also possible.

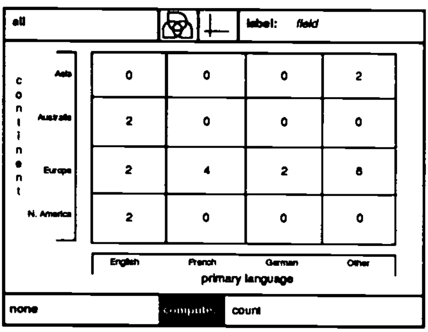


Fig. 11: In this case, performing groupwise computations produces a crosstabulation.

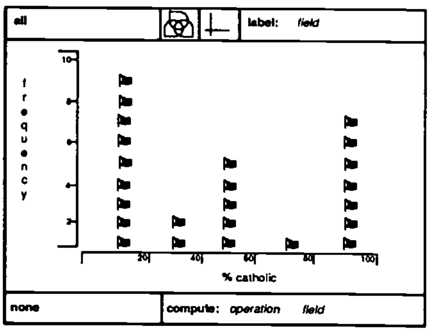


Fig. 12: Putting "frequency" on one axis causes the icons to pile up, producing a frequency distribution graph.

Figures 2-12 are snapshots of the screen's appearance during a sample session with the Tabletop. It is left to the reader to imagine what happens between snapshots. The general flow of interaction is that the user imposes structure on the screen space -- by setting up a circle with a constraint, or by associating the vertical or horizontal axis with one of the fields -- after which the icons move to take up positions dictated by the new structure. The icons move simultaneously and quickly, but smoothly, so that each individual's movement can be tracked. If one "drags" an icon to another part of the screen, it will hasten back to its proper place (a favorite feature of younger users). The icons can be made to carry labels based on any field in the database, and summary computations can be generated over subsets of the data. From these simple, pseudo-physical principles emerge scatter plots, frequency histograms, crosstabulations, Venn diagrams and a number of other "plots," less familiar but equally informative. All of these plots are built by the user in a way that helps to make their meaning clear. They can all be incrementally modified (changing "less than" to "less than or equal" in a Venn circle, for example) with immediate feedback. And they are all open to further querying: one can always change the labels on the icons, or examine an interesting one in detail by double-clicking (as in figure 2).

This software is new in many ways; there is much to discuss and more to investigate concerning its potential educational uses. Many questions relate to the intelligibility of the representations. In our piloting we have found that students as young as eight can interpret many of the representations. In fact, the enthusiasm and engagement with which many youngsters explore the system's behavior seems to hint that the Tabletop's representations resonate somehow with issues (categorization, inclusion and exclusion, ranking by order) which are of special concern at this age. The success of these children in interpreting and constructing scatter plots (as well as their initial misinterpretations and misconstructions) suggest, among other things, that the animated, manipulable scatterplot of the Tabletop may be an ideal context in which to learn the basic principles of two-dimensional graphing, well before encountering the extra complications of graphs of continuous functions.

As a data analysis tool, the Tabletop offers a number of important advances over the software currently in use in schools. It provides an unprecedented range of analytical operations together with representations that help to make them intelligible. Its flexibility and rich feedback stimulate exploration rather than holding students to a predetermined path of analysis. It invites students to look for unanticipated patterns, and to cross-check their first results. Time and effort previously devoted to simply generating graphs, charts and summary data can now be shifted to thinking about what they mean. But to predict the Tabletop's role in the complete data modeling process we must first consider the nature of that process.

Data Modeling

We begin with a truly naive and preliminary breakdown of stages of one basic kind of computer-aided data modeling. Of special interest to us is how the "phases" act, not as sequential components of a temporal process (which they can be on a trivial level), but rather as sets of "fingers" which grow into one another and mutually influence one another. Indeed, an important

measure of overall data modeling competence might be the degree to which the mutual influence of these phases is internalized in a way that structures planning and execution of each stage.

I. Early Phases of Data modeling: Designing the Data Model
- A. Problem Specification
  1. Finding or accepting the problem (determining a goal structure or understanding and accepting a given goal structure)
  2. Refining the problem's goal structure by refining the problem's elements:
     - a) Defining data structures (choosing one or many variables, establishing the unit of analysis. More advanced students might consider hierarchical or relational data structures).
     - b) Defining categories
     - c) Defining relations among categories
- B. Solution Specification
  1. Determining a solution method (e.g. a survey, observations, a data search, etc)
  2. Defining solution instruments or resources; establishing conventions for measurement and coding
  3. Designing sampling and control strategies
  4. Testing or piloting the solution

II. Middle Phases of Data Modeling: Data Gathering and/or Measurement
- A. Administering data gathering instruments or executing data search, etc.
- B. Coding data for computer entry
- C. Entering data into a computer database

III. Late Phases of Data Modeling: Data Analysis
- A. Exploring and describing the data (with the help of the computer) through charts, graphs, summary statistics, etc.
- B. Analyzing and interpreting relationships in the data
- C. Drawing conclusions, e.g. about causal relationships or the best course of action
- D. Presenting arguments for the conclusions, supported by data displays

To help give the flavor of how these issues actually arise during student inquiry projects, and to illustrate the significant challenges that come up in the design phases of data modeling, we present examples drawn from clinical work in progress.

One group, which we have seen for four one-hour sessions at the time of writing, consists of three fifth-graders of normal ability, experienced with computers but not with our software. They have been investigating student practices and preferences at lunchtime. In an initial brainstorming session the students (with some help from us) drew up a questionnaire which they would administer as interviewers. The first question asked whether the interviewee bought the school lunch, brought his or her lunch from home, or had no lunch. Additional questions established the contents of the meal, which parts were preferred and which parts, if any, were thrown away. The students planned to interview about ten students each on a day when the school cafeteria offered a popular lunch, and again on a day when the lunch was unpopular. The students' initial grasp of the logic of this kind of study was tentative, however:

> In the first session they felt that it was important *not* to interview the same students on two days (they had collected questionnaire data once before, but not in a varied-condition context). When they came to the second session with data for one day's lunch, Kamal and Glenn remarked that they had spoken to students who ate no lunch and had decided, contrary to the now-forgotten original plan, not to record them, since most questions on the questionnaire did not apply. Tyesha, however, had recorded one such case, marking all the other questions with a line. Of course this data would have made it possible to compare proportions of non-lunch-eaters on different days. But after two session devoted to entering the first day's data and beginning to learn how to manipulate it with the Tabletop, the

students seemed on their own initiative to have developed a renewed interest in and a somewhat sturdier comprehension of how aggregating the questionnaire data might help to reveal the difference between the two days.

The initial batch of completed questionnaires included many incomplete or idiosyncratic notations, from which the students (sometimes consciously, more often unconsciously) reconstructed the facts they needed. The data entry phase helped bring many of these issues to light:

> The first question on the lunch questionnaire, for example, asked "bought lunch, brought lunch or no lunch?" Glen tried to make sense of a questionnaire that appeared to say "no" lunch but went on to list favorite parts, etc. It turned out that Kamal had coded answers to this question as "yes" or "no," meaning bought or brought, respectively (he had disregarded those with no lunch). In another questionnaire, rather than enumerate the parts of the meal, another student had recorded simply "school lunch." The students reconstructed by recalling what had been in the standard school lunch that day. Reconstruction was less successful in the case of a lunch brought from home, the contents of which had been recorded as "lots of things." The students also found that they had written "bread" in several places, sometimes meaning the sub roll of a meatball sub and at other times meaning the slice of bread that came with meal. They resolved this ambiguity by coding the latter as "bread and butter." The researchers offered to enter the next round of data for the students, but warned that they would type exactly what they saw on the questionnaires. The second set of questionnaires had far fewer ambiguities.

Some of these issues in the recording and coding of data seem clearly allied with, if not identical to, what writers on literacy have called decontextualization. This is not surprising: once students begin to work with data in situations of purposeful activity with coordinated action by many participants, the appropriate use of the symbols and processes involved must clearly constitute a kind of literacy (Gee, 1989).

Our fifth-graders have not yet had to grapple with category design issues. They did encounter some questionnaires that showed evidence of students having brought some lunch from home and bought more at school. These prompted discussion, but no one has yet suggested modifying the bought/brought/none classification scheme. An older group of students, however, has proved more able to anticipate problems of categorization before collecting data. These three eighth-grade students of above-average ability began by defining the problem: to study the relations between their peers' musical tastes and other personal and academic characteristics.

> As they began to think about the design of a questionnaire a number of category design issues arose, mostly relating to music (categories of students -- "preppy," "burnout," etc. -- seemed less subject to debate). Is rhythm and blues a form of rock? Is there a difference between light rock and classical rock? Does a song need to be old in order to be a "classic?" How do you determine whether a song is hard rock or heavy metal? Is heavy metal a subcategory of hard rock? Interestingly, the word "subcategory" did not appear in the first session, so the last question was actually formulated as "is heavy metal hard rock?" However, that they were thinking in terms of hierarchy is clearly reflected in their use of indentation when writing notes.

More data design issues were forced as they actually began to draft the questionnaire:

> It occurred to them to gather additional data by asking about favorite songs and favorite artists. An important question arose when they needed to decide whether they would request a rank-order of music preferences or use a Likert-like preference scale. After initially deciding to have students list their preferences, they began to worry about how long the form would take to administer. This forced reconsideration of their decision. Their

first revision of the idea was to ask whether the respondent listened to a given category and then only request rank order of those catogories and their subcategories. The messiness of this approach led them to switch to a preference scale which would be used only if the student first responded "yes" that they listened to that type of music. The phrasing of the preference question was briefly problematic, turning on the question of whether to ask how much the student listened to this type of music, or how much they liked it. They decided on the amount of listening. This stimulated a further decision to request the respondent's favorite radio station.

This data is extremely preliminary, given especially that the activities described are still in progress. However, it is already possible to see the importance of junctures in the data modelling process where reflection is provoked. The eighth-graders began to think more carefully about categories when they actually tried to write the questionnaire questions. The fifth-graders began to reflect on the problems in their coding as they tried to enter it into the computer, and they seemed to develop a clearer idea of their multiple-day data collection strategy during a period of "messing about" with their first day's data on the computer. We are beginning to see the iterative process of data modeling, not as a grand loop from data design to data collection to data analysis and back to the beginning, but as a tangle of loops small and large, connected at many such junctures. For experienced data modelers, the issues of each phase are implicitly present at every other phase, with the net effect that looping is minimized (but not eliminated, of course). The less experienced the student, however, the more these loops need to be actually traversed. One challenge to curriculum development is to insert junctures that trigger the right data modeling questions authentically and quickly, so as to conserve students' time and enthusiasm.

The Tabletop's improved support of data analysis should help students become better data designers by helping them better understand what they would like to do with the data, perhaps using the Tabletop's visual representations as guiding images of the desired end result. Procedurally, the Tabletop eliminates previous bottlenecks in the data analysis phase and thus facilitates iteration throughout the entire data modeling process (Data entry is a remaining bottleneck which we hope to widen with future extensions to the software). But perhaps its role is best understood by thinking of the Tabletop not as a tool for doing data analysis, but as a medium for embodiment of the data model itself. It provides a near-transparent interface to the formal system constituted by the data together with the formally allowable operations on the data, which is precisely what we have defined the data model to be. Since all phases of data modeling are concerned with the data model, the central role of the software is thus no surprise. This perspective on the role of technology in inquiry, namely as a medium for the embodiment of models and theories, has guided the design of our software. We hope to test and refine it as our clinical work continues.

REFERENCES

Gee, J.P. (1989). What is literacy? The Literacies Institute tech. rep. #2. Newton, MA: Education Development Center.

Goodman, N. (1978). *Ways of worldmaking*. Indianapolis: Hackett.

Mathematical Sciences Education Board of the National Research Council (1990). *Reshaping School Mathematics: a philosophy and framework for curriculum*. Washington: National Academy Press.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics* . Reston, VA: NCTM.

Taba, H. (1967). Teacher's handbook to elementary social studies. Reading, MA: Addison-Wesley.

Unified Sciences and Mathematics for Elementary Schools (1976). *The USMES guide*. Fourth edition. Newton, MA: Education Development Center

Table 1

Percentage of children giving correct answers for changes to total caused by changes to parts

	TASK												
	1 - BLOCKS			2 - BUTTONS			3 - ZOO			4 - EQUATIONS			MEAN
Age	+	-	-/+	+	-	-/+	+	-	-/+	+	-	-/+	
4	83	100	83	42	75	58	17	58	50	0	0	0	47
5	100	100	75	67	67	67	92	75	83	0	0	0	61
6	100	100	100	100	83	92	92	67	75	25	25	17	73
7	100	100	100	92	100	92	100	92	100	92	75	92	95
Mean for task			97			78			75			27	

\+ object added
\- object removed
-/+ object moved

There was not much difference between Tasks 2 and 3, despite expectation that seeing counted parts might make it more difficult for children to attend to the whole. There was no significant difference between children's accuracy on the different operations, although the youngest children showed a somewhat better understanding of the effect of decreasing a part than they did of the other operations.

Table 2 shows the percentage of children who were able to provide explanations for each operation on each task. As expected, children could provide correct answers before they could give explanations. A similar pattern to that of Table 1 shows that children's ability to explain the operations increased with age, and that it was easier to explain the effect of changes to parts of uncounted quantity than to explain changes to counted quantity or derived equations. In addition, changes due to compensation were significantly more difficult to explain than were the other operations ($p<.01$).

Table 2

Percentage of children giving appropriate reasons for changes to total caused by changes to parts

	TASK												
	1 - BLOCKS			2 - BUTTONS			3 - ZOO			4 - EQUATIONS			Mean
	+	-	-/+	+	-	-/+	+	-	-/+	+	-	-/+	
Age													
4	83	92	58	17	33	33	8	33	17	0	0	0	31
5	83	92	50	50	50	50	67	50	50	0	0	0	45
6	100	92	100	100	67	83	75	58	58	8	17	0	71
7	100	100	100	92	100	92	100	92	100	58	75	42	88
Mean for Task			88			73			67			17	

+ object added
- object removed
-/+ object moved

Examples of rationale given by even the youngest children showed a clear understanding of the effect of operations on the total.

D(4) for moved block *"Because just moving them apart doesn't mean you've got - it's not the same"*
J(5) for added block *"because you've got not much... and I've got more"*
A(7) for removed block *"Because you took one of mine away and that means you've minused a number away from mine and you haven't from yours"*

Similar reasoning was given for changes to counted quantities.

D(4) for moved button. *"Because even though I sneaked it into this hand it doesn't make that it's just a different number...if you just put them altogether"*

Compensation on equations was the most difficult to explain, even for seven-year-olds. Clear explanations showed children to be visualizing the numbers as sets.

K *"..if you know that 10 and 10 are 20, what do you think 11 and 9 would be?"*
W(7) *"20"*
K *"Why?"*
W *"Because 11 and 9 it's still the same number, because if you put one of the group of 11 over to the 9 group it would - they would both be 10 and that means it must be 20."*

Discussion

The results of this study support Resnick's proposition that children understand the relationship of changes to the parts to changes to a whole, before they start school. All children, from four years up, clearly understood the effect on the whole of removing something from one of the parts. In contradiction to Resnick's expectation, not all of the children of four and five years understood the effect adding to one of the parts or moving something from one part to another part, but all children understood all three operations by age six, and could explain them. Their understanding of similar relationships with counted quantity and with equations increased with age, presumably with their understanding of the number system. Explanations for these changes were similar in nature to those given for changes in uncounted quantity, suggesting that they were based on common schemas.

The children's formal school curriculum and informal pre-school activities were investigated and observed, to see if any of these concepts were taught. There was no evidence of any systematic teaching of the protoquantitative concepts. It can therefore be assumed that children constructed these protoquantitative schema from their general experience.

There was systematic teaching between the ages of six and seven which would support the construction of a schema for compensation with numbers when children studies "all the names of a number", i.e. 8 is the same as 7 and 1, 6 and 2, etc. There were fewer experiences that would support the concept that an addition or subtraction to one of the parts would have an effect on the whole

although studying the words "more" and "fewer" add to this. In discussion with teachers it appeared that they had no regular way of assessing the schemas that children came to them with or incorporating these schemas in their teaching program.

The children interviewed did appear to be integrating what they were taught with their existing schemas, but without the conscious help of their teachers. These where children who shared the same culture as their teachers, and therefore communication with their teachers was likely to have been good. It could well be that children who did not share the culture of the school would be less likely to integrate their learning in this way. It would then be particularly important for teachers to understand these underlying schemas, so that they could help children construct schemas for operating with equations which were integrated with their pre-school, protoquantitative schemas for part/whole relations.

References

Carpenter, T. P. & Moser, J. M. (1984) The acquisition of addition and subtraction concepts in grade one through grade three. Journal of Research in Mathematics Education. 15 179-202.

Gelman, R. & Gallistel, C. R. (1978) The child's understanding of number. Cambridge MA: Harvard University Press.

Putnam, R. T., deBettencourt, L. U. & Leinhardt, G. (in press) Understanding derived fact strategies in addition and subtraction. Cognition and Instruction.

Resnick, L. B. (1986) The development of mathematical intuition. In M. Perlmutter (Ed.) Perspectives on intellectual development: The Minnesota Symposia on child psychology. Vol 19, pp 159-194.

Resnick, L. B. (1989) Developing mathematical knowledge. American Psychologist, 44, 162-169.

Steffe, L. P. & Cobb, P. (1988) Construction of arithmetic meanings and strategies. New York: Springer-Verlag.

FACTORS AFFECTING CHILDREN'S STRATEGIES AND SUCCESS IN ESTIMATION

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In this study, secondary children were asked to give estimated answers to problems where solutions could be found by a single use of either multiplication or division involving whole numbers and decimals. Equivalent problems were presented either purely symbolically as "sums" or in simple contexts as "word problems". It was found that the "word problems" were generally answered more successfully than the equivalent "sums". The difference was particularly marked in those cases for which children find it most difficult to identify the operation required to solve the word problem. Interviews revealed that provision of a context appeared to enable children to avoid inappropriate algorithmic strategies for estimating and to use informal knowledge and strategies successfully.

"Front-end" methods, in particular rounding, are the estimation strategies most generally taught in schools but, while these may be used successfully by children with good arithmetical skills and thorough understanding of number and operations (Reys et al 1982, Rubenstein 1985), many children have difficulties caused at least in part by 'deficiencies in the conceptualisation of multiplication and division and failure to think in terms of place value' (Levine 1982).

Unlike most traditional "school" mathematics, estimation does not have clear cut right or wrong answers. The level of accuracy provided by "front-end" methods may not be appropriate in many everyday contexts where only a general idea of the size of the answer is needed. The acceptability of an estimated answer, and hence the method used to make it, depend on the context and the purpose for which the answer is to be used (Usiskin 1986). Developing a variety of strategies and flexibility in their use may help children to estimate successfully in familiar contexts despite deficiencies in their computational skills and understanding. This study (Morgan 1988) aimed to investigate the factors affecting the difficulty children have with estimation and the strategies available to them.

The test design

A test was designed to investigate children's responses to being asked to estimate answers to questions which could be solved by a single operation of either multiplication or division of whole numbers and decimals. It was intended that the numbers involved should be too difficult for most of the children to be able to compute answers mentally in the time allowed for the test in order to ensure that their answers would be estimated.

The question variables included: the type of numbers involved (two whole numbers, two decimals, one of each); whether multiplication or division was required for the solution; the inclusion of numbers less than one either in the question itself or in the solution. In addition, each type of question was presented twice: in the form of an abstract "sum" and as a "word problem" in a simple context.

eg *6.23 - 8.85* and
"The cost of 6.44 metres of ribbon was £4.86. What would the price of one metre of ribbon be?"

Two parallel versions of the test (A and B) were used with half the children each in which the numbers were interchanged between the "equivalent" questions in and out of context to ensure that they could be compared; only one pair of "equivalent" questions was found to have been answered by significantly different proportions of children on the two tests.

Following the initial analysis of the test results, 18 of the 199 children were interviewed using some of the questions from the test in order to gain some insight into the strategies used. None of these children had had any recent instruction in estimation and only three of them could remember having been taught a method.

The test results

Examination of the distribution of the answers given by the children to the estimation test revealed for each question a group of children giving answers clustered around the correct answer. The clusters included more widely spread answers than just those which might have been obtained by conventional estimation techniques, indicating not only that other methods were used to gain many of the answers but also that the children's ideas of what constitutes an acceptable estimate may not be the same as those determined by "experts". For the purposes of this study, an answer within such a cluster was considered to be "successful". This included all answers which could have been achieved by conventional methods but also allowed for answers which might be achieved by using a non-computational concept of "reasonableness".

The facility of the questions was on the whole low (mean facility 40.9%) with only 4 of the 16 questions answered successfully by more than half the children. In every case the question presented in context was answered more successfully than the equivalent question presented out of context. Most of the children answered more questions successfully in context than out; some were successful with all eight questions in context but only two or three out of context. There was, however, a minority which was more successful out of context.

Multiplication v division

On the whole, division questions proved to be easier than multiplication questions and whole numbers easier than decimals, but there were exceptions to this, especially when numbers less than 1 were involved. (See Table 1.) The low success rate on questions 10&13 and 3&16, the questions involving long multiplication of whole numbers and of decimals respectively, was not expected, particularly when compared to the results for questions 5 and 12, which involved division of decimals.

All the children tested had been taught an algorithm for long multiplication of whole numbers, even if they were unable to use it correctly, but very few had experienced long division of decimals without using a

calculator. This may even have been a source of difficulty on the multiplication questions.

Table 1

Question	% successes
whole number multiplication	
10&13 (48*31 and 32*56)	
10 (no context)	34
13 (context: *A sweet manufacturer puts 32 sweets into each bag. How many sweets will be needed to fill 56 bags?*)	36
decimal multiplication	
3&16 (41.8*8.37 and 37.9*6.41)	
3 (context: *My car can go 41.8 miles on each gallon of petrol on the motorway. How many miles can I expect to travel on 8.37 gallons of petrol?*)	35
16 (no context)	31
decimal division	
5&12 (92.3/2.54 and 71.8/2.54)	
5 (context: *A table is 92.3 centimetres long. About how many inches is this? (1 inch is 2.54 cm.)*)	57
12 (no context)	41

When interviewed, several children attempted to implement the long multiplication algorithm mentally with mixed success:

> eg Kamaljeet, 2nd year girl
> question 16: 37.9*6.41 no context
> "Well first I'd get like, in my mind I put 37.9 just to one side. Then I get the first part of 6.41 and, you know kind of multiply 1 by each of those which'll stay the same. Just put 9. Then get 4, do 4 times 9 which is 27 .. 36. Then get 6, 6 times 7 which is 42 plus 3, 45 ...."

The answer which Kamaljeet eventually arrived at was 900. The complexity of the algorithm she was using prevented her from thinking about whether what she was doing made any sense. None of the children interviewed, however, attempted a long division algorithm. The most common strategy for questions 5&12 (92.3/2.54 and 71.8/2.54) was either to round 2.54 to 3 or to truncate it to 2 and then adjust the answer by guesswork. This process transforms a difficult calculation into a relatively easy one. Where an exact calculation was attempted for one of the long division items it was done by multiplication; for example, in question A1 (391/23 context: daffodils) trying to find a number to multiply 23 by in order to make 391.

The relatively low facility of the multiplication items does not confirm Rubenstein's (1985) finding that estimation involving multiplication was easier than division. It does, however, underline the point that estimation is a different process than calculation, as the APU (1980) results showed that children performed multiplication calculations more successfully than division calculations involving the same types of numbers. Although some of the sources of difficulty in calculation also appear to cause the same sort of

problems when estimating, it cannot be assumed that relatively easy types of calculations will also prove relatively easy to estimate as they may not encourage efficient strategies.

Large numbers

All the questions involving long multiplication by numbers greater than one resulted in underestimation by a considerable number of the children. For each of these questions, at least three quarters of the children gave answers smaller than the exact answer. This can be seen (Table 2) by comparing the exact answer to the upper quartile of the answers given by the children:

Table 2

Test A	question	answer	upper quartile
3	41.8*8.37	350	294
10	48*31	1488	1488
13	32*56	1792	1692
16	37.9*6.41	243	240

While some of these answers, particularly for questions 10 and 13, can be explained by place value errors which put them in the hundreds rather than the thousands, even those children who gave answers of the correct order of magnitude tended to underestimate. The effect of multiplication on the magnitude of an answer when large numbers are involved may be conceptually difficult for children. If they rely on guesswork or a vague sense of the size of a number in context in order to get an answer, they are likely to be less successful when the numbers are larger and hence more difficult to visualise because further from their experience; smaller numbers may be chosen as answers because they are more familiar and more meaningful to the children.

Numbers less than 1

The three questions on which the children were least successful illustrate the areas which have been found to cause most problems both in calculating and in identifying the required operation (Hart 1981): multiplication and division by numbers less than one and division of a smaller number by a larger. Examination of the answers to these questions reveals many children giving answers which indicate misconceptions about the nature of the operations involved. For question 2 (88.2*0.68 and 93.4*0.72 no context) there are answers slightly greater than 88.2 and 93.4 which imply a belief that multiplication always makes things bigger. Question 4 (4.86/6.44 and 6.23/8.85 no context) gave rise to answers which could have been obtained by dividing the wrong way. (See Table 3.)

Table 3

Some answers arising from misconceptions

A2	93.4*0.72	(no context)	23% between 93.4 and 105
B2	88.2*0.68	(no context)	22% between 88.2 and 91
A4	6.23/8.85	(no context)	17% between 1 and 2
B4	4.86/6.44	(no context)	25% between 1 and 2
A6	3/0.06	(no context)	35% = 0.02
B6	2/0.04	(no context)	35% = 0.02

These answers indicating misconceptions were not apparent when the equivalent problems were presented in context.

The effect of context

Each of the questions presented as a word problem was answered more successfully than the equivalent question out of context, confirming the findings of Reys et al (1980). The magnitude of this difference, however, varied widely from question to question. The largest differences occurred in those types of question for which the numerical form of the question had the lowest success rate: multiplication by a number less than one, division by a number less than one and division of a smaller number by a larger.

i) **multiplication by a number less than one:**
A7. *The price of cheese is 88.2 pence for each kilogram. What is the cost of a packet containing 0.68 kilograms of cheese?*
42% successful answers
A2. *93.4 * 0.72* 22% successful answers

ii) **division by a number less than one:**
A11. *A baker has a piece of dough weighing 2 kilograms. How many rolls weighing 0.04 kilograms each can be made from this dough?*
42% successful answers
A6. *3 / 0.06* 10% successful answers

iii) **division of a smaller number by a larger:**
A15. *The cost of 6.44 metres of ribbon was £4.86. What would the price of one metre of ribbon be?*
41% successful answers
A4. *6.23 / 8.85* 22% successful answers

These types of word problems are also the ones for which children have been found to have greatest difficulty with identifying the operation (Hart 1981). The test results suggest that many of the children are using strategies to estimate answers to the questions in context which do not involve identifying and using the required operation. This was confirmed during the interviews. Many of those interviewed achieved successful answers to these and other word problems without calculating and used their knowledge of the context to help them. Shoba (4th year girl) identified the operation required for question 7 (*The price of cheese is 88.2 pence for each kilogram. What is the cost of a packet containing 0.68 kilograms of cheese?)* incorrectly but achieved a reasonable answer by what she called "guessing:

S Eighty-eight point two divided by zero point six eight ... it'll be eighty-eight divided by point seven ... that's about three ... about ... seventy pence.
I And how did you get seventy?
S Well, eighty-eight ... it's about eighty-eight pence for one kilogram and point six of a kilogram would be much less. So I just guessed it.

Of the 9 children presented with the above question during the interviews, 5 did no calculation but used some "guessing" and achieved successful answers. Clearly this type of "guessing" involves some understanding of the number system as well as of the context but it does not involve any computation.

Question 15 (*The cost of 6.44 metres of ribbon was £4.86. What would the price of one metre of ribbon be?)* was "guessed" by 4 of the 6 children presented with it, in all cases successfully. Unlike the case of question 7, the operation was identified correctly by 3 of the 4 "guessers" but was not actually carried out. Rupermjit (2nd year girl) explained her answer of 70 pence in terms of rounding the numbers and dividing them but admitted that she had not actually done this:

I Where did you get seventy from?
RB Dunno. First I made the four pound eighty-six into five pounds then I made the six pound forty-four into .. the six point forty-four metres into seven metres then I sort of divided it.
I Did you actually do the dividing?
RB Not really.

Her use of the words "pound" and "metres" throughout her discussion of this question shows how strongly her thinking about the question was linked to the context.

Table 4
Use of "guessing" during interviews

"guesses" leading to	successful answers	unsuccessful answers
in context	14	3
out of context	2	1

A form of "guessing" was used in context by children with a wide range of scores on the estimation test. One boy who had answered 14 out of 16 questions successfully on the test used "guessing" to answer 3 out of the 4 questions with which he was presented during his interview. The one he did not "guess" was out of context. As can be seen in Table 4, "guessing" was frequently a successful strategy but was hardly ever used out of context.

The provision of context for a question also appeared to help children to avoid using inappropriate algorithmic strategies. The questions had been designed to be too difficult to calculate exactly mentally but in some cases this appears to have been the only strategy available.

The question which was most commonly approached algorithmically was question 16 (37.9*6.41 no context); possibly the children's confidence that they know the long multiplication algorithm encouraged them to use it for this question. (Question 10 (48*31 no context) was only given to one child during the interviews who also used an algorithmic strategy to answer it.) Only one high scoring 4th year boy used a front-end strategy for question 16; the other four children who were given this question all attempted to use all the digits to calculate an answer - only one of them was successful. When the same children were given the parallel question 3 (*My car can go 41.8 miles on each gallon of petrol on a motorway. How many miles can I expect to travel on 8.37 gallons of petrol?*) one of them failed to understand the context and gave no answer but the others all used front-end strategies.

This extract from Kulvender's (2nd year boy) answer to question 16 gives a flavour of the calculation he was attempting out of context:

"... Put one there, now leave four there. Then that'll be eight, eight plus four which gives you twelve. Two there, keep one there. Nine plus one, nine plus one is ten. Zero there and one there."

Trying to do a lengthy "written" calculation in his head gave him the final answer 10.21. He could not judge if this was sensible because "It's just these points and all that. That's what confused me."

In contrast, he very quickly gave the answer 341 to question 3 by multiplying 8 by 4 and adding on a bit more to get nearer the exact answer because "I couldn't do the lower numbers because it would take much more time, just did the front numbers."

The presence of a context seems to have made some of the children more able or more willing to use just "the front numbers" and hence able to gain more successful answers because of the simpler calculations involved. A feeling of uneasiness about using decimal numbers was expressed by several of the children during their interviews. Possibly a lack of clarity about the relative significance of numbers before and after the decimal point makes them unwilling to risk ignoring any of the digits and therefore leads to an attempted algorithmic strategy. In context it may be easier to understand that 8.37 gallons is only a little more than 8 gallons or that it does not make much difference whether you use 41 or 41.8 or 42 miles. A strategy which uses only the front-end of the numbers appears more acceptable in context.

Implications

There was a strong feeling among the children interviewed that, if possible, an exact answer would always be preferable to an estimate. Estimating the answer to a "sum" is a pointless exercise when a calculator can quickly give the exact answer. This presents a problem for motivating the learning of estimation strategies. It is vital that estimation should be made meaningful by providing appropriate contexts. Asking children to estimate out of context appears to prompt many of them to use inappropriate and inefficient algorithmic methods. Even the artificial, trivial contexts provided in this study allowed children to avoid this by using front-end methods or by considering the properties of the numbers and the context. Discussion of the purposes of estimation and of alternative strategies should give greater validity both to estimation as a mathematical activity and to successful "guessing".

References

APU, 1980, *Mathematical Development Secondary Survey Report 1* Hart,K., 1981, *Children's Understanding of Mathematics: 11-16,* John Murray

Levine,D., 1982, "Strategy use and Estimation Ability of College Students", *Journal for Research in Mathematics Education* *13(5)*

National Council of Teachers of Mathematics, 1986 Yearbook, *Estimation and Mental Computation*

Morgan,C., 1988, *A Study of Estimation by Secondary School Children*, Unpublished MSc dissertation, Institute of Education, London

Reys,R.,Bestgen,B.,Rybolt,J.,Wyatt,J., 1980, *Identification and characterisation of computational estimation processes used by in school pupils and out of school adults*, Washington DC: National Institute of Education

Reys,R.,Bestgen,B.,Rybolt,J.,Wyatt,J., 1982, "Processes Used by Good Computational Estimators", *Journal for Research in Mathematics Education 13(3)*

Rubenstein,R., 1985, "Computational Estimation and Related Mathematical Skills", *Journal for Research in Mathematics Education 16(2)*

Usiskin,Z., 1986, *Reasons for Estimating*, in NCTM 1986 Yearbook

ASSESSMENT IN PRIMARY MATHEMATICS: THE EFFECTS OF ITEM READABILITY

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A major cognitive task for students undertaking mathematics tests is to understand written questions. But achievement in tests is usually presumed to be a measure of computational ability: reading capability is seldom considered. The purpose of this study was to investigate the effects on student performance when some reading factors of test items were modified. Subjects (n = 186) were matched for reading ability. Higher achievement was associated with modification of technical terms, simplification of sentence structure and the elimination of extraneous reading (p<0.001).

The need for assessment tools to be action-oriented so that they can inform pedagogical practices has often been recognised (e.g. Bates,1984; Clarke,1989; Eggleston,1984). There is also a growing body of evidence that 'objective' tests do not provide a measure of only mathematical achievement or capability because performance is affected by socioeconomic, race, gender, language and other individual and group interactive variables. Failure is political, according to Mellin-Olsen (1987), with the language of teaching and testing as well as the mechanisms of the exam system playing their parts in the reinforcing mathematics success among the privaleged classes.

Yet tests are still administered to pupils for selection, placement and streaming purposes, thus affecting self-concept, personal expectations and teacher expectations as well as future educational opportunities. Mathematics tests thus continue to play a key role in the selective assessment which leads to cultural reproduction (Broadfoot,1979). This piece of research, for instance, used a test originally designed for allocating 186 potential high-school students to different Year 7 classes which would then experience different curricula.

The purpose of this study was to examine the effects on student performance of systematically rewording mathematics problems. This was a pilot study, intended only to examine some language factors seem important in mathematical cognition and test performance, and therefore worthy of further enquiry. It was hypothesised that pupils would write more correct answers to mathematics test questions which had been modified by altering or avoiding some technical terms, writing numerals in symbolic form, simplifying sentences and by eliminating extraneous reading.

Theoretical framework

Reading capability has been demonstrated to affect word-problem cognition. For instance, Newman (1977), in studying errors made by 124 low-achieving sixth grade children, found that at least 35% of errors on mathematics achievement tests were caused by reading problems; so many pupils failed to arrive at a point where they could apply the relevant computational skills. It seems linguistic factors affecting the readability of test items prevent pupils transforming word problems into appropriate mathematical representations. Thus mathematically capable students may be penalised in mathematics classes and tests because of their reading capabilities.

Not only does the reading of mathematics require basal reading competence, but students need to have developed specialised reading skills unique to mathematics. Aiken (1972), in examining the interactive effects of intelligence, reading ability and general mathematical aptitude maintained that verbal ability was the key to the positive correlations between these three cognitive fields, but recognised that mathematical texts make extra logico-linguistic demands on readers. Karlin (1972) differentiated between general reading ability and 'critical' reading ability, with reading competence and confidence in one context area not transfering to the reading of mathematics word problems. Hater (1975) and Fitzgerald (1980) with extensive studies of mathematics text materials, found their readability levels were often too high for their intended readership.

Because mathematics text is different in many ways from narrative prose (Ballagh & Moore, 1986; Cohen & Stover,1981; Klare,1978; Munro,1982), a detailed study of each readability variable would require extensive research. Readability of text is affected by many factors, only some of which can be measured objectively in quantitative form. These factors include semantic structure, syntactic structure, interest and motivation level, textural features, rarity and structure of vocabulary, connotation levels, content familiarity, and legibility. They are interactive, and more compex than the three most common factors of word difficulty, sentence length and sentence complexity included in many readability formulae (e.g. those reviewed by Klare,1963).

Methodology

The dependent variable for this study was the number of correct answers to original and modified questions on mathematics tests administered to 186 sixth grade pupils. The subjects were required to take a completion-type mathematics test of twenty questions, administered by the researcher with photocopied sheets.

Original questions were the twenty questions supplied by the local secondary school, usually used (along with a reading comprehension test and a general IQ test) by the high school to stream children into different classes for all curriculum areas. In modifiying the questions, attempts were made to leave computations unaltered. Pairs of questions were created, with modification of the following readability factors:

(a) the use of original test vocabulary; or its avoidance or modification

In modifying terms, attempts were made to use more common terms, avoid possible multiple meanings, or to eliminate technical terms without affecting comprehensibility or mathematical content. For example :

> Original: Rearrange this number sentence to demonstrate that addition and subtraction are inverse operations. Use the same numerals in any order. 7 + 2 = 9
> Modified: Change 7 + 2 = 9 to a subtraction equation. Use the same numbers in any order.

(b) numerals written in word form; or numerals written as symbols

This involved simple substitution of figures for words. For example:

> Original: There were one hundred and twenty-eight children in a school. Fifty percent of them went on a bus trip. How many were left at school?
> Modified: There were 128 children in a school. 50% of them went on a bus trip. How many were left at school?

(c) the use of original sentence structure; or of simpler sentences

The simplification of sentences involved shortening sentences, usually through the use of a full stop (period) to eliminate logical connectives. Paranthetical phrases were also eliminated. Care was taken not to increase the density of questions by artificialy lowering the number of words per question. For example:

> Original: Because Danny bought 6 paddlepops, he had only $4 left from the $7 he had taken to the shop. How much did each paddlepop cost?
> Modified: I had $7 to spend. I bought 6 paddlepops. My change was $4. How much was each paddlepop?

(d) the inclusion of extraneous reading; or its exclusion

Extraneous reading was omitted, resulting in more compact problems which retained only essential information. For example:

> Original: A dress pattern my mother bought calls for 4 meters of soft material. The material is oo sale for $2.36 a metre. How much will Mum spend in purchasing the material?
> Modified: 4 metres of material. $2.36 per metre. How much will it cost?

The effects of reading ability were controlled for the purpose of comparing results on original and modified questions through a matched subjects design. Reading ability was quantified by the Neale analysis test, with reading ages ranging from 7 years, 0 months to 13+ ($x = 11.07$, SD = 1.7 yr). Subjects were matched by reading age to enable the use of data within pairs for comparison of original and modified questions. The difference of the means of reading ages of the two goups was less than two months ($t\ (91) = 0.22\ p > 0.01$). Students were also paired within sexes and within classes (i.e. mathematics and reading teachers).

Original and modified questions were presented on two test forms, test A and test B (after Gardner,1972):

Question	Test A	Test B
1	modified	original
2	original	modified
3	modified	etc.

Answers were marked either correct or incorrect, with no allowance for computational errors.

In order to determine that tests were of approximately equal mathematical standard, questions were first trialled with a class ($n = 28$) from another school. Achievement of Test A and Test B was found to be of equivalent standard, both in the trial ($t\ (13) = 0.19\ p < 0.01$) and the research test ($t\ (91) = 0.18\ p < 0.01$). Internal test constistency, as calculated with the Kuder and Richardson Formula 20 (1937, pp.151-160) was high. No significant difference was found between achievement on groups of original questions ($r_{K-R}20 = 0.62$) or between their modified forms ($r_{K-R}20 = 0.64$).

To reduce the compounding variables of socio-economic status, schooling background, curriculum content and teaching styles, subjects were selected from only one school. They were 186 Year 5 and 6 students enrolled in an Australian primary school. Ages, midway through the school year, ranged from nine years, four months to twelve years, five months ($x = 10.7$, SD = 0.77 yr).

Decsystem 20, M9.1 was used in the analysis of data. Probabilities of chance were expected to be less than 0.01. Two-tailed tests were used. Frequencies were calculated for student achievement on test A and test B, each pair of questions, all original questions and all modified questions, and for each group of original or modified questions, grouped according to the four independent variables. *t* -scores were calculated to compare means of results for test A with test B, all original questions with all modified questions, original with modified questions for each independent variable and the reading ages of children doing test A with those doing test B.

Results

When technical tems were avoided or modified, questions were answered correctly more often than the original questions ($t (91 = 3.54\ p<0.01$). Simplifying sentence structure produced better test results ($t (91) = 3.37\ p<0.01$). Eliminating extraneous reading resulted in better test results ($t (91) = 4.05\ p < 0.01$). However, altering numerals from word form to symbols made very little difference to test results ($t (91) = 1.14\ p>0.01$). Apparently this factor does not affect readability for Year 5 and 6 children who have had extensive practice with numerals in both verbal and symbolic forms.

Overall, modifying some of the reading elements of the mathematics test led to a significant improvement in test results ($t(91) = 4.8\ p<0.01$). These reults are summarised in table 1, below:

Table 1: Means, standard deviations and t-test comparisons between original and modified questions

	Original		Modified		t-scores
	$\bar{x}$	s	$\bar{x}$	s	
Technical terms	2.36	1.55	4.00	0.87	3.54 *
Numerals	2.50	1.24	2.32	1.29	1.14
Sentence Structure	1.64	1.22	3.64	1.65	3.37 *
Extraneous reading	2.35	1.28	3.57	0.76	4.05 *
Overall	8.50	3.57	13.28	3.5	4.81 *

Number of pairs = 92

* Significant at .01 level

Pearson correlation co-efficients were calculated for results of all original questions with all modified questions and for each group of questions (grouped according to the four independent variables) with each other group.

Pupils who performed well on original 'technical terms' questions, and who coped well with complex sentence structures, generally performed well with all original questions ($r = 0.70$, $r = 0.71$). They were also able to quickly locate and put aside irrelevant information ($r = 0.79$). These results may have been achieved because of greater reading, mathematical or general ability. However, there was also a strong correlation for students who achieved better results with modified technical terms and sentence structure with their performance on all modified questions ($r=0.73$, $r= 0.88$), as opposed to the original questions ($r=0.47$, $r=0.16$). For these children, test performance was enhanced significantly by language modification.

Discussion

This paper does not argue that technical terms should never be used in mathematical texts. Participation in and access to higher mathematics depends heavily on competence in mathematical language. Children must develop meaningful general, technical and symbolic mathematical vocabularies, together with their abilities to think, speak and read each, or to translate from one vocabulary to another. Similarly, they need practice in isolating essential information from word problems and learning to handle complex sentence structure. Facility with the appropriate language is essential for the conceptualisation of messages contained within mathematical problems. In using mathematics text books, students usually need to read instructions, explanations and completed examples, as well as to read and solve practice examples. Thus context-area reading becomes an important activity in the development of mathematical concepts.

However, if learning from a book is to be achieved, or word problems are to be completed accurately, the text must be written at an instructional level. This suggests that authors, publishers and teachers should be concerned with text readability. Mathematical text presented to the child in any medium (e.g. textbook, worksheet, chalkboard), situation (group or individual work, tests or homework) and form (including graphs, charts, formulae and other more unusual forms of text) should be comprehensible.

This also suggests that the necessary facility with vocabulary should be developed orally, with the child capable of using it in both expressive and receptive modes (Del Campo & Clements, in press) before being expected to cope with mathematical terms in written text, particularly in

individual test situations. Emphasis on oral work to allow construction of shared understandings for technical terms should precede the reading of those terms. In practical discussions a variety of language can be used in small group work situations, focussing on the development of language and concepts. Explicit teaching of appropriate textual genres and features can be undertaken.

Mathematics language needs to be consciously taught and learned. Research (e.g. Ballagh & Moore, 1986; studies quoted by Earp, 1970) indicates that instruction in mathematical vocabulary and specific reading skills results in greater mathematical achievement. Newman (1977) suggests a technique of interviewing children about their reading, comprehension and solution of mathematics problems. She lists questions which may not only aid in the diagnosis of reading problems but also assit students to understand where their own problem-solving difficulties lie. In introducing any new text, mathematics teachers need to train children not only in the appropriate mathematics skills and knowledge; but also in reading vocabulary and techniques appropriate to the textural genre, mode and form. Teachers can use deliberate strategies to teach common features of mathematical vocabulary, such as polysemous words, the importance of prefixes and suffixes, and the importance of distinguishing between similar words (e.g. *section* , *segment* and *sector*). With metalinguistic training, students can be alerted to structural features such as passive voice, inverted order, logical connectives and negative form. Problem representation techniques to aid in translation from the textual to the pictorial or symbolic can be practised.

It is important that further research be carried out to identify those mathematical language factors which need careful introduction and use. Sentence structure, as well as terms used and the amount of reading involved should be considered, as should other readability factors not tested in this study, such as semantic structure, syntactic structure, interest and motivation level, rarity and structure of vocabulary, connotation levels, content familiarity and legibility.

When testing, if a teacher's objective is to test mathematical rather than reading ability, it would seem necessary to word problems concisely but simply, and to provide assistance in reading for some students. A major implication of this study relates to the fact that the original test was to be used, along with two other tests even more dependent upon reading skills, for the placement of pupils into streamed classes for all curriculum areas. As such, the test was meant to be using mathematical competence (and possibly mathematical potential). Without entering into discussions

regarding the ill-effects of streaming, or the advisability of using such tests to evaluate either achievement or aptitude, one could question the validity of measuring mathematical skill without considering the interference of extraneous factors such as reading ability.

References

Aiken, L.R. (1972). Verbal factors and mathematics learning: A review of the research. Journal of Research in Mathematics Education, November, 304-313

Ballagh, A. and Moore, R. (1986). Access and success in mathematics in the post compulsory years. In N. Ellerton (Ed.) Mathematics: Who Needs What? Papers presented at the 23rd MAV annual conference, December 1986. Mathematical Association of Victoria, Melbourne,167-172

Bates, R.J. (1984). Educational versus Managerial Evaluation in Schools. In P. Broadfoot (Ed.) Assessment, Schools and Society. Methuen, London,127-143

Broadfoot, P. (1979). Assessment, Schools and Society. Methuen, London

Cohen, S.A. and Stover, G. (1981). Effects of teaching sixth-grade students to modify format of variables of math word problems. Reading Research Quarterly, 1981, 16, 2, 175-200

Del Campo, G. and Clements, M.A. (1989). Expanding the modes of communication in mathematics classrooms. Journal fur Mathematik-Didaktik (In press)

Earp, N.W. (1970). Observations on teaching reading in mathematics. Journal of Reading, April,*13*, 529-532

Eggleston, J. (1984) School Examinations - Some Sociological Issues. In P. Broadfoot (Ed.) Assessment, Schools and Society. Methuen, London,17-34

Fitzgerald, C.G. (1980). Readability of the Fry sample procedure. Journal of Reading, 15, *4*, 489-93

Gardner, P.L. (1972). Words in Science. ASEP, Melbourne

Hater, M.A. and Kane, R.B. (1975). The close procedure as a measure of reading comprehensibility and difficulty of mathematical English. Journal for Research in Mathematics Education, March, 121-127

Karlin, R. (1972). Teaching Reading in High School. Merril, Toronto

Klare, G.R. (1963). The Measurement of Readability. Iowa State University Press, Iowa

Klare, G.R. (1978). Assessing Readability. In L. Chapman (Ed.) Reading from Process to Practice. Routledge and Kegan Paul, London, 62-102

Kuder, G.F. and Richardson, M.W. (1973). The theory of the estimation of test readability. Psychometrica, 2,151-60

Mellin-Olsen, S. (1987). The Politics of Mathematics Education. Reidel, The Netherlands

Moodie, M. A. and Clarke, D. (1989). Successful assessment - documenting classroom observations. In B. Doig, Everyone Counts: Papers presented at the 26th MAV annual conference, December 1989. Mathematical Association of Victoria, Melbourne,153-163

Munro, J. (1982). Reading symbolic mathematics statements. In P. Costello (Ed.) Aspects of Motivation. The Mathematical Association of Victoria, Melbourne, 57-70

Neale, M. (1980). Neale Analysis of Reading Ability. 3rd edition. Macmillan, New York

Newman, M.A. (1977). An analysis of sixth-grade pupils errors on written mathematical tasks. Research in Mathematics Education in Australia, *1*, 239-258

SOCIAL SENSE-MAKING IN MATHEMATICS; CHILDREN'S IDEAS OF NEGATIVE NUMBERS

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A narrative story-telling methodology was used to tap children's use of a debts and assets model to support calculations with negative numbers. Children showed superior performance on problems posed in the context of the story, in contrast to their ability to solve isomorphic problems presented as formal equations. Those children whose performance was most enhanced were unschooled children from India, who were very familiar with the social situations and problems depicted. The debts and assets analogue appeared to encourage the use of a Divided Number Line model, resulting in difficulties when children had to perform calculations involving crossing over the zero amount from a debts to an assets status.

This paper explores children's intuitions concerning negative numbers. There is a growing body of research indicating that people's intuitions about mathematical concepts are grounded in the goals, physical properties of entities, and social situations that they encounter in their everyday lives (e.g., Resnick, 1989), and that in many cases, their mathematical sophistication in concrete life situations may exceed their ability to solve isomorphic problems posed as "school math." However, in the case of negative numbers, it is not entirely clear what everyday experiences could serve as the basis for the development of relevant concepts. Historically, an arithmetic based on negative quantitites developed slowly and late. Although an early form of negative numbers was noted in China in the second century BC, it was not until the thirteenth century that modem bookkeeping was developed. Bookkeeping, the first systematic application of negative numbers, used Arabic numbers and double entry, and formalized a different status for debts and assets, with balance as a key concept and subtraction as the key operator. Practically, then, debts and assets served as the motivation for the development of negative and positive numbers, but little is known about whether and how children use an understanding of debts and assets to support arithmetic operations with quantities involving negative numbers. Peled, Mukhopadhyay, and Resnick (1989) interviewed first, third, fifth, seventh, and ninth grade children as they solved various forms of equations including negative numbers, and concluded that the models children had were quite abstract, rather than based on experiences in manipulating concrete objects or materials. The youngest children appeared to have no model at all. Among the older children, understanding of negatives progressed from a Divided Number Line Model, in which children computed by partitioning problems into moves toward and away from a balance point at zero, to a Continuous Number Line model that treated integers, both positive and negative, as coherent ordered entities. Neither the younger nor the older children spontaneously referred to natural analogues or practical situations like finances.

In this study, we investigate children's ability to interpret a natural social situation, depicted in a narrative story, and to use their understanding of that situation to generate and apply a mental model of debts and assets in solving problems including negative quantities. Our sample includes elementary school children who have not yet received formal instruction in negative numbers and a group of unschooled Indian children whose everyday experience includes considerable activity in buying and selling, owing and paying. To tap knowledge that children may have about this social context, we use a story-telling methodology, asking children to help reconstruct and interpret events in an extended narrative concerning the financial situation of a fictitious character.

Method

The U.S.A. subjects were 51 students enrolled in a parochial school in a predominantly middle class, suburban community; the Indian subjects were 5 boys in Calcutta. The American children were approximately half boys and half girls, 10 second-graders, 12 third-graders, 17 fourth-graders, and 12 fifth-graders. The Indian children attended school only occasionally (average schooling, two years) and were all occupied as houseboys. Each was between 10 and 13 years old (Indians not from the middle class are often uncertain of their exact age).

Each child participated individually in a two-part procedure. American children were interviewed in a schoolroom by one of two female interviewers; the Indian children were interviewed in a private home by a female Indian interviewer who spoke Bengali. The first task, the story interpretation, was to reply to a series of questions about a story concerning the financial difficulties of a character named Sam who lived on a farm and made a living by raising animals and crops. The story (which had an equivalent Bengali translation) was an extended narrative, approximately five pages of text, recounting a series of events that resulted in fluctuations in Sam's debts and assets. Children were told that the interviewer had heard the story a long time ago, and that they would be asked at various points to help reconstruct events.

The interviewer read the story aloud and asked the child's help in filling in "the missing parts of the story" by solving practical problems concerning negative numbers that were raised in the story context. The questions posed were deliberately ambiguous, that is, of the form, "What is Sam's situation?" rather than, "How much money does Sam owe?" This ambiguity elicited different kinds of answers, some more informative than others. As we will see, some children produced answers that primarily addressed the semantics of the story and the problems of the characters, whereas other children focused on and calculated quantities. The story presented four categories of target problems. Question 1 asked children to explain the consequence of a set of events leading up to the *creation of debt*. Questions 2 and 3 addressed the *cumulation of debt*. Questions 4, 5, and 6 concerned the *reduction of debt*. The

final item, question 6, concerned the *re-establishment of assets.*

The second task, equations solution, was to solve a total of 16 equations by adding or subtracting negative numbers. The equations paralleled the signed number problems presented in the story situation, and also included a few additional problems, including forms that have no real-life analogue. Each equation was presented in three mathematically equivalent notations, some including terms in brackets. The child was presented these equations one at a time in counterbalanced order and asked to solve each and explain the solution. Paper and pencil were provided. The Indian children were reluctant to try these problems, and therefore the equations solution task was not pursued with this sample. For the U.S.A. sample, order of presentation of the story situation and equations solution procedures was counterbalanced within grades.

Results

In this section we report results for each category of problems presented in the story situation and compare children's performance on these problems with their answers to the corresponding problems in the equations solution task.

Creation of Debt. The opening events in the story establish that Sam, starting with a given sum of money, receives some bills, including one from the carpenter, whom he cannot afford to pay. At this point, children were asked, "So, what's Sam's money condition?" Answering this question depends upon understanding, either intuitively or mathematically, that one can subtract a quantity from a lesser quantity to yield a negative quantity. Children's responses to this question, as well as all subsequent questions, were classified into a rough hierarchy ordered by the amount of semantic, quantitative, and mathematical understanding that they demonstrated. The least sophisticated responses mentioned only the immediate practical consequences of the story events. Somewhat more sophisticated were responses that focused on one or more of the relevant numerical quantities in the situation. The most mathematically sophisticated responses included calculations that combined these quantities as described by the events in the story. In Table 1 (and subsequent tables as well), categories of responses are indicated by the column headings; responses are arranged in order of sophistication from left (most sophisticated) to right.

The majority of the responses are in column 1, indicating both that children understood that Sam now owed money and that, in addition, they had performed a calculation to specify the amount of money owed. Smaller percentages of children, almost exclusively in the lower grades, gave answers that were correct but focused on only one of the relevant quantities. For example, responses in column 2 indicated that Sam did not have enough money (the word "enough" explicitly refers to the sum of [+a] - [+b], without calculating or estimating the debt). Responses in column 3 said that Sam needed more money (focusing only on -[+b]). Finally, responses in column 4 contained arithmetic errors and misinterpretations of the situation. Table

1 also establishes that although most children answered Question 1 fully and correctly, very few at any grade were able correctly to solve any of the forms of the corresponding equations. Notably, although none of the Indian children could perform the equations solution task, all five answered the story problems with responses tabulated in column 1, the most sophisticated type of answer.

Table 1: Responses to Story and Equations for Creation of Debts

	Story Question 1 (% each response)				Equations (% correct)		
Grade	1	2	3	4	a - b	[+a] - [+b]	[+a] + [-b]
Second (N = 10)	60	0	30	10	0	0	0
Third (N = 12)	83	8.5	8.5	0	17	17	17
Fourth (N = 17)	88	0	0	12	6	6	0
Fifth (N = 12)	92	8	0	0	8	8	17
Indian (N = 5)	100	0	0	0	-	-	-

Cumulation of Debt. In questions 2 and 3, children were asked about events that resulted in Sam's debt becoming greater. Responding to these questions involved understanding the addition of two negative quantities, or in the context of the story, the cumulation of money owed.

Table 2: Responses to Story and Equations for Cumulation of Debts

	Story (% each response)						Equations (% correct)		
	Question 2			Question 3					
Grade	1	2	3	1	2	3	- a - b	[-a] + [-b]	[-a] - [+b]
Second (N = 10)	40	50	10	50	50	0	0	0	0
Third (N = 12)	67	33	0	42	50	8	8	0	0
Fourth (N = 17)	65	25	10	47	41	12	6	12	0
Fifth (N = 12)	75	25	0	75	25	0	8	17	0
Indian (N = 5)	100	0	0	100	0	0	-	-	-

As Table 2 indicates, performance drops for both these questions, probably because children find it confusing to use the *add* operation to achieve a larger negative result. That is, the

arithmetic operation that is appropriate (adding) seems intuitively inconsistent with the objective (compile a greater negative quantity). In the most sophisticated responses to these questions (column 1 in Table 2), children point out that Sam owes more money and also go on to calculate the new quantity. Responses in column 2 mentioned the additional bills and noted that money was owed, but did not show any attempt to cumulate the debt. A few responses (under column 3) referred to unrelated issues in the situation. Once again, only the Indian children produced responses that were all classified as the most sophisticated. As in question 1, the U.S.A. sample did very poorly on the equation versions of these problems (recall that the Indians did not attempt them at all).

Reduction of Debt. Questions 4, 5, and 6 ask the children to explain Sam's situation after each of three events in which he receives money that he can credit toward reducing his total debt.

Table 3: Responses to Story and Equations for Reduction of Debt

	Story (% each response)											Equations (% correct)		
	Question 4			Question 5			Question 6							
Grade	1	2	3	1	2	3	1	2	3	4	5	- a + b	[-a] + [+b]	[-a] - [-b]
Second	90	10	0	70	30	0	0	50	10	20	20	0	0	0
Third	100	0	0	83	17	0	50	17	17	0	17	17	17	17
Fourth	100	0	0	100	0	0	5	18	24	41	12	6	0	6
Fifth	100	0	0	92	8	0	0	0	50	42	8	17	25	25
Indian	100	0	0	100	0	0	0	0	40	20	40	-	-	-

As Table 3 shows, even the youngest children performed very well on questions 4 and 5. To answer them, children had to realize that a negative quantity was being decremented by the introduction of a smaller positive quantity. Column 1 responses noted that Sam's debt had been reduced and, in addition, included a calculation of the amount of the reduction. Responses in column 2 mentioned only that Sam was still in debt or remained in debt. The younger children, in contrast to the older ones, were more likely to give column 2 responses, which summarized Sam's practical situation without trying to re-evaluate the quantity of the debt. In contrast to questions 4 and 5, children did very poorly on question 6, which is worded somewhat differently. Here, Sam has sold some farm animals and says, "At least this will help me out in my current crisis." The question asks, "What does Sam mean by 'help'?" Apparently, children found the word "help" misleading and believed that it implied a major change in Sam's circumstances. In contrast to questions 4 and 5, very few children responded to question 6 by trying to calculate the amount of debt reduction (column 1). Column 2 responses indicated that Sam's debt was almost erased; that he had only a little left to pay back.

However, the amount to be paid was not calculated. Many children mentioned only the (+b) quantity; that is, the amount of money that Sam received for selling his farm animals. Some children noted that this amount could be credited toward the debt (column 3); others simply noted the the quantity by pointing out that the sale had generated money without noting that this amount could be applied against the debt (column 4). Column 5 responses are particularly interesting; they represent misinterpretations of the story situation, generated, we believe, by the word "help" in the story question. These responses indicate that Sam had now erased his debt and had some money left over. Performance on this question, relative to questions 4 and 5, was most seriously disrupted for the fourth- and fifth-graders and the Indian children. The Indian children were especially skillful in using the story cues as a model for supporting their problem solution. In general, they were better than any of the American children at following the events of the story and keeping track of Sam's changing fortunes. However, in this case, when the semantic cue in the story implied a "happy ending," their performance, like that of the school children, also disintegrated; in particular, 40% of their responses were the misinterpretations in column 5. The school children's poor performance on the equations solution task confirms that they did not have sufficient formal training to override their misinterpretation of the story situation.

Re-establishment of Assets. Sam sells enough blankets to cancel his debt and have some money left over. Question 7, which asks about Sam's situation after this event, requires children to add a larger positive quantity to a smaller negative quantity, yielding a positive sum.

As Table 4 shows, children's performance in response to this question is approximately equivalent to their performance on the cumulation of debt. Column 1 responses indicate that Sam's debt has been erased and he now has money once again. Column 2 responses focus on the fact that he has money from the sale but do not mention that some of it must be credited to cancelling the debt. Column 3 responses are misinterpretations or misreadings. In contrast to results in the equations solution task that we have seen so far, children solved a substantial number of the equations that corresponded to this question, usually by simply inverting the positive and negative terms of the equation to form a simple subtraction problem. However, for no group did performance on the equations equal or exceed performance on the story problem.

Table 4: Responses to Story and Equations for Re-establishment of Assets

	Story Question 7 (% each response)			Equations (% correct)		
Grade	1	2	3	- a + b	[-a] + [+b]	[-a] - [-b]
Second (N = 10)	50	30	20	38	0	38
Third (N = 12)	67	25	8	50	17	58
Fourth (N = 17)	65	35	0	47	0	53
Fifth (N = 12)	58	42	0	42	17	42
Indian (N = 5)	80	0	20			

Discussion

In this study, we find a sizeable difference in children's performance with the solution of signed number problems in two different contexts. When they encounter these problems embedded in a narrative story about the social context of debts and assets, their performance is far more complete and competent than when they encounter the isomorphic problems presented as formal equations with mathematical notation. Furthermore, those children who are most familiar with the particular social context presented--that is, the older American children and the Indian children--show the greatest enhancement in performance. This study, like other work in everyday reasoning (e.g., Carraher, Carraher, and Schliemann, 1985; Schliemann & Acioly, in press), finds that people show a superior ability to use and understand mathematical ideas when the relevant concepts and operations are introduced in a contextualized, familiar social situation.

A related methodological point is that we find story-telling an effective device for eliciting informal knowledge in elementary-school children. Although children's ability to solve short mathematics word problems has been much studied, the use of a narrative with an extended and believable storyline is an unexplored methodology for assessing mathematical understanding. Children's ability to perform with understanding in this assessment situation was enhanced, we believe, by their engagement in the story. We note here that story-telling is a particularly familiar and valued medium in the culture of nonschooled Indian children. The Indian children in our sample were deeply involved in the story of Sam's tribulations. Clearly they were not interpreting the interviewer's questions as a request for school answers. Instead, they often spent as much time thinking about practical strategies for solving Sam's problems as they spent on the calculation of quantities. For example, the Indian children worried about whether Sam should try to get a loan from his brother, and considered at length the merits and disadvantages of financial obligations with a family member. However, the story did not invariably enhance mathematical performance. When story cues were misleading, children--including the Indians--lost their track on the situation and made misinterpretations inconsistent

with the quantitative information being presented.

A final issue concerns the extent to which people engage in or avoid arithmetical calculation in their everyday reasoning (e.g., Lave, Murtaugh, & de la Rocha, 1984). Large percentages of the children answered the story questions not only by describing the social consequences of Sam's situation, but also by performing calculations to quantify Sam's current debts or assets. Children appeared to believe that calculating and tracking the amount of Sam's debt was relevant to the practical problem of helping him improve his situation. Some of these calculations appeared to be more difficult than others. For example, using the *add* operation to calculate a greater negative balance appeared confusing. Crossing the zero boundary to re-establish assets was also difficult relative to those problems in which Sam's finances fluctuated but remained within the debit or negative category. These results are consistent with those of Peled, Mukhopadhyay, & Resnick (1989), who described children as using a Divided Number Line model. It is likely that the use of this model is reinforced by the story context, since debts and assets are different kinds of financial entities with different social implications. We conclude that the effects of a meaningful social situation on mathematical performance may be complex, sometimes enhancing and sometimes disrupting performance, and perhaps differentially encouraging the use of particular mental models at the expense of others that children may have in their repertoire.

References

Carraher, T. N., Carrher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology, 3*(1), 21-29.

Lave, J., Murtaugh, M., & de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B. Rogoff & J. Lave (Eds.), Everyday cognition: Its development in social context (pp. 67-94). Cambridge, MA: Harvard University Press.

Peled, I., Mukhopadhyay, W., and Resnick, L. B. (1989). Formal and informal sources of mental models for negative numbers (pp. 106-110). In G. Vergnaud, J. Rogalski, and M. Artique (Eds.), *Psychology of mathematics education, Actes de la 13e Conference Internationale* (Vol. 3). Paris, France: Laboratoire PSYDEE.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist*, 44(2), 162-169.

Schliemann, A. D. & Acioly, N. M. (in press). Intuitive mathematics and schooling in understanding a lottery game. *Cognition and Instruction*.

CHILDREN'S PRE-CONCEPT OF MULTIPLICATION : LOGICO-PHYSICAL ABSTRACTION

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This paper reports on a pilot study investigating first and second graders' understanding of pre-multiplication. Pre-multiplication is defined here as the child's perception of multiplicative structures generated by the iteration of one-to-one or one-to-many correspondences. Subjects were presented with several tasks dealing with the invariance of a multiplicative structure with respect to a changed configuration or a regrouping of the subsets. Marked differences were observed between the children in the two grades. These differences were also evident on tasks involving the decomposition of a whole into different but equivalent arrays, as well as on tasks dealing with pre-commutativity and pre-distributivity.

At last year's meeting of PME, we presented a first epistemological analysis of the early beginnings of multiplication (Nantais & Herscovics, 1989). A more refined version was communicated at the meeting of the CIEAEM (Nantais, Boulet, Bergeron & Herscovics, 1989). These analyses were based on a model of understanding identifying two tiers, the first one involving the physical pre-concepts on which a mathematical conceptual scheme can be built, and a second tier describing the understanding of the emerging mathematics (Herscovics & Bergeron, 1988).

This year, we have concentrated on designing and experimenting tasks related to the first tier, that of the understanding of the pre-concept of multiplication. At the physical tier, we consider that a **situation is perceived as being multiplicative when the whole is viewed as resulting from the repeated iteration of a one-to-one or a one-to-many correspondence.** Three levels of understanding can be associated with the pre-concept of multiplication: intuitive understanding, procedural understanding, and logico-physical abstraction. Pilot studies on procedural understanding are dealt with in a related paper (Beattys, Herscovics & Nantais, 1990). The present paper deals with logico-physical abstraction.

Logico-physical abstraction refers to the construction of logico-physical invariants with respect to some transformations, the reversibility and composition of logico-physical transformations, and generalizations about them. We also include here the properties of pre-arithmetical operations. The tasks we have designed to assess logico-physical abstraction have been explored with 5 first graders and 4 second graders in a local primary school.

Conservation of plurality and quotity

The first task we suggested to our subjects did not deal with multiplication but verified if they conserved plurality (Piaget's classical 'number' conservation test) and quotity (Greco's comparable

1 Research funded by the Quebec Ministry of Education (Fonds FCAR EQ 2923)

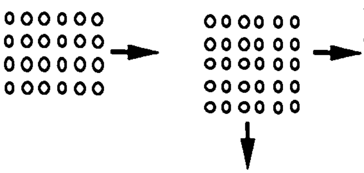
test including counting). Two of our first graders and three of the second graders conserved plurality . Regarding quotity, this was conserved by four first graders and three second graders.

Invariance of a multiplicative situation with respect to configuration

We have used four tasks to assess the children's perception of the invariance of a multiplicative situation with respect to configuration. Two variables involved in the design of these tasks were the elongation in one or two direction, and the presence or absence of a comparison set.

For the first task, we used 24 chips arranged in a 4 x 6 rectangular array that was stretched in the horizontal direction. In the second task, we used 30 chips arranged in a 5 x 6 array and stretched both horizontally and

Task 1 Task 2



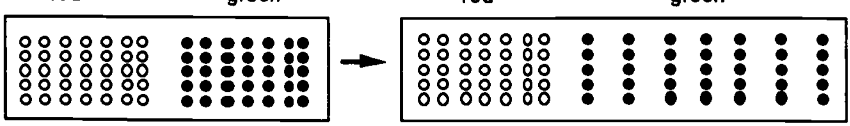
vertically. For both tasks the questioning was identical:

Can you tell me if all the rows are equal? Look at what I'm going to do. (After stretching the array) **Can you tell me if now there are more chips, less chips, or the same number of chips as before I stretched the rows, or do we have to count them in order to really know? Why do you think so?**

<u>Responses</u>: The results obtained were quite different in the two grades. All four second graders succeeded on both tasks. Among the five first graders, two succeeded on the first task and three on the second one. Most surprising, among these children three of them had not shown that they conserved plurality.

The next two tasks involved the same transformations but in the presence of a comparison set. Based on prior work dealing with the understanding of plurality, it was thought that this would make the tasks more difficult due to the interference caused by the visual differences between comparison sets and the elongated sets. For the third task related to this criterion we used a 5 x 7 array of red chips and another one of green chips. The chips had been cut out of felt and were disposed on a felt board. The array of green chips was then stretched horizontally. The fourth task involved another board with two 5 x 7 arrays of yellow and green felt chips respectively, the green array now being stretched horizontally and vertically. For both tasks, the questioning was similar: **Here is a set of red chips and here is another set of green chips. Can you tell me just by looking at them if we have the same number of red and green chips? Look at what I'm going to do.** (The interviewer then stretched the green array). **Just by looking, can you tell me if there are more, less, or the same number of green chips as red chips? Or must we count them all in order to know?**

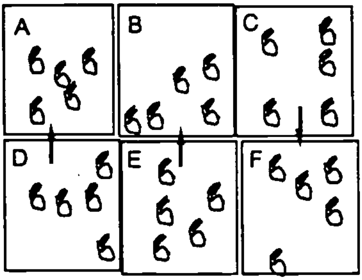
red green red green



Responses: To the initial query about the two sets having the same number of elements, all children answered affirmatively, most of them basing themselves on the visual similarity Other children explained it by counting either the rows or the columns.but not both. Our subjects had a somewhat better success rate than on the first two tasks. Four of our five first graders succeeded on task 3 and three succeeded on task4. All the second graders who were given the tasks handled them successfully. (The interviewer forgot to ask task 3 from one subject and skipped task 4 with two subjects).

Invariance with respect to a regrouping of the subsets

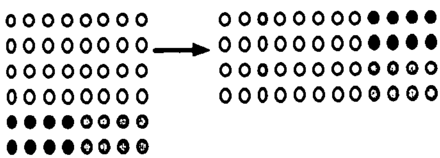
Regarding the invariance of a multiplicative situation with respect to the regrouping of the subsets, four different tasks were used. The two variables involved were the randomness of the elements in the subsets and the presence of a comparison set. For the first task we used a cardboard on which six rectangles were drawn, each one containing 5 little felt rabbits. The children were told: **Here is a farm and this is a barn where rabbits are**



kept. But the farmer has to repair three cages. Thus he must move some of the rabbits. Look at what I'm going to do: While I repair this cage(D), **I will put these rabbits in this cage** (A). (Similarly, the rabbits where then transferred from E to B and from C to F). **Do you think that now in the barn there are more, less, or the same number of rabbits as before? Or do we have to count them in order to know? Why do you think so?**

Responses: Children were almost evenly divided in their response. In each grade, two children thought that the regrouping of the rabbits into three cages resulted in more rabbits in the barn. Clearly, these subjects were focusing on only one aspect of the multiplicative situation, that of the number of elements in each group. They were not compensating for the smaller number of groups. Those children who thought that the number had not changed justified it in terms of "nothing added, nothing taken away from the barn".

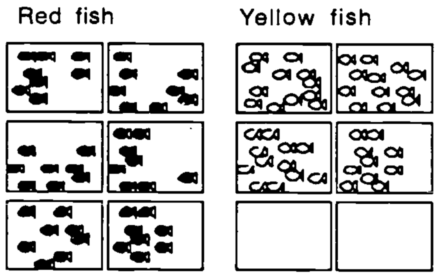
In the second task dealing with this aspect of invariance we used a 6 x 8 array of identical chips and transferred the bottom two rows as follows: the bottom chips on the left side were aligned along the first two rows, the bottom chips on the right hand side were then aligned with rows 3 and 4. The different shadings shown in the diagram are there simply to help visualize this transformation. The questioning then proceeded as in the prior task.



Responses: This transformation seemed to have a greater impact on the first graders. Two of them thought that the total number had changed whereas none of the second graders did.

The third and fourth task verifying this invariance where variations on the first two since the only change was the addition of a comparison set. In task 3, children were presented with two felt cardboards representing two pet shops, one selling yellow fish and the other selling red fish. Each "pet shop" contained 6 fish tanks and each aquarium contained 8 little fish cut outs in felt. Two of the tanks with yellow fish were then emptied and the fish redistributed into the other yellow tanks. The questioning proceeded as follows: **Here are two pet stores that sell fish to keep in an aquarium. One store only sells red fish and the other one sells only yellow fish. There are eight fish in each aquarium. Do you think that there is the same number of fish in the two stores?** Following an answer, the subject was told: **But in the pet store selling yellow fish, two tanks are leaking and we have to move the fish in the tanks. Look at what I'm going to do.**

(The interviewer then removed the yellow fish from the bottom tanks and distibuted them one at a time into the other yellow tanks so that the child would be assured that the fish had been distributed equally). **Now, do you think that there are more, less, or the same number of yellow fish as red fish, or that we have to count them in order to know? Why do you think so?**



Responses: Most of our subjects did not succeed on this task. Among the five first graders, one of them perceived the invariance of the total number of yellow fish, while three children felt the the numbers were no longer the same; we could not proceed with one little girl since she thought right from the start, before any transformation that the number of fish in the two pet stores was not the same. Among our four second graders, two children perceived the invariance whereas two of them did not.

Results on task 4 dealing with this invariance were much better. In this task children were presented with two 4 x 7 arrays of red and green chips respectively. The green array was then transformed into a 2 x 14 array. Three of the five first graders and all the four second graders thought that the total number of chips in the two sets had remained the same.

Decomposition into equivalent products

Whereas in the previous tasks we had started with multiplicative situations, either sets subdivided into equal subsets or sets displayed in rectangular arrays, the task we envisaged here was to verify if children could perceive the possibility that the same quantity could form two different but equivalent multiplicative situations. To this end we displayed to the second graders two rows of 36 pink and 36 green chips. We then transformed the row of pink chips into a 4 x 9 array and the green

chips into a 3 x 12 array. With the first graders we used rows of 24 that were transformed into arrays of 3 x 8 and 2 x 12 respectively. The questions were **Here are two rows of chips. Do you that we have the same number of pink chips as green chips**

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Look at what I'm going to do. With the pink chips I make four rows of nine and with the green chips I make three rows of twelve. Do you think that the four rows of nine will give me the same number as the three rows of twelve, or that we have to count them in order to know? Why do you think so?

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Responses: The results indicate differences between the two grades. Two of the five first graders and three of the four second graders thought that the two arrays had to have the same number. The other subjects did not. It is interesting to note here that all the children who conserved plurality on the Piagetian test also conserved it in this task.

Pre-commutativity

In order to verify if children perceived the commutativity of a multiplicative situation (with respect to the total amount of objects), three different problems were presented. The first one was purely verbal. The children were asked: **If I have six bags of marbles and nine marbles in each bag, and you have nine bags and six marbles in each bag, can you tell me if you and I have the same number of marbles, or if we don't have the same number of marbles, or if we would have to count them all in order to know? Why do you think so?** Children were presented with a sheet of paper on which the information was written in the form of two columns:

Me	You
6 bags	9 bags
9 marbles	6 marbles

In the second problem children were told a similar problem but the objects were put out in front of them (five bags of eight red chips vs eight bags of five red chips) and the third problem involved the comparison of a 5 x 9 array of circles vs a 9 x 5 array.

Responses: The differences between the children from the two grades were remarkable. None of the five first graders perceived the commutativity of the multiplicative situation. On the other hand, for each one of these problems, three of the four second graders thought that the quantities were the same.

Pre-distributivity of multiplication over addition

In order to verify if children had some inkling about the distributivity of multiplication over addition, our subjects were shown two arrays of white circles (4 x 5 and 4 x 6) and a 4 x 11 array of black circles and asked if the total number was the same .

Responses: The responses followed almost the same pattern as for pre-commutativity. Three of the four second graders thought the total number of white circles was the same as that of the black circles, but only one of the five first graders. The justifications were straightforward: "They're the same. Because these are separated and the others are not. If we put these together..."

The following table provides an overview of the responses for each child.

Task	Do	Va	Pie	Maj	J.F.	Ch	Juli	Mar	Ale
Conservation of plurality	1	1	0	0	0	1	1	0	1
Conservation of quotity	1	1	0	1	1	1	1	0	1
Invariance wrt configuration									
task 1: elongating 1 array in 1 direct.	0	0	1	1	0	1	1	1	1
task 2: elongating 1 array in 2 direct.	1	1	0	1	0	1	1	1	1
task 3: elong. 1 of 2 arrays in 1 direct.	1	1	1	1	0	1	1	?	1
task 4: elong. 1 of 2 arrays in 2 direct.	1	1	0	1	0	1	?	?	1
Invariance wrt regrouping of subsets									
task 1: rabbits in 6 cages to 3 cages	1	1	1	0	0	1	1	0	0
task 2: 6 x 8 array into a 4 x 12 one	1	1	0	1	0	1	1	1	1
task 3: regroup 1 of 2 sets of pet fish	1	0	0	?	0	1	1	0	0
task 4: regroup 1 of 2 arrays	1	1	0	1	0	1	1	1	1
Decomposition into equiv. products	1	1	0	0	0	1	1	0	1
Pre-commutativity									
verbal problem	0	0	0	0	0	1	1	1	0
problem with concrete objects	0	0	0	0	0	1	1	0	1
array problem	0	0	0	0	0	1	1	0	1
Pre-distributivity	0	1	0	0	0	1	1	1	1
Total score	10	9	3	7	1	15	14	6	12

The first five columns indicate the results obtained with the first graders, the last fo ur columns, with the second graders. The number 1 indicates success on the given task, the number 0 indicates a failure, while the question mark shows that the task could not or was not presented to the subject.

The total score does indicate that two (J.F. and Pie) of the first graders are barely on the verge of perceiving the existence of multiplicative structures, that Do and Va are well on their way, followed closely by Maj. However, these scores are in marked contrast with those of the second graders, with

the exception of Mar. As expected, the test dealing with the conservation plurality is a determining factor in the child's mathematical evolution.

By way of conclusion

We have defined children's conception of pre-multiplication in terms of multiplicative structures, that is, the perception of multiplicative situations generated by the iteration of a 1:1 or a 1:n correspondence to create a whole. Thus, in all our tasks we tried to focus our subjects' attention on the whole set. However, we are not quite sure that this was achieved with the type of questions used on the first two tasks testing for the invariance with respect to configuration. It is not at all sure that by asking "if all the rows are equal" we achieve consideration of the whole at the initial stage. However, when the child is asked to compare the quantity following the transformation (elongation), there is little doubt that the whole set is the one being discussed. Of course, when two sets are present, then the comparison of the transformed set with the other one is in terms of global quantity.

As mentioned earlier, the conservation of plurality (assessed through the classical Piagetian test) seems to be a determining factor in the child's readiness for multiplication. This can be readily explained. If anything, the child's failure on the Piagetian task indicates an inability to compensate for the visual elongation of a row by the corresponding decrease in the density of the row. This inability to account for two complementary variables must also be a critical cognitive factor in the recognition of multiplicative situations, since these always involve two factors: the number of groups and the elements per group.

The general data bear out the importance of the conservation of plurality on the success rate achieved with the multiplication tasks. However, there seems to be one exception to this statement. Four children who did not succeeded on plurality should not have succeeded on the tasks dealing with the invariance of configuration but three of them did (Pie, Maj and Mar). Clearly, some other factors are involved here.

The results on the tasks dealing with regrouping of subsets indicate that for first graders, when randomly disposed subsets are regrouped into larger subsets, the presence of another unchanged set does interfere with the child's reasoning. This is shown by the different results on the rabbit task and on the pet fish task. However, when into comes to regrouping rectangular arrays, the presence of a second set does not seem to alter the results.Nor does such a presence seem to affect the success rate of the second graders.

The task involving the decomposition into two different but equivalent arrays was remarkably related to prior conservation of plurality. The tasks dealing with the axioms of pre-multiplication separated the children in the two grades. It should be noted that although the three tasks on pre-commutativity were expected to be unequally difficult, the results did not bear this out.

The exploratory work reported in this paper shows there is little doubt that second graders are ready to learn arithmetic multiplication. However, this does not mean that one should follow the philosophy of existing programs and stress almost exclusively the development of arithmetic procedures. Of course, these procedures are of prime importance. However, the usual tendency to over emphasize them is achieved at the expense of conceptualization. As this pilot study and the related one on logico-physical procedures (Beattys et al.,1990) show, it is possible to develop many tasks related to the concept of multiplication without necessarily quantifying the total sets. This may yet provide us with a better definition of arithmetic multiplication. Until now, the definition needed to be procedural: "multiplication is repeated addition" However, this only tells us how to answer the question "How many?". Viewing multiplication as the mathematization of multiplicative structures may bring us to consider it as an operation as vital and primitive as addition, subtraction or division.

References

Beattys, C., Herscovics, N. & Nantais, N. (1990), Children's pre-concept of multiplication: Procedural understanding, to appear in **Proceedings of PME -14**, Oaxtapec, Mexico

Herscovics, N. & Bergeron, J.C. (1988) An extended model of understanding. In M.Behr, C.Lacampagne, & M.Wheeler (Eds.) **Proceedings of the 10th Annual Meeting of PME-NA**. Dekalb, IL, 15-22

Nantais, N. & Herscovics, N. (1989), Epistemological analysis of early multiplication, in G.Vergnaud, J.Rogalski, & M. Artigue (Eds.), **Proceedings of PME-13**, la Sorbonne, Université de Paris, France, vol.3, 18-24

Nantais, N., Boulet, G., Bergeron, J.C. & Herscovics, N. (1989), Compréhension logico-physique de la multiplication chez le jeune enfant, in A.Warbeq (Ed.), **Actes de la 41e rencontre de la CIEAEM**, Bruxelles, Belgique

BUILDING ON YOUNG CHILDREN'S INFORMAL ARITHMETICAL KNOWLEDGE

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This paper reports some learning outcomes of a classroom teaching experiment aimed at facilitating the development of children's informal arithmetical knowledge through increasingly abstract levels of understanding. The experimental curriculum is built around students' construction of their own conceptually-based algorithms as a problem-solving activity, supported by a classroom atmosphere of discussion, negotiation and interaction. Results indicate qualitatively better understanding of computational procedures, intellectual autonomy, and a substantially faster rate of progress. New insights about the mechanisms and prerequisites for students' transition to higher levels of conceptual and procedural understanding are discussed.

INTRODUCTION

Researchers generally agree that young children enter school with a wide and rich repertoire of informal problem-solving strategies and self-generated algorithms that express and are based on their conceptual understanding (Carpenter & Moser, 1982; Ginsburg, 1977; Steffe, Cobb & Von Glasersfeld, 1988; Steffe, Von Glasersfeld, Richards & Cobb, 1983). Despite this available conceptual basis, many children have severe difficulties learning school mathematics. The transition from children's informal arithmetical notions to formal school arithmetic is a critical stage in children's learning of mathematics (Carpenter & Moser, 1982). Instead of ignoring or even actively suppressing children's informal knowledge, and imposing formal arithmetic on children, instruction should recognize, encourage and build on the base of children's informal knowledge (Carraher, 1988; Resnick, 1989).

Our research group is engaged in an ongoing research and development project on the mathematics curriculum in the first three grades of school, trying to build on children's informal knowledge and studying and facilitating the development of their conceptual and procedural knowledge. The project has much in common, both in content and approach, with the Cognitively Guided Instruction Project at the University of Wisconsin (Fennema, Carpenter & Peterson, in press) and the Second Grade Project at Purdue University (Cobb & Merkel, 1989; Cobb, Yackel & Wood, 1988). We are currently in the second year of a classroom teaching experiment (cf. Cobb *et al*, 1988) in the first three grades of eight schools (Human, Murray & Olivier, 1989). Research is being conducted on teachers' beliefs and development, classroom social interactions, and on students' beliefs, attitudes and conceptual growth. In this paper we briefly report some first results concerning student learning outcomes. To place these outcomes into perspective, however, we first supply a brief outline of our research basis and the main characteristics of the experimental curriculum.

RESEARCH BASIS FOR THE EXPERIMENT

From our baseline study (Murray & Olivier, 1989) we formulated a semantic model (cf. Baroody & Ginsburg, 1986), describing the development of students' computational strategies for the basic arithmetical operations through four increasingly abstract levels, each level associated with its prerequisite understanding of number and numeration. We briefly describe the four levels:

Level 1. The ability to count a number of objects and a knowledge of the number names and their associated numerals, without assigning meaning to the individual digits of a numeral. The typical computational strategy at level 1 is *counting all*.

Level 2. The ability to conceptualize a given number as an abstract unit item with a meaning independent of physical referents or counting acts (Steffe *et al*, 1983), i.e. the numerosity of the number has been acquired. Level 2 understanding of number is characterized by *counting-on* and *counting-down* computational strategies.

Level 3. The ability to use the additive composition property of numbers to replace a given number with two or more *numbers* that are more convenient for computation, e.g. to interpret 34 as 30 + 4 (note: *not* 3 tens and 4 ones). This provides the child with the conceptual basis to use *thinking strategies* (Cobb & Merkel, 1989; Steffe *et al*, 1988), i.e. solving a computation by relating it to other known results.

Level 4. The ability to interpret a two-digit number as consisting of *groups* of tens and some ones (e.g. 34 as 3 *tens* and 4 *ones*), without losing the meaning of the number as a *number*. Level 4 understanding of number is a prerequisite for the meaningful execution of the standard written algorithms in its most sophisticated form.

The study indicated that many children are highly creative in inventing their own powerful non-standard algorithms based on sound level 3 understanding of number and numeration. The study also indicated that the traditional primary mathematics curriculum fails to capitalize on the rich informal mathematics that children bring to instruction. Standard algorithms are taught at a syntactic level based on surface characteristics of the numeration system, thereby fostering a perspective of mathematics as instrumental understanding. Children relying on the school-taught standard algorithms had a low rate of success, demonstrating an impoverished conceptual base.

These observations motivated us to implement an alternative approach to early arithmetic, aimed at facilitating children's constructions of the number concepts and computational strategies of the first three levels in our model, in contrast to teaching them level 4 understanding as does the traditional curriculum.

THE EXPERIMENTAL CURRICULUM

We briefly outline some of the more distinguishing characteristics of the experimental curriculum and the rationale for some of its elements.

A constructivist framework. There is a genuine attempt to implement a teaching practice reflecting a constructivist viewpoint of learning, in the acceptance that conceptual knowledge cannot be transferred ready-made from one person to another, but must be actively

built up by every child on the basis of his own experience. The teacher therefore becomes less a dispenser of knowledge, but rather a facilitator of learning, supporting and guiding the child to construct his own knowledge. Children's ideas are respected and valued, and the child is seen as an active participant in the learning situation, not a passive receiver of knowledge. In the experimental curriculum this means that computational procedures are not imposed on students, but students are encouraged to construct their own conceptually-based algorithms.

Child-generated strategies. Regarding content, the main feature of the experimental curriculum is that the standard vertical paper-and-pencil algorithms for arithmetical computations are not taught at all, and do not form part of the curriculum for the first three years of school. Our rationale for abandoning the standard algorithms is two-fold. Firstly, our baseline research showed that number and numeration concepts develop slowly and that children need extensive experience of the first three levels of understanding as developmental underpinnings for true level 4 understanding. During the first three grades students do not have adequate level 4 understanding as a basis for relational understanding of the standard algorithms. In essence we are therefore limiting instruction to procedures that can be related to children's existing conceptual knowledge. At the same time, however, students' conceptual knowledge must be enriched to support the acquisition of more advanced procedures; we do not believe that an instrumental understanding of the standard algorithms can contribute to students' conceptual knowledge. On the contrary, premature introduction of the standard algorithms at the syntactic level actively inhibits the development of adequate semantic number concepts and is detrimental to children's perspective on mathematical activity. On the other hand we are quite certain that most students are able to develop sound level 3 conceptual knowledge on which to base their own invented thinking strategies, and that these strategies enrich their conceptual knowledge and facilitate mental computation, computational estimation and flexible computational procedures. Secondly, in view of the advent of the calculator and changing objectives for computation, we do not think the standard algorithms have a legitimate place in the curriculum, at least not in the first three grades (Olivier, 1988).

Children's self-generated strategies are therefore viewed as important in their own right, and not as transitional procedures towards the standard algorithms. The experimental curriculum does not represent an alternative route to the traditional syllabus – the objectives have changed. In particular, automaticity is not an objective and drill does not feature as an important activity in teaching. Our cognitive objectives for computation are to facilitate an understanding of number and the properties of operations, and to develop algorithmic thinking as a process, including the ability to execute and explain given algorithms, to compare and evaluate the efficiency of different algorithms, and to design and modify algorithms for particular purposes.

Didactical contract and classroom processes. The constructivist perspective and the objective of children inventing their own computational strategies necessitate the renegotiation of the traditional didactical contract in the classroom (the assumptions, expectations and obligations that teachers and student have about their roles in establishing mathematical knowledge). Instead of the teacher demonstrating computational methods, the teacher presents all mathematical activity as problems to solve, challenging and expecting students

to make some progress in solving the problem in their own way. Children have to accept this challenge and take responsibility for their own learning.

Our didactical approach is further inspired by socio-constructivism, emphasizing the role of negotiation, interaction and communication between teacher and students and between students in the evolution of their cognitive processes. Problems are therefore set to students in small groups. Students are expected to demonstrate and explain their methods, both verbally and in writing, with the teacher providing needed support with respect to notation and terminology. Children are also encouraged to discuss, compare and reflect on different strategies, trying to make sense of other students' explanations, thereby learning from each other. Teachers spend a lot of time listening to pupils, accepting their explanations and justifications in a non-evaluative manner, with the purpose of understanding and interpreting children's available cognitive structures. This enables the teacher to provide appropriate further learning experiences that will facilitate the child's development.

Number concepts, place value and manipulatives. Children's computational strategies are built on their number concepts, and the curriculum therefore makes systematic and thorough provision for facilitating the development and transition of number concepts through the three sequential levels mentioned earlier. Number concept development goes hand in hand with children's construction of computational algorithms, and we make little distinction between the two. Traditional place-value instruction emphasizes verbal rules for assigning values to the digits in numerals based on their position (e.g. ones place, tens place). This is accomplished through empirical abstraction from grouped collections, i.e. place value is taught as physical knowledge, often resulting in children interpreting the digits in a numeral at its face value without recognizing ten as composed of ten ones. From a constructivist perspective, however, the meaning of a symbol like 34 consists of the child's interpretation of the symbol based on his available knowledge structures, i.e. the child sees what he understands, while traditional instruction assumes that the child understands what he sees. Understanding place value therefore involves reflective abstraction.

We do not teach the standard algorithms in the experimental curriculum, therefore there is no need for acquiring level 4 understanding of place value. Level 3 understanding of place value is more than sufficient to support children's semantic thinking strategies. We explicitly try to avoid a too early empirical abstraction of place value, because it will inhibit the development of adequate level 3 understandings. Our approach is to help children to construct increasingly sophisticated concepts of different units, including ten, and to build these concepts on children's counting-based meanings by encouraging increasingly abstract counting strategies and child-generated computational algorithms (cf. Cobb & Wheatley, 1988). We hypothesize with Kamii (1986) and Richards & Carter (1982) that multiplication may be a prerequisite for level 4 understanding of place value.

In the experimental curriculum we avoid representations that may foster a too early syntactic meaning of numerals. We therefore avoid manipulatives that embody non-proportional representations of number, e.g. positional abaci, Cuisinaire rods and coloured chips representing different values and also Dienes blocks. Instead, children use loose counters, collect them into groups of ten, and count 10, 20, 30, 31, 32, 33, 34. Children have two sets of numeral cards: multiples of ten and ones. To represent the numeral, they take the 30 card and the 4 card and place the 4 over the zero of 30. The representation of two-digit numerals is therefore handled as the juxtaposition of two numbers from the start.

SOME OUTCOMES

We briefly report first on some aspects of insights into children's construction of arithmetical knowledge and then on some achievement data.

Self-generated strategies. Our expectation that if given the opportunity children can devise their own increasingly sophisticated thinking strategies as their number concepts develop has been more than justified: the vast majority of students have rapidly progressed to level 3 strategies, with different students displaying a large variety of strategies and individual preferences, depending on their level of mathematical thinking. Students' computational strategies mature towards shorter, more abstract strategies as their number concepts grow and their knowledge of specific techniques increases, sometimes within the space of days. For example, Etian was observed using the following strategies on three consecutive days:

96 + 16:	96 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 – 5 → 16 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 – 1 → 0 5 + 1 = 6
94 + 13:	13 + 13 + 13 → 39 + 13 + 13 → 65 + 13 + 13 → 91 1 + 1 + 1 + 1 + 1 + 1 + 1 → 7 rem 3
83 + 13:	13 + 13 → 26 + 13 → 39 × 2 → 78 1 + 1 + 1 + 1 + 1 + 1 → 6 rem 5

Students are remarkably flexible in their thinking and use a variety of strategies for different problems involving the same operation. Their choice of strategy seems to be governed by a careful analysis of the properties and relationships between numbers, as illustrated by Claude's strategies in the following examples:

236 + 325:	200 + 300 → 500 + 50 → 550 + 6 → 556 + 5 → 561
163 + 399:	101 + 399 → 500 + 62 → 562

Children's thinking strategies fall into two broad categories. In the first, which we have called *accumulation* or iterative strategies, a transformation on one of the operands is followed by a compensating transformation on the answer, leading to a gradual approach to the final answer by a series of increasingly better "approximations", as illustrated by the following examples:

28 + 35:	20 + 30 → 50 + 8 → 58 + 5 → 63
83 – 27:	80 – 20 → 60 – 7 → 53 + 3 → 56

In the second category, which we have labelled *replacement* strategies, transformations are made on both operands in the original problem before any attempt at computation, resulting in an equivalent numerical expression which is easier to solve, e.g.

28 + 35:	20 + 30 = 50; 8 + 5 = 13; 50 + 13 = 63
83 – 27:	86 – 30 = 56

It would appear that most students naturally construct accumulation strategies, and that replacement strategies develop later. However, it may not be a natural construction; pupils' procedures are probably constrained by the previous emphasis on counting-on strategies and by the institutionalized practices of the classroom community (Cobb *et al*, 1988). Children's use of accumulation strategies would seem to inhibit an understanding of

equivalent expressions and foster an operational rather than a relational meaning of the equals symbol. This is one reason why we introduced the arrow notation for written exposition.

Conceptual and procedural knowledge. Although the relationship between conceptual and procedural knowledge is not simple, we have ample evidence of their mutual support of each other in a programme that emphasizes both conceptual and procedural knowledge.

We have consistently found that advances in conceptual knowledge lead to advances in children's procedural knowledge. For example, if a child solves 26 + 37 by counting on, providing the child with appropriate experiences to establish part-whole relationships leads to the child spontaneously constructing level 3 thinking strategies. However, a certain level of conceptual knowledge does not guarantee a corresponding level of procedural knowledge in our model. Although children's computational procedures are based on and express conceptual knowledge, a certain level of procedural performance does not necessarily define a child's optimal conceptual knowledge. It appears that children "pass through" the levels in our model several times. We cite two examples.

Firstly, we have often observed that a child may use level 2 counting-on strategies for adding small numbers, yet revert to level 1 counting all for larger numbers. We explain this regression by the fact that these larger numbers are outside the child's range of constructed numerosities: a child does not acquire the numerosities of all the two-digit numbers simultaneously; although he may have acquired the numerosities for a certain range of numbers, the numerosities of numbers beyond this range have also to be acquired before he has secured level 2 number understanding to support counting-on strategies for this range of larger numbers. Indeed, we have found that helping the child to construct the required numerosities enabled him to advance to counting-on strategies. The size of numbers therefore has a marked influence on the development of children's conceptual and procedural knowledge, and students must be carefully guided through increasingly larger ranges of numbers. We have also found many cases of children employing level 3 thinking strategies for larger numbers, yet using counting-on for smaller numbers, often in the same problem, for example in finding $27 + 35$ as $20 + 30 \rightarrow 50 + 7 \rightarrow 57 + 5 \rightarrow 62$ the last addition is typically done by counting on. It seems that this inconsistency can be explained in terms of perceived cognitive economy: children view counting on with small addends as an efficient enough strategy, which they can perform with little effort, while the same cannot be said for larger addends. This shows that computations with larger numbers need not be postponed until students have mastered the basic facts at the immediate recall level.

Secondly, it seems that children pass through the levels anew for each new operation, albeit at an increasingly faster rate. For example, children's initial computational strategies for multiplication are typically at level 2. Although they may have the prerequisite number concepts in place to support abstract strategies, they do not immediately employ level 3 strategies. The properties of the operations (e.g. the distributive property for multiplication) must first be developed.

There is also clear evidence that procedural knowledge can contribute to conceptual knowledge. For example, some second graders devised a guess-and-check strategy for sharing division: for $99 \div 3$ they counted by 30's to 90, deducing that $90 \div 3 = 30$. Then, at some stage, instead of checking a new guess, they shared the remainder: $9 \div 3 = 3$. This

then spontaneously evolved to an efficient division strategy based on useful decompositions to exploit known number facts, e.g. 76 ÷ 4: 40 ÷ 4 = 10; 36 ÷ 4 = 9; 10 + 9 = 19. Using this strategy significantly contributed to the students' number concepts by stressing many different relationships that were not apparent in their mainly decimal decompositions (e.g. 76 = 70 + 6) for addition and subtraction. This strategy may also facilitate an understanding of equivalent expressions and the equals symbol as a relational symbol.

Some achievement data. Direct observation and the results of written tests and clinical interviews with students show substantial improvements in the learning outcomes of the experimental curriculum over the traditional curriculum. Regarding achievement (correct answers) the experimental group outperformed control groups in all aspects of computation and word problems, and progressed at a substantially faster rate (Malan, 1989).

Of more importance to us than higher scores and faster progress is the evidence of the experimental group's high qualitative understanding of number and their computational strategies. For example, Malan (1989) found an average drop of 25% in the test scores of the grade 2 control group for subtraction exercises requiring regrouping against those that did not require regrouping, whereas the scores of the experimental group remained virtually unchanged. Of course we do not know what procedures children used, but if children in the experimental group did use thinking strategies, this result is hardly surprising, because the most popular thinking strategies do not require any regrouping, e.g.

$62-36: \quad 60-30 \rightarrow 30-6 \rightarrow 24+2 \rightarrow 26$

Malan also found evidence that the use of thinking strategies drastically reduces the incidence of systematic errors (bugs). For 62 – 36 44% of students in the control group exhibited systematic procedural flaws, compared to 8% in the experimental group. This would suggest that the children in the experimental group have a much better understanding of their computational algorithms. We confirmed this observation in clinical interviews conducted with all the grade 2 students in one experimental and one control school. The table below shows some data on students' strategies for solving 26 + 37.

Percentage of students using different strategies for 26 + 37[a]		
	Experimental Group (N = 48)	Control Group (N = 66)
Percentage correct	75	53
Cannot/will not	2	5
Level 1: counting all	4 (50)	3 (0)
Level 2: counting on	10 (20)	15 (20)
Level 3: thinking strategies	83 (85)	36 (92)
Standard algorithm	0	41 (41)

[a]Entries in parenthesis are the percentage of correct responses in that cell.

Of the 27 students in the control group using the school-taught standard algorithm only 11 were successful, while the others all demonstrated typical systematic errors, reflecting the

use of garbled syntactic rules applied to meaningless symbols: 7 students obtained a result of 53, "forgetting" to carry; 4 pupils obtained 513 or 531, showing scant semantic understanding; 2 students added digits across the numbers (adding ones and tens) obtaining 99. No systematic errors were found in the experimental group: for children using thinking strategies all errors were due to simple basic fact errors. This applies also to children in the control group who used thinking strategies. One is struck again by the high incidence of students using self-generated strategies in spite of being taught differently, and the substantially greater success rate of thinking strategies compared to that of the standard algorithm.

REFERENCES

Baroody, A.J. & Ginsburg, H.P. (1986). The Relationship Between Initial Meaningful and Mechanical Knowledge of Arithmetic. In J. Hiebert (Ed.), **Conceptual and Procedural Knowledge: The Case of Mathematics** (pp. 75–112). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Carpenter, T.P. & Moser, J.M. (1982). The Development of Addition and Subtraction Problem-Solving Skills. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), **Addition and Substraction: A Cognitive Perspective** (pp.9–24). Hillsdale, NJ: Lawrence Erlbaum Associates.

Carraher, T.N. (1988). Street Mathematics and School Mathematics. In A. Borbás (Ed.), **Proceedings of the Twelfth Annual Conference of the International Group for the Psychology of Mathematics Education** (pp. 1–23). Veszprém, Hungary.

Cobb, P. & Merkel, G. (1989). Thinking Strategies: Teaching Arithmetic through Problem Solving. In P.R. Trafton (Ed.), **New Directions for Elementary School Mathematics** (pp. 70–81). Reston, Virginia: The National Council of Teachers of Mathematics.

Cobb, P., Yackel, E. & Wood, T. (1988). Curriculum and Teacher Development: Psychological and Anthropological Perspectives. In E. Fennema, T.P. Carpenter & S.J. Lamon (Eds.), **Integrating Research on Teaching and Learning Mathematics** (pp. 92–130). Madison, Wisconsin: Wisconsin Center for Education Research, University of Wisconsin-Madison.

Cobb, P. & Wheatley, G. (1988). Children's Initial Understanding of Ten. **Focus on Learning Problems in Mathematics**. 10(2),1–28.

Fennema, E., Carpenter, T.P. & Peterson, P.L. (in press). Learning Mathematics with Understanding: Cognitively Guided Instruction. In J. E. Brophy (Ed.), **Advances in Research on Teaching**, Volume 1, Greenwich, Connecticut: JAI Press.

Ginsburg, H.P. (1977). **Children's Arithmetic: The Learning Process**. New York: D. van Nostrand.

Human, P.G., Murray, J.C. & Olivier, A.I. (1989). **A Mathematics Curriculum for the Junior Primary Phase**. Stellenbosch: University of Stellenbosch, Research Unit for Mathematics Education.

Kamii, C. (1986). Place Value: An Explanation of Its Difficulty and Educational Implications for the Primary Grades. **Journal of Research in Childhood Education**, 1 (2), 75–85.

Malan, I.G. (1989). Verslag van Tussentydse Toetsing van Sub. B– en st. 1–leerlinge. [Report of Interim Testing of Grade 2 and 3 Students.] (Junior Primary Mathematics Project Communiqué no. 5). Stellenbosch: University of Stellenbosch, Research Unit for Mathematics Education.

Murray, H. & Olivier, A. (1989). A Model of Understanding Two-Digit Numeration and Computation. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), **Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education**, Vol. 3 (pp. 3–10). Paris, France.

Olivier, A.I. (1988). The Future of Pencil-and-Paper Algorithms in the Arithmetic Curriculum. **Pythogoras**, 17, 11–16.

Resnick, L.B. (1989). Developing Mathematical Knowledge. **American Psychologist**, 44 (2), 162–169.

Richards, H. & Carter, R. (1982). The Numeration System. In S. Wagner (Ed.), **Proceedings of the Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education** (pp. 57–63). Athens, Georgia: University of Georgia.

Steffe, L.P., Cobb, P. & Von Glasersfeld, E. (1988). **Construction of Arithmetical Meanings and Strategies**. New York: Springer Verlag.

Steffe, L.P., Von Glasersfeld, E., Richards, J. & Cobb, P. (1983). **Children's Counting Types: Philosophy, Theory and Application**. New York: Praeger Publishers.

FROM PROTOQUANTITIES TO NUMBER SENSE

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The research described in this paper explores the efficacy of an early mathematics program that is aimed at developing number sense and is built enitrely on children's invented procedures and on their informally acquired quantitiative knowledge. In an effort to socialize children to think of themselves as reasoners about number, the classroom program routinely provided daily conversation about numbers and attention to quantitiative examples in everyday living situations. Results from the first year indicate that the program produced large improvements both in number sense and in computational competence across all ability levels.

In the U.S., as in many other countries, there are calls for early mathematics education that focuses less on computational skill and more on mathematical understanding and problem solving. Central among the objectives put forth in this new view of the goals of mathematics education is the development of number sense. According to Sowder (Sowder & Schappelle, 1989), number sense is a well-organized conceptual network that enables one to relate number and operation properties. It can be recognized in the ability to use number magnitude to make qualitative and quantitative judgments and in the use of flexible ways of solving problems involving number. Number sense is as much a habit of thought with respect to numbers and their relationships as it is any particular set of arithmetic facts or skills. It embodies a sense of confidence in one's mastery of numbers, a belief that there are many different ways to use numbers or to solve problems involving numbers, and a sense of empowerment with respect to the world of mathematics and numbers.

From this definition, it follows that a mathematics program aimed at developing number sense must be viewed not just as a program for developing particular forms of knowledge about numbers, but equally for developing dispositions to use this knowledge in flexible, inventive ways. This in turn means that it is not possible to simply add a number sense component to a curriculum that otherwise conveys to children that there are certain "correct" and expected ways to deal with arithmetic problems. Only if children come to believe that there are always multiple ways to solve problems, and that they, personally, are capable of discovering some of these ways, will they be likely to exercise--and thereby develop--number sense. For this reason, we believe, a serious commitment to number sense as an educational goal in mathematics requires that, for a considerable period of time, no specific computational algorithms be taught, but that children instead engage in massive practice in inventing computational and estimation procedures as well as in using them to solve everyday problems that they can understand.

This proposal is far from easy to adopt. Few parents are prepared to risk having their children not be taught to calculate. And teachers are wary of curricula that risk failure in traditional terms as a result of too much experimentation. The result is usually a compromise, at best, in which specific algorithms and

[1]The instructional program described in this paper was developed and implemented in Bill's classroom based on research and evaluation carried out by Resnick and Lesgold. The authors are grateful to the administration, parents, and primary grade children of the St. Agnes School, Pittsburgh, Pennsylvania, for their enthusiastic participation in this work.

procedures are taught (although sometimes with more discussion of their underlying rationales than used to be the case) and practiced, after a relatively short early period of more exploratory number activity.

Our goal was to test whether an early mathematics program built entirely on children's invented procedures could produce sufficient computational skill to meet current societal demands. The route we took was a calculated gamble. We thought that, by building explicitly on children's informally acquired knowledge about quantities and their relationships and by developing a classroom routine in which conversation about numbers was daily fare, we could provide so much contextualized practice in number use that all reasonable computational objectives would be met without instruction on specific computational procedures.

A Theory of the Psychological Origins of Number Sense

We were willing to take this gamble because a substantial body of research, accumulated over the past decade, suggested that almost all children come to school with a substantial body of knowledge about quantity relations and that children are capable of using this knowledge as a foundation for understanding numbers and arithmetic. Two earlier papers (Resnick, 1989; Resnick & Greeno, 1990) review and interpret the research leading to this claim and develop a theory of how informal knowledge of quantities and counting might develop into mathematical knowledge about numbers and operators. We summarize the essential elements of that theory here.

Protoquantitative schemas

During the preschool years, children develop a large store of knowledge about how quantities of physical material behave in the world. This knowledge, acquired from manipulating and talking about physical material, allows children to make judgments about comparative amounts and sizes and to reason about changes in amounts and quantities. Because this early reasoning about quantity is done without measurement or exact numerical quantification, we refer to it as <u>proto</u>quantitative reasoning. We can document the development during the preschool years of three sets of protoquantitative schemas. These are:

- <u>Protoquantitative</u> <u>compare</u>, a schema that makes greater-smaller comparative judgments of amounts of material. Using it, children express quantity judgments in the form of comparative size labels such as bigger, longer, and more. These comparisons are initially based on direct perceptual judgments, but they form a basis for eventual numerical comparisons of quantities.
- <u>Protoquantitative</u> <u>increase/decrease</u>, a set of schemas that allow children to reason about the effects of adding or taking away an amount from a starting amount. Children know, for example, that if mother removes a cookie from one's plate there is less to eat and that if nothing has been added or taken away, they have the same amount as before. These schemas are protoquantitative precursors of children's eventual construction of unary addition and subtraction schemas. They also provide the framework from which conservation of quantity schemas will develop.
- <u>Protoquantitative</u> <u>part/whole</u>, a set of schemas that organize children's knowledge about the ways in which physical material comes apart and goes together, which allows them to make judgments

about the relations between parts and wholes. Children know, for example, that a whole cake is bigger than any of its pieces. This protoquantitative schema is the foundation for later understanding of binary addition and subtraction and for concepts of commutativity, associativity, and the complementarity of addition and subtraction. It also provides the framework for a concept of additive composition of number that underlies the place value system.

Quantification of the schemas

The protoquantitative schemas become the basis for number sense when they become integrated with children's knowledge of counting. Gelman and Gallistel's (1978) seminal work showed that children as young as three or four years of age implicitly know the key principles that allow counting to serve as a vehicle of quantification. Even when children are able to use counting to quantify given sets of objects or to create sets of specified sizes, however, they do not necessarily think of counting as a way of solving problems involving quantity relations. Sophian (1987), for example, has shown that children who know how to count sets do not spontaneously count when asked to solve conservation and similar problems. This means that counting and the protoquantitative schemas exist as separate knowledge systems, isolated from each other.

A first task of the primary school curriculum is to nudge children toward the use of counting--thus exact numerical quantification--in the context of problems that they previously would solve only by applying their protoquantitative schemas. Through such practice, the children not only acquire competence in solving problems about amounts in terms of numerical measures, but they also learn to interpret numbers in terms of the relations specified by the protoquantitative schemas. Eventually, they can construct an enriched meaning for numbers--treating numbers, rather than measured quantities of material, as the entities that are mentally compared, increased and decreased, or organized into parts and wholes by the schemas.

The Instructional Program

With this research base as a grounding for our efforts, we set out to develop a primary arithmetic program that would engage children from the outset in invention and verbal justification of quantity and number operations. Our goal was to use as little traditional school drill material as possible to provide children with a consistent environment in which they would be socialized to think of themselves as reasoners about number. Six principles guided our thinking and experimentation.

1. Draw children's informal knowledge, developed outside school, into the classroom. An important early goal of the program was to stimulate the use of counting in the context of the compare, increase/decrease, and part/whole schemas. This was done through extensive problem-solving practice, using story problems and acted-out situations. Counting (including counting on one's fingers) was actively encouraged, rather than suppressed as it often is in traditional teaching.

2. Develop children's trust in their own knowledge. To develop children's trust in their own knowledge qua mathematics, our program stressed the possibility of multiple procedures for solving any problem, invited children's invention of these multiple procedures, and asked that children explain and justify

their procedures using everyday language. In addition, the use of manipulatives and finger counting insured that children had a way of establishing for themselves the truth or falsity of their proposed solutions.

3. Use formal notations (identity sentences and equations) as a public record of discussions and conclusions. The goal here was to begin to link what children know to the formal language of mathematics. By using a standard mathematical notation to record conversations that were carried out in ordinary language and were rooted in well-understood problem situations, the formalisms took on a meaning directly linked to children's mathematical intuitions.

4. Introduce the whole additive structure as quickly as possible. It is important to situate arithmetic practice in a general system of quantity relationships. This is best done, we conjectured, by laying out the additive structures (addition and subtraction problem situations, the composition of large numbers, regrouping as a special application of the part/whole schema) as quickly as possible and then allowing full mastery (speed, flexibility of procedures, articulate explanations) of the whole system to develop over an extended period of time. Guided by this principle, we found it possible to introduce addition and subtraction with regrouping in February of first grade. No specific procedures were taught, however; instead, children were encouraged to invent (and explain) ways of solving multidigit addition and subtraction problems, using appropriate manipulatives and/or expanded notation formats that they developed.

5. Talk about mathematics; don't just do arithmetic. Discussion of numbers and their relations within problem situations is a crucial means of insuring that protoquantitative knowledge is brought into the mathematics classroom. In a typical daily lesson, a single, relatively complex problem would be presented. After a teacher-led discussion analyzing the problem, teams of children worked together to develop solutions and explanations for those solutions. Teams then presented their solutions and justifications to the whole class, and the teacher recorded these on the chalkboard using equation formats. By the end of the class period, multiple solutions to the problem, along with their justifications, had been considered, and there was frequently discussion of why several different solutions could work. Mathematical language and precision were deliberately not demanded in the oral discussion. However, the equations provided a mathematically precise public record, thus linking everyday language to mathematical language.

6. Encourage everyday problem finding. Children should come to view mathematics as something that can be found everywhere, not just in school, not just in formal notations, not just in problems posed by a teacher. We wanted them to get into the habit of noticing quantitative and other pattern relationships wherever they were and of posing questions for themselves about those relationships. To encourage this, homework projects were designed to use the events and objects of children's home lives: for example, finding as many sets of four things as possible in the home; counting fingers and toes of family members; recording categories and numbers of things removed from a grocery bag after a shopping trip.

Results of the program

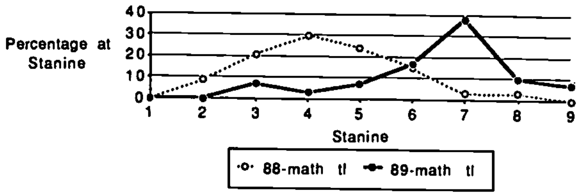
The program was initiated during the 1988-89 school year. The school served a largely minority and poor population. All children in first, second, and third grade classes in the school participated.

Because we had initially intended to introduce the program one year at a time, our data are most complete for the first grade. We interviewed all first graders three times during the year, focusing on their knowledge of counting and addition and subtraction facts, along with their methods for calculating and their understanding of the principles of commutativity, compensation, and the complementarity of addition and subtraction. At the outset, the children were not highly proficient. Only one third of them could count orally to 100 or beyond, and most were unable to count reliably across decade boundaries (e.g., 29-30, 59-60). The size of the sets that they could quantify by counting ranged from 6 to 20. About one third could not solve small-number addition problems, even with manipulatives or finger counting and plenty of encouraging support from the interviewer. Only six children were able to perform simple subtractions using counting procedures. By December the picture was sharply different. Almost all children were performing both addition and subtraction problems successfully, and all of these demonstrated knowledge of the commutativity of addition. At least half were also using invented procedures, such as counting-on from the larger of two addends (the MIN model), or using procedures that showed they understood the principles of complementarity of addition and subtraction. By the end of the school year, all children were performing in this way, and many were successfully solving and explaining multidigit problems.

Additional evidence suggests that the program was having many of the desired effects on children's confidence in their mathematical knowledge. Many of the children sang to themselves while taking the standardized tests that the school regularly administers. When visitors came to the classroom, they offered to "show off" by solving math problems. They frequently asked for harder problems. These displays came from children of almost all ability levels; they had not been typical of any except the most able children during the preceding year. Homework was more regularly turned in than in preceding years, without nagging or pressure from the teacher. Children often asked for extra math periods. Many parents reported that their children loved math or wanted to do math all the time. Parents also sent to school examples of problems that children had solved on their own in everyday family situations. Knowing that the teacher frequently used such problems in class, parents asked that their child's problems be used. It is notable that this kind of parental engagement occurred in a population of parents that is traditionally alienated from the school and tends not to interact with teachers or school officials.

To assess whether the computational aspects of the standard curriculum were being met, we examined data from the standardized mathematics achievement test that the school annually gives its first graders at the end of March. We used as a control group the children who had studied mathematics with the same teacher the preceding year. The following figure shows this comparison.

California Achievement Test
Math—Total Subscore



As can be seen, there is a massive improvement (equivalent to 1 1/2 standard deviations) from 1988 to 1989. Of particular importance, the statistical change was not achieved by improving the performance of the higher ability children, while leaving lower ability children behind. Rather, the entire distribution shifted upward. To check on whether these differences might have been due to a population difference, rather than an instructional program difference, we also compared scores on the "readiness" test that the school had administered at the end of kindergarten. The control group had performed slightly better than the program group in kindergarten.

The test scores of second and and third graders who were introduced to a modified version of the program partway through the 1988-89 school year provided additional evidence of the program's effectiveness. The second graders in the program showed an improvement equivalent to 1 1/2 standard deviations, the third graders to 1 standard deviation, compared with children who had not been in the program.

Conclusion

Our data show that an interpretation- and discussion-oriented mathematics program can begin at the outset of school by building on the intuitive mathematical knowledge that children have when they enter school. Such a program appears to foster the habits and knowledge that signal developing number sense. Our standardized test score data show that this kind of thinking-based program also succeeds in teaching the basic number facts and arithmetic procedures that are the core of the traditional primary mathematics program. Assuming that we can maintain and replicate our results, this means that a program aimed at developing sense can serve as the basic curriculum in arithmetic, not just as an adjunct to a more traditional knowledge-and-skills curriculum. Finally, our results suggest that an invention-based mathematics program is suitable even for children who are not socially favored or, initially, educationally able. This kind of program, if present in schools at all, has usually been reserved for children judged able and talented--most often those from favored social groups.

This apparently successful program presents some fundamental challenges to dominant assumptions about learning and schooling. Both educators and researchers on education have tended to define the educational task as one of teaching decontextualized knowledge and skills. An alternative view of the function of school in society is to think of schools as providing contexts for knowing and acting in which children can become apprentices--actual participants in a process that is socially valued, even though they are not yet skilled

enough to produce complicated performances without support. In this project, we were trying to create an apprenticeship environment for mathematical thinking in which children could participate daily, thus acquiring not only the skills and knowledge that "expert" mathematical reasoners possess, but also a social identity as a person who is able to and expected to engage in such reasoning. Several lines of theoretical work (e.g., Collins, Brown, & Newman, 1989; Lave, 1988; Rogoff, in press) inspire our thinking about learning as apprenticeship.

Our work addresses questions of how the apprenticeship metaphor can usefully inform early learning in a school environment. Among the problems to be solved is that of insuring that necessary particular skills and knowledge will be acquired, even though daily activity focuses on problem solving and general quantitative reasoning. Our first year standardized test results suggest that we have not done badly on this criterion, but we need to understand better than we do now what in our program has succeeded and what the limits of our methods might be. A second, even broader issue is the nature of the master-apprentice relationship. In traditional apprenticeship, teaching is only a secondary function of the master in an environment designed primarily for production, not instruction. Future work will analyze the role of the teacher in maintaining an apprenticeship environment specifically for learning purposes.

References

Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics. In L. B. Resnick (Ed.), Knowing, learning, and instruction: Essays in honor of Robert Glaser (pp. 453-494). Hillsdale, NJ: Erlbaum.

Gelman, R., & Gallistel, C. R. (1978). The child's understanding of number. Cambridge, MA: Harvard University Press.

Lave, J. (1988). Cognition in practice. Boston, MA: Cambridge University Press.

Resnick, L. B. (1989). Developing mathematical knowledge. American Psychologist, 44(2), 162-169.

Resnick, L. B., & Greeno, J. G. (1990). Conceptual growth of number and quantity. Unpublished manuscript.

Rogoff, B. (in press). Apprenticeship in thinking: Cognitive development in social context. New York: Oxford University Press.

Sophian, C. (1987). Early developments in children's use of counting to solve quantitative problems. Cognition and Instruction, 4, 61-90.

Sowder, J. T., & Schappelle, B. P. (Eds.). (1989). Establishing foundations for research on number sense and related topics: Report of a conference. San Diego, CA: San Diego State University, Center for Research in Mathematics and Science Education.

PROSPECTIVE ELEMENTARY TEACHERS' KNOWLEDGE OF DIVISION

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This study endeavors to contribute to the research base with respect to prospective teachers' subject matter knowledge of mathematics. The study investigates their knowledge of division through an open-response written instrument and through individual interviews. The written instrument was specially designed to assess understanding of division concepts. Results showed adequate procedural knowledge, but inadequate conceptual knowledge and sparse connections between the two. Weak and missing connections were identified as well as aspects of individual conceptual differences.

THEORETICAL FRAMEWORK AND SCOPE OF STUDY

This research is based on recent empirical and theoretical work in the area of teacher knowledge in general (Grossman et al., 1989) and mathematics teacher knowledge in particular (Ball, 1988). The work of these researchers and theorists has led to a view of teacher knowledge as an integrated whole made up of interconnected subdomains, all of which are essential to the effective functioning of the mathematics teacher. A research base with respect to prospective teachers' knowledge (and beliefs) is essential if we are to develop instructional interventions that will help prospective teachers extend and modify their knowledge. This study focuses on subject matter knowledge, one key subdomain, which has been characterized as knowledge both <u>of</u> and <u>about</u> mathematics (Ball, 1988).

Although subject matter knowledge is not made up of discrete subunits, it was nevertheless necessary to narrow the focus of the study. The study was focused as follows:

1. Focus on knowledge of rather than knowledge about mathematics. These two aspects of knowledge seem to be inextricably linked much as two systems of the body. Not only does the operation and condition of one affect the operation and condition of the other, but the structure of each necessitates certain compatible structures in the other. It is assumed to be the same with knowledge of and about mathematics. The structure of the knowledge in each domain shapes the knowledge in the other. It seems possible, however, and useful at times (as in this study) to focus on one while acknowledging their interdependence.

2. Focus on the conceptual field of division. As above, this is not an area of focus with neat boundaries. It is a cluster largely within multiplicative structures, whose connections go beyond multiplicative structures (e.g. subtraction, fractions). Division was chosen for two reasons. First, division is an important topic in the elementary curriculum. Second, previous work had been done with respect to prospective teachers' knowledge in this area. Graeber et al. (1986) and Harel et al.

(1989) focused on primitive beliefs about division. The latter work also identified solution strategies and the effect of various problem quantities. Ball (1988) looked at the connectedness of teacher knowledge in three applications of division.

Two questions were addressed in this study:

1. What should a prospective elementary teacher's knowledge of division consist of / how might it be assessed?

2. What is the nature of (current) prospective elementary teachers' knowledge of division?

METHODOLOGY

A set of seven problems was specially devised in response to the first question. The problems were designed to require more than an algorithmic knowledge and to assess a number of aspects of knowledge of division. (While these problems do not exhaust knowledge of division, they assess more broadly than had previously been done.) These open-response problems were administered to 33 randomly selected, prospective elementary teachers to explore broadly their conceptual knowledge of division. For each problem, the responses were sorted into groups of similar responses. The frequency of each response type and the implied understandings were considered.

In the second phase, eight prospective elementary teachers were interviewed as they worked on particular problems from the original problem set. The interviews were used to get a more in-depth view of the students' thought processes and understandings.

The subjects were elementary education students in a large, American, public university. At the time of their participation in the study, all subjects had completed the mathematics part of their teacher preparation program (three required courses) and had not yet participated in the student teaching practicum (the final semester of the program).

RESULTS

Because of space limitations, responses to each problem cannot be discussed in depth. Instead, each problem is presented followed by a brief summary of results of the written responses (% of correct responses and % of most common incorrect responses). Written responses to problems 5, 6,and 7 are discussed in greater detail. These three problems were used in the interviews. Based on analysis of the interviews, characterizations of subjects' knowledge were derived.

Problems and Responses

1. Write three different story problems which would be solved by dividing 51 by 4 and for which the answers would be respectively:

   a) 12 3/4

   b) 13

   c) 12

You should have three realistic problems.

Responses:

a. 72% correct

b. 39% correct

c. 75% correct

2. Inez remembered a rule that to divide by a decimal, you move the decimal point the necessary number of places to make the divisor a whole number and move the decimal in the dividend the same number of places. She did the following:

.51 .123

Does this rule work? Explain why or why not.

Responses:

9% adequate explanations

33% said that whatever you do to one # you must do to the other

3. Write a story problem for which 3/4 divided by 1/4 would represent the operations used to solve the problem.

Responses:

30% created a correct problem

33% created problems which would be represented by number expression other than the one that was given

4. Does 24 divide into 42^{72} (42 times itself 72 times)* without a remainder? Explain.

Responses:

3% considered the factors of 24 and the factors of 42 raised to a power and determined correctly that 24 goes into 42^{72}.

6% were able to answer the question correctly based on computations that they performed (42/24, 42x42/24, and 42x42x42/24)

33% said that because 24 does not go into 42, 24 does not go into any power of 42.

9% said that because 2 is a factor of 24 and 2 is also a factor of 42 that 24 goes into 42 to the 72nd power.

5. How could you find the remainder of 598,473,947 divided by 98,762 by using a calculator? The calculator does only the four basic operations, add, subtract, multiply, and divide (there is no remainder key). Note that dividing on the calculator gives a decimal answer. Describe how you would use the calculator to find the remainder and why that would work.

Describe a second method if you can.

Responses:

- 0% were able to solve this problem in two different ways.
- 24% were able to offer one method of finding the remainder, 2 subjects by method one (subtracting the product of the whole number part of the quotient and the divisor from the dividend) and 6 subjects by method two (multiplying the decimal part of the quotient by the divisor). It is interesting to note that method one is essentially what these subjects had practiced each time they did long division; however only 6% of them were able to generate the method in response to this particular problem.
- 33% said that a remainder could be obtained by putting the decimal over a power of ten (e.g. .35562 = 4/10 or 36/100)
- 12% said that you can read it from the decimal (e.g. it's the first digit to the right of the decimal, or the decimal answer is the remainder)
- 12% said to divide the decimal part of the quotient by the divisor.

6. Serge has 35 cups of flour. He makes cookies which require 3/8 of a cup each. If he makes as many such cookies as he has flour for, how much flour will be left over?

Responses:

- 15% correct
- 30% claimed that there was 1/3 of a cup of flour left over. (The 1/3 was part of the 93 1/3 which was the quotient of 35 cups of flour divided by 3/8 of a cup per cookie and thus referred to the number of cookies that could be made.)

7. In long division carried out as in the example, the sequence "divide, multiply, subtract, bring down" is repeated. Explain what information the "multiply" step and the "subtract" step provide and how they contribute to arriving at the answer.

```
         59
e.g.: 12)715
        -60
         115
        -108
           7
```

Responses:

Subjects' responses were devoid of references to the meaning of division. Their responses were generally of two types. The first indicated that the operation was necessary to get the next number in order to continue the long division algorithm. This response was equivalent to "that is how you do it" and had no explanatory value. The second type of response was based, not on the meaning of division, but rather on the local goal of the procedure at the given point. Subjects would say that the multiplication step checks to see that the number selected for the quotient was not too large. (Since 5 was selected as how many times 12 goes into 71, the 60 derived from 5 x 12 indicates that 5 was not too large.) They further explained that the subtraction step checked to see that the number that they had chosen was not too small (since the difference is less than the divisor).

These responses seem to suggest:

1. The subjects do not search for an explanation in the idea that division is a separation of a quantity into groups. Rather their explanations were procedural without connections to the concept.
2. For many, it seems that the knowledge that they were able to bring to bear on this problem had not changed since they were perhaps nine or ten years old. At that age, when they were first learning the algorithm, they were concerned with using it accurately. The problem was to select the next correct digit for the quotient. These subjects, though they are now an average of twenty-two years old, have not developed insight into the connection between the design of the long division algorithm and the meaning of division.

Results of Interviews

Problems #5, #6, and #7 were used for the interviews. The problems were always administered in this order so that work with problems #6 and #7 would not improve performance on # 5. Interviews were analyzed in terms of weak or missing concepts. A summary of this analysis follows. (Because of space limitations, transcript data are deferred to the oral presentation.)

The subjects' seemed to be unable to connect the meaning of division (i.e. the separating of a quantity into equal size groups) with the symbolic division carried out by the calculator or represented by the long division algorithm. The whole number part of the quotient, the fractional part of the quotient, the remainder, and the products and differences generated in long division generally did not seem to be connected with any concrete notion of what it means to divide a quantity.

Whereas Graeber et al. (1986) found evidence of primitive partitive and quotitive models among prospective elementary teachers, subjects did not seem to call on primitive models to make sense of the numbers and procedures involved. It should be noted that the numbers used in problems #5 and #7 did not violate these primitive models (e.g. quotient is always less than dividend). Thus, the primitive models would have been useful in this context. Subjects, instead, seemed to search for meaning within a narrow procedural space, reiterating rules for the procedure and comparing it to procedures with simpler numbers that had become more automatic.

Silver (1986) hypothesized that the lack of connection between the meaning of division and the symbolic operation (i.e. long division) might be traceable to early experiences with division. He observed that elementary students develop a partitive model of division and then are presented with a division algorithm based on a quotitive model ("How many 12's in 715?").

Connections between the semantic content of the problem and the computation performed were weak. In solving the word problem (#6), subjects often chose to perform appropriate multiplications and divisions, but lost the connection between the numbers generated and the physical quantities being manipulated in the problem.

The connection between multiplication and division may be a weak one. Betty's first step to find the remainder using the calculator was trial and error multiplication using the divisor as one of the factors. This was an attempt to find the largest such product less than the dividend. Carl multiplied the divisor by the whole number less than the quotient and the whole number greater than the quotient. He was initially surprised that the difference between the two products was the divisor. He was able to use his understanding of multiplication to make sense of this. However, while in the context of division, he did not seem to have a sense of what he had done.

Some subjects were able to make sense of division in the context of a word problem, but were unable to connect what they know in these contexts to an abstract level of thinking about division. Betty was unable to interpret the decimal part of the quotient in problem #5, but was able to interpret without difficulty the decimal part of the quotient in word problem #6.

Other Conceptual Weaknesses

Although not explored in depth through interviews, several other conceptual weaknesses seemed to be suggested by the written data.

-- Lack of connection between the meaning of whole number division and the meaning of division involving fractions

-- Inability to understand divisibility (could be related to the weak connection between multiplication and division)

-- Inability to perceive the division relationship in a fraction or ratio

-- Inability to think flexibly or consciously about division as partitive _and_ quotitive

Individual Differences

Analysis of the interviews revealed that individual differences in knowledge of division were more complex than the having or not having of particular knowledge or the connectedness of that knowledge. What also emerged as important was what I will call the _connectability_ of the knowledge. Four levels of connectability were observed: (It may be useful to think about connectability as a continuum.)

1. The information seems to be well connected. The subject has the relationship immediately available.
2. A relevant task and the demand for verbalization of thought processes seems to result in the subject's making the connection during the interview.
3. The connection does not seem to be made until a critical experience is initiated by the interviewer. (For example, Emily was not able to make sense of the long division procedure until she was asked to make up a word problem for 715 divided by 12.) The "cognitive restructuring" that takes place in such situations is reminiscent of the constructivist teaching experiment (Cobb & Steffe, 1983).
4. The connection does not seem to be able to be made given the subject's current knowledge.

Each of these levels of connectability suggests a different instructional intervention.

CONCLUSIONS

The prospective elementary teachers in this study exhibit serious shortcomings in their knowledge of division. Their _procedural knowledge_ seems generally adequate while their _conceptual knowledge_ is lacking. Procedural knowledge consists of "the formal language, or symbol representation system, of mathematics" and "the algorithm, or rules, for completing mathematical tasks". Conceptual knowledge "can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert & Lefevre, 1986).

The subjects seemed to have appropriate knowledge of the symbols, vocabulary, and algorithms associated with division. However, many important connections seemed to be lacking leaving a very sparse "web of

knowledge." Silver (1986) emphasized that procedural and conceptual knowledge need to be interconnected. The lack of such connections was apparent as subjects failed to connect their understandings of division and the semantic features of the word problems to the procedures which they employed to divide. This view of prospective elementary teachers' knowledge suggests the need for a mathematical education that focuses on facilitating the making of cognitive connections much more than the imparting of additional information.

The analysis of data suggests specific connections which would likely prove important in the creation of a dense web of knowledge. The absence of such connections would likely prevent not only adequate knowledge of division, but also the development of understanding in areas such as combinatorics, probability, and statistics which are the foci of recent curriculum reforms.

This picture of prospective elementary teachers' mathematical knowledge as procedural and sparsely connected may turn out to extend to many areas other than division. In addition, we can speculate about the adverse effect of this state of knowledge of mathematics on their knowledge about mathematics. (This is not to imply that cause and effect here is unidirectional.)

It is an empirical question as to how the knowledge assessed in this study affects teacher decision-making and performance and ultimately student learning. Earlier research of this type suggests that the effect may be considerable (Thompson, 1984).

Note: * The wording was selected for its common parlance, not for its mathematical exactness.

REFERENCES

Ball, D. (1988) The subject matter preparation of prospective mathematics teachers: Challenging the myths. Research report 88-3 of the National Center for Research in Teacher Education, Michigan State University.

Cobb, P., & Steffe, L. (1983) The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14, #2.

Graeber, A., Tirosh, D., & Glover, R. (1986) Preservice teachers' beliefs and performance on measurement and partitive division problems. Procs. of the Eighth Annual Meeting of PME-NA.

Grossman, P., Wilson, S., & Shulman (1989). Teachers of substance: Subject matter knowledge for teaching. In Maynard Reynolds (Ed.) Knowledge base of beginning teachers. Washington, DC: AACTE.

Harel, G., Behr, M., Post, T., & Lesh, R. (1989) Teachers' knowledge of multiplication and division concepts. Unpublished.

Hiebert, J. & Lefevre, P. (1986) Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert, ed. Conceptual and procedural knowledge: The case of mathematics. Hillsdale, NJ: Lawrence Erlbaum Associates.

Silver, E. (1986) Using conceptual and procedural knowledge: A focus on relationships. In J. Hiebert, ed. Conceptual and procedural knowledge: The case of mathematics. Hillsdale, NJ: Lawrence Erlbaum Associates.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. Educational Studies in Mathematics, 15, 105-127.

RELATIVE AND ABSOLUTE ERROR IN COMPUTATIONAL ESTIMATION

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ABSTRACT

One topic necessary for a deep understanding of computational estimation is the distinction between absolute and relative error of estimates, particularly for products. Ten seventh-grade students received instruction on absolute and relative error, and their performance on both the post-instructional interview and the retention interview (five months later) was greatly improved. The instruction was not direct, but rather marked by group discussions and explorations of problems. This instruction also seemed to have a small but positive effect on students' propensity to estimate by rounding without compensating for the large errors introduced.

The purpose of this study was to examine the effects of instruction on relative and absolute error of products of whole numbers. In particular, we wished to assess the level of difficulty students exhibit in attaining understanding of these types of error, and to study the influence of such instruction on the increased use of compensation when estimating products.

Background and Rationale

Computational estimation is a complex task involving conceptual components including the recognition that approximate numbers are used and that an estimate is itself an approximation, the recognition that the appropriateness of an estimate depends on the context and on desired accuracy, and the acceptance of multiple values as estimates and multiple processes for obtaining estimates (Sowder, 1988). In addition, a good estimator is able to reformat the numbers used to compute by such means as rounding, truncating, or changing the form of the numbers; translating the structure of the problem into a form where computations are easier to perform mentally, and compensate appropriately either during or after obtaining an estimate. (Reys, Rybolt, Bestgen & Wyatt; 1982). In a study of the development of these concepts and skills (Sowder & Wheeler, 1989), it was found that these understandings and skills are slow to develop. Even by ninth grade, students

prefer to compute then round, rather than round then compute, although they understand that when asked to estimate they are to compute with approximate numbers. Not until ninth grade did the students in this study attempt to compensate, even when the compensation was straightforward and would have obviously led to a better estimate. In pilot testing estimation items with eleventh and twelfth graders, Case and Sowder (in press) found that none of their students could deal with relative error in estimations of products of computations, even though researchers agree that this is a fundamental notion of a true understanding of estimation (Sowder & Schappelle, 1989). None of the students in either the Sowder and Wheeler study nor the Case and Sowder study had had any instruction on estimation. The effectiveness of instruction on these topics, and in particular whether instruction on relative and absolute error would increase students awareness of the need for compensation on most estimates of products, was unknown.

The study reported here was part of a larger study investigating the interrelationships existing among number size concepts, mental computation, and computational estimation. In particular we were looking for long-term cognitive change resulting from instruction on these topics. The use of instruction as a research tool has become an accepted methodological approach to the study of cognitive development (Belmont & Butterfield, 1977). Doyle (1983) has indicated that one purpose of much of the research on cognitive processes has been to identify ways of improving the design of instruction. For example research on cognitive processes used by experts often results in recommendations for structuring academic tasks so that students are explicitly taught how to perform in the same manner as the experts. Instruction of this type on the topic of computational estimation has met with some success (Reys, Trafton, Reys, & Zawojewski, 1984). It seems unlikely however that the type of expertise acquired from such instruction would lead individuals to think in the same manner as the experts being emulated. Hatano's (1988) distinction between routine experts and adaptive experts clarifies this difference. Routine experts are able to solve familiar problems quickly and accurately, but are not able

to "invent" new procedures because they lack the rich conceptual knowledge of the adaptive expert. Hatano points out that routine expertise is not without value, and in many instances is sufficient. It is only when novel or unusual problems are posed that a lack of conceptual knowledge becomes a serious handicap. Doyle calls this type of instruction direct instruction, in contrast to indirect instruction where carefully structured opportunities are provided for discovering rules and inventing algorithms.

The students in this study received instruction on mental computation, on comparing and ordering numbers, and on computational estimation. The description of the computational estimation instruction and the results on selected problems on relative and absolute error and on compensation are included here.

Method

Sample: Fourteen seventh graders from a small private school participated in the study. However, we include here only the ten male students on whom our data was complete.

Instructional Unit: The estimation unit consisted of seven lessons, which took nine class periods to complete. In the introductory lesson students discussed newspaper articles in terms of whether or not numbers used were estimated or exact values, and the usefulness of estimation. Absolute error and percentage of error were introduced in the second lesson, and students discussed how the size of absolute errors was related to the "goodness" of the estimate. Students then played a measurement estimation game in which points were first given based on accuracy of estimates, then on the percentage error of estimates. Lesson 3 focused on estimates of addition and subtraction using whole numbers and rational numbers. Lessons 4 and 5 focused on estimation of multiplication first via a game in which students were to choose the closest estimate for products of two-digit numbers (e.g., for 36 x 42: 40 x 30, 40 x 42, or 36 x 40), then on finding range estimates and on using compensation to get closer estimates. Lesson 6 dealt with estimates of products and quotients using rational numbers, and Lesson 7 was on the effects of

changing one or both operands in estimating products and quotients. The instructor used an indirect form of instruction with discussions of all problems worked, and no rules were given for any type of estimation.

Assessment: Students were individually interviewed three times: before the instructional unit, two weeks after the instructional unit, and five months after the instructional unit. The five months included a three-month holiday and two months of algebra instruction during which no estimation problems were discussed. Interview item were the same or very similar from one interview to the next.

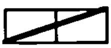
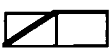
Results

Students performance on three typical items is shown in the glyph table. (A glyph is a pictorial representation of data [Sacco, Copes, Sloyer, & Stark, 1987]). The selection of glyphs to portray this data was made on the basis of their ability to portray a large amount of data in an efficient and easy-to-interpret manner.

Item 1: "The closest estimate for 22 x 84 is (a) 20 x 80; (b) 20 x 84; (c) 22 x 80."

This item required students to weigh the relative errors introduced when rounding one factor rather than another when estimating products. No student was successful in the pre-instruction interview. Eight of the ten students used an "additive" strategy, that is, they considered only the difference between the original and approximate numbers, and did not consider the effect of the change on the product. A typical answer was: "20 x 84, because 20 is only 2 off of 22, and 80 is 4 off of 84; 2 is closer than 4." The other two students selected 20 x 80 "because that's how you estimate." Yet all were able to compare the errors in both post-interviews. This time, a typical answer was: "22 x 80. 20 x 84 is 2 x 84 away, but 22 x 80 is only 4 x 22 away." A computational or interpretational error was made in six of the twenty post-instruction and retention answers, but students in all six cases indicated some understanding of the relative errors. For example, one student chose 20 x 84 "because 2 x 84 is 168 and 4 x 22 is 88, so we take more off in 20 x 84."

Glyph Table: Pre-instruction, post-instruction, and retention performance on selected items.

Student #	Items		
	Closest estimate for 22 x 84 is a) 20 x 80 b) 20 x 84 c) 22 x 80	34 x 86 estimated as 30 x 80 has error 344, 496 x 86 estimated as 500 x 86 has error 344 Which is better? (a) First (b) Neither (c) second	Estimate 42 x 34 (beyond 40 x 30)
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

Explanation of glyphs:

The intersections of a heavy line with the vertical segments of a rectangle represent performance on an item on the pre-instruction interview, the post-instruction interview, and the retention interview, respectively. The intersection of a heavy line with the lower horizontal segment and a vertical segment represents an incorrect or unacceptable answer, while the intersection of a heavy line with the upper horizontal segment and a vertical segment represents a correct answer with acceptable rationale. An intersection halfway between the lower and upper levels represents an incorrect answer due to incorrect calculation, but with a correct rationale. (The intersection was also placed midway for Student 3 on Item 3 because compensation was used, but in a grossly inadequate fashion.)

Example: The glyph for Student 8 on Item Type 1 is 

On the pre-instruction interview, this student claimed that 20 x 84 was the closest estimate for 22 x 84, because it was "only two numbers off" whereas 18 x 90 was "four numbers off." The student received the lowest marking for this incorrect answer. On the post-instruction interview and again in the retention interview, the student said 22 x 80 was a closer estimate than 22 x 80 because 2 x 84 = 168 and 4 x 22 = 88, so that 20 x 84 was further off. The student received the high marking the last two times.

(continuation of text)

Item 2: "If 34 x 86 is estimated as 30 x 86, then the exact answer is 2924 and the estimate is 2580. The difference between these two number is 344. If 496 x 86 is estimated as 500 x 86, the exact answer is 42,656, the estimate is 43,000, and the difference is again 344. Which of these would your choose: (a) the first estimate is better; (b) they are the same; (c) the second estimate is better."

In this item, two estimates of products had the same absolute errors but different relative errors, and students were to select the estimate with the smaller relative error as "better." Again, no students was successful on the pre-instructional interview. All ten students said that the estimates were "the same, they're both 344 off." By the post-instructional interview, 60% were successful, and 90% were successful on the retention interview. Typical reasons given for correct answers include: "The same error with bigger numbers is not as bad"; "344 is a lot off for small numbers like in the first one."

Item 3: "Estimate 42 x 34"

On this last item, students were simply asked to estimate a product. Note that 40 x 30 is usually considered an acceptable estimate. However, we wanted to see whether or not the work on relative error would have an effect on estimating products such as this one, and results given here indicate whether or not students went beyond a simple estimate of 40 x 30. Two students used compensation (e.g., it will be bigger than 1200) in the pre-instructional interview to go beyond 40 x 30. By the post-instructional interview, three students solved the problem by multiplying 40 x 34, one by multiplying 40 x 35, and one by multiplying 42 x 30. All five were considered to be successful. Of the five successful strategies on the retention interview, three continued to use technique of rounding one of the numbers, while two found 40 x 30 then compensated up.

Discussion

It may seem that finding the estimate of a product should be easier than dealing with relative and absolute error, but the difficulty with the third item lay in the fact that students had to recognize, on their own, that rounding to the closest multiple of ten gave a gross estimate deserving of some sort of compensation. Such recognition is very difficult to teach. The increase in this type of recognition is some measure of success of this instructional study, but performance was still lower than for the other items. Estimating products by rounding and then calculating was a process students had long found to be satisfactory, whereas dealing with relative and absolute error was new, with no "unlearning" required.

This study showed that middle-grade students can, through instruction, develop an understanding of the relative error of estimates. This understanding does not seem to develop naturally, as was noted earlier. That this was true understanding rather that a dependence on remembering rules formulated during instruction can be ascertained by the high performance level of students on the retention interview.

References

Belmont, J. M. & Butterfield, E. C. (1977). The instructional approach to developmental cognitive research. In Kail, Jr., R. U. & Hagen, J. W. (Eds.) *Perspectives on the development of memory and cognition* (pp. 437-481). Hillsdale, NJ: Lawrence Erlbaum.

Case, R., & Sowder, J. T. (in press). The development of computational estimation: A neo-Piagetian analysis. *Cognition and Instruction.*

Doyle, W. (1983). Academic work. *Review of Educational Research, 53,* 159-199.

Hatano, G. (1988). Social and motivational bases for mathematical understanding. In G. B. Saxe, & M. Gearhart (Ed.), *Children's Mathematics* (pp. 55-70). San Francisco: Jossey-Bass.

Reys, R. E., Rybolt, J. F., Bestgen, B. J., & Wyatt, J. W. (1982). Processes used by good computational estimators. *Journal for Research in Mathematics Education, 13,* 183-201.

Reys, R. E., Trafton, P. R., Reys, B. B., & Zawojewski, J. (1984) *Developing computational estimation materials for the middle grades.* Final report of NSF Grant No. NSF 81/13601.

Sacco, W., Copes, W., Sloyer, C., & Stark, R. (1987). *Glyphs: Getting the picture.* Rhode Island: Janson.

Sowder, J.T. (1988). Mental computation and number comparison: Their roles in the development of number sense and computational estimation. In J. Hiebert, & M. Behr (Ed.), *Number concepts and operations in the middle grades.* Hillsdale, NJ: Erlbaum & Reston, VA: NCTM.

Sowder, J. T. & Schappelle, B. P. (1989). *Establishing Foundations for Research on Number Sense and Related Topics: Report of a Conference.* San Diego: Center for Research in Mathematics and Science Education, San Diego State University.

Sowder, J.T., & Wheeler, M. M. (1989). The development of concepts and strategies used in computational estimation. *Journal for Research in Mathematics Education, 20* (2), 130-146.

A Child Generated Multiplying Scheme[1]

Leslie P. Steffe

My argument in this paper is that children's multiplying schemes are modifications of their counting schemes and their concepts of multiplication are constructed as a result of modifying the operations of their counting schemes. To make the argument, I show how Zach, an eight-year-old child, interpreted and solved what to me was a multiplication task. In this context, I characterize his counting scheme and explain an important modification he made in that scheme. Finally, I show how Zach established his first multiplying scheme and explain the operations he used in establishing it.

Solving a multiplication task

Zach was a participant in a two-year teaching experiment that was devoted to the study of children's multiplication and division (Steffe & von Glasersfeld, 1985). At the beginning of the teaching experiment, Zach had constructed a number sequence, numerical concepts for "two" and "three", but he did not take a pair or a trio of items as countable unit items. In the following protocol, Zach counted when solving the involved multiplication task, which illustrates my claim that children's multiplication schemes are modifications of their counting schemes. Zach assimilated the task using his initial number sequence, but he did not modify that sequence when solving the task to include what I took to be a unit of two.

T: (Places a red piece of construction paper in front of Zach and a pile of blue pieces cut so that six blue pieces fit exactly on the red piece. After Zach fit six blue pieces on the red piece in two rows of three, one of the six blue pieces is removed and the five remaining blue pieces and the red piece on which they are placed are covered using a paper of the same dimensions as the red piece. Two orange pieces are then placed

on the visible blue piece, which they cover exactly.) How many orange pieces fit on the blue piece?

Z: Two.

T: How many orange pieces do you think will fit on the red piece (the two orange pieces are left on the blue piece in full view of Zach)?

Z: (Sits silently, slightly turning his head back and forth as if scanning a re-presented piece of paper) twenty-three!

T: How did you calculate that?

Z: (Waving his hand over the paper) Because, um, I put the orange pieces on the mat and counted them up.

T: Tell me how you counted them up.

Z: I went (sequentially touches the screening paper along the bottom, where his points of contact form a row) 1, 2, 3, 4, 5 (repeats touching the screening paper in rows, each above the other) 6, 7, 8, 9--10, 11, 12, 13--14, 15, 16, 17,--18, 19, 20, 21, 22, 23 (Zach continued counting until he reached 23. He stopped because he thought he had filled the space along the uppermost row with orange pieces.)

Because of the way in which Zach moved his head back and forth when he was sitting silently and because he said that he "put the orange pieces on the mat and counted them up", his reenactment of counting seems to reflect what he did when he solved his problem. My interpretation is that he took an orange piece of paper as a countable unit and it was his goal to find how many of these countable units would fit on the red piece of paper. There was no indication whatsoever that he took the two orange pieces of paper as an occasion for making a composite unit that he could then use as a countable item.

The situation as interpreted and solved by Zach could not possibly have been multiplicative. Nevertheless, it was a meaningful situation for him. I look to the estimates that he made of how many orange pieces would fit across the blue piece and of how many such rows of orange pieces he could make as well as his counting activity to indicate the nature of his meaning. Because he did count, I take one of his meanings prior to counting as an

this sequence symbolized counting the orange pieces as if they were already placed on the red piece. His counting scheme was anticipatory[3] because there was no countable collection of orange pieces in his immediate visual field that he could count. His goal was to establish the numerosity of these orange pieces, which he did when he put them "on the mat" while counting. This is solid indication that his counting scheme was a number sequence rather than a sensory-motor or figurative scheme[4].

In that he did not take the two orange pieces as a countable unit is consistent with my interpretation that the words of his number sequence did not refer to the operations that were necessary to make a unit of units. Although there is nothing in the protocol to indicate it, his number words did refer to the verbal number sequence up to and including a given word, but they did not refer to taking that verbal number sequence as one thing--as a unit. "Seven", say, was a symbol for "one, two, three, ..., seven", but it was not a symbol for creating a unit containing that verbal number sequence.

An experiential composite two

Zach's failure to make a unit of two and use it as a countable item is consistent with the interpretation that he had constructed an initial number sequence but no more advanced number sequence. Just a few days later Zach modified how he used his number sequence to include making a composite unit of two.

T: (Places a pile of blocks in front of Zach) Pretend that you count out twelve blocks by two. Could you find out how many times you would count by two?

Z: (Points to the pile of blocks with his right index finger and middle fingers in an obvious attempt to count blocks by two.)

T: Would you like to count them by two? Go ahead.

Z: (Without moving any blocks, Zach successively points to pairs of

blocks using a finger pattern for two).

T: (Covers the blocks) Pretend that you counted those blocks by two. Can you find how many times you would count?

Z: (Places the same two fingers--his right index and middle fingers--on the table, folds them, and then puts up his right thumb as he silently utters number words. He then puts the two fingers on the table again, folds the middle finger and raises the index finger. He then raises the two fingers and wiggles the middle finger. Continuing, he raises the two fingers and then places his right ring finger on the table and then repeats raising the two fingers and places his little finger on the table. Following this, he moves his left hand only slightly.) Six!

Zach used his finger pattern for two when he initially counted to twelve, and this served as a basis for abstracting a unit of two from his repeated use of the pattern. He then used this unit of two to create composite units in the activity of counting. His modification of counting was a candidate to be called a multiplying scheme. Its most important aspect was that it occurred in a reenactment of an immediate past counting experience. Having just repeatedly instantiated his numerical pattern for two using his right index and middle fingers, Zach isolated this pair of fingers as recurrent. Repeatedly pointing to pairs of blocks was an action that he introduced and it was this immediate past experience that he reenacted. However, the reenactment served a new goal--to find how many times he pointed. This goal, coupled with reenacting the pointing action, served as an occasion for Zachary to apply his uniting operation to the results of his reenacted pointing action--his instantiated finger pattern--and the result was his countable item

My inference that Zachary did indeed take each pair of fingers as one thing is based on his recording repeatedly placing his right index and middle fingers on the table. The strongest indication is when Zachary took each finger of the pair as a record of making the pair. He seemed to be aware of the pair of fingers as an entity as well as of each finger of the pair as an item in its own right apart from the pair.

Taking the fingers of an instantiated numerical finger pattern as one thing is not the same thing as re-presenting a numerical finger pattern and taking those results as one thing. I call the results of the latter operation an abstract composite unit and the results of the former operation an experiential composite unit. Experiential composite units are made out of sensory material and appear to the actor in experience. They are not available to the actor prior to experience and that is why I do not consider Zach's modified counting scheme to be a multiplying scheme.

A multiplying scheme

There is a period of time when a child can establish an abstract composite unit to reach a particular goal, yet the unit is not available prior to operating. In the following protocol, Zach established an abstract composite unit for "three". This was the first time I had observed him establish such a unit and it occurred in the context of interactive communication. The only part of the interactive communication that is reported concerns how he established and used the abstract composite unit.

T: If you had six rows of blocks, I wonder how many blocks that would be (Zach understood that three blocks would be in each row)?

Z: (Using his index finger, he traces a segment on the table and then touches the table three times, where his points of contact form a row. He makes six such rows) eighteen.

It is important to note that Zachary could take the numerosity of the rows as a given. He knew how many rows he wanted to count but could see no rows in his immediate visual field. Enacting a row by tracing a segment on the table, and then repeatedly touching the table three times in a row while keeping track of how many rows he made until reaching six, indicates that he made an abstract composite unit of three prior to acting.

I view his tracing a segment on the table as an instantiation of his organizing activity that occurred prior to counting, an organizing activity

that was essential for him to carry out his rather complex counting activity. In the organizing activity, I infer that Zachary created a linear pattern of three abstract unit items as a result of re-presenting a row of blocks and then reprocessing that row using his unitizing operation. The result of unitizing a figurative item can be thought of as a slot that can be filled by sensory material of any kind--an abstract or interiorized unit item. The figurative item is "left behind" when the unit pattern is abstracted. These linear patterns can be used in re-presentation to create a visualized image of three blocks and it is my hypothesis that Zachary took this visualized image as one thing, as indicated by his continuous tracing action.

Having created a row of three blocks as an interiorized object, Zachary then coordinated instantiating that object six times with counting the unit items he created in the instantiations to establish their numerosity. Because he used the abstract composite unit of three sequentially, I call the resulting scheme a sequential integration multiplying scheme. The word "integration" refers to the act of uniting the results of counting three times--say, "7, 8, 9"--into a composite unit. Because his concept of three--the interiorized row of three blocks--was active prior to counting "7, 8, 9", I do not call the resulting unit he created in experience an experiential composite unit because it was an instantiation of the operations that constituted his concept. It was an instantiated abstract composite unit.

Zach's classroom teacher had chosen him to participate in the teaching experiment because she believed he was among the least able of her students in mathematics. Yet, I have demonstrated that Zachary could engage in productive mathematical activity, making abstractions that are crucial for

genuine mathematical progress. Unfortunately, productive mathematical activity is difficult to foster and it is currently confined to the most mathematically talented students. I believe that this unfortunate state of affairs is due to how productive mathematical activity is viewed. Rather than ask the deceptively simple question "What might constitute a child's productive mathematical activity from their frame of reference?" the attitude has been that productive mathematical activity is relative to an adult's conception of mathematics. Realizing that productive mathematical activity can be a result of interactive mathematical communication does provide an important alternative because reserving productive mathematical activity only for what is usually taken as the mathematically talented student is discriminatory and excludes almost all students from mathematics.

Notes

[1]The research on which this paper is based was supported by the National Science Foundation under Grant No. MDR-8550463. All opinions and findings are those of the author and are not necessarily representative of the sponsoring agency.

[2]When a child recreates an experience of counting, they may recreate the experience of saying number words in sequence, which is what I call the verbal number sequence.

[3]A counting scheme is anticipatory if the child can re-present results of using the scheme prior to counting.

[4]To be a number sequence, a counting scheme must be an operative or interiorized scheme.

Project Reference

Steffe, L. P., & von Glasersfeld, E. (1985). Child generated multiplying and dividing algorithms: A teaching experiment. NSF Grant No. MDR-8550463.

SALIENT ASPECTS OF EXPERIENCE WITH CONCRETE MANIPULATIVES†

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A current hypothesis among many mathematics educators is that it is helpful, and perhaps necessary, for students to be able to represent mathematical ideas in several equivalent ways. This sentiment is embodied in Dienes' Multiple Embodiment Principle.From a constructivist perspective, if manipulatives are effective, it is due to the their constraints on students actions and thereby the greater number of occasions where students are prompted to reflect on their understanding in relation to their goals. This study investigated the hypothesis that the more pronounced in students' experience is the constraining nature of a notational system, the more likely they are to conceive of notational algorithms as deriving from adaptations to the system's constraints.

A current hypothesis among many mathematics educators is that it is helpful, and perhaps necessary, for students to be able to represent mathematical ideas in several equivalent ways. This hypothesis is embedded in Dienes' Multiple Embodiment Principle (Bruner, 1963; Dienes, 1960) and is thematic in recent expositions of multiple, linked representational systems (Kaput, 1986). Dienes' rationale was that to develop an abstract concept, one needs more than one example which embodies it and from which the concept is to be abstracted.

From a constructivist viewpoint, the role of "embodiments," including concrete manipulatives, is that they provide situational constraints on students' activities, and hence provide occasions for students to make real for themselves the mathematical constraints that constitute the boundaries and glue of a concept. In encountering a constraint, students are blocked from doing something they want to do. Such disequilibria may foster reflection and abstraction of the mathematical constraints intended by the designer of the materials. For example, the concept of addition is elementary: it is to combine two quantities. The difficulty occurs in *naming* the resulting quantity's value. The task of naming the value of a quantitative operation's result is made even more difficult when all aspects of a naming process are constrained to occur within a specific representational system, such as decimal numeration. Dienes' base-ten blocks make the constraints of decimal numeration explicit. If students are asked to solve, say, a subtraction problem with Dienes' blocks they have occasions to reflect upon, and interiorize, the impact that constraints of decimal numeration have upon methods to name a difference.

The hypothesis of this study was this: The more pronounced in students' experience is the constraining nature of a notational system, the more likely they are to conceive of notational algorithms as deriving from adaptations to the system's constraints. To study this hypothesis, the first author developed a computer microworld that incorporates multiple, linked notations for decimal numeration and compared students' use of it with students' use of Dienes base-ten blocks.

The computer program used in the study was a mathematical microworld for decimal numeration. The genre of mathematical microworlds is described elsewhere (P. Thompson, 1985, 1987). This particular microworld, called BLOCKS, runs on a Macintosh.

MICROWORLD

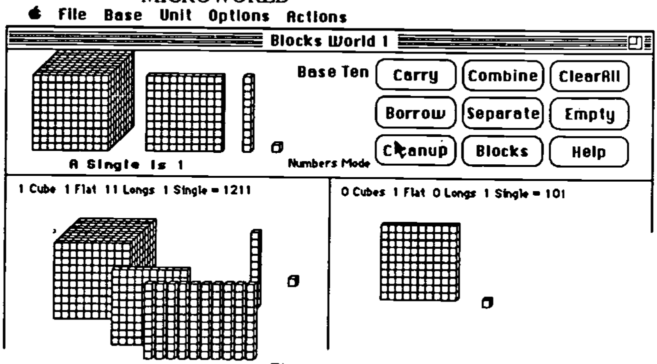

Figure 1

BLOCKS presents students with a supply of Dienes blocks (Figure 1) and regions where blocks may be stored. Two notational systems corresponding to displayed blocks are presented: expanded notation and traditional numeral. Boxed words (called "Buttons") in the display's top right corner will be explained later.

Students create collections of blocks by using a mouse to drag copies of blocks from the source region to the region in which the blocks are to be stored. Subtraction is done by dragging blocks from one storage region to another. Addition is done by creating collections in the two storage regions and by clicking the **Combine** button (Figure 2). Clicking **Combine** causes BLOCKS to remove the vertical dividing line and consider all blocks as one collection. Clicking **Separate** causes the vertical line to be redrawn, splitting the single storage area in two.

Interaction

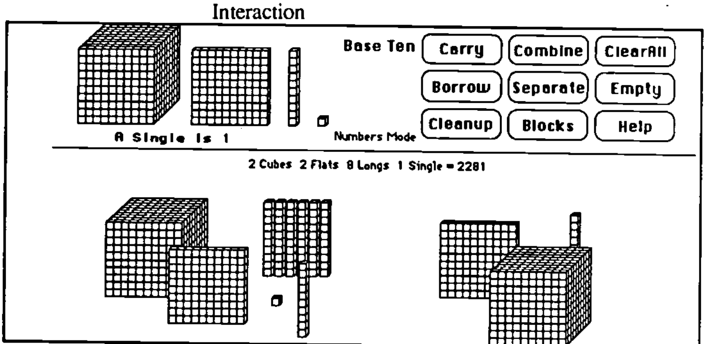

Figure 2

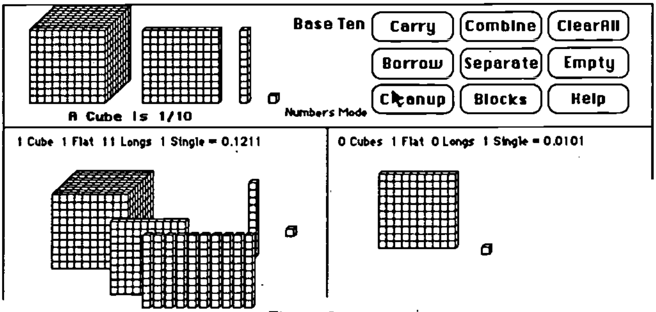

Figure 3

The buttons **Borrow** and **Carry** effect transformations of the digits in a numeral and thereby effect transformations of the blocks in a collection. Transformations of digits are effected by clicking on a digit in the numeral's expansion, then clicking **Borrow** or **Carry**. **Borrow** causes one block of the kind corresponding to the clicked digit to be unglued into 10 of the next smaller kind (with the exception of a single). **Carry** causes 10 blocks of the kind corresponding to the clicked digit (if there are at least 10) to be glued into one block of the next larger size (with the exception of a cube). The transformation is enacted with blocks and the result is reflected in the numeral's expansion.

The ***Unit*** menu contains options for what stands for one. The options range in sequential powers of ten from "A single is 1000" to "A cube is 1/1000". The numeral display reflects the user's choice of unit. Figure 3 repeats the display presented in Figure 1, except a cube denotes 1/10.

METHOD

Subjects

Twenty fourth-grade students enrolled in a midwestern university laboratory school were subjects of the study. Ten students were male; 10 were female. The laboratory school's enrollment is chosen to represent the geographic region's population academically and socio-economically. Average percentile ranks for subjects' Iowa Test of Basic Skills scores were: Concepts-70, Problem Solving-73, Computation-60, and Total Math-71.

Procedures

Students were assigned to two treatments: microworld instruction and wooden-block instruction.

Assignment to treatments

Students were matched according to their scores on a 19-item whole number computation, place value, fractions and decimal fraction pretest (test-retest correlation = .83). Item scores were entered into a stepwise multiple-regression analysis with total test score as the dependent variable. The analysis ended with six items ("six best items") being included in the regression equation. Sums were computed on those six items to give a pretest subscore; these "six best items" subscores were ranked in descending order. Pairs were formed by taking adjacently ranked sums. Members of pairs were assigned at random to experimental conditions through the use of a random

number generator. Pairs were further grouped by pretest score: low (3 pairs), medium (5 pairs), and high (2 pairs).

Two procedures were used to test the validity of the rankings. First, students' "six best items" subscores were correlated with their total pretest scores (Pearson's $r = .92$). Second, item scores were analyzed by factor analysis (orthotran-varimax). Two factors emerged: Representations and procedures. Factor scores were computed for each student and correlated with their "six best items" subscores (Pearson's $r = .91$).

Posttest

The posttest was in two parts: the pretest (as given before treatments) together with items on ordering decimals, decimal representations, appropriateness of method, and decimal computation. Items were scored for correctness of result and validity of method. Following the posttest, eight students were interviewed: the two pairs scoring highest on the pretest and the two pairs scoring lowest on the pretest. All interviews were videotaped and transcribed.

Instructors

The students' regular 4th-grade teacher taught the microworld group. A research assistant taught the wooden-blocks group. The regular 4th-grade teacher had never used this instructional approach before, nor had she used a microcomputer in instruction. The research assistant was an experienced teacher who was thoroughly familiar with the aims of instruction and with the computer program being used by the microworld group. We assigned the research assistant to the wooden-blocks group so that any "teacher expertise" bias would favor wooden-blocks instruction.

Instruction

Each instructor worked from a transcript written for the microworld treatment. The wooden-blocks instructor modified segments appropriate only for the microworld so that they were appropriate for wooden blocks. In-class activity sheets and homework sheets were identical for both groups. All microworld instruction was videotaped; field notes were taken during blocks instruction.[1]

Instruction was in three segments: Whole number addition and subtraction, decimal numeration, and decimal addition and subtraction (see below). Instruction on whole number addition and subtraction emphasized place-value numeration, transformations of numerals, the creation of methods for solving addition and subtraction problems, and the recording of actions done while applying a method. An emphasis was placed on students' freedom to create schemes for operating on blocks to solve addition and subtraction problems, with the provision that they had to represent in notation each and every action in their scheme, whether it be a change of representation or an arithmetical operation. How students denoted their actions was in large part left up to them. Alternative action schemes for solving problems were discussed frequently, as were alternative notational schemes for any given action scheme.

RESULTS

Testing

Students' performance on the pretest before and after instruction was stable. Each group's average score increased somewhat after the instructional period, with the exception of the High Blocks group, whose performance deteriorated (Figure 4). The deterioration in the High Blocks group was focused predominantly in the items having to do with concepts of decimal fractions. The increase in both Medium groups was across all items, the greatest increase occurring on items having to do with concepts of decimal fractions.

Students' performance on the posttest's decimal concept and skill items is shown in Figure 5. The disordinal interaction between pretest performance levels (Low, Medium, and High) and treatments (Microworld and Blocks) is obvious. The Low Blocks group outperformed the Medium and High Blocks groups on each of the areas of decimal representations, decimal computations, and decimal ordering; The Low Blocks group outperformed all groups on decimal computation (though all but Low Microworld scored about the same). The Medium Microworld group outperformed all Blocks groups on decimal representations, decimal ordering, and appropriateness of method. The High Microworld group outperformed all groups on each of the concept tasks.

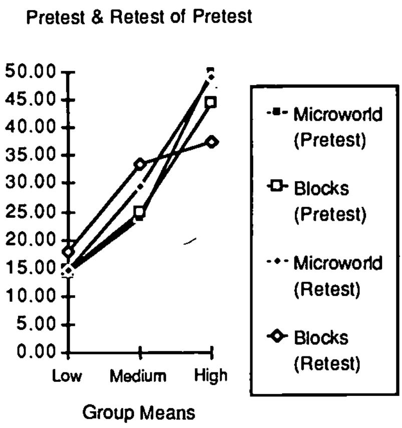

Figure 4

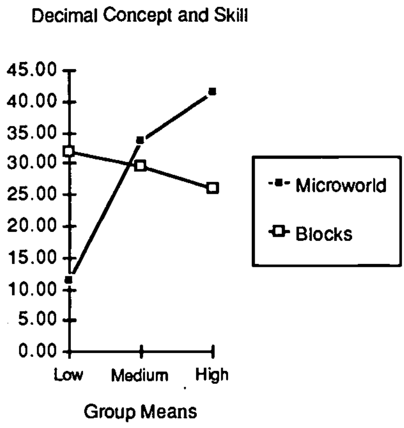

Figure 5

It is noteworthy that all Blocks students used either standard computational algorithms, or used addition and subtraction algorithms based on a method introduced early in instruction as just one example of unconventional approaches to calculation sums and differences. The Microworld group showed a variety of methods, although no student used more than one.

On one item showing the record of a method not used by any students during instruction, all but one Blocks student agreed with the statement that "the answer is correct, but it's not the right

way to do it." Among the Microworld group, 3 of the 4 Low students agreed with the statement, while 4 of 5 Medium students and 2 of 2 High students disagreed with the statement.

DISCUSSION

Evidently, the experiences had by the two treatment groups were very different. A review of field notes taken during wooden-blocks instruction and of videotapes recorded during microworld instruction suggests the nature of the difference.

Both groups began instruction expecting to be told "how to do it." Students in the Blocks group showed great resistance throughout instruction to entertaining alternative methods of solving addition and subtraction problems. On one occasion a student asked, "Can we just do it the old way?" The Microworld group also resisted discussing alternative methods, but only initially. One Microworld student, when asked after class one day how he liked what the class was doing, replied: "I really like it, this way of doing it if it makes sense to you. But I'm afraid to do it, really." His fear was that "Next year the teacher might mark it wrong."

On several occasions it was apparent that wooden blocks were of little value in constraining students' actions and thinking relative to mathematical concepts. For example, one in-class activity said "select a block to stand for one, then put blocks out to represent 3.41." One student selected a cube to stand for one, then looked back at her paper, reading "three hundred forty-one." She put out 3 flats, 4 longs, and 1 single, looked at the paper and back at the blocks, then went to the next task. A student in the Microworld group started similarly, selecting "A Cube is One" from the Unit menu, and then putting out 3 flats to make "three hundred forty-one." After putting out 3 flats, the Microworld student looked at the screen and said, "Point three? That's not what I want. ... Oh! A cube is one!"

Though the wooden-blocks teacher frequently oriented students toward correspondences between what they did with blocks and what they might write on paper, the Blocks students showed little evidence of *feeling* constrained to write something that actually represented what they did with blocks. Instead, they appeared to look at the two (actions on blocks and writing on paper) as separate activities, related only tangentially by the fact that the written symbols could have reference in the world of wooden blocks.

Except for the Low students, students in the Microworld group repeatedly made references to actions on symbols as *referring* to actions on blocks. One reason for this might be that their attention was always oriented toward the symbols, *even when their intention was to operate on blocks*. That is, the Microworld group acted on blocks by acting on their symbolic representations. Thus, the relationships between notation and manipulatives was always prominent in their experience.

Wearne and Hiebert (1988) outlined a local theory of competence with written symbols. In their theory manipulatives serve as referents for symbols, and actions on manipulatives serve as

referents for actions on symbols. The idea of manipulatives-as-referents is not in contrast to the issue of interiorization of constraints. Rather, it highlights one aspect of interiorizing concrete materials as embodiments of a mathematical system. The contribution of interiorized constraints to a student's thinking is that they provide the principles by which the system works.

Constraints are what make situations problematic, and it is overcoming constraints that constitutes problem solving. Students must conceive of notation (literal or manipulative) as *representing* something. Notations themselves cannot be the object of study. Also, students must construct an equivalence between notational systems. Multiple, linked representational systems do not make these achievements easy. Rather, they can have the effect of orienting students' attention to the issues of representational equivalence.

FOOTNOTES

† Research reported in this paper was supported by the National Science Foundation (Grant No. MDR 87-51381) and by Apple Computer, Inc. Any conclusions or recommendations contained in the paper are those of the author and do not necessarily reflect positions held by the National Science Foundation or by Apple Computer.

[1] Blocks instruction could not be videotaped. Two special education students were present during instruction, but not part of the study. State and university policy does not allow videotaping or photography of special education students without parents' permission.

REFERENCES

Bruner, J. (1963). *The process of education*. New York: Vintage.

Dienes, Z. (1960). *Building up mathematics* (4th edition). London: Hutchinson Educational.

Kaput, J. (1986). Information technology and mathematics: Opening new representational windows. *Journal of Mathematical Behavior, 5*, 187-207.

Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. A. Silver (Ed.), *Learning and teaching mathematical problem solving: Multiple research perspectives* (pp. 189-236). Hillsdale, NJ: Erlbaum.

Thompson, P. W. (1987). Mathematical microworlds and intelligent computer-assisted instruction. In G. Kearsley, (Ed.), *Artificial intelligence and instruction*. Addison-Wesley.

Wearne, D., & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for Research in Mathematics Education, 19*, 371-384.

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