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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following research papers: "Children's Connections among Representations of Mathematical Ideas" (A. Alston & C.A. Maher); "Algebraic Syntax Errors: A Study with Secondary School Children" (A. Avila, F. Garcia, & T. Rojano); "The Development of Conceptual Structure as a Problem Solving Activity" (V. Cifarelli); "From Arithmetic to Algebra: Negotiating a Jump in the Learning Process" (A. Cortes, N. Kavafian, & G. Vergnaud); "Continuous Analysis of One Year of Science Students' Work in Linear Algebra, in First Year French University" (J.L. Dorier); "Avoidance and Acknowledgement of Negative Numbers in the Context of Linear Equations" (A. Gallardo); "Introducing Algebra: A Functional Approach in a Computer Environment" (M. Garancon, C. Kieran, & A. Boileau); "LOGO, to Teach the Concept of Function" (D. Guin & I.G. Retamal); "The Concept of Function: Continuity Image versus Discontinuity Image: Computer Experience" (F. Hitt); "Acquisition of Algebraic Grammar" (D. Kirshner); "Embedded Figures and Structures of Algebraic Expressions" (L. Linchevski & S. Vinner); "A Framework for Understanding What Algebraic Thinking Is" (R.L. Lins); "Developing Knowledge of Functions through Manipulation of a Physical Device" (L. de Lemos Meira); "Students' Interpretations of Linear Equations and Their Graphs" (J. Moschkovich); "An Experience to Improve Pupil's Performance in Inverse Problems" (A. Pesci); "Algebra Word Problems: A Numerical Approach for Its Resolution: A Teaching Experiment in the Classroom" (G. Rubio); "Children's Writing about the Idea of Variable in the Context of a Formula" (H. Sakonidis & J. Bliss); "Observations on the 'Reversal Error' in Algebra Tasks" (F. Seeger); "Generalization Process in Elementary Algebra: Interpretation and Symbolization" (S.U. Legovich); "Effects of Teaching Methods on Mathematical Abilities of Students in Secondary Education Compared by Means of a Transfer Test" (J. Meijer); "On Long Term Development of Some General Skills in Problem Solving: A Longitudinal Comparative Study" (P. Boero); "Cognitive Dissonance versus Success as the Basis for Meaningful Mathematical Learning" (N.F. Ellerton & M.A. Clements); "Time and Hypothetical Reasoning in Problem Solving" (P.L. Ferrari); "The Interplay between Student Behaviors and the Mathematical Structure of Problem Situations: Issues and Examples" (R. Herschkowitz & A. Arcavi); "Paradigm of Open-Approach Method in the

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# International Group For the Psychology of Mathematics Education



## PROCEEDINGS

# Fourteenth PME Conference

With the North American Chapter
Twelfth PME-NA Conference

(July 15-20)



México 1990

**VOLUME II**

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# Algebraic Thinking and Functions

# Children's Connections Among Representations of Mathematical Ideas

Alice S. Alston and Carolyn A. Maher
Rutgers University

*Analysis of the written problem protocols and videotape segments of 11 children for seven problem-tasks containing common structural elements over a 6 day period is made in order to gain insight into the development of the representations built by the children and the connections made between and among the representations. A more detailed analysis is given of the mathematical behavior of two children. The observations provide descriptions of the process by which children construct representations of ideas in cooperative group problem-solving settings and in individual written assessments. This information lends insight to the results of a larger study in which highly significant gains in understanding were made by participants in these activities as compared with a control group.*

Much attention has been directed recently to the need to study the processes by which learners build-up systems of representation of mathematical ideas and relate them to other systems (Davis, 1984; Kaput, 1987). One approach to assessing understanding of a mathematical concept is recognition of that idea embedded within qualitatively different representational systems (Lesh, Post, and Behr, 1987). The building-up of meaningful experience may come about by being aware of the structure of the activity and reflecting on it (Steffe & Cobb, 1983). Another dimension of the building-up of representations is task involvement by the learner (Cobb, Yackel, & Wood, 1989). Lave (1988) urges us to consider, in studying the transfer of knowledge among representations, the learner's social interaction or other factors that motivate problem solving. Wood and Yackel (1990) demonstrate in their work the importance of peer group dialogue so that learners have an opportunity to make sense of each other's interpretations and serve mutually supportive roles. Brown, Collins, and Duguid (1989) also direct us to consider learning that arises out of shared activity by other learners in a context in which representations of ideas are constructed and discussed together. Our view is that learning mathematics is facilitated in an environment that provides for social interactions in small group problem solving tasks that enable learners to build-up representations, over time, of the structure of the idea(s). (Maher, 1987; Maher, Alston, & O'Brien, 1986).

## Background

As part of a larger investigation to measure the mathematical behavior of 12 and 13 year old 7th grade children, a teaching experiment was conducted in which 84 children from two schools (one, a public school in a blue-collar community and the other, an independent school in an affluent suburban community) were given a series of 7 problem-tasks in a natural classroom setting over a five day period of time. The purpose of the study was to observe and analyze children's mathematical thinking as they were engaged in tasks dealing with the properties of closure, identity, inverse and commutativity and to assess their ability to make connections among various representations of each of the concepts. The population was stratified into three groups, the first two from a public K-8 school, and the second from an independent middle school: high ability prealgebra students from the public school; students enrolled in regular seventh-grade math (students in this heterogeneous group ranged from remedial to average in ability) from the public school; and (non-honors) prealgebra students from the independent school. Children from each school population were randomly assigned to two comparable groups for five class periods, one, experimental and the other, control. The children in the experimental group were given three nonnumerical problem-tasks each based on a different concrete embodiment of the properties for the purpose of constructing solutions to the problems posed without any teacher intervention, while those in the control group were similarly engaged but were given problems that dealt with different content. In a final class session, each child was given three final written assessments. Each of these was a numeric problem task that was structurally isomorphic to one of the concrete tasks.

A logistic regression analysis was used to examine the relationship between problem solving success and the factor of experimental versus comparison group for each of six mathematical assessment categories across the three written postassessments. The categories were defined as (a) Closure, (b) Identity, (c) Inverse, (d) Order, referring to commutativity, (e) Transfer, that is generalization to other mathematical ideas, and (f) Total, referring to total success on the preceding five sections for the particular written assessment. The analysis of the Transfer category gave the probability of success for those children who had participated in the experiment to be 97% for the high ability experimental students; 79% for the average and heterogeneous experimental groups; and 32% for the comparison groups for all three of the postassessment tasks.

This report provides a description of the mathematical behavior of a group of 11 experimental students. Because of limitations of space, a more detailed analysis is given of only two of the children to provide insight into how they developed the concepts in their small groups and made connections to the ideas across the task activities.

In particular, the study sought to investigate, for representations built by all eleven children, the following three questions:

(1) Did the children make meaningful connections from the ideas considered in each of the seven problem tasks to other mathematical ideas?

(2) Did the children make connections from one task to another?

(3) What references, if any, do children make to mathematical ideas that are not specific to their task activities?

For the two more detailed analyses of children's mathematical behavior, two additional questions were also addressed:

(4) What references, if any, do children make to structural similarities and differences among representations?

(5) What connections, if any, do children make among the three concrete representations? Among the three numeric representations?

## Methods

Each child was given a written preassessment task (WPA). During the five following sessions the children met in experimental or comparison classes. The members of each experimental class were partitioned into groups of two or three children to work together to construct solutions of the three nonnumerical problem tasks. The structure of Task One (T1) was a Klein group using two small wooden figures, a boy and a girl. The elements of the set were the four possible 180 degree turns of the two figures taken together and the operation was one turn followed by a second. Task Two (T2) had a lattice structure and the elements of the set were cards, each of which had cut out a different polygonal shape. The operation was placing one card on top of a second to form a resulting polygonal shape. The third task (T3) had a cyclic group structure based on index cards, called Road Cards, each of which had a different set of four straight lines from beginning points on the left side of the card to end points on the right. The result of the operation, in which one card was followed by a second, was the single card with with the beginning points of the first card and the end points of the second. Within each experimental class, two groups were randomly chosen to be videotaped during all five sessions. On the final day to the teaching experiment the children returned to their regular class groupings and each child was given the three final written assessments (FWA1, FWA2, and FWA3). In each of the seven problem-tasks, the children were asked to construct a table of results for the set of elements and the given operation and then to answer a series of questions concerning closure, identity, inverse, and commutativity for that particular operational system. The concluding question of each task was whether the problem called to mind any other problems or ideas about mathematics, and, if so, to describe them.

The seven boys and four girls considered in this investigation were four of the cooperative problem-solving groups. The data included the children's solutions to each of the four written assessments, each child's written solution to the nonnumerical problem tasks, transcripts of the videotapes, and observer notes of the group problem-solving sessions. (Note 1)

## Results

Table 1 presents a summary of the data for the eleven children, In each task, evidence of the presence of a meaningful connection is indicated by a **Y**, its absence by **N**, and when the presence of a connection was in doubt, by **U** (e.g., student appeared to go along with group consensus).

Categories of connections were also indicated. A reference to one of the concrete tasks was coded as **c**; reference to a written numeric assessment was coded as **a**; references to mathematical ideas that were not specific to the task activities were coded according to their context (e.g., arithmetic operations (**o**); numbers (**n**); fractions (**f**); and geometric ideas (**g**). Whenever children made specific reference to a particular property, the connection was coded as **p**.

The table indicates that all children made some connections during the duration of the study. Two of the eleven children made connections in the preassessment, one to the property of zero and the other to the arithmetic operation of addition. In the final assessment, all but one made connections to a variety of representations. An analysis of two children's mathematical behavior, Ed (C1) and Joe (C7), follows the

Table to illustrate the nature of the connections made among the various tasks and to other mathematical ideas.

**TABLE 1: CHILDREN'S CONNECTIONS AMONG REPRESENTATIONS**

| | Problem Task | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **CHILD** | **WPA** | **T1** | **T2** | **T3** | **FWA1** | **FWA2** | **FWA3** | **Connections** |
| C1 | Y | Y | Y | Y | Y | Y | Y | p,a,n,f,o,g,c |
| C2 | N | Y | Y | U | Y | Y | Y | a,f,c,p,o |
| C3 | N | U | U | U | Y | Y | Y | a,f,c,o,p |
| C4 | N | U | Y | Y | N | Y | Y | a,g,o |
| C5 | N | Y | Y | Y | Y | Y | N | g,a,o |
| C6 | N | Y | Y | Y | Y | N | Y | a,g,c |
| C7 | N | Y | Y | Y | Y | Y | Y | p,o,c |
| C8 | N | Y | U | U | Y | Y | Y | o,p,n,c |
| C9 | Y | Y | U | Y | Y | Y | Y | o,p,c,a |
| C10 | N | Y | Y | Y | N | N | Y | n.p,c,a |
| C11 | N | Y | U | Y | Y | Y | Y | p,c,a |

Note: Y = presence, N = absence, U = uncertain of connections.
p = properties (i.e. commutativity, identity), a = assessment problem tasks, c= concrete tasks, o = arithmetic operations, n = numbers, f = fractions, g = geometric ideas

**Case 1:** Ed (C1) worked on the classroom cooperative group tasks with two girls, Trish (C2) and Natasha (C3), all children from the independent school. He was one of the two children who indicated in WPA that the problem reminded him of another mathematical idea. His written response stated that the problem reminded him of **problems about zero because zero and any other number is the same result.** In his written work for T1, he wrote: **the problem task reminded him of WPA,** commenting aloud that the **commands were like numbers.** He also wrote that the **problem was like fractions -- like knowing how to cancel out the common factor.** While working with his group to figure out inverse elements for the set, he announced to the others: **It's like fractions; like 1/2 times 2/1; they cancel.**

His reference to numbers was continued in T2. Ed wrote that this problem reminded him of **addition, subtraction, and multiplication with charts.** He then referred again to WPA: **I did a chart like this with numbers that followed sort of the same pattern.** In discussing this problem with the group, Ed stated: **It's the commutative property; like 3 + 1 = 1 + 3. It's like adding.** While solving T3, Ed pointed out to his group: **This problem is like the other ones; this card is the special card** (identity element) **like Nobody Turns** (the identity element in T1). Ed wrote in FWA1 that the task reminded him of **all of the problems that we have done because all have a procedure to get the result and the results all follow a pattern.** In FWA2 he wrote that the problem reminded him of **problems about 1 because** (in the task) **1 and any of the numbers turns out to be 1 and in multiplication 1 times any number equals 1.** Finally in FWA3, C1 wrote: **We did another problem almost exactly like this a few days ago.**

Case 2: Joe (C7) worked with Dave (C8), both students in the heterogeneous public school group. Joe wrote in WPA that nothing about the problem reminded him of any other mathematical ideas or problems. In T1, however, he wrote: **It reminds me of addition and multiplication,** and in discussion with his partner he stated: **This is like the commutative property. OGT and OBT or OBT and OGT** (two elements of the set). **Either way, they equal BT** (a third element). **It's like** ***please, my dear Aunt Sally;*** **it's like addition and subtraction; no, I don't think subtraction works -- only addition and multiplication.** In T2 Joe wrote that the problem reminded him of **the commutative property and the property of 1.** He had pointed out to Dave earlier: **It's** (referring to Card D, the identity) **like 1 and E** (another element of the set) **is like 4. And D on E is E just like 1 times 4 gives you 4 because if you put D on anything nothing happens. It's just like Nobody Turns** (the identity element in T1). In his response for T3, Joe wrote: **It reminds me of the other two problems and problems about numbers because the cards are like numbers.** In discussing with Dave he said: **Card A** (the identity element) **is like 1 for multiplication. The property of 1 - You know! - 1 times anything is the other number.** In FWA1 Joe wrote: **It reminds me of the problems that we just did.** Then in FWA2, Joe continued: **The problems about the commutative property.** Finally, in FWA3 Joe concluded: **They were about Roads and stuff.**

## Conclusions

The analysis indicated that each of the eleven children made at least one connection in at least three of the 6 tasks (not including WPA) with the mode being five. All children made at least one connection between tasks. Also, all children made references to at least two different kinds of other mathematical ideas. Both Ed and Joe made comparisons between the concrete tasks and operations with numbers. Both indicated recognition of commutativity, comparing the concrete elements with number representations. Both recognized the property of the identity in the concrete tasks and each referred to the identity with numbers and in the other concrete tasks. Both boys indicated in the final written assessments that the numeric problems reminded them of the concrete tasks that they had done because of properties such as commutativity and identity. The detailed analysis of the mathematical thinking of Ed and Joe reported here is representative of the cases developed for the other children.

A limitation in the study is that data were often obtained from video taped episodes in which some children were more verbal than others. A design that includes follow-up interviews could provide insight into the nature of the uncertainty category as well as an opportunity to probe for meanings that are unclear or inconsistent in written statements. For example, Ed's consistent recognition of the identity among the seven tasks would lead one to expect that what he meant to have written in his final assessment was that in multiplication, one times any number equals that number rather than what he actually wrote (1 times any number equals 1).

The detailed description of the representations articulated by the children and the connections among them supports the statistical analysis of the larger study in which the experimental group scored significantly better than the control. It is important to understand how children build-up their mathematical ideas so that appropriate task activities can be provided in classrooms to facilitate learning.

Note 1. For a detailed analysis of the children's problem-solving behavior in the group activities as well as a description of the nonnumerical tasks (See Alston & Maher, 1988; Alston, 1989).

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# ALGEBRAIC SYNTAX ERRORS: A STUDY WITH SECONDARY SCHOOL CHILDREN

Alfonso Avila(*), Francisco Garcia(*) and Teresa Rojano(**)

(*) Unidad de Apoyo Didáctico-Secretaría de Educación, Cultura y Bienestar Social del Estado de México.

(**) Centro de Investigación y de Estudios Avanzados - Programa Nacional de Formación y Actualización de Profesores de Matemáticas, MEXICO.

**ABSTRACT.** - *This paper reports on a study carried out whith 221 Secondary School children of the State of Mexico. Paper and pencil tests were administered in order to detect the presence and frequency of albebraic syntax errors previously reported in other studies (Booth [ 1 ], Matz [ 8 ], Kieran [ 5,6 ], Kückemann [ 7 ], Collis [ 2 ], Trujillo [ 9 ], Filloy/Rojano [ 3 ]). Some of these results were confirmed for the mexican data, particularly those concerning interpretation and manipulation of algebraic symbols, symbolization of generalizations, equation solving and word problem solving. Other kinds of errors appeared which can be interpreted as teaching effects.*

Introduccion. - De acuerdo a las investigaciones de L. Booth [ 1 ] y D. Kückemann [ 7 ], los niños entre 11 y 16 años de edad pueden tener distintos niveles de interpretación de los símbolos literales, cuando éstos aparecen en expresiones algebraicas (por ejemplo, la letra como objeto, como incógnita específica, como número generalizado o como variable). Estas interpretaciones con frecuencia conducen a tipos específicos de errores en el desempeño de tareas algebraicas. Por otro lado, el trabajo de M. Matz [ 8 ] proporciona elementos teóricos que sugieren la presencia de procesos mentales tales como la extropolación y la generalización, capaces de generar en el álgebra, tanto las respuestas correctas como las incorrectas. Tal es el caso de la bien conocida tendencia a aplicar linealmente todo tipo de operadores (por ejemplo, en $(a + b)^2$ se obtiene $a^2 + b^2$), al carecer de criterios de discriminación entre un dominio de extrapolación válido y uno

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de los items del examen de álgebra del estudio Strategies and Errors in Secondary Mathematics (Booth [ 1 ]), para cubrir la sección correspondiente a aritmética generalizada. También se incluyeron series de items para abarcar los temas de simplificación de expresiones algebraicas, resolución de ecuaciones y resolución de problemas verbales, para lo cual se adoptaron preguntas de los trabajos de Kieran [5], Matz [8] Trujillo [ 9 ] y Filloy/Rojano [ 3 ] y se elaboraron items exprofeso para la parte de sistemas de ecuaciones lineales. Se completaron 23 preguntas (42 items) para la versión definitiva del cuestionario, algunos de los cuales se incluyen a continuación , a fin de ilustrar los temas considerados.

1. ¿Qué significa $mn$? Subraya todas las respuestas que creas son correctas:

a) $m$ y $n$

b) $m \times n$

c) $m + n$

d) $25 + 26$

e) $25 \times 26$

f) Si tienes otra respuesta, por favor escríbela

______________________________________________

2. ¿Cómo escribirías 3 aumentado a $5y$ ?

______________________________________________

5. Reduce, cuando sea posible, las siguientes expresiones

a) $a + a + 3b + 5a =$ ______________________

b) $4 + 3y =$ ______________________

c) $2a + 5b + 3a =$ ______________________

d) $5y - 2t =$ ______________________

e) $(a - b) + b =$ ______________________

9. ¿Se puede obtener de la ecuación 1 la ecuación 2 ?

Escribe Sí o No

1) $2x - 6 = 4$

2) $2x - 6 + 6 = 4 + 6$ ______________________

1) $3a + 5 + 4a = 19$

2) $12a = 19$ ______________________

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no-válido.

En otro orden de ideas, estudios teóricos y empíricos han revelado que el tránsito de la aritmética al álgebra, requiere que cambios profundos en el nivel conceptual tengan lugar (Freudenthal [4], Filloy/Rojano [3], Kieran [6] ) Ya que, de no lograrse tales cambios, el anclaje en la manera aritmética de pensar genera cierto tipo de operaciones aberrantes en álgebra (como por ejemplo, la operación defectuosa de la incógnita, en $16x = 3x + 5 \rightarrow (3 + 5) = 8) \rightarrow 16 \div 8 = 2 \rightarrow x = 2$; o la redistribución del error, al afirmar que $x + 37 = 150$ tiene la misma solución que $x + 37 - 10 = 150 + 10$).

Uno de los propósitos del trabajo que aquí se expone es el de verificar, hasta qué punto, los errores algebraicos más frecuentes, reportados en los estudios mencionados anteriormente, están presentes en la población estudiantil de las escuelas secundarias de una parte del sistema educativo mexicano y confrontar los resultados obtenidos en este contexto y nivel escolar con los resultados de otras investigaciones de la misma naturaleza. Otro de los propósitos es llevar a cabo un análisis de los tipos de error detectados, en términos de a) la interpretación de los símbolos y operaciones algebraicas, por parte de los estudiantes; b) los procesos mentales que pueden generar las respuestas erróneas; c) los efectos que la enseñanza puede llegar a tener en la generación y/o rectificación de los errores típicos.

Ya que los trabajos señalados con anterioridad se complementan unos a otros, en cuanto a dar explicaciones plausibles de la presencia y uniformidad de los errores algebraicos, algunos aspectos de dichas investigaciones se tomaron en cuenta para conformar un marco teórico para el análisis de los datos recabados en el estudio aquí reseñado.

## METODOLOGIA Y PASOS DE LA INVESTIGACION

### Elaboración y Aplicación del Cuestionario

Para la elaboración del cuestionario, se adoptaron algunos

14. ¿Cuándo es verdadera la siguiente expresión?

$$L + M + N = P + N$$

Siempre   Nunca   Algunas veces

18. Resuelva las siguientes ecuaciones

a) $3x - 4 = 8$

b) $6x-3 = 2x + 1$

---

20. Plantea la ecuación que conduce a la solución del siguiente problema; no se requiere que des la solución.

a) El doble de u n número disminuído en *12* es igual a *26* ¿Cuál es ese número?

22. Simplifica, cuando sea posible, las siguientes expresiones:

$\dfrac{2x}{x} =$   $\dfrac{2x + 3y}{x + y} =$

Para una administración preliminar del cuestionario a una población pequeña de estudiantes, se hicieron ajustes de algunos items en relación al programa de estudios vigente y al lenguaje utilizado en los libros de texto usuales en la región. Una versión definitiva del examen fue aplicada a 221 niños del segundo grado de la enseñanza secundaria, provenientes de cuatro escuelas de zonas urbanas y sub-urbanas en el Estado de México. Se verificó que, en ese momento, los niños ya hubieran estudiado los temas del cuestionario, incluyendo la resolución de sistemas simples de ecuaciones lineales.

**Análisis de los resultados.**

Se llevaron a cabo dos tipos de análisis de los resultados, uno, cuantitativo, el cual permitió clasificar los items del cuestionario en cuatro niveles de dificultad: en el primer nivel se incluyen los items de menor dificultad, con un porcentaje de error entre 19` y 44%; en el segundo nivel, los items con porcentaje de error entre 48 y 64%, en el tercero, los de porcentaje de error entre 67 y 84%; y el cuarto nivel, el de mayor dificultad, los de porcentaje de error entre 86 y 100 %..

Los items que resultaron "más fáciles" para esta población corresponden a la resolución de enunciados verbales tipo abbaco ("encuentra un número tal que su doble sea...") que corresponden a ecuaciones lineales simples con una ocurrencia de la incógnita; resolución de ecuaciones de "un solo paso" ($6x = 4 \rightarrow x = 4/6$) y simbolización de operaciones con números y letras (4 sumado a 3n). En las franjas de "dificultad media", se encuentran items de substitución numérica de la "variable" en expresiones simples; reducción de expresiones, agrupando términos semejantes; verificación de equivalencia de ecuaciones; equivalencia de expresiones con notación literal; resolución de ecuaciones entre un número y un binomio, traducción a ecuaciones de enunciados verbales simples; interpretación de expresiones como 5n; expresión de perímetros de polígonos con uso de letras. Finalmente, la franja de mayor dificultad, la conforman 19 de los 42 items considerados para el análisis cuantitativo e incluye tareas de reducción de expresiones como $a + a + 3b + 5a$ y $4 + 3y$; simbolización de "perímetros generalizados"; simplificación de expresiones racionales como $\frac{2x}{x}$ ó $\frac{2x + 3y}{x + y}$

ó $\frac{x}{2x - x^2}$ ;

resolución de sistemas de ecuaciones como $\begin{array}{l} 2x + y = 1 \\ x - 2y = 8 \end{array}$

ó $\begin{array}{l} x = 1 \\ y = x + 3 \end{array}$ ; operaciones entre binomios.

Además del cuantitativo, se llevó a cabo un análisis cualitativo de los tipos de error cometidos en cada item.

Del análisis de las respuestas erróneas más frecuentes encontradas en este estudio se desprenden, básicamente, dos hechos:

a) La aparición reiterada, en la mayoría de los items, de respuestas que manifiestan dificultades reportadas en los estudios de Booth, Matz, Kieran, Trujillo, Filloy/Rojano. A continuación se muestran algunas de tales dificultades, exhibiendo ejemplos, para cada una de ellas.

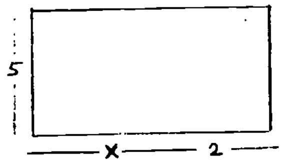
- Concatenación de símbolos: $mn = m + n$

$$5n = 5 + n$$

$$J + P = JP$$

- Interpretación de las letras:

+ Letra no usada




$A = 5 \times 2$

+ Asignación de valores a las letras según el alfabeto:

$$mn = 25 \times 26$$

+ Asignación de valores específicos diferentes a letras diferentes:

Nunca $L + M + N$ es igual a $L + P + N$

- Aplicación de la regla "sumar números y anotar las letras":

$$3 + 5y = 8y$$

$$4 + 3y = 7y$$

$$2 + 5a = 7a$$

$$4 + 3n = 7n$$

- Ausencia de significado para los paréntesis:

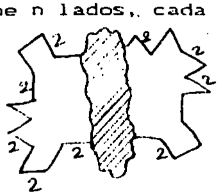
$$5(2a + b) = 10a + b$$

- Ambigüedad notacional:

$4a = 43$, si $a = 3$

- Incapacidad de generalización: -

Este polígono tiene n lados, cada lado mide 2.




$P = 20$

$P = 2^{10}$

- Empleo de métodos primitivos en la solución de ecuaciones y problemas, como ensayo y error y hechos numéricos.

- Traducción algebraica deficiente de enunciados:

  + Expresan 3 aumentando a 5y como $3^5$, $3^{5y}$, $3^5y$
  + Expresan m + 5 multiplicado por 3 como m + 5 x 3
  + Expresan el "triple" de un número disminuído en 18 es igual a ese mismo número como 3x − 18 = 18.

b) La manifestación repetida del empleo exagerado de expresiones en forma de potencias:

$$a + a + 3b + 5a = 5a^3 + 3b$$
$$2a + 5b + 3a = 5a^{2+}5b$$
$$h + h + h + h + t = h^{4+t}$$
$$x + x + 5 + 5 + 6 = x^2 + 5^2 + 6$$
$$2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 2^{10}$$

lo cual puede ser atribuido a la enseñanza reciente de la notación exponencial.

## CONCLUSIONES

En relación a los propósitos del estudio, puede decirse que la confirmación de la presencia de dificultades ya reportadas en la literatura de investigación, nos remite a las explicaciones teóricas de distintos autores, tales como la existencia de niveles de interpretación simbólica en álgebra; el anclaje en la aritmética cuando se abordan tareas de resolución de problemas y ecuaciones; la presencia de procesos mentales de extrapolación, que producen entre otras cosas, la hipergeneralización de la linealidad entre operadores; y la necesidad de una semántica de la producción de simbolos compuestos (letras y números) en las tareas de traducción al álgebra de problemas verbales. Explicaciones que, en una primera aproximación a los datos recolectados, pueden ser aceptadas como plausibles. Esto, por otro lado, no cancela la posibilidad de plantearse el atacar estas dificultades, en el nivel de la enseñanza, ya que la población estudiada la conforman niños que se inician en el estudio del álgebra y para quienes puede tener sentido crear acercamientos de enseñanza que contemplen los aportes que las investigaciones recientes nos han proporcionado.

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# THE DEVELOPMENT OF CONCEPTUAL STRUCTURE AS A PROBLEM SOLVING ACTIVITY

Victor Cifarelli
University of California at San Diego

This study examines the development of conceptual structures in problem solving situations. Nine college freshmen were interviewed as they solved a set of similar algebra word problems. All interviews were videotaped and written transcripts of the solvers' verbal responses were prepared. Analysis of the solvers' solution activity yielded four increasingly abstract levels of structural knowledge.

## INTRODUCTION

The notion of conceptual structure underlies many goals for instruction in mathematical problem solving. Mathematics educators who have as their goal the development of "intellectual autonomy" (Kamii, 1985) in the problem solving actions of their students view conceptual structures in terms of their interpretive qualities, as a means by which solvers can organize their problem solving experiences "with a view to making predictions about experiences to come" (von Glasersfeld, 1987) (e.g., making conjectures about one's potential solution activity in new situations).

Despite universal agreement about the importance of solvers developing such structural knowledge, current work in situated cognition suggests the need to reexamine the traditional view of conceptual structures as "decontextualized formal concepts" which are transferred across learning situations (Brown, Collins, and Duguid, 1989). The idea that learning and cognition are situated suggests that learners build up their conceptual knowledge in the context of ongoing activity. As a result, concepts continually evolve with each occasion of use, "because new situations, negotiations, and activities inevitably recast it in a new, more densely textured form". According to Lave (1988), a solver's articulation of structure in a problem solving situation generates learning oppportunities in which exploration of "the plausibility of both procedure and resolution in relation to previously recognized resolution shapes" can lead to a restructure of one's prior solution activity. This paper will argue that solvers construct such conceptual organizations while performing mathematical problem solving activity and that further development of the structure is a process of reconstruction resulting from subsequent problem solving activity.

## OBJECTIVES

The purpose of the study was to acquire an understanding of the processes of constructing conceptual knowledge during mathematical problem solving. The study focused on the internal activity of the learner with particular emphasis on the ways that learners elaborate, reorganize, and reconceptualize their solution activity while engaged in mathematical problem solving.

Solvers face problematic situations in their mathematical activity when they can't see any way to achieve their goals (Pask, 1985). When problem solving is related to one's goals as such, a variety of situations qualify as genuine problem solving situations. For example, solvers might face a problematic situation when they attempt to make sense of or understand statements that describe a specific algebra word problem. Alternatively, the solver's problem might be to understand why a particular solution method led to unanticipated success or why two different solution methods led to the same result. These situations arise in the course of goal directed activity and can serve as learning opportunities for solvers (Pask, 1985; Cobb, Yackel, & Wood, 1989). Successful resolution of such situations can be viewed as the construction of conceptual understanding in the context of ongoing activity (Vergnaud, 1984; Lave, 1988) with the result being that the solvers build structure for their current solution activity (or restructure their prior solution activity). Hence, the goals of the study were to provide clarification for these ideas by observing solvers as they experience and resolve a range of problem solving situations and to characterize their subsequent growth in structural knowledge.

## METHODOLOGY AND DATA SOURCE

Subjects came from calculus classes at the University of California at San Diego. Nine subjects participated in the study. Subjects were interviewed as they solved a set of similar algebra word problems (see Table 1). The interviews were videotaped for subsequent analysis. In addition to the video protocols, transcripts of the subjects' verbal responses as well as their paper-and-pencil activity were used in the analysis.

Table 1: SET OF LEARNING TASKS

TASK 1: Solve the Two Lakes Problem

The surface of Clear Lake is 35 feet above the surface of Blue Lake. Clear Lake is twice as deep as Blue Lake. The bottom of Clear Lake is 12 feet above the bottom of Blue Lake. How deep are the two lakes?

TASK 2: Solve a Similar Problem Which Contains Superfluous Information

The northern edge of the city of Brownsburg is 200 miles north of the northern edge of Greenville. The distance between the southern edges is 218 miles. Greenville is three times as long, north to south as Brownsburg. A line drawn due north through the city center of Greenville falls 10 miles east of the city center of Brownsburg. How many miles in length is each city, north to south?

TASK 3: Solve a Similar Problem Which Contains Insufficient Information

An oil storage drum is mounted on a stand. A water storage drum is mounted on a stand that is 8 feet taller than the oil drum stand. The water level is 15 feet above the oil level. What is the depth of the oil in the drum? Of the water?

TASK 4: Solve a Similar Problem In Which the Question is Omitted

An office building and an adjacent hotel each have a mirrored glass facade on the upper portions. The hotel is 50 feet shorter than the office building. The bottom of the glass facade on the hotel extends 15 feet below the bottom of the facade on the office building. The height of the facade on the office building is twice that on the hotel.

TASK 5: Solve a Similar Problem Which Contains Inconsistent Information

A mountain climber wishes to know the heights of Mt. Washburn and Mt. McCoy. The information he has is that the top of Mt. Washburn is 2000 feet above the top of Mt. McCoy, and that the base of Mt. Washburn is 180 feet below the base of Mt. McCoy. Mt. McCoy is twice as high as Mt. Washburn. What is the height of each mountain?

TASK 6: Solve a Similar Problem Which Contains the Same Implicit Information

A freight train and a passenger train are stopped on adjacent tracks. The engine of the freight is 100 yards ahead of the engine of the passenger train. The end of the caboose of the freight train is 30 yards ahead of the end of the caboose of the passenger train. The freight train is twice as long as the passenger train. How long are the trains?

TASK 7: Solve a Similar Problem that is a Generalization

In constructing a tower of fixed height a contractor determines that he can use a 35 foot high base, 7 steel tower segments and no aerial platform. Alternatively, he can construct the tower by using no base, 9 steel tower segments and a 15 foot high aerial platform. What is the height of the tower he will construct?

TASK 8: Solve a Similar Simpler Problem

Green Lake and Fish Lake have surfaces at the same level. Green Lake is 3 times as deep as Fish Lake. The bottom of Green Lake is 40 feet below the bottom of Fish Lake. How deep are the two lakes?

TASK 9: Make Up a Problem Which has a Similar Solution Method

The nonstandard format of the tasks provided opportunities to observe solvers as they faced problematic situations. For example, even though solvers might construct a solution to Task 1, they could conceivably face problems while solving later tasks despite recognizing that similar solution methods are involved (e.g., solvers could face a problematic situation while solving Task 3 if they try to do exactly the same thing as they did in solving the earlier tasks). Hence, such situations provide opportunities for solvers to develop greater understanding about their solution activity. In addition, the similarity among the tasks allowed opportunities to observe how the solvers' newly constructed conceptual knowledge influenced subsequent solution activity in similar situations (i.e., development of control of solution activity).

Using the written and video protocols, the analysis proceeded from detailed observation of the ways the solvers resolved situations they found to be genuinely problematic while solving the tasks. The solvers were inferred to have experienced such situations when their initial anticipations of what to do in solving a particular task proved incorrect when solution activity was carried out and novel activity was required. In this way, the analysis focused on qualitative aspects of the solvers' solution activity (i.e., changes in their anticipations and reflections) which indicated that constructive activity had occurred. Based on the results of the qualitative analysis, detailed written case studies of the solvers' performance were prepared.

## RESULTS

Analysis of the solvers' solution activity indicated a gradual building up of their structural knowledge as they solved the tasks. Procedures constructed while solving the earlier tasks were elaborated upon as solvers solved later tasks. This constructive activity was characterized in terms of distinct levels of solution activity. Four increasingly abstract levels of solution activity were inferred from the solvers' performance. The levels are summarized in Table 2.

Table 2: Levels of Solution Activity

| LEVEL OF ACTIVITY | DEFINING ATTRIBUTES | EXAMPLES |
|---|---|---|
| Structural Abstraction | Solver can "run through" potential solution activity in thought and operate on its results | Solver can draw inferences from results of potential activity without the need to carry out solution activity |
| Re-Presentation | Solver can "run through" prior solution activity in thought | Solver can anticipate potential difficulties prior to carrying out solution activity |
| Recognition | Solver encounters new situation and identifies activity from previous tasks as relevant for solving current task | Solver recognizes diagrammatic analysis activity as appropriate for solving Tasks 2-9 |
| Instrumental | Solver demonstrates fragmented, unreflective solution activity | Solver uses mechanical coding activity as part of a translation strategy |

The following paragraphs include episodes from the case study of solver MB and serve to illustrate examples of the different levels of structural knowledge demonstrated by the solvers.

The solver's performance during the interview can be summarized in the following way. The solver struggled to construct a solution to Task 1. She initially pursued a strategy where she coded all information contained within the problem statements. When this approach did not lead to a solution, she pursued an alternate solution method incorporating a geometric approach (i.e., diagrams of the lakes were constructed and relevant lengths from the diagrams were translated to a vertical axis which served as a reference aid in constructing relationships). This solution activity led to a correct solution and resulted in the construction of an initial recognitionary structure. Solution activity performed while solving Tasks 2-9 enabled the solver to elaborate and refine the initial structure, achieving higher levels of abstraction and control with each successive task. The following paragraphs describe this development.

The solver's initial attempt to solve Task 1 could be described as an unreflective, instrumental approach (i.e., she did not appear to reflect on or think about the nature of potential solution activity prior to carrying it out). She initially interpreted the task as a routine algebra word problem and proceeded to code all information without attempting to develop a deeper understanding of the situation.

> S: That strikes me as an algebra problem with 2 variables. So the first thing I should do is assign variables to everything that is important.

She constructed a diagram and proceeded to generate all possible algebraic relationships. Symbols representing variables were manipulated in a mechanical fashion as the solver tried to code and relate everything in the problem without reflecting to the extent necessary to consider whether such assignments were relevant in finding a correct solution to the problem. This activity resulted in the generation of algebraic equations which she later found to be inappropriate.

> S: I have 4 unknowns and 3 equations. And that's not good enough for me to solve an algebra problem.

The solver realized she faced a genuine problem and proceeded to pursue an alternate method of solution. She abandoned her unreflective approach (where symbols were manipulated mechanically without regard to possible relationships) in favor of a more relational approach (where reflection upon entities signified by the symbols led to the construction of a viable solution method). This reflective approach was indicated by the solver's conscious intention to use the drawing as an interpretive tool that would aid her conceptualization and elaboration of potential relationships.

> S: I am going to look for a geometrical relationship for my drawing which I am going to redraw because this is not accurate.
> S: This is the bottom, this is the surface of Blue Lake and this is the bottom of Blue Lake. This distance is 12 and this distance is 35. And this whole distance is twice that whole distance. (LONG PERIOD OF REFLECTION HERE)
> S: Okay, if I label this whole distance X ... I can say ... that 12 plus X plus 35, which is the height of Clear Lake, is going to equal twice X. And that's the relation in one variable I can solve.
> S: And the relation I was missing here is the fact that I'm looking at differences in height, not absolute height.

This constructive activity culminated with the generation of an appropriate algebraic equation for the problem, albeit an incorrect one (i.e., she made an error in her diagram). This algebraic relationship expressed a viable cohesive solution method rather than isolated relationships that corresponded to fragments of the problem statement. Upon discovery of an error in her diagram, the solver reconceptualized the problem and generated a new algebraic equation which led to a correct solution.

> S: The bottom of Lake, ... and this lake is 12 feet above the bottom of that lake. So I didn't draw it that way. I drew it 12 feet below.
> S: That means that my geometrical solution is probably off.
> S: So, the distance between these two is still 35. The distance between these two is 12.
> S: Yeah, but X doesn't mean the same anymore.
> S: So, 35 plus X equals 24 plus 2X. So 35 minus 24 equals ... X.
> S: So Clear Lake is equal to 35 plus X which is 46. And Blue Lake is equal to 12 plus 11 which is ... 23. That's the solution!

The solver's solution activity for Task 1 involved the construction of novel relationships which expressed an initial conceptual structure. This activity was novel in the sense that it involved meaning making activity in genuinely problematic situations. The result of this novel activity was that the solver structured her solution activity. Given this initial implicit initial structure, solution activity performed while solving Tasks 2-9 gave rise to opportunities for the solver to elaborate and reconceptualize the relationships she constructed while solving Task 1.

To say that the solver constructed a conceptual structure for her solution activity while solving Task 1 is evident from her initial anticipations as she solves Task 2. At this point her structure was primitive in the sense that while she could recognize the appropriateness of using similar solution activity, she could not anticipate a potential problem suggested by the additional information contained within the problem statements.

> S: The first thing that strikes me is that this problem is alot like the previous one.
> S: And ... I think it would serve me well to start off in this one by just drawing a picture.

The gradual discovery of the superfluous information puzzled the solver, suggesting that her initial anticipation was based on a <u>recognition</u> of the relevance of activity similar to that which she had just completed (i.e., at best she could only recognize diagrammatic analysis of the type performed in Task 1 as appropriate to the new situation and could not anticipate potential difficulties). She paused to reflect on the situation.

> I: What are you thinking?
> S: I'm thinking that this line drawn due north doesn't seem to have anything to do with the problem.

While the situation appeared to constitute a minor problem for her, she was not able to state with certainty that the added information was indeed irrelevant. She eventually chose to ignore the information ("So I'll just look at the other relationships first") and constructed a solution.

Solution activity performed in Task 3 indicated that additional constructive activity had occurred and that the solver had reorganized her structure. After reading the problem statements, she proceeded to construct a diagram. The solver initially anticipated that she would use the same procedures that she had used while solving earlier tasks. However, she anticipated a potentially problematic situation soon after constructing a diagram yet prior to carrying out the solution method.

> S: And here's the water level, here's the oil level.
> S: And the water level is 15 feet above the oil level.
> S: So solve it ... (ANTICIPATION) ... the same way. ...(ANTICIPATION) ... Impossible!

The suddeness with which she was able to anticipate potential difficulty suggests that she had attained a level of reflective activity not demonstrated while solving prior tasks (more precisely, she had "run through" the potential solution activity in thought and could "see" the results as being problematic). Further, this reflective activity served as a driving motivation for subsequent solution activity.

S: It strikes me suddenly that there might not be enough information to solve this problem. So I better check that. (LONG PERIOD OF REFLECTION HERE)
S: I suspect I'm going to need to know the heights of one of these things.
S: But I could be wrong so ... I'm going to go over here all the way through.

The solver spent much time and energy pursuing the elusive information. She finally concluded that the problem, as stated, could not be solved.

Tasks 2 and 3 presented opportunities for the solver to reflect on, elaborate, and generalize the procedures she developed while solving Task 1. In each case, the solver gave initial meaning to the task she faced by assimilating the new situation to a conceptual structure that functioned at the level of recognition (i.e., she recognized that the activity she performed in solving Task 1 was relevant for solving Tasks 2 and 3). In resolving problematic situations while solving Task 3, the solver was inferred to have reorganized the structure at a higher level of abstraction (i.e., at the level of Re-Presentation). The solver appeared to further develop her structure as indicated by her solution activity in subsequent tasks. The solver demonstrated this more abstract structure while solving Tasks 4 and 9.

Task 4 required the solver to construct a problem she could solve. In constructing a problem to solve, the solver reflected on potential solution activity in a powerful way which was not evident in earlier tasks.

S: The things they could ask for are things like ... (ANTICIPATION) ... the height of one of the buildings but ... (ANTICIPATION) ... there's not enough information to get that. ... (ANTICIPATION) ...
S: The only thing we have information about is ... (ANTICIPATION) ... Ah, the relative heights of the two facades.
S: So, if I were ... if somebody wanted me to solve any problem, that's probably what they're asking for.

This episode illustrates the solver's developing flexibility and control of her solution activity. This development continues throughout the remainder of the interview. The solver's solution activity in Task 9 indicates that she had reorganized her structure (i.e., at the level of Structural Abstraction) to the extent that she could reflect on her potential solution activity and anticipate its results without the need to carry out the activity. The task required the solver to construct a novel situation which had a similar solution method to the prior tasks.

S: Okay, ... (ANTICIPATION) ... I'm thinking of something with different heights.
S: Oh, ... (ANTICIPATION) ... bookshelves in a bookcase.
S: No, ... (ANTICIPATION) ... that's no good. ... How about hot air balloons!

The solver ran through potential solution activity for the particular situation she proposed (i.e., bookshelves) and anticipated its results (i.e., that it would not work for "bookshelves" but that she could solve it for "hot air balloons"). So, her structure allowed her to run through potential solution activity in thought, produce its results, and draw inferences from the results. She routinely constructed appropriate algebraic relationships and completed the task.

CONCLUSIONS

The study was exploratory and future work needs to focus on the following areas. First, the characterization of conceptual structures as actively constructed by solvers suggests the importance of self generated solution activity. Problematic situations were not given to solvers. Rather, they were self generated in the sense that they arose as solvers tried to achieve their goals and purposes. In addition, the solvers' ability to transform initial conceptual structures into more abstract forms was made possible by the solvers' ability to generate new material to reflect on when they faced such situations. Second, the results of the study suggest a relationship between cognitive and metacognitive activity. The cognitive act of expressing their structure in new situations and the ways that they resolved problematic situations that they faced along the way had a powerful influence on the solvers' subsequent solution activity performed while solving later tasks. More precisely, they were able to anticipate what it was they were to do and the result of doing it before they carried out the activity. In metacognitive terms it can be said that planning and monitoring activity (i.e., anticipations about potential activity) developed as a result of the solvers performing specific cognitive acts (i.e., the expressing of their structural knowledge in new situations and the resolution of problematic situations in which they found themselves). The crucial point here is that their developing ability to monitor and plan their solution activity was made possible by their cognitive advances. This calls into question the notion that metacognitive skills can be treated as a separate level of cognitive functioning (Brown, 1988).

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# FROM ARITHMETIC TO ALGEBRA: NEGOTIATING A JUMP IN THE LEARNING PROCESS

Anibal CORTES, Gérard VERGNAUD, Nelly KAVAFIAN

How can teacher and students negociate the move from arithmetic to algebra during the very first phase of introduction to algebra. The first problems that can be put into equation and solved by algebraic means can also be solved by arithmetic. Therefore some scaffolding and tutoring must be offered to students for them to accept to deal with equations; unknowns, and the transformation of equations.

## INTRODUCTION

The learning of algebra constitutes a significant epistemological jump for secondary school pupils. By this we mean that the pupil has to shift suddenly from one state of mathematical knowlege to another by rapidly assililating new notions and procedures: (unknown, variable, equation, function, graphic representation...) which build on previously acquired knowlege but which require entirely new types of thinking. Passing from elementary arthmetic to algebra, pupils will have to substitute for the iterative treatment of problems stated in natural language, the manipulation of algebraic expressions according to explicit rules ( a procedure which gives rise to a succession of equations).

How to start teaching algebra? with which types of problems? The answer is not immediate. In the course of a previous experiment, we set pupils simple problems which put in the form of equations; led to equations of the type a+x=b, ax=b and ax+b=c. These problems are, in fact, easily solved through arithmetic. Therefore, putting problems into equation form and the algebraic treatment of equations is initially a response to the teacher's request. Pupils learn, for sure, but the introductory process is slow and rests entirely on the pupils acceptance of the didactical contract.

Algebra takes on a much clearer meaning in the solution of problems which are insoluble or difficult to solve trough arithmetic. Problems with two unknowns are generally good examples by may be it would be setting too high a hurdle to start the study of algebra with this type of problem.

Problems with one unknown which in the equation form require an equation where the unknown appears on both sides (of the type ax+b=cx+d) gives rise to serious difficulties for beginners (we shall treat them only in the second didactical sequence). Consequently we have chosen an intermediate approach by starting the study of algebra with a problem with one unknown giving rise to an equation of the type ax+b=c followed quickly by problems with two unknowns. <u>The present paper es concerned only with the very first phase of introduction to algebra. We propose to make a detailed analysis of the observed processes in 7th-grade (28 pupils) and 8th-grade (30 pupils).</u>

**Which are the conceptual difficulties encountered when first working with algebra?**

The concept of the equation: the fist step in solving a problem algebraically is to express it as an equation. This consists in making explicit the mathematical relationship between the unknown and given data in order to find a value for the unknown. This tool-like characteristic of the equation may be visible to the pupil when the equation is put in the form x=... it is not visible when x is incorporated in the analytical expression of the relation ax+b=c.

Most pupils are not familiar with the concept of the equation. For them an equation is an abbreviated way of writing the terms of the problem: a summary. The purpose of the equation largely escapes them. Arithmetic formulations are generally used as a mnemonic device for arithmetic calculations. This cannot easily be applied to algebra since it es necessary to work with an unknown.

The concpt of the unknown: The concept of the unknown is closely related to the concept of the equation. These two concepts are constructed in parallel. One gives meaning to the other and vice-versa. A broad definition of the unknown would be: "what is not known in the terms of the problem". But one tacitly calls "unknown" something that was to be calculated by jumping over the problem of intermediate unknowns. In algebra the unknown is symbolised by a character which represents an unknown number (in the solutions of problems one should rather speak of an unknown magnitude). One can see that characters written by pupils can sometimes symbolise an object or a unit rather than a number or a magnitude.

The meaning of the "=" sign. The "=" sign may have several meanings:

a) It introduces a result. The "=" key of a pocket calculator carries this meaning (it serves the purpose of introducing the result by making it appear). Similarly; in the most common usage of formulas, in V=L.l.h, for example, the "=" sign introduces the way to calulate V. For many pupils the "=" sign exclusively carries this meaning, which can sometimes lead to writing incorrect equalities. For example, in 70-25 = 45+47 = 92-52 = 40, the number following the "=" sign is the result of the algebraic sum expressed on the left.

b) Equivalence. In algebraic equations the "=" sign has the following meaning: what is on the right of the "=" sign is equivalent to what is on the left for an appropriatly selected value of the unknown. This meaning takes shape at the same time as the concept of equation and unknown.

c) Identity: For example, in the transformation of literal expressions.

d) Specification or definition. For example, in f(x)= 2x + 52, the "=" sign introduces the analytical expression of this function.

The homogeneity of the equation: in the expression of a problem (of physics for example) the homogeneity of the written terms of the equation is controlled. Now, it is not at all obvious to secondary school pupils that the terms of an equation must be homogeneous; addition of values of the same kind (same units, same meaning). We chose to ask our pupils to write the units of the data from the very first session of study; this constitutes a first approach to the control of homogeneity. Later these pupils will be confronted with problems in which control of the units is not sufficient: for example, one does not add prices and profits, neither weights and prices.

Numbers, the treatment of numbers: An average 7th-grade pupil is supposed to be capable of operating within the D+ set; he hardly knows fractions and directed numbers. In the 8th grade, pupils are supposed to have become acquainted with fractions and directed numbers. Nevertheless they also have many problems. Now, the processes of putting into equation form and solving equations algebraically call for a thorough grasp of operations with numbers, especially with directed numbers. For example, it might be necessary to multiply or divide by a negative number.

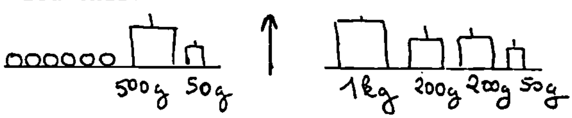
Algebraic calculation - the "detour" behaviour. One of the most important aspects of the jump between arithmetic and algebra is the acceptance of the "detour" behaviour: the pupil must accept not to attempt immediatly to calculate the unknown or intemediate unknowns (to put the problem into equation), accept to forget the meaning of values and relationships represented by algebraic expressions (succession of intermediate equations) accept to rely on operations on written symbols which may not have an arithmetical meaning, and nonetheless trust that the solution thus found is both interpretable and correct: conservation of the equality and the solution throughout the algebraic calculation.

## FIRST SITUATION

Putting into equation the first problem and solving it algebraically require the use of concepts and procedures which are barely understood or entirely misenderstood by our pupils (equation, unknown, succession of equivalent equations, conservation of the equality...). The gap between the problem to be solved and the pupil's knowlege creates a paradox which can only be resolved through the teacher's tutorial activity.

Tutorial activity: here the tutorial activity offered to pupils consists in braking down into stages the process. This way of providing guideposts for the task is designed on the one hand to help define the steps in the process and on the other hand to discourage the search for an arithmetic solution. At each stage the pupil will be faced with a particular difficulty while the observers will have the opportunity of establishing a discussion with him. During the experience each class is distributed into groups of four pupils (five groups in 7th-grade and four en 8th-grade); a larger group is under the responsability of the principal teacher (8 pupils in 7th-grade and 14 pupils in 8th-grade).

The terms of the first problem are the following: On a pair of scales in equilibrium, we have identical marbles and weights labelled thus:



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Item a) Write the equation which you think represents this perfectly balanced pair of scales. The unknown in the problem which we are going to calculate is the mass of a marble.

Most pupils launch themselves into an arithmetic solution. The observer-teacher then ask the pupils in both classes to write the equation. Most pupils then produce (and think of) the equation as a summary of the terms of the problem. To the question "what is the use of an equation?" most pupils reply "it translates a text", "it simplifies like a shorthand"... Nonotheless, a few pupils recognise the equation's properties as a tool: "the unknown can be put on one side and it is possible to calculate it more quickly". The results obtained are the following:

| Algebraic equations | 8th | 7th |
|---|---|---|
| 6x + 500g + 50g = 1Kg + 200g + 200g + 50 g | 7 | 6 |
| 6x + 550g = 1450 g | 9 | 2 |
| 6x + 500 + 50 = 1000 + 200 + 200 + 50 | 2 | |
| 6x + 550 = 1450 | 1 | |
| TOTAL | 19 | 8 |

Neither 8th-grade nor 7th-grade pupils have ever solved problems through algebra. However, 8th-grade pupils are more capable of writing algebraic equations. The degree of elaboration of the written mathematical expression is greater in the 8th-grade as well: 10 pupil write a reduced form of the equation.

We cannot be sure that the written characters (x,y,z,a) have a correct meaning in all cases. Indeed the pupils can use such symbols due to acquired training without for all that having a correct representation of the unknown (the mass of a marble). We have noticed that the pupils who write "x" have a <u>tendency to read</u> their equations as an equivalence between values (equivalence of masses in our case). On the other hand, pupils who use a symbol which is "closer" to the object referred in the terms of the problem ("marbles, "m" or a drawing) seem to interpret their equation as a simple juxtaposition of objects of different kinds. For example:

| "Juxtaposition" of objects | 8th | 7th |
|---|---|---|
| 6m + 500g+50g=1Kg+200g+200g+50g | | 3 |
| 6 marbles+500g+50g=1Kg+200g+200g+50g | 2 | 5 |
| ooo ooo +500g+50g=1Kg+200g+200g+50g | 2 | |

These symbolisations of the unknown can prove operational in an arithmetical treatment (which is performed closer to natural language) but can produce a shift of meaning in an algebraic treatment. For example, the pupil whose algebraic calculation results in: "one marble = 0,15 Kg" wants to signify that "the mass of a marble is equal to 0,15 Kg". This pupil has navigated between the object (marbles) and the property of the object that we wish to calculate (the mass of a marble).

| Equivalence of masses | 8th | 7th |
|---|---|---|
| 6 masses of a marble +500g+50g=1Kg+200g+200g+50g | | 3 |
| 550g (6x) = 1Kg450g | 1 | |
| 6x+500g+50g is equal to 1Kg+200g+200g+50g | | 2 |
| 6y+(M=500g) +(M=50g) = (M=1Kg) + (M=... | 1 | |
| "... 1,450Kg. On the other we have 550g and 6x | | 1 |

The first equation is an equivalence of masses and very probably the others are too; but algebraic notation is missing. The second equation serves as the base for an arithmetical calculation; in the third, the pupils resort to natural language to express the equivalence. The new meaning of the "=" sign is unknown to them. The fourth and fifth expressions are close to natural language.

| Absence of the coefficient of the unknown | 8th | 7th |
|---|---|---|
| x +500g+50g = 1Kg+200g+200g+50g | | 1 |
| x 550g ! 1Kg450g | 2 | |
| x = 500g+50g+1Kg+200g+200g+50g | 1 | 1 |
| o + 550g = 1450g | 1 | |

The "x" in the first equation represents the mass of six marbles: the pupil is therefore using an intermediate unknown. The second expression resembles a "reduced" drawing and serves as the base for an arithmetical calculation. The third line is particularly interesting because the expression resembles a formula: on one side there is the unknown that we wish to calculate; on the other are all the terms of the problem; in the middle is the "=" sign which introduces a result. In order to solve the problem, this pupil has to write a mathematical expression which he is not familiar with; he prefers to write the mathematical expression he knows while neglecting the meaning of the equality.

Four 7th-grade pupils calculate the value of the unknown and then write a numerical equality: 900+550 = 1450. These pupils have been able neither to make use of the adult's tutoring nor to calculate the unknown by algebraic means. This demonstrates the relevance of our thesis: it is necessery to discourqge solutions by arithmetical means. Finally, a 8th-grade pupil writes a false equality because he does'nt state the units of the date: 6x + 500 +50 = 1 + 200 + 200 + 50

In their intuitive approach to the concept of equation as a shortened of the problem, most pupil state units. We have tried to include writing units in the didactical contract: with the aim of intoducing control of the homogeneity of the equation.

Item b) Express all the terms of the equation in Kg.

The observers point out to the pupils that the equation has to be reduced in order that it can be used to calculate the unknown and that the reduction of the equation requires that all its terms be expressed in the same units. We chose to calculate the unknown in Kg in order to emphasize the constraint imposed by homogeneity by means of a conversion of units which requires a certain degree of elaboration. Several pupils make mistakes in converting which we will not mention. The observers also point out that the equation represents an equivalence of masses. The pupils then write the following equations:

| Algebraic equations | 8th | 7th |
|---|---|---|
| 6x+0,5Kg+0,05Kg = 1Kg+0,2Kg+0,2Kg+0,05Kg | 13 | 5 |
| 6x+0,55Kg = 1,450Kg | 8 | 7 |
| 6x+0,5+0,05 = 1+ 0,2+0,2+0,05 | 2 | 2 |
| 6x + 0,55 = 1,450 | | 1 |
| 6m+ 0,5Kg+ 0,05Kg = 1Kg+0,2Kg+0,2Kg+0,05Kg | | 4 |
| TOTAL | 23 | 19 |

Compared to the previous item there is a certain elaboration of the mathematical expression, expecially in the 7th-grade: a larger number of pupils write algebraic equations, in particular reduced equations.

The meaning of the unknown (mass of marble) in discussed within the groups of pupils. We notice that those who were using the "m" symbol do not change it. On the other hand three 7th-grade pupils doo not use the word "marble" anymore; they now write: 6 masses of one marble + 0,5Kg+0,05Kg=1Kg+0,2Kg+... (five 7th-grade pupils at all).

Tutorial activity is more effective in small groups of four pupils than in larger ones which are under the responsability of the principal teacher. The following expression belong to pupils from larger groups.

| | 8th | 7th |
|---|---|---|
| 6 marbles+0,5Kg+0,05Kg=1Kg+0,2Kg+0,2Kg+0,05Kg | 2 | 2 |
| ooo ooo+0,5Kg+0,05Kg = 1Kg+0,2Kg+0,2Kg+0,05Kg | 2 | |
| 6x = 0,9 ; x=0,15 | 2 | |
| 0,5Kg+0,05Kg = 1Kg+0,2Kg+0,2Kg+0,05Kg | 1 | |
| 6y+(500g=0,05Kg)+(50g=0,005Kg) = 1Kg+(200g=0,2Kg... | | 1 |

Some pupils retain symbols which are "close to the object in spite of the teacher's remarks about the meaning of the unknown. Two pupils who had previously solved the problem arithmetically in grams write teir calculations in Kg (third line). One pupil connot manage to write an equivalence with an unknown (he had previously written x=500g+50g+1Kg+200g+200g...). One 7th-grade pupil introduces the conversion of units into his expression; he remains close to natural language.

The equation with units ( which could be called a physical equation since it expresses a relation between magnitudes) raises the problem of the treatment of units in algebraic resolutions. It es necessary to be able to proceed to equations without units.

Item c):Write the equation of the problem expressing each term in Kg without stating units. Let "z" (for example) be the unknown. The unknown number.... stands for....expressed in...

Proposing the letter "z" to denote the unknown gives rise to a debate about the relevance of the symbols used; non-algebraic expressions ("marbles", "mass of a marble" and drawing) are replaced by "z". The effect of the terms in which this item is stated goes beyond what is desirable since many pupils who had symbolised the unknown by a letter also change.

The expression "unknown number" draws the attention of the pupils to the fact that the unknown is a number. The mayority of the pupils complete the blanks in the sentence by: The unknown number <u>z</u> stand for <u>the mass of a marble</u> expressed in <u>Kg</u>.

| Algebraic equation | | 8th | 7th |
|---|---|---|---|
| 6z + 0,5 + 0,05 = 1 + 0,2 + 0,2+ 0,05 | and | | |
| 6z + 0,55 = 1,450 | | 11 | 9 |
| 6z + 0,55 = 1,450 | | 14 | 16 |
| | TOTAL | 25 | 27 |

The passage to an equivalent reduced equation is not obvious; several pupils link the two equations with an"=" sign at the end of the first. There is therefore a shift of meaning: the two distinct equations become a succession of transformations; a kind of algebraic sum. The conservation of the equality (the passage to another equivalent equation) gives rise to serious difficulties:

| | 8th | 7th |
|---|---|---|
| 6x+0,55 = 1+0,2+0,2+0,05 = 6x+0,55 = 1,45 | 1 | |
| 6z+500+50 = 900+500+50 = 1,450 | | 1 |
| 6z+0,5+0,05 = 1+0,2+0,2+0,05 = 0,55+1,45 = 1,85 | 1 | |
| 0,5+0,05 = 1,0,2+0,2+0,05 = 0,55 = 1,45 | 1 | |
| 6x+0,5+0,05 = 6x+0,5+0,05 = 0,55 ; x = 0,55/6 | 1 | |
| 0,9+0,55 = 1,450 | 1 | |

Both equations are written on the same line (first line). A 7th grade pupil (second line) writes the left hand side again replacing the unknown with its value in grams, and comes to a "result" in Kg (the right hand side). One pupil (third line) groups together all the reduced numerical terms after his equation and comes to a number: a shift towards an algebraic sum. One pupil (4th line) is not capable of reducing his equation in the presence of the unknown. One pupil (following line) is also unable to operate on the numbes in the presence of the unknown; he detaches the numerical part and puts forward for "x" the value which solves for the equation 6x = 0,55. One pupil writes a reduced equality without unknown (last line).

Algebraic solution of the equation. Item d): By substracting 0,55 from each side of the equation a new equation is obtained. Which one?

The required notation (6z + 0,55 - 0,55 = 1,45 - 0,55 ; 6z = 0,9) leaves a trace of the algebraic working, it allows the pupils more easily to check their work; <u>it permits the construction of a script-algorithm which is used to provide guidance in the very beginning, when resolution strategies are lacking</u>. This notation does not appear naturally it has to be required; it has to be constructed.

After the definition of the word "side" and "term" the observers justify the algebraic operation (mathematically or by refering to the scales) which the pupils can check by arithmetic. The work on the equation, the passage to an equivalent equation and the required notation raise problems (the observers occasionally have to write it). Some pupils use this notation, others write the resulting equation directly:

| | | 8th | 7th |
|---|---|---|---|
| 6z+0,55-0,55 = 1,45-0,55 | ; 6z = 0,9 | 15 | 8 |
| 6z = 0,9 | | 6 | 9 |
| 6z = 1,45-0,55 | ; 6z = 0,9 | | 1 |

Besides the notation, some pupils have problems writing two distinct equations:

| | | | |
|---|---|---|---|
| 6z+0,55-0,55 = 1,45-0,55 = 0,9 | ; 6z = 0,9 | 3 | - |
| 6z+0,55-0,55 = 1,45-0,55 = | ; 6z = 0,9 | 1 | 4 |
| 6z = 1,45 - 0,55 = 0,9 | | 2 | 1 |

Many pupils propose to treat each side of the equation separately, some of them write it as :

| | | |
|---|---|---|
| 6z+0,55-0,55=6z ; 1,45-0,55 = 0,9 ; 6z = 0,9 | 2 | 4 |

Two pupils substract 0,55 from the coefficient of the unknown:

| | | |
|---|---|---|
| 5,45 x = 0,9 | | 1 |
| 5,45 Z + 0 = 1,395 | 1 | |

This type of error (related to a misunderstanding of the order in which operations must be carried and to the weakness of the concept of the unknown) occurs frequently in the beginning and desappears rapidly afterwards.

item e) Divide each side of the equation by 6. What new equation is obtained?

The proposed notation (6z/6 = 0,9/6 ; z = 0,15) and its justification are quickly accepted because they shed lighjt on the step to the final equation: z = 0,15. The equation 6z = 0,9 is easily solved through arithmetic; pupil may therefore relý on it as a mean of checking their work. Most of the pupils write required notation others do not.

| | 8th | 7th |
|---|---|---|
| 6z/6 = 0,9/6 ; z = 0,15 | 17 | 21 |
| z = 0,9/6 ; z = 0,15 | 2 | |
| z = 0,15 | 11 | 6 |
| 6z/6 = z ; 0,90/6 = 0,15 ; z = 0,15 | | 1 |
| TOTAL | 30 | 28 |

Later, in the course of the experiment, the majority of the pupils adopt this notation.

**CONCLUSION:** We have dealt with the detailed analysis of the first hour of the sequence rather than treating the whole of the 15 hours superficially. This first problem shows the beginning of a conceptual construction which stretches over several years of learning.

It seems important to point out the part played by the organisation of the mathematical concepts in a conceptual field (here, we have treated a small part of the whole). The study of the conceptual field together with the epistemological approach and the data concerning the pupils'difficulties allowed us to analyse the mathematical contents as proposed to the students and viewed by them; it also helped us to understand their behavior and to build didactical sequences while keeping control.

A good collection of papers can be found in "The ideas of algebra, K-12 ; 1988 ; National Council of teachers of mathematics USA, (especially the paper of Carolyn Kieran and the paper of Zalman Usiskyn). We have also used the ideas of Y. Chevallard and E. Filloy.

# CONTINUOUS ANALYSIS OF ONE YEAR OF SCIENCE STUDENTS' WORK, IN LINEAR ALGEBRA, IN FIRST YEAR OF FRENCH UNIVERSITY

DORIER JEAN-LUC

Equipe de Didactique des Mathématiques, GRENOBLE (France).

**Abstract**

*Linear algebra is one of the newest fields students discover in their first year at university. Its abstract nature is often a problem for them. We wanted to know if notions in basic logic are prerequired to succeed in linear algebra, and if yes what kind of previous abilities are needed. We wanted as well to have a better appreciation of what teaching linear algebra consists of and what kind of effects it produces on students. In the following article we describe the methodology employed to analyse results of standard first year science students in a pretest about basic logic and algebra notions, and in all the tests given to them in linear algebra during a year. Then we try to answer the questions raised above with help of this analysis and its statistical results. Finally we will propose a new organisation of teaching linear algebra according to our hypotheses.*

## 1- Introduction

The research presented in this paper is based on the analysis of results of the tests given all the year through, in the field of linear algebra, to students in their first year at a French university.

Our main goals were :

- To better determine what teaching linear algebra consisted of, especially through the analysis of tasks proposed to students, within the questions given in the tests.

- For a standard section of first year science students with a fairly standard teaching of linear algebra, we wanted to determine the methods, procedures and mistakes of students in relation to the tasks proposed to them and relatively to their individual previous abilities in basic logic and algebra notions.

- Being then able to draw a diagnosis on the different effects of this teaching, we may propose some hypotheses for its possible change.

## 2- The methodology and the hypothesis

We analysed copies of eight different tests.

We first took eighty-four copies from a pretest on basic notions in logic and algebra. This had been given to students in their first weeks at university, before any specific teaching in these fields. The evaluation of this test gave us the individual level of acquisition to what we thought may be prerequired for linear algebra.

For the analysis itself, we used a methodology introduced by A. Robert and F. Boschet, in their work on the acquisition of real analysis notions in first year at a science university ( [1] et [2] ).

Among the questions on the test, we sorted out four main types : quantification (QA), implication and equivalence (EQ), numerical algebra (AN) and algebraic structure (AS). For the first two types we distinguished three different levels, in the tasks induced by the questions. The first one is a purely formal setting, but seen from the outside, since it is asked to say whether a proposition expressed in formalised language is true or false (QA1 and EQ1). The second one is formal as well, but the task is this time internal, since it is asked to give the negation of a formalised proposition.Only the QA-type of questions appeared at this level (QA2). The last level corresponds to an interplay between the formal setting and another setting, in the meaning introduced by R. Douady [3]. The questions, this time, consist in translating a proposition from a formalised language into an every-day or a graphic formulation, or vice-versa (QA3 and EQ3).

So we obtained seven different types of questions, which we can associate to seven different bodies or "blocks" of knowledge.

The hypothesis we made and which is induced by Piaget's work, is, in outline, that the acquisition of new knowledge is usually made possible by the destabilisation of old knowledge followed by its reorganisation through complex cognitive mecanisms of destabilisation/reequilibration. The necessary destabilisation is not usually part of the explicit teaching, and the process described above is then of course unconscious. Yet, R. Douady [3] showed that (at least for primary school pupils), didactical situations, in which a notion, meant to be taught, may be seen in at least two different settings, in which the pupils have different levels of ability, is suitable to start this dialectical process in good conditions.

After A. Robert's and F. Boschet's work ([1] and [2]), we think that former knowledge, efficient in different settings, may be a better guarantee for the acquisition of a new notion. More precisely we may raise such questions as : will a student who is very good at formal logic (EQ1, QA1 and QA2) but not very good at dealing with the interaction between formal and natural languages, learn linear algebra less well than a student , who is globally of the same level in logic, but having more homogeneous abilities ? For every prerequired block of knowledge, is there a minimal threshold of acquisition beyond which the probability of success is much higher ?

To be able to answer these questions, we have defined for every block, three different states : full (2), half-full (1), empty (0), according to a mark given to the questions related to it. We have also considered the parameter B, giving the number of empty blocks, which mesure the number of gaps in previous abilities

We then obtained nine different variables (the global mark of the test, the seven blocks, and the number of empty blocks), which evaluate the level of acquisition of basic logic and algebra notions, for each student. A statistical study of the results of the tested population, led us to build a new block Q, with the QAi, to summarise the level of different abilities in questions dealing with quantification.

We finally kept only the following nine variables, which we show to be the most significant ones : the global mark N, the number of empty blocks B and seven variables (being 1 or 0) for the blocks : EQ1 (2) (full), EQ3 (2), EQ3(0) (empty), Q(2), Q(0), AN(2), AS(2). This seemed to be, with minimal loss, the best way to keep information compact enough and suitable to our further purpose.

Nevertheless in addition to the specific methodology developed here, some restrictions about the test itself, are to be considered, to give the real value of this evaluation, which is of course only a partial way of considering the contents as well as the level of acquisition of preriquired notions in basic logic and algebra. Indeed, if the questions about logic, in the test, seem to be suitable, although necessarily incomplete, the ones about algebra appeared to be less satisfaying : numerical questions were a bit too imprecise to give a good evaluation and the ones about structure were too "cultural" to give a real idea of the level of acquisition (for instance : asking someone to give an example of a group is not enough to evaluate his knowledge about groups).

The seven other tests were : four "ordinary" two-weekly tests, the mid-term and the final exam, and a special mid-course true/false-test. Except for the last one all these tests included questions on real analysis subjects.

For each of these tests, we made an a-priori analysis, which includes a explanation of the tasks induced by the questions and the different procedures that could possibly be developed by students. We then gave the statistical results, with marks given to every question and codes to identify special procedures, which we gathered in a table, whose arrays represent the students. We also obtained a global mark for each test. We divided every sample into three categories, according to these marks, we managed to balance the distribution numerically.

We analysed, in this order, thirty-nine papers from the first ordinary test (T1), seventy-four from the mid-term exam (E1), forty-six from T2, fifty-eight from the true/false-test (TF), fifty-eight from T3, fifty from T4 and seventy-three from the final exam (E2). Apart from the mid-term and final exams, none of these tests were compulsory, besides we had to photocopy the papers in the short time while the correctors had them ; those two material reasons explain the difference in numbers of papers analysed for each test. In the end, we got unfortunately only nineteen complete sets of papers of the eight tests.Each paper analysed corresponds to a student whose pretest we have analysed anyway, so that the students analysed at each test form a sample of the main population analysed for the pretest.

For each test, we made a short analysis of the new data obtained for the pretest with the new sample. We compared the mean-value of marks, their standard deviation, the percentages of students having EQ1(2), EQ3(2), EQ3(0), Q(2), Q(O), AN(2), AS(2), $B=0$, $B\leq 1$; $B\leq 2$, with the equivalent data for the whole population. In each case, we noticed only little variations, which always have rather obvious explanations. The samples imposed on us under material circumstances seem then to be representative enough of the whole population, to give a certain validation to our general conclusions.

For every test we finally gave a crossed table, giving for each of the three different groups of students defined by the mark of the test, the mean value and standard deviation from the pretest, and the distribution of the same ten variables as above. We gave a table with percentage on the line and one with percentage on the column, which gave an easily read representation of the correlations between each test and the pretest.

Finally, we analysed more precisely the results of all the tests (including the pretest) for the nineteen students, whose eight papers we had . We made several factorial analyses (Analyse en Composantes Principales) of some of the different characters definied on the sample, although the small number of students did not allow us to make a real statistical analysis. Nevertheless, we got quite a lot of information on every students, which would not have been possible with too many students. More over, we took the results as they appeared in a real teaching situation, with all its complexity. This kind of analysis, for linear algebra had not been made before, as far as we know, in France. So we claim that our work, was a necessary step before carrying out a statistical analysis over many more students. To be able to look at the correlations between the different components of the knowledge in linear algebra over a large statistical population (a few hundred), we have to be more familiar with the contents of the teaching, the different tasks and procedures involved in linear algebra, and we must be able to draw some hypotheses that will help us to build tests according to them. We hope that the kind of analysis, we propose, meets these aims.

### 3 - The results

#### a - Global analysis of the contents of the teaching

In most French universities, first year students in science classes follow a two-hour-a-week course over one semester, which represents more or less a fourth of their annual teaching in mathematics. The course usually starts with the axiomatic definition of a vector space, and finishes with the results about diagonalisation of matrices. This is of course an average estimation. In fact linear algebra having completely disappeared from secondary teaching, even for geometry, a new tendacy consists, in first year at university, of teaching a bit less abstact linear algebra and a bit more linear algebra for geometry.

The abstract part of this teaching is usually feared by students, because of its esoteric nature and by teachers, because of the bare obviousness of most reasonings, which leaves them without arguments faced with their students' incomprehension.

On another hand, a historical study (cf also [4]), has confirmed us in the idea that linear algebra is a simplifying and unifying concept. For this reason, it is usually very difficult, if not impossible (?), to find "the suitable problem" to introduce a notion related to it, as we would like to do according to G. Brousseau's "Théorie des situations" [5] or R. Douady's "Dialectique outil/objet" [3]. There is no problem, except a few, far too complicated for students, for which linear algebra is an absolute necessity. Besides, even if linear notions give a more elaborated or a more general answer to a problem, it is often too subtile for students to realise, because they already have many difficulties in using concepts, which they are not familiar with, to be able to have a critical look at their work.

This nature, quite specific to linear algebra, leads to a dichotomous attitude in teaching, which is reflected in two different kinds of problems.

The problems of the first kind present applications of linear notions to questions about polynomials, functions or series... They include interplay between different settings, and change of point of view. Most of them are both real problems and good illustrations of the simplification and generalisation given by solutions using linear algebra, but only to someone who has first no difficulty in using linear notions and who is secondly quite familiar with the subject involved. For instance most of the problems of interpolation with polynomials have very elegant and generalisable solutions with use of the theory of vector spaces, but one needs to have quite a lot of calculations to do, to see the simplification given. Besides, in those problems one usually needs to obtain a lot of results, before being able to reach the first questions really concerning linear algebra. So if such problems are given to students, one may have to deal with the following two difficulties :

1) The use of linear algebra will be only an effect of the didactical contract, as it is not absolutely necessary to solve the problem and the students cannot appreciate the simplification it provides. Students will follow the process of resolution induced by the questions even if they see a solution not using linear algebra.

2) The first questions necessary to approach the linear questions may need so many abilities in different fields that only a few students will manage to answer the questions dealing with the notions of linear algebra. The evaluation of the result of such problems is then more on these questions than on linear algebra.

In the second kind of problems, linear concepts are used in a formal setting without interplay with any other setting. Those might be either very formal and difficult questions about subtile notions, such as supplementary spaces..., or on the contrary mechanical use of algorithms such as the search of eigenvalues and eigenvectors of a matrix... In the first case they give useful results, although very hard to obtain, in the second case they are only training for calculation and easily evaluated contents for tests!... These problems do not use "real" vector space, but very general ones, mostly $\mathbb{R}^n$.

In our analysis, tests T1, E1 and T2, are of the first kind.

E1 is a typical example. The goal of the problem was to obtain Gregory's formula, which gives a polynomial in terms of the values of the P(n+1) - P(n) (n=0 to deg(P)). There is a very attractive solution, through the study of the operator D : P --> Q s.t. Q(X)=P(X+1)-P(X). Yet it is quite long and difficult, it is then really useful only for theoritical reasons or if you need to calculate quite a lot of polynomials. In fact, most of the students didn't succeed in proving all the steps leading to the formula, mostly through lack of technical ability in algebraic calculations. But when they were asked in the last question, to find the polynomials of degree less than three, whose values in 0, 1, 2 and 3 were given, although Gregory's formula had been given, they used a direct method and solved a system of four linear equations with the four coefficients of the polynomial as unknown!

In T1, the question was to find the polynomials of degree less than four, whose values in 0 and 1, as well as the ones of the derivated polynomial were given. The solution induced by the test, was to first find the polynomials, whose all four known values are 0, and then to deduce the general solution by addition of any solution, for instance the one of the third degree. Of course the first question is obvious, for such polynomials can be divided both by $X^2$ and $(X-1)^2$, but most of the students did not realise that, and again solved a system of four equations with five unknowns! As they have to solve another system to find the solution to the third degree, they ended with more calculations, plus a theoritical proof, than if they had directly solved the system of four linear equations given by the conditions.

In T2, the questions preparing the linear solution were so technical (they used polynomials with two variables), that hardly no students had a chance to answer any question about linear algebra.

It is clear that there is a real difficulty here.We think that such problems should be introduced by explicit metamathematical approach and that the "technical" points they raise in the field of algebraic calculation or logical reasonning should not be under-estimated.

TF, T3, T4 and E2 are of the second kind.T3, T4 and E2 are mostly applications of numerical algorithms about the search for eigenvalues and eigenvectors, diagonalisation or reduction to a triangle form of matrices ... But in T3 and E2, we find also some very theoritical questions. The true-false test is typically about abstract notions, although nearly all of them refer to $\mathbb{R}^3$. It would be too long here to develop all the results to this quite specific test, it shows in outlines that formal questions about basic notions of linear algebra such as linear independance, generating subsets, supplementary etc... bring to light some sharp misunderstandings from students.

Generally one of the most obvious difficulties for students in all tasks about linear algebra is to be able to keep control of what they are doing. This goes from the confusion between variables and parameters in the resolution of linear systems and leads to one of the best illustrations of it : in E2, students were asked to find an orthogonal basis of eigenvectors, after they found three eigenvectors of two different eigenvalues, they proved, in all details that they were independent, without shortening the proof to the independance of the only two of same eigenvalue, and then that they were orthogonal.

**b - The main statistical results**

The factorial analysis on the seven series of marks (all but the pretest's) for the nineteen students reveals two sets of tests : T1, E2 and T4 on one side and T2, TF and T3 on the other side, E2 being just between those two groups.

This is a different distribution to above. These two groups separate the calculating tasks from the more conceptual tasks. Indeed in the first group of tests there were quite a lot of resolutions of linear systems, asked explicitly (T1 and E2) or appearing as the suitable ways to solve questions (determinations of polynomials, eigenvalues or eigenvectors...). T4 consists mainly in the use of algorithms for calculations with matrices. T1 and E2 use also algebraic calculation notions for polynomial or integral. On the other hand T2 and T3 and mostly TF deal with more conceptual problems. The final exam seems to be quite a well-balanced compromise of the two.

This separation is given by the second factorial axis of the analysis, the first one separates the globally successful students from the ones who failed. The distribution of students on the first factorial plan is quite harmoniously spread out, which seems to induce that both numerical and conceptual abilities are useful, but independent, to succeed in linear algebra. For instance, it shows that students can globally succeed in linear algebra, without having a good conceptual basis. For instance they can find the triangle form of a matrix without having a good knowledge of the concept of supplementary subspace, although it is a basic notion for the theory of matrices' reduction. This points out a contradiction in teaching linear algebra. The choice made in most French universities' curricula to teach linear algebra from the definition of a vector space to the diagonalisation of matrices all in one year, induces a restriction in the teaching of basic concepts to the benefits of more easily taught and evaluated notions such as reduction of matrices. This is of couse the effect of the difficulties and the failure encountered in the teaching of abstract notions.

**c - Correlations with the pretest**

The correlations with the pretest are globally quite strong. The main correlation appears with the number of empty blocks. This confirms our hypothesis about the existence of a minimal threshold in the acquisition of previous abilities beyond which chances of success in linear algebra are greater. The AN and AS blocks are not very correlated, and among the blocks related to logic, Q is the most correlated of all. These results seem to show that a certain level of previous abilities in basic logic, mainly abilities in the use of quantification, is required to reach a minimal success in linear algebra.

But some results of the more detailed correlations are a bit surprising.

For instance in the true/false test, there were the two following propositions given for any linear map $f : \mathbb{R}^3 \dashrightarrow \mathbb{R}^3$ :

- If (U,V) are two independent vectors of $\mathbb{R}^3$, then (f(U), f(V)) are also independent.
- If (f(U),f(V)) are independent, then (U,V) are also independent.

Many students got mixed up in the use of the definition of independent vectors so that they say exactly the contrary of what was true. It first seemed to be a difficulty related to logical notions about quantification and implication. But it appeared that it is only slightly correlated with the results to the pretest. Other similar phenomena may be noticed in our analysis. This leads us to think that logical difficulties specific to linear algebra might exist, and cannot be solved by any former teaching in logic.

The last step in our analysis was to reorganise our data in terms of several tasks as well as procedures in the different fields of linear algebra. We defined 23 variables and made factorial analyses on several groups of them. This gave us answers or enlargement to local hypotheses and the results, which could not be easily summarized, and would take too long to be developed here. We'll try to fill in this gap during the oral presentation.

**4 - Conclusions - Outlook**

Presented in so few pages, this work may seem very disorganised and partial. It compiled quite a large amount of data and had to deal with a field, which was nearly unexplored by didacticians, so if the conclusions it drew are incomplete, it is nearly by necessity.

We have now to answer our last goal. If notions in basic logic seem to be prerequired for linear algebra, it seems that prerequistes extend to more general abilities in different areas of algebraic calculations such as polynomials, integral or differential calculus etc... which are not necessarily part of standard mathematical teaching for first year science students, and may have specific aspects in linear questions. As some logical problems seem to be specific to linear algebra as well, we propose the following reorganisation of the teaching of linear algebra.

The first step would be to teach only basic notions but over a longer period and quite separated in time from the calculations with matrices, which could be only a further part of the teaching, not necessarily in the same year.

This first part should be illustrated through the solving of several problems dealing with varied vector spaces. In those problems, there should be an explicit metamathematical approach, in which the student should have an active participation (like comparing two or more different ways to obtain the same solution with or without linear algebra). It should include as well the explicit teaching of logic and algebraic calculation notions useful to solve the problems.

We think that this could help to reduce the difficulties raised by abstraction as logic would be part of the explicit teaching. Finally it seems to be a more satisfaying approach from the epistemological point of view, as basic concepts would really appear as unifying and simplifying notions used in various fields in which the students will be given sufficient abilities.

The content of this paper is developed in our doctoral thesis, that should be defended and published by the end of the year. This thesis will also include a historical presentation of the emergence of linear algebra basic concepts and some elements for a new teaching approach.

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# AVOIDANCE AND ACKNOWLEDGMENT OF NEGATIVE NUMBERS IN THE CONTEXT OF LINEAR EQUATIONS.

AURORA GALLARDO and TERESA ROJANO
Centro de Investigación y de Estudios
Avanzados del IPN - México

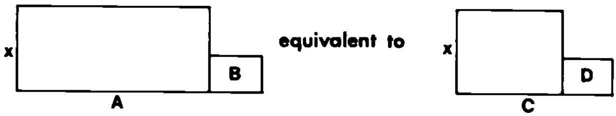
**Abstract.**- This paper reports on the difficulties with negative numbers displayed by 12-13 year olds in a clinical study on linear equation solving. As a methodological counterpart, we describe acknowledgment and avoidance manifestations concerning negative numbers in two cultures: the chinese and the greek one. In the conclusions, a first hypothesis is outlined with regards to possible causes of avoidance and conditions under which acknowledgment arises in individuals.

**Introduction.**- In the last few decades, outstanding efforts have been made in the research field trying to elucidate the problem of misconceptions and operative deficiencies associated to negative numbers. Such efforts have developed in different directions, towards: the teaching field [e.g. Bell, A. 1,2], the psychology [e.g. Resnik, L. et al 13], the history [e.g. Glaesser, G. 9]. In the present work we appeal to history in order to find out explanative elements of the observations made in the clinical study "Operating on the Unknown" concerning negative numbers. Thus, once this history analysis is carried out, the methodological cycle: epistemological level-clinical observation- history level will be completed. This methodology characterizes our research since its very beginning, seeing that the clinical stage was - preceded by a history investigation about the pre-symbolic algebra methods (XIII-XV centuries) for equation solving [3]. This article comprises three sections:
1.- Difficulties with negative numbers in the study "Operating on the Unknown". 2.- Negative numbers in two antique civilizations. 3.- Conclusions.

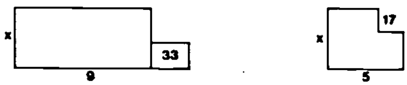
## 1.- Difficulties with negative numbers in the study "Operating on the unknown".

This clinical study was carried out in the period 1981-1986 and analyses transition phenomena from arithmetic to algebraic thought [4,5,6]. It consists of 22 videotaped interviews with 12-13 year olds, who face for the first time simple linear equations with occurrences of the unknown on both sides. A previous classification of children in three pre-algebraic proficiency levels was made (upper, medium and low level). Among the results reported in other papers [6,7,8] we will refer to those related to different difficulty areas in the learning of algebra, in particular, we will focus on the specific area of negative numbers in which, manifestations of avoidance are present, for instance when a negative solution is not conceived in an solving equation process. Such an avoidance appeared in different ways in the interviewees; for example:
1) Interprating the symbol 'X' as a positive number, cancelling in this manner the possibility of solving equations of the type X+1568=392 (children of all levels). 2) Changing the equation's structure: when the written equation (in the item above-mentioned) is read as "I have to find out a positive number such that added

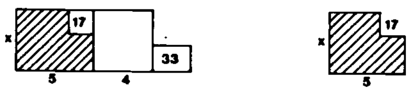
to 1568 sums 392", some children tended to replace the operation sign '+' by another one. 3) Assigning different numerical values to different occurrences of 'X' in the same equation. For example, with the aim to achieve to a numerical identity in 4x+6=2x some children anticipated that the 'X' on the left hand side would be smaller than the 'X' on the right (children of the medium level). Besides the avoidance manifestations, cases of avoidance - acknowledgment of negative numbers were detected, for example: a girl (of the upper level) was taught to solve equations of the type Ax + B = Cx + D (where A, B, C and D were particular natural numbers) by means of translating the equation's elements into a geometric situation, where figures with equivalent area were - involved:




Once the geometric model was understood by the student, she - spontaneously extended it to other mode of equations, including those with negative constant terms (Ax + B = Cx - D). She interpreted the "negative term" as an action of "removing a piece of the figure with an area equivalent to D". This interpretation corresponds to considering those terms as subtrahends.
Thus, the geometric translation of 9x + 33 = 5x - 17 Was:




The pupil carried out the following actions in the model:




which led to the reduced equation (with one occurrence of 'X'): 4x + 33 + 17 = 0. Nevertheless, at this point of the solving process, the student showed an avoidance syntome, she kept quiet for a few minutes because of the presence of a negative solution. This, as it can be seen in this case, although the elements of the equation are provided with geometric meanings and the negative terms as well as its transposition are interpreted as removing, adding and composing actions, this does not result in a total acknowledgment of negative numbers. There exists an essential difference between interpreting a negative number as subtrahend (a - b) and conceiving it as an isolated number (possibly, as a negative solution of an equation).

## 2.- Negative Numbers in two Antique Civilizations

We start with the fiu zhang suanshu (Nine chapter of the mathematical art), one of the earliest mathematical text in China [11]. The various mathematical concepts and techniques embodied in the nine chapters of the text were, in fact, the culmination of knowledge and practical experiences of chinese mathematicians prior to the beginning and the early year of the Christian era.* Let us examine the eighth chapter entitled fang cheng. Just like the other chapters in the text, the present version of the fang cheng chapter contains a number of problems together with their respective solutions. Firstly, we find in the chapter the use of negative numbers, showing that the ancient Chinese had a clear concept of them and were able to apply it in mathematical considerations as we would do nowadays. Secondly, the fang cheng chapter shows the formulations and solution of simultaneous linear equations of up to five unknowns. Thirdly, the fang cheng chapter introduces the methods of solving equations by tabulating the coefficients of the unknowns and the absolute terms in the form of a matrix on the counting board, thereby - facilitating the elimination of the unknowns, one by one.

In explaining the content of the chapter, we shall be using modern notation. However, it must be emphasized that ancient civilization had no ready made sets of notations. Conceptualizations were in a verbalized form, though the Chinese took a forward step when they used rod numerals to convert concepts onto the counting board.

**An overview of the fang cheng chapter.** There are only two methods in this chapter. The first, called fang cheng or calculation by tabulation, is on solving a set of equations. The second method called the positive - negative rules (sheng fu shu), comprises of rules for the subtraction and addition of positive and negative numbers.

**Fang Cheng - The method of calculation by tabulation.** Lui Hui defines the term fang cheng as the arrangement of a series of things in columns for the purpose of mutual verification. The number of columns to be set up is determined by the number of things involved. Each column has two sections; the top section consists of the quantities $a_{ij}$ $(i,j = 1,2,...,n)$ of the various things while the bottom one shows the absolute terms $b_i$ $(i = 1,2,...,n)$. Such an arrangement on the counting board can be shown as follows:

*Neither the author non the date of the composition of this text is known to us. Liu Hui's commentary remained the principal source of the study for the fiu zhang.

| | Left | | Right | |
|---|---|---|---|---|
| Top | $a_{n1}$ | $a_{21}$ | $a_{11}$ | Thing 1 |
| | $a_{n2}$ | $a_{22}$ | $a_{12}$ | Thing 2 |
| | $a_{nn}$ | $a_{2n}$ | $a_{in}$ | Thing n |
| Bottom | $b_n$ | $b_2$ | $b_1$ | Absolute terms |

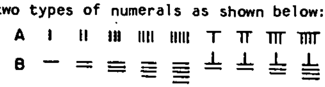
The whole process of operation is done on the counting board - using the rod numerals to represent the various quantities** The unique place - value feature of this method of computation renders the use of symbols unnecessary. In each column of things of the counting board, the space between $a_{1m}$ and $b_m$, has the - implicit function of an equal sign.

The former matrix arrangement is transformed in such a way that all the numbers in the upper side of the main diagonal are equal to zero (only columns are operated on). This transformed matrix corresponds to a diagonalized set of equations, from which all the unknowns are successively determined.

One can see that it is essentially the usual method in present day algebra.

**Zheng fu shu - The positive - negative rules.** Since the process of the fang cheng solution is the succesive elimination of numbers - through mutual subtraction of columns, there could be cases when a number to be subtracted from in one column is smaller than the corresponding one in the other column. The opposite result obtained has to be indicated and certain rules have also to be established in order to continue the eliminating process. This gives rise to the creation of names: the term fu to indicate the resulting opposite amount to the term zheng for the normal - difference. The concept of zheng and fu seems to have evolved from such ideas as "gain" and "loss" as clearly shown in Problem 8 which reads: "By selling 2 cows and 5 goats to buy 13 pigs, there is a surplus of 1000 cash. The money obtained from selling 3 cows and 3 pigs is just enough to buy 9 goats. By selling 6 goats and 8 pigs to buy 5 cows, there is a deficit of 600 cash. What is the price of a cow, a goat and a pig? "The text considers

** There are two types of numerals as shown below:



The A type numerals is for representing units, hundreds, ten thousands, etc., while the B type is for tens, thousands, etc.

the selling price zheng because of the money received and the buying price fu because of the money spent. The surplus amount is considered zheng and the deficit fu. Liu Hui points out that these terms are merely names to indicate the nature of numbers. For the purpose of computation, numbers described by these terms have to be transcribed into a concrete form. He tells us that there are two ways of doing this with rod numerals. If different colored rods are used, then red ones represent zheng and black ones represent fu. Alternatively, if the rods are of one color only, the fu numeral is indicated by an extra rod placed diagonally across its last non-zero digit. He explains: when a number is said to be negative, it does not necessarily mean that there is a deficit. Similary, a positive number does not necessarily imply that there is a gain. Therefore, even though there are red and black numerals in each column, a change in their color resulting from the operations will not jeopardies the calculation. Liu Hui's exposition on negative numbers shows that he conceptualizes them as a class of numbers in the mathematical sense that is familiar to us today. The concept of positive and negative, which initially evolved from opposing entities such as "gain and loss", "add and minus" and "sell and buy", is now detached from linguistic associations. Its development has resulted in negative numbers being regarded as one group of numbers with properties which are connected with the other group of "normal" or positive numbers. These properties are defined by these positive-negative rules [12] which may be represented in modern symbols as follows:

Suppose $A > B > 0$, then for

substraction $\pm A - (\pm B) = \pm (A - B)$,

$\pm A - (\mp B) = \pm (A + B)$,

$0 - (\pm A) = \mp A$

for addition $\pm A + (\pm B) = \pm (A + B)$

$\pm A + (\mp B) = \pm (A - B)$

$0 + (\pm A) = \pm A$

Problem 8 involves selling and buying which equate to the concept of positive and negative respectively. The corresponding set of equations in tabulated form becomes:

| | | |
|---|---|---|
| -5 | 3 | 2 |
| 6 | -9 | 5 |
| 8 | 3 | -13 |
| -600 | 0 | 1000 |

As it can be seen, the fin zhang has provided substantial evidence that, by the first century, the Chinese not only accepted the validity of negative numbers but understood their relationships with positive ones and were able to formulate rules and to compute with them. Outside China, the recognitions of negative numbers as a separate class of numbers came much later. The first mention of these numbers in an occidental work is in the Arithmetica of Diophantus [10], where the equation $4x + 20 = 4$ is spoken of as absurd, since it would give $x = -4$. On the other hand the greeks knew the geometric equivalent of $(a - b)^2$ and of $(a + b)(b - a)$; and hence, without recognizing negative numbers, they, knew the results of the operations $(-b)\cdot(-b)$ and $(+b)\cdot(-b)$. In fact, we could assert that with the greek culture, a history of avoidance of negative numbers is initiated and, it was not until the 15$^{th}$ century that these numbers were gradually accepted in their own right.

**Conclusions.**- The eight chapter of the fiu zhang gives the earliest general method of solving a system of linear equations. By tabulating numbers in an array, the Chinese invented a notation and raised this branch of algebra from a rhetoric form to a notational one. When the fan cheng method was applied to the various problems, it was inevitable that this led to the concept of a class of numbers different from the class of numbers that was known. Thus, the negative numbers emerged from this computational language, freed from the concrete meanings that they used to have in the context of specific word problems. On the other hand, in the greek culture, the numeric domain of the solutions of an algebraic equation was restricted to positive numbers (probably due to their geometric interpretation of the elements of the equation). This led to avoidance manifestations. Nevertheless it can be said that the greeks had a partial acknowledgment of these numbers, since the geometric language admitted the subtraction of areas. The different conceptions extracted from these two cultures provide elements to build up hypothesis worthy of being proved at the level of individuals, concerning the possible causes of avoidance and the conditions which may further a full acknowledgment of integer number. Considering the clinical evidence as well as the history findings, up to the moment, we can conclude that the kind of language conferred to the elements of the equation determines the acceptance of a negative solution. Further studies at both, the clinical and the history levels, will provide new elements of analysis to the problem initially stated.

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# INTRODUCING ALGEBRA: A FUNCTIONAL APPROACH IN A COMPUTER ENVIRONMENT[1]

Maurice Garançon, Carolyn Kieran, and André Boileau
Département de mathématiques et d'informatique
Université du Québec à Montreal

*In the first phase of a long-term project, we have been studying a functional approach to introducing algebra in a computer environment. The 13- and 14-year-old students have been learning to represent algebraic word problems in the form of computable algorithms, which serve as intermediate representations in the process of developing standard algebraic representations, and which also permit guess-and-test numerical strategies. In a study of trial-and-error numerical strategies in a computer environment, we found that the students: 1) do not refer to the context of the problem to help them in their numerical search, 2) operate on the implicit hypothesis that the function is increasing, and 3) rely on partial pattern-matching of the digits. A second study that investigated the algorithms used by students to represent algebra problems showed that a functional approach, based on separating the situation from the question, was extremely accessible to all students; it also helped to avoid some of the difficulties that are traditionally exerienced by students when translating problems of the type* $ax \pm b = cx \pm d$ *into equations.*

## Theoretical Framework

As the first phase of a long-term project, we have been studying for the past few years a functional approach to introducing algebra in a computer environment. At PME-XI we described part of the theoretical framework supporting this research (Boileau, Kieran, Garançon, 1987); however, since 1987, we have not presented any update at international PME conferences. We have therefore decided to present at this time a summary of our work over the past three years.

The students have been learning to represent word problems as computable algorithms, a form of representation which we believe constitutes an intermediate step in the development of standard algebraic representations. A characteristic of this approach is that the development of the algorithms is based, at first, on the students' operational knowledge of arithmetic and that the resulting sequence of instructions is also operational, in that it can be executed in the computer environment.

To support this approach, we may cite several recent studies which have shown that, for a given concept, operational representations are more accessible to novice students than are structural representations.

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Une version française de cet article est aussi disponible auprès des auteurs.

In a theoretical paper, Sfard (1989) presents an analysis of different mathematical concepts (number, function, and others) to show that abstract notions may be conceptualized in two fundamentally different ways: structurally (as objects) and operationally (as processes). For example, the notion of natural number may be conceptualized as a counting process, or at the other extreme as the cardinality of a class of equivalent sets. Likewise, a function may be seen operationally as a calculation process or structurally as a subset of the Cartesian product of two sets. Sfard also shows that from an historical perspective, operational conceptions generally preceded structural ones, and suggests that there may be a parallel development at the psychological level. In fact, in a study (Sfard, 1987) of sixty 16- to 18-year-old students who had a good knowledge of the concept of function, and in particular its structural definition, she found that the dominant conception remained operational rather than structural. In another part of the same study, 96 students aged 14 to 16 years were asked to translate four word problems into equations, and also to give verbal algorithmic descriptions to calculate the solutions to the same problems. Results showed that the students were much more successful in the verbal description task.

In the same vein, the work of Clement, Lochhead, and Soloway (1980) shows that for certain word problems, students find it easier to arrive at a correct algorithmic representation in a computer language (BASIC in this case) than to formulate an algebraic equation. The authors attribute the difficulties encountered with algebraic equations to the absence of a procedural interpretation of these equations.

Representing word problems by computer programs presents other attractive features. It enables the students to use trial-and-error and successive approximation techniques, both of which are linked to the arithmetic experience of beginning algebra students and which also favor a functional view of algebra. Such techniques have been recommended for the early teaching of algebra (Fey, 1989).

**The Computer Environment**

1. Writing the algorithms

Our approach to introducing algebra uses CARAPACE, a computer environment specially created to meet our research objectives. CARAPACE is a tool to aid in the solving of algebra problems, allowing the writing and computing of algorithms.

To be computable in CARAPACE, an algorithm must be specified in terms of independent variables (input variables) and the functional relations among variables. These functional relations must be ordered such that a variable needed to evaluate another variable must have been previously identified as an input variable or evaluated by a preceding relation. For writing these algorithms, CARAPACE provides a screen divided into three parts. In the first part, entitled "Ask for the following values," the input variables are entered, so that upon execution CARAPACE will ask for trial values of these variables with which it will execute the algorithm. The second part of the screen, "Carry out the following calculations," is for entering the functional relations, ordered from top to bottom, one per line. The third part is called "Display the following results," and indicates the names of the variables which will be displayed at the end of the execution.

It should be noted that the only restrictions on naming a variable are that it not begin with a number, nor contain any arithmetic operating symbols. For example, "The price of the 3rd house" is perfectly admissible. This allows for the creation of algorithms that retain a significant portion of the semantics of the original problem.

The writing of the functional relations is done without using the equal sign, in order to avoid confusion with its different meanings (indicating an equivalence or the result of a calculation). Instead we use the word **gives** (accessible with one touch of a key), to indicate that this is a calculation which gives a result. The functional relation, then, is a calculation involving variables, constants (the givens of the problem), and operators (+, -, x, ÷, exponent, parentheses), followed by the word **gives** and the name of a variable. An example is presented in Figure 1.

<u>The problem</u>

<u>The situation</u>: The town of Verval has twice the number of inhabitants as Beaubourg, and Beaubourg has 87 654 inhabitants fewer than Montclerc.

<u>The question</u>: If the total population of these three towns is 567 890, what is the population of each town?

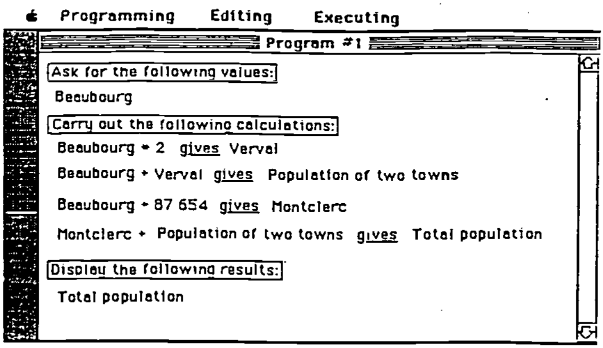


**Figure 1.** A problem (left) and the corresponding algorithm entered by the user

Once the algorithm has been written, the syntax and algorithmic structure may be verified by CARAPACE, and if the algoritm is not "executable", CARAPACE gives a message indicating the nature of the error.

CARAPACE offers six levels of use which impose restrictions on the accepted level of generality of the algebraic expression. The first level allows only one operation per line (in addition to **gives**). The students are therefore required to name (using semantically-laden variable names) the intermediate results of every calculation. Difficulties associated with the order of operations are therefore temporarily delayed. The first level is the only one that we have actually used in our experimentation. Therefore, a description of the other levels will be very brief:

- The second level allows more than one operation per line on the left hand side, as long as the expression is completely parenthesized.
- The third level allows partial use of parentheses. For example, a + b + c is admissible, whereas at the second level, we would have to write a + (b + c) or (a + b) + c. On the other hand, a + b - c is not accepted and would have to be written (a + b) - c or a + (b - c).
- At the fourth level, the traditional order of operations is observed.
- At the fifth level, implicit multiplication before parentheses is accepted.
- Finally, at the sixth level, implicit multiplication is accepted throughout, on condition that one-letter variables are used. This is a necessary restriction in order to distinguish between the product ab and

the variable name ab.

These levels have been enumerated according to an increasing order of generality such that an algorithm which works at one level will also work at any of the higher levels, with the exception of Level six which requires single-letter variables.

2. Executing the algorithms

Once an algorithm has been written in CARAPACE, its execution may be presented in two different modes, one a detailed calculation and the other a table of values. The execution always begins with a request for values for the input variables. The detailed calculation mode simultaneously presents a step-by-step rewriting of the algorithm and an evaluation of the variables and expressions of the functional relations, taking into account the order of operations and parentheses, if necessary. The second mode of execution presents a table of values with the variable names as column headings. As soon as an input variable is entered, a new line is added to the table. No calculations are shown.

At any time in the execution, the user may switch from one mode to the other, and the input and output values (as well as all intermediate values that may have been named in the "Display the following results" zone) of the last 15 executions are always available for review in the table of values. (See Figure 2.)

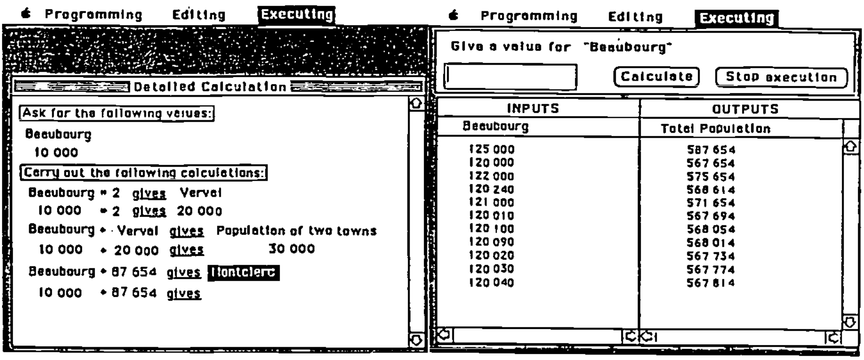


**Figure 2.** The execution of the algorithm of Figure 1. On the left : a partial view of the detailed calculation mode in which the user has entered the value of 10 000 for "Beaubourg". At a signal from the user, the value of "Montclerc" will be calculated, before going to the next line. On the right : the tabular mode, in which a new value for "Beaubourg" is awaited. The goal is to obtain 567 890 for "Total population" (see the problem of Figure 1).

## Pedagogical approach used in this environment

1. Separating the question from the problem

Consider a word problem which may be symbolized as F(X)=Y. On one hand, we have a situation involving X and Y and the relations between them symbolized by F, and on the other hand we have a question. If the question is to find X given Y, we have an algebraic problem which involves (if possible) inverting F. If, however, the question is to find Y given X, we have an arithmetic problem which simply requires the evaluation of F using X.

It is this simple remark that forms the basis of our approach to initiating students to the representation of word problems by algorithms and to their solution by successive approximation. The main idea is to separate the question from the problem situation, to model the situation by posing questions which generate arithmetic problems, and to represent these solutions in a progressively more generalized way.

Consider this problem as an example: "Corinne works part-time selling magazine subscriptions. She earns 20$ per week, plus a bonus of 4$ for each subscription sold." The question is: "How many subscriptions must she sell in order to earn 124$ in a week?" During the first session, the situation is presented without the question, and the "interviewer" asks questions like "How much will she earn if she sells 3 subscriptions?,... 5 subscriptions?,... 8 subscriptions?,... etc." Note that in order to answer these arithmetic questions, the student must possess an operational understanding of the implied functional relations. The next step is to have the student formalize these functional relations. He/She is asked first to verbalize and then write his calculations, line by line, as follows (for 8 subscriptions):

**8 X 4 <u>gives</u> 32**

**32 + 20 <u>gives</u> 52**

(Note that from the start we encourage the use of **"gives"** to demarcate the result of a calculation.)

After a few calculations of this type, with different numbers of subscriptions, we ask the student to create a table of all the trial values and the corresponding values calculated. The goal of this exercise is to encourage the student to consider names for the table headings by engaging him/her in a discussion aimed at recognizing and naming variables; these variable names in turn serve as input and output variables in CARAPACE. Typical variable names in our example might be "number of subscriptions," "salary," and "salary with bonus."

Once this exercise is done, the students are asked to write a series of generalized instructions (with no given value for the number of subscriptions), using the table headings as names, to arrive at something like this:

**number of subscriptions X 4 <u>gives</u> salary**

**salary + 20 <u>gives</u> salary with bonus**

We now have an algorithm which may be entered into CARAPACE and executed, once we have identified "number of subscriptions" as an input variable.

2. The numerical search for solutions

Once the algorithm has been written in CARAPACE, we can now return to the original problem and ask the student the planned question: "How many subscriptions must she sell in order to earn 124$ in a week?" As it stands, the algorithm allows the calculation of "salary with bonus," given the "number of subscriptions." The question may now be reformulated as follows: For what value of "number of subscriptions" will "salary with bonus" have the value of 124? The student then tries different values for "number of subscriptions," with CARAPACE calculating the corresponding "salary with bonus." The resulting tabular display shows the student how different values of "salary with bonus" are functions of "number of subscriptions." Usually the target value of 124 is not achieved on the frst try; the student must reevaluate his choice before making another guess. The students' strategies become apparent as they gradually refine their guesses to ultimately arrive at their goal. This is the solving process that we call numerical trial-and-error search.

**Study of Numerical Searches Used in CARAPACE (Kieran and al., 1988)**

During the 1987-88 school year, we worked with two 13-year-old students of slightly above average mathematical ability. Our goal was to document both their numerical search strategies and the influence of a known problem context on these strategies. We used two methods. In the first, the student began with a word problem and created his own algorithm in CARAPACE before proceeding to a solution search. In the second method, the problem and corresponding function were hidden, and the student attempted to arrive at a target output with a series of trial input values. For both methods, the students used the tabular display mode.

The results:

Effect of Context: When the students began to solve their problems on paper, their input values reflected their knowledge of the context of the problems (with respect to both external semantics, like the price of an object, and internal semantics, like choosing an even number when there was a division by two and the result had to be a whole number). However, once the work was at the computer, we noticed no difference in the numerical search strategies, whether the context of the problem was known or hidden.

Search strategies: We recorded two main strategies which we call increasing and decreasing. An increasing strategy is: If the result is too small (or respectively, too large) with respect to the target value, then increase (or respectively, decrease) the trial value. This strategy was associated with such substrategies as: bisection, comparison of variation, asymmetry, digit-by-digit, additivity, and partial additivity. Certain expected strategies were not employed, notably proportionality, interpolation and reliance upon the given relations of the word problems.

A decreasing strategy is: If the result is too small (or respectively, too large) with respect to the goal, then decrease (or respectively, increase) the trial value. The students generally had difficulty using this strategy, and even in the case of decreasing functions, frequently returned to their preferred increasing strategy.

Our subjects were reluctant to choose a truly random value in their first trial. They tended to choose a number that was the solution to a previous problem, or they would calculate a value from the givens of the problem, knowing full well that their guess was most likely incorrect.

**Study of Algorithmic Representation Processes in a Functional Approach (Kieran and al., 1989)**

In this study during the school year 1988-89, our subjects were 12 seventh-grade students of average mathematical ability. We divided the research into three phases, according to the algebraic structure of the word problems. In the first phase, the students were introduced to CARAPACE using the functional approach described earlier in this paper. The problems were structured in the form $a x \pm b = c$, and the problem situation was presented without the question. The major finding which emerged from this phase was the facility with which the students were able to develop ordered algorithms and transpose them to CARAPACE. On the other hand, as soon as the students were given the question which completed the algebra problem, most of them attempted to use inverse operations to solve the problem directly. The algorithm which they had just developed did not seem to provide any useful purpose for them.

In the second phase, the students were presented with the complete word problem (both situation and question). Problem types included: $a\,x + x = c$, $b - (d\,x + e.a\,x) = c$, $x + a\,x + b\,x = c$, $x + a\,x + (x + b) = c$. At the beginning, we noticed that many students persisted in using inverse operations (but this time unsuccessfully) to attempt to find the solution directly from the text of the word problem, bypassing completely any intermediate written representation. After several sessions, however, they began to realize that their success with inverse operations was diminishing as the problems became more difficult. They then decided to use CARAPACE to represent the problem situations and to help them in their numerical search.

For the third phase, complete problems were presented, and most were of the type $a\,x \pm b = cx \pm d$. Note that in order to solve these problems using CARAPACE, where equality does not exist, the student is required to produce two functional representations of the forms $a\,x \pm b$ and $c\,x \pm d$, and give trial values for $x$, with the goal of obtaining the same result for both functions. Representing these problems in algorithmic form presented no difficulties for the students, and with these problems, no one attempted to use inverse operations.

Our results go in a different direction from those of Filloy and Rojano (1984), who have proposed the existence of a "didactic cut" between problems of the type $ax + b = c$ and the type $ax + b = cx + d$; they have suggested that the first type can be solved arithmetically (with inverse operations), whereas the second type require an algebraic representation involving direct (or forward) operations. Our students represented both kinds of problems with equal facility using the algoritmic approach of CARAPACE.

**Research to come**

In her analysis of the passage from operational conceptions to structural conceptions of a mathematical notion, Sfard (1989) identifies three phases which she calls interiorization, condensation and reification. Our goal in 1989-90 is to study the phases of interiorization and condensation for the concepts of variable and algebraic expression. By working at the higher levels of CARAPACE with word problems of increased algebraic complexity, the subjects of our study will gradually approach the formal algebraic notation of equations. In particular, we will be observing the roles of a) parentheses, b) the ordering of operations, c) the shortening of variable names to arrive finally at single-letter variables, and d) implicit multiplication, in making the transition from procedural representations to more standard algebraic representations. We will also be looking at the nature of those situations which provoke the use of inverse operations directly from the word problem statement, as well as those situations in which the use of inverse operations would be useful.

In 1990-91, we plan to study a representational form which traditionally has presented difficulties to students: Cartesian graphing of algebraic relations. To provide continuity with our current projects, we will be adding a graphing module to CARAPACE which will give an algorithmic and dynamic character to graphic representation. In this new environment, the student will be able to draw a point or series of points on the graph, with the possibility of specifying and following step-by-step the calculations and geometric constructions carried out, at the student's own chosen level of detail. In this way, we will be able to study the effects of this student-controlled construction mechanism on the qualitative and quantitative interpretation which the students bring to the graphs they produce. The data will be analyzed in terms of operational and structural conceptions of graphs (Sfard, 1989), in addition to the nature of the links made by students among graphical, procedural, and algebraic representations.

In 1991-92, we will examine the problem of algebraic manipulations. Often these manipulations are carried out mechanically, without any consideration of the numerical models on which they are based. We plan to create a computer environment which will allow the students not only to do algebraic manipulations but also to interpret them numerically. From this, we hope to acquire a better understanding of the interaction between meaningful manipulations of expressions/equations and students' conceptions of these mathematical objects. We also hope to be able to document the kinds of learning engaged in by beginning algebra students when using our modified form of symbolic manipulator.

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# LOGO TO TEACH THE CONCEPT OF FUNCTION .

**Dominique GUIN** et Ismenia GUZMAN - RETAMAL
*Institut de Recherche de Mathématique Avancée,*
*10 rue du Général Zimmer ,*
*67084 Strasbourg Cedex , France*

*Summary : Our aim is to elaborate teaching situations with a constructivist approach underlying the various aspects of the function notion on one hand , on the other hand the relationships between them.We assume that a computer environnement can give an efficient contribution to our goal in two different ways: programming activities in an applicative language LOGO , and using microcomputer as a tool .We present some teaching situations and first issues on a comparison of productions with a traditional teaching in an other classroom : there are significant differences between the two groups according to the concerning aspects of the function concept . After these first results , we have altered our didactical situations for a new experiment : we discuss similarities and difficulties about it . The research concerns twenty 14-15 years old students , having a LOGO experience prior to the study .*

## OBJECTIVE :

Objectives of french curriculum on functions are:

- **Recognition** of a function in various situations ( graphical, algebraic , common life ).
- **Manipulating** functions .
- Applications to **equations** of lines .

Our objectives are to contribute answering the questions raised in [ Tall D.( 1987 ) ] in the restricted domain of teaching the function notion :

" - In what ways can multiple links representations be integrated into the curriculum for learning, teaching problem solving ?

- In what ways can computer environments be designed an used to provide intelligent support to the learning process ?

- In what way are programming and the use of prepared software ( computer as a **tool**) complementary , and what constitutes an optimum combination of the two in terms of understanding and efficiency ( time on task ) ? "

To find an answer for these questions , it is necessary to **elaborate** and **experiment** teaching situations underlying :

- the various aspects of the notion of function ,

- the **relationships** between them .

A computer environnement can give an efficient contribution in two different ways :

- **programming activities** in an applicative language LOGO ,

- and using microcomputer as a tool .

**LOGO language** has been chosen for its good adaptability to programming activity for problem solving because of its procedures , and to work on functions because it is an applicative language. **Functional programming** languages such as Logo are very close to the language of mathematics [ Klotz S. (1986) ] .

## THEORETICAL FRAMEWORK :

Vinner already introduced a distinction between two aspects of the function notion : concept-**definition** and concept-**image** [ Vinner S. ( 1983 ) ] .

A **function** is not just a :

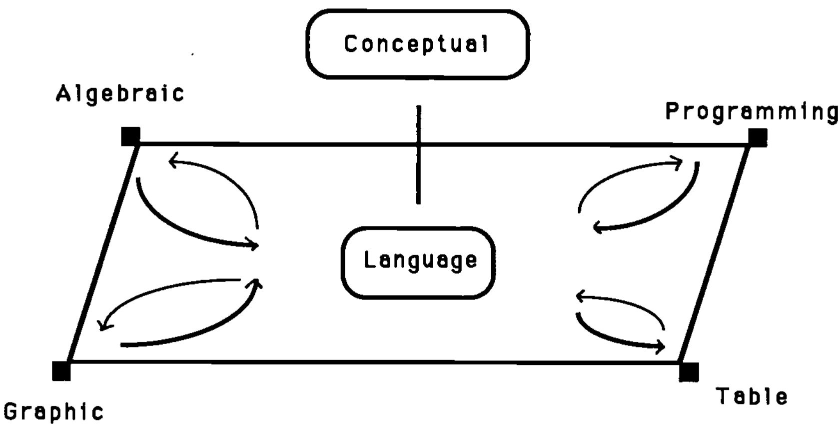
a table of values

a graphical representation

a formula

a correspondence

it is **all of them** at **once** . It seems necessary to define different registers that I. Guzman [Guzman I. ( 1989 ) ] displays in this way :



a) registers of **processing** on the same plane : the algebraic (formulas ) , the graphic, table (of values ) and programming activity LOGO .

b) registers of **conceptualization** ( relationship , correspondence ) and **language** which allows communication between registers .

We have noticed that usually , in France , the teacher gives the function definition (relationship or correspondence between two sets ) . Then , the exercises are essentially calculus on algebraic expressions and graphic representations of functions . **Links** between different registers are rarely explicited ( for example , the link **graphic <--> algebraic** ) and the use of these registers to analyze **empirical** situations is left out .

**Our cognitivist and didactical hypothesis :**

1) Constructivist hypothesis : mathematical knowledge is constructed by **problem solving** .

2) The use of a **programming activity** in mathematical teaching requires a real **alphabetisation** in Computer Science [ Rogalski J. ( 1985 ) ] .

3) An appropriate use of computer **as a tool** can **underlie links** between different registers.

Several studies were carried out in this way :

- **Link** **algebraic** register <--> **graphic** register , [ Dreyfus T. , Eisenberg T. (1987)], [Goldenberg E. ( 1987 ) ] , [ Zehavi N. , et alii (1987)].

- **Link** **algebraic** register <--> **LOGO** **programming** register , [ Leron U., Zazkis R. (1986 ) ] .

## METHOD

### The LOGO project ( 87 ) :

In french curiculum , it is the first teaching on the function notion .We had to build didactical situations favouring **interactions between different registers** optimizing the use of a microcomputer environment :

1) Introduction to the function notion as a procedure linked to a table of values .This activity deals with the **table** and LOGO **programming** registers dealing with the **conceptual aspect** (correspondence).

- introduction of vocabulary on fonctions : image , domain of definition , etc.. .
- large scope of examples ( **empirical situations** ).

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**F : length of a word**

| X | Cat | Chicken | Dog | Elephant | Stork | Marmot |
|---|---|---|---|---|---|---|
| F ↓ F(X) | | | | | | |

*****************************************************************

2) Experiments with the function notion :

- with various examples according to the curriculum of such students ( constant , linear and affine functions ) .

- by investigations on **hidden functions** [ Leron U., Zazkis R. ( 1986 ) ] : students have to find out the LOGO procedure-function with an experiment on different values of the variable.This activity deals with **table , algebraic , LOGO programming** registers , **language** and **empirical situations** .

3) Introduction to graphic representation of a function as an execution of a procedure representing the set of points with coordinates ( x , f ( x) ) . This activity deals with **table , algebraic , graphic** and **LOGO programming** registers , **language** and **empirical situations** . This process points out the role of parameters a and b in the expression $f ( x ) = a x + b$ through a **systematic variation** of these parameters .

**Experiment environment :**

This experiment took place in a classroom ( 20 students, 14-15 years old ) with the participation of the mathematic teacher. The fifteen sessions occured within 4 weeks . There was 7 micro-computers in the classroom . Functionnal programming requires a good level of programming abilities . It is the reason for having choosen students with a LOGO experience of more than one year :

1 - Basic teaching of LOGO programming turned to structured programming [ Dupuis C., Egret M.A., Guin D. ( 1988 ) ] ( 6 months ) .

2 - Teaching on recursion [ Dupuis C. , Guin D . ( 1988 ) - Guin D. ( 1986 ) ] , ( 6 months).

**First results :**

A test was submitted to a reference classroom [ Guzman I. ( 1989 ) ] , leading to the following observations :

__1) first group :__ The answers which are better in the reference classroom are essentially related with **conceptual** register and its link with **language** register :

- In the **recognition** task of a function from **empirical situations** , the vocabulary on functions (as domain of definition , image ...) is rarely used in the experiment classroom : It points out a **weakness** between **conceptual** register and **empirical situations** .

- In the **recognition** task of a function from a **graphic representation** , the students of the reference classroom used a graphic criterion : the function notion was introduced in this classroom from conceptual and graphic registers . In the experiment classroom, the link **graphic** register <--> **conceptual** register was not enough emphazised .

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During the mathematics lesson , the teacher asks for examples of functions and receives the following answers from his students :

- Alain proposes **weight** as **function** of **size** for all the students of the school .
- Bernadette proposes **weight** as **function** of **age** for a child between 0 and 5 years old .
- Claude proposes size as **function** of **age** for an adult between 0 and 40 years old .

In your opinion , are these examples good or bad and tell why?

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- In the experiment classroom , the function notion is seen as a **way to calculate** one object from the other , if there is not such a possibility , there is no function : its points out an unbalanced practice between **LOGO programming** and conceptual registers .

- Common **language** was not used for **recognition** of a function : students do not handle **key-words** related to function concept , for example : **dependence** , although they have feeling of it . A specific work on **conceptual** register and **links** with **language** and **graphic** registers in



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**empirical situations** seems to be lacking . For this it is important in analysis of tasks to locate the **words** linked to conceptual aspects ( **conceptual** register ) necessary to build models on empirical situations .

2) second group : The answers which are better in the experiment classroom are essentially related to **production tasks** . We can conjecture that these results come from an active practice on **empirical situations** , **functional programming** , **table** registers and **links** between them wich are rarely tackled in traditional teaching : for example , there are no attempts from students in the reference classroom to find a correspondence between graphic situation and algebraic expression ( link **graphic** register <--> **algebraic** register ).The functional vocabulary as " linear or constant function " etc... is more used in experiment classroom . The answers which are better in the experiment classroom are also related to **application tasks** using properties requiring that a **relationship** or a **correspondence** be set up , but also in **producing** or **recognizing algebraic expressions** .

## The graphic project ( 88 ) :

According to the first results , we are aiming to alter our didactical situations :

1) Simultaneous introduction in **graphical register** ( as in traditional process ) and **table** register of function **notion** .

2) Introduction in **graphical register** of functional **vocabulary**

3) **Recognition** activities which point out the **correspondence** aspect of a function in **table** and **graphical** registers which allowed students to find out by themselves the **graphical** criterion .

4) Investigations on **hidden** functions with microcomputer : during this second experiment , students had **too weak** a level in LOGO for **programming activities** , they only used microcomputers as a **tool** .

5) Investigations on **graphical representations** with microcomputer .

## A comparison between the graphic project ( 88 ) and the logo project (87):

Between these two experiments there are **differences** and **similarities**. The differences which are external to the structure of the project are relative to the **training** of the students : class 88 was weaker and had not enough **abilities in LOGO** programming. Therefore, the programming register has only been implemented as a **tool**, especially in the work sheets " hidden function " and " graphic representation ": **Programming activities requires a real alphabetisation in LOGO** .

The two projects have the same aim as far as teaching is concerned : to bring into play **all the registers** involved in the concept of function. External factors were the cause of a modification in methodology. In order to present the concept of function , **the logo 87 project** has highlighbed the **programming** register and has presented a function as a procedure. The table register has also been simulaneously brought in , with a work on functional vocabulary . On the other hand , **the graphic 88 project** presents the concept of function in the **graphic** register pointing out intuitively the **correspondence aspect** by developing a graphic criterion used to identify a function. Therefore , the first work sheet about "Functions" is different in the two projects. The "Hidden Functions" work sheet and the "Graphic Representation" work sheet are almost the same one . The fourth work sheet which is entitled "Affine Functions and Lines Equations" is different because the lines equations in 1988 were taught before the function chapter .

## Discussion

- Results are only relevant on the **qualitative level** because the populations in the two experiments were very different .There are still a few variables that we did not grasp such as training of students for example . From a qualitative view point the following facts were pointed out :

- A **slight improvement** at the **conceptual level** in the experimental class 88 in relation to the experimental Class 87.

- The **link** between the **graphic** and **algebraic** registers **did not improve** during the **first** experiment ( in 1987 ). The behaviour of students in the experimental and control class was the same one . In others words, their behaviour was the classical behaviour of students at this level. We do not have a didactical explanation for this fact. During the **second** experiment ( 88 ) , we have discovered that students have great **difficulties** in **linking** the **algebraic** and **graphic** registers . They **stumbled** in interpreting the **algebraic** expression of a function and **translating** it using tables and graphs. Nevertheless , the **inverse operation** of interpreting graphs and graphic representations was more **successfull** that in the traditional process . It is obvious that the **graphic strategy** used by the students of class 88 has not been **transferred** to the others registers.

To conclude :

- In the first experiment ( hight level ) , there were very **good results** for **manipulations** and very **poor results** at the **conceptual** level .

- In the second experiment ( low level ) , there were an **improvement** at the **conceptual** level and very poor results for **manipulations** .

This leads us to ask the following **question** : Is there an independence between the **understanding** of the conceptual aspect of **correspondence** of a function and the **manipulation** of this **correspondence** in the others registers and their links ? To answer this question , we have to find good conditions of comparison .

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# THE CONCEPT OF FUNCTION : CONTINUITY IMAGE VERSUS DISCONTINUITY IMAGE (Computer experience)

Fernando Hitt
Sección de Matemática Educativa del CINVESTAV, PNFAPM, México.

## SUMMARY

**With the increasing use of computers in the classroom and also of graphics packages, a concept of functions is being encouraged. This is the concept of function defined by a single formula, a continuous function. Here we shall consider functions expressed by more than one formula in a "paper, pencil and Logo" context, with the aim of developing in secondary school pupils a broader notion of function than that which is generally possessed by such pupils.**

## INTRODUCTION

In previous studies of the concept of function, we find that the problem of obstacles to understanding has been approached in a number of ways. One such approach is that of Tall and Vinner [1981] and Vinner [1983] who interpret certain obstacles in terms of a lack of interaction between what they call "concept definition" and "concept image".

Markovits, Eylon and Bruckheimer [1986] report answers given by pupils in relation to a graph of a discrete function that it was not a function, since "*the points are not connected*". In the same study, the authors state: "*Only one student drew the graph of the following function correctly*

$f:\begin{Bmatrix}\text{natural}\\ \text{numbers}\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{natural}\\ \text{numbers}\end{Bmatrix}$ ; $f(x) = 3$ . *Most answers subconsciously replaced the natural by the real numbers*".

Another study, this time with teachers [Hitt, 1989], we can see that the teachers showed a strong tendency to think in terms of continuous functions (spontaneous behavior). Their concept image is linked to an idea of function continuity expressed by a single formula.

Our activities related to this work, are designed to provide the pupil with problems and exercises whereby the different variables mentioned above can, in a structured way, play a positive role in developing a concept image.

MATHEMATICS TEXTS (definition of function and images used)

As is well known, different authors introduce the concept of function in different ways. Generally, we can classify them into four main definition types (Hitt [idem]). The experience of teachers and researchers has shown that some definitions are not all equivalent in an educational context [Malik, 1980].

THE COMPUTER AND THE GRAPHICAL EXPRESSION OF FUNCTIONS

With the use ever-increasing use of the computer and the production of new graphics software, a new concept image is being generated which is different from the one being purveyed in textbooks. We use the phrase "*new concept image*", because, unlike textbooks, which show graphs of discontinuous functions, the software that has been available up to now does not allow graphical functions defined by more than one formula.

Let us consider another problem. If we take, for example, the functions: $f(x)=\begin{cases} 3 & \text{if } x<0 \\ 10 & \text{if } x>0 \end{cases}$; and $g(x)=4x+6$, if $x\in\mathbb{N}$. The two functions are continuous in their definition domain. However, when the graphs of the functions are shown to the pupils, they interpret each one in a different way. Some will think that the functions are not continuous ("*The pencil was taken off the paper when drawing the curve*"). Our primary intuition [Fishbein, 1987] prevents us from seeing them as continuous. What can we do to develop a secondary intuition, in Fhisbein's sense?.

Another problem is that because of the limitations of computer screens, functions such as $f(x)=(1/x)\ \sin x$ ,for $x\in\mathbb{R}-\{0\}$, where the limit at 0 on the left is equal to the limit at 0 on the right, appear to be continuous functions in $\mathbb{R}$. In some cases, for example in Tall [Supergraph, 1985], when the computer is drawing the graph of the function and finds that it is not defined at one point, the computer makes a beep. However, the problem with the image on the screen remains.

To summarize, an unsolved problem with currently available software is that the concept (definition and image) which they are implicitly reinforcing can be described as :

* Function defined by a single formula
* Function-continuity
* Continuous domain (connected)

These problems may be resolved in the near future. That is, new versions of the graphics packages will be produced with the capacity to overcome these kind of problems. Much work needs to be done on reconciling textbooks with computer languages and educational software.

DESIGNING ACTIVITIES WITH PAPER, PENCIL AND COMPUTER

Our aim was to concentrate specifically on graphically expressing functions, both continuous and discontinuous: "Functions and Graphs, Logo graphic tasks" [Hitt, 1989]. We also wished to place special emphasis on the domain and set image of functions.

It will be seen that the proposed activity is attempting to build a bridge in the pupils, between the concept of function and their mental image of it by using the computer. A further objective was to provide the mathematics teacher with "paper, pencil and computer" activities for use in the classroom. Thus, we have attempted to link a language, Logo, and a concept, namely, that of function.

The development of a secondary intuition in Fishbein's sense [ibid] would have to be developed through mathematical activities before making use of graphic software which would introduce obstacles into the change from one level of intuition to another.

It is our hypothesis that the proposed activities will help to bring about this change in the level of intuition. In our experiment, we only try to prove that no knowledge of Logo required by the pupil in order to work with it.

Two approaches were adopted in the experiment: a laboratory approach, and a normal working situation in the classroom. The sample consisted of five pupils. The interview with pupil called 1 was undertaken entirely in the University laboratory (two and a half hour session). He was 14 year old (end of 3rd year in secondary school). He had little previous experience with Logo in the context of Turtle Geometry.

The remaining pupils, aged between 16 and 17, were given activities to work on in a normal individual work session in their computer laboratory (2 hours). These pupils were in the 6th year of secondary school. They already had some knowledge of Basic but had not previously worked with Logo.

We now set up the activities that the pupils were required:

1. Write down the concept of function.
2. Read the definition of function which we would use in our context and our examples.
3. To write again about the concept of function. 4. Draw graphs of $f_1, \ldots, f_7$ and write down their set image.
5. To discover functions FUNONE, FUNTWO,...,FUNSEVEN
6. To write down the function related to the graph showed.
7. To write once again about the concept of function.

ANALYSIS OF THE RESULTS

I. Definition of function Activities 1, 2, 3 and 7.

Pupil 1 wrote that the concept of function was a process and that function can be represented by a graph. He also gave an example, $3 \xrightarrow{\times 2} 6$. This answer suggests that a formal definition was lacking. The experiment had a strong effect on the concept image of this pupil. In fact, at the end (activity 7) he wrote three pages and provided three graphs of continuous functions, one graph of a discontinuous function, and two examples of functions expressed by more than one formula.

Pupil 2 did not remember what the definition of function was.

Pupil 3 The word "*operation*" was used by this pupil in the three definitions that were written by him. In his final definition, we can see that the functions expressed by more

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than one formula had an influence on his original idea, as evidenced by the use of the phrase "*combination of operations*". Furthermore, he added to his definition the words "*there is only one image for each number*".

Pupil 4. This pupil provided a definition of function which was associated with activities of differential calculus and locating maxima and minima.

Pupil 5. This pupil's definition was in terms of '*operation*' and '*mathematical process*'. In his second definition there was momentary break influenced by our presentation. At the end he returned to his original idea, adding the words '*function contain different operations*'.

In the light of the foregoing paragraphs, pupil 1 can be seen to have assimilated the ideas that were presented to him much more fully than did the other pupils, owing to being at an '*intuitive stage*' . That is, his knowledge of the concept of function was intuitive rather than formal.

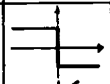
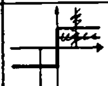
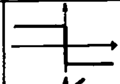
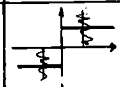
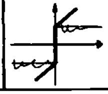
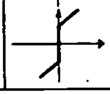
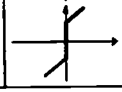
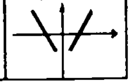
Surprisingly, none of the students, in their first definition, included any of the graphs they had previously acquired in the course of their studies.

II. Transfer from algebraic to graphical form. Activity 4.

Pupils were then asked to do some tasks with paper and pencil, transferring functions written in algebraic form to the graphical form, and giving an image set for each one.

The five pupils were new to compound functions, saying that they were used to having functions in the form y =_______. I suggested that they analyze the function in parts. They answered that this would be sufficient to know what to do.

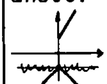
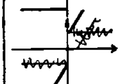
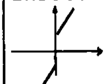
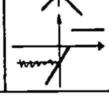
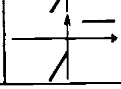
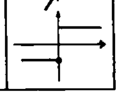
The main problems arose with functions $f_5$ and $f_6$. Generally pupils coped with $f_1$, $f_2$, $f_3$, $f_4$ and $f_7$ without difficulty. In the case of pupils 2 and 4, the concept image is linked to function continuity, with the result that the curves they drew were not functions. Pupil 3 shows considerable confusion, drawing a correct graph of $f_4$ but then changing this to a vertical straight line. This same pupil drew an incorrect graph for $f_5$ (drawing two straight lines which did not represent any function), and then corrected himself, giving another wrong answer. It is possible that he failed to read $f_7$ correctly and inverted the two parts of the function. Pupil 5 followed the

| TRANSFER FROM ALGEBRIC TO GRAPHIC FORM | PUPIL No. 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f_5(x)=\begin{cases} 2 & \text{if } x\le 0 \\ -2 & \text{if } x>0 \end{cases}$ | correct answer |  |  |  |  |
| $f_6(x)=\begin{cases} x-2 & \text{if } x\le 0 \\ x+2 & \text{if } x>0 \end{cases}$ |  |  | correct answer |  |  |

same procedure as the previous pupil with $f_5$. When this pupil drew the graph of $f_6$, he must have thought that $f(x)=x-2$, for $x\le 0$, would have to pass through the point $(-2,0)$ instead of $(0,-2)$. And for $f(x)=x+2$, $x>0$, he thought that the line should pass through $(2,0)$. It is important to note that this pupil realized that at $x=0$ there was a "*change in the function*". It can in fact be seen that he left a gap in the graph.

III. Transferring an algebraic form (inside the computer) to an algebraic and graphical form. Activity 5.

At the end of the activity 4, the pupils were asked to work with functions FUNONE, FUNTWO,..., FUNSEVEN (paper, pencil and computer).

| GUESS MY FUNCTION AND MAKE A GRAPH | PUPIL No 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f_5(x)=\begin{cases} -1 & \text{if } x\le 0 \\ 1 & \text{if } x>0 \end{cases}$ | correct answer | correct answer | correct answer |  | correct answer |
| $f_6(x)=\begin{cases} x-1 & \text{if } x\le 0 \\ x+1 & \text{if } x>0 \end{cases}$ | correct answer | correct answer |  |  |  |
| $f_7(x)=\begin{cases} x-1 & \text{if } x\le 3 \\ 1 & \text{if } x>3 \end{cases}$ | correct answer | correct answer |  |  |  |

Pupil 3 made several attempts to find a graph for $f_6$, finally producing a one which was not a graph of a function. Pupil 4 tried out several relative whole numbers ($\mathbb{Z}$), and in defining functions $f_5$, $f_6$ and $f_7$ he did not take account of the interval (0,1). Here it is worth mentioning that this pupil wrote down the functions wrongly, committing syntactic errors. It is possible that these errors had been committed in the past, but had been corrected when they came up in the course of activities, but in coming up against a new situation (quite complicated in some cases) the error appeared again in a new context.

$$f_5(x)=\begin{cases} x > 1 = 1 \\ x \le 0 = -1 \end{cases} ; \quad f_6(x)=\begin{cases} x > 1 = x + 1 \\ x = 0 = -1 \\ x < 0 = x - 1 \end{cases} ; \quad f_7(x)=\begin{cases} x > 1 = 1 \\ x \le 0 = x-1 \end{cases}$$

A further difficulty with function $f_7$ is that pupils 3, 4, and 5 interpreted the graph for $x \le 3$ as if it had to be positioned beneath the x axis. It may be that when pupil 5 drew the graphs of functions composed of more than one expression, was expecting that the two graphs would be of the same type.

IV. Transferring functions from the graphical form to the algebraic form. Activity 6.

Pupil 1 was the only one who did this part completely (committing one error), the others did not have enough time to write down the answers to the functions $f_{11}$, $f_{12}$, $f_{13}$ and $f_{14}$.

## Conclusions

Functions defined by a single formula did not seem to cause any great problem in a connected domain (except as far as the subconcept image set was concerned). The results show that it is possible to work with functions expressed by more than one formula and that pupils could become better at handling these if they were given more practice with them.

A knowledge deficiency will resurface as errors with the passage of time. The definition which we provided did not, except very briefly, replace the definition that had been acquired by the pupils in earlier years. The results show that in only one case were new elements incorporated and only in one case did the definition help a pupil to recall parts of his definition that he had forgotten.

If we compare the results of the activity in which functions were transferred from the algebraic form to the graphical form (paper and pencil), with the results of the "guess my function" activity (paper, pencil and computer), it is seen that learning has taken place in the course of these activities. The drawing of straight lines not representing the graph of any function virtually ceased. Some of the corrections which the pupils made and the graphs of discontinuous functions that they produced lead us to believe that some progress in the mental image of these pupils was made.

The emphasis that we gave to the subconcepts domain and image set on the coordinate axes (when graphs were shown) did not appear to have any positive effect on the behavior of the pupils.

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# ACQUISITION OF ALGEBRAIC GRAMMAR

*David Kirshner*

Louisiana State University

*A new rule of algebra is proposed and evidence for its psychological reality, presented. A model for acquisition of this rule is explored.*

Usually it is presumed that the rules of the algebra game are explicitly available; having first germinated in a mathematical mind, next been transplanted to textbook and teachers manual, and finally been harvested in the classroom for distribution to the populace. For instance the focus of Carry, Lewis and Bernard (1980) (following Bundy, 1975) on strategic decisions of selecting and sequencing rules suggests that the character of the rules themselves is relatively unproblematic. For Wagner, Rachlin and Jensen (1984) the available rules can be captured by "rote memorization of formulas and algorithms" (p. 7). Others, (e.g. Matz, 1980) postulate intermediate processes between the available rules and the rules actually used in solving problems. She proposes that *extrapolation techniques* may be required to bridge the gap between the *base rules* of the curriculum and problem contexts for which no available rule exactly fits. For instance she describes how the new situation $ax + ay + az$ might be handled by deriving the needed rule $A \cdot (B + C + ... + W) = (A \cdot B) + (A \cdot C) + ... + (A \cdot W)$ from the given rule $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$ (p. 104). In all of these instances, however, the rules underlying successful algebraic performance are introspectively obvious.

Elsewhere (Kirshner, 1987a; 1987b; 1989) I have proposed rules of algebra that are not introspectively obvious, and argued for a reassessment of traditional assumptions about the nature of algebraic knowledge. The present paper also proposes a new rule of algebra and offers support for its psychological reality; however, the focus here is on the possible processes of acquisition of the rule, and on such characteristics of the human cognitive system as can be inferred from the acquisition processes.

## THE GENERALIZED DISTRIBUTIVE LAW

The Generalized Distributive Law (GDL) presented here as a psychological theory was first introduced by Schwartzman (1977) (in a slightly less rigorous form), but as a pedagogical technique. It is based upon a simple hierarchy of *operation levels* which groups together inverse operations:

Level 1 operations are addition and subtraction

Level 2 operations are multiplication and division

Level 3 operations are exponentiation and radical

(If "*" is an operation, then "|* |" represents its level.)

Using this convention, the GDL can be simply stated:

$$(a \mathbin{\&} b) * c = (a * c) \mathbin{\&} (b * c), \text{ whenever } |*| = |\&| + 1$$

Note that this generalized rule subsumes eight other rules usually presumed to be discrete entries in the rule system of algebra:

*Level 2 over Level 1*

$$(a+b)c = ac + bc$$

$$(a-b)c = ac - bc$$

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

*Level 3 over Level 2*

$$(ab)^c = a^c b^c$$

$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$$

$$\sqrt[c]{ab} = \sqrt[c]{a}\,\sqrt[c]{b}$$ [1]

$$\sqrt[c]{\frac{a}{b}} = \frac{\sqrt[c]{a}}{\sqrt[c]{c}}$$

The claim is that the GDL is not just an interesting formalism, potentially useful as a pedagogical rule, but rather an integral part of the knowledge that is acquired in the development of algebraic skill.

Support for the psychological reality of the GDL involves analysis of the oft reported errors of the form $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ (Budden, 1972, p. 8; Schwartzman, 1977, p. 595; Laursen, 1978, p. 194; Davis & McKnight, 1979, p. 37 and p. 98; Matz, 1980, pp.

[1]The rules involving the radical operation appears in surface form to be left-distributive; however, Kirshner (1987, p. 93) argues that the deep representation of the radical operation is reversed from its surface form.

98-99; and Smith, 1981, p. 310). These errors can be seen to satisfy the *Overgeneralized Distributive Law*,

$$(a \ \& \ b) * c = (a * c) \ \& \ (b * c), \text{ whenever } |*| > |\&| ,$$

proposed here as a developmental precursor to the GDL. The present analysis is that the Overgeneralized Distributive Law represents a phase of covert 'experimentation' with contextual constraints on the application of distributivity. In this account, the GDL results from honing down distributivity to its maximally permissible context of application: $|*| = |\&| + 1$.

Matz (1980) also has attempted to account for the $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ errors as overgeneralization of distributivity rules; but without postulating an introspectively unobvious rule like the GDL. In her analysis, some normally useful processes for extrapolating from base rules to new situations has gone awry.

There are a number of deficiencies with Matz's explanation of these errors that are avoided in the present account. Firstly the extrapolation techniques that are presumed to have gone awry in the overgeneralization errors are not explicated in her theory. She gives illustrations, but does not detail the actual mechanisms at work. As a consequence of this lack of specificity, Matz's theory can be used to describe errors that *do* occur, but provides no theoretical basis to predict which error *should* occur. In contrast, the present theory predicts exactly the observed errors.

What is more, the present theory can extend in its prediction to a range of data that Matz's theory cannot explain even after-the-fact: Matz (1980, pp. 98-99) notes the occurrence of other linearity errors including $a(bc) = ab \cdot ac$, $a^{mn} = a^m a^n$, $a^{m+n} = a^m + a^n$, and $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$. These errors can be described as fulfilling yet more elementary versions of the Overgeneralized Distributive Law in which right or left distributivity holds, or operations distribute over themselves. For lack of a more precise analysis, call this *Open Context Distributivity*. The present theory, therefore, predicts that the first set of errors should prove more tenacious than this latter class, since as the student progresses from wider contexts to narrower contexts, *open context* $\rightarrow |*| > |\&| \rightarrow |*| = |\&| + 1$, the latter class of errors falls away before the former. This prediction, I believe, corresponds with the *facts*[1]; facts that Matz's framework

[1] Unfortunately, longitudinal records of student behavior are completely absent, and even systematic crosssectional data are scarce; nevertheless, the anecdotal evidence is strong.

cannot account for.

The greater specificity, predictive rather than descriptive adequacy, and greater range of applicability of the present account would seem to make it a far stronger explanation of the $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ errors than the account of Matz (1980). But the existence of introspectively unobvious rules like the GDL raises a host of new questions, among them questions about rule acquisition, to which we now turn.

## ACQUISITION

A variety of approaches to the question of rule acquisition are possible ranging from a Chomskyan innatist model in which the processes of induction are presumed to lie far beneath conscious cognition, to Anderson's (1983) ACT* (ACT STAR) theory of learning in which it is presumed that "all incoming knowledge is encoded declaratively; specifically, the information is encoded as a set of facts in a semantic network" (Neves and Anderson 1981, p. 60). For the purposes of coming to terms with the relatively radical notion of introspectively unobvious rules of algebra, it seems prudent to select the framework that is most compatible with usual assumptions about mathematical knowledge; namely the ACT* theory.

ACT* has been applied extensively to learning from direct instruction in such domains as geometry proof (Anderson, 1983b), computer programming (Anderson & Reiser, 1985), and word processing (Singly & Anderson, 1985). The approach taken in the theory is to trace processes of *proceduralization* and *composition* whereby new knowledge which enters the system in declarative form --for instance as text book rules or teacher instructions-- is *compiled* into automatically executed procedures as skill is developed.

Generally speaking, knowledge compilation results in the evolution of less abstract rules as more general declarative structures gradually becomes adapted to the specific conditions of the task environment. But ACT* does invoke *inductive tuning* for the creation of more abstract rules. Inductive tuning involves the complementary processes of *generalization* and *discrimination*. In generalization, conditions on the applicability of a rule are relaxed, resulting in a new version that applies to a broader range of contexts than the original rule. Discrimination tightens up rules that have been overgeneralized. Anderson (1986) illustrates the tuning mechanism with an example from language acquisition:

> Suppose a child has compiled the following two productions from experience with verb forms:
>
> **IF** the goal is to generate the present tense of KICK
>
> **THEN** say KICK + S
>
> **IF** the goal is to generate the present tense of HUG
>
> **THEN** say HUG + S
>
> The generalization mechanism would try to extract a more general rule that would cover these cases and others:
>
> **IF** the goal is to generate the present tense of X
>
> **THEN** say X + S
>
> where X is a variable.
>
> Discrimination deals with the fact that such rules may be overly general and need to be restricted. For instance, this example rule generates the same form, whether the subject of the sentence is singular and plural. Thus, it will generate errors. By considering different features in the successful and unsuccessful situations and using the appropriate discrimination mechanisms, the child would generate the following two productions:
>
> **IF** the goal is to generate the present tense of X
> and the subject of the sentence is singular
>
> **THEN** say X + S
>
> **IF** the goal is to generate the present tense of X
> and the subject of the sentence is plural
>
> **THEN** say X
>
> These learning mechanisms have proven to be quite powerful, acquiring, for instance, nontrivial subsets of natural language (J. R. Anderson, 1983). (p. 205)

These processes of generalization and discrimination can be applied to model acquisition of the GDL. Assume the existence of rules of the following form which capture students applicative knowledge of the $(a+b)c = ac + bc$, $(ab)^c = a^c b^c$, and other "distributive" rules of the curriculum:

**IF** the goal is to generate an expression that has addition as its dominant operation

and the current expression has multiplication as its dominant operation

and the next-most-dominant operation is the goal-dominant operation (addition)

and the dominant operation is to the right of the next-most-dominant operation

**THEN** create a new expression with the goal-dominant operation (addition) as dominant

and the previously dominant operation (multiplication) as next-most-dominant

(and assign the subexpressions appropriately)[1]

**IF** the goal is to generate an expression that has multiplication as its dominant operation,

and the current expression has exponentiation as its dominant operation

and the next-most-dominant operation is the goal-dominant operation (multiplication)

and the dominant operation is to the right of the next-most-dominant operation

**THEN** create a new expression with the goal-dominant operation (multiplication) as dominant

and the previously-dominant-operation (exponentiation) as next-most-dominant

(and assign subexpressions appropriately)

Generalization across operations would produce the new production:

**IF** the goal is to generate an expression that has & as its dominant operation

and the current expression has * as its dominant operation

and the next-most-dominant operation is the goal-dominant operation (&)

and the dominant operation is to the right of the next-most-dominant operation

**THEN** create a new expression with the goal-dominant operation (multiplication) as dominant

[1]The *dominant* (or least precedent) operation of an expression is the last one to be performed if variables are assigned values and the expression evaluated. The *next-most-dominant* operation is second-to-last to be performed in evaluating the expression. The *goal-dominant* operation is the dominant operation in the goal expression.

and the previously-dominant-operation (exponentiation) as next-most-dominant

(and assign subexpressions appropriately)

With such a generalized but undiscriminated rule in place, the student, faced with an expression like $(a+b)^2$, and having as a goal to generate an expression that has "+" as its dominant operation, but not yet having mastered the appropriate rule for achieving this goal, will call upon the (over)generalized rule to derive $a^2 + b^2$. Eventually processes of discrimination would constrain such overgeneralization, achieving the appropriately constrained rule corresponding to the GDL:

**IF** the goal is to generate an expression that has & as its dominant operation

and the current expression has * as its dominant operation

and the next-most-dominant operation is the goal-dominant operation (&)

and the dominant operation is to the right of the next-most-dominant operation

and $|*| = |\&| + 1$

**THEN** create a new expression with the goal-dominant operation (&) as dominant

and the previously-dominant-operation (*) as next-most-dominant

(and assign subexpressions appropriately)

Such derivations go some way toward explicating the development of rule structures; however they are incomplete. A complete theory also must account for the presence of the original rules, in their given form. In Anderson's (1986) linguistic example (above), the goal structure "to generate the present tense of HUG" (p. 205) implies that the category of *present tense* previously has been abstracted from the child's linguistic experience. Thus a full account of the present-tense rule would have to explicate this process of abstraction. (It seems plausible that tense differentiation could be motivated in terms of pragmatic communication needs of the child; but such an account must be given for the above explanation to be complete.)

In the GDL derivation it can be observed that the operations in the given and derived expressions of the curricular rules are linked (recall the reference to "goal-dominant operation" in both the condition and action statements). If this were not so, and the rules were presented as

IF the goal is to generate an expression that has & as its dominant operation

and the current expression has * as its dominant operation

and the next-most-dominant operation is &

and the dominant operation is to the right of the next-most-dominant operation

THEN create a new expression with & as dominant

and * as next-most-dominant

(and assign the subexpressions appropriately)

with no mechanism to link together occurrences of & or *, the structure would be too complex for generalization to occur.

Can the richer representation required for generalization be justified in the terms of traditional cognitive (computer science-inspired) theory? Perhaps not! In standard applications (e.g. Bundy, 1975; Carry, Lewis & Bernard, 1980), computational production rules are condition/action pairs with no necessary rational association between the condition and the action. It is perhaps a peculiarly human form of representation that results in the perception of

$$(a+b)c = ac+bc$$

not as a rule for writing a new expression from an existing one, but as a rule for re-forming a single expression. In this sense, distributivity may be an emergent property of formally fixed expressions, rather as movement in a motion picture (movie) emerges from individually fixed stills.

This analysis is suggestive, not conclusive. It appears that the GDL could not easily be induced by a computer programmed with unembellished condition/action productions, but that it might be induced by a cognitive system endowed with richer representations. Furthermore, the human cognitive system would seem to be predisposed to such rich representation, either as an artifact of visual/perceptual functioning, or perhaps as a result of vast natural-language experience with syntactic forms identical in structure to algebraic rules:

e.g. (*Dogs* and *cats*) are *animals*) → (*Dogs* are *animals*) and (*cats* are *animals*)

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# EMBEDDED FIGURES AND STRUCTURES OF ALGEBRIC EXPRESSIONS

Liora Linchevski and Shlomo Vinner, Hebrew University Jerusalem

In this paper we try to clarify some relationships between success in the well known embedded figure test and success in what we call a hidden algebraic structure test. We claim that both tests require certain visual-analytical abilities. The visual-analytical ability required for the hidden algebraic structure test is probably a major component of the ability to handle high school mathematics and therefore there is a high correlation between success in the hidden algebraic structure test and a common mathematics test. Analysing our present research data one can hypothesize that the ability to identify hidden algebraic structures does not depend on age but does depend on the immediate algebraic experience in the period prior to the day on which the test was taken.

In many algebraic tasks it is crucial for the student to identify certain structures in given algebraic expressions, structures that sometimes cannot easily be seen on the surface. For instance, when a student is asked to add $1/(a^2 - b^2) + 1/(a^4 - b^4)$ it is more than helpful if he or she realizes that $a^4 - b^4$ can be considered as $(a^2)^2 - (b^2)^2$ and therefore can be written as $(a^2 - b^2)(a^2 + b^2)$. If he or she should solve: $(x + 1)^2 - 7(x + 1) + 12 = 0$ it will be much easier to consider $x + 1$ as one "entity" and solve $z^2 - 7z + 12 = 0$. It is only natural to hypothesize that this ability to identify "hidden structures" in algebraic terms is one of the components of success in school mathematics. Therefore, it is natural to expect a certain correlation between this ability and success in common mathematics achievement tests. On the other hand, if you want to think of a general ability from which the above particular ability is derived, it seems that this general ability should be the ability to identify certain simple figures hidden or embedded in a complex configuration. This ability is measured by the well known embedded figure test (EFT).

Because of the common way to report about psychological research (namely, without including the test items), a mysterious predictive power is associated with the EFT. The reason for this is that the EFT correlates with too many "things", especially with intellectual achievements (Witkin (1977), McNaught (1982), El-Famamaury (1988) and many others). However, a simple analysis of the EFT items indicates that it measures the ability to distinguish certain details from their context. This is a general analytical ability and when implied in particular situations, which require analytical ability, results in high success. This was already implied by Witkin and Goodenough (1981) who associated "field-independence" with being analytical. We, here, do not wish to involve the entire theory of field dependence with our research questions. It will be totally redundant. What we said above will be enough to explain our findings. Furthermore, we do not wish to elaborate on the dispute whether field-independence (high success in EFT) is a cognitive style or ability. We will call it ability being aware that Witkin and researchers consider it as a cognitive style. Our first task was to construct a mathematical test that will measure directly the algebraic ability mentioned above, namely, the ability to identify certain hidden algebraic structures in given algebraic terms. We denote it by HAST. Kieren (1988) related to the above situation using different terminology. She speaks about the surface structure and the systematic structure of a given algebraic term. The surface structure is, more or less, what you see on the surface. The systematic structure is "all the equivalent forms of the expression according to the properties of the given operations (p. 434). Thus, theoretically, according to Kieren, the systematic structure of a given algebraic expression is an infinite set of equivalent algebraic expressions. However, from a practical point of view, we are not interested in the set of all equivalent expressions. We are interested only in one or two expressions which are relevant to our algebraic task. That is the reason we prefer to speak about hidden algebraic structures and not about the systematic structure. In addition to that, a hidden algebraic structure in our approach can be a surface structure in Kieren's approach. For instance, when considering $5 + 3(x + 2)$(Kieren (1988), p. 434) in a certain context we can claim that $5 + 3z$ is a hidden structure of $5 + 3(x + 2)$ and preserving this structure while carrying out a certain algebraic task can be very helpful. For Kieren $5 + 3Z$ is the surface structure. Kieren, of course, formed her terminology for theoretical purposes different than ours.

Our research questions were:

1. Is there a statistically significant correlation between our HAST and EFT?

2. Is there a statistically significant correlation between our HAST and a common mathematical achievement test?

3. Is success in HAST correlated with age?

The first two parts of HAST included items which are taught already in grade 8. Hence, if the ability to identify hidden algebraic structure is mainly innate and it is formed basically by the introduction of the relevant algebraic topic, then elder students are not expected to do better than younger students. In other words, algebraic experience will not have a major role in the formation of the above ability.

## Method

1) Sample: Our sample included four grade levels (grade 8 - grade 11), each of them devided into two groups, high level and low level students. The division was done by the school and as every division it is not hundred percent reliable. Groups 1, 3, 5, 7 are the low level students of grades 8, 9, 10, 11 respectively and groups 2, 4, 6, 8 are the high level students of grades 8, 9, 10, 11 respectively.

2) Questionnaire: Our hidden algebraic structure test (HAST) had three parts. In the first part, after seeing one example, the student was asked whether a given algebraic expression could be obtained from $a + b$ by substitution. If the student answered positively, he was asked to state exactly what should be written instead of a and what should be written instead of b in order to obtain the given algebraic expression. Only positive answers with correct substitutions were considered as correct answers. The algebraic expressions of part A were:

1) $3x + 2y$ 2) $a^2 - b^2$ 3) $x^2 + 3y + z$

4) $a + b + c$ 5) $-b + a$ 6) $x^2$

7) $b + 5(c - d)$ (the items will be denoted by A1 - A7).

In part B the situation was quite similar with the only difference that the expression in which the student was supposed to substitute was: $(a + b)(a - b)$. The algebraic expressions of part B were:

1) $(1 + x^2)(1 - x^2)$ 2) $(6 + x)(x - 6)$

3) $(-b + a)(-b - a)$ 4) $(a + b + c)(a - b + c)$

5) $(b + \sqrt{b^2 - 4ac})(b - \sqrt{b^2 - 4ac})$

These items will be denoted by B1 - B5.

The third part of the questionnaire was administered only to grades 10 and 11, since its items were related to the solution formula of the quadratic equation, a topic which was studied only by the 10th and the 11th grades in schools where the questionnaire was administered. The third part started with the following introduction:

It is common to present a quadratic equation with the unknown x as: $ax^2 + bx + c = 0$, where a, b, c are the coefficients that can be specific numbers but also letters. It is known that the solutions of the quadratic equation, if exist, are given by:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In the following there are several equations. For which of them it is possible to substitute in the above formula in order to find the soultion. When it is possible, please make the appropriate substitution. You are not asked to calculate the final solution.

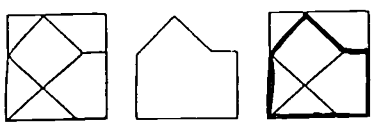
The equations were:

| | |
|---|---|
| 1) $x^2 + 2x + 1 = 0$ | 2) $x^2 + mx - 5 = 0$ |
| 3) $x^2 + ax + cx = 0$ | 4) $x^2 - 2x + 1 - m^2 = 0$ |
| 5) $9x^2 - 3mx - 6x - m + 1 = 0$ | 6) $-x^2 + 7x = 0$ |

These items will be denoted by C1 - C6. It is worthwhile to mention that part B and C included some items that have not been reported above. These were items with negative answers. Many weak students who could not see the appropriate substitutions in the positive cases claim also about these items that it is impossible to obtain them from the given expression or it is impossible to solve the equation. We considered these answers as "false positive" and excluded them from our analysis. The items were: $(1 + x)(1 - y)$ and $(-x + y)(x - y)$ in part B and $ax + b + 1 = 0$ in part C.

The second test which was meant to be administered to our sample was a simple group form of the embedded figure test. Because of administrative difficulties only 120 students of the entire sample ($N = 322$) wrote this test being a partial population of groups 3, 4, 5, 6 (the 9th and 10th graders). A representative item of the EFT can be the following:

There is a simple figure (Fig. B). Does this figure appear in configuration A and where?



A B C

The student is supposed to draw his answer as in Fig C.
There was also a common achievement test that was administered to all the 10th graders. It was a classification test which was designed to distinguish between the low level and the high level students at the end of the school year.
Three typical questions out of 10 in this test were:

I A two digit number is given. Write 1 at its right side, then write 3 at its left side and then add the two numbers which were formed. The result is 598. What is the given number?

II Find the area of a quilateral triangle if the diameter of the circle inscribed in it is 8 cm.

III Find a, b if it is known that a, b are integers such that $(a + b)(a - b) = 41$.

Results

Before bringing the data which relates directly to our 3 research questions we would like to bring some information concerning the HAST. In the following we will list the questions of each part of the test according to their difficulty order. In parantheses we will note the percentage of the correct answers. The number of respondents to parts A and B is 322. The number of respondents to part C is 179.

A1(88.5), A5(71.5), A7(50.0), A2(47.0), A3(42.0), A4(35.5), A6(32.5)
B1(69.5), B3(52.5), B5(41.5), B2(17.0), B4(2.0)
C1(70.5), C2(55.5), C6(22.5), C4(22.5), C3(13.0), C5(9.5)

Note that only 1/3 of the respondents could see that $x^2$ can be obtained by substitution from a + b (for instance, $a = x^2$, $b = 0$). The fact that the plus sign does not appear in the expression was probably the cause that so many students did not see the hidden structure a + b in it. The items in part B were harder and even much harder than the items in part A. Only 17% could see that $(6 + x)(x - 6)$ is equal to $(x + 6)(x - 6)$ and therefore can be obtained by substitution from $(a + b)(a - b)$. Only 7 students out of 322 that answered the questionnaire could see that $(a + b + c)(a - b + c)$ is equal to $((a + c) + b)((a + c) - b)$ and therefore could be obtained from $(a + b)(a - b)$ by substitution. It is interesting that C6 and C4 had the same degree of difficulty in our sample. The difficulty in each of them is related to the free coefficient of the quadratic equation. In C6 the problem is to identify 0 (which is not written) as the free coefficient. In C5 the problem is to identify $1 - m^2$, a complex expression, as the free coefficient, which in the schema of the quadratic equation is denoted by a single letter. The three above sequences are Gutman scales (roughly speaking, almost everybody who answered correctly a given item, had answered correctly all the items which are easier than this item). The coefficients of reproducibility are 0.84 for the first sequence and 0.94 for the second and the third sequences. Additional information about part C is that only 45.5% thought that the solution formula of the quadratic equation is inappropriate for solving ax + b + 1 = 0 (an item which we only mentioned in the previous section but did not include in the above sequences). Taking into account the information in the third sequence it can be claimed that at least 3/4 of the respondents cannot identify the structure of the quadratic equation when its form is "much" different from the common prototypical form (items C6, C4, C3 and C5). Only 70.5% of the students know how to use the solution formula even in the simplest case (C1). This is amazing because this topic is a central one in the curriculum and the solution formula itself was given in the questionnaire. In C2, where a very simple parametric form of the quadratic equation appeared, the success level dropped to 55.5%.

The coefficients of correlation between HAST and EFT and between HAST and the mathematics classification test are given in Table 1.

Table 1 Coefficients of correlation between the various tests

| | Mathematics classification test | EFT |
|---|---|---|
| HAST | N = 81<br>r = 0.85<br>p = 0.001 | N = 120<br>r = 0.75<br>p = 0.001 |

Thus the first two of our research questions are answered positively. The answer to the third question is not so clear and in order to relate in a non-superficial way we would like to present to the reader some tables.

Table 2 Percentages of correct answers and means to questions A1 - A7 in groups 1 - 8

| | A1 | A2 | A3 | A4 | A5 | A6 | A7 | mean |
|---|---|---|---|---|---|---|---|---|
| 1(N = 18) | 78 | 11 | 17 | 11 | 56 | 6 | 6 | 26.4 |
| 2(N = 31) | 97 | 38 | 42 | 32 | 71 | 32 | 65 | 53.9 |
| 3(N = 30) | 93 | 43 | 33 | 33 | 87 | 18 | 20 | 46.7 |
| 4(N = 64) | 97 | 69 | 45 | 44 | 83 | 61 | 56 | 65.0 |
| 5(N = 82) | 76 | 39 | 32 | 22 | 60 | 21 | 44 | 42.0 |
| 6(N = 59) | 93 | 61 | 71 | 63 | 86 | 39 | 78 | 70.1 |
| 7(N = 17) | 82 | 24 | 12 | 12 | 35 | 29 | 12 | 29.4 |
| 8(N = 21) | 95 | 38 | 52 | 38 | 67 | 24 | 67 | 54.4 |

Table 3 Percentages of correct answers and means to questions B1-B5

| Question / Group | B1 | B2 | B3 | B4 | B5 | mean |
|---|---|---|---|---|---|---|
| 1 | 28 | 0 | 11 | 0 | 0 | 7.8 |
| 2 | 74 | 29 | 45 | 0 | 36 | 36.8 |
| 3 | 53 | 7 | 50 | 7 | 3 | 24.0 |
| 4 | 86 | 22 | 77 | 3 | 48 | 47.2 |
| 5 | 59 | 12 | 39 | 0 | 34 | 28.8 |
| 6 | 90 | 27 | 73 | 2 | 81 | 54.6 |
| 7 | 41 | 0 | 18 | 0 | 6 | 13.0 |
| 8 | 76 | 19 | 52 | 10 | 66 | 44.6 |

Table 4 Percentages of correct answers and means to questions C1 - C6 in groups 5 - 8

| Question | C1 | C2 | C3 | C4 | C5 | C6 | mean |
|---|---|---|---|---|---|---|---|
| 5 | 70 | 49 | 11 | 13 | 5 | 20 | 28.0 |
| 6 | 86 | 78 | 12 | 39 | 15 | 25 | 42.5 |
| 7 | 18 | 6 | 6 | 0 | 0 | 12 | 7.0 |
| 8 | 71 | 57 | 29 | 29 | 19 | 33 | 39.7 |

Table 5 Means of correct answers to parts A and B of HAST

| Group | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Mean | 18.6 | 46.8 | 37.3 | 57.3 | 36.5 | 63.7 | 22.6 | 50.3 |

Table 6 Means of correct answers to the entire HAST

| Group | 5 | 6 | 7 | 8 |
|---|---|---|---|---|
| Mean | 33.7 | 56.6 | 17.4 | 46.8 |

If we look only at the means of part A, the easiest part of HAST, we discover that there is almost no difference between either good or weak students of the 8th grade and the either good or weak students of the 11th grade. There is an improvement in grades 9 and 10 but there is almost no difference between grades 9 and 10. Our guess is that this improvement is due to the fact that in grades 9 and 10 a lot of attention is given to the manipulation of algebraic expressions. Hence, the experience with algebraic expressions contributes quite a lot to the ability to identify hidden algebraic structures. On the other hand, after the period of intensive manipulations on algebraic expressions is over, the ability decreases and stabilizes around the level it was in the 8th grade.

As to the means of part B, the picture is even more complicated. There is an improvement from grade 8 to 9 and from grade 9 to 10. However, there is a regression from grade 10 to grade 11. These results can be explained in a similar way to the previous one. The items of part B belong to a repertoire of exercises which appear very frequently in grades 9 and 10. These exercises usually disappear from the 11th grade repertoire. Thus, the ability to discover hidden algebraic structures

decreases and stabilizes somewhere between the level of the 8th grade and the level of the 9th grade. The same picture is discovered if you look at the means of part A and part B together (Table 5). The above arguments also explain Tables 4 and 6. Namely, immediate experience with algebraic expressions improves the ability to identify hidden algebraic structures but the moment this experience stops, the ability decreases and stabilizes quite close to the point of its function. Note that we analysed our data as if it were developmental data whereas, what we really did was comparing different groups of different ages. This is quite common in educational research when comparing age levels and it is based on some reasonable assumptions, however it should be noted.
In order to neutralize the effect of immediate algebraic experience on the ability to identify hidden algebraic structures, perhaps a different and more sophisticated research design is needed.

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# A framework for understanding what Algebraic Thinking is

Romulo C. Lins
Shell Center for Mathematical Education / CNPq-Brasil

On this paper a framework (the **Numerical-Analogical** framework) is proposed in order to provide a reference for investigations (both theoretical and experimental) on the nature of Algebraic Thinking. The framework is described and its adequacy is demonstrated by examining: experimental evidence from students' work (both new and previous findings), the historical development of algebra and algebra as a subject-matter in Mathematics. A characterisation of Algebraic Thinking on the basis of the Numerical-Analogical framework is provided. The belief that Algebraic Thinking can only happen in the context of algebraic symbolism is shown to be erroneous and misleading.

> *"But neither of them was able to prove the theorem, and Waring confessed that the demonstration seemed more difficult because no* notation *can be devised to express a prime number. But in our opinion truths of this kind should be drawn from notions rather than from notations"*
>
> C.F. Gauss, on Wilson's theorem, in *Disquisitiones Arithmeticae*

## 1. Introduction

Until now, a substantial amount of information has been gathered on the learning of school algebra (eg, Collis,1982;Küchemann,1984; Wheeler & Lee,1987; Bell,1987), but nevertheless, a clear characterisation for "Algebraic Thinking" is still missing (Kieran,1989; Lee,1987).

As a whole, that research has been strongly focused on investigating Algebraic Thinking as the mode of thinking that goes with "doing algebra" (either interpreting or manipulating algebraic statements or using algebra to solve problems and explore situations), rather than the mode of thinking that allows the development of algebra.. A consequence of this "content-driven" approach is that the students' "informal" solutions have been characterised more in terms of misinterpretations and failure to "understand" and less in terms of what they are actually doing .

L. Booth suggested that the sources of those misunderstandings (or lack of understanding, as it might be more adequate) are to be found in an incompatibility between the "informal" methods used by the students and the methods of algebra rather than in developmental obstacles (in the sense of Piaget) (see, for example, Booth, 1984). We strongly share this point of view, and investigating the nature of those "informal" solutions, at the same time we investigate the **nature** of "algebraic" solutions, has been the central objective of a set of studies carried out by the author for the last two years, aiming at identifying possible source(s) for that incompatibility.

A framework that helps us to understand the twofold nature of this question, is one that enables us to handle the different meanings that can be attached to the elements involved in the situation that is being dealt with by the students: numbers, operations and arithmetical and algebraic symbolism (where they are involved), but also the imagery suggested or provided by the situation or used as a support for reasoning (the context of "realistic" problems, diagrams, etc). In speaking of "meaning" we are inevitably led to **referentials**, and this is what our framework has to provide in the first place: a description of **different fields of reference** in which **different interpretations** of those elements produce **solutions of different nature**.

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...er is a result of the work being carried out by the author as part of his PhD studies, under the ...on of Dr Alan W Bell, at Nottingham University.

A first important consequence of thinking in terms of distinct fields of reference within which the elements of a situation are interpreted, is that our approach is not content-driven: the same framework can be applied to the analysis of solutions of "realistic" and "purely numerical" problems, problems set in algebraic language and "verbal" problems. Also of considerable importance, such framework can be applied to the analysis of the algebra of the "ancients", and this might shed some new light onto a possible paralellism between the historical development of algebra and the aqcisition of algebraic thinking by individuals.

In the next four sections such a framework is sketched and support for its adequacy is drawn from three sources: the historical development of algebra, algebra as a theoretical discipline and empirical evidence from investigations on students' solutions. On the last section we return to the framework and its characteristics are fully described.

## 2.THE NUMERICAL-ANALOGICAL FRAMEWORK

Our framework distinguishes between two basic fields of reference: the **Numerical field of reference** and the **Analogical field of reference**.

To operate within the **Numerical field of reference** means that only the "arithmetical" environment is relevant to the process of manipulating or exploring a situation. If it is the case of solving a problem, the problem is solved through the manipulation of the numerical relationships contained in or described or allowed by it, and this process is guided by the arithmetical structure of those relationships and by the principles that are recognized as governing the arithmetical environment.

To operate within the **Analogical field of reference** means that a situation is manipulated or explored by manipulating features of the situation itself. Arithmetical operations are used to **evaluate** parts, and the choice of operation to be used is made on the basis of a **qualitative** analysis of the situation or problem that is being examined.

The framework we propose here has two fundamental characteristics:

(i) it rejects the idea of a "**pre**-algebraic" mode of thinking, something that when extended or further developed leads to an "algebraic" mode of thinking; we use instead the idea of a "**non**-algebraic" mode of thinking; the "meaninglessness" pointed out by students is interpreted not in terms of the "meaninglessness" of algebra itself , but in terms of the *shift of referential* that is necessary to operate within the Numerical field of reference.

(ii) the N-A framework is concerned with the process of solution, not with the problems to be solved or the situations to be structured. As a result, the use of algebraic (literal) notation does **not** characterise any of the two modes. Although solving a "purely" algebraic problem using algebra (eg, formally solving an equation written in symbolic notation) is certainly an activity that develops within a Numerical field of reference, the same "purely" algebraic problem might be solved within an Analogical field of reference (for example, modelling it with a scale balance). Also, the general description of the number of, say, dots on a geometrical pattern "using letters", for example, is typically Analogical, because the choice of operations to be used in the description depends only on the way in which the pattern is visually perceived, but a "purely arithmetical" problem can be handled in a typically Numerical way (eg, $[157+157+157+157+157] \div 5 = 157$ because there are five 157's, etc.).

## 3.From the historical development of Algebra

Westernly, the historical development of Algebra has been referred to as a succession of three phases: rethorical, syncopated and symbolic (Joseph, 1988, is a brief but excelent appraisal of Eurocentrism in Mathematics). The first phase is associated to pre-greek 'algebra', the second with the work of Diophantus and the third with the work of Viete and Descartes. (eg, Hogben, 1957). This description clearly corresponds to a development of algebra as a subject-matter, given our modern definition of Algebra as a form of "symbolic calculation", and this is thoroughly expressed on the usual assertion that Viete was the first to produce "truly" algebra.

Jacob Klein's work (Klein,1968, originally published between 1934 and 1936) ' departs from this line of analysis. It shows, based on a deep reading of Greek classical d on a careful study of Viete's work and of the cultural and conceptual context surrounding him, that Viete's deeper achievement was not simply the development of a symbolic notation (his, after all, was to some extent still "syncopated" and full of geometrical suggestions...), but shifting algebra from "solving problems" to "a method for solving problems" . Viete himself comments on his work saying "TO LEAVE NO PROBLEM UNSOLVED". The way in which Viete achieves his goal is by bringing the solution of the problems entirely into the context of numbers and for this reason his work is about how to proceed within a (general) numerical context. Klein's work, however, does not consider similar developments outside the Diophantus-Viete axis.

The work of arabic mathematicians from al-Khwarizmi (c.800) onwards share the same Numerical character of Viete's, and if in many instances careful attention is paid to the process of 'translating' the problems into a suitable Numerical form (Rashed,1984, p20), this does not mean that "solving problems" was the 'raison d'etre' of their work . In fact, the arabic algebra extends itself over "algebraic" powers, operating with polinomials, normal form of an equation, polinomial equations of higher degree, and a number of topics in Number Theory, a body of knowledge that makes Viete's "Introduction to the Analytical Art" look like a first book in school-algebra . **It has to be stressed however, that until at least the 12th century the arabic algebra is totally "rethorical", and even the work of al-Qalasâdi - 15th century - is still in a "syncopated" form** (for example, the use of distinct symbols for x and $x^2$).(Cajori,1928, items 115,116,118,124)

The nature of the mode of thinking that generates such knowledge is partially explained in the words of an arabic mathematician – As Samaw'al (12th century) – who said that algebra was concerned with "...operating on the unknown using all the instruments of arithmetics, in the same way in which the arithmetician operates on the known [values]" (Rashed, p27). This comment is better understood in the context of the process of "arithmetisation" which algebra underwent after the pioneer work of al–Khwarizmi, a process that consisted in restricting the methods of algebra to those of "arithmetics" (Rashed, p32, but also analysed in many other places in the book. It is particularly interesting to consider the link that Rashed establishes (p25) between al-Khwarizmi restricting himself to equations of the 1st and 2nd degrees and his conception of proof [to a great extent geometrical]). The process of "arithmetisation" undergone by algebra in this period corresponds, in the context of the epoch, to the process of "abstraction" that algebra underwent during the 19th and 20th centuries: the substitution of a collection of procedures for solving "classes" of problems (later: a collection of results about specific systems, "arithmetical" and "non-arithmetical") by a method that allows us to attack problems in any of those classes (later: an "abstract" system the results from which can be applied to all those particular instances of systems). Algebra becomes an autonomous discipline (later: Abstract Algebra becomes an autonomous discipline).

A less explicit – but equally distinctive – aspect of the arabic algebra, is the fact that once a "contextualized" problem is represented in terms of arithmetical relationships, the process of solution develops entirely within the Numerical field of reference. It is for this reason that careful attention is given to the process of "translation": from that point on, the "context" would not provide a source of reference; if the arithmetical relationships do not accurately correspond to the problem, the algebraic method could not detect the mistake and the Numerical process of solution would result in a waste of time (to say the least). This "internalism" is made possible by the development of algebra as a "theoretical" discipline (Rashed, p20) – already clear in al-Khwarizmi's use of normal forms of equations – at the same time it makes possible further developments in algebra. As Klein points out throughout Part II of his book, this kind of "internalism" was not possible in Diophantus, especially because of his conception of number (the conflict between the "pure" number and the "number of things" and the concept of *eidos* as the only possible form of "general number").

Those two principles – "arithmeticity" and "internalism" – are also characteristic of Viete's work, and to such an extent implicitly taken by him that they become almost transparent by staying always in the background of the symbolic invention. However "hidden", these are exactly the principles that support Viete's creation of a "symbolic calculus". (for those who wishfully think that Vieta's algebra is totally context-free, let us remember that he had different symbols for subtractions where one number was known to be greater than the other and subtractions where this was not known)

What becomes evident with this picture in view, is that a content-driven approach to [...]nding the Algebraic mode of thinking leads us to miss the point that the "symbolic calculus" of algebra was but a consequence of the development of a body of knowledge that already embodied the calculus (*hisab*, for al-Khwarizmi) that is **progressively** made "symbolic".

We think that it is totally adequate, then, to characterise **Algebraic Thinking** as **the mode of thinking that produced – from the arabic mathematicians on, to our knowledge – the "theoretical" discipline we know as Algebra**. As a consequence, "arithmeticity" and "internalism" are features of thinking algebraically. As we said before, "abstraction" would replace "arithmeticity" in a more general characterisation, but we will keep the latter for two reasons:

(i) Our primary interest is in the development of an algebraic mode of thinking; school-algebra is an algebra of numbers, as Algebra was for a very long period of time ;

(ii) We think that by using "abstraction" one reinforces the idea of an absolute "lack-of-meaning", which we deny as misleading.

## 4. From Algebra as a subject-matter in Mathematics

A simple way of defining Abstract Algebra is to say it is "the study of algebraic systems", an algebraic system being composed by a set, one or more algebraic operations defined on it and a set of axioms which have to be satisfied by the operations. An algebraic operation on a set A however, is a function from $A^n$ onto A, and this means that the set A is mentioned separetely not because its elements are relevant in any sense, but because we want all the operations to refer to the same set. This is, in a sense, the result of the evolution of the "internalism" mentioned in the previous section: the operations are defined internally and they all refer to same set of elements; no other reference is needed. Because we do not want to refer to anything else "external" (particular), the elements are "abstract", and the only way to do any kind of manipulation within this system is on the basis of the properties of the operations. This allows us generality, as operations are "globally" defined. In a very similar way, if one is solving an equation in a "purely numerical way", one has to do it on the basis of properties of the arithmetical operations.

This characteristic of Algebra means that **in Algebra operations become objects, ie, they are a source of reference, they have properties**. This is true both for "number algebra" as it is for Abstract Algebra.

When dealing with school-algebra, it is usually useful to think in terms of operators (eg, "+2") instead of in terms of bynary operations (Kirshner, 1987), but this does not essentially alter our point, because the operators are built from the arithmetical operations. Moreover, as a consequence of Algebra being used as a method, ie, generally applicable, we are left in fact with only four arithmetical operators (viz., **+a, -a, xa, ÷a**).

This analysis of Algebra as a subject-matter helps us to understand an aspect central to much of the discussion about Algebraic Thinking: that of meaning.

When a problem or situation is modelled in terms of arithmetical relationships, the objects that provide information on "what can be done to manipulate those expressions" are, as we saw, the operations and their properties, this corresponding to an algebraic treatment. On the other hand, when an Analogical model is used the numbers are associated, as "measures" (or operators operating on "measures", eg, "3 buckets"), to some other object; if one is dealing with a "purely numerical" problem, the numbers might be associated, for example, to parts and wholes; those other objects and their "qualitative" structure are the elements which provide us with information on "what can be done to solve the problem". One knows which operation to perform and with which numbers because each operation corresponds to an evaluation and the numbers are "attached" to the parts involved.

What is "lost" in a Numerical process of solution is exactly this Analogical reference on "what to do with the quantities", and this is the meaning of "meaningless" that could be applied to an algebraic solution. ("it is meaningless" <=> "I can't see how those elements tell me this is what I should have done")

## 6. The N-A framework and research on Algebraic Thinking

(I) Harper (1987) analysed solutions to the problem "If you are given the sum and the [illegible]e of any two numbers, show that you can always find out what the numbers are", and identified three groups of answers that correspond to "rethorical" (totally verbal), "Diophantine" (or "syncopated"; symbols only for the unknowns) and "Vietan" (or "symbolic"; symbols for the unknowns and for the given [general] values) answers.

One has to notice however, that all three kinds of solutions are general, in the sense of being generally applicable to any sum and difference given **and they are thus undistinguishable from that point of view.** Moreover, Viete's answer to the problem (p88) totally corresponds to the "rethorical" answer presented on p81, apart, of course, the use of letters (and this is correct even to the extent that Viete's answer is $^1/_2$D – $^1/_2$B and not $^1/_2$ [D-B] ). Whenever a correct "rethorical" answer is not accompanied by an explanation as to how the result was obtained (as it is the case with the one presented on p81), one has to consider that the process of "thinking out the problem" (p80) could correspond to anything, including Viete's method.

The important point here is that although lacking symbolic generality, "rethorical" and "Diophantine" solutions might eventually involve much of the same mode of thinking that a "Vietan" solution does (we emphasised the "eventually" because an Analogical solution is also possible on all three 'styles' ).

Harper's classification of solutions is certainly useful to describe differences in the use of mathematical symbolism, but by itself it does not provide a framework that enables us to distinguish different **modes of thinking.**

As a consequence we are again led to the necessity of a framework that takes into consideration the ways in which solutions are produced, ie, which are the sources of reference used, and this is exactly the focus of attention of the N-A framework.

(II) Lesley Booth's follow-up study of the CSMS survey (Booth, 1984) produced a number of important findings. Although primarily concerned with situations that involve the use of letters, Booth's conclusions point out to the necessity of understanding children's "informal" methods if we are to understand the nature of the gap between non-algebraic and algebraic modes of thinking.

Of particular interest to us is her characterisation of the "child methods"(p37): "(1) intuitive, ie, based upon instinctive knowledge: not systematically reflected upon and not checked for consistency within a general framework; (2) primitive, ie, tied closely to early experiences in mathematics; (3) context-bound, ie, elicited by the features of the particular problem; (4) indicative of little or no formal symbolized method; (5) worked almost entirely within the system of whole numbers (and halves)".

If those "methods" are seen as based on a qualitative analysis of the situation presented (an Analogical approach), the first four characteristics follow as a consequence: context-bound because the solution depends on understanding a particular situation and the possibility of manipulating its elements to perform evaluations; non-systematiized because of the obvious "one-off" (or even "few-off") character of the solutions; intuitive because non-systematic, but also probably because the knowledge required to perform the qualitative analysis is not seen as mathematical knowledge; little or no symbolization both because the strategies actually used to "think the problem out" – comparing, decomposing and recomposing wholes, for example – are easily and accurately described by verbal statements, and because "thinking the problem out" (using the strategies) is of a different nature than "working the problem out" (the actual evaluations, the performance of the operations). Symbolic notation might be used to describe but this does not contribute to the process of solution itself . [This is not the case with a Numerical solution, because the operations are at one time the source of reference and the instruments used to manipulate the information: a concise and homogeneous notation which is intended to be manipulated is adequate and possible] Those "methods" are primitive because the operations can retain their original role, that of being **tools for evaluation**. (the latter idea is also conveyed, in a slightly different form, in the assertion that children see operations as "something to be performed" [Booth,*op. cit.*, pp90-91]).

Three of Booth's research findings (pp85 and following) also provide evidence that an Analogical approach is probably preferential to those students (the item numbers correspond to the original text):

(1.c) " Some children are confused over the distinction between letters as representing (s) or number(s) relating to a measure or object, and letters as representing the measure itself. ...". From the point of view of the N-A framework, this could be interpreted as a consequence of the students operating Analogically, ie, as the numbers are "numbers of things" and as those "things" are the source of reference on what to do or on how it works (more specifically, the qualitative structure involving those "things"), it would be more natural to represent primarily the "things" and not the numbers that correspond to them.

(3.b.i) "The context of the problem determines the order of operation" and (3.b.ii) "In the absence of a specific context, operations are performed from left to right". Those two points indicate the extent to which the operations are **not** constituted as **objects** and their use remain subjected to other sources of reference.

(III) On a exploratory study carried out in Nottingham, England, in 1989 and reported in Lins(1990), two groups of 3rd year secondary school students and a group of 4th year primary school students were asked to solve a set of five "verbal" problems and to explain why they did it that way. Both correct and incorrect solutions, together with the explanations, were then analysed to determine – whenever possible – the source(s) of reference used by the students to work the problems out.

**Two of the problems used:**

Carpenter: The stick on the top is 28cm longer than the one in the bottom; altogether they measure 160cm. How long is each of them?

Buckets : From a tank filled with 210 liters of water I took 3 full buckets: Now I have only 156 liters left. How many liters go into a bucket?

The analysis showed that in many cases the solutions were Analogical (eg, "to take 156 away from 210 to determine how much was taken by the 3 buckets" on BUCKETS), but it also showed that in those cases where only the calculations were provided they corresponded – in all but one instance – to those that would be used with the simplest Analogical solution (for example, when solving the Carpenter's problem, to begin with 160–28 but not with 160+28 and never representing the difference as the result of a subtraction [as in **x–y=28**] ). The overall result of the analysis suggests that: (i) the use of an Analogical approach, as we define it, is experimentally generally verifiable, and (ii) those students used mainly an Analogical approach.

The following fragments of an interview from another study (Laura , 10yrs5mths) provide a clear example of the use of an Analogical approach: (the problem is "George and Sam have £1.60 altogether, but Sam has 38p more than George does. How much does each of them have?"; the emphasis on the transcription is ours)

**Int**:... how did you know that you had to take 38p away and not to add 38p?
**Laura**: If you added 38p... then... ahnn... if you added 38p then you wouldn't have, ahnn... you would have more than £1.60 to start off with... and it says you only have £1.60.
..............
**Int**: But if you take 38 away, then you have less than you had...
**Laura**: yeah... I think I was just trying to get the 38p out of the way for a bit ! and then...

Features of the situation act as constraints and source of reference in the process of solving the problem.

(IV) In another study, we investigated the sources of reference used by six postgraduate students in the University of Nottingham to validate given symbolic representations as correctly describing a verbally given situation (a brief discussion is in Lins,1988). One of them was the well known "students and professors" situation ("In a school there are six students for each professor,...", etc.). Two basic strategies were identified: (i) always to refer back to the verbalised situation, and (ii) to determine one correct symbolic representation and from it to derive the correctness or incorrectness of the others on the basis of algebraic manipulation. One of the students, who otherwise always referred back to the text and adopted as correct the – "wrong" – representation 6S=P, when faced with the item 18P=3S simply divided both sides by 3 to obtain 6P=S and concluded it was not in agreement with the verbal description. Moreover, she proved quite able to solve formally set equations and had no difficulty with the CSMS item "... is greater: 2n or n+2". The information gathered by this exploratory study suggests that ... Analogical approach (in that above case modelling the situation by putting "blocks" into correspondence) is **not** necessarily the result of an inability to deal with "unclosed" or "symbolic" expressions, but rather the result of structuring the situation using a referential that is different from the referential that would produce a representation in terms of arithmetical relationships.

(V) Friedlander et al.(1989) investigated, among other things, differences between visual and numerical justifications, a distinction that corresponds – in the context of the problem analysed, a "geometrical" problem – to our N-A distinction.

## 7. Conclusion

The N-A framework was developed as part of our effort to provide a clear characterisation of Algebraic Thinking. On its foundation is the assumption of two distinct fields of reference (**Numerical** and **Analogical**).

| Numerical | Analogical |
| --- | --- |
| Operating within the Numerical field of reference means that only the arithmetical structure is relevant. | Operating within the Analogical field of reference means that the relevant information is provided by the "qualitative" structure (eg, bigger/smaller, decrease/increase, wholes/parts). |
| the objective of any manipulation is to derive new arithmetical relationships that, because of its form, bring with it new information about the initial relationships; In doing so, one is guided exclusively by the operations involved and their properties. **Operations can have properties because they are OBJECTS.** | the objective of any manipulation is to make **evaluations** possible; this is done through the manipulation of the elements of the situation; comparing wholes and decomposing wholes and rearranging the parts thus obtained are typical Analogical strategies. **Operations are the TOOLS with which the evaluations are carried out.** |
| because the guiding principles apply irrespective of the particular arithmetical structure dealt with – with the few canonical exceptions that also apply to arithmetics, like division by zero, etc – **operating within the Numerical field of reference has a strong character of method**; meaning belongs thus to the process as a whole. (A METHOD TO SOLVE PROBLEMS) | operating within the **Analogical field of reference** is an activity bound by the specific "qualitative" structure, and thus presents itself as **a procedure**; meaning belongs to each step of the solution process, as the "qualitative" structure changes with each new evaluation. (TO SOLVE A PROBLEM) |
| Limits of the context are taken as limits for the answer but not for the process of solution | Limits of the context are taken as limits for the process of solution |

Other important general features of the N-A framework are:

(1) The use of symbolic notation is not characteristic of operating within any of the two fields of reference; nevertheless, a symbolic notation that is intended to be manipulated is possible and adequate when operating within a Numerical field of reference but not when operating within an Analogical field of reference.

(2) The central distinction being made is between ways of interpreting the elements of problems and situations and **not** between the consequences of different interpretations;

(3) It avoids the idea of "pre-algebraic" and "algebraic" modes of thinking that is inherent to the content-driven arithmetical-algebraic distinction; this offers us a perspective of analysis of the learning process different from that of developmental stages.

From the point of view of the N-A framework, Algebraic Thinking is naturally defined as the mode of thinking that enables one to operate within the Numerical field of reference.

Nevertheless, Algebraic Thinking applies to fields of reference other than the Numerical (applied to sets it might lead for example to Boolean algebra); for this reason it is adequate to use Numerical instead of Algebraic field of reference, once we are examining the development of Algebraic Thinking in the context of school-algebra, which is certainly an algebra of numbers.

Also, algebra being the study of the properties of an algebraic system (as defined in section 3) Algebraic Thinking is the mode of thinking that leads to the development of algebra, and the symbolic system that corresponds to the calculus embodied in the ideas of algebra is a possible consequence of thinking algebraically, **not** a characteristic of it.

The N-A framework enables us to examine the development of an algebraic mode of thinking in more depth, both because it links Algebraic Thinking to a field of reference (and then – as a consequence – to what is possible and necessary when thinking algebraically) and because non-algebraic thinking is characterised in itself and not as "inability-to-think-algebraically". This positive characterisation of a non-algebraic mode of thinking is essential if we are to understand the "misconceptions", "failures" and "rejections" related to the learning **and** use of algebra. Also, the N–A framework provides a non-circumstancial explanation for the inadequacy of "algebra as a language", by exposing the impossibility of a "translation" producing by itself the required *shift of reference* that takes one into the Numerical field of reference.

Because Numerical and Analogical fields of reference are distinct, operating within one of them cannot be reduced to operating within the other. This means that each of them provide distinct approaches that are more or less adequate depending on the task in hand; non-algebraic approaches are not weaker *a priori* (see, for example, Janvier, 1989 and Fischbein,1988) and the fact that this conclusion follows from the way in which our definition for Algebraic Thinking is built is certainly an indication of the its adequacy.

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# DEVELOPING KNOWLEDGE OF FUNCTIONS THROUGH MANIPULATION OF A PHYSICAL DEVICE

Luciano L. Meira

*Mathematics Education*
*University of California, Berkeley*

**Abstract**

*In this paper, I discuss the usefulness of algebra instruction that provides students with dynamic physical systems as models of algebraic notations, and a curriculum that profits from their intuitions about mechanisms and causality. I analyze one student's emerging understanding of linear functions and algebra as he uses mathematical concepts, principles, and symbols as modeling tools to explore a simple winch machine.*

## Introduction

By the end of middle school, children are typically introduced to new levels of mathematical abstraction in the study of algebra and functions. Current mathematics instruction at that grade level too often over-emphasizes *symbol manipulation* in ways that obscures children's understanding of the objects, both mathematical and concrete, that the symbols are about (Kaput, 1987; Schoenfeld, in press; Greeno, 1988; Brown, Collins & Duguid, 1989). *Physical referents* of mathematical abstractions are typically overlooked, under the claim that symbol manipulation promotes robust "context-free knowledge." This study examines learning of algebraic functions fostered by physical operations on a mechanical winch. I will argue that mathematics instruction characterized by manipulation of dynamic physical systems provides students with a sense of mechanism and causal relation that facilitates learning (White & Frederiksen, 1989), and that helps to engage learners in meaningful mathematical activity.

The analysis describes one student's emerging understanding of linear functions and algebra as he uses mathematical concepts, principles, and symbols as modeling tools to explore a physical event. The activity discussed in this paper involves the student's attempts to write equations that model the functioning of a simple "winch machine."

## Background to the Analysis

The subject was a 12 year old 7th grade student, named CC. Nearly 5 hours of interviews were conducted in which CC solved problems and manipulated a winch mechanism. The device consists of two spools fixed in a common axle (Figure 1). As the axle is turned, the spools drag small blocks, labelled A and B, along a numbered track. The spools circumference and the blocks initial positions at the track can be set to several values. Mathematically, the relation between block position and number of spool turns map to the function $y = mx+b$ as "Position$_{(final)}$ = Spool * Turns + Position$_{(initial)}$."

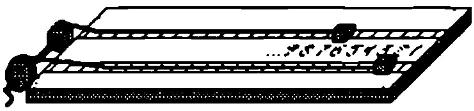

Figure 1 - The winch mechanism.

Before the study, CC had done some simple work solving one or two-step equations of one variable, but had not studied intensively either word problems or modeling of the type described here. The following is an example of the problems in the learning curriculum: "[The equations "embodied" in the winch were A: $y = 4x+8$ and B: $y = 6x+3$] *Would there ever be a point at which block B is ahead of the other block? (If 'yes':) After how many turns? (If 'no':) Why not?"* The curriculum did not include teaching interventions that explicitly dealt with topics such as formal algebraic structures or strategies to record and/or interpret experimental data. A micro-developmental analysis of the student's work on the learning curriculum was conducted. Pre and post-tests were employed to contrast the student's incoming and final knowledge states. All sessions were video taped.

Protocol analysis focused on obtaining a microscopic trace of the understandings developed by the student. The following activities were considered: (1) generating equations, graphs and tables; (2) handling the physical device; (3) describing properties and relationships observed in the device. Two questions guided the analysis of the reasoning processes underlying the genesis and evolution of conceptual understanding:

(1) What aspects of the situation did the subject represent? and

(2) How did these representations evolve?

The segment of protocol discussed below focuses on the student's use of algebraic notations to record and interpret a physical event. The analysis illustrates mathematics understanding as a gradual process that depends on connecting pieces of physical, arithmetic and algebraic knowledge constructed in activity.

## Analysis

The child first worked a winch word problem (which served as a pre-test) and wrote a symbolic expression designed to capture the situation described. He then worked on the physical device solving many practical problems and managed, by means of the dialectic between his naive algebraic knowledge and his perception of the device, to evolve an expression close to the correct equation. I describe below the episodes that formed the basis of the subject's initial grasp of algebraic notations and meanings in the physical situation.

In the pre-test, CC correctly solved winch word-problems that described a scaffold used by window cleaners on a building. The problems were similar to those used in the learning curriculum. His solutions were empirical, using number sequences. The underlying equation in the described situation was $\underline{y = 3x+2}$. The subject was then asked to "*write an equation to show the relationship between number of spool turns and height of the board in each scaffold.*" The following summarizes his answer:

> CC: "I know I just have to add 3 onto the next answer... (Writes $\underline{n+3 = n}$) because on my 6th spool turn I found that it was 20 meters high, and so with that 20 meters I add another 3 meters and that's my next answer."

In the following session the student worked with the actual winch machine, which was set up to embody the equation $\underline{y = 4x+8}$, correctly solving many practical problems. For example, when given the question "*How many turns will it take for the block to be at the 72 mark?*" the subject mentally computed 72 minus 8 equals 64, estimated 16 as the number that multiplies 4 to yield 64 (writing down '4*16 = 64'), and gave 16 as the answer. Requested then to write an equation to show the relationship between "*number of turns and where the block is at*", he wrote **4n = n**:

> CC: "Four times the number of turns you want to go/ should I put n in there?... n, equals n. (It goes) 4 inches every turn, so 4 times the number of spool turns equals the answer."

Note that the use of a multiplicative relationship is already an advance over the answer for the word problem (n+3 = n). Note further that the use of the same literal, n, with different index values in the situation did not seem to confuse CC, as he managed to keep distinct the assigned meanings. The problem, however, is to specify the assigned meaning to what he names "the answer." Given the problem context at this point and other segments of the protocol, "the answer" seems to indicate 'how far the block goes.' Yet, it remains to be known whether the subject meant 'displacement' or 'position' of the block.

There also seemed to be important links between the equation 4n = n and the arithmetic procedures used to solve practical problems with the winch. Immediately following the segment above, CC provided the explanation transcribed below:

> CC: "(...so 4 times the number of spool turns equals the answer) ...just like down here I did 16 times 4, this will be/ the 4 is right there (points to 4 in '4n = n'), 16 is the n (first from left to write), equals 64 and that's the answer."

Having interpreted the reference as including the whole procedure (72-8 = 64; 4*16 = 64), I pointed out to CC that his equation did not include the subtraction operation. He then proceeded to revise the equation:

> CC: "Oh, yeah! ...N would be the place where it's starting, minus (writes **4n = n-n**)... wait, you have to do it first... in order to find how far we want it to... I'm thinking if I put minus n, the n would be how far it starts out at... you'd have to find out how far it would go to be able to minus how far that is..."

Not satisfied with 4n = n-n, he then suggested the expression **4n = n+n**:

> CC: "I just found that if you added like... This (4n =n) gives you the answer of how far it would go like this (points to 4*16 = 64), but then we could add the place where it starts; put another n right there (completes expression 4n = n to 4n = n+n) and that's the number of inches times the number of turns you want to go,

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equals the number that it's gonna go, and then... in order... well... and then add the 8 (block initial position)... to it, and that's your answer. Because like this (4*16=64) it went 64 inches but since it started at 8 it went 72 inches, because you added the 8 to the 64."

LM: "You said 'that's your answer.' What the answer is? Is it the number of turns? What is it?"

CC: " Where it would stop; where it would be."

At this point, CC seemed to realize that he had focused on distance travelled (as in $4n = \underline{n}$ or $4n = \underline{n}-n$) rather than final position (as in $4n = \underline{n+n}$). Indeed, he then made the first spontaneous reference to final position:

CC: "(4n = n+n) is 4 times the number of turns, equals how far it will go plus how far it started off, and that gives you where the block will be at."

This reasoning appears to be robust and sensitive to the situation, though misleading from a strictly formal stance. The subject is then asked to "*check with the apparatus whether the equation 4n=n+n works*." He turns the spool 6 times, which makes the block arrive at the 32 mark. His reaction is transcribed below:

CC: "Ok, so it's at 32, and then... so you did, 4 times my 6 equals thirty... yeah, no... yeah, 4 times 6, no... equals 24!... and then... you added the 8 and that's 32..."

LM: "What are you thinking?"

CC: "...I said 4 times 6 turns equals 24; but I want it to 32, in 6 turns... but wait... this equation can work with this (device) but you have to say you did the 6 turns and then added 8 on; so it's like saying 4 times 6 and you added 8 to it... it doesn't seem right, because it started at 8!"

In this segment, we see a clash among the student's understanding of equations, of the arithmetic procedures that worked in practical problems, and his model of the physical mechanism. The equation is then rewritten as $\underline{8+(4n) = n}$. This time, the last 'n' in the equation is labelled "*the overall answer*." Note the match between the position of terms in the equation and the sequence of states and events in the physical device: (1) block starts at position 8; (2) handle linked to a 4 inch spool is turned n times; (3) block arrives at position n, the "overall answer." We observe that, save for the use of 'n' to represent both the number of turns and where the block finally lands, his final equation is correct. Moreover, it was generated by a heavy reliance on the mechanism of the situation, which did not appear in the paper-and-pencil winch problem in the initial test.

The change in CC's understanding of the modeling task can also be detected through a contrast between his initial and final assessment tests. Figure 2 shows CC's answers to the scaffold word problem discussed earlier. The question read as follows: "Draw a graph and write an equation to show the relationship between number of spool turns and height of the board in each scaffold (A and B)."

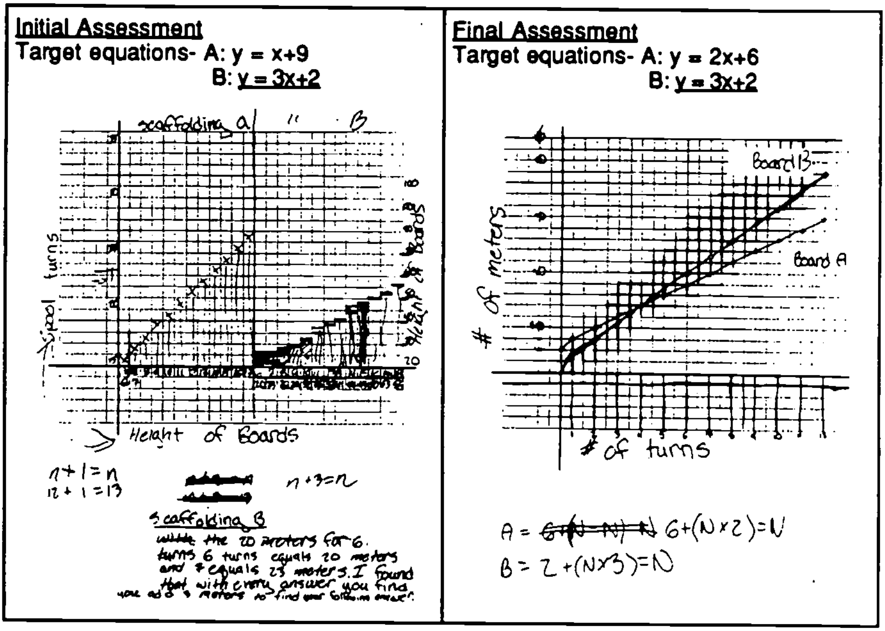


Figure 2- Pre and post-test answers to the scaffold problem.

Two advances are notable: (1) CC's graphs in the final test are far more comprehensible and sophisticated than in the initial test; in particular, he has evolved from bar to line graphs based on data points inferred from the described situation; (2) he is clearly able to apply the algebraic knowledge developed during the study and analyzed above —e.g., compare $\underline{n+3 = n}$ to $\underline{2+(N*3) = N}$.

## Conclusion

In the segments of protocol presented, we observe three elements of the subject's evolving understanding of the situation: (1) a mental model of the physical mechanism, inferred from his overt simulations of the functioning of the device or verbal accounts of its mechanism; (2) arithmetic knowledge, in the form of calculations of unknowns in specific problems directly involving the apparatus; (3) algebraic knowledge, used to annotate quantitative (and physical) relationships observed in the situation. The excerpts above present CC's mathematical understanding as constituted of pieces of physical, arithmetic and algebraic knowledge.

I interpret the subject's evolving algebraic knowledge as fostered by his perception of the physical winch. The device provided the means by which CC could manipulate quantities (as opposed to symbols) and test his intuitions about patterns and algebraic structures. This case study lends support to White & Frederiksen's (1989) claim that science learning proceeds from an understanding of causal principles:

> "The evolution of knowledge can be captured as a progression of increasingly sophisticated causal models that are qualitative early on but that can later be mapped into quantitative models as students' understanding progresses." (p. 94)

School mathematics too often over-emphasizes symbolic manipulation and the symbol systems taught have only other symbols as referents. As an alternative for algebra instruction at the middle-school level, I suggest the value of dynamic physical systems as powerful aids in promoting students' understanding of symbol systems and concepts.

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## Students' Interpretations of Linear Equations and Their Graphs

**Judit Moschkovich**, University of California at Berkeley, U.S.A.

*This study examines data from two algebra classrooms identifying common student interpretations of linear equations and their graphs. These interpretations are consistent with previous research in this content area, and arise even after direct instruction and experience graphing lines. Peer discussions of these alternative interpretations are also analyzed for evidence of resolutions of alternative conjectures. Lastly, suggestions are made for iencouraging the transfer of authority fror. the teacher to peer discussions.*

### Introduction

Within a constructivist framework, the mistakes students make as they learn mathematics are a crucial aspect of instruction. As Lampert (1986) puts it, students "need to be treated like sense-makers, rather than rememberers or forgetters" (P. 340), or mistake-makers. Students' alternative interpretations should be taken into account in two ways. First, we need to identify common alternative interpretations that students generate in different content areas. Second, we must develop instructional methods to address specific interpretations. This paper explores how students interpret linear equations and their graphs in different ways than experts do. Previous research shows that student interpretations include: changing the y intercept moves lines horizontally (Goldenberg, 1988), and using a three-slot schema which includes the x intercept for equations of the form y=mx+b (Schoenfeld , Arcavi, and Smith, in press).

Within a Vygotskian framework that views knowledge as socially constructed, classroom discussions of students' interpretations are crucial contexts for students to develop meaning for the mathematics they are engaged in. Many researchers have proposed learning through social interaction, and peer collaboration specifically, as an important element in constructing classroom environments where students can make sense of mathematics (Resnick, 1989; Brown and Pallincsar, in press). Combining aspects of the constructivist and Vygotskian frameworks, this study examines how students in two classrooms generated alternative interpretations of equations and lines, and how their discussions did or did not resolve conflicting viewpoints.

### Linear equations and their graphs

After typical school instruction, students may or may not be able to perform standard procedures such as graphing equations, solving for variables, or changing the forms of equations. However, experts' knowledge of functions extends beyond procedural competence (Schoenfeld et al, in press; Moschkovich, 1989). For example, an expert's view of this domain includes treating curves as conceptual objects which can be manipulated and seeing changes in the parameters of equations as having corresponding changes in the curves. Both of these aspects of expertise are crucial for using functions in advanced mathematics courses such as calculus. Guided exploration of functions, their equations, and graphs with graphing software, as opposed to direct instruction or pencil and paper tasks, has been proposed as a useful tool in the development of this elaborated view of functions (Schoenfeld, in press).

**Subjects and Methods**

The site for classroom observation was an urban high school in California where a pilot version of Math 9, a college preparatory course, is being tested. The students in the two classrooms observed are 9th and 10th graders following a college track curriculum; that is, they are neither remedial nor honors students. The curriculum was designed to encourage exploration, discovery, and discussions of alternative understandings. Thus, the classrooms are an excellent environment for exploring students' active construction of mathematical knowledge through interaction with peers or teachers. The curriculum unit observed lasted approximately five weeks. The chapter included modeling of real world situations with equations and graphs, interpreting graphs, use of graphing calculators and computer software, and student group work with some class discussions. The lessons that will be discussed in detail involved graphing on a calculator (Day 13), producing lines on the computer screen to match lines in a handout using Superplot (Day 14); playing Green Globs, a game where students use straight lines to shoot random globs (Day 15); and graphing on a calculator (Day 16). In each of the two classrooms I observed peer group work and teacher-student interactions. I observed and audiotaped students working in groups during four lessons, and videotaped four pairs of students working on a computer using Green Globs (Dugdale, 1982) and Super Plot. The classroom notes, audiotapes, and videotapes were analyzed in terms of two themes: students' alternative interpretations of linear equations and their graphs, and instances of peer discussions of these interpretations.

**Analysis**

Following previous research on students' knowledge of linear functions (Schoenfeld, Arcavi, and Smith, in preparation), my own pilot work, and recurring themes during these five lessons, I focus on four areas of common alternative interpretations: the role of the y intercept, separating x and y intercepts, slope as location, and separating slope and y intercept.

1. The role of the y intercept is not obvious

In Classroom B, on Day 13 of the chapter, the teacher directed a whole class discussion on the role of the slope and the y intercept in the equations and graphs of straight lines. The students had graphed the equations y=2x+1, y=2x-1, and y=2x on their calculators and the teacher graphed these three lines on the board. The teacher then posed the following question:

Teacher B: Can anyone tell me what the significance was of that number there? (pointing to the 1 in y=2x+1).Does anybody know where this plus one and this minus one came in to play on these graphs? (silence) Can you see it on there?
Student: Yeah...(silence)
Teacher B: To make along story short, there are little blip marks on the x and the y axis, right?
Mt: Yeah
Chorus: yes
Teacher B: Which little blip mark did this graph go through?
M: Two, the second one.
Students: Two and two...
Teacher B: This one you drew, which blip mark did it go through?
Student: Negative two
Chorus: Negative two
Teacher B: Negative two?
Mt: Yes
M: And positive two.
Student: And negative one...
Teacher B: That one went through here didn't it (pointing to (0,1) on the graph)?
Mt: Yeah
Student: Through the middle
Chorus: It went through the middle.

The teacher proceeded to ask the students what y values were produced for different x values. However, the students never resolved the question of what the role of the +1 or -1 in the equations was in the graphs during this lesson. This could be explained by the fact that the students had not yet had enough experience graphing lines, and thus were not yet ready to discover the role of the y intercept. However, episodes from subsequent days show that even with more graphing experience and direct instruction, the role of the y intercept remained problematic.

In Classroom A, on Day 16, students worked on graphing lines with the same intercept or the same slope on their calculators. They were asked to answer two questions in their groups: "What does the number in front of the x do to the lines?", and "What does the number being added or subtracted do to the lines?" The three previous lessons had involved graphing lines with the same slopes or intercepts on the calculator (Day 13), reproducing on the computer computer screen lines given in five exercises (day 14), playing Green Globs (day 15), and a summary by the teacher on how to rotate or translate lines (Day 15). It seems reasonable that by this lesson students should have noticed what the y intercept does, and used this knowledge to either produce lines or hit globs. However, the two students that were audiotaped were again unable to resolve the second question.

<u>2. The x intercept is important (when using the form y=mx+b)</u>

In Classroom B, on Day 15, the students played Green Globs. The teacher's introduction to the activity summarized how to translate lines up and own by changing the b in the equation, and rotate lines by changing the m in the equation. The following is an excerpt from the video transcript of two students who had played five games using mostly horizontal and vertical lines.

(Game 6: M and K have tried the equations: y=x-3, y=x-2, y=x=y-3, and x=y-3.)
Mt: Negative y...OK (he types in the equation x=-y-3 and then x=-y)
K: X minus, y equals x minus....y equals x minus 1,2,3,4 (counts along the x axis and keep s his finger on (4,0)).
Mt: Four...ah yep. Y equals x...
K: Minus four
Mt: X minus...Sure? That won't be up here? (traces a line from the IV to the II quadrant)
K: No..(shakes his head)
Mt: (Types in the equation y= x-4)

In this episode K used the x intercept (4,0) to generate the equation y=x-4. Unfortunately, the line y=x-4 did hit the globs they had selected. The x and the y intercept for the line they wanted to produce were respectively (4,0) and (0,-4). Thus, K's use of the x intercept to generate a line was not challenged by the result on the screen.

After class, on Day 16, I worked with two students (M and C) who had questions about the previous lessons and their homework. For one of the problems they were working on they had generated and graphed three lines on the board: y=x, y=x+3, y=x-2 on the blackboard. I asked the two students if they could write an equation for the top line, y=x+3. One student looked and pointed at the x intercept, saw it was (-2,0) and generated the equation y=-2x. When I suggested using a table of values to check whether that was the right equation, the three of us showed that y=-2x didn't work for the line in question (using a table for y=-2x gave a different line).

<u>3.Slope and location are related.</u>

Returning to Classroom A, Day 16, students D and C tried to determine the role of the m in their equations.

D: Ok, one question. The number before x . Hey, what does the number before x do?
C: See, this is y, right, and then this is x...
D: Yes
C: So y equals x is over here, is five. I don't know how to explain it! I know what it means.
D: Try it
C: See here is 1,2,3,4,5 so it's all the way there and you'll run all the way over there. No, that's not how I would say it. Something like that, I don't know. See like here's the spaces. Here I think I got it.
See here it is. Over here it's like y equal to negative three x, and it's over here, so this side is negative and so this side is positive. So over here it's five, like that.
D: Oh, the number before gives it the side like the positive, it starts from the positive side, right?
C: Yeah, like that!

These two students appear to have figured out something about the sign of the slope. However, they refer to lines as "starting from" somewhere. This is a reasonable result of their experience with graphing calculators and software. On both screens lines are graphed starting from left to right. This means lines with positive slope "start" in the III quadrant, and lines with negative slope "start" in the II quadrant. There is nothing inherently wrong with this, if what they are talking about is a student version of "lines with positive slope rise to the right, and lines with negative slope rise to the left". However, in the subsequent whole class discussion, this is not the interpretation of the slope that D presented, or that the rest of the class supported.

After graphing lines on their calculators and discussing the questions in their groups, one student from each group (7 in all) went to the overhead projector to give their answer to the question "What does the number multiplying the x (in y=mx+b) do?". Three different students insisted that the line for y=5x looked like the line for y=x. They graphed it on the overhead projector by counting from the origin to (-5,-5) and to (5,5) and connecting these two points. Answers to the question included "it tells the calculator where to draw the line", "it tells where it starts from", "it directs where the line should start". When prompted to explain why their lines were y=5x, several students pointed to (-5,-5) as the starting point. None of the students generated any other ways to prove or disprove this proposition. The teacher then suggested checking what y values were produced for different x values when using the equation y=5x. Subsequently, two students used this process to show that the lines they had graphed were in effect y=5x and y=2x.

4. Separating slope and y intercept

In the example presented above in section 2 (Day 16, M and C), not only did the student try to use the x intercept in the equation, she also tried to place the x intercept in the m slot. Thus, separating the role of the m and the b in the equation is also problematic. On Day 16, while two students in Classroom A used Superplot, they also faced the issue of separating the role of the slope and the y intercept. On this day students were given a series of lines to reproduce on the screen (Exercise 1: y=4, y=2, y=-1, y=-5. Exercise 2: y=x, y=2x, y=3x, y=-3x, y=-2x, y=-x. Exercise 3 : y=2x+3, y=x+3, y=.5x+3, y=-2x+3, y=-x+3, y=-2x+3.) Students S and E worked together on exercises 1 through 3. They successfully produced four horizontal lines to match the ones in Exercise , and the six lines for Exercise 2. However, they were stumped when they came to Exercise 3. They asked the teacher for help in taking the line y=3x and "putting it more up", that is changing the y intercept from (0,0) to (0,3). The teacher helped them to notice that all the lines went through the point (0,3). After the teacher left, S and E tried the equations y=3.3x and y=4.4x. The teacher returned and suggested they try "plus or minus something". Next, they tried the equations y=0.3 +5x and y=3+5x. They matched one line in Exercise 3 with this last equation and attempted to change the slope of this line by changing the b (y=3.5 +5x, y=3.8+5x, y=3.7+5x) until they realized that this was not affecting the slope of their lines. They finally moved to trying the equations y=3+x, y=3+2x, and y=3+3x, as the way to translate lines up and down. Thus, for these two students, slope and intercept were initially neither independent on the graph nor did they show up in different places in the equation.

**Peer Discussions**

Some of the students I observed were engaged in "finding the right answer" (Lampert, in preparation). They looked to see what other students were doing for inspiration, and asked other students, the teacher, or the researcher for answers. In their case, conflicting interpretations did not generate peer discussions. Instead, these students accepted conflicting interpretations or results, and moved on to another activity. Other students were attempting to make sense of the mathematics for themselves. That is, they looked for patterns, generated conjectures, propositions, or questions, and searched for explanations.

For example, S and E above discussed every decision they made, attempted to justify their own viewpoints when they disagreed, used the computer to check their conjectures, and then modified their conjectures to be consistent with the computer feedback. In their case, at least at the level of generating equations, the computer feedback played a crucial role in resolving conflicting viewpoints. As far as providing explanations of why equations and lines behaved as they did, however, the computer proved insufficient. For example, while D and Mi where playing Green

Globs (Day 15), they asked each other to explain why lines with positive and negative slopes looked the way they did several times. Neither student attempted to provide an explanation or use the computer to explore this question further.

**Summary**

Expert interpretations of equations and their graphs are packed with meaning. Experts know that the variables and parameters in an equation are relevant in different situations (Goldenberg, 1988) and they know that the m and the b in the equation are the relevant parameters for comparing equations of the form y=mx+b (Moschkovich, 1989). The classroom and peer discussions outlined above show that even with experience graphing lines and some direct instruction, students generated alternative interpretations. The most common ones were: the x intercept is important (i.e. it should show up somewhere in the equation); m and b are not independent (i.e. if you change one in the equation,the other might change in the graph; if you want to translate a line, change m or b; if you want to rotate a l ine change m or b); slope and location are related (i.e. the line for y=5x starts from the point (-5,-5)). Students did not seem to parse equations of the form y=mx+b as y=*x +Δ, that is with m and b as the relevant parameters that rotate or translate lines lines and make the equations different.

Peer discussion was a good context for generating conjectures, but not for choosing between different alternatives. In terms of the examples presented above, the fact that b is the intercept because (0,b) satisfies the equation y=mx+b and (0,b) lies on the yaxis, or the Cartesian connection (Schoenfeld, Smith, and Arcavi,in press), could have resolved conflicting interpretations of the role of the y intercept. Again, using a table of values generated by a proposed equation could have resolved conflicting interpretations of the role of the slope. As students are introduced to the definition of slope as directed line segments, another element of the Cartesian connection between the graphical and algebraic representations, students could also use this piece to resolve conflicts involving slope. These are the sorst of mathematical tools that Lampert (in preparation) suggests "enable students to make arguments of a substantially different sort than they would be able to make without them (p. 17)."

Beyond providing students with specific methods such as these, instruction also needs to provide students with legitimate processes for exploring parameters and choosing between alternative conjectures. As Lampert (in preparation) proposes, doing mathematics and thinking mathematically involves mathematical tools, activities, such as gathering information, organizing it strategically, generating and testing hypotheses, and producing and evaluating solutions, and discourse processes. While mathematical tools are a crucial element of doing mathematics, discourse processes froe evaluating conjectures through discussions are also essential.

The examples presented above are not meant as evidence of the poor performance of these students or teachers. On the contrary, many of the students observed were engaged in "sense-making" (Schoenfeld, in preparation). The teachers encouraged students to talk about their interpretations and conjectures, and tried to address them in subsequent lessons. Students should be expected to construct alternative interpretations, even if these interpretations look mathematically "wrong". Moreover, instruction needs to include not only a discussion of alternative interpretations but also tools, activities and discourse processes for choosing between alternative conjectures. If students are to move from seeing mathematical knowledge as something that the teacher possesses and magically transmits into students' heads to evaluating their own conjectures, we need to consider in detail the activities and discourse processes students can practice to become "authorities" in the process of constructing mathematical knowledge for themselves and with each other.

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# AN EXPERIENCE TO IMPROVE PUPILS' PERFORMANCE IN INVERSE PROBLEMS (*)(**)

A. PESCI, Dept. of Mathematics, University of Pavia, Italy.

*SUMMARY*

*The experimental work intends to improve solution strategies in problems with inverse procedures with 11-12 year old pupils. The meaning of "inverse problems" and the steps necessary to solve them are described. The significance of the use of arrows planned to face inverse problems is also described. The present didactic plan is intended particularly to help weak students understand a problem through different representations of the situation itself.*

## 1. INTRODUCTION

This work is framed in a research, started three years ago, which plans to study how to improve, in 11-14 year old pupils, solution strategies in problems with inverse procedures (inverse problems). Referring to literature on reversible thought, Piaget gave a lot of importance to the concept of reversibility: the reversibility of thought operations, which requires the mobility of mind in the forward and reverse directions, is placed, in the mental development of the child, in the period of formal operations. The mastery of such ability is considered essential for the exsperimental and logical-mathematical thought (Piaget, p. 334). From a mathematical point of view there are many activities which require to reconstruct the direction of a mental process and then to change the direction of the train of thought: for instance, when we deal with direct or inverse arithmetics operators, with direct or inverse theorems or with a formula which is to be read from left to right or from right to left. The psychological basis in these situations is considered the same (Kruteskii, p. 143).

Several psychological studies proved moreover that the skill to reconstruct, in a train of thought, two directions, direct and

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(**) The psychologist M.G. Grossi collaborated in this research.

reverse, is essential to master many situations, not only mathematical and it is not easily reached by all pupils (Kruteskii, p. 288).

In our work by inverse problem we mean a problem which requires us to go backwards in a given succession of (arithmetical or not) operators, when the result of such a succession is known. In simple cases the succession consists of just one operator.

The solution of an inverse problem requires, mainly, the understanding of the succession, given in more or less explicit way, the awareness of the need to go backwards and the skill to inverte, in the right order, the given path. In what follows we will see, in a synthetic way, the work plan realized in the last year for 11-12 year old pupils and what we are doing now for the same age-group.

## 2. THE DIDACTIC PROPOSAL

The didactic itinerary proposed to 11-12 year old pupils, described in details in Pesci, deals, essentially, with the concept of arithmetical operators (+k, -k, xk, :k) as binary relations, with the composition of arithmetical operators and with the inversion of a composed relation. All that with the essential use of language of arrows.

The main objectives linked to the didactic plan are the following:

- to use the concept and the visualization of a binary relation and its inverse to face the usual inverse problems (in arithmetic, in geometry, in proportionality problems,...);
- to stress everytime the structure of a problematic situation, without taking into account the nature of a numerical data (numbers with or without point, greater or less than 1,...), so to avoid misconceptions' influence on the choice of operations (Bell et al.; Fischbein et al.; Mariotti et al.).

As far as the use of the language of arrows is concerned, it is important to mention that it has been thought at two different levels which we call "concret" and "abstract" respectively.

At the first level the arrow represents a situation where it is clear, in a logical-temporal sense, the starting point, the point of arrival and the operator in use.

At this level, the dynamics of the situation, explicited on the paper, is handling in a concret way and it simplifies the reasoning to go back to the starting point (when is known, obviously, that +k and -k, xk and :k respectively, are one the inverse of the other).

At this "concret" level the scheme with arrows is a "diagrammatic model" which semplifies mental processes (Fischbein, pp. 165-166).

The language of arrows is also important in order to improve the construction of mental images ("abstract" level).

With arrows a particular simple scheme may be constructed: it can be transferred into mind as it is and it can be enriched by many other meanings, at more abstract and more formal levels.

The psychologist widely recognize the power of images' code and its partial autonomy from the verbal one (Cornoldi, p. 91). Hence the importance to develop and train abilities of visualization as a basic skill in young pupils (Lean-Clements; Bishop) and the importance, therefore, of schemes which for their simplicity can be internalized as mental images.

## 3. RESULTS FROM THE FIRST VERIFICATION

To study the influence of the didactic proposal (mentioned in 2.) on the strategies of solution of inverse problems, three questionnaires were given to 2 experimental classes (33 pupils in all) and to 1 class of control (21 pupils).

Every questionnaire has 8 problems, 4 direct (as distractors) and 4 inverse. The text of the questionnaire, the way of administration and othe details are in Pesci.

Here I would complete with the final results obtained in the last year, emerged from the comparison between the exit of the first questionnaire (before the didactic proposal) and the third one (at the end of the scholastic year).

In the two following tables, S is for experimental group and C is for the control group. In table 1 there are the percentages of correct solutions in direct and inverse problems. But it is more significant, in order to not take into account the initial situation of the classes, to look at percentage variations of the correct problems in the third questionnaire with respect to the first one (see table 2).

TABLE 1

| | | DIR. PR. | INV. PR. |
|---|---|---|---|
| I Q. | S | 45% | 18% |
| | C | 61% | 26% |
| III Q. | S | 63% | 42% |
| | C | 61% | 42% |

TABLE 2

| | | DIR. PR. | INV. PR. |
|---|---|---|---|
| I-III Q. | S | +38% | +129% |
| | C | 0% | + 59% |

Here I limit myself to underlining the positive influence (+129%) of the didactic proposal on the solution of inverse problems.

Since the result is only indicative, for the low number of tested pupils, an analogous experiment is now in course, as descibed below.

4. THE PRESENT PLAN

In the present scholastic year the work-plan for pupils of the same age-group (11-12) has the two following aims:

- to reconfirm the positive influence of the didactical proposal in 10 experimental classes (about 200 pupils) through the same questionnaires mentioned above;
- to place particular attention to pupils with difficulty of learning who have been identified, beyond the teacher's judgment, by a double tests (see 4.1).

The objective of the activities planned for those pupils (see 4.2) is to strengthen their ability to represent problematic situations, working in such a way as to arrive at the necessary skills to use the language of arrows.

The hypothesis which is to be verified is whether the use of arrows, as said above, simplifies the reasoning and allows good performance, also to weak pupils, in inverse procedures.

We think that the work with problematic pupils may make it possible to characterise better the potentialities of image language versus verbal language and the complementarity of one language with respect to the other.

4.1 Test to identify weak pupils

The main objective of the double test, presented to the experimental classes, was to identify the least able pupils with reference to the basic skills which are required to face our didactic proposal centered on inverse procedures and on the use of arrows.

The first part of the test has 9 items: the first three deal with logical-temporal sequences, 4 and 5 with the symbolisation and spatio-temporal abilities, the last three the astraction, namely regularities' identification and production.

The first 9 items are the following:

1) Every morning Luca, before arriving at school, does the following actions. Put them in time order, numbering them from 1 to 5: he leaves home, he pays for the bun, he wakes up and gets dressed, he goes to a bakery and buys a bun, he arrives at school and greets his friends.

2) Look at the following pictures and put them in time order, numbering them from 1 to 6. *(For sake of brevity pictures are omitted).*

3) Write, in the right order, 5 actions you do when you wash your hands.

4) I have invented a secret code for phone numbers, here it is:

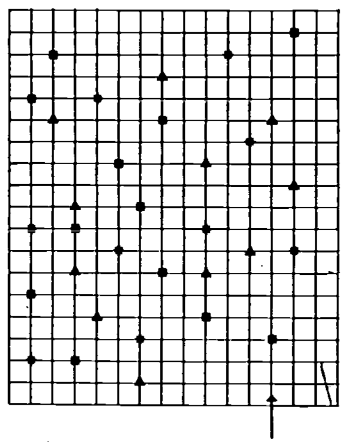

What phone number is this  ?

5) Invent your own secret code and write "we have won".

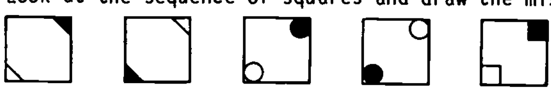
6) In a game there are obstacles of different forms: *.▲.■.
When a ball meets one of these obstacles it changes direction as indicated below and it goes on until it meets another obstacle.

*: ↑   ▲: ↓   ■: ←

Draw the route of a ball which enters as indicated by the arrow:

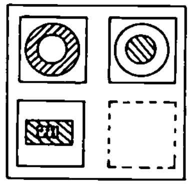


7) Look at the sequence of squares and draw the missing square:

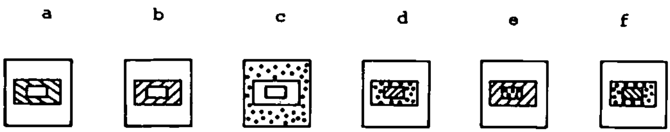
 ..........

8) Explain, with your own words, how the following sequence has been constructed: 5, 8, 11, 14, 17, 20

9) Look at the following picture:



With which of the following little square would you complete it?




The second part of the test has 8 problems, each of them requires only one operation. They are the following:

1) Today my parents have given me 10.000 lire pocket money and yesterday my grand-parents gave me 15.000 lire. If in my wallet I already had 7.500 lire how much have I got now?

2) To make a cake you need 3 eggs. How many eggs do you need to make 7 cakes?

3) To go on holiday I drove 355 Km in my car. Coming back I came a different way and I drove only 317 Km. How many kilometers did I save?

4) I have to put 120 books on shewes of equal dimensions, each of which contains 15 books. How many shewes do I need?

5) I would like to buy some pens and I have 8400 lire. If a pen costs 600 lire how many pens can I buy?

6) A cork weighs 3.2 gr. How much do 25 corks weigh?

7) For a ring you need 2.5 gr. of gold. How many rings can you obtain with 35 gr. of gold?

8) In Carla's wardrobe there are 8 skirts and 12 yumpers. In how many different ways could Carla dress herself?

Table 3 shows the percentages of correct solutions in the first and in the second part of the test respectively. Tested pupils were 215.

| TABLE 3 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| I PART | 98% | 83% | 89% | 94% | 77% | 59% | 86% | 64% | 37% |
| II PART | 98% | 97% | 90% | 80% | 80% | 88% | 63% | 44% | |

Given 0 points to every wrong or omited item and 1 point to every right item, the average score is 6.94 in the first part and 6.45 in the second one. Pupils for whom reinforcement activity is planned (see 4.2) are those who obtained a score less than 4 at least in one part of the test and who have been also considered weak by the previous judgment of the teacher.

4.2 Reinforcement Activity

It can be described, synthetically, in the following way:

a) activity aiming to understand a given text, with the explicit request to represent in different ways (figural, symbolic or verbal) the situation given in verbal way or, viceversa, with the request to construct a text around a situation given in non-verbal way;
b) activity aiming to strengthen logical-temporal abilities;
c) activity aiming to improve awareness of symbolisation moment.

In every class the most weak pupils (in the sense before mentioned) have been placed in the same group and an "average" pupil has been put in their group, not with a leader function but with the aim to favour the work itself of the group.

Even the rest of the class, divided in groups, works with the activity described in a), since we consider essential, in learning, to work with translations from one mode of representation to another (Janvier, pp. 27-32). In every class the groups of pupils are etherogeneous and the proposed activities are sometimes differentiated.

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# ALGEBRA WORD PROBLEMS: A NUMERICAL APPROACH FOR ITS RESOLUTION (A TEACHING EXPERIMENT IN THE CLASSROOM)

Guillermo Rubio
Universidad Autónoma de México. CCH Sur
Mexico

This paper reports the findings of a teaching experiment carried out in the classroom. It is a proposal for teaching how to solve algebra word problems using numerical approaches. The research had two prime aims. One of them was to investigate if the teaching approach made students get better results than those obtained in a pre-test. The other one was to analyze the changes produced in the student's thinking by the teaching scheme in the algebrization of the word problems. In general terms, the proposal was succesful. The students acquired more flexibility to interpret and to translate word problems to equations, the latter not being, as usual, in a literal form. The following was another important observation of the study: if the students do not keep track of the operations carried out and the meanings linked with those in a proper numerical interpretation procedure the algebrization process of the word problems is not possible.

The purpose of this paper is to communicate some of the findings of a study carried out with a group of 28 Senior high School level students (15-17 years old), during the academic year 1988-1989. The investigation was focused on the modification of the student's algebra knowledge acquired at the secondary school; in particular, on the development of their ability to algebrize word problems using a numerical approach to solve them. Underlying the teaching proposal's outline is the intention to intermix the arithmetic background with algebraic elements in such form that the teaching process enables to go to-and-fro between both interpretations. The aim of this procedure is to visualize and to solve problems, which treated in a conventional way would need working out a great deal of algebraic sintaxis and semantics. Within the teaching experiment described in this paper, the last assumption of the proposal's outline was not fully verified. A further investigation will be carried out for that purpose.

## Background and Theoretical Framework of the Proposal

Cervantes and Rubio, in 1983, investigated the posibility to implement an algebra course taking into account the ideas discussed by Piaget (1979) and Aleksandrov et al (1956) related with the formation process of the scientific knowledge of Humanity. Later, within the structuring procedure of the present proposal, it was considered that, an adaptation of such ideas to the teaching process is linked with the constructivist psycological current related to the need to face the individual with problematical situations that will enable him to construct meaningful knowledge (Rubio, 1987; 1988; 1989; 1990).

The principal aspect of the teaching proposal, which at the same time is not common, is the need to make a numerical interpretation of the word problems. The latter must lead to a proper algebraic interpretation and its solution. The assumption is that this numerical interpretation—which involves an iterative procedure trial,interpretation,error-is connected in an epistemological way to the first two stages described by Piaget (1973). These stages are related to the assimilation process of the real world to the logical-mathematical way of thinking within the development of contemporaneous physics. Piaget argues that those stages preced the data translation to a system of equations. The numerical interpretation captures both stages. The first one, the establishment of facts or data from the real world is not independent of mathematics modeling such as: classification, relationships, correspondences, measurement, etc. The second one, refered to the building of intuitive and qualitative schemes constitute a core guide towards formalization. The third stage, the algebraic interpretation underlies the numerical interpretation of the problem.

Likewise, this teaching approach takes into account the students' spontaneous pre-algebraic trends to solve word problems (see Bell, 1977; Trujillo, 1987). Those trends could be a consequence of the iterative use of numerical values (and its operations), made since the primary school, as abstract representations of physical and geometrical magnitudes of the real world within the student's environment. In an attempt to solve a word problem those numerical values become "thought concreteness". It is a belief that such process of concreteness enables, in many cases, the formation and adquisition of the operations' meanings, which are established between the unknowns and the problem's data. In this way, the numerical approach can provide a means to enable the student to face and solve a problem with a new conceptual framework. This structure is an organization of the preceding student's conceptual system.

## The Teaching Proposal

The following phases are differenciated in the teaching proposal:

1. **The understanding of the problem.**
2. **Numerical approach of the solution (trial and error process).**
3. **The interpretation of operations and relationships.**
4. **The obtention o the equation derived from the pattern determined by the trial and error process.**
5. **Algebraic and/or numerical resolution of the equation.**

*Illustration using two problems solved in the classroom*

**Problem 1.** A teacher hands out 120 chocolate bars and 192 sweets between the students in a classroom. Each student recieves three sweets more than chocolate bars. How many students are there in the classroom?

Phases 1 and 2 *Number of students* $\overbrace{\quad 20 \quad}^{\text{Number posed as a solution}}$

**Phase 3 :**
*Number of chocolate bars received by each student* 120/20 = 6
*Number of sweets received by each student* 192/20 = 9.6

$$¿\ \underbrace{6 + 3}_{\text{numbers of sweets}} = \underbrace{9.6}_{\text{Numers of sweets}}\ ?$$

The methodology establishes a process to recover the operations carried out, that is:

¿ 120/20 + 3 = 192/20 ?

After the analysis of similar trials, a letter is posed as the precise solution of the problem ( **Phase 4)**

120/ y + 3 = 192/y

where y = number of students.

The folllowing shows the different types of equations found after the numerical approach to the problem:

$$120/y + 3 = 192/y$$
$$(120/y + 3) = 192$$
$$120/y + 120/y + 3 = (120 + 192)y$$
$$192/y - 3 = 120/y$$

**Problem 2.** CCH has twice as much students as the Colegio de Bachilleres and the latter has 87654 students less than the UAM. The total of students in the three institutions is 567890. How many students does each institution has? Observation. The problem does not have a whole number solution.

**Phase 1.** Setting up the unknowns.

Number of students in CCH.
Number of students in Colegio de Bachilleres
Number of students in UAM

**Phase 2.** Numerical values were chosen for one of the three unknowns and afterwards the values of the other two were computed. The three interpretations which emerged in the classroom are described in the following paragraphs.

*First case.* A value for the number of students in CCH is posed.

$$¿\underbrace{\overbrace{400000}^{\text{quantity posed}}}_{\text{Students of CCH}} + \underbrace{\frac{400000}{2}}_{\text{Students of Bachilleres}} + \underbrace{\frac{400000}{2} + 87654}_{\text{students of UAM}} = \underbrace{567890}_{\text{Total number of students}}?$$

$$\underbrace{\qquad\qquad\qquad}_{\text{Total number of students}} = \underbrace{\qquad}_{\text{Total number of students}}$$

After several trials, the process lead to an equation of the following type:

$$x + \frac{x}{2} + \frac{x}{2} + 87654 = 567890$$

where $x$ = number of students of CCH

*Second case:* A value was posed for the number of students of Colegio de Bachilleres.

$$¿\ 2(90000) + \overset{\text{quantity posed}}{90000} + 90000 + 87654 = 567890\ ?$$

$$\left\{\begin{matrix}\text{Students}\\ \text{of CCH}\end{matrix}\right\} + \left\{\begin{matrix}\text{Students}\\ \text{of}\\ \text{Bachilleres}\end{matrix}\right\} + \left\{\begin{matrix}\text{Students of}\\ \text{UAM}\end{matrix}\right\} = \left\{\begin{matrix}\text{Total}\\ \text{number of}\\ \text{students}\end{matrix}\right\}$$

A process several trials conduced to the following equation:

$$2(x) + x + x + 87654 = 567890$$

where $x$ = number of students of Bachilleres.

*Third case.* A group of students in the classroom posed a value for the number of UAM's students.

$$¿\ \overset{\text{quantity posed}}{120000} + (120000 - 87654) + (120000 - 87654)2 = 567890$$

$$\left\{\begin{matrix}\text{students}\\ \text{of UAM}\end{matrix}\right\} + \left\{\begin{matrix}\text{students of}\\ \text{Bachilleres}\end{matrix}\right\} + \left\{\begin{matrix}\text{students of}\\ \text{CCH}\end{matrix}\right\} = \left\{\begin{matrix}\text{total}\\ \text{number of}\\ \text{students}\end{matrix}\right\}$$

The following equation was obtained from the trial and error procedure:

$$x + (x - 87654) + (x - 87654) = 567890$$

where $x$ = number of students of CCH.

## The Word Problems

In the class sessions, twenty two word problems were solved using the numerical approach. Sixteen of them were linear problems with one, or more unknowns. The latter type could be reduced to equations with one unknown. The other six comprised areas of rectangles, which generated a quadratic equation. The selection of the problems and the order of presentation took into account aspects such as: a) the number of times the unknown appeared in the equation associated with the word problem, b) the side of the equality in which the unknown appeared in the equation: on both or only on one side, c) in equations with more than one unknown, considerations were made with respect to the difficulty of expressing an unknown as a function of the others, in order to get a first order equation, d) equations in which the unknown is a divisor, e) the use of parenthesis to group properly the terms of an equation, f) the difficulties to determine the equivalence relationship.

It is a belief that the preceding elements are related to semantic and/or sintactic difficulties caused by the word problem's textual entities. The former type of hindrances are difficult to characterize however, it was observed that students found less obstacles when faced with problems linked more directly with their "realworld". The words used in the problem were "meaningful" for the pupil. Furthermore, it was considered thatalgebraic semantic difficulties are also prompted by the number of composite unknowns which are necessary to be generated in a problem (Trujillo (1987)) (e.g., if x = number of students, "120/x = number of chocolate bars per student" is a composite unknown

resulting from the interrelationship of two quantities with different meaning). Supposedly, this last aspect is linked with the structure's complexity level of the operations established beteween the unknown magnitude and the data; aspect which is originated by the conditions given in the word problem.
The area problems were thought to be sufficiently meaningful for the pupils, though, in almost all the levels, they had several geometrical difficuties. A lack of development of the ability to "imagine" and represent in a numerical or algebraic way the dimensions of a rectangle, which are being increased or diminished, was shown.

**The classification tests.** Three paper and pencil tests were applied: a pre-test, a post-test and a delayed posttest. The post-test was applied immediately at the end of the teaching sesions (12 weeks), during three week periods (two of them of 50 minutes and the third one of 100 minutes). The delayed post-test was applied 33 days afterwards; during this period of time no academic activity was carried out. The tests used were of the same type. Each of those was composed by three axis or subthemes of algebra: operativity (29 items), equations (22 items) and semantic-resolution of problems and interpretation of word formulations-(24 items). The experimental group was classified with respect to each subtheme of the three tests. Pupils were ordered according to their performance and items to the order of difficulty. The classification was a means to prepare tables to record the changes of the students' results in the three tests. Likewise, to enable the identification of interrelationships between the three axis in each test, a further classification was carried out.

**The data.** The data comprises: the students' answered tests (pre-test, post-test and delayed post-test), the classroom annotations of two pupils, the teacher's daily observational notes, the sequence of 22 written problems worked out in the teaching sessions during the term and the aforesaid classification tables.

**Development of the Proposal**

The group was subdivided in subgroups of 4 pupils and a problem was posed. The dynamic method of the teaching proposal consists of encouraging pupils to express the unknown in a written form (during the course of the development of several problems pupils were reticent. In many cases, the prime difficulty encountered in solving word problems was to have a clear and explicit idea of what had to be found). Pupils are then urged to pose random numerical values for one of the unknowns in order to get them involved in a verifying process of the hypothetical value as a solution. This procedure comprises a new reading of the problem and a search for relationships between the mentioned value and the data, using "meanigful" arithmetic operations linked with the conditions pointed out in the problem's formulation and the analysis of the element's unities which intervene in each operation (e.g., multiplying 120 chocolate bars by 24 students would have no sense in the sweet's problem to indicate how many chocolate bars would each student receive).

Once this first mental interpretation is carried out, pupils are encouraged to perform operations and interpret the elements of the operations and its results, as well as, the potencial equivalence relationship which emerges from the comparative process derived from the problem's conditions (as shown in problem 1). In each trial, pupils are asked to write horizontally each operation carried out-to enable an insight of the actions involved. Furthermore, they should observe the role of the hypothetical solution value in the operation. After several trials, students are asked to search for a proper pattern, for all the trials performed (by marking the numerical value posed as solution to the problem). Finally, pupils are asked to use a letter to represent the "exact" solution of theproblem, by this process an equation which models it the word problem is obtained. At this point, two different alternatives are used. For the first problems of the sequence only a numerical solution is required; the procedure is to give different values for the literal and to verify, making further estimations, the obtention of an equality. In a progressive manner, the algebraic resolution of the equation using an Eulerian type process (carrying out the same operations on both sides of the equality) is taken over.

## Results

**Results related to the selection of the unknowns.** An attempt was made, in the class sessions, to aid students the least. The strategy lead pupils to choose unknowns in a problem to start working out the method trial-interpretation-error. It was observed that students acquired more flexibility to interpret a problem without restricting themselves to literal translation. In several cases, they started to pose numerical values for the first unknown which appeared in the problem's formulation. However, in some of the preceding problems, the pupils selected that unknown which comprised less mathematical difficulties—in a numerical and sintactical sense—when a value was given to it. Understood the problem's semantics, the pupils got intensively involved in the numerical resolution. They showed good competence to operate positive decimal numbers; a vast number of students approached the solution using various decimal cyphers even without a calculator.

**Results related to the search of an equivalence relationship.** A qualitative analysis of the problems worked out by the students shows that, the numerical approach becomes an aid to build up a comprehension of the equivalence relationship between two sets of operations necessarily compared while solving a problem. However, this was not an homogeneous process; the whole population did not achieve that goal. For example, in problem 7, 30 percent of the pupils continued comparing the operations separately, that is, without relating them to the equal sign (the latter was being used twice as a connective). Nevertheles, in a progressive manner, more students established an equivalence relationship. It is a belief that the numerical approach, enabling to go to-and-fro from the numerical interpretation (which gives rise to a visualization of the equalness between two numbers) to the algebraic one, generates consciousness of the equivalence between two expressions. It is easier to capture that understanding by this procedure since it allows a comparison between numerical values of two algebraic expressions (meaningful for the students) derived from the word problem, which at the same time is, actually solved in a numerical way.

**Results related to the setting up and the resolution of the equation.** Arithmetic operations are not frequently presented in a horizontal way. At the begining of the course, that situation caused some resistance of the pupils to write or rewrite the arithmetic operations relating the unknown with the data Similarly, in the process to capture in a written form all the operations carried out to solve a problem, in order to assign meaning to each operation and its results, it was observed that some students left out some of those operations (particularly, when those were mentally done). As a consequence, pupils obtained equations which did not represent the problem properly. In the first phase of the teaching experiment, an algebraic resolution of the equation derived from the problem was not required. The purpose was to obtain an algebraic representation. The next phase aimed at a gradual involvement of the students in a Eulerian type process showed that the understanding of this method required a reconsideration of the knowledge acquired in preceding courses, when these have emphazised a mechanical use of transposition. From this observation the following hypothesis emerges: those pupil who are not able to give off the transposition method face serious difficulties to understand the equation as an equivalence relationship and therefore to establish and assign meaning to the equations derived from the word problem.

**Some quantitive results.** One of the fundamental aims of the teaching proposal, mentioned at the begining of this paper, concerns the semantic aspect of word problem solving. In quantitive terms, an improvement of the student's performance for the semantic axis items was achivied. The comparison of the results between the pre-test and the delayed pos-test showed: a) an increase of the percentage, between 40 to 80 percent for 15 items, b) an increase between 15 to 38 percent for 9 items and c) no improvement in only one item. The error percentage of the population's mean decreased 36 percent. Even though, a great impact on the other aspects-operativity and equations-was not expected in this first trial, the comparison between the results obtained for the items of these axis in the pre-test and the delayed post-test showed that the error percentage of the population's mean decreased 18 percent.

**Further studies related to the project.** In the preceding parragraphs, a first analyses of the axis has been presented. However, a further analysis searching for the interrelatioships between different axis should be carried out to complete this first stage of the investigation. A second trial is planned, using this teaching proposal as a basis in order to further understand the building up process of composite unknowns and the efficient use of the arithmetic operations immersed in an algebraic setting.

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# CHILDREN'S WRITING ABOUT THE IDEA OF VARIABLE IN THE CONTEXT OF A FORMULA

H. SAKONIDIS and JOAN BLISS

King's College London, University of London

ABSTRACT: *This paper presents the analysis of the responses of 394 pupils from 13 to 16 years old to three questions which consider aspects of the algebraic idea of a variable via its role in a formula. The results show a tendency of the pupils i) to focus on the operation rather than on the variables of the formula ii) to give explanations either through utilitarian considerations or by focusing on the operation or by restating the given information and iii) to refer to the idea of a variable at different levels, possibly dependent on pupils' cognitive level and the nature of the given task.*

## INTRODUCTION

The concept of a variable is one of the keystones of the discipline of mathematics because its understanding is decisive for the comprehension and appreciation of a considerable number of mathematical ideas. Much research has been concentrated on the idea of a variable and how it is understood in the context of school mathematics. This research shows that children have considerable difficulties not only with the idea itself but also with its representation, or both (Kuchemann, 1981, Booth, 1984). Piaget's work also dealt with variable in the scientific sense - and pupils' methods of isolating and controlling variables - which is seen as a formal level ability. He (1958) argued that children can only conceive of a variable in late formal operational stage and that this becomes apparent when they start to reflect upon reciprocal relationships between several variables.

Although the idea of a variable has been investigated, variable set within the algebraic idea of a formula has been given little attention. However, the importance of this idea and its particular characteristics are far from any doubt since the development both of the concept of variable and the interdependence of variables take place in the context of a formula.

This piece of research considers aspects of the concept of a variable as it is taught in the context of school algebra, examining ways in which pupils write about it through a consideration of its role in a formula.

## THE STUDY

The focus of this paper is on three questions about different aspects of the algebraic idea of formula which have been taken from a larger questionnaire on children's ideas about algebra. 394 pupils between the ages of 13 to 16 years old were involved in the study, taken from four urban schools: one boys, two girls and one mixed. There were 155 3rd year pupils ( 90 boys and 65 girls), 153 4th years ( 73 boys and 80 girls) and 86 5th years ( 44 boys and 42 girls). All the subjects had at least one year of formal teaching of algebra. The schools were banded for mathematics and a top and a middle group were taken from each school in the 3rd and 4th years and a top group only in the 5th year.

In the following, an analysis of the responses in each of the three questions is given both in terms of the type of mathematical focus of the response and the explanation given by the pupil for the response. Examples of the pupils' written explanations are given for each question. Finally, some discussion and conclusions are provided. For each of the three items, the analysis includes two components:

(i) Mathematical focus: (a) Focus on Variable: when the child focuses on the variable (s) or the representation of the variable(s) of the problem; (b) Focus on Operation: when the child refers to the operation relating the constituents of the given formula and (c) Dual Focus: when the response focuses on both the variable(s) and the operation(s) involved.

(ii) Content focus: Categories specific to each question are described in the relevant part of the presentation below.

We illustrate below examples of the three types of mathematical focus described in (i) above. These are all taken from the three questions which are the focus of this study. It is suggested that reader refers to these when familiar with the question:

**Focus on operation:** *"C is the biggest number because the result must be bigger if p is being added to something else first" (q. 2); "It tells us that to find the time travelled you have to divide the distance by the speed"* (q. 3).

**Focus on variable:** *"The least helpful answer is Tom's because letters can stand for different things in formula"* (question 1); *"C is the largest number because we are adding a positive number to an undefined number so the result must be larger than the original"* (q.2).

**Focus on both operation and on variable:** *"The new formula tell us that you* *have to know the values of d and s and then you divide them to find the time"* (question 3); *"C has to be bigger because whatever value is given to p, you have to add 2 to get C"* (q.2).

## QUESTION ONE

The concept of a formula becomes object of focus through its representation. Therefore, the understanding of what constitutes an appropriate representation of a formula is of interest. This question presents to the pupils a well known formula, that of the area of a rectangle, in three different forms: 1) area = width x length 2) A = a x b and 3) a x b, and children are asked to make a choice of the most helpful (part one) and the least helpful (part two) formula, giving each time the reason for their choice.

(i) choice of formula

In part 1, formula 1 is the most frequent choice by the majority of 4th and 5th year pupils, however for the 3rd years their choice splits almost evenly between formula 1 and formula 2. In part 2, for the 4th year the choice of the least helpful is clearly formula 3 (67%) whereas for the 3rd and 5th year the choice is divided between formula 2 and formula 3 (formula 2: 35% and 38% respectively and formula 3: 37% and 49% respectively).

(ii) The mathematical focus

The analysis of the data shows that none of the responses focus on operation. A number of children refer to the variables of the formula in both parts, with a small increase in the four year ( for the 3rd year approx.25% in both parts; for the 4th and 5th years about 35% of the responses in part 1 and about 40% in part 2).

(iii) The content of the explanations

The analysis of the data showed that the reasons for choice could classified in the following 3 categories based on a consideration of whether or not the formula is: (a) confusing or misleading, (b) sufficiently explicit and (c) efficient, that is, "it allows you to do things".

In part 1 the responses in the efficiency category are a little more predominant than those in the explicitness category and this difference between the two increases in the 5th year (ratio of responses: 3rd year: 7 : 5; 4th year, 6: 5; 5th year 2:1). In part 2 the efficiency and explicitness responses have very similar profiles for the 3rd and 5th year pupils, with a frequency of response of around 40% for both. In the fourth year responses appealing to the explicitness criterion are a little more predominant than those referring to efficiency in a 3:2 ratio.

Examples of the children's writing to illustrate these categories are given below:

**Efficiency:** *"The most helpful answer is area = width x length because you can work it out very easily"; "The most helpful answer is A = a x b because it is better expressed and easier to rearrange."*

**Explicitness:** *"The most helpful answer is area = width x length because it is very detailed, all the information you need is there"; "The most helpful answer is area = width x length because people can understand what the three components of the formula are immediately because they do not have to remember what any substituted letter stands for."*

**Misleading:** *"The least helpful answer is A= a x b because the two a's can be confusing."*

Summarising for question one, in both parts, pupils, particularly the older ones, choose formula 1 to be the most helpful and formula 3 to be the least helpful. The only type of mathematical focus is on variable (maximum 40%) and the majority of the explanations are based on criteria of "explicitness" and "efficiency" in both parts. In their explanations pupils talk about variables as concrete objects where, for example, the variable for length "a" is a "thing name".

## QUESTION TWO

The focus of this question is on the role of the constituents of a formula - the variables - and their interrelationships. Pupils are given a problem where the relationship between two variables C and p is expressed through the formula $C = p + 2$. They are presented with an answer given by a child and which they are told "is wrong". The problem concerns which variable in the formula represents the bigger number and the imaginary pupil replies "C because it's on the left-hand side." Pupils are asked to imagine explaining the problem to the pupil who is wrong.

(i) The mathematical focus

In all three years, the responses where the focus is on operation, are the most frequent, with a frequency which is very similar across the years (approx. 45%). However about 30% of the responses in all years focus in some way on variable.

(ii) The content of the explanations

Four possible explanations - which were in fact procedures for comparing the size of variables - were identified: (a) The response compares C and p+2 but the expression p+2 is seen as a whole and not in terms of its constituents related by an operation, (b) The response compares C with p+2, based on the operation which relates p and 2, (c) The answer compares C and p+2 by focusing on the relative size of the variables and (d) The response compares C and p+2 either using by substituting values or or by making generalised statements about the nature of formulae.

In all three years, the most frequent type of explanation is that which compares the size of the variable through the operation, with about half of the pupils giving this type of response. The next most frequent type of explanation relies on the relative size of the variables and is given by approximately 15% of the pupils in each year. Examples of the two most frequent types of answers for (ii) are:

**Explanations using operation for comparison:** *"C is the biggest number because if the reverse formula is used (C=p-2) C is bigger because you take off 2 to get C" ;"She is wrong because when you add P and 2 together then you find out the answer to C".*

**Explanations using relative size:** *"C represents the largest number because C is 2 more than P"; "P is always 2 less than C".*

In summary, the majority of the responses focus on operation, with a frequency which is similar across the years. However a focus on variable is observed in about a third of the responses. The emphasis on operation is reiterated in the children's explanations because the majority of pupils discuss the interdependence of the variables in terms of the operation which relates them. In those explanations where variable is referred to it is seen as a 'varying number' in relation to other 'varying numbers'

QUESTION THREE

In this third task, the focus shifts to the effects of manipulating the representation of a formula in certain ways. In particular, the rearrangement of a formula is the subject of this question. The formula given is a well known one, that of the relation between speed, time and distance. The formula **d=st** was presented followed by the rearrangement: t=d/s and the pupils were asked to explain "what the new formula tell us ".

(i) The mathematical focus

The "focus on operation" type of responses are by far the most frequent, with a frequency which peaks a little in the 4th year (72%), and which is very similar in the 3rd and 5th years approximately 65%. A focus on variable is only about 15% in all years.

(ii) The content of the explanations

Inspection of pupils' responses gave rise to the construction of the following four categories of types of explanation: **(a)** Static approach : When the child does not add anything new to the given information about the formula, **(b)** Pragmatic or functional approach : When the child sees the formula as having a functional purpose, **(c)** Inter-relational approach : When the child considers that there is a relationship among the variables of the formula and **(d)** Logico-mathematical : When the child sees the numerical solution as dependent on knowing the values of the other variables.

The category of "static" responses are the most frequent in all years; their frequency is similar in the 3rd and 5th years and decreases in the 4th year (57%, 46% and 55% respectively). The"logico-mathematical" type of responses are the next most frequent their incidence peaking in the 4th year, but staying approximately the same in the 3rd and 5th years (10%, 28%, 13% respectively)

Examples of the pupils' written responses in these two categories are as follows:

**Static:** *"The time taken can be found by dividing d by s"; "Time is equal to the distance covered over the speed"; "It tells us that t for time is being made equal with distance and speed divided."*

**Logico-mathematical:** *"It tells us the time to travel a certain distance at s speed, where s must be known in order to get t"; "It tells us that if you know what d and s equal you then can find t".*

Summarising, the algebraic focus of the answers is again on operation. Also in children's explanations operation plays a role since in about half of them there is simply a repetition of the information in the formula which is held together by the operation. There is a small minority of pupils in all years who use explanations of an "if... then" type of reasoning. When the pupils refer to variables in their written explanations they treat them again as "thing objects" denoting either abstract but familiar entities, e.g. time , or more concrete ones such as distance and speed.

## DISCUSSION AND CONCLUSIONS

The main characteristic of the three tasks is that they all deal with the notion of variable through its role in a formula. Despite this, the results above show a strong persistence on the part of the pupils to avoid any concern with variables and as in the two last questions to concentrate on operations. This fairly consistent absence of reference to variables could, perhaps, be understood in terms of the way in which pupils interpret letter symbols in an algebraic context.

Although the letter symbols used in all three questions can be seen as variables, the nature of the given tasks may determine the approach to the notion of variable adopted by pupils. We would suggest that when the task "takes away" the difficulty and the abstraction of the notion of variable by providing the means to handle it either as a concrete or familiar entity or as a "number object", pupils reject the abstract idea of variable, adopting a surrogate object to deal with it. Thus in question 1 they use the "thing-objects" to approach the task pragmatically and since all the given formulae have the same linear expression, they do not have to worry about the relationships between the variables. The same could be the case for question 3, but now the "thing objects" are linked by an operation which has to be considered. However, when the task does not provide any means of avoiding the notion of variable at an abstract level as in question 2, pupils seem to be forced to take a step towards using it . They adopt a "variable-like" approach, considering the letter as "a varying number in relation to other varying numbers" .

The above would suggest that in the attempt to overcome or avoid difficulties with the abstract notion of variable, children are likely to rely on those elements of the context that will allow them to go back to previous representations. Piaget argued that concrete thinking remains essentially attached to empirical reality, whereas formal thought deals with verbal statements substituted for objects. Mathematical variable is apparently more than a verbal statement replacing an object; in fact, it is a symbolic statement replacing a value which makes the notion more abstract. Therefore, it would seem reasonable for children to attempt to find ways of embedding this idea in a reality which has more meaning. The results support this general proposition of Piaget in that pupils are seen to be approaching this notion by resorting either to pragmatic or to familiar aspects of the context of the problem in order to cope with the abstractness of the idea of a variable.

Clearly it is difficult to compare the content of the explanations since the three questions are different. However, it appears that pupils see the relationship between the variables in a formula mainly via the operation which relates them and not in relation to one another. Pupils only consider variables in relation to one another when the question provides a framework for doing so. Question 2 is the only question which actually sets out such a framework of reference but even then pupils do not see the overall nature of the relationship between variables but rather how in their terms "one variable influences or operates on another and how that, in turn, influences the next". Furthermore, pupils show little appreciation for the fact that the mathematical manipulations of a formula give successive equivalent mathematical statements rather they see a rearranged formula in isolation from the given formula (question 3).

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# OBSERVATIONS ON THE "REVERSAL ERROR" IN ALGEBRA TASKS

Falk Seeger
Institut für Didaktik der Mathematik
University of Bielefeld - West Germany

Summary: *The aim of the present contribution is first to present new empirical findings on the wellknown "reversal-error" in algebra tasks. Following this the implications of these findings are discussed for the more general problem of tasks in algebra and empirical work. The new empirical findings indicate that "semantic confusion" might not give an adequate explanation of what is happening when students adopt the reversal-error-strategy and that the concept of "variable" did not play an important role. Most of the 549 students in our sample were able to come to a correct solution with the "wrong" algebraic equation when asked to apply it to a task that required arithmetical operations. The implications of these findings for the role of tasks in educational as well as in research settings are discussed.*

The empirical studies that will be reported in this paper started from the wellknown observation of Clement & Kaput (1979) given in their "Letter to the Editor". Their observation gave rise to a bunch of follow-up studies (e.g. Clement 1982; Clement, Lochhead & Monk 1981; Clement, Narode & Rosnick 1981; Cooper 1986; Fisher 1988; Kaput & Sims-Knight 1983; Lochhead 1980; Rosnick & Clement 1980; Wollman 1983). These studies generally confirmed the first impression by Kaput and Clement that a large proportion of students couldn't give a correct solution to the following task:

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Write an equation using the variables S and P to represent the
following statement: "There are six times as many students as
there are professors at this university". Use S for the number
of students and P for the number of professors.
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The solutions to the above task were showing that approximately 50% of the answers were algebraically "wrong" having the form of "6S=P". This was called the reversal error. That a similar proportion of reversal errors could be found not only with students but also with faculty members made this observation still more astonishing.

In the first part of the present contribution I would like to report some findings from own empirical studies. In the second part I will add some observations and theoretical speculations about the role of tasks used in empirical studies and tasks in math education in general.

## The empirical study

The goal of the empirical studies was to test the following idea: the "reversal" strategy from the point of view of the subjects that follow it cannot be understood as totally "wrong". Actually, the reversal strategy makes sense from an arithmetical point of view: if the equation is understood as establishing a relation between the set of students and the set of professors.

The most interesting point now was, to what extent students adopting the reversal strategy were able to carry out arithmetical operations with the "wrong" equation. Our hypothesis was, that actually the "wrong" equation had nothing to do with their ability to correctly perform on arithmetical tasks that were related to the original problem. In order to test the above hypothesis we administered written test questions to a sample of 549 students. The age of the students was between 13 and 24, 70% being 15 to 17 years old. The tests were completed in class, the teachers were distributing them, collecting them and mailing them back to us. So the test situation was very similar like a written examination, but the teacher was told to explain the purpose of the test to the students.

## The tasks

There were four different tasks that were imbedded into a common task context: the relation of students to teachers as actually recorded in one of the "Länder" of the Federal Republic of Germany and as projected in educational planning. In the first task the "classical" question of Kaput & Clement was put, the second task asked for an application of that equation, the third task required some arithmetical operations, while the fourth task asked for the equation that expressed the student-teacher relation in the third task. So, the first two tasks should represent the case of having an equation and applying it to a certain arithmetical context, in short: "equation" (task 1) -> "arithmetical application" (task 2), while the third and the fourth task should represent the inverse case, "arithmetical application" (task 3) -> "equation" (task 4).

**Task 1** "There are twenty times as many students as there are teachers in Northrhine-Westfalia. Find an equation for this situation, where S is the number of students and T is the number of teachers"

**Task 2** "There are 1.400.000 students in Northrhine-Westfalia. How many teachers are there then? (use the equation from task 1)"

**Task 3** "Educational planning in 1973 was assuming that 1985 there should be 17 students for each teacher. According to this, how many teachers should there be in the following Länder in 1985? (Fill in the number of teachers below with the help of your pocket calculator!)"

| | students | teachers |
|---|---|---|
| Hessen | 385.000 | |
| Niedersachsen | 627.000 | |
| Nordrhein-Westfalen | 1.400.000 | |
| Rheinland-Pfalz | 290.000 | |
| Saarland | 79.000 | |
| Schleswig-Holstein | 224.000 | |

**Task 4** "Find an equation that expresses the relation of students and teachers given in Task 3, where S is the number of students and T the number of teachers."

## Results

The results for Task 1 and 4 ("equation" context) showed a variety of different forms of equations:

| Task 1 | | Task 4 | |
|---|---|---|---|
| equation | % | equation | % |
| 20T = S | 43.7 | 17T = S | 30.2 |
| T = 20S | 28.8 | T = 17S | 18.4 |
| T:S = 1:20 | .7 | T:S = 1:17 | 1.3 |
| S:T = 1:20 | .2 | S:T = 1: 17 | - |
| S:T = 20:1 | 1.1 | S:T = 17:1 | 2.4 |
| T = S:20 | 14.8 | T = S:17 | 23.5 |
| S = T:20 | 1.3 | S = T:17 | 1.3 |
| other | 5.7 | other | 12.2 |
| no answer | 3.6 | no answer | 10.6 |

The results showed a considerable decrease of the reversal error from **28.8%** in **Task 1** to **18.4%** in Task 4. A correct answer was given by **57.4%** in **Task 4** compared to **60.3%** in **Task 1**. The equation T = S:20 was used by 14.8% in Task 1 and T = S:17 by 23.5% in Task 4. For **Task 2** it was not very suprising that **89.6%** of the students gave a correct answer. Having used the equation with the reversal error was no obstacle for coming to a correct arithmetical result. For

**Task 3** the task difficulty across the six subtasks was still lower with **91.4%** correct answers.

In the following table the differences between the use of equations aT = S, T = aS, and T = S:a are shown in absolute frequencies:

| | **Task 1** | **Task 4** |
|---|---|---|
| aT = S | 241 | 166 |
| T = aS | 158 | 101 |
| T = S:a | 91 | 149 |
| other | 39 | 74 |
| no answer | 20 | 59 |
| N | 549 | 549 |
| Chi-Square = 70.48<br>DF = 4<br>$p < 0.001$ | | |

The adoption of one of the different equations listed above could be seen as reflecting the use of different "strategies" by the students. The results indicate that the use of a certain strategy depended to a considerable extent on task-context. In our study "task-context" was playing a role in two dimensions: first, as an overall "applied" context that was relevant for all four tasks leading to a relatively high proportion of correct answers (as compared to related studies from other authors), second as a subcontext resulting from the different requirements in task 1 -> task 2 (equation -> application), and task 3 -> task 4 (application -> equation). In subcontext A (task 1 -> task 2) the equation aT = S was used by 43.7%, going back to 30.2% in subcontext B (task 3 -> task 4). The "reversal error" also was reduced from 28.8% in subcontext A to 18.4% in subcontext B. The reduction of the "reversal error" as well as the reduction of the algebraically correct equation in subcontext B was largely due to the increased use of equation T = S:a. These equation obviously is very close to the arithmetic procedure that was required in the "applied" tasks, because it literally describes the order of procedural steps to take: given a number of students and a multiplication factor to calculate the number of teachers by dividing the number of students by the multiplication factor.

Even the students that start with the "reversal error" equation are finding highly creative - albeit mathematically incorrect - ways to transform the original equation into the form where the number of students is divided by the multiplication factor. For reason of space only one example from the test is taken to illustrate the fundamental clash between the (incorrect) algebraic concept and the procedural concept:

$L = 5 \cdot 20 \;|\; : 20$

$L = \frac{5}{20}$ $\quad L = \frac{5}{20}$ $\quad L = \frac{1\,400\,000}{20}$

$L = 70\,000$

There was only one case out of 549 where a student was "correctly" filling in the numbers in the "reversal error" equation reaching a total of 28 million teachers for 1.4 million students. Maybe some kind of wishful thinking was involved here.

## Task and context

If we try to explain - as we did - the findings as influenced by the variation of task-context, it must be said that "context" as an important theoretical construct was in our study only dealt with on the level of the task itself. That is to say, that the context or the situation of working on the tasks was not controlled in our study. If already on such a restricted level, task context is an important factor, it should be more important on a larger scale. This lead to a critical evaluation of the design of our study and the tasks that were used. In the following I would like to sketch some of the apparent shortcomings of our study relating them to important issues of empirical research in math education. The focal point of interest here is how tasks are employed in empirical research.

1. We have to ask ourselves how the design of our standard test restricted the interpretability of the obtained results. It is quite clear that we can say only very few things about the processes that underly or accompany the solution of the tasks. Theses processes could only be dealt with in an indirect way, whereas a clinical interview study or a transcript/protocol analysis could have told more about that. However, what is gained with one method seems to be lost with the other one: the insight into the distribution of certain solutions on a large scale could not be gained when the reconstruction of solution processes via clinical interview or transcript/protocol analysis is the aim of a study. These two approaches basically differ in relation to time.

2. A closer look at the distribution of solution processes in the of the reversal-error was instructive. It could show that the situatedness of thinking (cf. Brown, Collins & Duguid 1989; Lave 1988; Suchman 1987) about algebra tasks could not be understood as a "misconception". Context-boundedness in clinical interview studies often was seen as a major obstacle to algebraic thinking (cf. Booth 1984, p. 37). A central problem in clinical studies often seemed to be how children make sense of the interview situation and consequently on the tasks that were presented to them. One central problem of standard test situation is that it nearly automatically is identified with an examination situation followed by the positive implications on the motivational level this has for some students while it has negative consequences for others.

3. The results obtained in this study confirm the view that a change in the very notion of "task" is overdue. The tasks presented to children cannot be understood as "objective" stimulus conditions being the same for each child. It rather must be seen that children actually work on tasks that differ from the given task and from the tasks other children work on. Even for the seemingly simple case of the division algorithm Newman et al. (1989) could show that children turn the same "objective" task into very different "personal" tasks. They understand tasks as "strategic fictions" that arise in social interaction.

4. If our notion of the "task" should change this also entails the notion of "error". There is important evidence from cognitive psychology (Norman 1987; Norman & Draper 1986; Seifert & Hutchins 1989) and from the psychology of work (Wehner & Mehl 1986; Wehner & Stadler 1988) that a strategy that is designed for the avoidance of errors might no be as effective as a strategy designed to exploit the vital importance of errors (cf. Bromme, Seeger & Steinbring 1990). Errors should be understood as productive and creative achievements. Consequently, tasks and task systems should be designed "user-centered" or"user-friendly" instead of following the philosophy to minimize and avoid errors. The idea is that errors should not lead to a system crash. Repair strategies (Brown & van Lehn 1980) seem to be a suitable means in this context.

## Concluding remarks

To reconcile the standard test procedure and the methods of clinical interviews and transcript/protocol analysis is an important issue for research in math education (cf. Ginsburg 1981). Rather than seeing the different methods as belonging to different research paradigms, it could be tried to project them onto different levels of the process-structure of math education. Obviously, in addition to that, new methods could be adopted that allow for the empirical research of the "situatedness" of learning.

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# GENERALIZATION PROCESSES IN ELEMENTARY ALGEBRA: INTERPRETATION AND SYMBOLIZATION.

SONIA URSINI
Sección de Matemática Educativa
CINVESTAV-IPN

*The results obtained through a written test concerning the symbolization of situations involving a generalization process and the concept of generalized number, are reported. 65 children, 11 - 14 years old, starting with the study of algebra were tested. One of the most interesting results was the regularity of the answers obtained, together with the instability of a particular individual's answers. This study is a part of a wider project concerning the feasibility of diminishing, in a computational environment, the difficulties children have with the different characterization of variables.*

Children have great difficulties and insecurities when faced with expressions that involve literal symbols ([1], [3], [5], [6], [9], [11], [12], [14]). There are a lot of data concerning the most common errors they commit [1] and very interesting results on how they interpret literal symbols [6]. The focus of this article is on the way children, starting the study of elementary algebra, symbolize, on their own, situations involving a generalization process and the concept of generalized number. The main objective of the study was to find out: 1) How children symbolize such situations and if they use literal symbols for it; 2) If their answers present some kind of classifiable regularity; 3) If the answers given by a particular child are stable in a certain class.

<u>Methodology.</u>

To answer the questions mentioned above, a questionnaire was designed where children were asked to: 1) Interpret literal symbols representing unknown or generalized numbers, (14 items); 2) Symbolize situations that involved unknowns or generalized numbers, (16 items).

The questionnaire was partly based on [2], [4], [8], [10] and [11]. It was applied to 65 students, aged 11 - 14, entering the first year of Secondary school in Mexico City and other mexican towns. None of them had had previous instruction in algebra. Overviews of results of questionnaire indicated however that almost the totality had some notion about the use of literal symbols and considered them as representing unknown numbers.

For each item an analysis of the answers obtained was done, and for each student the answers given to all the items were analyzed. The results suggest a classification of the answers obtained, that we will verify further with a wider population.

The main topics of the study, along with the rationale for each topic and some examples of the items we used to approach them, are showed in the table (see next page).

<u>Results.</u>

For each topic it has been possible to classify the answers obtained.

<u>Symbolization of simple verbal statements involving an unknown or a generalized number</u>. (7 items)

The answers given pointed out that there were children who:

1) Could not symbolize algebraically and gave a numerical answer to these items; (21/65)

2) Answered writing a single letter; (4/65)

3) Were able to symbolize simple statements that implied writing an equation where they have only to add to or multiply a literal symbol by a number; (31/65)

4) Could do (3) and also symbolize statements that implied writing an open expression, eg. 8*(3+X), using letters, numbers and brackets; (9/65).

<u>Interpretation of the symbolization of a generalization.</u> (5 items)

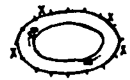
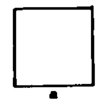
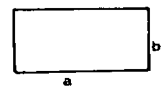
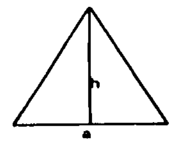
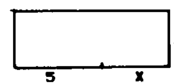
When asked to interpret a letter that represented a generalized number in an expression, we found that there were children who:

1) Could not interpret the letter in any way; (19/65)

2) Interpreted it as 'letter evaluated' [6], assigning it an arbitrary but specific value; (26/65)

3) Interpreted it as 'specific unknown' [6], and without giving the value they specified that it can have only one value; (2/65)

20 of the 65 subjects showed inconsistency in their answers; in similar circumstances, they did not interpret the letter in the same way. Some of them could not interpret it consistently and, for some items, gave no answer (7/65); others interpreted it as 'letter evaluated' but also as an 'object' [6] (12/65); only 1 child interpreted it as 'letter evaluated' and gave also a range of variation interpreting the letter as 'generalized number' [6]. The interpretation of 'letter as an object' appeared

| | | | | | |
|---|---|---|---|---|---|
| TOPIC | Symbolization of simple verbal statement involving an unknown or a generalized number | Interpretation of the symbolization of a generalization | Generalization processes and symbolization of a generalization | Interpretation of a functional relation; solution of equations with one or more appearances of the unknown | Interpretation and symbolization of some known geometrical concepts. |
| RATIONALE | How do children symbolize verbal statement that implies the use of unknowns and generalized numbers? Do they use letters? | How do children interpret the symbolization of a generalization? | How do children symbolize a given generalization? How do they generalize and express it? | How do children interpret a functional relation? Can they solve simple equations: with one unknown; with more than one unknown? | How do children interpret letters that are indicating figures' dimensions? How do they symbolize an area and a perimeter of a known geometric shape? |
| ITEMS | Write a formula which means:<br>An unknown number multiplied by 13 is equal to 127.<br>..................<br>Write a formula which means:<br>8 multiplies the sum of 3 and an unknown number. | In the following expressions write down all the values you think X can have:<br>X + 2 = 2 + X<br>5 + X<br>..................<br>Write the meaning of the following expressions using words:<br>2 * p<br>X * 7<br>2 | (10)A racing track is divided in 16 parts of equal length. Each part is X kilometers long. Write a formula to express the total length of the racing track.<br><br>....................<br>(11)The following shape is not completely visible. We do not know how how many sides it has; we will say it has N sides. Each side is 2 centimeters long. Write a formula to calculate the perimeter of the shape.<br><br>....................<br>(12)A road which was X kilometers long was extended by 25 kilometers. How long is it now? | In the following expressions write down all the values you think X can have:<br>3 + X = Y<br>X + 5 = X + X<br>X + 3 = 5 | Write a formula for the area of the following figures:<br><br><br><br><br><br><br><br> |

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clearly when there were letters different from X and it was possible to relate them with another subject, eg. geometry: 2*p was thought of as 2 times the perimeter.

Generalization processes and symbolization of a generalization.

(7 items)

To see how children generalize and symbolize a generalization, two groups of items were given:

a) An already generalized situation with the primary symbols given (items 10, 11, 12) was presented and children were asked to conceive the general situation and create a new symbol using the given one. There were children who could not symbolize any item (20/65). The others could symbolize one or more items correctly: 40 of the 65 subjects symbolized correctly the item 10; 19 the item 11; 30 the item 12. The items 10 and 12 were apparently of the same degree of difficulty, but while the item 12 has only a verbal explanation, the item 10 has also a drawing schema. It seems that the presence of the drawing helped many children in their process of generalization and symbolization. But the number of correct symbolizations diminished substantially for the item 11, where a partially hidden figure were shown. This caused confusion in many children, whose answers to items 10 and 12 were correct. Answers such as L+L or N+N , which were trying to give the total number of sides (visible and not) of the figure, were given; some children ignored the hidden part of the figure, multiplying the number of visible sides by 2; others gave N*2/2 as the answer.

b) A sequence of geometrical shapes with a sequence of numbers in correspondence (eg. number of sides) (4 items), were presented. Children were asked to find out and symbolize a general rule, which will produce for any further figure of the sequence its corresponding number. There were children who:

1) Could not answer any item; (5/65)

2) Could generalize only by drawing; (26/65)

3) Could generalize by drawing and by numbers; (8/65)

4) Could symbolize algebraically the simplest item; (8/65).

25 of the 65 children showed inconsistency in their answers, giving one or another of the previously mentioned answers. Some of them, besides symbolizing by drawing could also, for some items, go on with the numerical sequence, but only when small numbers were involved. (15/65)

In the answers to these items as well as to those referring to the symbolization of simple verbal statements, it was clear that when faced with a process that involved more than one operation, the majority considered only one of them. We may compare this behavior to the partially executed procedures found by Matz [9], however remarking that in this case we are referring to the symbolization and not to the solution process.

<u>Interpretation of a functional relation; solution of arithmetic (one appearance of the unknown) and non-arithmetic (more than one appearance of the unknown) equations.</u> (7 items)

When asked to interpret a functional relation, there were children who:

1) Gave no answer; (28/65)
2) Assigned a unique value to X and calculated the corresponding value for Y; (33/65)
3) Gave arbitrary and unrelated values to X and Y; (2/65)
4) Gave a range of values for X; (1/65)

Some children tested had difficulties when solving an arithmetic equation. No one could solve the non-arithmetic one; when trying to do it, almost all of them assigned different values to the different appearances of the unknown [3].

<u>Interpretation and symbolization of some known geometrical concepts.</u> (2 items)

In primary school, in Mexico, children are faced with the use of literal symbols when dealing with general formulae for the area and perimeter of geometric figures. To see the extent of their understanding of these formulae and the meaning they attached to the literal symbols presented in them, they were faced with known geometric figures the dimensions of which were indicated:

1) With literal symbols; 2) Combining numbers and letters.

In both cases they were asked to symbolize the area and the perimeter.

When faced with figures where the dimensions were indicated only by letters, there were children who:

1) Could not give any answer; (15/65)
2) Remembered the general formulae already learned and wrote them using letters different from those indicated; (25/65)

3) Assigned an arbitrary value to the letters and calculated the area or perimeter, or assigned an arbitrary value directly to the area or perimeter; (11/65)

4) Were able to symbolize using the given dimensions; (13/65)

Only 3 of the 65 children tested showed some inconsistency: sometimes they wrote a general formula learned by heart and sometimes considered the letters given.

When faced with shapes where the dimensions were indicated by letters and numbers, there were children who:

1) Could not give any answer; (7/65)

2) Assigned an arbitrary value to the letter and calculated the area or perimeter, or they assigned an arbitrary value directly to the area or perimeter; (6/65)

3) Remembered the general formulae and ignored the indicated dimensions; (8/65)

4) Considered the letters as generalized numbers and were able to manipulate them. They considered the indicated dimensions and were able to adapt their previous knowledge to the new circumstances; (3/65)

35 of the 65 children tested showed inconsistencies in their answers. For similar questions they gave different answers of the type listed above. Sometimes they assigned an arbitrary value to the letter and sometimes they tried to manipulate it without giving a value and instead invented their own way of doing it (5/65). Sometimes they assigned an arbitrary value to the letter and sometimes they ignored the letter and considered only the numbers (20/65). Sometimes they ignored the letters and sometimes they manipulated them as generalized numbers (7/65). It seems that many children when evoking the general formulae, were unable to consider as a symbol, the letter that indicated the dimension of the figure. It was clear that for the majority, the letters that appear in the general formulae did not represent a generalization but were considered as labels.

Conclusions.

1. The answers children gave confirm some results already found in other studies: they had difficulties with the use of brackets [1]; there existed confusion between the signs of addition (+) and multiplication (*) [3]; they assigned different values to the same letter when it appeared several times in an expression [3]; they were unable to interpret the conventional

algebraic notation [5]; some of the categories established by Kuchemann [6] appeared when interpreting the literal symbols.

2. All the children used literal symbols to express themselves. The great majority interpreted them as specific unknown when they were asked to interpret a given symbolization. It was not always so when they were asked to symbolize; in this case some of them even were able to state algebraically a generalization (items 10 and 12). However, there was no clear evidence that they could interpret the letter as generalized number, nor that they were capable of interpreting the expression written by themselves, as a general expression and operate on it.

3. The detailed analysis of the answers obtained led to a classification of these answers concerning capability in manipulating situations that involve generalization processes and their symbolization, and the use and interpretation of letters as generalized numbers. In spite of the fact that all the answers given by the children tested fitted into the given classification, the answers of a particular child were not stable in any one class. We therefore have a classification of responses not of children.

4. Because the sample tested was quite general, including children from Mexico City and other Mexican towns, and because we found: a) the presence, albeit in an inconsistent way, of the literal symbols in children's answers for an unknown or for symbolizing a generalization; b) the presence of some correct answers although also incorrect ones to similar items, showing inconsistencies in children's answers; we consider justified the hypothesis that children entering the first year of secondary school, in Mexico, belong to a 'zone of proximal development' [13] concerning their capability of dealing with the concepts of unknown and generalized number and their symbolization. We will investigate this hypothesis in a LOGO environment specially designed for this purpose, that is we anticipate that in such an environment in contrast to the common school algebra environment (see (1)), children can come to learn generalized number by provision of carefully structured and sequenced activities.

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# Assessment Procedures

# EFFECTS OF TEACHING METHODS ON MATHEMATICAL ABILITIES OF STUDENTS IN SECONDARY EDUCATION COMPARED BY MEANS OF A TRANSFERTEST

Author: Joost Meijer
Center for Educational Research
University of Amsterdam
Grote Bickersstraat 72
1013 KS Amsterdam, The Netherlands

## ABSTRACT

*In the research project[1] to be described here, a transfertest for mathematics was constructed wherein optional help was provided by means of hints, that were presented on the screen of a micro-computer. The first aim of the project was to investigate the claim, that the scores on the items, where hints were consulted, would contribute to the predictive validity of the test. This implies that such scores could be weighted in an objective way and that this procedure would render a more refined measurement of mathematical transfer capacity, compared to a test where all items have to be solved independently.*
*The second aim was to compare the effects of different teaching methods by means of this test. The results show that offering help in testing situations seems to be a valid approach and adds to the amount of observed variance in transfer capacities between students, thus indicating more refined measurement. Only marginal differences in effects of teaching methods were found, which also seem to depend on student characteristics.*

## Introduction

At the 1985 PME conference we presented preliminary results of a project involving the measurement of mathematical learning potential of students in secundary education in the Netherlands (see Meijer et. al., 1985). This project finished in 1987. We will report about its outcomes at this conference (the delay is due to various circumstances).
The central task which we took upon ourselves at the start of the project was the composition of a transfertest for mathematics, wherein, contrary to ordinary mathematics tests, students could ask for guiding information ("hints") if they were unable to solve a test-item independently. In this way we hoped to arrive at more refined measures of mathematical transfer capacity than measurements resulting from the use of conventional tests, which contain only "all-or-nothing" items. Because transfer of mathematical knowledge and skills is a very difficult task, such tests usually result in very crude distinctions between the transfer capacities of subjects.
The idea was originally derived from Vygotksy's theory (1964), which states that, since learning is an interactive, social process, accurate measurement of (mathematical) performance level can occur only if a subject has access to help in any way. In other words, Vygotsky assumed that measuring *independent achievement* is insufficient for predicting future performance. Taking into account performance levels of subjects given hints when needed will contribute to the predictive validity of a measuring procedure, if certain methodological requirements, such as reliability and equality of "information impact" on each subject, are satisfied. The difference between the level of performance one can achieve independently and the level of performance one can achieve with help, be it from elders, peers, books, charts or even computers, was dubbed "the zone of proximal development" by Vygotsky. It may be conceived of as "thinking structures in embryonic form" (Ginsburg, 1983).
In the inquest of Krutetskii (1976) concerning the structure of mathematical abilities of schoolchildren such a procedure was applied. However, since help for subjects was available from experimenters in individual testing situations, the reliability of this procedure for measuring learning potential is questionable. If rigid psychometrical demands are taken into account, it is essential that every student requiring help receives the same information. Moreover, this information should have equal value for each subject, i.e. all subjects should be pushed equally further to the solution of the item by every hint, independent of their initial ability.
The results show that such strict psychometrical demands could hardly be met. Nonetheless the experiment proved succesful because its outcomes highlight the possibility of educational use of tests with availability of hints and shed light upon the factors influencing mathematical performance.
A secondary goal of the research project was to assess the differential effects of teaching methods for mathematics. Recently, so called "realistic" methods for teaching mathematics have been propagated in the Netherlands (see a.o, Treffers, 1987). These methods are more loosely structured than conventional methods and emphasize "reinvention" of mathematical principles by students in stead of extensive explanation by teachers.

Concluding, the goal of the research project to be described here was twofold:

1. Development of a test wherein help can be obtained by students if they cannot solve the mathematical problems independently;
2. Comparison of the effect of teaching methods for mathematics on the mathematical ability of students.

## Method

Attainment of the first goal of the project implied the development of a test for mathematical ability with a sufficient level of difficulty, so that the effect of offering help could be assessed. Since transfer of mathematical knowledge and skills is one of the most important goals of mathematics education and usually hard to measure, it was decided to construct a *transfertest* for mathematics, containing hints. This leads to the opportunity to measure transfer in a more refined way than conventional tests, because the inability to solve transfer problems independently can be alleviated by offering help. Partial credit can be given for correct answers given after consultance of hints.

[1] The project was partially sponsored by the Foundation for Educational Research, The Hague, The Netherlands, grant no. 1128. It was conducted at the Free University of Amsterdam, De Boelelaan 1115, Amsterdam.

The question of the usefulness of offering help during a testing situation confronted us with the question of the predictive validity of the test to be developed. Obviously, measuring mathematical performance when help is available only makes sense if such measurements contribute to the validity of the measurement of independent mathematical performance level.
In order to be able to judge the usefulness of the testing-procedure, a *criterion-test* was constructed that was administered six months after the administration of the transfertest.
A necessary condition for transfer of mathematical knowledge is availability of preliminary knowledge. Therefore, it is important to control for mastery of taught subject-matter. This was done by construction of a *mastery test*. Items in this test only called upon knowledge of the subject-matter that was taught, i.e. knowledge structures did not have to be transformed to find the solution. Most of the items in this test were derived from a test, constructed by the Dutch National Institute for the Development of Educational Tests.

If one wants to compare the effects of teaching methods in a pre-posttest design it is important to control for initial mathematical aptitude of subjects. No clear operationalization for this latter concept could be found, neither in the form of a definition (see Meijer et. al., 1985) nor in the form of a test, specifically designed for this purpose.
In view of the lack of tests for initial mathematical aptitude it was decided to administer two tests for cognitive ability, that correlate high with mathematical achievement. Two subtests of a test for measuring learning progress in secondary education, constructed by Horn (1969) were used for this purpose. The subtests concerned were a) a test consisting of *series of figures*, b) a test consisting of *series of alphanumeric characters*. In both tests the element that did not fit in the series had to be eliminated by the subject. Aurin (1968) reports high correlations of both tests with mathematics achievement.

Many problems in the test for measuring mathematical transfer capacity required recognition of the underlying mathematical structure in a situation that is described in common language. It was assumed that the capability to recognize this structure is influenced by field-dependency (Witkin et. al., 1977). It was established by several investigations that expertise correlates with the ability to type domain-specific problems in terms of domain-specific principles in stead of superficial characteristics of the problem-statement, which novices use to categorize a problem (Chi, Feltovich & Glaser, 1981). This is similar to the ability to distinguish figure and background, a feature that the Embedded Figures Test pretends to measure. Therefore, the *Group Embedded Figures Test* was administered in order to be able to correct for initial differences in this ability between students.

The usefulness of offering help during a testing situation can be tested by investigating the contribution to the predictive validity of scores on items where hints were consulted, compared to a test wherein independent achievement is assessed only. That means that if we add learning-potential scores to a regression equation containing the criteriontest scores as a criterion and a measure for independent achievement as a predictor, the beta-weight for this added predictor should be statistically significantly greater than nought.

Differences between the effect of teaching methods on the abilities of students should strictly be studied by imposing rigid restrictions on an experiment, in order to prevent plausibility of alternative explanations. In educational research this is hardly possible. In this research project, we compared the mathematical abilities of students in secondary education, learning mathematics from different series of textbooks. The contents of each of three commercial series was studied carefully by experts on the didactics of mathematics (see De Leeuw, Meijer, Groen & Perrenet, 1988). It was concluded that the first series of textbooks could be characterized as highly structured, but at the same time only teaching algorithms, "mechanistic", as De Lange (1987) types it. The second series clearly aimed at insight, but on a very formal, abstract level ("structuralistic"). The final series was most similar to the "realistic approach", as mentioned in the introduction. In these methods context-problems are used to introduce mathematical concepts as well as for application of taught concepts. The teachers role should be to build on the intuitive notions of students and creating conditions that allow students to discover mathematical solution methods for different kinds of realistic problems.
It was hypothesized that students using the last method would score highest on the transfertest because they should be most used to using mathematical solution methods for unacqainted problems. Some of the items in the transfertest required generalization of known mathematical principles, which was called "vertical transfer" by De Leeuw (1983). It was expected that students learning from the structuralistic method would show highest performance on these items. Transfer-scores should be corrected for mastery of subject-matter in testing these hypotheses, since the availibility of knowledge is a prerequisite for transfering it.
Finally, it was hypothesized that students characterized by a high need for prestructuring subject-matter would benefit most from highly structured teaching-methods and would be disadvantaged by loosely structured teaching-methods. The "realistic" teaching method for mathematics can be typed as relatively unstructured, because much emphasis is put on proper construction and inventions of students, i.e. students have to discover mathematical structures, which are only implicitly present in the context-problems that serve as subject-matter, by themselves.
On the other hand students that are low in need of prestructuring may be disadvantaged by highly structured methods, because they will be continuously disturbed by their lack of freedom to impose structure by their own effort. It was assumed that field-dependency is a good measure for need of prestructuring, since distinguishing figure from background depends on structuring activity. In other words, an interaction between teaching method and field-dependency in their effect on mathematical ability was expected.

## Procedure

### *1. Development of the transfertest*

The transfertest for mathematics contained relatively difficult items. We started out with a test of eighteen items, each supplied with six hints. All items concerned the subject-matter of functions, linear as well as quadratic. All hints were open indications, but gradually increased in specificity, building on Selz' ideas (1935) about "kleinst moegliche hilfe", i.e. we did not want to offer solutions for the problems, but only structuring information. Hints could be made visible by tearing of pieces of paper on the answer sheet. It appeared that most items were too difficult for our subjects, and that the effect of the hints differed greatly. Also, the hints did not yield very many extra solutions. The advocated scoring method was to give partial credit for correct answers, depending on the number of hints used. The main problem was however that there was no way to verify whether a hint had been

understood by the subject. This implied that credit points would be subtracted unjustfully if subjects did not assimilate or already were aware of the information that was contained in the hint.
A solution for this problem was found by changing the format of the hints. In stead of offering open ended indications, a question was being asked with four answering possibilities. As in most multiple choice questions, only one of these answers was correct, but only two were incorrect. The remaining answer was "I don't know". This was added to avoid guessing, because subjects automatically received extra explanation if they chose a wrong answer. In this way verification of assimilation of the content of the hint was made possible. It was assumed that subjects that answered the hint-question correctly needed no further information. In this case, only one credit-point was subtracted. If a subject gave a wrong answer or admitted he/she did not know the right answer, extra explanation was given and two credit-points were subtracted.
Also, it was attempted to make the test-problems easier, mainly by deleting items with very low p-values and adding simpler ones. Sometimes test-problems were simplified by incorporating the content of a hint in the problem-statement or by clarifying the structure of the problem in advance.
This procedure was tried in individual testing situations, where experimenters recorded the sessions on audio-tapes. Subjects were instructed to think aloud while trying to solve the test-items, during independent solving efforts as well as after having been offered a hint by the experimenter. Hints were presented on slips of paper. Experimenters did not offer any extra explanation by their own initiative.

After trying the new procedure on this small scale, is was repeated collectively with 210 students in grade four of secondary education. The items in theirselves had clearly become easier, the percentage of test-problems that was solved independently rose from 12 to 44. Although the percentage of items that was answered incorrectly notwithstanding consultation of hints declined from 56 to 24, it could not be shown that the effect of the hints had actually increased. The percentage of items that was solved after using one or more hints remained 16.
The pilot-researches resulted in 15 items, that were deemed suitable for the final version of the test. During the pilot-experiments it was also made clear that it took quite long to solve the test-problems. One the one hand, this was due to the difficulty of the problems, one the other hand it obviously takes time to read and process a hint. As a consequence, the total test was split up in seven partially overlapping subtests of five items each.
The final version of the test-problems consisted of open-end questions, presented on paper sheets, that served as answer sheets at the same time. Hints could be obtained on the screen of a micro-computer. The new procedure for presenting hints was retained, i.e. depending on their answer on a hint-question subjects received extra explanation. If the answer was correct only one point was subtracted from the score for the item, if extra explanation was offered after a wrong answer or choosing the alternative "I don't know" one more point was subtracted. If an item was solved without help five points were obtained. Only two hints could be consulted, so asking for both hints and answering them incorrectly, but giving the right end-solution resulted in a score of one point. Wrong solutions to test-problems always yielded a zero score, regardless of hint usage. The final experiment was conducted on eight schools, twelve classes were involved, totalling 325 students.

### *2. Administration of the reference tests for mathematical ability*

The test for measuring actual mastery of subject matter was administered in the same period as the transfertest. Half of the subjects first completed the test for actual mastery and then worked on the transfertest. The order was reversed for the other half of the subjects. This was necessary because there were only sufficient micro-computers available to serve half a class of students. The test consisted of 19 items, all except one were multiple-choice items. The only open ended item, that was scored by judges, concerned the quality of a drawing of a graph of a quadratic function.
Both tests were administered in june 1986, when subjects were at the end of their third year in secondary education.
The criteriontest for mathematical ability was administered between december 1986 and january 1987, about half a year after the first testing session. Because of the delay between the testing sessions, the predictive validity of the transfertest could be assessed in a meaningful way.

The criteriontest consisted of 10 items. Two parallel versions were constructed in order to avoid possible fraude, because students were arranged in adjoining seats. The test was carefully designed so that items represented subject-matter taught between the period of administration of the first and second battery of tests. Solutions given by subjects were scored by judges based on guidelines that were established in advance.

### *3. Administration of the reference tests for cognitive ability*

The test for fieldindependency was administered at the same time as the test for mastery of mathematical subject-matter and the transfertest. A period of four lessons hours (50 minutes each) was available for this first testing period on all schools. Two lesson-hours were reserved for completion of the transfertest, while the other two were devoted to four tests in total, i.e. the Group Embedded Figures Test, the mathematical mastery test, a mathematics attitude test and a test measuring achievement motivation and fear of failure. On the relationship of these latter tests with mathematical performance I will not divert here; but a report of this is in progress (Meijer, paper submitted to the Journal of Educational Research, Meijer, in preparation).
The Group Embedded Figures Test consists of several example-items and 13 test-items. In every item the problem is to identify a simple geometrical figure in a rather complex drawing. Subjects must answer the problems by stressing relevant lines in the drawing.

The other tests for measuring cognitive ability (series of figures and alphanumerics, subtests of the PSB) were administered simultaneously with the criteriontest for mathematical ability. One of the reasons to use these tests was that their administration-time is very limited (respectively 8 and 5 minutes). Since there was only slightly more than one lesson-hour available for all tests, it was important to save time.

Concluding, the following table shows which tests were administered when:

**Table 1 Time table for test-administration**

| | june 1986 | january 1987 |
|---|---|---|
| mathematics tests | test for actual mastery<br>transfertest | criteriontest |
| cognitive ability | embedded figures test | series of figures<br>series of alphanumerics |

## Results

The hypothesis that performance on items where help could be obtained contributes to the predictive validity of a test which also accounts for independent performance was confirmed.
Scores on the transfertest were divided in the proportion of correct solutions arrived at independently and the proportion of correct solutions found after consulting hints. Since only five items were administered, independent achievement was operationalized as the amount of items solved correctly without help, divided by 5. Learning potential scores (operationalized by performance level when given assistance) were calculated by dividing scores on items where hints were used by 5. Therefore, the maximum learning-potential score to be obtained was 4 (i.e. 5 items times 4 = 20 divided by 5 = 4). Subjects solving all items without consulting hints were excluded from the analysis. In this context, the problem with the impeccable performance of such eminent students means per definition that their learning-potential is nil, which is not very plausible.
Initially 325 students took part in the experiment. Except four students, all of these completed the tests administered in june 1986 (actual mastery test, transfertest and embedded figures test plus some other tests that are not of interest here).
Because the second testing period took place after the selection of students between their third and fourth year in secondary education, 86 subjects could not participate in the second testing period, due to non-promotion to fourth grade or dropout.
Complete data (on the actual mastery test, the transfer test and the criteriontest) were available for 224 subjects. Table 2 contains the results of a regression analysis wherein independent achievement and learning-potential scores predict criteriontest performance:

**Table 2 Prediction of criteriontest performance**

| predictors:<br>transfertest | criterion:<br>criteriontest for mathematical ability | | | | |
|---|---|---|---|---|---|
| | R | $R^2$ | ß | F | p |
| 1. items solved without help | .329 | .108 | .289 | 26.95 | <.001 |
| 2. items solved with help | .354 | .125 | .137 | 4.34 | .038 |

Obviously, when using the proportion of items solved correctly without consulting hints as a predictor in the first place, the score on items solved correctly after consulting hints turns out to contribute to the amount of explained variance in the criteriontest scores significantly. This means that measuring performance when help can be obtained is not only a valid predictor for future performance (as independent performance), but also shows sufficient discriminative validity. If the correlation between criteriontest performance and independent achievement is partialled out, there still remains a portion of criteriontest score variance, that can be explained by learning potential scores. However when using actual mastery test scores as a third predictor, the effects of learning-potential scores are no longer statistically significant:

**Table 3 Prediction of criteriontest performance revisited**

| predictors:<br>transfertest<br>actual mastery | criterion:<br>criteriontest for mathematical ability | | | | |
|---|---|---|---|---|---|
| | R | $R^2$ | ß | F | p |
| 1. items solved without help | .325 | .105 | .227 | 25.81 | <.001 |
| 2. actual mastery test scores | .383 | .147 | .197 | 10.49 | .001 |
| 3. items solved with help | .392 | .154 | .091 | 11.85 | .175 |

We must conclude that independent transfer capability of mathematics knowledge and skills is the best predictor for future mathematics performance, even after controlling for initial mastery of mathematics subject-matter. Measurement of transfer capability in situations where help can be obtained ("learning potential") can contribute to the predictive validity of tests, albeit marginally. Apparently partial credit can be given in a very objective fashion.
The results of the final experiment also show that the effectiveness of the hints had increased. The results of the second experiment showed no improvement in the effectiveness of hints compared to the first experiment (both experiments resulted in 16 % items, solved correctly after consulting hints). In the final administration this percentage increased quite dramatically to 38.
This is probably partly due to the fact that the second pilot experiment took place in grade four, while the final experiment was conducted in grade three. Because of this, the difficulty of the test problems increased comparatively. Items were relatively easy for subjects in the second pilot experiment (illustrated by the fact that 44 % of the items was solved without help), while they were quite difficult for subjects in the final experiment (only 23 % of the items was solved without help). The results show that a scoring method where the amount of consulted hints is taken into account can be devieed in an objective way and that this procedure renders a more refined measure for mathematical ability than measuring independent mathematical achievement only.

We will now turn to the comparison of the effect of teaching-methods. Three commercial teaching-methods (i.e. series of text-books for mathematics) were compared (see Method).
Method A can be characterized by a highly structured approach, leaning heavily on algorithmization, stronly emphasizing practice in recognizing known problem-types and using the appropriate solving-algorithm.

Method B is typified by a relatively high level of abstraction, aiming at insight. Although algorithmization is also important in this method, theory is explained extensively before application. Practice in solving many similar problems consecutively is stressed.
Method C relies heavily on the "reinvention" principle (Freudenthal, 1978), i.e. the capacity of students to construct mathematical principles and rules by themselves. Context problems are used for developing mathematical solving methods as well as for applying such methods. The role of students is very important: teachers should appreciate solving attempts of students and instruction should be based on the constructions put forward by students.

The mean scores on the three measures of mathematical ability of students using these three methods are displayed in table 4.

**Table 4 Mean scores on mathematics tests per teaching method**

| Teaching methods | Mathematics tests | | |
|---|---|---|---|
| | Mastery | Transfer | Criterion |
| A | 9.4 | 10.4 | 12.2 |
| B | 9.7 | 10.5 | 14.2 |
| C | 10.8 | 11.5 | 11.9 |

Although students using method C score highest on both tests administered in grade three, their mean score on the criteriontest is lowest.
Because it was hypothesized that method C would score highest on transfer, after controlling for initial ability (measured by actual mastery) a covariance-analysis was performed wherein transfertest-scores were the dependent variable, teaching methods were conceived of as quasi-experimental treatments and actual mastery of subject-matter was the covariate. In order to make sure that effects of teaching-methods would not be confused with the effect of school-environment or the grouping of students in classes, groups of students in the same class within teaching methods were used as a nested factor. In this way the effect of teaching method can be distinguished from the effect of the grouping of students. Table 5 outlines the results.

**Table 5 Effects of teaching methods corrected for initial mathematical ability**

| | SS | df | MS | F | p |
|---|---|---|---|---|---|
| Within cells variance | 9948 | 308 | 32.3 | | |
| Mastery test | 2043 | 1 | 2043 | 63.3 | <.0001 |
| Teaching methods | 3.4 | 2 | 1.7 | .05 | .948 |
| Classes | 240 | 9 | 26.6 | .83 | .593 |

In can be seen that practically all variance in mathematical transfer capability can be explained by initial mathematical knowledge of subjects; teaching methods and grouping of students in classes hardly matter. The fact that classes hardly make any difference also points out that the effect of teachers on the development of mathematical ability of students should not be overestimated, since all 12 classes had different teachers.

One of the assumptions of covariance analysis is that dependent variables regress with equal slopes on covariates in every cell of the design matrix. This implies that the regression of mathematical on cognitive ability should be equal for all students in all three teaching methods, in order to find out if one of these methods results in a higher mean mathematics achievement of students, after controlling for cognitive ability.
It was found that several measures for mathematical ability show quite different strengths of the relationship with measures for cognitive ability in method C, compared to methods A and B. Contrary to our expectations it appears that mathematical learning results for students using the most loosely structured teaching method hardly correlate with their cognitive ability.
This means that the results of the covariance analysis indicating that there is no difference between the effect of the three methods should be interpreted with great care. In order to shed more light on the relationship between mathematical and cognitive ability, and on the differential effects of the teaching methods, an analysis-method proposed by Jöreskog and Sörbom (1986) was applied.
A computer program developed by the authors mentioned first (LISREL, short for Analysis of Linear Structural Equations) was used to estimate: 1) the regression of mathematical on cognitive ability for each method, 2) the difference in means of mathematical ability for each method, corrected for mean cognitive ability of students using every method.

On the basis of the covariance matrix and the vector of means of the mathematical ability measures (mastery, transfer and criterion) and the cognitive ability measures (the Group Embedded Figures Test and the Series of Figures) for each method, the linear structural model, depicted on page 6, was analyzed.
Eta$_1$ and eta$_2$ are latent (unobserved) variables, respectively representing mathematical and cognitive ability. Lambda$_{11}$, lambda$_{21}$ and lambda$_{31}$ are the regressions of the three tests used for mastery, transfer and criterion performance on "true" mathematical ability; lambda$_{42}$ and lambda$_{52}$ are the regressions of the scores on the Group Embedded Figures Test and the Series of Figures on "true" cognitive ability.
The variable ksi$_1$ is a dummy variable. The parameters gamma$_1$ and gamma$_2$ represent the difference between means on the latent variables for the students using the three teaching methods. Since no absolute value for the mean of a latent variable can be estimated, the mean of one method for these variables is set to nil, so that only deviations from this standard have to be estimated. Beta$_{12}$ represents the regression of latent mathematical ability on latent cognitive ability.
For estimating the differential effects of the teaching methods on mathematical ability, corrected for cognitive ability, the parameter gamma$_1$ is of interest. If this parameter differs significantly for teaching methods, it may be concluded that students educated with these methods differ in mathematical ability. The restriction is however that other parameters (the observed means on measurements of mathematical and cognitive ability and the regression of observed on latent variables) do not differ. Only differences in the means of the latent variables (unobserved mathematical and cognitive ability) are of interest.

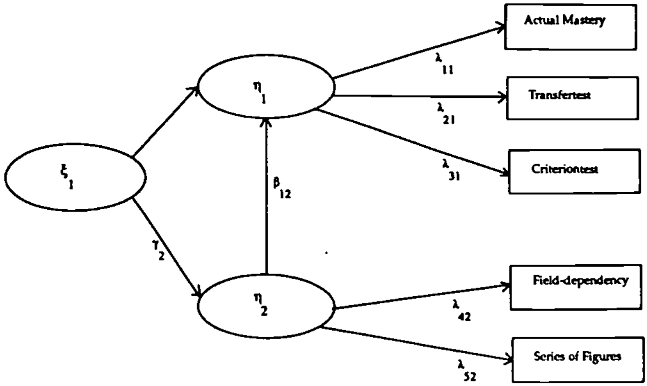


**Figure 1 A linear structural model for estimating differential effects of learning methods**

These restrictions do not meet the data. Because of the suspection that the regression of mathematical on cognitive ability was different for teaching method C and the fact that students using this method also scored lowest on the criteriontest, $beta_{12}$ and $lambda_{31}$ were estimated separately for method C. Results are summarized in table 6:

**Table 6 Estimated parameters per teaching method for the LISREL model, depicted in figure 1**
(standard errors between brackets)

| Teaching method: | B | | A | | C | |
|---|---|---|---|---|---|---|
| Parameter: | | | | | | |
| $lambda_{11}$ | 1 | (.000) | 1 | (.000) | 1 | (.000) |
| $lambda_{21}$ | 2.334 | (.349) | 2.334 | (.349) | 2.334 | (.349) |
| $lambda_{31}$ | 1.836 | (.289) | 1.836 | (.289) | .632 | (.332) |
| $lambda_{42}$ | 1 | (.000) | 1 | (.000) | 1 | (.000) |
| $lambda_{52}$ | .558 | (.144) | .558 | (.144) | .558 | (.144) |
| $beta_{12}$ | .689 | (.198) | .689 | (.198) | .126 | (.127) |
| $gamma_1$ | .576 | (.425) | -.583 | (.463) | .000 | (.000) |
| $gamma_2$ | -.395 | (.576) | .305 | (.562) | .000 | (.000) |

($chi^2$ = 48.98, df = 24, p = .002, Goodness of Fit = .885)

Although the model shows only very moderate correspondence to the data, it is obvious that the regression of mathematical on cognitive ability is not significant for students using method C, while teaching methods A en B show significant influence of cognitive on mathematical ability. Also the regression of criteriontest performance on latent mathematical ability is much smaller for method C than for methods A and B.
There appears to be no difference between mathematical or cognitive ability between the three methods, although it seems that students educated with method B show slightly superior performance. At the same time, they seem to have lower mean cognitive ability.

However, these differences are statistically insignificant and open to doubt because of the differing values of $lambda_{31}$ and $beta_{12}$ for method C, compared to methods A and B.

Since the regression of criteriontest performance on latent mathematical ability and the regression of latent mathematical ability on latent cognitive ability appear to be similar for mathods A and B, an adequate comparison of the effects of these methods on mathematical ability by means of this method of analysis may be made. The results of this analysis are summarized in table 7:

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**Table 7 Estimated parameters per teaching method for the LISREL model, depicted in figure 1**
(standard errors between brackets)

| Teaching method: | B | | A | |
|---|---|---|---|---|
| Parameter: | | | | |
| $lambda_{11}$ | 1 | (.000) | 1 | (.000) |
| $lambda_{21}$ | 2.153 | (.322) | 2.153 | (.322) |
| $lambda_{31}$ | 1.763 | (.269) | 1.763 | (.269) |
| $lambda_{42}$ | 1 | (.000) | 1 | (.000) |
| $lambda_{52}$ | .546 | (.145) | .546 | (.145) |
| $beta_{12}$ | .711 | (.204) | .711 | (.204) |
| $gamma_1$ | 1.224 | (.442) | .000 | (.000) |
| $gamma_2$ | -.698 | (.565) | .000 | (.000) |

Method A was used as the standard in stead of method C here, i.e. $gamma_1$ and $gamma_2$ were set to nil for this method. All other parameters were set equal for both teaching methods. The model fits the data remarkably well ($chi^2 = 10.53$, $df = 15$, $p = .785$).
Obviously for both methods mathematical achievement is quite strongly influenced by cognitive ability, as measured by the score on the Group Embedded Figures Test and Series of Figures. Again, it seems that students using method B start out with a slight handicap in cognitive ability, but this difference is not statistically significant. In spite of their initial disadvantage these students perform significantly better on all mathematical tests taken than students educated with method A.

## Discussion

The results of the experiment described in this paper show that there is definite perspective in constructing transfertests for mathematics wherein the testee can obtain help if needed. This is of great importance in the light of the fact that transfer of domain-specific knowledge and skills usually occurs on a very small scale. It is often only observed in the case of great similarity between problems.
The lack of evidence for transfer may very well be due to the poor discriminative power of transfertests, wherein items have to be solved independently.
It was established in this research-project that obtaining help during testing-situations can be scored in an objective way. Therefore, basic methodological objections against such procedures hardly seem valid. Equal help was available for all subjects, offered by means of a computer, without the intervention of an experimenter.
Although questions concerning equal impact of information contained in every hint for each subject still remain unresolved, we think that the development of this type of test could be of major importance for education.

Since there are so many confounding variables involved (for example: social economic background of students, quality of teachers and school-climate), comparison of the effects of teaching methods is a very difficult matter. Dubin & Taveggia (1968) even contend that no differences can be found in the effects of teaching students individually, in small groups or in lecturing groups, involving rather large amounts of students. Therefore, chances of finding differences in mathematical achievement between students educated by different series of textbooks seem remote. The influence of school environment, teachers, student population, must be a greater potential source of influence compared to what school mathematics book is being used.
In this study no support could be found for the hypothesis that a loosely structured teaching method (method C) renders relatively bad learning results for students low in cognitive ability. On the contrary, cognitive ability appeared hardly relevant for students educated by such a method. This may imply that loosely structured teaching methods are indeed beneficial for *all* students.
Furthermore a significant advantage was revealed for students using a "structuralistic" teaching method (e.g. method B), compared to students educated with a "mechanistic" teaching method (method A).
Therefore, we may conclude that structuralistic teaching methods should be preferred over mechanistic teaching methods. However, the effect of structuralistic teaching methods depends on cognitive abilities of students equally strong as the effect of mechanistic methods. In contrast, the relationship of the effect of realistic teaching methods with cognitive ability of students appears hardly important. In order to ensure optimal results for *all* students, the application of such teaching methods seems very promising.
At present we are conducting a preliminary investigation into the effects of realistic teaching methods for mathematics in the Netherlands in different types of secondary education. We hope to be able to present results of this study at future conferences.

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# Didactical Analysis

# ON LONG TERM DEVELOPMENT OF SOME GENERAL SKILLS IN PROBLEM SOLVING: A LONGITUDINAL COMPARATIVE STUDY

Paolo Boero, Dipartimento di Matematica, Università di Genova

*In this report I relate upon a research performed on written texts of 65 grade II to the grade IV primary school pupils, concerning the development of their hypothetical reasoning skills in problem solving. Some comparisons are made on the nature of the problem situations in which these skills appear for the first time: mathematical or non-mathematical problems; situations contextualized or not in some "experience field" of our project.*

## 1.Introduction

In the researches concerning applied mathematical problem solving performed during the next fifteen years,a growing importance is attributed to the "content" of the problem situations,to the "context" in which they take place and to the children's involvement in the "context " , in order to interpret both the conceptual and the procedural acquisition of the pupils, and the difficulties they meet in some circumstances (Carpenter & Moser,1983 ; De Corte & Verschaffel,1987; Nesher,1980 ; Lesh,1981 ; Lesh,1985 ) . Carraher's (1988) results concerning the acquisition ("in the street situations") of important problem solving skills of "strategical" type for problems concerning "money" give further insights in the same direction.

In (Boero,1988) I examined the pupils' "sensitivity" to the "content " and to the "context "of some problem situations taking place in the Genoa group's project for the primary school.In (Boero,1989) I proposed a theoretical framework for the "context" problem , in relation to the experiences of the curricular projects developed by my group and current researches. In that conference, I advanced the hypothesis that the choice of suitable "fields of experience" might influence the development of general problem solving skills concerning the "representation" processes,hypothetical reasoning and metacognition.

Ferrari (1989) considered the hypothesis (derived from the classroom observations of many teachers working in our group) that the child *"who uses properly hypothetical reasoning in mathematical problem solving is already able to use it properly in other settings".*

In this paper I will provide more precise elements to support the hypothesis that children's involvement in suitable "contexts", in classroom activities, may influence the development of some general problem solving skills.These elements might also contribute to clarify the relations existing between the development of general problem solving skills in mathematical and in non-mathematical problem situations.The research I relate upon concerns some of the skills involved in generating and managing hypotheses during the construction of strategies in problem solving . I wish to compare the moments at which these skills appear in written form during non-mathematical activities,and in contextualized or noncontexualized mathematical problems, for children followed from the age of 7 to the age of 10.

## 2.The educational context

As described in more details in (Boero,1988 and Boero,1989 ) ,the educational context to which I will refer is a curricular project concerning mathematics and the other main subjects taught in the Italian primary school.This project consists in a systematic work in suitable "experience fileds"(Boero,1989) concerning the natural and social reality (for instance: "productions in the classroom", "history of the family", "economical exchanges", "the sun shadows", and so on). Most of the mathematical problem situations proposed to the children are inserted in these "experience fields " .However,the work in each "experience field" also concerns many non-mathematical activities (for instance,performing experiments and writing reports about them; studying historical and geographical topics about the experience fields; and so on).These activities naturally produce many non-mathematical problem situations.

Particular importance is given in the project to the verbalization processes and to the activities aimed at developing verbal competencies.

Each year,the controlled experimentation of the project concerns about 40 classes of each grade from the age of 6 to the age of 10.For each grade 4 classes are "observation classes", and these classes are followed from the first to the fifth grade, usually by the same teachers (according to the Italian tradition) who collect all the pupils' written texts with detailed information about the conditions in which they were produced.For the observation classes we have at our disposal, amongst the others materials, individual written texts concerning: non-mathematical activities performed in the "experience fields"; mathematical problem solving contextualized in the "experience fields"; decontextualized problem solving. For each child belonging to an "observation class" we collect, during five years, about 1000 written texts (about 1 each day) About 40% of these materials concerns (partly or completely) mathematical activities.

## 3.Posing the research problem

The structure of the project and the materials we collect in the "observation classes" allow us to perform some systematic analyses concerning the development of general problem solving skills and the influence of the "experience fields" on it.However,it is necessary to distinguish carefully the analyses which may really give a reliable insight to the questions considered .For instance,the materials at our disposal allow us to establish when and in which context some skills appeared in written form,for the first time,in order to study the transfer to other contexts and situations, and so on.Then we must focus on the skills whose development we wish to analyze.With regard to the hypotheses quoted in the introduction, we must choose skills involved both in non-mathematical and mathematical problem solving activities (and, in this case, in contextualized and noncontextualized situations).For a first longitudinal study,I consider in this paper some skills concerning "generating and managing hypotheses in problem solving " in order to ascertain the influence of the context,of the content of the problem situation and of the teacher's request on their appearance and transfer.These skills may be described in more detail by considering the following two kinds of performances:

TRIALS-type performances : the pupil makes (heuristic) trials and,after analyzing their outcome, plans the following activities, in order to reach the solution ( *"I make the assumtion that...; than I see that...; and so I must take..."*)

BONDS-type performances : the pupil discovers the bonds inherent to the problem situation and builds a strategy leading to the solution according to these bonds (*"I must take into account that...; and then if ....else..."*)

These kinds of performances are relevant in many problem solving situations (Ferrari,1989),and their importance is growing in the computer age.As we will see in the examples,frequently both performances need to be taken into account in the same problem solving process and in many cases they are interwoven.

The requests activating the production of the written texts, which reveal the skills we are considering, may be of three kinds:

(VD) Verbalizing the reasoning "During" the performance*(" write down what you are thinking ")*

(VA) Verbalizing the reasoning immediately "After" the solution has been attained *("relate about your resolution of this problem")*

(VG) Verbalizing a "General" method of resolution *(" explain to a friend of yours how he may solve this problem")*

It is necessary to point out the fact that under one of these requests many children combine elements referring to the others.For instance in the texts produced under a (VG)-request they frequently combine (VG) -pieces and (VA)-pieces (especially if children have already solved the problem in a particular case).In the texts produced under a (VD)- request they frequently insert (VA)-pieces (because many children prefer to get a partial solution and then to relate upon the method utilized to get it).We may observe also that in the same problem situation the nature of the request may pull towards TRIALS-type or BONDS-type performances: in many problem situations a (VG)- request pulls towards BONDS-type performances which are less important under a (VD)- request.

As typical problem solving situations demanding the skills we are considering,we may quote (from our project):

(EXAMPLE 1) *To set a wood table in an horizontal position over an "irregular" ground, utilizing a spirit level.*. It is a problem situation which only partially refers to mathematics ( it will be classified later as a "mixed situation"); in our project this problem situation is contextualized in the "experience field" of the "sun shadows"; TRIALS-type and BONDS-type performances are required, the latter especially under (VG)-requests.

(EXAMPLE 2) *To relate upon an experience of production in the classroom, demanding to perform some "controls" and choices consequent to the outcomes of these controls.* It is a problem situation without (or with poor) mathematical content, contextualized in the esperience field of the "class productions", and relevant especially for BONDS-type performances (particularly under a (VG)-request: *"write down the recipe to prepare the cake..."*)

(EXAMPLE 3) *To divide an expense (like 107000 liras) amongst some children (for instance,* *34 pupils)* : before a written calculation technique has been taught, this problem situation (of mathematical content) demands TRIALS-type performances (Boero, Ferrari & Ferrero, 1989); the problem may be posed as a completely noncontextualized one,or as a question naturally contextualized in the classroom activities.

Referring to these examples and to the conceptual tools which are utilized during problem solving processes, we may distinguish (but see also §6 ) three kinds of problems: "non-mathematical problems", like in EXAMPLE 2; "mixed problems", like in EXAMPLE 1; and "mathematical problems", like in EXAMPLE 3 .

## 4.Available materials and utilized materials

I have considered the individual written texts collected in four "observation classes" (73 pupils) followed by the same teachers during the school years 1983/84 to 1988/89 . I have restricted my analyses to grade II,III, and IV (where the activities aiming to develop hypothetical reasoning are more frequent), and to written texts referring to mathematical and non mathematical problem situations in which TRIALS-type and/or BONDS-type performances are demanded. With these restrictions, I have taken into account 5510 texts referring to 128 problem situations (many of them common to 3 or 4 classes) which may be classified as follows: about 75% contextualized in some "experience field"(and 25% non-contextualized); about 32% of non mathematical type; about 20% of mixed type; about 48% (23% contextualized, 25% non-contextualized) of prevailing mathematical type. All the non-contextualized problem situations are of mathematical type.This distribution is almost uniform for every grade and for every class considered,and almost uniform during each year.This "uniformity" may be explained by the fact that the classes adopt the same project and that the project maintains, at every grade, an equilibrium amongst the different kinds of activities we are considering.

On every written text considered the teacher noted the conditions in which it was written. The texts were produced during the first or the third phase of the usual work of our classes on a given problem situation:

- phase I:individual work,with individual writing of a text (under requests of VD,or VA,or VG-type) ; we will refer to these texts as "Autonomous Without Discussion" (briefly, AWD-) texts
- phase II: discussion of the strategies proposed by some pupils (who illustrate them to the other pupils); if there are no valid strategies,the teacher gives some suggestions and points out the inadequacy of the proposed strategies
- phase III: individual work, with individual writing of a text.The pupils having already produced a good AWD-text are asked some further question (not considered for the aims of this analysis); the others pupils produce "Autonomous After Discussion" (briefly, AAD-) texts, or need the individualized support of the teacher in order to produce " Supported texts"
- phase IV: comparison,correction and enregistration on the individual "copybook" of the whole work performed in the classroom on the problem situation considered.Incidentally I observe that the analysis of these "copybooks" may be very useful to reconstruct, after some years, the conditions in which the texts at our disposal were written and to complete any information

eventually lacking or ambiguous on the written texts collected.
Examining with the teachers (M.G.Bondesan,A.Carlucci,E.Ferrero, G.Pontiglione, A.Rondini) the records of the 73 pupils ,we excluded 8 pupils for the following reasons:1 was frequently absent at grades II and III and followed the activities of another class during five months in grade IV; 3 had a record containing many scarce and/or confused "traces" (expecially in grades II and III);4 revealed a complete mastery of hypothetical reasoning in mathematical and non mathematical situations already at the beginning of grade II.
Considering the other 65 pupils,we had 4975 written texts at our disposal; by eliminating 1312 texts (unintelligible; or too scarce; or containing strategies not aimed at solving the posed problems- the greatest majority of these texts were "Autonomous Without Discussion" texts), I considered the remaining 3663 texts.It must be pointed out that these texts may contain "good" strategies, or also partly wrong resolutions .The reason for this choice is the fact that a partly wrong resolution may contain a good approach to hypothetical reasoning .
TABLE 1 gives some information about the distribution of the texts which have been taken into consideration for the following analyses. The percentage is evaluated on the data of each line.

TABLE 1 - distribution of the 3663 written texts utilized (65 pupils)

| | CONTEXTUALIZED PROBLEMS | | | NON CONTEXT.PROBL. |
|---|---|---|---|---|
| | NON MATH.PROB. (32%) | MIXED PROB. (20%) | MATH.PROBL. (23%) | (MATH.PROB.) ( 25%) |
| AWD-texts | 463 (35.5%) | 298 (22.8%) | 283 (21.7%) | 261 (20.0%) |
| AAD-texts | 452 (36.0%) | 265 (21.1%) | 261 (20.8%) | 278 (22.1%) |
| supported texts | 326 (29.6%) | 210 (19.1%) | 268 (24.3%) | 298 (27.0%) |

It may be observed that the first line contains only 1305 texts, with a distribution which leaves to mathematical problems (48% of the total) only 41.7% of the texts; in fact, mathematical problems (especially the non contextualized ones ,25% of the total) give a larger contribution to "scarce" or "completely wrong" strategies). It must be pointed out,however,that non-contextualized mathematical problems are generally not more "difficult" (as far as the mathematical concepts and procedures involved are concerned) than the contextualized problems proposed at the same time.Then we find here a first element which indirectly supports the hypothesis that the "content" and the "context" of a problem situation influence the quality (autonomy..) of the performances related to hypothetical reasoning.

**5.Some analyses performed and their results**

We tried to get information pertinent to the problem posed in § 3 from the 3663 selected texts.

A first analysis was performed on the <u>first occurrence</u> of an hypothetical reasoning (TRIALS-type or BONDS-type performances) in the "Autonomous Without Discussion" (AWD) -texts <u>or</u> in the "Autonomous After Discussion" (AAD)-texts produced by the pupils.The data obtained is shown in TABLE 2 (2 pupils out of 65 had never produced an hypothetical reasoning in problem solving activities up to grade IV):

TABLE 2 : first occurrence of hypothetical reasoning

| | CONTEXTUALIZED PROBLEMS | | | NON-CONTEXT.PROB. |
|---|---|---|---|---|
| | NON MATH.PROB. | MIXED PROB. | MATH.PROB. | (MATH.PROB.) |
| AWD-texts | 7 | 4 | 2 | 0 |
| AAD-texts | 23 | 13 | 10 | 4 |

Some remarks:

- the 13 pupils of the first line are all considered by their teachers to be "good problem solvers" in any kind of problems
- amongst the 50 pupils of the second line,we have considered the 19 pupils who needed more than 3 individualized interventions (registered in "supported texts") before producing an AAD-text revealing the presence of hypothetical reasoning; they are considered to be "poor problem solvers" by their teachers.They produced their first hypothetical reasoning in a non-mathematical contextualized problem in 12 cases, and in a non-contextualized (mathematical) problem in 2 cases.

Another analysis was performed on the 59 pupils (out of 65) who produced an AWD-text revealing the presence of hypothetical reasoning before the end of grade IV

TABLE 3 : first occurence of hypothetical reasoning in AWD- texts

| | CONTEXTUALIZED PROBLEMS | | | NON-CONTEXT.PROB. |
|---|---|---|---|---|
| | NON MATH.PROB. | MIXED PROB. | MATH.PROB. | (MATH.PROB.) |
| AWD-texts | 28 | 16 | 13 | 2 |

We have also performed an analysis about the time delay separating the first hypothetical reasoning performance from the transfer to other kind of situations in which it occurred; we got the following results:

- transfer from "non- mathematical" to "mathematical" problems in AWD-texts: 26 cases ; mean value of the delay, 3.4 months (standard deviation: 1,4 months);transfer not realized in 2 cases
- transfer from "contextualized" to "non-contextualized" problems in AWD-texts: 45 cases; mean value of the delay: 7.1 months (standard deviation:3.1 months);transfer not realized in 12 cases

Other analyses performed concern:

- the age at which pupils,on average, reveal for the first time an hypothetical reasoning in an

AWD-text (63 cases,adding the 4 cases already revealing the mastery of hypothetical reasoning at the age of 7,evaluated with the age of 7) : we have found an age of 8 years and 5 months, with a standard deviation of 8 months

- the kind of requests under which pupils produce their first written hypothetical reasoning : for the reasons considered in § 3, it is not easy to answer this question; considering only 28 pupils producing a text perfectly coherent with the kind of request, we see that in 24 cases the request "forcing" the hypothetical reasoning is a (VD)- or an (VA)-request .This result may be explained in two manners: by the fact that in our classes the (VG)-requests are more frequent at the IV grade than at the II grade; and/or by the fact that the (VG)-requests are more" difficult" than the others (pupils need to take into account also "cases" not experienced while solving the problem in a particular situation).

## 6.Discussion

The results seem to prove that:

i) the classroom work contextualized in the "esperience fields" of our project anticipates (in comparison with non-contextualized problem situations) the development of the skills of hypothetical reasoning considered in this paper.This result agrees (for those problems demanding TRIALS-type or BONDS-type performances) with the general results quoted by Lesh (1985) concerning the success of good problem solvers "experts" in a given domain *"who tend to use powerful content-related processes"*, and the failures of pupils *"who do not have relevant ideas in a particular domain"*.

ii) the classroom work in non-mathematical,well contextualized problem situations favours some anticipation in the development of hypothetical reasoning in comparison with problem situations strongly referring to mathematical contents

Possible limitations to these results may depend on:

- the arbitrary classification of problems,both concerning the distinction between "mathematical", "mixed" and "non-mathematical" situations, and the distinction between "contextualized" and "non-contextualized" situations (for instance,a problem like EXAMPLE 3 in § 3 may be proposed as a "contextualized" problem in a class working on "class productions",or as a "non-contextualized" problem one month later...but children may easily refer the problem to their past experience...).This limitation appears to be intrinsic to the research.
- the small number of pupils involved (but this limitation might be overcome by extending the analysis to the other "observation classes" of our group); I observe however that all the results of the analysis performed on this group of 65 pupils agree with i) and ii)
- the "didactical contract " (Brousseau,1984) taking place in the non-contextualized situations ; *often* they are "evaluation tests",and then many pupils consider them *in any case* as "evaluation tests" (also if they are proposed without this aim); this might reduce the "acceptance of risk" which favours the TRIALS-type performances . However we observe that many pupils do not

transfer the skills already extensively revealed in contextualized situations of lower level of difficulty to non- contextualized situations

- evaluation of the solving strategies: generally the evaluation is not difficult for the good problem-solvers ; on the contrary, it may be arbitrary in many cases concerning the low level of the classes (confused verbal "traces", interference of suggestions coming from the schoolmates
- interpretation of the verbal productions of pupils: generally the syntactic analysis of the verbal "traces" of the solving strategies is not sufficient to ascertain the presence of an hypothetical reasoning (which frequently is expressed without an hypothetical period); then it is necessary to make a semantic type analysis,and this implies (in some cases) rather arbitrary choices.

Keeping these limitations in mind, the internal coherence of the results of the analysis performed, in agreement with the "impressions" of many teachers of our group who have observed the same phenomena in their classes , and the external coherence with the results of other researches on problem solving seem, however, to enhance the validity of the conclusions (i) and (ii).The next step of the research will be to deepen the analysis of the factors which allow the "experience fields" to act on the development of the skills concerned in this paper : my present opinion is that the "motivating" factor is not the most important one, and that ,on the contrary, to discover the most relevant factors it is necessary to analyze the mental processes which bring the pupil to a total mastery of the "experience fields" .In any case, the impact of this kind of research on mathematical education is not negligible, due to the fact that TRIALS-type and BONDS-type performances are of great importance for problem solving with the computers; and that most of the problem situations proposed to pupils in our primary school are of non-contextualized, mathematical type.

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# COGNITIVE DISSONANCE *VERSUS* SUCCESS AS THE BASIS FOR MEANINGFUL MATHEMATICAL LEARNING

Nerida F. Ellerton and M.A. (Ken) Clements
Deakin University, Australia

**Abstract**
*Cognitive dissonance theory has often been advocated as a guide to mathematics teachers interested in creating stimulating learning environments for their students. This paper contrasts cognitive dissonance theory with the success-based theory of Karmiloff-Smith, and argues that the latter theory is more compatible with natural learning environments in the mathematics classroom. A case study which involved 28 pupils and 2 teachers in a primary school is outlined. The natural learning environment created by the teachers enabled significant worthwhile mathematical learning to occur - learning that contrasted with that arising out of more traditional mathematics classroom environments.*

## Learning as a Result of Cognitive Conflict *Versus* Learning Based on Success

*Cognitive Conflict Theory*

There is a considerable body of data supporting the idea that children best learn mathematics by being exposed to their misconceptions before actively resolving their inner cognitive conflict (see, for example, Bell 1986; Bell and Bassford, 1989). Typically, Piaget's equilibration principle, with its twin notions of assimilation and accommodation, is called upon to provide a theoretical basis for this conflict-resolution theory. Concerning the role of the teacher, Piaget (1975) himself wrote:

> The teacher as organiser becomes indispensable in order to create the situations, and construct the initial devices which present useful problems to the child . . . he [the teacher] is needed to provide counter-examples, that compel reflection and reconsideration of over-hasty solutions. (p. 16)

Vygotsky's (1986) notion of a zone of proximal development is also invoked to support the theory (see, for example, Brown and Campione, 1984, pp. 145-146). The zone of proximal development is said to refer to the distance between the level of performance the child can reach unaided and the level of participation the child can accomplish when guided by someone else who is more knowledgable in that domain. For a particular child in a certain domain, this zone may be quite small; that is to say, the child is not yet ready to participate at a more mature level than his/her unaided perfomance would indicate. For another child in the same domain, however, the zone of proximal development can be quite dramatically large, indicating that, with teacher assistance, and sometimes minimal assistance at that, the child can participate much more fully and maturely in the activity than one might have supposed. In traditional terms, these are notions of "readiness," and Vygotsky's theory is invoked to justify the teacher manipulating a learning environment so that the child will experience cognitive dissonance and, as a result, move rapidly within his/her zone of proximal development.

*Learning Mathematics Naturally*

Michael Cole and his colleagues (Laboratory of Comparative Human Cognition, 1983) have appealed to Vygotskian theory to develop a theory of learning which differs from cognitive conflict theory. Focusing on Vygotsky's notion of expert scaffolding, and in keeping with his long-standing in "natural learning," Cole points out that children of many cultures are initiated into adult work-activities gradually, and without explicit instruction. The adults simply get on with doing their work, and the children participate, first as spectators, then as novices (responsible for very little of the actual work), and then increasingly as maturing participants. They become capable of performing more complex aspects of the work that they have seen modelled by adults many times before (Brown & Campione, 1984, p. 146). In this situation, the main agenda is the natural one of getting the task done, and the idea of helping children learn is less important than the work activity itself; there is no suggestion of a teacher-learner relationship in the situation, yet the natural learning environment is powerful, perhaps even more powerful than a cognitive dissonance teacher-guided environment.

Mathematics educators have long taken the idea of a natural learning environment for mathematics seriously (see Clements & Del Campo, 1987, pp. 4-39), and there is evidence to support the osmosis theory of mathematics learning (Ellerton, 1988). It needs to be recognised, however, that the natural learning thesis differs sharply from cognitive dissonance theory. As Peled and Resnick (1987) pointed out:

> The natural-environmental approach suggests that understanding of a concept emerges from dealing with real world situations; therefore the exemplifications should be the situations themselves, rather than a representation of the abstract mathematical entities. The structural approach, on the other hand, treats the abstract mathematical entities and their mathematical senses as the reference of the exemplification. Real world situations, according to the structural approach, should be introduced instructionally only after the formal system has been established. (p. 185)

Peled and Resnick (1987) went on to argue that the natural approach is not necessarily superior, from a learning perspective, to the more structured approach. They outlined an investigation in which they defined numbers and operations first, and only later introduced real world situations (p. 189).

*Success rather than cognitive dissonance as the basis of learning.* The cognitive developmental psychologist, Annette Karmiloff-Smith, is another who has questioned some aspects of the cognitive dissonance theory of learning. Karmiloff-Smith (1984, p. 40) has argued that neither failure nor economy are the major motivations of developmental change. In her view, cognitive change, in an individual, stems from a constant motivation for control, both over the external environment and over one's internal representations - in this respect she is in line with cognitive dissonance theory. However, she focuses on the gradual process of gaining such control, demonstrating how children spontaneously go beyond initial success, achieved through their adaption to environmental feedback, to working subsequently on their internal representations as a form of problem solving in its own right. Success, and not failure, she argues, is the essential prerequisite of fundamental developmental change; in fact, failure is rarely the prerequisite of representational change. According to Karmiloff-Smith (1984):

> Failure generates behavioral change during which the system evaluates, and narrows the distance between, the child's goal and the child's present output. By contrast, representational changes are the result of representational reorganizations, the prerequisite for which is not failure, but procedural success. (p. 40).

Thus, Karmiloff-Smith (1984, p. 40) claims, "once children have obtained a robust initial success, they go beyond it and try to understand why certain procedures are successful, unpacking what is implicit in them, and unifying separate instances of success into a single framework."

### Cognitive Conflict *Versus* Success-Based Mathematics Classrooms

The previous discussion outlining the differences between learning theories based on cognitive dissonance and "success" theories has important implications for mathematics classrooms.

*Cognitive dissonance classrooms*. In these classrooms, it is the role of the teacher to arrange for learners to experience cognitive conflict situations at just the right time. It follows that teachers need to be constantly assessing what experiences need to be provided for different learners, and trying to ensure that the appropriate conditions for these experiences are present in the classroom.

A likely consequence of the cognitive dissonance approach is that teachers will make most of the decisions concerning what, how, and when individual children should learn. In such circumstances, it could hardly be claimed that the children "own" what they are learning; rather, the teacher and the textbook writer are likely to be seen by the students as the controllers of their mathematical destinies. The quest to understand becomes an individual pusuit, and is therefore more likely to be a competitve, rather than a collaborative act.

*Success-based classrooms.* Here it is likely that teachers will be more accepting of what students do. Consequently, adult-imperfect responses will be accepted as appropriate provided they are consistent with perceived cognitive growth. The teacher's focus will be more on what children *can* do, and less on what they *cannot* do. The classroom environment is likely to be closer to a so-called "natural" learning environment. In a success-based classroom, the teacher will not feel the need to assess constantly what the child is ready to learn, or what conflict situation needs to be imposed to ensure maximum learning. Therefore, learning from other children, both through the process of osmosis and through discussion, is more likely to take place. The "ownership" of knowledge is not seen as resting with the teacher or the textbook, and the quest to understand can be seen as deriving from a combination of individual and group activities.

*Strathbogie Primary School: An Example of a Success-Based Mathematics Classroom*

Videotapes were made of 28 children, aged 5 to 12, involved in mathematical activities with two teachers in two composite grades (covering the whole range of early childhood and primary schooling) in a small rural school in Strathbogie in north-eastern Victoria. The teachers (one male, one female), each with eight years' teaching experience, were totally committed to a success-based approach. Our observations of the children doing mathematics in their classrooms convinced us that, not only were the teachers respected by the children, but the children realised that they themselves were responsible for their own learning of mathematics. The children themselves decided what aspect of mathematics they would investigate on any particular day, whether they would work individually or in a group, whether they needed equipment, what books they would use, and how they would record their findings (no student questioned the need to record). The students also decided whether they wanted to consult with their teacher about what they were doing.

Importantly, both teachers believed that their role was to be seen doing mathematics that was relevant to them (i.e. the teachers) personally (see Waters & Montgomery, 1989, for a more detailed statement of how the teachers saw their role). They worked quietly on their own mathematical problems (on a particular day, one teacher worked on the costing for a school camp that was imminent, and the other on the meaning of the Richter scale for earthquakes). The children did not expect the teachers to move around the classroom asking them questions. There was a definite impression that they, and not the teachers, owned the mathematics they were doing.

We interviewed some of the children on videotape, and in every case their responses were delightfully fresh, creative, and unihibited. They were unashamed of learning from others either by questioning or by observation; there was no sense that it might be "cheating" to watch how someone else approached a problem.

We do not wish to give the impression that initially, at least, some of our observations did not cause us concern. We wanted to correct the child who consistently reversed digits (though she read them correctly); we wanted to intervene and create cognitive conflict (in fact, we attempted to do so on a few occasions); and we continually asked ourselves "how well do these children know their basic mathematical facts?"

Yet, on reflection, we are certain that, if we had attempted to find out how well the children their basics by administering a standard, norm-referenced, pencil-and-paper test of basic mathematical skills, we would have been raping a system which had given children confidence to explore, to ask, to cooperate, to feel comfortable, and to learn mathematics. Probably, such a norm-referenced test would have found the class below the mean. However, we believe that the international mathematics education research community needs to question the validity of such an assessment for classroom situations such as the one we have described. Validity, of course, is related to objectives, and we believe that a willingness to explore, ask, and cooperate in mathematics, and a spontaneous enjoyment reflect higher order objectives than do skill-based objectives.

*The Teachers' Perceptions of their Task*

In a jointly authored article about the Strathbogie mathematics program, the two teachers commented that they had drawn together what they knew about learners and what they knew about mathematics in developing strategies aimed at assisting the children in their care to become better mathematicians (Waters & Montgomery, 1989, p. 81). In describing the program, they explained that some children work in pairs or in groups, but most work individually; there is constant discussion, often between an older and a younger child. The children ask each other for assistance regularly, and usually "some children are working in the classroom, some in the corridor/kitchen area, some are in the office, and some are outside" (p. 82). After stating that the children generated their own mathematical tasks, the teachers gave examples of activities that took place. These included:

* Drawing shapes with a ruler and measuring the corners with a protractor (Tristan, aged 6)
* Measuring the dimensions of a football (Jason, aged 7)
* Trying to cram a matchbox with the maximum number of different items (Roslyn, aged 11)
* Using a bead-frame to record sets of counting (1, 2, 3, ..., 10) while timing a minute using a stopwatch (Ben aged 6, and Shannon aged 7)
* Writing a description of what is understood of a short division algorithm (Loretta, aged 10)
* Measuring the distance from one set of goalposts to the other [in the school grounds] (Maren, aged 6)
* Making a scale model of the monkey bars using wire (Robyn aged 9, and Terry aged 10)
* Recording subtraction equations that give a negative number display on a calculator (Ryan, aged 7)

Waters and Montgomery (1989, p. 81) added that at 10.30am the children pack their equipment away and write individual descriptions of their work. The teachers also do this.

Among a series of individual "snapshots" provided by Waters and Montgomery (1989, p. 84) was the case of Bernadette, aged 11, who wanted to investigate how much burning time one box of matches would provide. She worked at the sink with a box of matches, holding one match at the very end with a pair of tweezers. For a while she experimented, lighting some matches and seeing them burn. She then asked Roslyn (aged 11) to help her for a few minutes. Bernadette had Roslyn used a stopwatch to time the burning of one match. There was much discussion over when to start and stop timing. "From the time the flame starts till the time the flame dies." They ran five trials, and Bernadette recorded each time.

Bernadette took her workbook back to the classroom. She had five various burning times between 31 and 33 seconds. She used these two measurments as minimum and maximum burning times, and disregarded the other three times. She then multiplied both numbers by 50 (there are 50 matches in a box) using a standard algorithm. To convert her answers from seconds of burning time to minutes, she divided both numbers by 60. To do this she used a calculator. Bernadette found that a box of matches has a burning time of between 25.83 and 27.50 minutes.

After this, Bernadette wrote about the mathematics she had done. "Today I burnt a match right to the bottom with a pair of tweezers, and timed it to find out how long it would take to burn a whole box of matches, one after the other. "33 x 50 = 1650, 31 x 50 = 1550."

At the conclusion of their article on the Strathbogie mathematics program, the two teachers said they believe that to be mathematics teachers, they must be practising mathematicians. That is why they themselves always do their own mathematics alongside their students. By doing this, they model both the scope of the mathematics course, and what it is to be a healthy learner of mathematics - self-motivated, self-directed and self-regulated. They said that they attempt to create an atmosphere that is "risk-free": learners' attempts are valued. When their pupils talk to them about mathematics, they are particularly interested in whether the pupils:

- see themselves as mathematicians;
- want to take responsibility for their own learning and to make sense of what they are learning;
- use mathematics frequently without inhibition
- believe that making mathematical sense of the world, and learning more mathematics isn't hard work, but is engaging and exciting;
- are willing to seek help from the Strathbogie Primary School community of mathematicians (pupils and teachers) which responds to their challenges, frustrations and successes (p. 85 )

## A Concluding Comment

Recently we claimed that "despite the universal rhetoric about school mathematics being integrally linked to scientific, technological and economic development . . . the main lesson learned by most school leavers after year of being forced to study mathematics is that they can't do it" (Ellerton & Clements, 1989, p. vii). It is possible that all around the world there needs to be a reconceptualisation of what school mathematics should be about (the "why"), what mathematics should be studied in schools (the "what"), how it should be presented and how it should be assessed (the "how"). Our observations of mathematics being done in a small rural school, some of which we have captured on video (Ellerton & Clements, 1990), suggest that a success-based theory of mathematics learning, linked with "natural" classrooms, might offer children far more, so far as their future mathematical growth is concerned than do traditional cognitive-conflict, teacher-textbook-owned approaches to school mathematics.

Perhaps there needs to be a whole new approach to mathematics curricular design (see Steffe, 1989). We are concerned that mathematics teachers and educators around the world, bolstered by the high status accorded to the subject they teach, have burried their heads in the sand and therefore remain oblivious to the irrelevance of an adult-defined, adult-monitored, adult-assessed, middle-class, largely male-inspired school mathematics agenda.

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# TIME AND HYPOTHETICAL REASONING IN PROBLEM-SOLVING

Pier Luigi Ferrari, Dipartimento di Matematica, Università di Genova, Italia

*Hypothetical reasoning plays an important function in problem-solving and, in particular, in resolution processes of complex problems. Many clues suggest that hypothetical reasoning cannot be analysed without taking into account the role of time. For example, I have observed that the 'if P then Q' construction has, for children, a meaning which is contiguous to the meaning of constructions such as 'when P, Q' and, in problem-solving, the ordering of the steps of the solving procedure is related both to time and logical consequence.*

*I try to explain the role of time in hypothetical reasoning as regards both the interplay between the logical and the chronological structure of events and the situations which allow a more frequent production of hypothetical reasoning. This may be related to some findings regarding the role of space-time representations in problem-solving.*

## 1. Introduction

In [Ferrari, 1989] I discussed the role of hypothetical reasoning in problem-solving at the age of 8-10. It seems to be crucial as regards relatively complex problems, which need the construction of a strategy with 2 or more steps. Examining more closely this issue I have observed a lot of phenomena which emphasize the importance of the variable 'time' in problem situations and the role of children's mental time when constructing a procedure. In particular, many clues suggest that the management of hypothetical reasoning is strictly related to the management of mental time by the child.

At this regard, out of the phenomena I have noticed, I report:

$a_1$) Children very often use connectives related to time (as 'when', 'till when' and so on) in order to denote hypothetical reasoning.

$a_2$) In most of arithmetical word problems with all necessary data explicitly given in the text of the problem, children, when asked to record 'a posteriori' their procedure and reasoning, perform it with a wide use of other connectives related to time (as 'then', 'so', 'after' and so on), but without explicit hypothetical constructions.

$a_3$) The situations in which the explicit production of hypothetical reasoning seem more frequent are those which allow the children to reflect about their own (or other children's) reasoning and to work in conditions of cognitive detachment. Situations like those allow children to use time freely as a basis for simulation (leaving from real time).

These observations are included in a wider frame of phenomena regarding the role of time in problem-solving. I am not going to examine them closely in this report but I shall refer to them on many occasions:

$b_1$) more or less 'expanded' management of problem-solving strategies (in particular, division strategies); the child, after understanding an algorithm, uses it without reconstructing in his mind the 'expanded' procedure which has generated that algorithm (e.g. the Greenwood algorithm for division, which in our curriculum is introduced as a natural outlet for more and more organized and effective trial-and-error strategies - see [Boero, Ferrari, Ferrero, 1989]).

In this way there is a sort of time-contraction which seems to affect even children's reasonings as they are expressed.

$b_2$) Specific difficulties children usually get into in problems involving time as a physical variable and correlations between problem-solving skills and space-and-time-managing skills.

The main aim of this report is, in short, to explain the role of time in hypothetical reasoning as regards both the interplay between the chronological and the logical structure of events and the situations which allow a more frequent production of hypothetical reasoning (in a 'spontaneous' and explicit way). As far as this last aspect is concerned, I shall refer to some findings regarding question $b_1$ related to the effects that the contraction process seems to generate on children's behaviour, on the management of problem-solving procedures and even on the kind of reasoning recorded.

## 2.The context

In this section I provide some information essential to understand the paper. A further information on the educational frame in which the research is included can be found in [Boero,1988], [Boero,1989], [Boero - Ferrari - Ferrero, 1989], [Ferrari, 1989].

In our project, 'experience fields' are strongly stressed; in particular problem-solving is dealt with mainly in 'experience fields'; children are often asked to build, in a context providing meaningful constraints, strategies in order to calculate arithmetical operations (as division) and the management of trial-and-error strategies is strongly enhanced.

- A wide space is assured for activities such as verbalizing, reflecting on the meaning of connectives (if...then..., while, whereas, when, till when, ...), analysing and describing complex machines and procedures.

The didactical context provides, anyway, many occasions of cognitive detachment, as, for example, when comparing different strategies or distinguishing between "how the machine works" and "how we can use the machine",...

## 3. Methodology of the research

We have a lot of materials from 'observation classes' (from which we gather, from grade 1 to grade 5, all texts individually produced by each child) and 'assessment tests' (administered at the half and at the end of the school-year) from alla the classes. As regards connectives, most of the protocols are 'spontaneous' productions by children. By 'spontaneous' we intend to refer to texts produced freely by the child, without any direct intervention by the teacher or his schoolfellows, but in a learning context which is planned to guide him towards a wide usage of complex syntactical constructions.

When examining children's 'spontaneous' productions it is necessary (as will be shown) to deal with the problem of the relationship between children's thought and the text which represents it.

Furthermore, when selecting and analysing children's spontaneous productions, it is necessary to take into account the time when verbalization has been performed; particularly, when the verbalization is performed while a problem is being solved, a wider presence of hypothetical reasoning and other syntactically complex forms has been observed; if it happens after a resolution procedure has been found, I have observed a great amount of sequentially-structured texts, with a wide usage of connectives as 'so', 'then', 'hence', ..., which are referred to both time-ordering and logical consequence. The ways in which the verbalizations are usually elicited fulfils the conditions stated by Ericsson & Simon [1980] not to affect child's mental process.

As regards the kind of materials, I have selected and analysed:

- normal working protocols, referred to situations in which the child is at ease but may be influenced by the teacher, on the ground of verbalization, since in our curriculum in some occasions the teacher 'lends the words' to the child in order to support him in expressing his thoughts.
- assessment protocols, which are produced in somewhat unnatural situations but are useful, related to the usual working conditions in class, to analyse children's behaviour without direct and immediate influences.

In this study I shall refer to these materials:

$m_1$) materials related to arithmetical word problems, with verbalization performed by children while (possibly with a tape-recorder) or after solving a problem.

$m_2$) written descriptions of everyday-life processes (e.g. how to prepare a coffee) and of the working of a machine (e.g. a slot-machine);

$m_3$) reports of discussions performed in class about the strategies each child has built in order to solve a problem (not necessarily a standard problem);

$m_4$) non-numerical word problems administered as assessment-tests at the end of primary school;

$m_5$) non-mathematical texts (e.g.: "Describe everyday-life in the Middle-Ages and tell if you would like living in the Middle-Ages").

The study will refer to materials selected among those produced in 2 classes of grade 4 and 2 classes of grade 5; for any grade considered there is 1 class from the suburbs of a big town and 1 class from a little town in the neighbourhood of another big town. Nevertheless, part of the findings I am going to present are supported by a greater amount of data, as it will be specified everytime. For example, the findings related to materials $m_1$ are supported by data from about 20 classes.

### 4. Some data from the analysis of the protocols

In various situations children use connectives as 'when', 'till when' and so on to express hypothetical reasoning:

a) When describing the working of complex machines (e.g. a slot-machine ) children uses either 'if... then...' or 'when' to express conditional controls; this seems not to be random, because to test each coin all children who insert a conditional control at that step use the form 'if... then...' (*"if the coin is 'good', the machine ..."*), whereas to test the total amount of the coin already inserted some 80% of the children who insert a conditional control use 'when' (*"When the amount of the coin inserted is 400 lire, the machine ..."*). Such a different usage might be accounted in relation to the different meanings the two controls assume for children. But it is not possible to claim certainly, without further evidence, that the usage of different linguistic forms is a signal of different ways of thinking, even if many of the materials I have examined seem to exclude that the usage of either of the forms should be random. The problem of the relationship between children's thought and linguistic forms they adopt to report them will not be examined closely in this report.

b) When describing other processes (e.g. the preparation of a cup of coffee) about 40% of the children uses almost once constructions such as "when P, Q" or "P till when Q" (e.g., *"when the coffee-pot is ready, put it on the fire"* or *"put water into the coffee-pot till when it is full"*). About 40% uses almost once 'if P then Q' (e.g. *"if the water is not yet boiling, wait a bit"*). About 15% of the children (all good problem-solvers) use both constructions. These children, when take into account the final amount of coffee use 'if...then...' (*"if the coffee is not enough, I must put more water, if it is too much I must put less water"*), whereas 'when' is mainly used related to more 'intrinsic' steps of the process, which are more difficult to master from the outside (*"when water boils, I must put the fire out"*). About 35% of the children (generally, poor problem-solvers) do not use any of the constructions I have mentioned, not making explicit any conditional control but introducing constraints in other ways (e.g. *"you must put enough of water into the coffee-pot, in order to prepare the right amount of coffee"*).

c) The same children of example 1 and 2 have been invited to describe the aspect of Middle-Ages everyday-life most striking for them, and to tell if they should like living in the Middle-Ages. About 50% of the children uses properly the 'if P then Q' construction almost once, and about 60% uses properly a conditional form with 'when'; the first group is contained in the second. Among children who use either form I have observed that constructions with 'when' are mainly used to speak of normal or unavoidable facts (*"when there was a war, many peasants would be killed"*), and those with 'if...then...' mainly to speak of facts more dependent on free choice of people or related to everyday-life (*"if a slave did not work, they punished him"*).

d) When discussing in class children's strategies and reasonings, about 80% of them uses almost once a construction with 'if...then...'. About 45% uses also a construction with 'when'. A very interesting example is the following. Reporting a discussion (performed in class) on the criteria proposed by the pupils to know whether a pin picked into a wooden board is vertical, Simone (a 4th grader, good problem-solver) writes: *"It is not true that if the pin is vertical, then the 'shadow' is long; it is quite the contrary, when the shadow is long, the pin is not vertical."*

Here, if the verbal forms adopted by the child reflect his thoughts, it is likely that the first construction (*"if the pin is vertical... then the 'shadow'..."*) is referred to a proposition to be falsified and the second (*"when the shadow is..."*) to a procedure to be performed (*"everytime I find a long shadow, I can argue..."*).

A general remark which can be done correlating the analysis of the protocols with a general information on the children who have produced them is that the children who never use any of the constructions mentioned are generally poor problem-solvers.

Related to the linguistic constructions used in the protocols, I have not found significant differences between $4^{th}$ and $5^{th}$ graders.

When solving problems with numbers, explicit forms of hypothetical reasoning can hardly be found if children already know an algorithm they can apply effectively to compute arithmetical operations; in conditions like these, in almost all the problems examined, no child uses explicitly a construction with 'if...then...' or 'when'; only a 10% of children uses in more than one problem constructions such as 'P since Q', which could be related to hypothetical reasoning. In most cases, the text is organized in an inferential, not hypothetical way: true statements are inferred from true statements and almost never explicit hypotheses are stated. The style is mainly procedural: children mainly connect and organize their actions (*"then I do..., and so I find..., after that I compute..."*) and hardly connect properties of objects with other properties of objects (e.g. number facts). Nevertheless, it is likely that even an inferential organization of the text may hidden forms of hypothetical thinking.

## 5. Space, time and hypothetical reasoning in problem-solving

I have found that, in the resolution of problems with numbers, children use widely hypothetical reasoning when they are forced to invent a strategy to compute an arithmetical operation (in particular, division). Nevertheless, (as already remarked at the end of section 4) these forms are no longer used when children can apply more contracted algorithms, which need not rebuilding every time the complete reasoning, though they can understand the meaning of what they are doing.

In my report at PME XIII I provided some examples which may contribute to a better explanation of these phenomena. In example 3 [Ferrari, 1989, p. 262], a child designes a resolution strategy for a division problem (to represent on a wall of the classroom a given period of time), organizing, with a wide resort to hypothetical forms, a trial-and-error strategy in which the child's spatial representation of the situation play an important function, as well as the representation of the procedure in a sort of mental time the child can manage quite easily. Protocols like this suggest that making hypotheses is strictly related to the mastery of mental time which allows the child to simulate in mind the possible developments of the situation.

Another example is the following: at the end of primary school, children have been asked to design a general procedure to order alphabetically an arbitrary set of surnames. In the texts produced by the children I have found a wide presence of reasonings grounded on space and time, where the spatial representation of words (e.g. written from the left to the right) and the flexible management of time related to making different hypotheses (in a sort of mental experiment) play an important function.

The main findings of the analysis of the protocols of the primary school pupils who experiment our curricular project at this regard are the following:

= all good problem-solvers can design resolution procedures in which space and time dimensions and their interplay seem to play a major role, and manage very well this interplay in a sort of 'game of hypotheses';

= the ability of designing resolution procedures with hypothetical reasonings strongly grounded on space and time is strongly correlated to the ability of dealing, in an effective way, with problems in which time appears as explicit variable in the text.

With regard to this last finding, Boero, Ferrari and Ferrero [1989] have discussed some examples of phenomena of this kind; for example, with similar numerical values, it is much harder for children to state "how many times this <u>has become</u> as big as before" than to state "how many times this <u>is</u> as big as that", and the strategies adopted are clearly different.

Analogous phenomena have been observed also in the classes of comprehensive school which experiment the project we have designed for this kind of school.

## 6. Discussion

From the data and observations reported in sections 4 and 5 the following conclusions can be drawn:

= Explicit forms of hypothetical reasoning in problem-solving are not produced in a spontaneous and uniform way, but mainly in particular contexts which can induce 'cognitive detachment'. In these contexts the child must be able to manage consciously the procedure and to take into account different alternatives.

= In problems with numbers, the child seems to meet with difficulties in describing objective relations among the elements of the problem and prefers to describe the organization of his actions (*"I do..., then I compute..., so I find..."*). In other words, the relations and properties among the elements of the problem are implicit in his resolution procedure. Procedures are customly represented as chronologically ordered sequences of operations, and then it is quite natural for children to ground their reasoning (and even the 'logical' - or arithmetical or geometrical ...- relations they may have found) on time. Then a statement such as "I compute P and then I find Q" represents both a relation of logical consequence (computing P implies that Q can be found) and a chronological ordering (before I compute P and afterwards I find Q"). These meanings seem to be joined in the child's mind.

= Also in the descriptions of procedures or of the working of machines there is some connection among hypothetical reasoning and time. Children who use hypothetical reasoning clearly seem to prefer 'if...then...' in the situations in which both alternatives are noteworthy and the control is related to something external to the machine or the procedure. The fact that slot-machine could refuse a coin (or that the coffee could be not enough, and so on) is a real possibility for children. The control on the amount of money they have inserted into a slot machine is not regarded as meaningful before they have inserted all the coins requested: to insert the exact amount of money, if they have already it in their hands, is only a problem of time for them. In the same way, a child knows that if he put a coffee-pot on the fire, it must boil, sooner or later; it is only a question of time.

= The analysis of the texts on the 'Middle Ages' corroborates these results. Also in this context, among the children who use both constructions, forms with 'if...then...' are preferred to describe events which may or may not happen (as the premature death of a slave) or depending upon people's will (as the flight of a slave), whereas forms with 'when...' are preferred to describe events children regard as ineluctable or more strictly related to time (as wars, hunting, seasons).

= Two different forms of reasoning, which are referred to the presence of alternative hypotheses, could be seemingly singled out when analyzing the children's protocols: in the first form (strongly guided by the context), though in general there are two or more alternatives, the context allows to single out which is true, and then the others are not even taken into account. On the linguistic ground, this form of reasoning is verbally represented without explicit hypothetical constructions. The second form of reasoning can be found in situations in which both the alternatives must be taken into account. It is clear that the task (e.g. comparation of different strategies, simulations, ...) or the particular situation (e.g. a complex situation, with the necessary data not all explicitly given) prevent the 'contraction' of this kind of reasoning (i.e. the 'evaluation'of the alternatives and the elimination of those which do not happen).

Nevertheless, it seems clear that the important role of time (related to the reconstruction and development of alternatives) and of the context prevent any identification between the mastery of hypothetical constructions in verbal language and the formal management of the proposition 'P implies Q' based on classical propositional logic.

= The strong dependence on time and meanings of the management of hypothetical constructions can contribute to explain some difficulties widely reported in literature about the learning of conditional sentences (regarded as sentences defined by means of truth-tables) [e.g. O'Brien et al., 1971; Johnson-Laird, 1975; Markovits, 1986].

In particular, the findings of Markovits on the effect of pictures in some tests on implication can be regarded in this way, because pictures as far as they neglect time and point out the statical aspects, may hidden more than verbal language the chronological dimension.

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# THE INTERPLAY BETWEEN STUDENT BEHAVIORS AND THE MATHEMATICAL STRUCTURE OF PROBLEM SITUATIONS - ISSUES AND EXAMPLES

Rina Hershkowitz Abraham Arcavi
Department of Science Teaching
Weizmann Institute of Science
76100 Rehovot, Israel

*In Friedlander et al (1989) we analyzed the mathematical behavior of seventh graders in generalization and justification processes. The analysis of the data presented there and additional data led us to focus our attention on the interplay between aspects of the mathematical structure of the problem situations we designed and the spectrum of observed student behaviors in these problem situations. Our aim is to tackle this issue by analyzing some epistemological aspects of problem situations in the first part of this paper. In the second part, we analyze the "traces" of the mathematical structure of the problems on student behavior.*

## The epistemological aspect

There are several epistemological aspects in the light of which problem situations can be examined. We would like to concentrate on the relationships between single examples of the problem situation domain (and actions one can perform on these examples), and processes of generalization and justification. In order to sharpen our description we will proceed to make a distinction between two "extreme cases".

Type 1 problems.

*The general process of justification is based on actions and processes which are analogous to the processes and actions carried out on one single example.*

Suppose one asks the following question:

"Solve 1/2 - 1/3= , 1/3 - 1/4= , 1/4 - 1/5= . Do you observe any pattern? Can you generalize? Can you justify your generalization?"

The process of formal justification of the general pattern [by means of algebraic manipulations of 1/n - 1/n+1=1/n(n+1)] is completely analogous to the arithmetic process by which one solves any single example, as follows.





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Single example $\frac{1}{7} - \frac{1}{8} = \frac{8}{56} - \frac{7}{56} = \frac{1}{56}$

Formal justification $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)}$

Even when the generalization and justification is not expressed by means of mathematical symbols, it will still be no more than a reflection of one particular arithmetical process. In other words, the justification can be "carried on the shoulders" of the single example, if one just looks at the single example with "general spectacles".

Type 2 problems:
*The processes of generalization and justification in the problem situation are completely different from the actions on a single example of the problem situation.*
Suppose one asks the following question:
"What can you say about the numbers resulting from the differences between the third power of a whole number and the number itself [$n^3 - n$]?" (This problem is also used in the study by Fischbein and Kedem, 1982). By trying different numbers, one may notice that all the differences are multiples of 6. But, in order to provide a universal justification of this generalization (or, in other words, to prove this conjecture formally) one needs some *extra steps*, in this case: i) the appropriate algebraic manipulations, and ii) their interpretation.
In other words, to produce:
i) the factorization $n^3 - n = n(n-1)(n+1)$, and
ii) its interpretation: the factors are always three consecutive numbers; at least one of the three consecutive numbers will always be even (divisible by two), and one of them will be always divisible by 3, therefore the product will always be divisible by 6.
Here, the numerical examples, no matter how many of them one may produce, are not "transparent": they will not let the general mechanism be seen or appreciated. Again, as in the Type 1 problem described above, the algebra provides us with the appropriate "general" spectacles for the general justification, but this time it does more than that. It enables us to see and express general relationships between numbers by laying down the

structure of these relationships. Such a structure is invisible through numerical examples.
Most of the typical problems in an Euclidean Geometry course are of this type. Current educational software (e.g. The Geometric Supposer, the CABRI) are excellent tools designed to support for conjecture-making of general properties, an aspect much overlooked in traditional courses. These computerized tools indeed sustain conjecture-making by providing easy and "cheap" ways to experiment by measuring, constructing quickly and efficiently several examples, and manipulating those constructions. But in order to produce a general justification of the conjecture, one usually needs, also here, *extra steps*, which in this case consist of a deductive reasoning chain, probably some auxiliary construction of general validity, and perhaps a good dose of insight to put things together. These *extra steps* seldom arise during the empirical phase of conjecture-making.
Consider for example the sum of the internal angles of a triangle. One can easily measure different types of triangles and conjecture quite quickly that the sum is 180º (or about 180º). However, in order to prove this conjecture one has to take *extra steps*: an auxiliary construction (a line parallel to one of the sides of the triangle through the third vertex, and a translation of the three angles) not present in the conjecture making process.

Our above description deliberately utilized "extreme cases" for the purpose of clarification. Obviously each problem has its own peculiarities, but elements from the above distinction can be identified as intertwined in the problem's "fabric". Consider the following.

A- The same problem situation can be "attacked" in different ways.
For example, one may notice that the general justification is, by virtue of its generality, obviously reducible to any of the single examples. In the case of the sum of the internal angles of a triangle, it is certainly possible to make the auxiliary construction needed for the proof while playing with a specific triangle. As a matter of fact, the general proof is usually accompanied by the drawing of "any" triangle (which can always be regarded as one specific example) and applying the "general" construction to it. However, in the case of $n^3$-n, it is quite unlikely to "discover" the structure of say $7^3$-7 as 7(7-1)(7+1) and thus to have a glimmer of the general justification.
Here we need to notice one central difference between generalization and justification processes in school algebra and school geometry. General

processes in geometry rely on seeing one single example as representative of a whole class, whereas algebra resorts to a symbolic language. This language enables handling "general" patterns and also laying down the structure of relationships, which are invisible from single examples.

B- There can be a more or less "smooth" transition from working on single examples to the general justification of the generalization.
Consider the following problem situation: "Find the sum of the internal angles of a polygon". One can proceed to work with, say, pentagons first, and start measuring the angles, and arriving at the conclusion in the same way we described for the triangles. If one generates the *extra steps* of dividing the pentagon into three triangles (whose sum of angles is already known) by means of the diagonals from one vertex, the general justification for all the pentagons arises: one has to multiply 180º by the number of triangles. And if one wishes to generalize further for the case of all polygons, an additional *extra step* is required, whose outcome will relate the number of sides (or vertices) of the polygon to the number of triangles (or diagonals) created.

We suggest that, in spite of the complexity raised by our distinction, the elements observed in the "extreme cases" are useful in shedding additional light on understanding student behaviors.

Student behaviors

The situations we describe in this section are borrowed from existing studies. We expect to bring to our oral presentation additional data from the study we are currently running.

I - *The example as "judicial evidence".*

A typical student behavior, while making general arguments and trying to justify them, consists of placing a single instance as "judicial evidence": like, in court, clear-cut evidence is necessary and sufficient to convince a jury about the certainty of an event*. In other words, a convincing justification is regarded as the presentation of a fact (an example) which confirms the general claim at stake. K., one of the seventh graders in our study, expressed

* This can be reinforced by the language. For example, the Hebrew word for "proof" is used both in mathematics and in law (as evidence). Seventh graders, who have rarely met the mathematical connotation of proof, may associate its meaning with that of "legal evidence", a well known word frequently heard in everyday life.

it very clearly: "I think that to prove something means to show some examples".

In such cases, the students confer to the single example, as a "prover" of a general claim, the same status conferred by mathematical standards to a counterexample as a "disprover" of a general claim.

In a more elaborated version of the example as "judicial evidence", we found students who check evidence from different domains of examples (small versus large numbers, different types of triangles, etc.) in order to make the "evidence" more convincing for themselves and/or others.

In any case, when students are requested to justify a general claim (given by others or even produced by themselves), they return to an example, or to domains of examples, which by their mere existence provide the justification requested.

This use of the example as "judicial evidence" is a justification tool in the hands of many students regardless of the epistemological type of the problem situation on which they work.

II. *Beyond "judicial evidence" - the role of examples in Type 1 problems.*

Once a generalization of a Type 1 problem (like the subtraction of two consecutive unit fractions) is achieved, and the student is requested to justify, (s)he does not resort to the example as a confirmation, but as a prototype from which a general property or mechanism can be abstracted. The following quotation is from a pre-algebra student (a seventh grader), attempting to justify verbally the general statement $1/n - 1/(n+1) = 1/n(n+1)$, by "riding on the shoulders" of the single example $1/8 - 1/9 = 1/72$.

"...each time we have, let's say, 72 divided by 8 we get 9, and when we divide by 9 we get 8... one number less the other is always 1... that's clear".

The phrases "each time", "let's say, 72 divided by 8...", and "one number less the other" are indications that, for lack of any other appropriate tool, the particular values of the example are not invoked as such, but as tokens or potential placeholders involved in a general mechanism. For some students, this use of the example is an intermediate step towards an example-free verbal formulation of the justification.

In sum, the very nature of Type 1 problems seems to encourage the use of examples as generators of justifications.

III. *Beyond "judicial evidence" - the role of examples in Type 2 problems.*
The following vivid anecdote will illustrate this point. During an in-service teacher training course in our department, teachers were presented with the curious equality 62x39=26x93. Generating more examples of two-digit multiplication convinced them very quickly that reversing the digits of the factors does not always lead to the same result. Puzzled by this rarity they were invited to investigate for which two-digit numbers the phenomenon will take place. Among those who did realize that they need to resort to algebra was one teacher who seemed the brightest of the group and who had no trouble producing the algebraic manipulations and reaching the right conclusion (the equality will take place when the product of the tens equals the product of the units of the two-digit numbers). However, when we noticed that she was checking for additional numerical examples after she completed the proof, we were puzzled. When asked, she told us that she was not checking in order to see whether she obtained the right result with the numbers, because the general validity of the algebraic tool was obvious to her. However, in light of the algebraic justification she had produced, she needed to feel what happens when one translates the mechanism of the algebraic general justification "to actual numbers", namely how do they combine and how do they behave as compared to letters.

It seems that mathematically-able people, in their search for meaningfulness would often use examples in this way, in order to get a feeling of those *extra steps* that they themselves (or somebody else) were able to generate.
We conjecture that this use of examples would be rarely encountered in Type 1 problems, in which the general proof is based on a repetition of the mechanism of a single example.

IV. *The struggle for the extra steps.*
In Friedlander et al (1989), we described a pair of pre-algebra students working with the following problem. They were given a calendar sheet of a given month, for example:

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 27 | 28 | 29 | | | | |

The students were requested to observe general patterns and to justify them. One of the students conjectured that, in any 2x2 cell arrangement, the difference between the two products of the numbers located in the diagonals is 7. In the process of trying to produce a general justification, they were able to express the products of the diagonals using letters [namely ax(a+7+1) and (a+1)x(a+7)]. At this stage they substitute different numbers for a and confirmed empirically their conjecture.

One of the students was quite happy with the algebraic "machine" they had created, and since this "machine" generated the expected numerical result, he though that the task was completed. He considered the creation of the algebraic "machine" as the justification sought. However, the other student expressed his dissatisfaction by saying that he still wanted to "show it [the justification of the general pattern] with letters". Since this is a Type 2 problem, and the *extra steps* needed required algebraic manipulations, he could not make progress. The lack of knowledge of algebraic manipulations needed for producing such justification, namely to prove that ax(a+7+1) and (a+1)x(a+7) differ by 7, did not prevent him from feeling its necessity.

## Epilogue

We suggest that aspects of the epistemological nature of a problem may turn out to be crucial variables in understanding and possibly predicting student behavior. If further studies confirm this view, the findings can have important instructional implications regarding the kinds and timing of the problem situations students should encounter in order to foster the need for general justifications and proofs.

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Paradigm of 'Open-Approach' Method in the Mathematics Classroom Activities
---Focus on Mathematical Problem-Solving---

Nobuhiko NOHDA, University of Tsukuba, Japan
Jerry P. Becker, SIU at Carbondale, USA

Summary

Our study on analyzing students' strategies and difficulties in problem solving is considered indispensable to improve teaching and learning in mathematics classroom activities. It seems that these strategies and difficulties are influenced greatly by some social and cultural factors, such as languages, symbols and representations etc.. This study is planned in order to make exact the effects of teaching and learning of teacher and students who engage in problem-solving by means of the 'Open-Approach' method, particularly with reference to share mathematical ideas of problem and use of mathematical patterns involved in problem solving. We have to become more aware of the information processes which consist of the communications and interactions between the teacher's explanations and pupil's approach to problem-solving.

The sixth-grade class (Male; 18, Female; 22, Totale; 40) we used in this study was composed of pupils in a rural elementary school near Tsukuba City. Ms. K. Mashiko is an excellent teacher, who had come to University of Tsukuba for studying mathematical problem-solving for about three months, three years ago. The lesson was held on January 26, 1987.

1. Mathematics Classroom Activities

Several difficulties concerning problem-solving are, in our opinion, due to the narrow and isolated conceptions of the basic didactical category on 'problem-solving'. We, therefore, attempt to reveal the global and relational character of 'problem-solving', which is to call attention to the necessity of dealing with a broad spectrum of activities related to Japanese culture and society.

These new demands can be found in Christiansen and Walter (1986), which necessitate changes in the teacher's role and moves:

1. changes in the distribution of emphasis on the different types of activity,
2. changes in the types of teacher's moves and in the sequencing of these in the teaching process,
3. changes in the ways in which the teacher serves as a mediator of mathematical meaning.

The process of problem-solving becomes evident when teaching is seen as a process of interaction between the teacher and learner-and among the learners-in which the teacher attempts to provide learners with access to mathematical thinking in accordance with given problems. This teaching/learning process (like all processes between learners) is influenced by a number of social and developmental aspects and factors which can be included in problem-solving. The communication between teacher and learner is thus not only conditioned by formal decisions about goals, content and teaching methods, but it is also strongly dependent on even more informal aspects in initiative stages of problem-solving, such as the teacher's words and explanations to the problem-solver, and the students' motivation to solve the problem and to be concerned with it.

We will cite an example of the problem-solving activities between teacher and learners (Fig. 1). Some of the roles of the teacher at different stages of the teaching/learning process are: instructor to teach mathematical knowledges and skills (Top-Down); teacher to help students in problem-solving (Bottom-Up); and decision maker to judge whether teaching goes ahead or not. The teacher's explication of such roles is integrated with his specific actions and serves in establishing his background and context for the interactions between his students' actual and inner activities in connection with their subjective words.

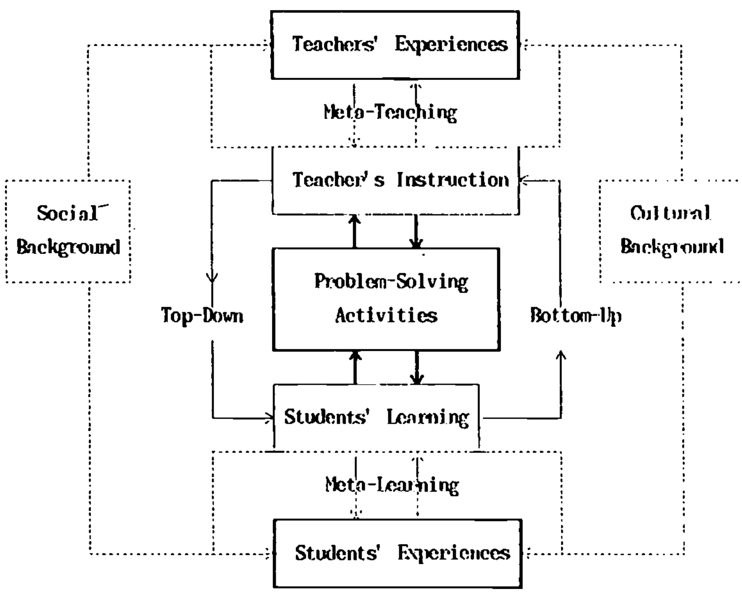



Fig. 1 Problem-Solving Activities

The above sentences illustrate the essential and relational character of communications between teacher and learners. Accordingly, communication through 'problem-solving' as an organizing principle in Japanese mathematics learning calls for meta-learning under the teacher's support. This communication is considered mathematics classroom teaching as controlling the organization and dynamics of the classroom activities for the purposes of sharing and developing mathematical thinking.

2. What is the open-approach?

The aim of open-approach instruction is to foster both the creative activities of the students and the mathematical thinking in problem solving simultaneously. In other words, both the activities of the students and the mathmatical thinking must be carried out to the fullest extent. Then, it is necessary for each student to have the individual freedom to progress in problem solving according to their own abilities and interests. Finally, it allows them to cultivate mathematical intelligence. Class activities with mathematical ideas are assumed, and at the same time students with higher abilities take part in a variety of mathematical activities, and also students with lower abilities can still enjoy mathematical activities according to their own abilities.

By doing so, it enables the students to perform the mathematical problem solving. It also offers them the opportunity to investigate with strategies in the manner they feel confident, and allows the possibility of greater elaboration within mathematical problem solving. As a result, it is possible to have a richer development in mathematical thinking, and at the same time, foster the creative activities of each student. This is the idea of the 'open-approach', which is defined as an instruction in which the activities of interaction between mathematics and students are open to varied problem solving approaches.

Next, it is necessary to make clear that the meaning of the activities of interaction between mathematical ideas and students' behaviors are open in problem solving. This has been explained from three aspects:

(1) Students' activities are developed by the open-approach.

(2) A problem using the open-approach involves mathematical ideas

(3) Open-approach should be in harmony with interaction activities between (1) and (2)

3. Characterizations of the 'Open-Approach' problem and method

We hope to become more aware of the information processes which consist in the 'Open-Approach' of relationships between the problem and method. We use here 'Open-Approach' problem as like non-routine problems: problem situations, process problems and open search problems (Christiansen & Walter, 1986). In actual practice, each teacher will have to take his or her own classroom conditions and teaching objectives into consideration. The method we use in 'Open-Approach' depends on the problems which consist of problem situations, process problems and open-ended problem, and procedures of these problems including classroom conditions and teaching objectives (Nohda, 1983, 1986).

We use here the problem: problem situations, process problems and open-ended problems. We define a problem as follows: A problem occurs when pupils are confronted with a task which is usually given by the teacher and there is no prescribed way of solving the problem. It is generally not a problem when it can be immediately solved by the students. Problem situations, process problems and open-ended problems are defined as follows:

Figure 2. Open-Approach Problem and Method

On going Lesson

A. Begining of Lesson — B. Solving of Problem — C. End of Lesson

Original Problem(s) ⇒ ⇒ Solution(s) ⇒ ⇒ New Problem(s)
→ Solution(s) → New Problem(s)
→ Solution(s) → New Problem(s)

Problem Situations — Process Problems — Generative Problems

Open-approach teaching differs from ordinary problem solving teaching. Here we use the problems as open-approach problems mentioned above. Treatments of these problems will depend on the teacher's intentions for his/her objectives:

A. What kind of problem does the teacher want the students to formulate from the given problem situations ? (Relation with Problem Situations)

B. How many ways of thinking does the teacher want the students to solve the problem given ? (Relation with Process Problems)

C. What kind of advanced problem does the teacher want the students to make from the original problem ? (Relation with Generative Problems)

4. Actual problem-solving activity in sixth-grade classroom

In every day life, pupils are confronted with many problem-solving as problem situations where they can take a variety of solution. The method for solving the problem of daily life, seems to include some regular rule or procedures.

To foster their mathematical thinking, mathematics teacher should emphasize problem-solving, in which pupils would discover better way of thinking through discussions of their various solutions of the problem.

Here is used the process problem. The sixth-grade class (Male: 18, Female: 22, Totale; 40) we used in this study was composed of pupils in a rural elementary school near Tsukuba City. Ms. K. Mashiko is an excellent teacher, who had come to me for studying mathematical problem-solving for about three months, three years ago. The lesson was held on January 26, 1987.

(1) Teacher's plan of problem solving

(a) Original problem in Japanese textbook

Squares are made by using small bars as shown in figure 3. When the number of squares is 8, how many small bars are need ?

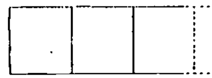


Fig. 3

(b) Change Problem to Process Problem

Squares are constructed by using small bars as shown in figure 4. When the number of squares is 10, how many small bars are used ?

Find the lots of ways of counting of the small bars, as possible.

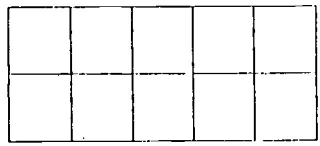


Fig. 4

(2) Actual lesson of problem-solving: First class, sixth-grade

(a) She started as follows: (5 minutes)

Each pupil was given a picture of 'small bars' and the teacher asked the pupils "How many squares are there in this figure?" and she put the real small bars on the blackboard. She explained some notions to them: "Arrange the small bars to shape two squares like Figure 4 and count the number of small bars one by one" as follows:

(b) Give hints to help pupils of lower abilities understand the problem. (5 minutes)

T: When the number of squares is 2, how many small bars are used ?

P: 7 bars

T: When the number of squares is 4, how many small bars are used ?

P: 12 bars

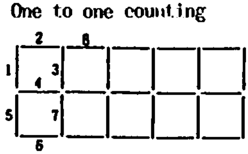
T: O.K., you come up with various ways of solving the problem.

After she explained the problem, pupils worked on the problem individually.

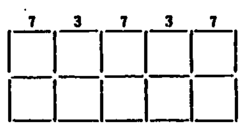
(c) Let's find various ways of counting on the sheet given.

Answer: 27 bars (15 minutes)

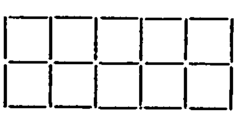
(i) One to one counting



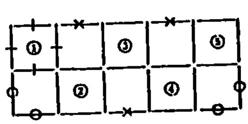

(ii) $7\times3+3\times2$



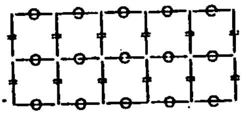
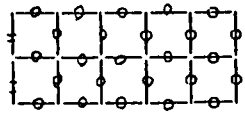
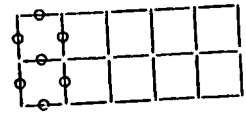

(iii) $4\times10$ (Wrong)



(iv) $4\times6+3$



(v) $5\times3 + 6\times2$    *(vi) $2 + 5\times5$    *(vii) $7 + 5\times4$

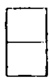
(d) Discuss pupils' ways of counting. Which one do you think is the best way ? And Why ? What happens when the number of squares increases ?(15 minutes)

T: Which is the easiest way to count when we have 20 squares ?

P: *(vi) $2 + 5\times5$ or (vi) *(vii) $7 + 5\times4$

(e) Formula expressed in words: (5 minutes)

[Two squares with 7 bars] + [5 bars] × [Sets of increase number] = [Total number]

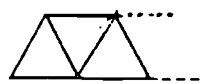
 : 7 +  $5\times(5-1) = 27$

Ans. 27 bars

(f) Give open-ended problems to pupils: (Homework)

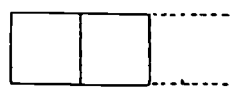
(i) When the number of equilateral triangles is 8, how many small bars are used?

Ans. $2\times8+1=17$



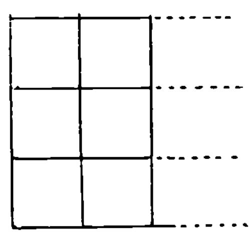
(ii) When the number of squares is 7, how many small bars are used?

Ans. $3\times7+1=22$



(iii) When the number of squares is 15, how many small bars are used?

Ans. $7\times4+10=38$



| Problem | variations of formula | Male(18) | Famele(22) | Totale(40) |
|---|---|---|---|---|
| (i) | 2 ×8 +1 | 2 (2) | 8 (8) | 10 (10) |
| | 3 +2 ×7 | 14 (14) | 9 (9) | 23 (23) |
| | 3 ×8 --7 | 1 (1) | 0 | 1 (1) |
| | others | 1 (0) | 5 (1) | 6 (1) |
| (ii) | 3 ×7 +1 | 3 (3) | 9 (9) | 12 (12) |



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| | | | | |
|---|---|---|---|---|
| | $4+3\times6$ | 14 (14) | 8 (8) | 22 (22) |
| | $1\times8+7\times2$ | | 2 (2) | 2 (2) |
| | others | 1 (0) | 3 (1) | 4 (1) |
| (iii) | $7\times5+3$ | 3 (3) | 4 (4) | 7 (7) |
| | $10+7\times4$ | 8 (8) | 5 (5) | 13 (13) |
| | $5\times4+3\times6$ | 1 (1) | 0 | 1 (1) |
| | others | 6 (0) | 13 (0) | 19 (0) |

Note ( ): Number of corrects

(3) Disscussion on sixth grade problem-solving

The variations of strategies used in this problem-solving were as follows: No pupil used one by one though they often used strategy in the lesson. Most pupils used the equations adjusted to the structure of the problem. This was dependent on the following communication between the teacher and the pupils: The teacher advised the pupils who could find the suitable configuration of the problem. We are impressed that pupils have a real appreciation of sharing the mathematical structures through communications among pupils themselves under the teacher's orientation and understand the mathematical formula through the results of pupils' homework. Dr. Jerry P. Becker gives the comment as follows: I have found that challenging problems bringing students together in thiking about the situation, searching to understand the problem, and then trying to solve it. Sometimes I almost sense that a "spirit of community" ensues with students reflecting and building on each other's ideas - this is a helthy state of affairs.

5. Instructional implications

The instructional implications from our study of the elementary school level in the context of problem-solving in the mathematics classroom consist of the following:

a. In the study of pupils' strategies and difficulties in problem-solving, we should concentrate on both the structure of the problem and the mode of the pupils' acts of problem-solving. We suggest here that the pupils need to initially act by themselves to solve the problem and then through communication establish mathematical structures in the modification of their initial acts: for example, pupils make the equation as like $7+5\times4$ from $7\times3+3\times2$.

b. Some excellent pupils can solve the problem by finding the mathematical structures underlying in the problem. The teacher has to support these pupils to promote their more advanced solution after they use the teacher's primitive method. They are willing to independently find the advanced solutions. The excellent communication is the most important for the teacher. Thus, they become the good problem solvers for the future.

c. Many normal pupils cannot solve the problem at hand. In these situations, the teacher has to advise them to be ready for their famillier problems which they have solved in the past. After they feel an appreciation for carrying out the problem-solving individually, they are able to solve the problem in the near future. This is the effect of the classroom activity supporting them by the teacher.

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(Jenuary 25, 1990)

# REFLEXIONS SUR LE ROLE DU MAITRE DANS LES SITUATIONS DIDACTIQUES A PARTIR DU CAS DE L'ENSEIGNEMENT A DES ELEVES EN DIFFICULTE .

Marie-Jeanne PERRIN-GLORIAN
IREM Université PARIS 7

*We worked in mathematics with classes mainly made up of pupils (9 - 13 years old) meeting with difficulty at school in many subjects, most of them coming from lower classes. Basing ourselves on results coming from cognitive psychology, sociology, social psychology..., we try to interpret difficulties in mathematics of these pupils in the theoritical frame of didactics of mathematics as it has extented in France. In particular, we draw the constrainsts bearing on negociation of the "didactic contract", and coming from teachers as well as from pupils. This leads us to reconsider the part of the master in the theory of didactic situations. We study more specially processes of "devolution" and "institutionnalisation" of knowledge. This study induces us to identify a kind of situations, called here "recall situations" that seems to us particularly important for pupils who are in trouble at school. The matter is, for pupils, to account orally for a situation of action which has taken place in a previous session, when action is no more possible*

## Introduction et cadre théorique de référence.

Nous nous plaçons dans le cadre théorique de la didactique des mathématiques tel qu'il s'est développé en France et a déjà été évoqué dans les rencontres de PME (R. Douady, PME 9 et 11, C. Laborde PME 13). Nous utilisons plus particulièrement la théorie des situations didactiques élaborée par G. Brousseau (1987) et les notions de jeu de cadres et dialectique outil-objet introduites par R. Douady (1987). Cette théorie s'est développée en adoptant un point de vue épistémologique qui donne une grande importance à la résolution de problèmes, aussi bien dans la construction du savoir que comme critère du savoir. Cette position n'est pas toujours conforme aux conceptions sur l'enseignement et l'apprentissage des professeurs qui utilisent les résultats des recherches. Les élèves ne sont pas non plus toujours prêts à engager leur responsabilité dans une résolution de problème. Ces distorsions s'observent particulièrement lorsqu'il s'agit d'enseigner les mathématiques à des élèves en grande difficulté scolaire, issus pour la plupart de milieu social défavorisé. Pour les étudier, nous empruntons avec A. Robert et J. Robinet (1989) le cadre théorique de la représentation sociale (Abric, 1987).

Dans ce texte, nous donnerons d'abord les grandes lignes de l'interprétation que nous faisons, en nous servant de ce cadre théorique, de l'échec d'élèves en grande difficulté à l'école. En relation avec cette interprétation, nous analyserons ensuite des aspects qui nous paraissent importants dans le rôle du maître.

## Interprétation des difficultés des élèves.

*Absence de création de représentations mentales et de projet implicite de réinvestissement.*

Nous avons constaté qu'il y avait souvent, chez les enfants en difficulté, un divorce net entre les situations d'action qui devaient servir à donner du sens aux notions enseignées et l'institutionnalisation qui est faite ensuite par le maître : au cours de l'action, dans les premières situations qui permettent d'aborder une notion nouvelle, on ne voit pas beaucoup de différences entre élèves. En revanche la différence s'accentue très vite dès qu'il s'agit de réutiliser les connaissances nouvelles dans d'autres situations. Le savoir institutionnalisé par le maître, même dans le cas où il est mémorisé, semble coupé des situations d'action qui lui ont donné naissance et ne peut être utilisé pour résoudre de nouveaux problèmes.

Une des principales explications que nous avançons est que les élèves qui ne rencontrent pas ce genre de difficulté ont un projet, même implicite, de décontextualisation dès le moment où ils travaillent sur la situation d'action. Ils savent qu'il y aura peut-être lieu de réutiliser l'expérience acquise. Ils se créent des représentations mentales non seulement pour résoudre le problème posé actuellement mais pour pouvoir en rappeler et réutiliser des éléments dans d'autres occasions. Ceci leur permet de réinvestir partiellement une connaissance, même si elle n'est pas encore totalement identifiée. Pour d'autres enfants, ce "transfert" ne se fait pas parce qu'ils ne font que résoudre le problème posé, dans les termes où il est posé, sans avoir de projet de connaissance. Il n'y a pas création de représentations mentales qui ont déjà valeur symbolique et sur lesquelles on pourra travailler ensuite. Il n'y a pas non plus de mises en relation, "d'accrochage" à l'ancien pour le renforcer ou le remettre en question. Tout ceci empêche la capitalisation et la mémorisation des connaissances. Ainsi, chaque expérience est nouvelle, ou plus exactement, seul le contexte est reconnu : "on a plié des bandes de papier, on a découpé des rectangles"...

*Manque de fiabilité des connaissances anciennes.*

L'absence de connaissances antérieures solides auxquelles se référer contribue à ce manque d'organisation et d'intégration des savoirs nouveaux : pour certains enfants, rien n'est sûr, tout peut toujours être remis en question, puisqu'ils ont l'habitude de se tromper.

*Absence d'identification de l'enjeu des situations didactiques.*

Une autre cause nous paraît être la non reconnaissance du véritable enjeu des situations proposées en classe et l'absence d'identification de l'objet du travail proposé par l'enseignant : par exemple, si celui-ci propose des découpages de rectangles pour travailler sur les fractions alors que, pour l'élève, il s'agit d'apprendre à partager les rectangles, il n'y a pas de lien entre cette activité et le pliage de bandes de papier. Ainsi, les fractions utilisées dans les deux contextes n'ont pas de rapport entre elles, l'élève n'a donc pas de souci de cohérence.

Cela a des conséquences au niveau didactique, par exemple l'usure rapide des situations : les élèves qui identifient la situation à son contexte se lassent avant qu'on puisse avoir une identification et une décontextualisation locale des savoirs en jeu suffisantes pour permettre leur réinvestissement ultérieur. Nous allons voir que cette usure participe à l'enclenchement d'un cercle vicieux renforcé ensuite par les choix des maîtres.

*Simplification des situations et enclenchement d'un cercle vicieux.*

La difficulté de réinvestissement des élèves est particulièrement grande dans le cas de situations complexes où il y a à identifier un problème connu à l'intérieur d'une situation où interviennent d'autres éléments. Cela renforce l'idée qu'on facilite l'apprentissage en simplifiant le problème, en mettant des paliers intermédiaires. Cela entraîne aussi, chez les enseignants comme chez les élèves, le désir de recourir le plus possible à l'apprentissage de procédures de traitement stéréotypées, plus sécurisantes. En effet, les élèves en difficulté quêtent l'approbation du maître à chaque pas dès qu'ils sortent de la routine. Ils réclament des algorithmes. Par ailleurs, du côté des maîtres, on fait moins confiance aux élèves, on a tendance à les aider davantage et on pense leur donner ainsi des moyens de réussir au moins quelque chose.

Il est vrai que les algorithmes eux-mêmes sont souvent insuffisamment mémorisés par ces élèves. Ceci entraîne une charge en mémoire insupportable lors de la résolution de problèmes, leur fait perdre le fil de la résolution et encourage donc l'enseignant à donner plus de place encore à l'apprentissage des leçons et des algorithmes.

En outre, avec les élèves en difficulté, les professeurs ont tendance à se concentrer sur le cadre numérique en négligeant des activités géométriques ou graphiques qui pourraient donner d'autres références. Comme un changement de point de vue est toujours difficile, ils pensent généralement que, pour ces élèves, il faut faire le moins de mélanges possible. Cela contribue à accroître le déficit de connaissances solides dans des cadres différents et empêche le fonctionnement de jeux de cadres, ce qui diminue encore les occasions d'apprendre à mettre en relation différents savoirs.

On assiste ainsi à l'enclenchement d'un processus "boule de neige" : les élèves ne se représentent pas les actions, ne perçoivent pas les enjeux —> les élèves ne mémorisent pas —> le professeur se concentre sur l'apprentissage des résultats du cours et de savoir-faire algorithmisés —> les situations proposées aux élèves se résument à la répétition de problèmes de contrôle stéréotypés —> les élèves ne se représentent pas, ne mettent pas en relation —> ... et l'apprentissage se résume au renforcement d'algorithmes dont les situations d'utilisation ne sont jamais maîtrisées.

*Autres aspects*

Nous ne pouvons développer ici d'autres aspects importants dans l'enclenchement du cercle vicieux dont nous avons parlé :

*Les problèmes de langage*, expression et lecture, sont aussi bien sûr à l'origine de difficultés en mathématiques, de trois façons au moins : au niveau de la prise d'information, au niveau de la conceptualisation, au niveau des productions.

Une autre difficulté tient à la capacité d'interprétation du niveau de discours du maître. Dans le déroulement de l'enseignement, en effet, le maître utilise plusieurs niveaux de discours qui sont souvent assez imbriqués et que l'élève doit réussir à décoder avec leur signification dans la situation. Il doit être capable de repérer ces changements de niveau et de tirer profit du discours non mathématique - que nous appelons ici un peu rapidement, métamathématique mais qui recouvre de nombreux registres que nous n'avons pas la place de distinguer ici - pour s'approprier plus facilement le discours mathématique du maître et des autres élèves.

*Les situations du quotidien* avec lesquelles les élèves ont une certaine familiarité, utilisent souvent des modes de raisonnement non conformes à ceux qu'on attend dans un cours de mathématiques. Il peut ainsi s'installer un véritable malentendu et une communication absurde entre le professeur et certains élèves. Ceci ne veut pas dire que l'expérience des enfants dans la vie quotidienne ne peut pas être utilisée, mais il faut alors bâtir, comme le font par exemple certains chercheurs italiens autour de P. Boero (1989), des situations qui s'appuient sur la réalité familière et permettent de la dépasser en posant aux enfants de véritables problèmes théoriques.

*Rapport de l'élève à l'école, à son métier d'élève.* Plusieurs des explications que nous venons d'avancer sont à relier au plan plus général des attentes et des représentations sur l'école, à la présence ou à l'absence de projet général, et à ce que Y. Chevallard (1988) appelle le rapport à l'école, au métier d'élève... Par exemple, un élève peut trouver illégitime qu'on lui propose un problème dont on ne lui a pas enseigné la réponse et refuser cette responsabilité.

*Rapport de l'enseignant à son métier d'enseignant.*

L'idée que l'enseignant se fait de son métier et de l'évaluation des élèves peut parfois l'amener à vider l'enseignement de son contenu, en particulier dans le cas où il s'adresse à des élèves en difficulté. Il peut être ainsi conduit à "surinstitutionnaliser" des résultats ou des méthodes rencontrés dans des résolutions de problèmes et à remplacer le véritable enjeu de l'enseignement par des intermédiaires introduits pour faciliter l'accès à une connaissance.

*Les représentations des enseignants sur les capacités des élèves* se conjuguent avec celles qu'a l'enseignant sur la bonne manière d'apprendre, ce qu'est une formation mathématique, le contenu visé. L'enseignant choisit en fonction de ces représentations qui lui font estimer le coût par rapport à la rentabilité attendue, les méthodes qui lui paraissent convenir compte tenu du contenu et du public.

*Représentation de soi de l'élève.* Leur situation d'échec à l'école contribue à donner aux élèves en grande difficulté une image d'eux-mêmes dévalorisée. Cette image et la représentation qu'ils se font de leur place par rapport aux autres élèves de la classe ont des répercussions sur toute leur vie scolaire, y compris l'acceptation de certaines formes de travail (en groupes, notamment)

**Le rôle du maître.**

*Le rôle du maître dans la théorie des situations.*

Dans la théorie des situations didactiques telle que la développe G. Brousseau (1987) et que nous schématisons ici, le rôle du maître dans les situations didactiques se situe essentiellement à trois niveaux : choix d'un problème et d'une situation a-didactique et détermination des variables didactiques de façon à mettre en jeu la connaissance visée, dévolution de cette situation à l'élève et institutionnalisation des connaissances. La première phase n'est pas forcément entièrement à la charge de l'enseignant qui peut avoir recours à des travaux d'ingénierie didactique. Nous nous intéresserons dans la suite aux deux autres phases. Nous laissons volontairement de côté l'évaluation qui intervient aussi de façon importante dans l'institutionnalisation des connaissances et dans les représentations que les élèves se font des concepts mathématiques et des mathématiques en général.

*La dévolution.*

Pour que l'élève construise un savoir, il est nécessaire, d'après G. Brousseau (1987), qu'il produise ses connaissances, les fasse fonctionner ou les modifie comme réponses aux exigences du milieu et non au désir du maître. Pour cela, il faut que l'élève accepte que la résolution du problème soit de sa responsabilité, qu'il accepte de prendre en charge ce que Brousseau appelle une situation "a-didactique", c'est-à-dire une situation dépouillée de ses intentions didactiques.

L'élève doit faire sienne la question posée et chercher à la résoudre sous sa propre responsabilité, sans essayer de deviner les intentions du maître ni chercher à lui faire plaisir. La dévolution est alors un processus nécessaire parce que l'accès de l'élève à la situation a-didactique ne va pas de soi car elle est au départ très imbriquée à la situation didactique qui la contient : "La situation a-didactique finale de référence, celle qui caractérise le savoir (...) est une sorte d'idéal vers lequel il s'agit de converger" (p.50)

La question que nous nous posons est donc la suivante : qu'est-ce qui permet à l'élève de converger vers la situation a-didactique, qu'est-ce qui fait qu'il met un savoir mathématique en jeu en tentant de résoudre le problème posé par le maître ? G. Brousseau donne lui-même par avance une première réponse à cette question : "L'élève sait bien que le problème a été choisi pour lui faire acquérir une connaissance nouvelle mais il doit savoir aussi que cette connaissance est entièrement justifiée par la logique interne de la situation et qu'il peut la construire sans faire appel à des raisons didactiques" (p. 49). La dévolution est essentiellement ce qui le lui fait savoir et il est vrai que c'est une condition pour que l'élève fonctionne de façon scientifique et non en réponse à des indices extérieurs. Mais il nous semble que l'affirmation de la première partie de la phrase "l'élève sait bien" n'est pas évidente. Suivant leur origine culturelle ou leur expérience scolaire antérieure, certains élèves savent bien en effet qu'il y a toujours un objectif d'apprentissage dans ce qu'on leur propose et on a l'habitude dans l'enseignement de faire comme si cette évidence était partagée. Or nos observations sur les élèves en difficulté nous laissent penser qu'elle ne l'est pas. La question didactique qui se pose alors est de savoir de quel projet faut-il faire dévolution à l'élève avec le problème (ou avant pour permettre ensuite la dévolution du problème à l'élève), ou, en d'autres termes, comment faire dévolution à l'élève de la prise en charge de son propre apprentissage ? Cette question se pose, pour le maître, dans la négociation du contrat didactique à plusieurs niveaux : au niveau général de l'enseignement des mathématiques dans la classe considérée, au niveau de l'ensemble du processus d'apprentissage d'un concept donné et au niveau de chacune des situations composant ce processus.

*L'institutionnalisation.*

Ceci nous amène à considérer l'institutionnalisation comme un processus qui se déroule tout au long de l'enseignement, un moteur de l'avancement du contrat didactique et du temps didactique et non comme une phase en fin de processus où le maître fait son cours. L'institutionnalisation des connaissances commence pour nous dès le tout début de la dévolution puisqu'il faut déjà que le maître donne à l'élève, s'il ne l'a pas, le projet d'acquérir ces connaissances. Evidemment, nous trouvons là un des paradoxes du contrat didactique que Brousseau a mis en évidence : le maître ne peut pas parler de la connaissance nouvelle puisque c'est justement l'enjeu de l'apprentissage, il peut au plus dire qu'on va apprendre quelque chose de nouveau et éclairer les élèves sur les connaissances anciennes à mobiliser pour "accrocher" cette connaissance nouvelle. En fait le maître tend à l'institutionnalisation tout au long du processus : s'il veut que l'institutionnalisation puisse se faire pour les élèves dans de bonnes conditions, avec du sens, il ne peut aller droit au but mais l'a toujours présent à l'esprit pour ménager dès le départ et tout au long du processus d'enseignement les conditions qui vont lui permettre de négocier le contrat didactique dans ce sens.

Avec des élèves en difficulté, les contraintes qui pèsent sur l'institutionnalisation sont particulièrement visibles et on se sent comme un funambule sur son fil : si à la suite de la résolution d'un problème, aucune décontextualisation n'est amorcée par le maître, les élèves ne retiennent rien et ne peuvent parler que du contexte du problème et non de son enjeu, s'il y a décontextualisation par le maître, on assiste à un dérapage formel qui amène les élèves à prendre les écritures mathématiques sans les créditer du sens qu'elles pouvaient avoir dans le problème traité. L'équilibre est difficile à trouver. Nous en déduisons que, pour certains élèves au moins, l'institutionnalisation ne peut se faire que de façon très progressive avec de nombreux cycles contextualisation - décontextualisation.

Ceci nous amène à distinguer des étapes dans l'institutionnalisation :
- institutionnalisations locales dans divers contextes, au sens de R. Douady (1987)
- réinvestissement d'un contexte dans un autre
- cours construit par le professeur au sens traditionnel, donnant un statut d'objet mathématique à certaines des notions rencontrées.

Ces étapes ne correspondent pas entièrement à un ordre chronologique, le réinvestissement se plaçant tout au long, avec des degrés de décontextualisation différents : dès que les élèves ont rencontré une première situation sur la notion, ils peuvent réinvestir des pratiques en reconnaissant une analogie entre deux situations, jusqu'après le cours où ils pourront peut-être réinvestir le savoir en tant qu'objet mathématique.

*Les situations de "rappel".*

Un des temps forts dans le processus de dépersonnalisation et décontextualisation des savoirs construits en classe se situe au cours des bilans qui suivent une phase de recherche des élèves. Des chercheurs ont analysé le rôle du maître dans ces bilans et distingué à côté des moments d'institutionnalisation, des moments où le maître cherche à homogénéiser la classe et où s'effectue une première dépersonnalisation des procédures mises au point par les élèves dans la phase de recherche. Nous avons, pour notre part, identifié un autre type de situations qui nous semblent jouer un rôle important dans ce processus de dépersonnalisation et décontextualisation, à deux moments au moins : d'une part elles vont permettre d'adapter l'institutionnalisation locale aux conceptions actuelles des élèves ; d'autre part, avant le cours proprement dit, elles vont permettre d'accrocher les notions qu'on va exposer aux problèmes qui ont permis de leur donner du sens.

Nous les appelons pour le moment, et faute d'avoir trouvé un meilleur terme, des situations "de rappel". Il faut tout de suite dissiper un malentendu possible : il ne s'agit pas de révision ni de rappel par le maître de ce qui a été fait, il s'agit plutôt pour les élèves de se rappeler une ou plusieurs situations déjà traitées dans des séances précédentes sur un même thème, avec un peu de recul donc, de faire un retour par la pensée et la parole sur ces séances. En essayant de dire collectivement ce qui s'est passé, quel problème on a traité, les élèves sont amenés à repenser le problème, les procédures de traitement envisagées dans la classe. Les élèves qui ne se sont pas construit de représentation mentale au cours de l'action trouvent là une nouvelle occasion et une raison de le faire puisqu'ils vont devoir parler de ce qui s'est passé et le décrire sans pouvoir agir à nouveau. Il se peut que pour certains élèves l'action soit à nouveau nécessaire mais elle est alors placée dans une nouvelle perspective : il faut agir non seulement pour trouver une solution mais aussi pour pouvoir en parler.

D'une part il se produit alors une dépersonnalisation des solutions dans la mesure où elles sont reprises et exposées par d'autres élèves que ceux qui les ont trouvées, d'autre part il se produit une pré-décontextualisation : en reprenant à froid ce qui s'est passé, on élague les détails pour identifier ce qui est important. A cette occasion, le sens caché, le rôle pour l'apprentissage de l'un ou l'autre des problèmes posés peut se révéler à certains élèves. De plus, s'il y a une suite de problèmes sur un thème, chacun des problèmes traités est intégré dans un processus, il est intériorisé avec un sens nouveau. Au cours d'une telle situation, les formulations évoluent, on peut avoir des retours sur des débats de validation qui ont déjà eu lieu ou rencontrer la nécessité de nouveaux. On n'est pas à proprement parler dans une situation de formulation où il s'agit de produire un nouveau langage, ni dans une situation de validation, mais on retravaille les formulations et les arguments déjà produits. En même temps, par le retour réflexif sur l'action que ces situations supposent, elles favorisent la construction de représentations mentales par les élèves.

Dans ce type de situation, le rôle du maître est très important. Le choix de donner la parole à un élève plutôt qu'à un autre donne à la situation une signification toute différente : s'il veut que la fonction d'homogénéisation et de dépersonnalisation soit remplie, il va donner la parole aux élèves qui n'ont pas trouvé de solution ou qui n'ont pas abouti pour vérifier qu'ils suivent et reprennent à leur compte les méthodes utilisées, s'il veut avancer dans la décontextualisation et la formulation, il va davantage donner la parole aux "leaders", quitte à faire reprendre les nouvelles formulations du problème par l'ensemble de la classe dans le courant de la séance ou ultérieurement. On voit ainsi une évolution par rapport à la phase de bilan où ce sont plutôt les "leaders" qui exposent les méthodes de résolution qu'ils ont trouvées, les "suiveurs" se contentant d'écouter ou d'intervenir sur des points de détail qui sont dans le domaine de l'ancien. Ses marges de manœuvre se situent aussi dans le choix des questions, dans ce qu'il reprend ou non des interventions des élèves, dans ses commentaires.

Il peut agir sur ces marges pour ancrer "le nouveau" dans les connaissances anciennes et dans ce que les élèves ont réellement fait ou faire avancer la connaissance en s'écartant un peu du problème réellement traité, en proposant un début de généralisation ou de réinvestissement dans un contexte légèrement différent.

Le rôle du maître est essentiel dans le processus d'institutionnalisation, quel que soit le style d'enseignement. Il doit notamment choisir ce qui est à retenir dans chaque séance et décider en même temps quel "ancien" remobiliser, que reprendre dans les activités des élèves, jusqu'où aller dans la décontextualisation.

Ces décisions vont dépendre de ce que les élèves ont réellement fait et de l'évaluation qu'en fait le professeur : est-ce que ce qu'il considère comme ancien est réellement acquis par suffisamment d'élèves, est-ce que l'appropriation des méthodes de résolution est suffisamment généralisée dans la classe... Il s'agit là d'une évaluation globale, intuitive des élèves, qui a des liens avec l'évaluation officielle réalisée par ailleurs mais qui ne s'y réduit pas (cf Perrenoud 1986). Cette lecture par le professeur du travail des élèves va faire intervenir les représentations qu'il a sur le savoir visé ainsi que sur la manière d'apprendre, sur son rôle dans l'apprentissage des élèves. De leur côté, les élèves sont inégalement prêts à suivre le maître dans une décontextualisation de ce qui a été vraiment traité. Il revient encore au maître de laisser ou non la possibilité de refaire ce chemin à d'autres moments pour ceux qui n'étaient pas encore prêts.

## Conclusion

En analysant les difficultés d'élèves de 9 à 13 ans en échec scolaire et le fonctionnement didactique, nous avons rencontré des phénomènes qui peuvent provoquer, à notre avis, le renforcement de l'échec de ces élèves. Une question didactique importante est celle du choix de situations de complexité optimale pour ces élèves. Une autre, que nous avons commencé à étudier ici, est le rôle du maître dans le processus d'institutionnalisation des connaissances. A l'issue de ce travail, il nous paraît important d'approfondir cette étude, en particulier dans les situations que nous appelons "de rappel" car elles nous paraissent un lieu possible pour briser le cercle vicieux qui maintient ces élèves en échec.

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# Geometry and Spatial Development

# DIAGNOSIS AND RESPONSE IN TEACHING TRANSFORMATION GEOMETRY

Alan Bell and Derrick Birks
Shell Centre for Mathematical Education, University of Nottingham, UK

In a comparative teaching experiment involving two classes taught reflection geometry by the same teacher, a conflict and discussion method showed superior initial learning and good retention over two months, while a widely used scheme of individual guided-discovery booklets shows very poor retention. The development through piloting of the conflict method involved discarding easier tasks and retaining those harder ones which provoked discussion. These led to an analytic awareness of the essential properties of reflection as distinct form a global perception of 'balance'.

Among the topics treated in previous researches on pupils' developing mathematical concepts and on the design of diagnostic modes of teaching which make use of this knowledge, geometry has been somewhat under represented. The main part of this paper will help to redress this balance. It will bring into play a different set of didactic variables and a somewhat different way of designing tasks from, for example, the experiment reported last year on the teaching of fractions. (Bell and Bassford, 1989). In that study, the learning tasks consisted mainly of challenges embodying the fundamental notions concerning fractions - finding many ways of displaying half a square, finding all the fractions equivalent to a given one, finding how to compare $^3/_4$ and $^4/_5$, then any two fractions, finding how to add fractions - the essence of the challenge being to decide the meaning of the question and to choose suitable representations in which to seek an answer. The main didactic variable, which was manipulated to provoke generalisation, was the size of the integers comprising the fraction. The present study concerns the transformation of reflection; a quite different set of didactic varibles arises, and different ways of generating the field of examples.

The range of problems considered, and most of the misconceptions observed are illustrated by Figure 1, part of the last worksheet of the teaching sequence. The aim was the construction of plane reflections in a line and the identification of the lines of reflective symmetry in plane figures. Previous research by Kidder (1976), Schultz (1978) and Küchemann (1981) had identified as relevant variables

1. the direction of the mirror (horizontal, vertical, $45^0$, other)
2. the complexity of the figure being reflected
3. the presence of a grid
4. the size of the figures and distance from the mirror.

The first three of these were incorporated by Küchemann in a structured sequence of questions. He identified levels of response as global, semi-analytic, analytic and analytic-synthetic. In *global* responses the object is considered and reflected as a whole with no reference to particular parts, angles or distances; in *semi-analytic* responses, a part of the object, usually an end point, is reflected first and the rest drawn from it matching the original in shape and size. In *fully analytic* responses, the object is reduced to a set of key points, each reflected individually. These are connected and the result accepted even though sometimes the image looks wrong. In *analytic-synthetic* responses, the analytic and global responses are co-ordinated so that the final image is accurate and also looks correct.

The present study consisted of interviews and a pilot teaching experiment with a mixed ability secondary school class aged 11 to 12 years, followed by a comparative teaching study with two other parallel classes of the same age. We shall report here the main misconceptions found in the interviews, the design of the materials for the teaching experiment, noting particularly modifications made following the pilot work, and finally give the results of the comparative teaching.

*Pupils' Concepts*

The first group of misconceptions comprised beliefs that horizontal objects must have horizontal images and vertical objects vertical images or that horizontal objects have vertical images and vice versa. These can be seen in questions 2, 4, 5 of Figure 1.

Approximately 40% of the sample made errors corresponding to one or more of these misconceptions during the pilot testing. The next misconception consisted of associating reflecting with various pairs of opposites such as forwards and backwards, towards and away, left and right, upwards and downwards. For example, one pupil producing a response somewhat like that in number 2 of Figure 1 said,

> "this one is on the left and points up so that one must be on the right and point down".

The following worksheet was given to Edward Green for Homework. Mark the work, correcting all the mistakes. In your book, explain where Edward is going wrong

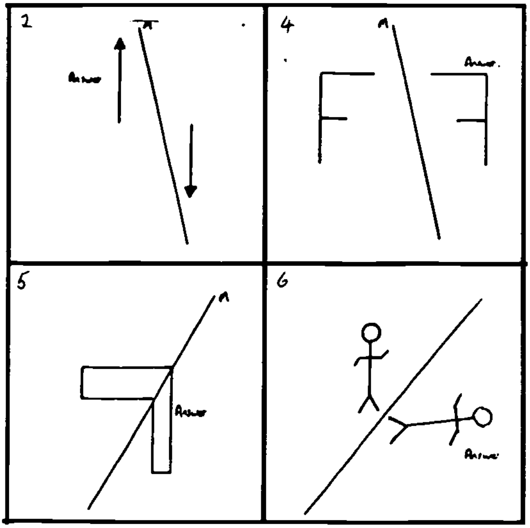



Figure 1

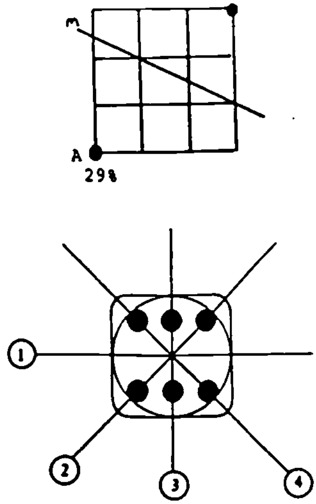



Figure 2

Such verbal descriptions of the relation between object and image which might be derived initially from some correct observation, are thus transferred to other situations to which they do not apply. In a similar way, the term 'straight across' was in some circumstances interpreted by some of the sample in a way dependent on the nature of the object and/or the presence of the grid and/or the slope of the mirror line. Thus the term might be used in item4 of Figure 1, and in cases like that shown in Figure 2, where axes 2 and 4 are taken as lines of symmetry.

The somewhat unexpected misconception that there could be more than one possible correct image was displayed by some 14% of the pupils interviewed; all of these were pupils who saw reflection as a mirror image, rather than as a folding and they often justified their conclusion by showing how the mirror could be moved, still standing on the same line, to produce a movement of the image. The same connection with physical mirrors rather than folding gave rise to another misconception, that the image might simply be similar to the original object and not necessarily congruent to it (Figure 1, No. 6). Over-generalisations of verbal statements were also used to justify erroneous placings of lines of symmetry on, for example, the letter N or as a diagonal of a rectangle; the shape was held to have a line of symmetry if it could be split into two equal parts even if they were inverted or displaced. Pupils said, for example, "it is the same on both sides". Another difficulty arose when a fairly complex figure might have symmetry if certain details were ignored. It would seem important that this type of example should be given, and pupils encouraged to give alternative statements about its symmetry according to whether or not various details are considered (figure 4).

## *Teaching Experiment*

The experimental teaching occupied 10 one hour lessons for each of the two groups. In the diagnostic method the pattern of each lesson was that pupils, in groups of about 4, discussed the problems on a given worksheet and arrived at agreed conclusions. Following this, there was a class discussion in which the conclusions from each group were contributed and defended and conflicts among the various interpretations were resolved. Extract from the worksheets for lessons 1, 3, 5 and 10 are shown in Figures 3, 5 and 1.

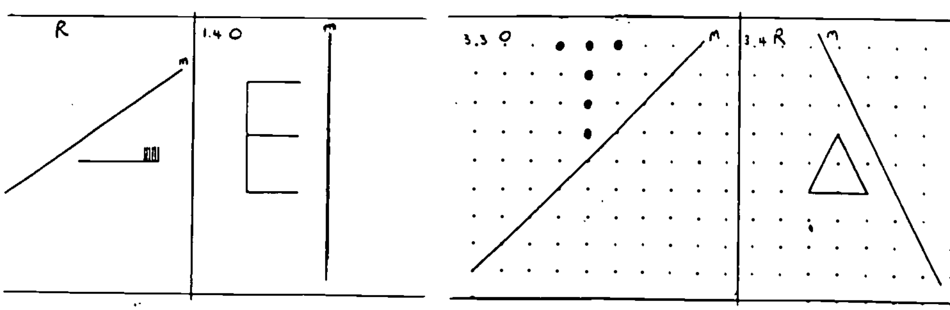



Figure 3

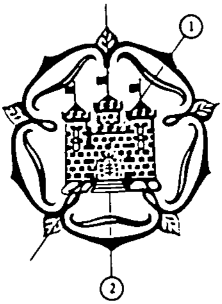
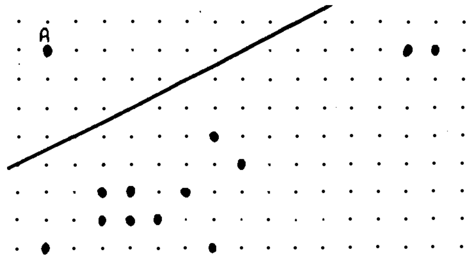



Figure 4




30 pupils were asked to find the point that was straight across from A. Ten different suggestions were given and are shown above. Circle the point that you think is correct or suggest another point if you do not agree with any of their answers

Figure 5

Some modifications were made to the sheets and to the mode of conduct of the lesson following the pilot work. These were as follows:

1. Certain easier tasks were omitted (marked O in Figure 3). These were answered correctly by most pupils, which left some of them with the impression that they were doing reasonably well and did not need to change their strategy even if they in fact possessed serious misconceptions causing errors on the harder questions.

2. The worksheet following Figure 5, which asked the pupils to write an explanation of why their choice in Figure 5 was correct. Some groups responded with rather weak explanations. In the modified lesson, the teacher intervened by playing 'devil's advocate' and so provoking them to produce more cogent arguments.

3. More time was taken at the beginning of the teaching to discuss the positive aspects of making errors, the importance of explanation and of listening skills, and the need for mutual respect of other's opinions.

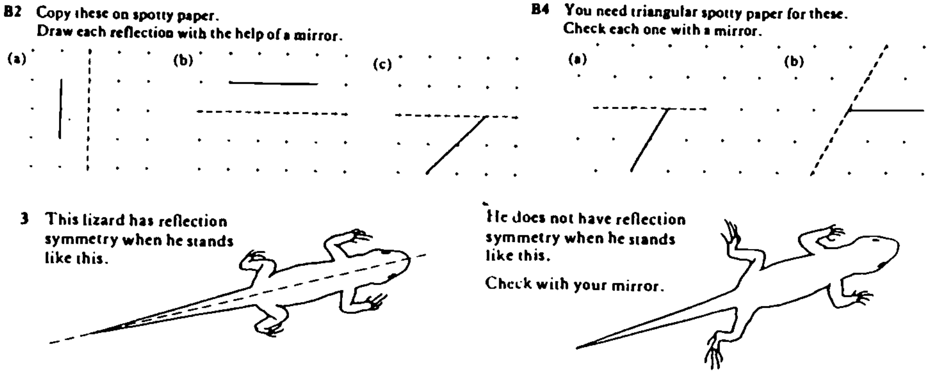


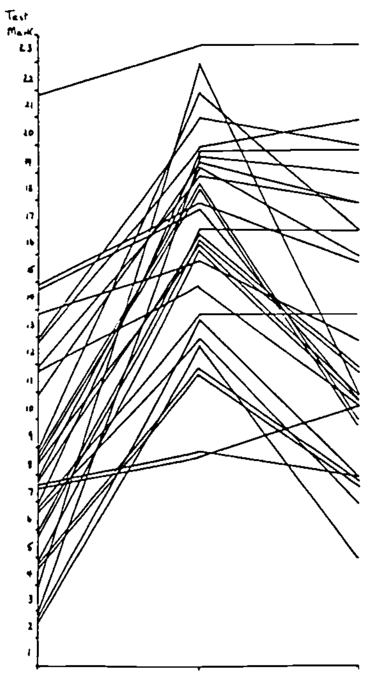

Figure 6

The alternative teaching method was based on two booklets on Reflections (from the SMP 11-16 course). These were in use in the school as part of an individual learning scheme during the first two years. The pedagogical method embodied in these is that of examples with explanation, followed by questions for individual practice. The first booklet concentrates on reflection as mirror images, the second booklet on folding. The questions become increasingly complex, but the learners are not asked to devise their own methods and none of the situations demands a high level of thought or enquiry. This contrasted with the diagnostic method, in which the aim was to lead pupils, through the discussion of difficult questions, to recognise and to state explicitly and carefully the general properties of reflection. The booklets were well received and enjoyed. In this group, the teacher was fully occupied in managing the issue of the booklets and administering the review and check tests, and in answering pupils' individual questions relating to the material. An example of the material is given in Figure 6.

## *Results*

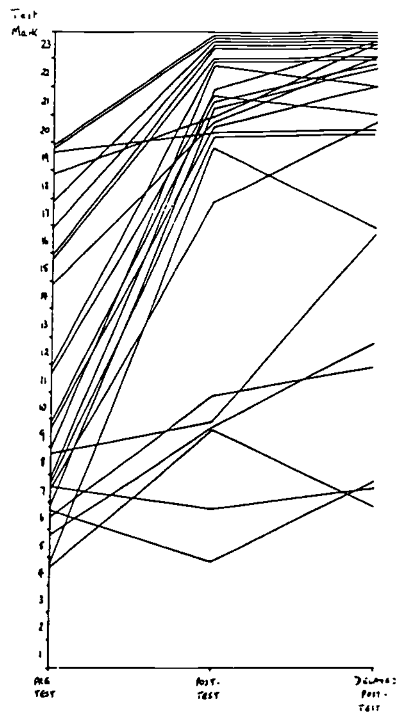
A 23 item test containing a mixture of items of the types illustrated here from both types of teaching was given to both groups before and immediately after the teaching and again 10 weeks later.

The graphs in Figures 8 and 9 show the performance of each pupil in each of the two groups. The superiority of the experimental method for retention, and correspondingly the long term inadequacy of the booklets teaching, is very evident. For a full report, see Birks (1987).



Scores of pupils in booklets group

Figure 7



Scores of each pupil in diagnostic group

Figure 8

*Implications*

Teaching of the type represented by the booklets method in this experiment is currently very common. It can best be characterised as 'guided discovery'. The initial explanation shows the pupils how to approach the questions, and as these are worked through different aspects of the embodied principles are called into play. Two elements commonly missing are (a) feedback and (b) awareness. Errors made are not generally discovered until sometime later when responses to the whole set of exercises are checked by the teacher or by reference to answers and at that stage, a score of 60 or 70% correct is regarded as satisfactory. Thus misconceptions brought into play by the remaining questions remain untreated, indeed, they are reinforced through the act of use. These materials also ignore the importance of making the correct principles, and the way in which they are manifested in various contexts, explicity through discussion. Other research has shown that what is actually learnt from studying a given piece of material is strongly influenced by the learner's orientation towards it, and this depends on the learner's expectation of the use to which this learning is to be put; for example, whether a factual recall test will be given or a test requiring comprehension of the material, or its application to fresh situations (Mayer and Greeno, 1972). In many current classroom environments, the expectation of future testing is minimal and, in some cases non-existent, and the pupil's orientation is towards the completion of assignments and the attainment of grades based on successful work. The distinction between *doing* and *learning* is often not made by pupils, nor sometimes by teachers, successful performance being what is rewarded rather than the acquisition of new knowledge or skills not possessed before, or the eradication of erroneous conceptions. These considerations suggest that metacognition, in the shape of pupils' awareness of their learning processes, is an important field for study and development at the present time. We intend during the next two years to make a study of pupils' learning concepts in a number of typical and innovative mathematics classroom environments, to develop approaches aimed at improving pupils' self awareness of learning, and to study the effects of the implementation of these.

# CHILDREN'S RECOGNITION OF RIGHT-ANGLED TRIANGLES IN UNLEARNED POSITIONS

Martin Cooper
University of New South Wales, Australia

Konrad Krainer
Universitat Klagenfurt, IFF, Austria

Are children better able to identify right triangles in orientations in which they have been learned or does the horizontal-vertical have an over-riding effect? Austrian primary school children were taught to recognize right triangles in particular orientations. On testing, they were able to identify them better when the shorter sides were horizontal and vertical, even when the triangles had been learned in other orientations.

In classrooms and geometry textbooks, right triangles are frequently presented standing on one of the non-hypotenuse sides - a "standard" orientation. Indeed, school children often experience difficulty in identifying figures such as squares when they are presented in non-standard orientations.

Cooper and Shepard (1973) found that when university students tended to take longer to say whether non-symmetric numerals and upper-case letters were "normal" or laterally inverted as the angle of inclination of the characters to their usual upright became greater. This suggested that people identify a tilted test character by mentally rotating an internal representation of the character into congruence with a long-term memory representation (a schema) of the character. Cooper (1975) found similar results with unfamiliar random angular forms. Herschkowitz <u>et al</u> (1987) have shown that children are able to recognize right triangles best when they are presented in "the upright position as usually drawn", have less success when the triangle is rotated through about 45°, with success decreasing "drastically" when the right-angle is "at the top". Eley (1982) found that children were able to identify letter-like symbols with more accuracy when the symbols were presented in orientations closer to the trained orientation.

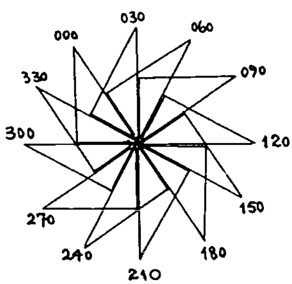
It may well be that when children are taught the definition of a particular figure by means of illustrations presented in a standard orientation they, too, form a mental image, or schema, of the figure in this standard orientation. When later faced with the same figure in a non-standard orientation, the child may then mentally rotate an internal representation of the figure into congruence with the already internalized standard-orientation representation.

Herschkowitz has suggested that the horizontal and vertical sides of the page on which a triangle is drawn may act as a "surrounding field" from which many people have difficulty in isolating the triangle. The effect is observed also in the case of isosceles triangles, the accuracy of recognition being greatest when the "base" is horizontal and at the bottom of the figure. Both Hart (1981) and Grenier (1985), mention this effect in relation to plane reflections when a vertical-horizontal grid is used as the background.

The purpose of the research presented in this paper was to investigate the variation in accuracy of recognition of a right triangle, learned in certain standard and non-standard positions, when later viewed in a set variety of orientations.

## METHOD

Twenty-four 35 mm slides were prepared for use in both the learning phase and the testing phase. Each slide depicted a clear circular disk set against a black background, the disk containing a triangle drawn in black. Half the triangles (Set R) were identical right triangles; the other half (Set N) were matching isosceles triangles having the same area and the same shortest-side length as the right triangles. The triangles in each set were oriented so that their <u>shortest sides</u> were respectively inclined at 0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300 and 330 degrees (clockwise) to the left-hand horizontal. For ease of reference, the slides were encoded as Rxxx or Nxxx, where the first character denotes the set and the 'xxx' stands for the angle of orientation. The R-set orientations are shown opposite




Fifty-five 7-8 year-old children from three primary schools in Klagenfurt (Austria) took part in the study. All were familiar with triangles and all had been taught to recognize right-angles. Children were interviewed individually.

In the initial, screening phase, the researcher questioned the child to ensure that he or she could recognize a right-angle. The child was then shown drawings of a right triangle and an isosceles triangle (both in standard orientation) and asked what the figures were called. All subjects said that they were triangles. The researcher then pointed to the right triangle and said that it was a special triangle because one of its corners was a right-angle (pointing to it). It was demonstrated that none of the corners of the other triangle was a right-angle. Again referring to the right triangle, the researcher told the child that a triangle having a right-angle in one corner was called a "right triangle". After the child had repeated the term, he or she was asked whether the isosceles triangle was a "right triangle" or not. All children gave the correct answer, many describing the isosceles triangle, later, as a "false" triangle.

After passing the screening phase, each child was randomly assigned to one of four methods in the training phase:

| Method | trained orientations | orientation group |
|---|---|---|
| A | 000, 090, 180, 270 [twice each] | "mod 000" |
| B | 030, 120, 210, 300 [twice each] | "mod 030" |
| C | 060, 150, 240, 330 [twice each] | "mod 060" |
| M | 180, 240, 300, 330, 150, 090, 060, 030 | (mixture) |

In each method, 16 slides (eight right triangles in the "trained orientations" mixed with the eight corresponding isosceles triangles) were projected on to the white wall in front of the child. For each of the first eight slides ("assisted"), the researcher told the child whether or not the triangle was right-angled, either pointing out the right-angle or demonstrating that none of the angles was a right-angle, as appropriate. After a short pause, the second sequence of eight slides ("unassisted") was screened, the child being asked to say whether or not the triangle was right-angled and, if so, to indicate the right-angle. If the child gave an incorrect response, the preceding slide was re-screened and the sequence continued from that point. If any child gave more than three incorrect responses, it was considered that sufficient learning had not taken place, and his or her results were discarded.

After a short pause, each child who successfully completed the training phase entered the testing phase and was shown the entire series of twenty-four slides in the following order:

R000, N090, R180, N270, N060, R150, R270, N120, R030, N300, R120, N180, R300, N030, N210, R240, R330, N150, R210, N000, R090, N330, N240, R060

For each slide, the child was asked to say whether or not the image represented a right-angled triangle and, if so, to indicate the right-angle. In this phase, neither assistance nor reinforcement was provided.

## RESULTS

### Training phase

The number of children entering each of the training phase methods and the number and percentage successfully completing it, were as follows:

| method | A | B | C | M | all |
|---|---|---|---|---|---|
| number entering | 10 | 15 | 17 | 13 | 55 |
| number (%) completing | 10 (100%) | 9 (60%) | 10 (59%) | 8 (62%) | 37 (67%) |

About two-thirds of the children successfully completed the training phase. There was a large difference between those who experienced Method A (100% completing) and those who experienced Methods B, C or M (about 60% completing).

### Testing phase

The numbers and percentages giving correct responses to the "R-set" slides triangles in the respective methods were:

| orientation | | 000 | 030 | 060 | 090 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Method | N | 10 | 4 | 3 | 8 | 5 | 4 | 8 | 5 | 5 | 9 | 4 | 5 |
| A | % | 100 | 40 | 30 | 80 | 50 | 40 | 80 | 50 | 50 | 90 | 40 | 50 |
| Method | N | 7 | 7 | 7 | 7 | 4 | 6 | 8 | 6 | 5 | 6 | 7 | 8 |
| B | % | 78 | 78 | 78 | 78 | 44 | 67 | 89 | 67 | 56 | 67 | 78 | 89 |
| Method | N | 10 | 6 | 7 | 7 | 7 | 8 | 8 | 4 | 6 | 8 | 6 | 8 |
| C | % | 100 | 60 | 70 | 70 | 70 | 80 | 80 | 40 | 60 | 80 | 60 | 80 |
| Method | N | 8 | 5 | 8 | 7 | 6 | 6 | 7 | 4 | 7 | 7 | 5 | 7 |
| M | % | 100 | 63 | 100 | 88 | 75 | 75 | 88 | 50 | 88 | 88 | 63 | 88 |
| total | N | 35 | 22 | 25 | 29 | 22 | 24 | 31 | 19 | 23 | 30 | 22 | 28 |

## INSPECTION OF DATA AND DISCUSSION

### Individual methods

<u>Method A</u> - trained orientations: mod 000 (000, 090, 180, 270)

The average success rate in the trained orientations was always 80% or greater. These orientations represent local maxima of performance, or "peak performances". For untrained orientations, this group identified between 30% and 50% of the right-angled triangles.

<u>Method B</u> - trained orientations: mod 030 (030, 120, 210, 300)

Average performance peaked at none of the trained orientations (although that for 030 was part of a plateau).

<u>Method C</u> - trained orientations: mod 060 (060, 150, 240, 330)

With the exception of orientations 000 (100%) and 210 (40%), all correct-identification percentages lay between 60% and 80%. None of the trained orientations shows a peak, although 060 and 150 belong to plateaux. The untrained mod 030 orientations have the worst results.

<u>Method M</u> - trained orientations:
030, 060, 090, 150, 180, 240, 300, 330

On average, the horizontal mod 000 (000 and 180) and trained mod 060 (060, 150, 240 and 330) orientations show good results (about 90%). No easily discernable pattern is evident.

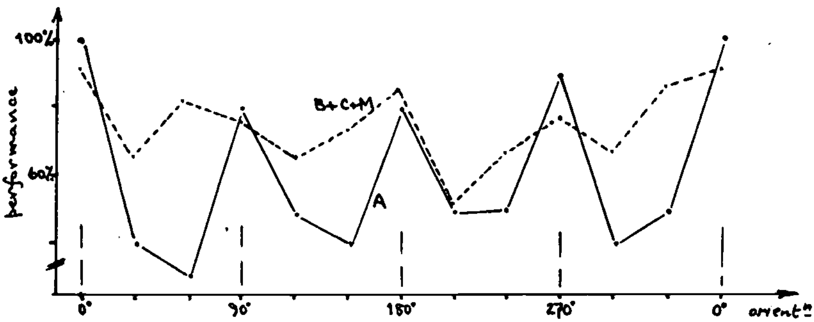
It is not surprising that when 7-8 year-old children are taught to recognize right-angled triangles in the standard mod 000 orientations, it is subsequently very easy for them to identify such figures in such orientations (100% of the research sample successfully completed the training). On the other hand, it is much more difficult for 7-8 year-olds to identify right-angled triangles in non-standard orientations, in spite of training in these positions under the same temporal conditions (only 60% completed the training). Thus, the training items were generally "easier" for children who experienced Method A than for those in Methods B, C or M.

Combinations of methods

Below, we present the numbers and percentages of correct responses for "non-A" method groups and for all groups:

| orientation | | 000 | 030 | 060 | 090 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Methods B+C+M | N | 25 | 18 | 22 | 21 | 17 | 20 | 23 | 14 | 18 | 21 | 18 | 23 |
| | % | 93 | 67 | 81 | 78 | 63 | 74 | 85 | 52 | 67 | 78 | 67 | 85 |
| Methods A+B+C+M | N | 35 | 22 | 25 | 29 | 22 | 24 | 31 | 19 | 23 | 30 | 22 | 28 |
| | % | 95 | 59 | 68 | 78 | 59 | 65 | 84 | 51 | 62 | 81 | 59 | 76 |

A graph of percentage of correct responses against angle of orientation is given below for Method A and for Methods B, C and M combined.




A glance at the table and graph given above suggests that, even for non-A methods, performance tends to be biased in favour of mod 000 orientations. For the Method A group, performance was very good at the trained orientations but relatively poor at all other orientations. The possibility of mental rotation between the mod 000 orientations cannot be ruled out. For all other method groups combined, subjects tended to perform better when the right triangles were presented in the "mod 000" orientations (apart from a small perturbation at orientation 090), even though these orientations were not trained ones. It is clear that the mod 000 orientations are more easily identified than any other group of orientations irregardless of which orientations were used in the training. The results for each of the mod-group orientations are summarized below:

| method | orientations mod 000 00,03,06,09 (A) | mod 030 01,04,07,10 (B) | mod 060 02,05,08,11 (C) | all combined (A+B+C) |
|---|---|---|---|---|
| A | 87.5% | 45.0% | 42.5% | 58.3% |
| B | 77.8% | 66.7% | 72.2% | 77.2% |
| C | 82.5% | 57.5% | 72.5% | 70.8% |
| M | 90.6% | 62.6% | 87.5% | 80.0% |
| B+C+M | 83.6% | 62.2% | 77.4% | 77.4% |
| A+B+C+M | 84.6% | 57.9% | 68.7% | 70.4% |

The last line in the above table suggests that, without regard to the method of training, the orientations may be classified into three groups: those which are most accurately, moderately accurately, and least accurately identified as right triangles.

The most accurately recognized group contains the triangles whose shortest and longest sides are horizontal and vertical. Of these, the triangles whose shortest sides are horizontal fare better than those whose shortest sides are vertical. In each of these subgroups, the triangle with the third vertex at the top is recognized with greater accuracy than that with the third vertex at the bottom. The "000" triangle and the "090" triangle - respectively the most and least easily recognized figures in this group - differ in both these characteristics; the "000" triangle has the shortest side horizontal and the

| response rate | orientation | horizontal or vertical side(s) | location of top or bottom vertex |
|---|---|---|---|
| 35 | 000 | short side horizontal | top |
| 33 | | | |
| 31 | 180 | | bottom |
| 30 | 270 | short side vertical | top |
| 29 | 090 | | bottom |
| 28 | 330 | no side horizontal or vertical | top |
| 25 | 060 | | top |
| 24 | 150 | | bottom |
| 23 | 240 | | bottom |
| 22 | 120 | hypotenuse horizontal | top |
| 22 | 300 | | bottom |
| 22 | 030 | hypotenuse vertical | top |
| 19 | 210 | | bottom |

acute angle at the top, whereas the "090" triangle has the shortest side vertical and the acute angle at the bottom.

The second most accurately recognized group consists of the triangles with no side either horizontal or vertical. Within this group, triangles with the acute angle uppermost fare better than those with the acute angle at the lowest position.

The least accurately recognized group consists of triangles having a horizontal or vertical hypotenuse. Within this group, one triangle is recognized less accurately than the others (which are all identified equally accurately). This triangle (orientation 210) has a vertical hypotenuse and the acute angle is the lowest position. Indeed, this triangle is the least accurately recognized triangle of the complete set.

In conclusion, the research shows that only a few 7-8 year old children can recognize right triangles in non-standard orientations. Even when they have been trained in these orientations, they can generally recognize right triangles better in the untrained, standard orientations than in the trained ones. It appears that children's natural body-axis direction (Piaget et al, 1972) over-rides any training of the sort given. "Upward-pointing" triangles tend to be easier to recognize than "downward-pointing" ones; again, children's natural standing orientation is upward from the base.

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# THE ROLE OF MICROWORLDS IN THE CONSTRUCTION OF CONCEPTUAL ENTITIES

Laurie D. Edwards
University of Washington

*This paper discusses the learning of a small group of middle-school children in a mathematical domain which was new to them. Their learning is described in terms of the construction of new "conceptual entities" which corresponded in important ways to the mathematical entities introduced during instruction. These mathematical entities, specifically, certain transformations of the plane (translation, rotation, reflection and dilation), were presented to the students in the context of an interactive computer microworld. By using linked visual and symbolic representations of the transformations in the microworld, the students were able to build their own partial understandings of these entities, and then go on to use them in problems-solving activities of various kinds. In addition, the microworld provided the feedback necessary for the students to "debug" or refine these conceptual entities so that they became increasingly close to the correct mathematical versions of the transformations.*

## Introduction

In a traditional textbook-based curriculum, students are often introduced to a new mathematical topic or domain by means of definitions and teacher-centered demonstrations. This kind of introduction is often followed by extensive practice with the new concept or procedure, and may culminate in actual applications or use of the concept in problem-solving. As an example, transformation (or motion) geometry is introduced in two recent textbooks as follows:

> Definition of Transformation
> A **transformation** is a one-to-one mapping whose domain and range are the set of all points in the plane (Bumby & Klutch, 1982, p. 440).
>
> The motion of an elevator is called a **translation**. You may think of a translation as a motion along a straight line without any turning (DeVault, Frehmeyer, Greenberg & Bezuska, 1978, p.290).

In the research reported here, children were introduced to transformation geometry in a very different way. Instead of presenting definitions and asking the students to apply these definitions, the researcher began by physically modeling simple motions of the plane in a very concrete manner (specifically, using cut-outs and transparencies on an overhead projector). She then elicited the students' <u>own</u> descriptions of the motions, and introduced a simple vocabulary for naming the transformations which was clearly related to what the students had seen and described. The remainder of the work with transformations took place as the students interacted with a computer-based "microworld" for transformation geometry. This microworld, called TGEO, linked the newly-introduced vocabulary (symbolic representation) to a dynamically-changing graphic display of the transformations (visual representation).

This paper will present the results of research carried out over a 6-week period with twelve middle-school children who used the TGEO microworld, and will discuss the role played by the microworld in the children's construction of new knowledge about transformations. Preliminary to presenting the details of the study and the results of the research, I will present a brief clarification of the terms "microworld" and "conceptual entity."

A microworld can be thought of as an embodiment of some abstract or idealized domain in a concrete or semi-concrete form which is accessible to new learners. Papert describes the microworld of the Logo graphics turtle as defining "a self-contained world in which certain questions are relevant and others are not" (Papert, 1980, p. 117). Microworlds typically present multiple, linked representations of the objects and operations in the domain. For example, in the Newtonian microworld known as the "dynaturtle" (diSessa, 1982), the motion of a graphical turtle on the computer screen is linked to "kicks" input from the keyboard. The turtle reacts to the kicks as if it existed in an idealized, friction-free Newtonian universe. Thus, the laws of Newtonian dynamics are embodied in the dynaturtle microworld; however, rather than being spelled out to the students as explicti laws, they are left implicit, waiting to be discovered. In creating an instructional context for using microworlds, activities must be designed which can help students to encounter the regularities in the domain, and to construct their own understanding of these regularities. It is the thesis of this paper that an important component in understanding a new domain lies in the construction and refinement of conceptual entities corresponding to the idealized mathematical or scientific entities of interest.

The term conceptual entities is introduced by Greeno (1983) in a discussion of problem-solving in mathematical and scientific domains. Greeno talks about the "ontology" of a domain, by which he means "the entities that are available for representing problem situations" (ibid., p 277.). Such mental "objects" are contrasted with attributes, relations, and operations which make use of the objects. As Greeno uses the term, "conceptual entities" refers to "cognitive objects that the system can reason about in a relatively direct way, and that are included continuously in the representation" of a problem or situation (loc. cit.). In the context of the research described here, the "system" is the learner; it is in the learner's mind that new conceptual entities are constructed. It is proposed here that when learners encounter a new domain, a significant part of their learning involves building new conceptual entities for the domain, distinguishing these entities from similar existing mental objects, and refining their understanding of the characteristics of the conceptual entities. The thesis of the research reported here is that a well-designed computer microworld can provide the conditions under which students can construct and "debug" conceptual entities in a new domain.

## Objectives of the research

The study was both an exercise in the principled design and evaluation of a new computer microworld for mathematics and a detailed qualitative investigation into children's learning in an intellectual domain which was new to them. The objectives of the research were to:

(1) design and implement a microworld for transformation geometry which would be effective in supporting students' learning in the domain; and

(2) to investigate the nature of children's learning as they interacted with the microworld. The aim was both to build a detailed qualitative model of what the students learned, and also to propose conceptual mechanisms which could at least partially account for the learning that occurred.

The students' learning was assessed both via quantitative measures (performance on paper and pencil worksheets and on a final exam) and by gathering extensive qualitative data (videotape and computer records of the students' interactions with the microworld, with the investigator, and with each other). This combination of quantitative and qualitative measures was intended to provide a sufficiently rich empirical base to at least begin to answer the following questions:

(1) Was the microworld and its associated curriculum of activities effective in helping the children to construct an initial understanding of the domain?

(2) How did the microworld support this learning? What were the characteristics of the microworld itself and the children's use of it which contributed to the students' learning in this new mathematical domain?

## Methodology

The study was carried out with twelve middle-school students, ages 11 to 14, from a private school in Oakland, California. There were nine boys and three girls in the group; of the group, one boy was Asian-American, one African-American, and the remaining children, Caucasian. The students worked in pairs after school in the computer lab one hour a week for a period of six weeks. Thus, their total exposure to the microworld was limited to about seven hours (including an initial introduction to the microworld in a whole-class setting).

The microworld itself is illustrated in Figure 1. Three euclidean (distance-preserving) transformations were instantiated in the microworld, called SLIDE, ROTATE/PIVOT and REFLECT/FLIP, as well as change of scale transformations (SCALE/SIZE).

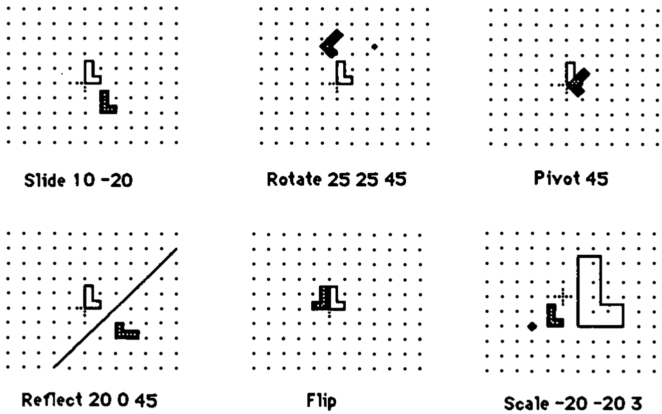


**Figure 1: Transformations**

The students were introduced to the transformations concretely, as described in the introduction. They were then immediately placed in a problem-solving situation, in which they were asked to use the transformations to play a game, called the Match game, on the computer. The purpose of the game is to apply a sequence of transformations in order to superimpose two congruent shapes on the screen. To succeed at the game, the students needed to understand each of the transformations, and in order to get the best score (by using the smallest number of moves) they also needed to compare the transformations with each other so as to find the most efficient sequence of moves.

Thus, the initial activity of the curriculum involved the students in problem-solving <u>using</u> the transformations (as contrasted with a more traditional approach of learning definitions and then practicing procedures with paper and pencil exercises). Later activities during the study asked the students to investigate <u>inverses</u> and <u>combinations</u> of the transformations and to use the vocabulary of transformation geometry to describe the <u>symmetries</u> of geometric shapes. The overall goal of the curriculum was to present the students with a range of increasingly-challenging contexts in which to <u>use</u> the transformations. This emphasis on <u>using</u> a concept as the initial step in learning is consistent with the Using-Discriminating-Generalizing-Synthesizing model proposed by Hoyles and Noss (1987). Not only was it hoped that these active, problem-solving contexts would be motivating for the children, but it was hypothesized that their understanding of the new mathematical entities would be richer and more flexible if they were constructed by the students themselves while solving problems.

## Results

The results of the study, in brief, indicated that the students were successful in using the microworld and the curriculum to build an initial and generally correct understanding of the euclidean transformations, and in applying this new understanding to problems in the domain. In the written final exam, which consisted of 12 tasks identical to those used in a British study (Hart, 1981), and 12 additional tasks, the students performed above the average for Hart's population on 10 of the 12 items. Thus, this group of students, who had a total of about seven hours of experience with the microworld, performed at a level comparable to the students in Hart's study, who were taught the topics of transformation geometry, as one part of their mathematics curriculum, over a period over several years.

In addition to this quantitative measure of the students' learning in the microworld, detailed protocol analysis was used to create a "learning paths chart" tracing the development of the students' understanding of the transformations. The students progressively discriminated more of the properties of each of the transformations as they worked through the curriculum, and they also showed development in their general and specific problem-solving strategies. The portion of the learning paths chart dealing with specific knowledge of the euclidean transformations is shown in Figure 2.

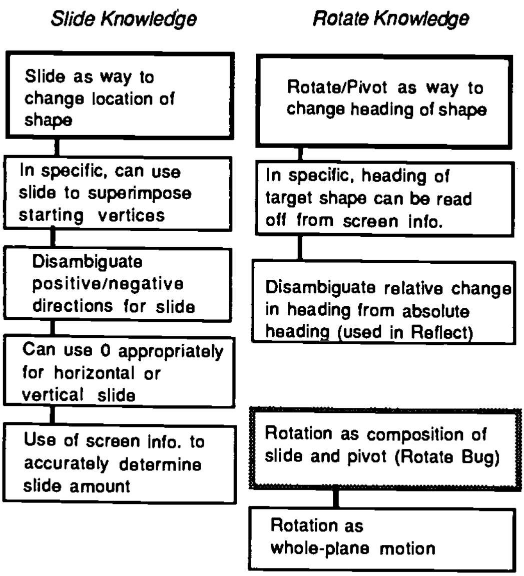


**Figure 2: Learning Paths Chart**

An example of a refinement of a conceptual entity is found in this learning paths chart, in the progression of the students' understanding of rotation is noted. An early misconception or alternative conceptualization of rotation (the "Rotate Bug") was found in 3 out of the 4 pilot subjects and 2 out of the the 12 main study subjects. In this conceptual bug, students believed that the rotation command was actually a combination of a translation and a pivot in place, rather than a turning of the whole plane around a single fixed center point. This bug was found much less often during the main study, when the transformations were introduced by using rotating sheets of acetate, rather than directly on the computer (as happened with the pilot group). In both cases, however, it is important to note that the students discovered and corrected this conceptual bug for themselves, when they found that their expectations about how ROTATE worked were not met in the microworld. In other words, they were able to use the visual feedback from the microworld in a process of conceptual "debugging" during which they refined their emerging conceptual entity for the rotation operation.

This process of constructing and refining conceptual entities is central to what makes the microworld effective as a learning environment. I will cite only one additional example of this process before turning to a discussion of some general characteristics of conceptual entities and microworlds.

In addition to constructing conceptual entities corresponding to each transformation, the students were asked to investigate new mathematical entities, including inverses, compositions and symmetries. The development of the students' understanding of inverse is another example of the construction of a conceptual entity, this time at a rather more mathematically-abstract level.

In the context of transformations of the plane, the inverse is the operation which "undoes" the previous mapping or motion. Thus, for example, the inverse of SLIDE 50 30 is SLIDE -50 -30, the inverse of a rotation would be a rotation in the opposite direction, and the inverse of any reflection would be the same reflection again. The term "inverse" was not introduced to the students until the second session, when they were asked to find the inverses for the various transformations, and to generalize by writing them as "formulas" (for example, the inverse of SLIDE A B would be written SLIDE -A -B). Even though inverse was not introduced explicitly in the early sessions, the students did use inverses while playing the Match game. If they missed the target shape, a common strategy was to invert the previous move, and re-enter a closer guess. In this case, the students were implicitly using the concept of inverse, but they showed no signs of being aware of inverse as a separate, identifiable conceptual entity. They did not have their own name for "undoing" operations, and they were unaware of any general characteristics of such operations. In other words, for them, the concept of inverse lacked both indexicality (having a name or way to refer to it) and internal structure.

When the Finding Inverses worksheet was given to the students, the opportunity was present for them to construct "inverse" as a conceptual entity. That is, once the students had worked at finding inverses explicitly, the idea of an "inverse" could be reasoned about directly, both in special cases and in its generalized form. The students were also able to determine the characteristics of inverses for each of the different euclidean transformations. They used the microworld to enter and test their candidates for inverses, and were successful at completing the worksheet and finding general versions for each of SLIDE, PIVOT, ROTATE, FLIP and REFLECT. In later activities, they were also able to use inverses in specific problem-solving situations. And finally, during a transfer task involving two new transformations, SIZE and SCALE, the children were asked to find the inverses for each operation. They understood this task immediately, and were able to carry it out easily (even though the inverses for size and scale were multiplicative rather than additive, as previous inverses had been).

Without a longer-term follow-up, it is impossible to assess the robustness of the students' construction of inverse. However, the children's work with this concept in the microworld showed a nice developmental sequence, from an implicit and informal use of inverses in playing the Match game, to an explicit focus on the term and its meaning for euclidean transformations, culminating in transfer of the idea to the context of a new set of transformations, SIZE and SCALE. Thus, in addition to constructing the individual transformations as a conceptual entities, each with its own name, internal structure and place in the children's reasoning processes, the students were able to construct and use an entity corresponding to an important and more mathematically-general concept, that of inverse.

## Discussion

In conclusion, I have proposed that it is through the construction of conceptual entities that learners made sense of their new experiences in the TGEO microworld. Greeno has stated that conceptual entities are objects about which people can reason directly, and which exist continuously in the representation of a problem situation. I would also propose that important characteristics of conceptual entities include indexicality, a representable internal structure, and a place in an emerging reasoning system. Conceptual debugging, in the context of the right set of curricular activities, is the process whereby students construct and refine conceptual entities. These entities, if they are productive, will be:

- useful in the immediate problem-solving or game-playing context;
- increasingly connected to other entities in the domain;
- ideally, more in line with standard mathematical entities; and
- the roots for the construction of new conceptual entities at the next level of abstraction.

The process of conceptual debugging is supported by a microworld environment because of a number of characteristics incorporated into its design. These include multiple, linked representations of the mathematical entities, which are presented in a way which is accessible to the new learner at his/her current state of understanding. The use of visual feedback is particularly effective in assisting students to "see" the differences between their emerging models of the transformations and the correct versions embodied in the microworld. Problem-solving activities centered around a computer microworld create a context where new entities are needed in order to succeed. And finally, a microworld which presents a symbolic system for representing mathematical operations and entities also provides students with a vocabulary, a way to attach names to the new conceptual objects which emerge as they interact with the microworld.

Paths for future research in this area will investigate commonalties among microworlds in different domains. Are the characteristics listed above essential for supporting learning in microworlds? What are the strengths and limitations of these interactive learning environments? In particular, an important issue concerns the difference between the kind of exploratory, inductive reasoning which is easy to do in a microworld, and an approach to discovery in mathematics which is more rigorous, analytic, and sensitive to the requirements of deductive proof.

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# THE COGNITIVE CHALLENGE INVOLVED IN ESCHER'S POTATO STAMPS MICROWORLD

Rina Hadass, Oranim, University of Haifa, Israel;
Nitsa Movshovitz-Hadar, Technion, Haifa, Israel;
Yehoshafat Give'on, Beit Berl, Kfar Saba, Israel.

## Abstract

This paper presents the preparatory stage of a study of the challenges embedded in computerized Escher-game environment. The instrument and the background are described, and results of a small scale pilot study of children operating in that environment are presented.

## Introduction

It is widely accepted that there are both practical and theoretical reasons for taking an interest in geometrical transformations. Usiskin (1974) has given a number of reasons for adopting a transformation approach to high school geometry. He claims that transformation approach is especially well suited for slower students. Küchemann (1981), who carried, within CSMS, a study of children's understanding of transformation geometry claimed: "The fact that the transformations can be defined in terms of actions (folding and turning), and their results represented in a very direct manner by drawings, means that the topic is ideally suited to a practical and investigative approach... in ways that are meaningful to most children." (Ibid p. 157).

The ultimate goal of our present work is to study slower students' intellectual functioning in an investigative environment of geometrical transformations. This is, within the framework of a curriculum designed for this population (Hadass and Movshovitz-Hadar, 1989; Movshovitz-Hadar, 1989).

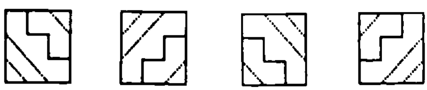
## The Environment

In developing the investigative environment, we were inspired by a game with engraved potato stamps invented in 1942 by the mathematician-artist M.C. Escher (Ernst, 1976). Escher's son, George, described a simplified version of this game, with which his father used to entertain him in winter evenings (Coxeter et als, 1986). We developed a computerized version of that game.

In our software, the potato stamps are replaced by squares. Each "stamp" has a different "engraved" pattern, and the various stamps are of such design, that when they are placed side by side, the lines connect to each other, and thus form beautiful "carpets". Twenty different stamps are stored in the computer memory (see appendix). Each can be called to the screen by a numerical code. The user can take a look at all the twenty stamps at one time, or see each of them separately.

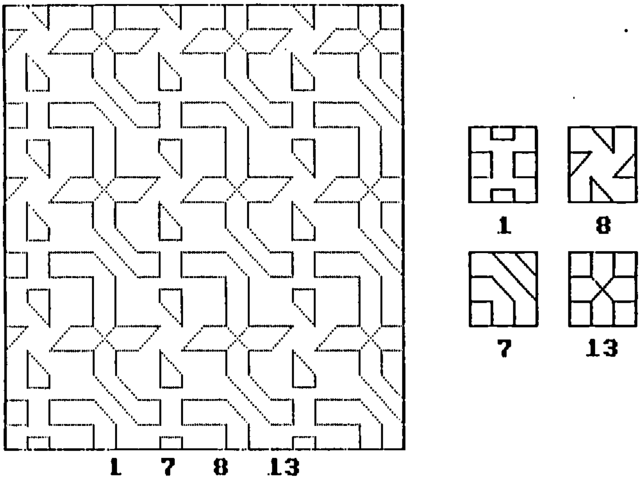
The software enables a variety of activities:

(a) Changing the basic stamps by transformations of rotations and reflections. E.g. by entering "4R" we get stamp number 4 rotated by $90^{0}$ to the right:

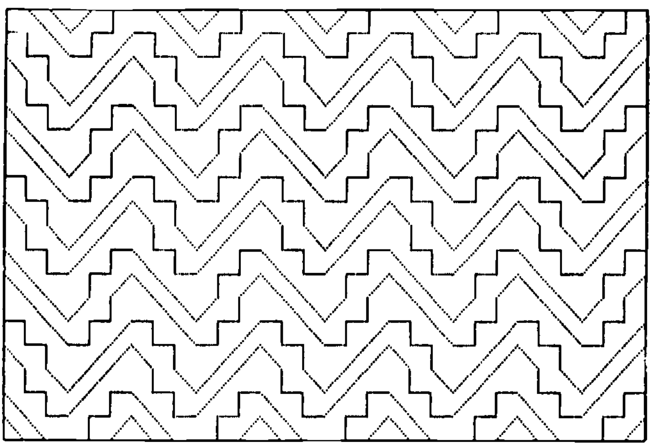


4 4R 4RR 4RRR

(b) Generating a carpet from a selected 4-tuple of the stamps, or their rotations and reflections, according to the user's choice. E.g.




(c) Detecting the basic stamps in a carpet, created randomly by the computer. E.g. Which 4-tuple of stamps generates the following carpet?



Operating the software can stimulate an investigation of many mathematical issues. For instance:

- Which basic stamps won't change under rotations or reflections and why?
- How many different stamps can be created from a given basic stamp by rotations (of $90^0$, $180^0$ and $270^0$) and by reflections?
- What is the influence of symmetry on the number of

distinct stamps obtained by rotations or reflections?

- How many different carpets can be created from four basic stamps by changing their order?
- How does the symmetry of the basic stamps affect the number of distinct carpets? The symmetry of the carpet obtained?

In addition, the software has the potential to introduce to students some basic notions of computer literacy.

## Preliminary Findings

We confine this report to findings obtained during the first exposure of nine children aged 11-17 to the software. Each child was given about 30 minutes to operate the software, in the presence of one researcher. All nine sessions were recorded. The reactions were then analyzed according to children's expressions, which can be attributed to intellectual functions. We bring here a few representative examples:

(1) Questioning the flexibility of the environment:
D. (17 year old), after looking the first time at the twenty basic stamps followed by a demonstration of a carpet created from 4 of them, by the researcher, asked:

- Is it possible to create new stamps?

Other children asked:

- Can I use more than four basic stamps to create a carpet?
- Is it possible to enlarge a given basic stamp?

G. (12 year old) commented also:

- It would be interesting if each stamp had a different sound..
- It would be nice if one could create additional stamps on the given ones or inside them.

A. (13 year old) suggested:
- It's worthwhile adding a colour to each stamp or to the whole carpet.

(2) Investigational behavior:
K. (13 year old) tried a carpet out of the basic stamps 6,7,6,7, and said:
- If we take 6,6,7,7, it will give another carpet, I guess.
She then confirmed it on the screen. Then she tested the differences between 4,4,7,7 and 4,7,4,7. She tried additional carpets and said:
- I am looking for non-symmetrical shapes. On the other hand, I think that with symmetrical stamps maybe the carpet will look nicer. That's why I keep trying.

(3) Concrete Observations:
After demonstration of a reflection of stamps 7 and 12, T (11 year old) was asked:
Q: What happens to stamp 15 by reflection?
A: You won't be able to notice any change, because the stamp is the same from all sides.
Q: And if I rotate stamp 15 by $90^0$?
A: You won't be able to notice anything either, because the stamp is the same from all sides.
Q: In which other stamps you won't notice a difference in rotation (by $90^0$)?
A: In stamps 2 and 20.
Q: What about stamp 1?
A: In reflection you won't see a difference, but in rotation (by $90^0$) you will see a difference, as it's long here and short there (pointing at the right places in the stamp, respectively).

(4) Generalization:
G. (12 year old) said:
- Symmetric stamps remain the same under rotation and reflection.

A. (13 year old) said:
- I know why each stamp can be connected to another one: it's because each side of the square is divided into three equal parts, and the connecting lines start out from two fixed points.

(5) Attitudes:
One girl, who seemed to be very practically oriented, said at the end:
- It's fun, but it lacks a defined aim.

After trying to discover the stamps from which a carpet was created, G. (12 year old) said:
- It's interesting, but difficult because the stamps interlock. If I spent one day playing with it, I would know whatever there is in it.

A. (13 year old) said:
- You can play with it as much as you like. There are 1001 possibilities.

K. (13 year old) said:
- You can make greeting cards out of it.
- You should write which combinations you prefer, so that you will be able to return to them, whenever you wish.

D. (17 year old):
- It's beautiful for textile designing.

## Summary

Our preliminary observations indicate that Escher-inspired computer environment, we created, provides fertile ground for a variety of intellectual activities. Moreover, it can make the topic of transformation geometry enjoyable, thus following Lesh (1976), giving a response to critics who charge that laboratory activities tend to "make fun topics important rather than making important topics fun".

The learning experience in this micro-world is different from the routine learning in school, and its advantages and limitations should be checked for the use of populations of students having different qualifications. It would be especially interesting to check whether low achievers, having a history of failure in school, can profit from this software. This study is in progress now.

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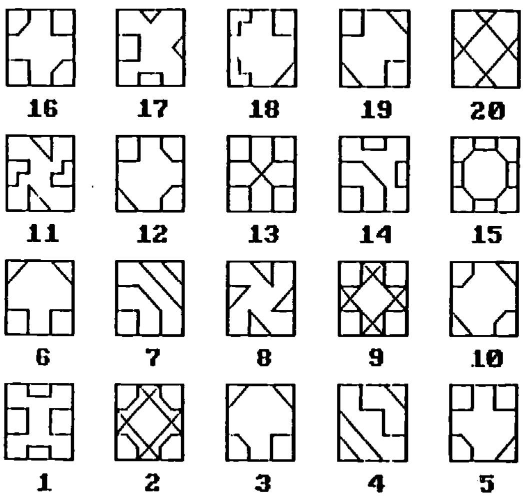
## Appendix

### The 20 Basic Stamps



A STUDY OF THE DEGREE OF ACQUISITION OF THE VAN HIELE LEVELS IN SECONDARY SCHOOL STUDENTS *

Adela JAIME and Angel GUTIERREZ

Depto. de Didáctica de la Matemática. Universidad de Valencia (Spain)

Abstract

*In this report we describe a method for assessing the* ***degree of acquisition*** *of every Van Hiele level of thinking by the students. Our core assumption is that for an accurate assessment of the students' level of thinking it is necessary to observe their way of using every thinking level.*

*We have administered a test on plane Geometry (namely on polygons) to a group of Secondary School students. The test has been analyzed according to the mentioned method, and we discuss the results. This way of working allows us to recognize different interesting students' behaviors.*

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The knowledge of the students' level of reasoning plays an important role in most research carried out on the Van Hiele model, as it provides a way for checking the theoretical hypothesis of researchers. It seems, therefore, important to define a method of evaluation which gives a good idea about the students' thinking level.

In this work, we have considered the Van Hiele levels 1 to 4, and we have excluded level 5. You can find a detailed description of the Van Hiele levels in several of the main references; see Gutiérrez, Jaime (1989) for a complete compilation of this references.

The existing literature about the Van Hiele model shows that, up to now, the methods of evaluating a student's thinking have resulted in the allocation of **one** level to the student. This has some problems, as there are students whose answers reflect the presence of various levels. In this paper we present a way of evaluation of the Van Hiele levels which considers such situation.

* This paper is part of a research project funded by the "Concurso Nacional de Proyectos de Investigación Educativa" (1989) of the Spanish Ministry of Education and Science.

There are two main points to consider:

1) The student's answers to several activities often reveal different Van Hiele levels of reasoning. This probably means that the acquisition of the levels is not absolutely linear (as stated in the theoretical descriptions of the Van Hiele model) but that the student is making progress within more than one level. We do not reject the hypothesis of the hierarchical structure of the Van Hiele level, but we propose to consider it in a wider meaning.

2) The acquisition of a thinking level by a student does not happen suddenly, but progressively. This progress can be recognized by the way how the student uses the thinking types specific to the level, from an initial period of lacking of awareness of the abilities of the level (no acquisition of the level of reasoning) to a complete mastery of the corresponding way of thinking (complete acquisition of the level), with several intermediate behavior patterns easily recognizable in the student's answers to problems. Of course, this comes in support of the hypothesis of continuity of the Van Hiele levels.

This progress towards the acquisition of the level is considered in our method of evaluation by means of the determination of a "**degree of acquisition**" of this level by the student. Therefore, the evaluation of a student's reasoning results in four values which reflect the student's degree of use of each Van Hiele level of reasoning. If we quantify the process of acquisition of a level of thinking, by representing it as a graduate segment from 0% to 100%, figure 1 shows the various periods of the progress through the segment divisions (acquisition of one level) that we have identified. The specific values of the partition are subjective and can be modified according to the researcher's point of view.

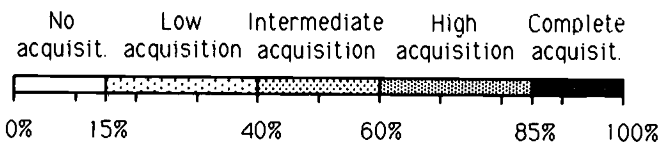


Figure 1

In order to determine a student's degree of acquisition of the Van Hiele levels, first we have to determine the **Van Hiele level** of each student's answer (levels 1 to 4). But the completeness of the answers and their mathematical accuracy should be taken also into account. We concrete these aspects of the answers by assigning each one of them to one of eight **types of answer**. To determine to which type an answer belongs, it is necessary to consider it from the point of view of the Van Hiele level it reflects; that is, the answer could be correct according to the requirements of a level, but incorrect according to the requirements of a higher level. Any answer to an open-ended item may be assigned to one of the following types:

Type 0: No reply or answers which cannot be codified.

Type 1: Answers which indicate that the learner has not attained a given level but which give no information about any lower level

Type 2: Wrong and insufficiently worked out answers which contain incorrect and very reduced reasoning explanations but give some indication of a given level of reasoning

Type 3: Correct but insufficiently worked out answers which contain very few explanations or very incomplete results but give some indication of a given level of reasoning

Type 4: Correct or incorrect answers which clearly reflect characteristic features of two consecutive levels of reasoning.

Type 5: Incorrect answers which clearly reflect a level of reasoning.

Type 6: Correct answers which clearly reflect a level of reasoning, but which are incomplete or insufficiently justified.

Type 7: Correct, complete, and sufficiently justified answers which clearly reflect a level of reasoning.

Consequently, these types of answer may reflect the various periods of acquisition of the Van Hiele levels of thinking represented in figure 1: Types 0 and 1 indicate no acquisition; types 2 and 3 indicate low acquisition of the level; type 4 indicates an intermediate acquisition; types 5 and 6 indicate a high acquisition; and type 7 indicates a complete acquisition.

Thus, we assign a **vector** (i,t) to each answer, where l is the level reflected by the answer and t is the type of answer (the component l is empty when t = 0). By weighting the types of answer t in terms of the percentage of acquisition of the reflected level of reasoning (from 0% to 100%) and by considering the vectors (l,t) of all the questions which could have been answered at each level, we obtain the student's degree of acquisition of the Van Hiele levels.

## Application of the method of evaluation to a specific test

As a part of an ongoing research project aiming the evaluation of the Van Hiele levels of thinking of students in Primary and Secondary Schools, we have administered a test on polygons to a group of secondary school students, and we have determined their Van Hiele levels with the method that we have just described.

Sample: The test was administered to 19 secondary school students (aged 15-16) in a Spanish Professional Training School.

The test: It was a paper and pencil test which consisted of nine open-ended items. Each sheet contained one item and they had a lot of blank space; furthermore the statement of the items encouraged the students to explain their answers. The questions dealt with several plane geometry topics: Triangles, quadrilaterals and polygons in general. Each student had a ruler and a protractor.

Items 1 and 2 are intended to identify specific sorts of figures: Regular, irregular, concave, and convex polygons in item 1; square, rhombus, and rectangle in item 2. The students were presented several figures and they had to identify them; they were also asked several questions aimed to know their ways of identification. Item 2 had also several questions about classification of figures, like: "Write whether there are quadrilaterals being rhombi but not squares. Justify your answer".

Item 3 began with formal definitions for "square" and "rectangle" that had to be used to answer to questions of identification and classification similar to those in item 2.

items 4 and 5 were based on the definition of a polygon (not a known polygon) called ANLA. Item 4 was similar to item 1, and item 5 consisted of questions about classification of ANLAs and other kinds of polygons.

In item 6 students were given a list of properties, and they had to select all those which were true in an obtuse triangle; they had also to select two minimal sets of conditions which enabled to define an obtuse triangle.

Items 7 and 8 were based on the sum of the angles of a triangle. In item 7 students were asked to prove that property for an acute triangle; students were provided with several hints so as to help them to write the proof. In item 8 students were given a complete proof for acute triangles and they were asked to prove the property for right and obtuse triangles.

In item 9 students were asked to prove that the diagonals of a rectangle have the same length and that the diagonals of a rhombus bisect and are perpendicular.

We do not evaluate the statement of the items (as done by Usiskin (1982) and Mayberry (1983)) but the students' answers (as done by Burger, Shaughnessy (1986) and Fuys et al. (1988)); then each item was assigned to a range of levels where it could be answered by the students (table 1). From our prior knowledge of the students, we suspected that most of them would have level 2 or perhaps level 3; therefore most items were intended to cover these levels. This assignation was first made by the researchers, and later it was improved by pilot testing.

| Item | Levels | Item | Levels | Item | Levels |
|---|---|---|---|---|---|
| 1 | 1, 2 | 4 | 2, 3 | 7 | 2, 3, 4 |
| 2 | 1, 2, 3 | 5 | 2, 3 | 8 | 2, 3 |
| 3 | 2, 3 | 6 | 2, 3 | 9 | 2, 3, 4 |

Table 1. Range of levels where the items can be answered.

The administration of the test: It took place in two sessions as a part of the class of mathematics. The students were allowed to take as long as they needed to answer the questions (time ranged from 25 to 45 minutes per session).

The evaluation of the test: We have assessed the students' level of reasoning by applying the twofold method described above (Van Hiele level and type of answers). First each researcher has assigned levels and types separately, and later we have compared our assignations, looking for a consensus when they were different.

To obtain the degree of acquisition of a given Van Hiele level, we weighed the student's answers to all the items which could have been answered in that level, according to the range of levels shown in table 1; that is, for level 1 we have considered items 1 and 2, for level 2 all the items, for level 3 items from 2 to 9, and for level 4 items 7 and 9.

Analysis of the results: Here we are not trying to generalize the results from the point of view of the general reasoning of the students, as the sample is restricted to one group of one High School. Our aim is rather to show how our proposal of considering the degree of acquisition of each Van Hiele level results in a more detailed information about the development of the students' reasoning than the classical assignation of one level of reasoning to the students.

The different patterns of acquisition of the Van Hiele levels obtained from the sample are shown in table 2, which depicts the acquisition of each level according to figure 1. Figure 2 shows graphs of the degrees of acquisition of the levels by students of groups A to F.

| | Level 1 | Level 2 | Level 3 | Level 4 | # of stud. |
|---|---|---|---|---|---|
| A | Complete | Intermed. | Low | ---- | 3 |
| B | Complete | Low | ---- | ---- | 9 |
| C | Complete | ---- | ---- | ---- | 2 |
| D | Intermed. | Low | ---- | ---- | 2 |
| E | Intermed. | ---- | ---- | ---- | 1 |
| F | Low | ---- | ---- | ---- | 2 |

Table 2. Number of students and their acquisition of each Van Hiele level.

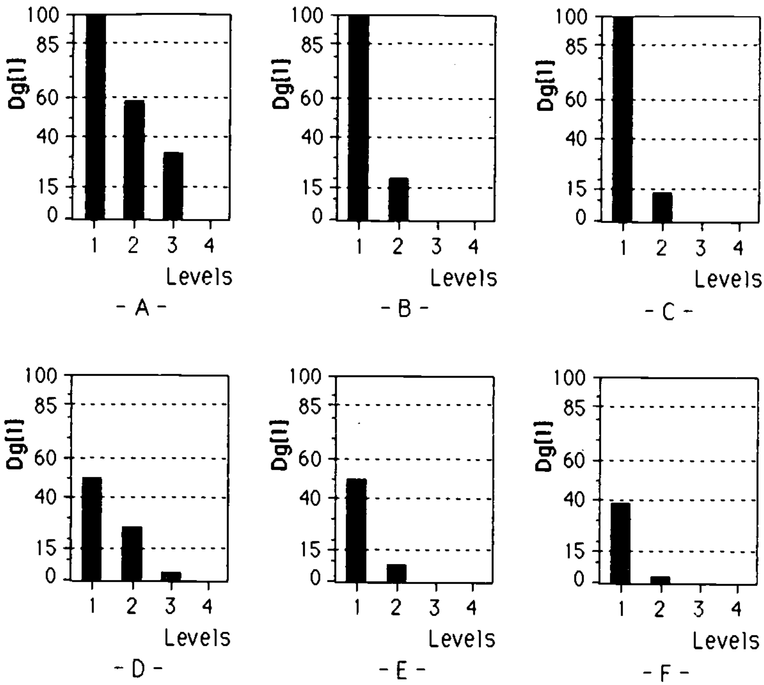


Figure 2. Patterns of the students' degrees of acquisition of the levels.

The classical assignation of a single Van Hiele level of reasoning would probably have resulted in the assignation of level 1 to the students in groups A to C and no level (or level 0) to the students in groups D to F; or, perhaps, it would have resulted in the assignation of level 2 to the students in group A, level 1 to the students in groups B to E, and no level to the students in group F. Anyway, it is clear that such kind of assignation implies an oversimplified view of the students' thinking abilities.

On the contrary, table 2 and figure 2 provide an evidence of important differences among the students: Some of them (group F) need strong instruction directed to the attainment of level 1, whereas other students (groups D and E) only need a reinforcement to complete the acquisition of level 1; the rest of the students (groups A, B, and C) has completely acquired level 1 and they need instruction for attaining level 2, some from the very beginning (groups B and C) and some from a more advanced point (group A).

Another interesting point that can be observed in table 2 is, referring to groups A and D, that a significant number of students begins the acquisition of the abilities of a thinking level before they have completely acquired the previous one. This is a situation certainly caused by the curriculum of mathematics and the way students have been taught, which has to be identified and investigated.

In short, this kind of assessment of the Van Hiele levels provides an accurate and detailed picture of the current students' thinking abilities.

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# Spatial Concepts in the Kalahari

by Hilda Lea

University of Botswana.

## Abstract.

Hunters and herdsmen in the Kalahari, who have never been to school and who have lived in very remote areas all their lives, were interviewed on two occasions to ascertain how far their spatial concepts have developed. When asked how they recognised animal footprints, and how they found their way in the desert, they were seen to have a very good visual memory, and to be aware of the minutest detail in recognising shapes. When given a visual thinking test, they performed with a high degree of skill on items related to their environment.

## Introduction.

Before Independence in Botswana, people's lives were largely untouched by the technological world, and many of those in remote areas of the Kalahari have been living all their lives in the same way and in the same environment as generations before them. It was therefore felt to be useful to investigate how far their spatial concepts have developed, as a result of the interaction with their particular environment.

Hunting in Botswana is carefully controlled, though Batswana can hunt non protected animals for food in areas where they live, during specific periods.
Seventy per cent of the national herd grazes in the Kalahari. Cattle do not normally stay near villages but are kept at cattle posts which are usually a long way off. Herdsmen spend their time in the sandvelt looking after cattle belonging to other people.
University students interviewed hunters and herdsmen known to them, at cattle posts in the Kalahari.

Spatial ability is a complex set of interlocking skills. Good visual memory requires an ability to retain, recall and manipulate information concerning shapes and spatial relationships. Visualisation depends on the degree to which the perception, retention and recognition of the configuration is seen as an organised whole. Orientation is an ability to manipulate a shape, to transform it mentally by moving or enlarging it, or seeing it from a different point of view.
Skills include aspects of distance, direction, perception, movement, and relationship of part to whole and objects to each other.

The Commonwealth Secretariat (1970) commissioned a review of research in different countries relating to difficulties students face in pictorial perception, in various cultural settings. Literature was reviewed on the subject and case studies discussed regarding the acquisition of particular skills. Eskimos were shown to have a high level of spatial ability (Berry 1966, 1974). In Papua New Guinea students from rural backgrounds were shown to have a highly developed visual memory (Bishop 1977). South Pacific studies on the navigation skills required when travelling by canoe among the islands, showed a highly developed sense of direction. Gladwin (1964) analysed the navigation skills of Trukese adults, which showed a concrete level of thinking. Navigation is by the stars, wind direction and wave patterns, and on a dark night by the sound of the waves and the feel of the boat. The Trukese knows where he is in relation to every island though cannot give a verbal account.

Lewis (1972) identified mental mapping in the orientation behaviour of Aborigines in Australia in finding the way. They seemed to have a dynamic mental map which was constantly updated in terms of time, distance and bearing, and realigned at each change of direction. The Aborigine also seemed to have "Dreamings" related in some way to paths which criss crossed the land, which ancestors had followed. They had great acuity of perception of natural signs and an ability to interpret them, and almost total recall of every topological feature of any country they had ever crossed.

Research shows that each society develops its own way of understanding and adapting to the environment. Different groups do not necessarily follow the same development path, since their particular goals and requirements are different. Visual memory is seen to be highly developed in many pre technological societies.

In Botswana, work has already been done on informal mathematics (Lea 1990), looking at mathematical activities in traditional daily life, and in the way of life of the Bushmen. This study looks at spatial abilities in the Kalahari to identify spatial skills acquired in the daily life of hunters and herdsmen, and to see how far these skills can be transferred to more structured situations.

## Experiment 1.

### Method.

During the Christmas vacation 1988, students carried out interviews with hunters and herdsmen in the Kalahari outside the game reserves, to ascertain a) how animal footprints are recognised b) how people find their way in the desert and c) if prints can be recognised when on paper, 42 subjects were interviewed of whom 26 were Bushmen.

### Results.

a) Though people live in an area where there are large herds of wild animals, it is still quite difficult to find them and to track them. The hunters explained with words, sand drawings and hand gestures how they know the difference between footprints. They said they sized them with their eyes and looked for distinguishing features such as presence of claw marks, distance between front and back prints, distance between toes and paw, distance between the two parts of a hoof, the depth and overall structure of the footprint.

Prints of hyenas and jackals are similar though differ in size. Their claws mark the ground but jackals' claws dig a little deeper. Leopards and lions have similar marks and do not show their claws. The leopard leaves tiny fur marks because its claws are hidden in their sheaths and covered with fur, and a lion's footprints are preceeded by a mark of fur as it tends to drag its paw. The general shape of the leopard's footprint is more circular than the lion's.

Hooved animals have similar prints as all have sharp pointed hooves except wildebeest and buffalo. Zebra prints are like a donkey's only larger. Buffalo and cows make similar marks but the hooves of the buffalo have an opening in the middle and make deeper marks. The four footprints of one animal never show the same mark.

They commented "Point to any spoor on the ground whether old or new, and the answer will be certain".

b) The Kalahari is a vast and desolate area, and those unaccustomed to the desert would find it apparently featureless. This is not so for the people who live there. Important features to be noted are particular trees, particular vegetation and vegetation under trees, and if travelling in unknown territory, these must be remembered in the correct order. The sun, shadow and direction of a breeze can help also.

Those interviewed said they would not get lost nor would they lose the track of animals they were trailing. Some said that if really lost, they would go to sleep and in the morning when the mind was refreshed they would know the way. Some said that a good method of finding the way back by donkey was to ride it without controlling it, and it would retrace the path followed earlier. Some said they would wait till night till the donkeys cry. Others said that if they were really lost when walking, they would look for a very straight tree with few branches, climb up and go to sleep, and in the morning the tree would tell them which way to go. Another said that when lost, shout "Beee.ee.ee." and if anyone hears he will come. Of those interviewed all had been to unfamiliar places and no one had ever been lost. One said "It may be easy to get lost in a city or village, but not out there, not in the wilderness".

c) Pictures of footprints were shown in three forms -- on sand coloured paper, solid black prints on white paper, and black outline on white paper. In each case there was no problem and all were identified. There was some argument over tiny detail for example that one print should be more pointed than another, or more curved at the edges.

## Discussion.

In any society, abilities best developed are those necessary for a way of life. Whilst most of those interviewed have little use for the printed word or picture, they were nevertheless able to recognise footprints irrespective of the context.

Mental mapping seems to be used in finding the way. Instead of referring to a map on paper at a particular time, positions and orientations are carried in the head, and these are realigned after every change of direction.

In Piagetian terms it would seem that thinking is at the concrete operations level, having no need to move to abstract thinking. In the concrete - iconic - symbolic mode of intellectual development, thinking is at the iconic stage because there is no need to move to the symbolic mode. The level reached is determined by the need of culture and environment.

**Experiment 2.**

## Method.

During the Christmas vacation in 1989, students gave the "K test on visual thinking" to 70 herdsmen in the Kalahari, from 8 different regions. The test contained 38 items requiring recognition and manipulation of shapes. Items 1 - 12 were chosen as having some relationship to the immediate environment, comparing lengths, tracing paths, following mazes, unravelling knots, and identifying right and left hands. Items 13 - 16 required the identification of animals from composite pictures from "Signs of the wild". Items 21 - 38 were more formal and were taken from Dale Seymour Set B visual thinking cards. Concepts tested were congruence, direction, geometric shapes, magnitude, part whole relationships, patterns, similarity, and rotation and position. All items gained 5 marks.

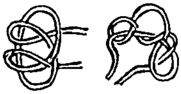
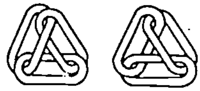
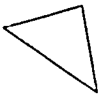
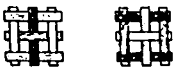
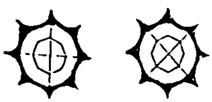
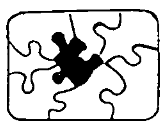
## Discussion.

Performance gave an overall of 56%. Questions related to the environment, Nos 3 - 18, 22 and 25 averaged 64%. Other questions which were more structured, many not having been encountered before, averaged 51%. This suggests that the subjects had a very good visual memory, and had acquired skills which were transferred to new situations.

As topological shapes do not have rules regarding length, number of sides and size of angles, they can probably be identified more easily, such as 32 simple figures, 29 key, and 25 irregular piece. 36 jig saw piece was more difficult as it had to be mentally rotated.

Knots questions were easy because a common activity is setting snares. 21 chain question would seem to be related, but this was not a high score. 22 arrows was done well though 35 arrows was not, as many of the subjects considered the arrows or the spots, but not both together.

7 and8 hands were easily identified though 9 and 10 to identify the odd one out was more difficult.
3 and 4 were following pathways by eye when ten paths intertwined. 23 faces would seem to have some similarity to footprint identification, but scores were not high. Perhaps the instructions were confusing.
20 embedded figures was well done.
18 orientation had quite a good score.
38 did not have a very high score, because unless turns were exactly through ninety degrees the cumulative effect of small errors gave a wrong direction.

| Item | Item |
|---|---|
| 12 82%  Will the knot pull tight. | 21 30%  From 8 find 2 the same. |
| 22 74%  From 12 find 3 same & direction. | 35 34%  From 15 find 2 exactly alike. |
| 32 78%  From 4 find piece like A. | 23 32% 34%  From 12 list pairs same shape; size. |
| 29 72%  From 6 choose 2 alike. | 25 58%  From 5 find piece like A. |
| 33 26%  From 7 same shape different size. | 20 78%  From 5 embedded figure like Y. |
| 24 86%  From 12 find 2 alike. | 30 36%  From 6 find 2 alike. |
| 31 26%  From 8 find 2 alike. | 18 60%  Identify slopes of A & B. |
| 38 34%  Face N turn L L R... End up. | 36 32%  From 6 choose correct piece. |
| 28 38%  From 6 choose piece to complete. | 27 58%  From 5 choose piece to complete. |

27 completing a lino pattern gained a higher score than 28 completing a zebra type pattern.
34 matching triangles gained a high score, but 33 to find triangles of the same shape and different size, was difficult.
13 - 17 animal pictures were enjoyed though not done particularly well. Two of these were composite pictures of animals in a group, where it was necessary to identify them in a holistic way.

Shapes with two variables were easily compared, though there was greater difficulty with three variables. With more than this, some sort of classification is needed to compare in a systematic way. Those interviewing said this was a problem with 31 where there were different arrangements of curved outline, circle, octagon and crossed lines. It was surprising that 24 had a high score as there were twelve pieces to choose from. These were coloured and this seemed to have made a big difference.

Items made use of spatial or visual imagery, and required the perception and retention of visual forms, and / or the mental manipulation of shapes, as well as a skill in making logical comparisons.

## Summary.

This paper attempts to show a relationship between cultural and ecological characteristics of a particular group of people, and the perceptual skills developed by that society. As in other cultures, certain perceptual skills must be developed for survival. The similarity to visual thinking of the Aborigines, Eskimos and others supports the idea that the development of perceptual skills is embedded in the individual's total environmental and cultural context.

It is clear that those who live in remote areas have highly developed spatial skills necessary for their way of life. In trying to measure the nature of these abilities, it would appear that they have excellent visual memory in identifying footprints, and the context in which these are presented is not important. They have a good sense of position and direction in their environment, and have a mental map which they can update quite easily. Performance was good on the visual thinking test, and it would be interesting to compare results on the same test given to secondary school pupils.

An abstract analytical way of thinking may be considered to be better in a technological society but not in a non technological society where visual thinking is very necessary. People tend to assume that those who have been to school are more intelligent, but this is not necessarily so. The type of intelligence differs and so does the experience.

## Acknowledgements.

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It would not have been possible to carry out this research without the help of students. Thanks are due to Mr Luckson Mabona, Mosiemamang, Namane, Petso, Kaisara, Sethapelo, Tinye and Tseleng who interviewed people known to them.

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# INTEGRATING LOGO IN THE REGULAR MATHEMATICS CURRICULUM. A DEVELOPMENTAL RISK OR OPPORTUNITY?

Tamara Lemerise, Ph.D.
Département de Psychologie
Université du Québec à Montréal

Summary

*Logo is at a turning point in its history. Certain crucial choices must be made in order to assure the development, if not the survival, of Logo in the current educational system. On the one hand, Logo still needs to penetrate more deeply into the educational milieu, multiplying its agents and its contributions. On the other hand, certain measures must be taken to assure that the fundamental link between Logo and mathematics be maintained. This report examines the extent to which the current trend favoring the integration of Logo into the school curriculum responds to these needs. Theoritical considerations based on experimental data are presented.*

## The adolescent period of Logo

It has now been nearly fifteen years that Logo has been known and used in primary schools for the purpose of creating an educational context that favors the development of mathematical thinking. Although it would be an exaggeration to say that Logo is currently undergoing an "adolescent crisis", it would not be farfetched to suggest, given the fundamental questions that one encounters nowadays, that Logo is very much in search of its identity. Among the many questions that have arisen, three in particular stand out.

(1) What is going to be the impact of Logo on the educational system? Will it be seen as the trigger that set off a <u>revolution</u> or as a factor of <u>evolution</u> in the educational system at large and especially in the field of mathematics?

(2) What is the connection between Logo and school mathematics? Is Logo <u>a new way of doing mathematics</u> to the exclusion of the old? a <u>mathematical alternative</u> that is complementary to the traditional one? or <u>a</u>

lifebuoy that has come to the rescue of a sinking traditionnal mathematics curriculum?

(3) Is the growing variety of ways of using Logo, which are actually observed in the schools, a sign of development and enrichment for Logo? a mere survival measure? or the premonition sign of an imminent death?

## The impact of Logo: revolution or evolution?

It is by now obvious that the initial goal of revolutionizing the educational milieu will not be achieved in a straightforward manner. It is more likely that any revolution that actually took place will be consolidated through the slow process of evolution: through successive waves of pedagogical changes that, step by step, embrace an ever increasing pool of active agents and significant educational topics. What data are available that support the hypothesis that Logo is a factor of evolution in the educational system?

On the one hand, there is clear evidence that Logo already went and is still going through important developments. Logo-Writer and Lego-Logo, to name but two recent innovations, are arms newly developed by Logo to help it reach out in the educational environment (Weir 1987). So in the last decade, not only have we witnessed the continuous technical sophistication of the early Logo but, more importantly, we have also seen Logo linked to different kinds of abilities that were not initially easely accessible to it: writing, mechanical and physical abilities with both theoretical and practical applications (diSessa, 1982, Weir, 1987, Weir, in press.). With this trend Logo tends to be more and more multidisciplinary slowly infiltrating its way into new fields after its initial start in programming and mathematics.

Parallel to this phenomenon of outward expansion, another kind of expansion can be observed, a more inward and subtle phenomenon, namely the development of cleaner and clearer connections with the mathematical universe. Such well-known authors as Hoyles, Noss (1987), Hillel, Kieran & Gurtner (1989) and Gurtner (in press) have recently underscored the pernicious possibility of children doing Logo without ever really getting in touch with mathematical entities or mathematizing the solution. There is a current today that favors a tighter and surer link to mathematical thinking. It is no longer enough to make loose associations between Logo and problem-solving abilities or turtle geometry; special care must now be taken to assure direct and solid connections with authentic *mathematical* types of solutions.

In its quest for rapprochement with the mathematical domain, Logo finds itself more and more in contact with elements that are already covered by the traditionnal mathematics' curriculum . A review of the literature points to an increase in the nature and in the frequency of the interaction. The extent to which these new liaisons with traditionnal mathematics can be beneficial to Logo -- or could constitute dangerous liaisons -- is an important question that need to be clarified in a general discussion of Logo's role in the evolution of the educational process.

**The relation between Logo and the traditionnal mathematics' curriculum: a risk or an opportunity?**

Simply stated, the risk of making tighter and tighter links between Logo and traditional mathematics is that Logo could be gobbled up by the traditionnal approach. This is often called the recuperation phenomenon. The old system annihilates the innovative approach by slowly adapting it to its own. Logo would then be treated, for example, as one exercice among others, an element of the curriculum mechanically "covered" by the teacher, which is what often happens to other mathematical topics. More tragically, the Logo spirit and philosophy could be muzzled for many years to come. Were Logo to be so ensnared, all hope would be lost for Logo as an active agent of change in the learning and teaching of mathematical thinking.

On the other hand, the opportunity that arises from forging tighter links to the traditional curriculum has to do with some Logo's contemporary needs. How, for example, could the insertion of Logo into the mathematics curriculum foster a real mathematical spirit and context when doing Logo. How could it favor the evolutionary role of Logo in contemporary education. Let us examine the opportunity and how the risks might be minimized.

*The link to the mathematics' curriculum. A way of mathematizing Logo*

As stated above, Logo does at times encounter difficulties in bringing children to think mathematically. Gurtner (in press) uses the metaphor of a tunnel to express how characteristics of Logo situations sometimes make "students miss nice view-points on mathematic and geometry"; and he asks that windows be opened in the Logo tunnels in order for children to have a perspective on related realities while working on specific Logo tasks. At the same time, Gurtner notes the need for bridges that permit students to go back and forth actively between Logo actions and basic mathematical principles, laws or notions. In order to avoid progressive isolation, Logo needs to be consolidated and enriched by significant links to the field of mathematics. As Côté (1989) emphasized when writing about his new microworld of "two turtles" -- "les deux tortues" -- many interesting connections can be made to concepts already in the primary and secondary mathematics' curriculum. Hoyles & Noss (1987a) have already start to work in this direction. Thus, from a general point of view, links with traditional mathematical content could be benefical to Logo.

In a way, Hoyles & Noss (1987a), de-dramatise the necessity for Logo to link up with mathematical concepts. The whole of mathematics' teaching seems to suffer from a similar but stronger malaise : "the separation of any sort of meaningful activity and the separation of pupil's conceptions from their formalisation". A first response to such a malaise resides in a general awareness of the need for links, concrete and abstract, for whatever problem-situation is being worked on. Many authors ( Coté & Kayler 1987; Côté, 1989; Gurtner, 1988; Hillel et al 1989; Hoyles & Noss, 1987a) have proposed the creation of mathematical microworlds as an interesting solution to this particular problem for Logo and to the more general problem of mathematics' education. The microworld notion can, of course, present subtle difference of definition from one author to the next, but what is most important is the view of working on a given topic from different points of view and with different kinds of tools (computer, paper and pencil - ruler, compass -, etc.). If that were done for all pertinent mathematical concepts (number, measure, area, variable, operation, function, etc.) the future of Logo and the future of mathematics would be in better hands! In sum, the confrontation of Logo with the mathematics' curriculum could be benificial to both, but especially to Logo given its chances of influencing the whole of mathematics' teaching.

*The link to mathematics' curriculum: A way to support the evolutionary role of Logo*

Historically, Logo has now reached the point where progress in the evolution of the learning and teaching of mathematics is, for the most part, in the hands of the teachers. In the beginning, Logo was actively supported by a nucleus of keyed up teachers and by a lot of researchers; then, after a short period of adaptation that in many cases brought along better infrastructural school support (more equipment, direct support in class, better information and training), a larger group of teachers became active in Logo. Today, with Logo having more direct links to a content that is known and judged important by teachers, a larger group could become positively involved with the Logo approach. This is the successive wave phenomenon mentionned above. Such a phenomenon is not particular to Logo and has often been observed in the past with other kinds of innovation. In the end what matters is that the spirit and crucial philosophy underlying the innovation be not lost in the successive phases of implantation and adaptation.

Authors who favor more direct links with the mathematics' curriculum have described the necessary conditions for not losing contact with the Logo philosophy (Côté, 1989; Hoyles, Noss 1987a; Hoyles, 1985; Gurtner, in press). What appears essential in implementing Logo in schools is not the form of presentation but the spirit in which it is presented, and the maintenance of specific pedagogical goals in whatever modality is chosen.

**The growth of variety in employing Logo : what counts?**

What evidence do we have that what counts is the nature of the goals pursued rather than the external means of presentation? An apparently "good" way of presenting something does not quarantee the respect of important goals: it is not because Logo is offered in an open non-directive environment that such developmental goals as the acquisition of autonomy, mathematical knowledge and thinking skills are neccessarily attained. Nor is it because Logo is offered in a relatively structured environment that such goals are <u>not</u> attained. As such ecologists as Bronfenbrenner (1979) and Garbarino (1982) have said : it all depends!

There is a wide variety of contexts in which Logo is offered today. A supervisor can choose basic Logo or opt for an expanded version such asLego-Logo. The context can be open, that is, centered on childrens' projects, structured in such a way that the situations are chosen in advance, or semi-structured, which alternates the two. It is possible to focus on aspects of visual art, programming and/or mathematics. Promoters of a mathematical framework can either choose to link it to mathematics curricula or find another way of assuring the process of mathematization .

Although it was once seen as heresy to do Logo in way different from what Papert (1979, 1980) first proposed, many now see it as a favor to Logo to vary its type of implementation. Of course it all depends on how it is done. Hoyles (1985a, 1985b) and Hoyles & Noss (1987a) have clearly described the needed conditions for an adequate integration of Logo in the school mathematics' curriculum. Research by Lemerise (in press), and Hoyles & Noss (1987b) has showed that a structured approach can facilitate the realization of many Logo goals. Côté (1989) and Weir (1987) even talk about "structured exploration" and "structured discovery" as a way of reaching some of Logo's goals; the pedagogical agent makes certain predetermined choices in order to favor exploration or discovery in a particular domain or situation. Some data now exist (Lemerise, in preparation) on how 4th, 5th and 6th graders behave in a specific microworld (Côté's (1989) "two turtles") that is tightly related to their school mathematics' curriculum. In a class' context where work on computer alternates with paper and pencil's work children construct, explore, compare and generate laws. Sometimes, of course, the way certain tasks are presented can trigger reactions of dependance, or guessing, but such problems arise in all contexts. What matters is that they are flushed out and dealt with intelligently.

Variety, in conclusion, is more often a strengh than a weakness, rigidity more deadly than flexibility. A revolution tends to be totalitarian, evolution more democratic.

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## Young Children Solving Spatial Problems

Helen Mansfield and Joy Scott
Curtin University of Technology
Western Australia

In our research with young children, we have begun to explore young children's actions when they are engaged in spatial problems. Our interest has not been so much in whether children are successful in solving the problems we pose to them, nor in tracing the success rates of children of different ages. Rather, we have been interested in exploring the procedures that children who are successful are able to use, and in trying to identify differences between the procedures employed by children who are successful in solving the problems and those who are unsuccessful.

Our research falls within the general framework of a constructivist view of learning. One aspect of this framework is the belief that the form taken by new knowledge constructed by learners is dependent on the form of the knowledge they already possess. Young children who already possess appropriate procedures and use them successfully on simple spatial problems seem likely to use those procedures more readily and successfully in subsequent more difficult problem-solving situations than children who do not already possess such procedures.

One of our basic assumptions is that children construct their own mathematical realities, which may differ significantly from the reality of the adult researcher. While there may be an inherently logical structure to a problem as it is perceived by adults, the structure the child imposes on the task may be different.

Lester (1983) has suggested that three main questions constitute the core of all mathematical problem-solving research:

(1) what the individual does, correctly, incorrectly, efficiently and inefficiently;
(2) what the individual should do; and
(3) how individual problem-solving can be improved.

The goal of the research reported here was to improve our understanding of how young children solve simple tangram-like problems. We were, therefore, interested in exploring the first of Lester's questions within the context of some simple spatial tasks.

In the tasks that we used, the children were requested to use two or three cardboard shapes to cover completely a region drawn on card. Clearly, these tangram-like puzzles require simple shape and size recognition and discrimination abilities. In particular, as well as recognizing the overall shape configuration of a region, children must be able to judge angles as equal and line segments as equal.

..e children may also draw on some planning skills to enable the tasks to be carried out successfully. Initially, there may be several ways that the first piece can be placed on the target 'egion. Usually, only one of these placements leaves a region that is the correct shape for the remaining pieces to cover it. Other initial placements cannot lead to a correct solution. The child then must recognize that the first piece prevents a solution and must be willing to remove it and place it again in a different position. The recognition that the first placement is unfruitful therefore requires further shape and size recognition and discrimination as well as willingness to remove and re-position a piece that seemed correct initially.

When children place a piece on the region, they may pick up the piece to be placed and by chance position it appropriately. If this does not occur, the child may be able to position the piece appropriately by removing it and trying again, or by rotating, reflecting, or translating the piece until it does fit satisfactorily. We speculated that children who are able to rotate or reflect the pieces before placing them on the region, or after they have been placed on the region, are more successful at solving the tangram-like problems that we presented to them. We also hypothesized that children who are able to recognize an incorrect placement and who are willing to remove a piece and try to find alternative placements are also more likely to be successful in solving the problems than those who do not display these behaviours.

The research reported here is exploratory. Our purpose was to observe children in a clinical situation as they attempted to solve a variety of tangram-like tasks, with a view to documenting the children's actions. We sought to identify the sequence of actions the children used, and to identify those actions that were efficient or successful. We also attempted to identify planning skills employed by the children.

**The samples**

We worked with three groups of young children.

1. Pre-school children. This sample consisted of 4 boys and 3 girls. These children were interviewed on one occasion. Their ages ranged from 45 months to 56 months, with a mean age of 52 months.

2. Pre-primary children. This sample consisted of 6 boys and 2 girls. These children were interviewed on two occasions. At the time of the first interview, they ranged in age from 55 months to 63 months, with a mean age of 59 months. At the time of the second interview, the mean age was 66 months.

3. Year one children. This sample consisted of 6 boys and 4 girls. These children were interviewed on one occasion. Their ages ranged from 70 months to 85 months, with a mean age of 75.5 months.

### Procedure

The two questions whose results are described here were questions three and four in a five question sequence. In the first two questions, we explored the children's knowledge of the names of common geometric shapes, presented as regions or as boundaries. In the fifth question, we explored whether the children were able to construct common geometric shapes from a variety of sticks of different lengths. Only the results of questions three and four are discussed here.

For questions three and four, the child was first presented with a set of geometric shapes made from pieces of card. The set consisted of congruent right isosceles triangles, congruent squares, congruent rectangles, and congruent equilateral triangles. Each square could be covered exactly by two of the right isosceles triangles, and exactly by two of the rectangles. The child was invited to handle the pieces and to sort them according to shape. Most children in fact did this without our asking them.

For question three, the child was then presented with four shapes drawn on card. Each of these shapes could be covered by two of the cardboard pieces which the child had already handled. There were divisions drawn on the shapes which represented the boundary between the constituent pieces and were intended to provide a clue to the placement of the required pieces. The child was asked to find the required piece from the set already sorted, or if the child did not know the name of the shape, the child was given the two pieces that were required. The following question was then asked: *Can you put the two shapes on top of this shape (shape on card indicated by gesture) so that they cover it exactly?*

A similar procedure was followed for question four. In this question, nine shapes to be covered were presented to the child on cards. The shapes were shown without divisions drawn on them to indicate the boundary between the pieces to be placed on them. Each shape could be covered by two or three of the cardboard pieces.

### Results

Reported here are the children's responses to questions three and four. Table 1 shows the number of children who were unable to complete the various problems presented in questions three and four.

Table 1

Numbers of Children who were Unable to Solve Problems in Questions 3 and 4

| | Question 3 | | | | Question 4 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Sample 1 | | 1 | | 3 | 6 | | | 3 | 2 | 5 | 2 | 4 | 3 |
| Sample 2(1) | | | | 2 | 2 | | | | | 1 | 2 | 1 | |
| Sample 2(2) | | | | | 1 | | | | | | | 1 | |
| Sample 3 | | | | 1 | 1 | | | | | 1 | 2 | | 2 |

The figures given in Table 1 suggest that the shapes that caused the most difficulty were shape 4 in question three, and shapes 1, 6, 7, 8, and 9 in question 4.

Shape 4 in question three consisted of a right isosceles triangle, presented with its longest side horizontal and at the base of the figure. The internal marking showed the boundary between the two smaller right isosceles triangles with which it had to be covered. Shape 1 in question four also consisted of a right isosceles triangle, with its longest side horizontal but at the top of the figure. Some of the youngest children's actions in attempting to solve these two problems are discussed in some detail below.

In question four, shape 6 was a parallelogram that could be covered by two right isosceles triangles, shape 7 a rhombus that could be covered by two equilateral triangles, shape 8 required a square and a right isosceles triangle, and shape 9 required a square and two right isosceles triangles.

The results summarized in Table 1 also show that the youngest sample (sample 1) had the greatest difficulty in completing the problems presented to them. While the table shows that most children in the other two samples were able to solve the problems, observation of the children as they attempted the problems showed that the older children did not necessarily find the problems easy to solve. Indeed, some of the problems proved to be quite difficult, but the children in samples 2 and 3 were very persistent in trying to reach a solution, and were also more prepared than the children in the youngest sample to remove and re-position a piece whose initial placement prevented completion of the problem.

A simple tally of success or failure at completing the problems also masks the quite different approaches the children used. The responses of four of the children from sample 1 as they attempted to solve the two isosceles triangle problems illustrate four quite different sets of actions.

Laura, who at 45 months was the youngest child interviewed, was successful at solving both the isosceles triangle problems, that is problem 4 in question three, in which the internal boundaries of the constituent triangles were shown, and problem 1 in question four, in which no boundaries were shown. In solving the first of these problems, Laura rotated one of the given right isosceles triangle pieces in the air until it matched one of the regions drawn on the target shape. She held the cardboard piece by its right angle, and appeared to match the other two angles of the piece to its region. In the second of these two problems, she was not told which two pieces to use, but picked up one of the correct right isosceles triangle pieces immediately. Again, she rotated the piece in the air until she was satisfied that it matched half of the target region before she placed it in position. She then easily placed the second piece.

Murray was unsuccessful at solving either of these problems. In the first problem, he placed a right isosceles piece so that its right angle matched the right angle of the target region. In doing this, he appeared to ignore the boundary drawn as a clue on the region. He then placed another triangle at the base of the target region, matching their two base angles. He was then left with an uncovered region for which there was not a matching shape. He chose a rectangle, and placed it so that it covered all of the remaining region but overlapped the boundary. It seemed that for Murray, the important objective was to cover all the region, without necessarily matching the shape of the target region. Apparently, the internal boundaries were no help to him. He also seemed to operate by matching congruent angles. In the second of these problems, he chose the same three pieces and placed them in the same way.

Jesse was successful in solving the first of these two problems but unsuccessful in the second. In the first problem, he initially tried an equilateral triangle, but after recognizing that the angles did not match, he chose a right isosceles triangle and was able to place it without difficulty. The second triangle piece was then easily positioned. In the second problem, he placed a triangle piece so that its base angle matched one of the base angles of the target region. He then considered the trapezoidal region that remained, and declared that it would not work, since there was not a shape like that available. He then re-positioned the first triangle piece so that its right angle matched that of the target region. Again, faced with a trapezoidal region remaining, he was unable to find a piece to match and made no further attempt at a solution. Jesse was able to recognize when a solution was not possible by considering the remaining region and attempting a match with the available pieces. He also attempted to re-position an initial placement, in contrast to Murray, who was unable to recognize that an initial placement did not enable a further correct placement to be made.

Teneka was also successful on the first of these problems but unsuccessful on the second. Teneka's approach seemed to be more arbitrary than that of the other children discussed here. In the first problem, she picked up the correct triangle in an orientation that by chance matched the target region. She seemed to recognize immediately that this piece was correctly placed. She then picked up and discarded a succession of triangle pieces until one that she picked up was in the correct orientation for the remaining region and she was able to complete the problem. She did not attempt to rotate the triangles, either in the air before placing them or after they had been positioned on the target region. In the second problem, Teneka made an initial placement of a triangle so that its base angle matched the base angle of the target region. She then tried a succession of triangle pieces, discarding those that did not fit in the way she had picked them up and selecting another. In this problem, she appeared to be matching the lengths of the second piece selected to the first piece she had placed on the target region. She appeared not to recognize that her first move could not lead to a correct solution, and was unable to re-position a piece that appeared initially to be correct.

The responses of these four young children show that they each used different sets of actions in an attempt to solve the two problems discussed here. One action that was used by Laura, who solved both problems, was rotation of the pieces. We ranked the children in all three samples according to how many of the problems presented in questions three and four they were able to solve. The children in the second sample, who were interviewed on two occasions, were entered twice in this ranking. Of the children who were ranked in the top ten according to the number of problems they solved, eight used rotation of the pieces in their solution attempts. Of the ten children who were ranked in the lowest ten according to the number of problems they solved, only two used rotations in their solution attempts.

The four children from sample one whose problem solving approaches we have described above used different procedures and had varying rates of success on these problems. From an adult perspective, Teneka's principal strategy of picking up and discarding pieces seemingly at random until they matched the target region was inefficient, and seems unlikely to enable her to solve the more difficult spatial problems with which she will be faced in her schooling. Yet Teneka was eventually successful at solving all the problems in question three and six of the nine problems in question four. There is no very compelling reason from Teneka's point of view to find a more efficient strategy. When we return to interview Teneka in a few months time, we will be interested to find whether she has retained this strategy or whether she has been able to or needed to find another strategy.

One question in which we were interested was whether young children continue to use the same procedures for similar problems over an extended period of time, that is whether the procedures they have developed well before starting school are persistent. It was to explore this question that we interviewed the children in sample two twice, seven months apart.

Our study of the actions of these children revealed some interesting features of their problem-solving strategies. At the time of the second interview, they were even more persistent than they had been in the first interview, and were generally willing and able to re-position a piece that would not allow a solution. As Table 1 shows, they were also more successful at solving the problems. We were also struck by the similarities that most children showed in their actions in the two different interviews.

In her first interview, Jessica was unsuccessful in the first triangle problem and successful in the second. Throughout her attempts at both problems, Jessica had both pieces she was trying to place on the puzzle in her hands. She would rotate and place one piece, then try to place the second. When that was unsuccessful, she would remove the first piece and rotate it to a new position. She appeared to be matching the lengths of the sides of the two pieces she was using, while also trying to cover the target region. At one stage in the second problem, she placed the first piece so that its right angle matched that of the target region and then placed the other triangle so that they formed a square.

In her second interview, Jessica was successful at solving both problems. Her actions were very similar to those she had used in her first interview. She had both pieces she was trying to place in her hands throughout, and placed first one and then the other, rotating them to try to place them. Again, she appeared to be matching the lengths of the two pieces, and she even constructed a square in the same way she had in the first interview. Perhaps the most obvious difference in her two interviews was that in the second, she did at one stage turn one piece over, which she had not done in the first interview.

While the similarities in the two interviews with Jessica were particularly striking, we noticed marked similarities in the actions of the other children in this sample. For example, Will more than any other child turned the pieces over several times in both interviews. He also placed the same pieces in the same inappropriate positions in the two interviews. However, by the time of the second interview, he also rotated the shapes in order to place them successfully.

### Conclusion

Our interviews with these young children showed that they had already developed some procedures for solving the problems we presented to them. Naturally, some of the children were more successful than others, and the youngest children were the least successful. The children who were successful showed an ability to recognize when a shape would not lead to a solution and a willingness to re-position pieces. Generally, the children were quite persistent in their efforts to solve the problems, with the children in the older samples being more persistent than those in the youngest.

Even though we interviewed only a small number of children, we found many quite different procedures used by the children. We also found that the actions of individual children who were interviewed twice were remarkably similar in the two interviews. The changes that were noticed were additions to the repertoire of actions that the children employed.

Our observations suggested to us that children who rotated the pieces were more successful at solving the problems than those who did not. Very few of the children actually turned the pieces over. It seems likely that children who try to rotate or turn pieces have developed actions that are particularly useful in these spatial problems. What we do not know is whether most children learn these actions for themselves or whether and how these actions can be taught.

Some children in all three samples used actions that from an adult perspective were inefficient. If we believe that children construct the procedures that they will eventually employ in solving spatial problems by trying out actions for themselves, then some children may retain their inefficient initial procedures over a long term, and may not construct the more efficient procedures that their peers are able to employ.

In the next phase of our work, we will be returning to interview the young children we interviewed here. We will be looking to see whether the procedures they display as they become older retain resemblance to the ones they have already used. We also want to work with older children to see whether the great variety of procedures we observed with these children are also observed with older children and whether older children are able to employ different procedures. We will also be exploring the procedures used by young children in a variety of other spatial problems, particularly three-dimensional problems. This work is directed towards gaining a greater understanding of how young children solve spatial problems and observing the genesis of the correct, incorrect, efficient, and inefficient strategies that older children use.

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# THE ROLE OF FORMAT IN STUDENTS' ACHIEVEMENT OF PROOF

W. Gary Martin
University of Hawaii at Manoa

*The two-column proof format is widely used in high school geometry in the USA. While many have suggested that alternative formats be used, little investigation of the impact of using other formats on students' ability to write proofs has been undertaken. The preferences of a class of 30 high school geometry students for two-column, flow, and paragraph formats were investigated, as was the relationship between the format used and success in writing proofs. This study suggests that, when given a choice, most students develop a marked preference for a particular format of proof, while others prefer to use a mix of formats. More students preferred two-column proofs than the other formats. The format of proof used was not found to be related to achievement in writing proofs.*

Developing students' ability to write and understand proofs has been one of the important objectives of high school geometry in the USA throughout the past century. However, for many students this objective is not being satisfied. Senk (1985) found that less than a third of high school students in proof-oriented geometry courses in the USA have "mastered" the ability to write proofs. Moreover, evidence exists that students frequently do not see a proof as a series of logical connections that guarantee the truth of a conclusion, given a set of hypotheses. For example, Martin and Harel (1989) found that many students base their judgment of the validity of a mathematical argument on whether it appears to be a proof, rather than on an analysis of its correctness. Fischbein and Kedem (1982) found that even students who accept a proof as being correct may not believe that this guarantees the universal truth of the statement.

One explanation for this lack of understanding, at least for students in the USA, may be the manner in which proof writing is presented. In the USA, proof has traditionally been presented to secondary geometry students using a rigid two-column format, in which the left column contains inferred statements and the right column contains reasons, generally definitions or theorems, to support each statement. Farrell (1987) describes the "awkward complexity" of the two-column proof. The typical high school geometry proof relies on *modus ponens*; $[(p \Rightarrow q) \wedge p] \Rightarrow q$. In the two-column format, the particular antecedent(s) (found in the preceding statements of the proof) are first presented, then a particular consequence (in a statement of the proof), and finally the general implication on which the inference is based (in the reason for

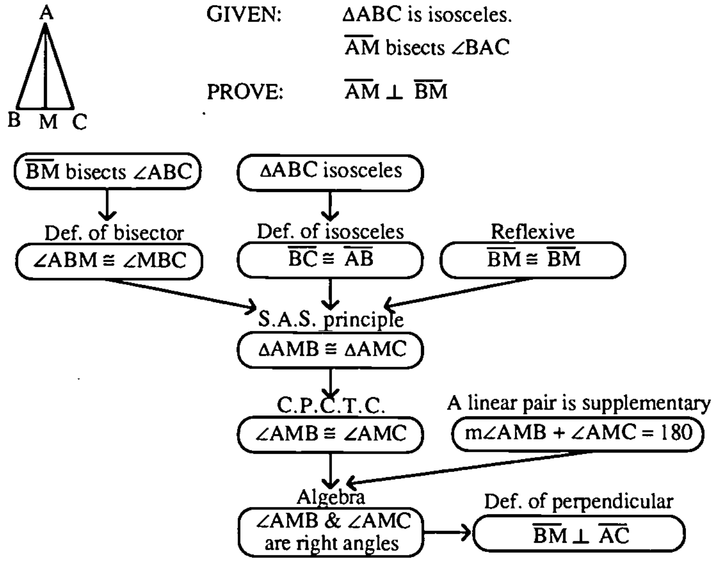


Figure 1. A sample flow proof, based on a student response.

the statement). This ordering does not follow the logical flow of *modus ponens*. Furthermore, the two-column format does not emphasize the logical connections between statements as the antecedent(s) for an inference are not specified but rather are buried in the preceding steps; this may prove confusing to students beginning the study of proof in geometry (MacMurray, 1978).

The use of two-column proofs has also been criticized on curricular grounds. The Conference Board of the Mathematical Sciences (1982) advocated "playing down" two-column proofs. The *Curriculum and Evaluation Standards for School Mathematics*, developed by the National Council of Teachers of Mathematics (NCTM, 1989), suggests that use of the two-column format should be greatly decreased. In particular, the document suggests that verbal paragraph proofs be emphasized and that the use of a particular format not be enforced.

Several alternative formats to the two-column proof have been suggested, in addition to paragraph proofs. For example, Retzer (1984) advocated adding numbers corresponding to the statements on which an inference relies to the "reasons" column of a two-column proof. MacMurray (1978) suggested the use of a "flow proof," in which logical connections are diagrammatically represented; an example produced by a student of this study is shown in Fig. 1.

Despite advocacies of increased attention to proof formats other than the two-column format, little attention has been given to the implications particular formats may have for students. If given a choice, will students prefer using paragraph or flow formats over the two-column format? Will they consistently use the same format? Why do they choose to use one format over another? Is use of a particular format associated with better achievement of proof? This study addresses these and other questions with respect to a class of students who were enrolled in a high school geometry course.

Subjects

A class of thirty tenth-grade students enrolled in a high school geometry class at the University of Hawaii Laboratory School (UHS) formed the subjects for this study. By state mandate, UHS is maintained for the purpose of curriculum research and development. In order to create curricula that are reflective of the needs of the students of the state, the student body is representative of all students attending public high schools in the state, based on intellectual ability, ethnicity, and socioeconomic level. The class participating in this study consisted of the top thirty of fifty-eight tenth-graders with respect to performance in previous mathematics classes at UHS and scores on standardized tests. Thus, students in the study were representative of average and above-average students of the state of Hawaii.

The class was conducted as a pilot study of an on-going curriculum research and development project in high school geometry. The emphases of this project include developing problem-solving processes (Rachlin, 1987), developing important concepts of geometry in accordance with the van Hiele levels (van Hiele-Geldof & van Hiele, 1984), and developing concepts of proof. Proofs were initially introduced the third week of the course by having students write informal paragraphs justifying why certain properties should be true for a given figure; this is consistent with the suggestions of the *Curriculum and Evaluation Standards* (NCTM, 1989, p. 144). Proof writing was not explicitly addressed until the fifth week of the course, when the given versus to-be-proven parts of a statement were discussed, along with general strategies for developing proofs. In the seventh week of the course, flow proofs were introduced. While two-column proofs were never formally presented in class and were never modeled by the classroom teacher, students were exposed to the two-column format in a textbook provided to them for reference. Students were initially required to use either the flow or paragraph format (or both formats) on several examples. They were later allowed to use the format of their choice on both homework and tests.

## Method

Three data sources were used in answering the questions of the study. First, four written assessments of students' proof writing were made at intervals throughout the school year; a follow-up assessment was made at the beginning of the second semester of the following year. The proofs were rated from two perspectives—the proof format used and the "correctness" of the proof. Correctness was rated on a 0-4 scale adapted from Senk (1985), as follows: 4—a proof reflecting all necessary aspects of the proof, with only minor omissions or errors; 3—a proof which is generally right, and has only one serious omission or error; 2—a proof which has a sequence of correct inferences but which is based on a faulty premise or fails to support the final conclusion; 1—a proof which includes one correct inference; 0—a proof with no correct inferences. Each response was independently rated by the investigator and a research assistant.

The second data source consisted of the complete written work of several students covering the entire school year. The work of five of these students was analyzed to provide a more detailed view of the impact of proof format on students' proof-writing. One proof was chosen for analysis from each week of the course in which proof was considered; the proofs were again analyzed by proof format and by correctness. In addition, the formats used in all homework problems throughout the school year were tabulated.

Finally, students were given a questionnaire concerning their preferred proof format in the third month of the course and again in the fifth month of the following year. In this questionnaire they were asked to identify their favorite proof format and why they liked or did not like each of the formats.

## Results

Responses from the five assessments of proof-writing were categorized by the format of proof used and by correctness; see Table 1. In the first assessment, taken early in their experiences with proof, most of the students still used paragraph proofs since this was the initial format introduced. By the second assessment, a distinct shift had taken place; the majority of the students were now using two-column proofs. This preference for two-column proofs was generally consistent throughout the remainder of the assessments, with relatively few students using paragraphs and almost no students using flow proofs. In the follow-up assessment the following school year, only five students used paragraphs and only one student used a flow proof. Note that missing scores are due to student absences from class.

Table 1. Achievement of Proof by Proof Format

| Testing date | Total | | Proof format: Flow Proof | | Proof format: Paragraph | | Proof format: Two-column | |
|---|---|---|---|---|---|---|---|---|
| | n | Mean | n | Mean | n | Mean | n | Mean |
| Year 1, Month 3 | 29 | 2.76 | 4 | 3.50 | 19 | 2.53 | 6 | 3.00 |
| Year 1, Month 5 | 25 | 3.04 | 2 | 3.00 | 5 | 2.20 | 18 | 3.28 |
| Year 1, Month 7 | 28 | 2.29 | 2 | 2.00 | 8 | 2.50 | 18 | 2.22 |
| Year 1, Month 9 | 30 | 2.63 | 2 | 2.00 | 8 | 2.50 | 20 | 2.75 |
| Year 2, Month 5 | 27 | 2.78 | 1 | 2.00 | 5 | 3.00 | 21 | 2.76 |
| Total Responses | 139 | 2.69 | 11 | 2.73 | 43 | 2.65 | 83 | 2.77 |

To provide a view of how individual students' usage of proof formats changed over time, the consistency of the formats they used in these assessments was also analyzed, as seen in Table 2. The minimum possible agreement is 40%, in which case the student would have used one format once and the other two formats twice. More than two-thirds of the class had a consistency of use of 80% or above, with a third being completely consistent in their use of a format. Thus, a picture of relatively consistent use of a particular proof format for a given student emerges, with most of the students using two-column proofs.

Table 2. Frequencies of Consistency of Use of Proof Formats

| Agreement | Total | Format: Flow Proof | Format: Paragraph | Format: Two-column |
|---|---|---|---|---|
| 60% | 8 | 1 | 0 | 7 |
| 80% | 10 | 0 | 1 | 9 |
| 100% | 12 | 1 | 4 | 7 |
| Total | 30 | 2 | 5 | 23 |

The correctness of the students' proofs does not appear to be related to the proof format used. In each of the assessments given, the mean scores for the formats appear to be very close to each other, as do the total mean scores for the formats. A similar result was found when comparing students' proof scores to the proof format they predominantly used; see Table 3. A Kruskal-Wallis analysis yielded H = 0.68, with $p > 0.71$, for this table.

Table 3. Relationship of Achievement of Proof to Preferred Proof Format

| Proof Format | n | Mean |
|---|---|---|
| Flow proof | 2 | 2.51 |
| Paragraph | 5 | 2.71 |
| Two-column | 23 | 2.71 |

The complete written work of five students was considered to obtain a more detailed view of the role of format in students' ability to write proofs. One proof from each week of the course was analyzed by format used and by correctness; see Table 4. Furthermore, the formats for all attempted proofs were tabulated. All the students initially used paragraph proofs, as this was the first format introduced. Several distinct patterns of use of and success with the various formats can be identified. Note that inferences relating format used to success must be viewed with caution since variables such as maturation may confound the inferred relationship.

Table 4. Use of Proof Formats by Individual Students

| Subject | Flow | | Paragraph | | Two-column | |
|---|---|---|---|---|---|---|
| | n | Mean | n | Mean | n | Mean |
| M. | 1 | 4.00 | 20 | 3.70 | 2 | 4.00 |
| S. | 0 | — | 12 | 2.67 | 8 | 3.38 |
| K. | 0 | — | 15 | 2.40 | 8 | 2.88 |
| L. | 2 | 3.00 | 13 | 2.54 | 3 | 2.67 |
| B. | 1 | 3.00 | 10 | 2.50 | 12 | 3.83 |

Three of the students showed a clear, long-standing preference for using a particular format. M. continued using paragraphs throughout the course, with infrequent deviations. M. was very successful in writing proofs; no differences in his use of the formats can be inferred. B. began using two-column proofs during the twelfth week of the course, after which he rarely used any other format. B. tended to be more successful with two-column proofs than with paragraphs. K. used paragraph proofs almost exclusively through the eighteenth week of the course, when he switched to primarily using two-column proofs, along with an occasional paragraph proof. K. tended to be somewhat more successful with two-column proofs. None of these three students used flow proofs more than four times in all of their written work.

The other two students used a more mixed set of formats. S. began using two-column proofs the twelfth week of the course and used them heavily for around a month; she also wrote a number of flow proofs during the twelfth and thirteenth weeks. By the sixteenth week she was again primarily using paragraph proofs, with an occasional two-column proof. She also sometimes used a mix of the two formats, a two-column proof in which reasons were presented as short paragraphs. She was somewhat more successful writing two-column than paragraph proofs. L. began using two-column and flow proofs during the thirteenth week of the course. Like S., L. continued to use some paragraph proofs. She rejected the use of flow proofs by the fourteenth week and never used another after that time. By the nineteenth week, she was again primarily using paragraph proofs. Unlike S., L. eventually completely abandoned the two-column format, never using it after the twentieth week. L. was equally successful in using the two formats. Both of these students experimented with flow proofs early in the course, but rarely used them later in the course.

The students were given a questionnaire the fourth month of the course and again the following year, asking them to identify the proof format they prefer and why. Their self-reports closely matched ($p<0.001$) the formats they used in their work—$\chi^2=21.86$ (df=4) and $\chi^2=35.96$ (df=4), respectively. Explanations for their preferences fell into the following categories. Preferences for two-column proof focused on organization ("Easier to organize my thoughts"), readability ("It's easy to read because it is like a list"), and understandability ("I can understand why what makes what"). For paragraph proofs, reasons focused on a flow of consciousness ("I just write it as I think it out") and a preference for writing ("I sort of like to write things out"). Several students expressed the opinion that flow proofs were especially good for short proofs; one student mentioned its adaptability ("Can adjust arrows, less erasing"), while others pointed to its quickness ("Much faster when you're short of time").

## Discussion and Conclusions

The data of this study suggest that, when given a choice, most students develop a preference for a particular format of proof, while others continue to use a mix of formats. Contrary to what might be expected, most students preferred to use two-column proofs. This is unexpected, both based on an analysis of the two-column proof from a student point of view and on the lack of emphasis on two-column proofs in their geometry course. Further, students produced plausible reasons for preferring the two-column format, including organization, readability, and understandability. The format of proof used was not related to achievement in proof-writing.

Many people have advocated a major deemphasis in the use of two-column proofs. This study seems to imply that such a change cannot be justified on the basis of student preferences or on the basis of achievement of proof-writing, although other reasons for the deemphasis (such as curricular considerations) may still be valid. In any case, given the strong preferences that students may develop for a particular format (such as M., who wrote "I love the paragraph, I like to thoroughly explain myself..."), the advice of the NCTM *Curriculum and Evaluation Standards* (1989) to not enforce a particular format for writing proofs seems particularly appropriate.

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# L'INFLUENCE DES ASPECTS FIGURATIFS DANS LE RAISONNEMENT DES ELEVES EN GEOMETRIE

A. L. Mesquita*
DEFCL
Lisbonne - Portugal

La géométrie est un domaine des mathématiques dans lequel on fait en permanence appel à trois registres, celui du registre figuratif, lié au système perceptif visuel, avec des lois d'organisation propres à ce sistème, celui du langage naturel, avec ses possibilités de description et d'explicitation du statut des énoncés et celui du langage symbolique, avec ses possibilités propres d'écriture et de recours à des formules. Parmi ces registres, ce sont surtout les deux registres de langage qui ont été étudiés dans les recherches en didactique des mathématiques (C. Laborde, 1982). Les recherches qui ont pour but analyser les difficultés soulevées par les problèmes en géométrie euclidienne se sont principalement focalisées sur les différences entre le mode de raisonnement des élèves et les exigences propres au raisonnement mathémathique. Tel est le cas de A. Bell (1976), de M. Stein (1986) et de N. Balacheff (1988), par exemple.

On a beaucoup moins prêté attention au rôle joué par la figure dans les problèmes en géométrie. Examinons brièvement ce rôle des figures. D'un point de vue mathématique, la question des figures semble être clair : un mathématicien sait ce qu'une figure peut lui apporter. Frenkel (1973) exprimait bien cet apport : "les figures permettent de mobiliser simultanément les multiples relations que la parole ou l'écriture - qui se déroule linéairement dans le temps - ne peuvent énoncer que successivement".

Les mathématiciens tout en reconnaissant ce rôle, reconnaissent aussi que les figures sont dispensables. "On s'en passe fort bien" (J. Dieudonné, 1964). L'absence des figures, prôné dans l'époque des mathématiques modernes est un "manque" toutefois bien suppléé par les mathématiciens. Bien entendu, les mathématiciens utilisent habilement ces trois registres et les articulent convenablement.

Pour les élèves, contrairement aux mathématiciens, le rôle des figures peut être ambigu [1]. En effet, a) ou bien les élèves ne parviennent pas à voir sur la figure ce qui peut amener à une solution, b) ou alors les figures attirent leur attention sur des pistes qui n'ont rien à voir avec le problème (et dans ce cas, la figure est un obstacle au développement du raisonnement) ; c) ou alors les figures remplissent un rôle heuristique pour le problème en question, et risquent de suggérer les démarches de raisonnement qui peuvent être selon les situations, correctes ou incomplètes.

D'un autre côté, les informations issues de ces trois registres ne sont pas nécessairement les mêmes. Le passage d'un registre à l'autre peut ne pas se faire directement. Il peut exiger alors une ou plusieurs transformations intermédiaires : c'est le phénomène de la non-congruence qui provoque un coût cognitif et qui constitue un obstacle pour les élèves, comme l'a montré R. Duval (1988a). D'un autre côté, due à la prégnance relative de l'information figurative, celle-ci domine naturellement l'information issue des autres registres (A. L. Mesquita, 1989a).

Nous présentons ici des critères d'un modèle d'analyse de figures qui vise à expliquer le pouvoir heuristique d'une figure dans un problème et qui cherche en particulier à determiner des facteurs qui pour une situation mathématique donnée font que les élements d'une solution soient plus ou moins visibles sur une figure ; par d'autres mots, qui cherchent à dégager les conditions de visibilité et de réorganisation d'une figure. Ces conditions de visibilité sont très variables, comment le suggèrent les recherches de J.-W. Pellegrino et R. Kail (1982). Ces auteurs ont mis en valeur le coût des opérations élémentaires (tels que la rotation et le déplacement) requises dans la recomposition d'une figure : les temps de réaction dans des tâches de reconnaissance sont variables, pouvant atteindre dix secondes dans le cas les plus complexes (ceux où la rotation et le déplacement sont mis simultanément en jeu).

D'un point de vue de la géométrie et des figures, la distinction suivante, déjà signalée par Merleau-Ponty (1945), à la suite des gestaltistes, a une importance fondamentale pour les traitements exigeant une réorganisation de la figure : "une ligne objective isolée et la même ligne prise dans une figure cesse d'être, pour la perception, la *même*. Elle n'est identifiable dans ces deux fonctions que pour une perception analytique qui n'est pas naturelle'(ibid., p.18). Cette distinction est à la base de notre modèle d'analyse de figures.

Pour notre analyse, nous prenons aussi en considération des critères liés a l'articulation entre les registres impliqués par les traitements. En particulier, nous mentionnons ici des critères liés à l'articulation entre le registre figuratif et le registre de langue naturelle. R. Duval (1988b) a montré l'importance de ce type de congruence, sémantique, entre ces deux registres, en mettant en évidence la différence de résultats obtenus dans une même tâche (présentée en deux versions, l'une congruente, l'autre non) : la tâche non-congruente obtient un taux de réussite moins élevé.

Cette articulation entre le registre figuratif et le registre de langue naturelle est à la base de deux critères importants[2].

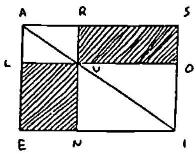
Un premier critère est le *rôle* de la figure. Il n'est pas le même dans tous les problèmes de géométrie. Ce rôle peut etre soit *descriptif*, quand il se réduit a une appréhension synoptique des propriétés en présence, soit *heuristique*, si la figure agit comme un déclencheur de démarches.

Le rôle de la figure est en général associé à ce que R. Duval (1988b ) appelle une *appréhension opératoire* de la figure, c'est-à-dire, à une forme d'appréhension centrée sur les modifications possibles de la figure et à sa réorganisation en des sous-figures autres que la figure de départ.

L'appréhension opératoire permet de mettre en évidence l'existence de figures fondamentales suggérant des traitements. Par exemple, dans le problème suivant, le rôle de la figure est heuristique:

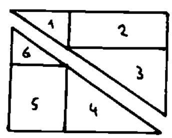
Dans la figure suivante, AI est la diagonale du rectangle ASIE.
Comparer les aires des deux rectangles hachurés OURS et LUNE .
(cocher la case correspondante à la réponse)

| OURS a l'aire la plus grande | Les deux aires sont égales | LUNE a l'aire la plus grande |
|---|---|---|
| ☐ | ☐ | ☐ |

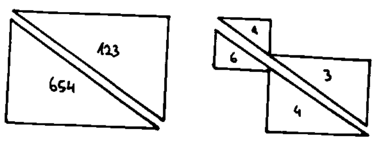



La reconnaissance de deux reconfigurations, l'incluante et la complémentaire, sont des facteurs indispensables pour une justification complète de l'égalité des aires (A. L. Mesquita, 1989a). La non-reconnaissance de ces deux reconfigurations est alors un obstacle heuristique.

Si la figure privilégie une certaine forme d'appréhension, cela peut mettre d'emblée sur des démarches de résolution, correctes ou non. Ainsi, nous avons vu (ibid.) que ce que nous avons appélé d'*identification analytique* de la figure -celle qui se centre sur les parties élémentaires- suggère naturellement l'opération de réunion (fig.1). Une *identification globale* -celle qui se fonde sur le partage de la figure- oriente plutôt vers le passage au complémentaire et les traitements soustractifs (fig.2).




(fig.1)




(fig.2)

Notons que l'utilisation de la réunion ne permet pas d'obtenir directement l'égalité des aires des rectangles hachurés. Un raisonnement par l'absurde ou par contraposition est alors nécessaire. Ces formes de raisonnement ont été utilisées par deux binômes de 10-11 ans :

| | |
|---|---|
| A: Je sais! ça ... c'est la même aire que ça ...donc ils sont pareils[3] les 2 rectangles (...) disons que ça, | réfs. 1 et 6, 3 et 4<br>réfs. 2 et 5 ; 3 et 4 |
| c'est 2, ça c'est 1, des deux côtés, donc ça fait (...) 3, il faut qu'en tout, ça fasse 4, disons...(...) il est | réf. 1 et 6 |
| obligé que les 2... fassent 1 cm ... sinon un est | réf. 2 et 5 |
| plus grand que l'autre... et vu que c'est bien divisé en 2... c'est forcément pareil des 2 côtés... | réf. R |

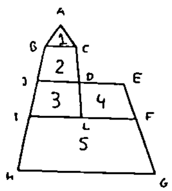
Un autre critère que nous avons considéré, le *statut* de la figure, est lié au type de traitement admis. La figure peut avoir un statut d'*objet* , si les relations géométriques utilisées pour sa construction peuvent etre reutilisées. Nous disons que la figure a un statut d'*illustration* quand on ne peut en extraire directement aucune relation géometrique. Même si certaines relations d'incidence et d'alignement, par exemple, semble respectées.

Ces distinctions ne sont pas automatiquement perçues par les élèves. D'où une non-congruence possible entre le statut de la figure tel qu il est engendré par la tâche, et l' interprétation de la figure telle qu'elle est perçue par chaque élève.

Par exemple, dans le problème suivant, le statut d'*illustration* de la figure n'est pas perçu par les élèves qui utilisent la mesure et la proportionnalité[4], procédures incorrectes pour une figure avec un tel statut :

Paul regarde, d'en bas, une cathédrale. Il fait le croquis ci-dessous, où il dessine ce qu'il a observé. Il note aussi les indications suivantes, prises aux archives de la cathédrale:

- 1 est un triangle équilatéral;
- 2 est un rectangle;
- 3 et 4 sont des carrés;
- la figure formée par 3, 4 et 5 est un carré;
- la longueur de AC est de 12m.




D'après ces indications, que peut-on dire des longueurs suivantes?
(cocher la case correspondante à la réponse)

| | C'est 12 m | Ce n'est pas 12 m | On ne peut pas savoir | CATHE1 |
|---|---|---|---|---|
| LF | ☐ | ☐ | ☐ | |

| | C'est 12 m | Ce n'est pas 12 m | On ne peut pas savoir | CATHE2 |
|---|---|---|---|---|
| FG | ☐ | ☐ | ☐ | |

| | C'est 12 m | Ce n'est pas 12 m | On ne peut pas savoir | CATHE3 |
|---|---|---|---|---|
| CD | ☐ | ☐ | ☐ | |

Les résultats des réponses des élèves à cette question le montrent bien :

*Tableau 1 : Les taux de réussite /échec [a]*

| | CATHE1 | CATHE2 | CATHE3 |
|---|---|---|---|
| Réussite | 64 | 50 | 37 |
| Echec | 31[b] | 42[c] | 57[d] |
| Non-réponse | 5 | 8 | 7 |

Note:
a en pourcentage
b 20% des réponses concernent LF≠12
c 24% des réponses concernent FG≠12
d 48% des réponses concement CD≠12

Le statut d'illustration est d'ailleurs la difficulté majeure de ce problème. Une fois dépassé cet obstacle, les substitutions nécessaires pour sa résolution sont facilement faites. Notons que la figure a ici un rôle descriptif : néanmoins, l'appréhension des propriétés exige une correcte interprétation du statut d'illustration de la figure.

*En guise de conclusion*

Ces critères nous donnent une base objective pour l'identification de certains obstacles liés aux problèmes. Ils apparaissent comme des critères efficaces pour une analyse de tâches et en particulier de leurs difficultés. Les aspects figuratifs ont, en effet, une influence dans le raisonnement et les critères réperés contribuent à l'éclaircir.

La distinction entre les types d'appréhension semble être un moyen indispensable pour effectuer une analyse utile des tâches géométriques. Cette distinction ainsi que les concepts utilisés constituent des premiers elements d'une théorie cognitive de la résolution des problèmes de géométrie. Ils se présentent comme des outils dont la finalité est double. D'un côté, ils permettent d'établir une gradation des difficultés de résolution de problèmes de géométrie en fonction du statut et du rôle des figures et des critères de congruence. Dans ce sens, une hiérarchie de difficultés peut être établie. D'un autre côté, en choisissant convenablement ces critères, on peut, par sa variation, s'attendre à que certaines questions revèlent des différences individuelles dans les réponses des élèves[5].

NOTES

1 Il n'est pas alors étonnant de constater que dans l'enseignement des mathématiques la place des figures a changé avec le temps. Ce changement reflècte le rôle variable et ambigu que les figures peuvent avoir pour les élèves.

2 Nous développons en A. L. Mesquita (1989b) le modèle d'analyse mentionné. Ici, nous nous occupons spécialement de quelques critères de ce modèle, et de leur influence sur le raisonnement.

3 Ont la même aire. A noter aussi que nous utilisons ici le codage introduit dans les figures précédentes.

4 Il s'agit de réponses d'élèves de 14 ans.

5 L'analyse de ces différences individuelles donne lieu à une typologie des comportements des élèves en géométrie, que nous décrivons en A. L. Mesquita (1989b).

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* Boursière de Fundação Calouste Gulbenkian
Departamento de Educação da Faculdade de Ciências de Lisboa
R. Ernesto Vasconcelos, C1- piso 3
P- 1600 LISBOA

CHILDREN'S UNDERSTANDING OF CONGRUENCE ACCORDING TO THE VAN HIELE MODEL OF THINKING

LILIAN NASSER

UNIVERSIDADE FEDERAL DO RIO DE JANEIRO - BRAZIL
KING'S COLLEGE - UNIVERSITY OF LONDON - ENGLAND

In this work, descriptors for the van Hiele levels of thinking in the concept of "congruence of shapes" are suggested. Four activities were designed and used in clinical interviews with English and Brazilian students aged 13 to 16 years. In Brazil congruence is taught formally, in a Euclidean approach, while English students learn congruence informally, through transformations. The analysis of the interviews give evidence that, despite the approach used, the responses fit the level descriptors suggested, and that the levels are hierarchical.

The van Hiele model of thinking in Geometry established by Pierre van Hiele and Dina van Hiele-Geldof in the late 50's (van Hiele, 1959) has been investigated according to several points of view in the last decade. Some research studies addressed the relation between the levels achieved by a student in different tipics of Geometry (Mayberry, 1983; Gutierrez and Jaime, 1987; Nasser, 1989). Usiskin (1982) and Senk (1985) investigated the relation between the van Hiele level achieved by a student at the beginning of the year with his (her) achievement on Geometry tasks during the year, while Fuys, Geddes and Tischler (1988) studied the effects of instruction modules on students' van Hiele levels. Burger and Shaughnessy (1986) provided a characterization of the van Hiele levels of development in Geometry based on responses to clinical interview tasks concerning triangles and quadrilaterals, and suggested that the same kind of investigation should be carried out concerning other geometric concepts.

This work is part of research to investigate if the learning and understanding of "congruence" by Brazilian secondary school students can be improved when the instruction is based on the van Hiele theory. To obtain a picture of how the concept of "congruence" is acquired, this work was developed with the purposes of:

(a) Suggest descriptors for van Hiele levels in "congruence";
(b) Develop activities fitting the descriptors in (a) to be used in clinical interviews;
(c) Through the analysis of the interviews, check if the level descriptors suggested in (a) are acceptable.

The topic of congruence is taught in different ways in England and in Brazil. In England, through a transformation approach, congruence appears informally, as "the same shape and size". There is no attempt to prove the congruence of triangles, but transformations that preserve length can be used to justify the congruence of shapes. On the other hand, in Brazil, through a traditional approach to Euclidean Geometry, congruence is taught using deductive reasoning. Students are asked to write proofs based on the cases of congruence of triangles (SSS, SAS, ASA) to justify other properties of shapes.

Analysing the descriptors of the van Hiele levels given for traditional Geometry by van Hiele (1959), Hoffer (1983), Burger and Shaughnessy (1986) and Fuys, Geddes and Tischler (1988), the following descriptors can be suggested for the van Hiele levels in congruence:

Basic level - Recognition of congruent shapes only based on appearance. Orientation is considered as a relevant attribute for congruence. Corresponding elements of congruent shapes are not yet perceived in isolation.

Level 1 - Recognition of congruent shapes relying on measurements and/or fitting on top of each other. Orientation is seen as irrelevant. Properties of congruent shapes are analysed (necessary conditions).

Level 2 - Establishment and understanding of sufficient conditions for the congruence of triangles. No attempt to justify formally the congruence of shapes.

Level 3 - Ability to reason logically, in order to justify the congruence of triangles. Informal proofs can be attempted (using the cases of congruence or transformations).

Level 4 - The importance of rigour in demonstrations is understood. Ability to write a formal proof using the cases of congruence or triangles or transformations.

Four activities were designed fitting the level descriptors above. The activities are described in detail bellow together with the expected responses to them.

Activity 1 - Recognition of congruent shapes:

The five cards on fig. 1 were shown to the student, who was asked if the pair of shapes in each card was "congruent" (or "the same shape and size") or not, and to explain.

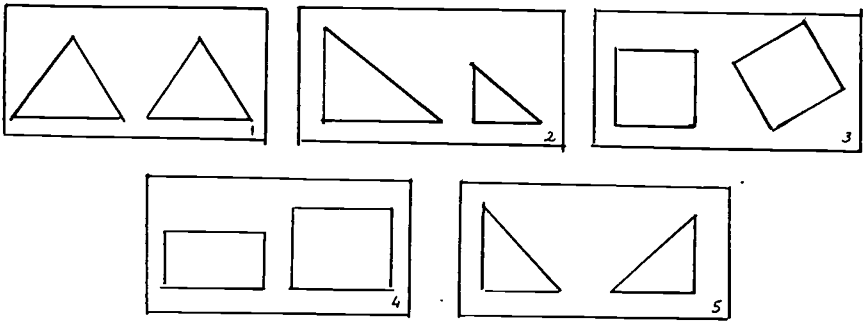


Fig. 1: Cards used in Activity 1

Students were offered measurement instruments or, if the possibility was mentioned, they could use tracing paper or fold the card to check if the shapes matched. Expected response: Basic level-recognition relying only on appearance, orientation considered to be relevant; Level 1 - recognition relying on measurements or transformations.

Activity 2 - Sorting congruent triangles:

The material for the second activity consisted of ten cutouts of triangles numbered, and with different colours in each face (fig. 2).

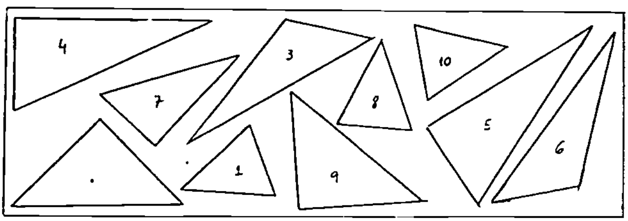


Fig. 2: Triangles to be sorted (Reduced size)

The student was asked to sort the triangles in groups of congruent triangles and to mention common properties exhibited by congruent triangles. Expected response: Basic level - sorting based only on appearance; Level 1 - the strategy of superimposing the shapes is used to justify the sorting; statement of necessary conditions for the congruence of triangles.

Activity 3 - Establishment of sufficient conditions:

In this activity, the student was asked whether it was possible to draw triangles with different shapes having the features shown in each card (fig. 3).

| Sides measuring 3 cm<br>5 cm<br>7 cm |
|---|

| A 60° angle and<br>a side measuring<br>5 cm |
|---|

| Sides measuring 4 cm and 7 cm<br>forming<br>a 50° angle |
|---|

Fig. 3: Cards Used in Activity 3

If the student could not answer promptly, (s)he was encouraged to try drawing one or more triangles fitting the features on each card and, then, answer the question. Expected response: Level 2 - at the end of the activity, sufficient conditions for the congruence of triangles (cases of congruence) could be established by the student, when required by the interviewer.

Activity 4 - Logical explanation:

This task was designed in order to evaluate students' logical reasoning and ability to justify the congruence of two triangles in a more complex figure. The task is shown in fig. 4.

Explain why $\triangle ABC$ is congruent to $\triangle CDE$.

Given: BC = CE

AC = CD

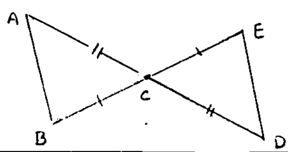


Fig. 4: Task requiring a logical explanation

Expected response: Level 4 - rigorous proof for the congruence of the triangles; Level 3 - informal, but logical explanation for the congruence. Acceptable answers at this level could vary from showing (with tracing paper) that one triangle was the image of the other after a rotation, or to observe that angles $A\hat{C}B$ and $E\hat{C}D$ were opposite angles, having the same measurement, and using activity 3 or the cases of congruence, conclude that the triangles were congruent.

Sample: The sample for the interviews was composed of 15 English and 10 Brazilian students of varying attainments aged 13 to 16 years.

Results of the interviews: Only two students have considered orientation as a relevant attribute for congruence (one English and one Brazilian). All the others could correctly solve the first activity. One English student used folding to verify the congruence of the shapes in cards one and five, while all Brazilian children used measurement. The sorting activity was easily solved by all students. The strategy of superimposition was used by all the English children, and by three Brazilian ones. The other seven measured the sides of the triangles. When asked about the necessary conditions for the congruence of triangles, all students mentioned the same lengths of sides. Some necessary but not sufficient conditions were mentioned: ten English and six Brazilian students mentioned the equality of the angles, and four English students said that the triangles had the same area. Activity 3 was more demanding for the students without a formal Geometry course. Only seven of the English children could come to a conclusion about the sufficient conditions for the congruence of triangles. For the seven Brazilian students which had already studied the cases of congruence, the answer should be easy, but three of these could not remember anything. On the other hand, two students without a Geometry course could reason based on the triangles they had drawn and conclude the cases of congruence. Only three students in the English sample could give an acceptable explanation for the congruence of triangles in activity 4. Some students attempted to explain using tracing paper, but failed to conclude the task. From the Brazilian sample, five students with a Geometry course tryed to answer activity 4, but only two succeeded. The others had some ideas of how the proof should be written, but could not organize them clearly.

Comments: The responses obtained seem to fit the level descriptors, despite the approach experienced by the student. Also, the levels appear to be hierarchical, since, in general, students performing at a certain level were sucessful in tasks demanding a lower level performance. The only exception were the two students who considered shapes with different orientations as non congruent, but could solve activity 2 at a level 1 performance. English and Brazilian students used different strategies to solve the tasks due to the approach to Geometry they had. The low familiarity of the Brazilian students with manipulations and concrete materials was shown by the strategies they used to solve the tasks. Although used to manipulations, the English students very seldom used transformations when solving the activities.

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# Measurement

PROSPECTIVE PRIMARY TEACHERS' CONCEPTIONS OF AREA

Cornelia TIERNEY, Leslie College, Cambridge MA, and Phillip Institute of Technology, Melbourne
Christina BOYD, Phillip Institute of Technology, Melbourne
Gary DAVIS, La Trobe University, Melbourne

*Area misconceptions of a population of prospective primary students were examined. Relations with other studies of area misconceptions are drawn, and use is made of the notion of cognitive "signpost" to explain what it is that the students do in their work on area. The emphasis in the student tasks was on comparison of regions by cut and past methods prior to judgements about numerical values of area.*

Area seems to be one of those concepts that is so intuitive and deeply embedded in everyday life that attempts to formulate carefully what is meant by it are seen as pedantry by a majority of elementary student teachers. Student teachers expect that what they learned, or imagine they were taught, in high school is an adequate base for their teaching of area concepts to children. However, as soon as one begins to probe their understanding of area one discovers that there are significant numbers of them who have no mental image of area at all, depend on memorized formulas, and incorrectly use linear measures for computations of area.

Deborah Ball (1988) writes about her concern that mathematics educators themselves take for granted that prospective teachers are well enough prepared in school mathematics content from material they dealt with when they last studied these topics in school. In practice these future teachers reveal some basic misconceptions that would make it difficult for them to to teach correct applications of formulas, let alone concepts. Ball uses as an example the prospective teachers' lack of knowledge of the relationship between area and perimeter. By asking students to respond to a hypothetical pupil's conjecture, she found that 80 percent of primary teacher trainees, and an even greater proportion of high school mathematics teacher trainees, believed that area always grows with perimeter and they were satisfied that a single example "proves" such a conjecture.

We have recently looked at the views of area and perimeter held by two populations of prospective primary teachers, those from a university where the entrance standards are reasonably high as well as those from a teachers' college where the entrance standards are somewhat lower. Responses of our university students to a question similar to that of Ball's was one of the things that drew our attention to the inadequacy of their conceptions of area and perimeter. Those of us who teach at the teachers' college had noticed many different occasions when our students demonstrated misconceptions about area in their college mathematics work or in designing curriculum for primary students. Some examples are:

- Students frequently generalized the formula for finding the area of a rectangle to plane figures other than rectangles.
- Many students think area is "Length by Width". When we asked our college students what they would teach a ten year old child about area, 80% of them drew a rectangle and wrote "L X W" or "L by W" near it. Some of these students placed arrows around a rectangle in a way which denoted perimeter

rather than area. The 20% of the students who did not mention Length x Width mentioned no formula: they were the only students to draw or name figures other than rectangles and they defined area as the space inside a figure.

• Many students who used the Length x Width formula for calculation did not move from linear units to square units. They either presented their answer in terms of the formula, without a numerical result, as in "the area of the board is 1.2 metres by 2.4 metres", or they labelled answers with linear units.

• Students often generalized changes in linear dimensions to changes in area. In responding to questions about the effect of halving or doubling the lengths of the sides of a square, most students said that the area was also halved or doubled.

• It was common for students to use whatever numbers were available (often lengths of sides when an altitude was appropriate) to get answers. When numbers were not available students would count something such as the nails or squares around the perimeter of a shape.

As we discussed our observations from the two populations of students, we realised that there was evidence to indicate that a very high proportion of elementary student teachers do not have an understanding of area which would support their teaching of it even with the aid of a reasonable textbook. In one class from the college, all of the five students who chose area as a topic to teach to children directly taught incorrect interpretations of area. For example a student-teacher demonstrated, to a grade six class, Length x Width as the way of finding area and then asked children to find areas of some rectangles and some non-rectangular parallelograms for which she provided measures of sides but not altitudes.

We believed that the situation is as Ball describes. However, the task of mathematics educators is not only to extend the concepts developed by students in their pre-college schooling but to undo much of what students mistakenly believe to be true. If our students had any informal primary school experiences which allowed them to develop concepts of area and perimeter, these have largely been replaced by strongly held beliefs based on the Length x Width area formula for rectangles.

## SIGNPOSTS

In talking about these misconceptions we began to call them "signposts". We chose the term signposts because it refers to the things on which students focus attention when they feel lost.

This notion of signpost is an analogy. It stems from our own experience that in travelling around a large and only partially familiar city some individuals are comfortable in heading in the general direction of where they want to be, discerning where they are in relation to their starting point by certain signposts. These signposts are most commonly landmarks, but may also be less obvious things such as the quality of housing in, the style of dress in, or general affluence of, a region. These signposts assist an individual in feeling comfortable that they are heading towards their goal and, just as important, that they can get back to where they started. For other individuals a detailed map is a necessary aid, and the route needs to follow a prescription worked out from the map. Similar signposts occur in all intellectual and physical activities. The question is one of orientation and a feeling of balance for an individual: that they can relate where they are to where they began and where they want to be, and that they do not feel unbalanced or out of control.

In our view it is this feeling of disorientation that students try to overcome in unfamiliar mathematical settings by reverting to familiar signposts, as irrelevant as these signposts might appear to an observer. The question of how close a student is to a solution of a mathematical problem is not something that can be judged objectively by an observer. The very notion of distance from a goal is a subjective notion, different for different people, and depends on how far someone feels that they have moved towards the goal and away from familiar signposts. Signposts for the teacher may have no use to the student. For example, to respond to a question about the area of a square formed by doubling the sides of a square of area nine, one student dutifully drew two squares, labelled the sides of one 3 and 3, and the other 6 and 6, but concluded, apparently without considering the drawings, that the new square would have an area twice that of the smaller square, or 18.

When students try to calculate the area of an irregular quadrilateral, for example, by using the length by width formula, what they are doing we postulate is returning to a familiar scene around which they feel comfortable and do not feel disoriented. It is for the student not at all a question of whether the formula is inappropriate, but rather a question of how, emotionally, they could move into uncharted territory without new signposts to help them orient themselves. This view of the way students approach problems has considerable implications for curriculum and the role of a teacher.

The signposts which the students most frequently in their area work were Length x Width, measures of sides of figures, and counting. We found we could not separate these misconceptions and deal with them individually. They were, and perhaps always are, interconnected into a unified construction of what area might mean to these students. For example, for many students:

- area is a measured by the formula Length x Width
- to apply the Length x Width formula , one needs to know the measures of the length and width (or of two adjacent sides) and as these linear measures are in centimetres or other linear units, so the answer is in the same units
- when measures are not given, one should count something (nails or pins, or squares around edges of figures, for example) in order to get numbers.

These signposts all represent actions which can be taken immediately - namely multiplying, measuring, counting - and they all result in a numerical answer. However their salient point is that they are actions with which the students feel comfortable: they are cognitive signposts that prevent them from feeling lost.

## THIS STUDY: PRE-NUMERIC AREA COMPARISONS

We were able to look more closely at the occurrences of the students' use of these signposts in a mathematics content course at the teachers' college. We spent approximately six hours on tasks related to perimeter and area of closed figures. After each session students wrote journals about what they had done and what they had learned and at the end of the work, they did a written 'test' paper. The observations discussed in this section are of the students' responses made in class and in their journals to the exercises described above and to the test questions.

The lessons were not planned from the beginning but in the way we usually plan our teaching, by continually responding to our observations of students. Instead of attacking each of these overused signposts separately, we tried to move students to a broader conception of area which would include representations built from experience as well as those learned as social conventions from previous teaching.

A mature quantitative thinker's view of the area of a planar region is as a number. This number is obtained by discovering how many standard unit regions - usually squares - exactly cover the region whose area is sought. Of course this might require infinitely many ever smaller copies of the unit, and so the number that gives the area measure might be quite a sophisticated number - such as $\pi$. A notion that is historically anterior, and seems to be also cognitively anterior, to this relatively sophisticated numerical notion of area measure is that of area comparison of two planar regions: one region has smaller area than another if it can be moved by rigid motions (or more generally by other area preserving transformations) so as to lie inside the other region. More generally one allows certain "cuts" of the first region before rearrangement: these cuts are in practice usually straight line cuts, but could, in principle, be something more general.

For many students the situation is often quite different. They want to find area as a number, but often any number at all will do. For example, one of our students told us that she was counting the nails around the sides of a geoboard so that she would have a number. Another told us that area had never existed for her because we have no tool to measure it. Of course calculating area is a number task: it is, as we have indicated above, the task of finding how many standard unit regions, or small copies of a unit region, it takes to cover the region whose area is sought. A major difficulty for students is that they confuse what it is that they are counting in order to find area, and they confuse what it is they count when they measure length. They also appear to have no mental perception that there is a region to be covered. Without this they seem to count or compute whatever they see. Thus, our first goal in this study was to encourage students to pay attention to the region whose area was sought, as well as to a measure of that region. We thought it plausible that if students learned to work with area directly by covering a region with unit squares or partitioning into regions of known area, they would not be so dependent on formulas which involved linear measures. We were really asking them to address the question: "Which of these regions has larger area?" without calculating the area of either region as a number.

We believe that in order to make sense of formulas, our students need to construct a mental image of the area as a region which they could focus on and talk about before attaching a number. We wanted them to change shapes and compare areas to see when the amount of area stayed invariant, when it halved, and which relationships of area to linear lengths stayed invariant. We wanted them to have the kind of experience generally provided for primary school children of finding areas directly without formulas by partitioning regions into measurable rectangles or triangles or covering them with unit squares and, when helpful, by moving portions of shapes.

We first posed a problem which we believed would require the students to attend to area without attending to the length of the sides. The problem was this: "Without using a ruler, put some cardboard shapes in order by area." The shapes which became the focus of attention were a square and a parallelogram which had the same area (and same base and height) but different perimeters. We expected that students would compare areas directly by placing one shape on top of another and by mentally "cutting and pasting" to test if the shapes were congruent.

STUDENT RESPONSES TO SOME COMPARISON TASKS

Instead of placing shapes on top of another, about half the students held the cardboard shapes edge to edge to compare perimeters. They were attempting to measure shapes not with unit squares as we suggest but with linear measure of the edges . They seemed to believe that if a shape has a larger perimeter it also has a larger area.

In trying to order the cardboard shapes by area, students who held shapes edge to edge noticed in a square and a parallelogram, with the same area, that the lengths of two sides of the parallelogram were a bit longer than the sides of the square. Other students showed the parallelogram and square to have the same area by placing them on top of one another and showing the congruence of the pieces which extended beyond the overlap. After much discussion, a number of students were convinced that a shape can be the same in area, and at the same time different in area, to another shape depending how they placed one shape on top of the other. While they agreed that two shapes appeared to have the same area, they thought that if one of the shapes were rotated ninety degrees, the two shapes would no longer have the same area. A student supported her theory that longer sides meant larger area pointing out that "one sticks out longer". Thus these students maintained their belief that if the side lengths were longer the area would be larger, while at the same time agreeing that if two shapes are held in a certain way one can see that they have the same area.

This activity exposed confusions between area and perimeter which we had not expected. These confusions were not just a matter of word usage. The students had not just reversed the two labels. I n fact, they had not distinguished for themselves two separate entities which would require two separate labels. Some students, like Robyn, commented on this: "This was a successful learning experience for me because before the class I could not distinguish area from perimeter." Others like Kylie didn't seem to have a concept of area at all: although she writes "area" we suspect that she is equating the word "area" with the boundary of the shapes. Kylie said: "It helped me understand how different areas can really equal the same area."

The disequilibrium created by trying to assimilate new notions of area is described by one student Cloe in her journal:

> "We were given a number of shapes and told to organize them according to their area. It was. difficult because we weren't allowed to calculate the area because of a few factors [*we did not allow students to use rulers* ] and sometimes when you thought you have them correctly ordered, you placed two shapes on one another and came up with a different answer. I also learnt that the perimeter calculations sometimes don't account for anything....I was still unclear and confused how to work out the area if you can't always calculate length of the object by the width. I walked out of the class knowing that Length x Width is not always true but was unsure what to do to come up with the answer to find the area of an object if the situation arose again."

The issue of sorting out differences in perimeter from differences in area came up again when students were asked to compare the areas of a set of different shapes with a unit square. The set of shapes included a parallelogram A which increased the length of sides and maintained area as in the previous example, and another parallelogram B which maintained perimeter instead. Students most often concluded that both parallelograms had the same area as the unit square since the first comparison activity "convinced" them of this.

To demonstrate the difference between A and B we put a "straw" square with pins in the corners onto an overhead projector and demonstrated what happened when the square was gradually collapsed to the smallest possible area. When asked at which point the area got smaller they agreed with one of the most confident students that it was half way down, but found it difficult to decide at which point the area became smaller in tilting the square a little. Students could see that the area was smaller at the extreme when there was an obvious visual difference.

As we read the students comments about their understanding, we noticed a pull between what they observed and what they expected or thought logical. Errors occurred when they were guided by only their observations or only what they thought logical. Effie, for example, doubted her perceptions when she measured and found that her ruler and school diary had the same perimeter, but the diary appeared to her to have a larger area: "I also learnt that shapes can be deceiving. The perimeter may be the same but the area of the ruler is more smaller or appears to be ? For example because of its boundaries." Later, when she had made a triangle, a trapezium, and a parallelogram from three identical rectangles of paper and measured all of the perimeters, Effie was convinced by her logic that the figures must have the same perimeter as well as the same area. She did not believe that she or her classmates had measured the perimeter accurately:

"It was good because we were involved in making these shapes and also finding out that all had the same perimeter [*we think she means area*] because they were originally the same rectangle. The perimeter was also the same apart from the errors in cutting the paper and having a slightly bigger or smaller piece of paper."

Stella, in responding to the same activity, tried to combine her sense that the shapes must be the same because they were made with the same size piece of paper, with her sense that some of them were bigger:

"The point of the activity was to make us decide whether the area of all the shapes were the same. The areas were the same, but some larger shapes, e.g. larger trapeziums, gave larger areas....The trapezium can be 32 square centimetres because the area is still there, it's just structured in a different way."

In the final written test, nearly all of the students described area as the minority of students had at the beginning--as the space inside a bounded figure. Although the students now expressed agreement with the instructors in viewing area as a region, some still used Length x Width as the way to compute area for all polygons. The majority however attempted to integrate their new notion of comparing areas of regions with area as a formula. Some created more complicated routines than multiplying length by width which however still did not work , for example they multiplied lengths of all the sides of a figure. Thus, although most students did not integrate notions of comparisons of areas of regions with area formulas, or correct their confusions about relationships between area and perimeter, they no longer accepted Length x Width as the all-encompassing solution to finding area. They had apparently given up some signposts without constructing new ones which would give them more success in finding area.

## DISCUSSION

It seems that three kinds of data are entering into students' responses to these area questions:

• social knowledge in the form of a formula their teacher taught them: Length x Width
• their visual perceptions
• their own supposed logical arguments.

The cut and paste argument makes sense to many students in that they see that if a piece is only moved rigidly then its area has not changed . This is a part of so called conservation of area as defined by Piaget and coworkers. Invariance of area under the group of rigid motions is a key basic feature of area. Indeed it was just this feature of area that Piaget took as the basis for his studies of conservation. What is not so obvious however, is that there is in fact no external reality of the concept of area to which we can refer, independent of its invariance under a group of motions, that tells us that it should be so invariant. In other words if people come to conserve area, as some seem to do, and if we take this conservation of area to mean its invariance under rigid motions, then they must have constructed this for themselves as part of their building of the concept of area. A curious question arises here however. If students accept "conservation of area" under rigid motions yet still imagine that shapes with the same perimeter have the same area, then is it really area, in the numerical sense, that they are conserving, or is it just a notion of equality of figures that are equivalent under rigid motions ?

When we saw that students tried to use perimeter to rank shapes by area, we posed questions to direct their attention to the lack of constant relationship between area and perimeter. Students seem to confuse area and perimeter and refer to "size" and "big" for both of them. It is entirely possible that the only perceptual cue they have is the perimeter. We believe that many of them cannot focus on the region whose area they have to compare or find, and they have learned to associate perimeter with a formula for area. An often expressed point of view is that the area of a region is the "amount of stuff in that region". This idea leads naturally for many people to a conviction that if the length of the boundary of a region is kept fixed as the shape of the boundary is changed then the area stays constant because: "the amount of stuff hasn't changed". This indicates a failure to understand area as the number of standard unit shapes needed to cover a region.

Hart (1984) deals with children's understanding of area in her chapter on measurement. She has a discussion of some nice experiments with 12 - 14 year old children that tease out apparent confusion between area and perimeter. These results must however be considered in light of the results in the doctoral thesis of Izzard (1979). He reports clear indications that perceptual boundary cues of shapes used in area and perimeter questions have a strong influence on a child's view of which feature is salient. Many of the secondary school children's misconceptions of area detailed by Hart are paralleled by our observations of student teachers' misconceptions of area.

This makes us suspect that we are seeing in these young adults a resurfacing of misconceptions that have lain dormant and unchallenged for a number of years. The recourse to numbers and formulas by these students is critical in their understanding, or rather their misunderstanding. It is critical in their ability to adapt their understanding of concepts taken for granted. New situations for learning require that students at this level synthesize intuition, experience, and logical deduction.The misconceptions have appeared, perhaps for the first time, in high school use of area formulas. This raises the issue of mathematical judgements depending on social authority, especially the authority of numbers and formulas, intuition and experience, and logical deduction. These students may not have been asked to consider area in primary school when it might have been acceptable for them to depend on their perceptions. Perhaps, on the other hand, they feel that only logical reasoning is acceptable, and if they can't develop their own then they must accept someone else's, in the form of what teachers tell them. They no longer test their assumptions with perception. Rather, they perceive what they decide to believe.

Hirstein, Lamb and Osborne (1978) discuss the results of interviews with 106 grade 3, 4, 5 and 6 children, designed to study how children incorporate numbers into their judgement of area. Several of the misconceptions they observed involve counting units without awareness of the unit's space-covering character. Analogously, children also counted lengths by counting marks between units of length, this time showing no awareness of linear units. These authors include a graph depicting the growth of the concept of area showing that children up to fourth grade gradually acquire a concept of the unit counting approach to determining area of a figure and children above sixth grade develop a multiplicative approach, but that there is a gap in between with no apparent connection made between these methods. Our work with prospective primary teachers indicates that this gap still exists. This then is a challenge for researchers and teachers - to understand the apparently different mental processes involved in these stages of dealing with area, and to devise problem solving situations that help students connect the actions and images of covering, cutting and pasting, and counting with the formulas for the various figures.

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Presenting author: Cornelia TIERNEY, Lesley College Graduate School, 14 Wendell Street, Cambridge MA 02138, USA.

Telephone: (0011 1) 617 868 9600 (work) and (0011 1) 617 864 5317 (home)

# Probability

# PROBABILITY CONCEPTS AND GENERATIVE LEARNING THEORY

Ole Björkqvist
Åbo Akademi, Vasa, Finland

*Earlier research findings on probability concepts are interpreted within the framework of the generative learning model. The interpretation is made with reference to the processes of the model, to the function of long term memory, or to compartmentalization of long term memory. The social constructivist philosophy is adhered to through emphasis on viable knowledge.*

## Introduction.

The generative learning model (Osborne & Wittrock, 1983) is an attempt to express constructivist views of learning, including some general aspects of the information processing models of the brain. The emphasis is on the meanings children (learners) actively construct for words and phenomena. The process of construction, located in short term memory, involves the generation of links to long term memory, which is a store of images, episodes, propositions, and skills. After successful testing of tentative links between sensory information and memory, meaningful understanding is reached, and the results are subsumed into long term memory.

The most significant use of the model is in science education where it proposes a frame for the study of alternative concepts of phenomena in science. It also links the study of conceptual structures to real world teaching.

In mathematics the subject area of probability shows similarity to elementary science. It involves concepts that have been constructed in an informal way before systematic teaching, common-sense thinking, counter-intuitive phenomena, a mixture of induction and deduction, and, above all, quantification of properties and relations that present themselves qualitatively in much of everyday life ("impossible", "more likely than", "good chance", etc). In most countries the mathematization of the concepts of probability belongs to the secondary school curriculum, and many students will not experience it systematically within the school buildings.

In the absence of a precise language in which the learner can describe his personal constructs, the alternative concepts of probability constitute a rich world to explore. One may start from the corresponding mathematical structures and a theory of their development (Piaget & Inhelder, 1975), from the application of generalized cognitive processes to probability (Fischbein, 1975), or from the application of probability as compared with other influences in human decision-making (Tversky & Kahneman, 1974, 1981). It would seem that the framework of the generative learning model is just as natural.

**Some philosophy.**

According to radical constructivism (Lerman, 1989) it is beyond the power of an outsider, such as a teacher, to know that a learner has come to an understanding of a concept. Understanding is subjective, a sense of freedom from contradictions sometimes coupled to a sense of completeness of knowledge. A learner therefore needs a way of comparing his knowledge with standards that he accepts, i.e., objectivity is located in the social domain rather than the transcendental. The process of teaching is essentially communication of world views and appropriate organization of the environment of the learner, to make active construction of meaning possible.

The remarkable homogeneity of world views in areas like science and mathematics can be explained with a darwinistic argument. The world views that have survived are judged to consist of viable knowledge (von Glasersfeld, 1987). "Viability" can be construed as a continuous variable, which in science and mathematics is almost dichotomized (viable knowledge equalling truth or a good model). The difference in terminology may be of less consequence to a scientist who is not philosophically inclined, but, interpreting alternative concepts in elementary science and mathematics, "viability", as assessed by the teacher, provides the criterion of quality sometimes needed to justify educational decisions.

On an individual scale, the generative learning model locates knowledge in long term memory. Viable knowledge is a set of images, episodes, propositions, and skills that will survive repeated testing when the individual actively generates links between sensory information and long term memory.

Interpretations of earlier findings.

The study of probability concepts has produced a vast amount of information in the form of test results and observational records, very often in connection with gambling. The analyses, even though starting from different theoretical viewpoints, have produced some similarities in the form of heuristics and biases that are repeatedly encountered (Hawkins & Kapadia, 1984; Hope & Kelly, 1983; Wagenaar, 1988; Walter, 1983). Many of those findings, even if paradoxical, can be given quite satisfactory interpretations within the framework of the generative learning model. The analysis will here be divided into interpretations that refer to the processes of the model itself, to the function of long term memory, and to compartmentalization of long term memory.

It can be noted that the generative learning model implies an intrinsic search for coherence, if meaningful understanding is to include freedom from inconsistencies at an individual level. This is akin to the "quest for certainty" basic to science itself. The possibility of reaching a state of coherence or certainty seems evident to anyone who has had intuitive cognitions, which have intrinsic certainty as one characteristic (Fischbein, 1987).

Probability concepts are sometimes classified as either intuitive or formal. This is not appropriate, as even highly formal concepts share many of the characteristics of intuitions, primarily the factors contributing to immediacy, namely visualization, availability, anchoring, and representativeness. However, probability concepts are not intuitive unless they exhibit intrinsic certainty, self-evidence, perseverance, and coerciveness (if you accept the definition of Fischbein), and this is rarely the case.

The *immediacy* of probability concepts is related to the process of testing tentative links between sensory information and long term memory. The extensive *use of heuristics*, whatever their nature, is similar in the sense that it speeds up the processing. Heuristics also tend to reduce uncertainty and thus give partial coherence. Obviously probability concepts are in no way unique here, neither with respect to immediacy nor to the function of heuristics.

Some of the specific biases discussed by Wagenaar (1988) in connection with gambling behavior can be interpreted with reference to the processes of the model.

*Confirmation bias*, the preference for information that is consistent with one's own views or even disregard of evidence to the contrary, exemplifies the search for coherence at the stage when sensory information is only being received. In general, a *biased learning structure*, the tendency not to analyse the existing alternatives according to their statistical or theoretical weight, reflects parsimony in the generation of links. Gamblers often develop their strategies from incomplete analysis of information that maximizes the hope for confirmation of conjectures.

In retrospect, people are not surprised about what happened, and even believe that they did predict the outcome. This is the *hindsight bias*, which can be associated with the last phase of subsumption in the generative learning model.

One of the best known common characteristics of probabilistic reasoning is the view that prediction of an event cannot be detached from the outcome of similar independent events in the recent past. In gambling this is the *sequential bias* or the *negative recency effect*. The dependency on recent empirical evidence is connected with the temporal correlations between sensory experiences which are so often found in physical science and in everyday life, and which also constitute the basis for causality. The tests of tentative links to long term memory follow time-dependent patterns.

Among the factors contributing to immediacy, *representativeness*, or the tendency to judge single events as exemplars of categories of events, can be seen as crucial to the generation of new links. Representativeness can thus be a positive heuristic. It can, however, be used to infer unwarranted properties in the hypothesized category of events.

Turning now to the function of long term memory, both representativeness and the other factors of immediacy, *availability, anchoring*, and *visualization*, imply the superiority of a rich store of images, episodes, propositions and skills. For example, a formal conceptualization of probability requires propositions usually acquired in school. In other cases, skills may have been acquired without formal education (Acioly & Schliemann, 1986). The specific *linguistic problems* often associated with probability concepts (Carpenter et al., 1981) may be seen as deficiencies in the extent or organization of the memory store, as may the *concrete information bias*, characterized by vivid or conspicuous incidents dominating abstract information, i.e., the attribution is to specifics rather than generalities.

The *problem framing* often has decisive influence when one is chooses a strategy for solution.The images and episodes saved in long term memory serve as sources of meaning when similar problems arise. *Reliance on habits* can be interpreted with reference to skills or episodes. Episodic memory also has a central role in the establishment of *illusory correlations*, exemplified in superstition, which is a frequent element in gambling situations or in layman weather forecasting.

Much of the memory store may be non-efficient, in the sense that the modeling of probability is mathematically deficient. For example, people tend to confuse conspicuous events with low probability events, and also to make probability estimates on the basis of absolute rather than relative frequencies. However, the memory store may represent viable knowledge on a personal level in spite of its mathematical shortcomings.

The compartmentalization of long term memory, finally, can be seen as a key factor in the interpretation of some other phenomena in accordance with the generative learning model.

In gambling, there is a tendency for people to have an *illusion of control* over the game, even if they on another level are completely aware of the negative mathematical expectations. Others exhibit *flexible attribution*, with successes due to their own skill and failures due to chance or bad luck. It has been shown (Wagenaar,1988) that many people regard skill, chance, and luck as three quite different concepts which together determine the outcome of gambling. As a consequence of this, people accept unfavorable bets or continue gambling after losing, which constitutes *paradoxical behavior*.

The *switching of problem solving strategies*, not specific to probability but very much in evidence, can be seen as the result of the attempts to generate links searching through one compartment after another in situations where the problem type is unfamiliar.

As a mathematical concept with three different theoretical starting-points (classical probability, empirical frequencies, and axiomatic probability) it seems logical that probability concepts should refer to more than one compartment in long term memory. There have been interesting test designs to study the interconnections of the conceptual aspects (Koops, 1981), but remarkably little is still known. More basic research in this area would be beneficial to the development of curricula and teaching methods.

## Conclusion.

This attempt to interpret earlier findings about probability concepts within the framwork of the generative learning model aims at a natural description in the spirit of constructivism with a social twist.

It is only applicable as far as the earlier research findings apply to a given person. In particular, many children do not have a rich memory store with which to link sensory information. The immediacy they presumably exhibit when interviewed about probability concepts may easily be forced immediacy, and in such case any tendencies observed are likely to be generalities with little specific connection to probability.

Alternative concepts are somewhat easier to accept in science than in mathematics. Alternative concepts in mathematics have primarily been studied to establish the nature of "misconceptions" and ways to overcome them, which in effect has become error analysis. To accept truly alternative concepts one must know enough about them to judge how viable they are. One needs a model showing how they are constructed. It is suggested that the generative learning model is a good one for concepts like probability, if and when the empirical foundation is comparable to the empirical basis for elementary concepts in science.

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# SOME CONSIDERATIONS ON THE LEARNING OF PROBABILITY

(Ages 10-11 and 14-16 years)

Ana María Ojeda Salazar*

King's College, London
Centre for Educational Studies, England
Sección Matemática Educativa del
CINVESTAV del IPN, PNFAPM, México

*In this work we examined the consistency of answers to groups of questions along the analysis of a discrete random situation posed to British children aged 10-11 and 14-16 years. In the experimentation, the situation was presented to the pupils in two slightly different contexts. For the older children, another variable was whether or not they had had a previous introduction to Probability. Among our general hypotheses there was the influence of the context in the answers to groups of questions concerning impossible, certain, complementary and compound events and conditional probability.*

This report concerns part of a study carried out in order to throw light on some aspects involved in the learning of Probability at pre-university levels.

Usually, the teaching of Probability is based on the solution of specific problems demanding an interpretation using the techniques taught, mainly in the basic grades of education. In this part of the study we did not ask the students different questions referred to different problems, and these last referred in their turn to different contexts. Instead, we posed a random situation, and through a (although very restricted) set of questions, pupils were led to its analysis.

One of the variables included in this study was the *context* in which the situation is presented (that is how the situation is referred). We proposed two discrete contexts differing in what one of them can be considered more familiar to pupils than the other. The contexts posed were the throw of two dice, usually known from everyday games, and the other

* This study was supervised by Dr. Kathleen Hart.

based on the urn model, which is generally introduced at school for didactical purposes. In particular we wanted to see whether pupils' preference for one result dominated their prediction of the most probable event.

We were also interested in pupils' answers according to whether they had received a previous instruction on Probability. Answers to questions about impossible, certain and complementary events were of interest to us.

Along the questions posed we were interested in verifying some results obtained from other research having different characteristics, such as the one concerning causal and diagnostic reasoning studied by Kahneman and Tversky [1982]. These authors found that people judging conditional probability assign higher probabilities to the conditioning event when this concerns the causes of a random result (*causal reasoning*) than when it refers to its effect (*diagnostic reasoning*, that is reasoning about *a posteriori* probability).

Some questions were posed about the idea of exchangeability (*échangeabilité*) for the analysis of compound events. This idea was studied by Lecoutre and Durand [1988], without the intervention of random variable.

Unlike the last study quoted, in this pupils were asked to consider the situation posed through a random variable. The main purpose was to look at the aspects pointed out above, not in a local perspective, but from a more general point of view. We wanted several notions of Probability and their interrelations to be considered in the analysis of the situation. Our framework was Heitele's [1975] proposition. This author considered ten fundamental ideas to be the guide for a curriculum in stochastics. These ideas are

* To assign a number from [0,1] to a random event to express its probablity of occurrence
* Sample space
* The addition rule
* Independence
* Equiprobability and symmetry
* Combinatorics
* The urn model and simulation
* Random variable
* The law of large numbers
* Sample.

Heitele suggested the introduction of these ideas by posing complex examples in which not only basic notions could be studied, but their interrelations as well (see Ojeda, [1986]).

Heitele's ideas constituted the framework for our study of the variables already mentioned involved in the learning of Probability.

## THE POPULATION AND THE METHODOLOGY USED

A questionnaire was given to 63 students aged 14-16 years and 23 pupils aged 10-11 from three different British schools. 29 of the older pupils had already received an introductory course of Probability and were considered as a top set according to their general mathematics performance at the school. The remaining pupils had not yet been taught Probability

The questionnaire consisted of two parts, but the aspects concerning this report correspond to only the first one.

Two forms of the questionnaire, called A and B, were designed by only varying the context of the situation proposed. In Form A the context was

> Two ordinary dice, one white and one red, are thrown at the same time. The number of dots on the top faces are <u>added</u>.

The situation in Form B was stated as

> Two cloth bags, one white and the other red, each hold six equal sized marbles, labelled 1 to 6.
> In each bag the marbles are well mixed.
> Without looking, two marbles are taken out, one from each bag. <u>The numbers drawn are added</u> And then, each marble is put back into its bag.

Within each form, all the questions we referred to corresponded to the same context. Other than the context, the situations, questions and their order were exactly the same for the same age.

Most of the questions were multiple choice although some open questions were posed for getting more information. The questionnaire was a little different for the younger pupils as we wanted to avoid additional difficulties due to fraction numbers in the options proposed rather than to probability notions. So, some questions posed to the older pupils were not included in the questionnaire for the younger children, who were posed 25 questions instead of 27. As the vocabulary used did not include technical terms it was expected to be understood by children aged 10-11 years.

The questionnaire was passed during mathematics class time in the Spring of 1989. The two forms were given in alternate form according to the registration list of the class.

GENERAL STRUCTURE OF THE QUESTIONNAIRE

Although the situation posed in the questionnaire may seem simple, in general its understanding requires a method for organising the ideas. So after a few questions with a qualitative approach, pupils were asked to fill in a table to produce the sample space so as to have it at hand. This was followed for a quantitative approach. There were several relationships between the different questions in order to see the consistency of the performance.

The distribution of the questions according to the fundamental ideas and some of the objectives are shown in Table 1.

| FUNDAMENT. IDEAS | QUESTIONS 10-11 | QUESTIONS 14-16 |
|---|---|---|
| Norming belief | 2,3,4,5 | 2,3,4,5 |
| Sample space | 6 | 6 |
| Addition rule | 18 | 8,9 |
| Independ. | 11,14 | 11,14 |
| Equidist. and symme. | 14 | 14 |
| Combinato. | 8,9,10,11,15,16,19,20 | 8,9,10,11,16,17,19,20 |
| Urn model and simul. | FORM B | FORM B |
| Stochastic variable | 8,9,10 | 8,9,10,17,18 |
| Law large numbers | 23,24 | 25 |

| | QUESTIONS 10-11 | QUESTIONS 14-16 |
|---|---|---|
| Impossible event | 3,12 | 3,12 |
| Certain event | 2,13 | 2,13 |
| Complemen. event | 9,16 | 9,18,20 |
| Exchange | 15,16,17 | 15,16,17,19,20 |
| Exchange-condition. | 19,20 | 21,22 |
| Causal-condition. | 21,22 | 23,24 |
| Diagnostic condition | 20 | 22 |

TABLE 1. Distribution of the Questions according to the Objectives.*

## RESULTS

Younger children did better on eighteen questions in Form B (urn model) than in Form A (dice), although the difference was bigger than 20% in questions 8, 11, 14, 17 and 22. Only the question 13 (certain event) was correctly answered by more than 20% of the pupils with Form A than with Form B.

The older children who were not taught Probability performed better in questions 8, 9, 10 and 11 (complementary event and sample space) with Form A than with Form B, in more than 20%. There was a significant difference in performance, better with Form B than with Form A, in questions 7 (sample space) and 13 (certain event).

Pupils who received an introduction to Probability did not show, in general, a difference in their answers according to the context, except to questions 10 and 13 concerning with the sample space and the certain event, which were better succeeded with the urn model.

The questionnaire began asking pupils for their preferred sum.

Question 27 (25 for the younger pupils), at the end of the analysis proposed, asked pupils to predict the sum of one trial of the situation posed. About 20% of the younger children answered this question correctly and almost all those with a previous instruction, both with Form A and B. There was a difference of 15% in the correct answering of this question, better Form A than B, in the older children without a course in Probability.

Table 2 shows the coincidence of the correct prediction, its justification (question 28) and the prefered sum:

| AGE → | 10-11 | 14-18 | |
|---|---|---|---|
| | | NO COURSE | COURSE |
| FORM A | 15% | 6% | 50% |
| FORM B | 0% | 21% | 36% |

TABLE 2. Coincidence between the most probable sum, its justification and the prefered sum.

Table 3 shows the success for the sets of questions concerning the concepts we were interested in. The changes in the numeration of questions according to the age of pupils were put into brackets. "COND" stands for conditional.

| QUESTIONS | | FORM A | | | FORM B | | |
|---|---|---|---|---|---|---|---|
| AGE | | 10-11 | 14-16 NO-COUR | 14-16 COURSE | 10-11 | 14-16 NO-COUR | 14-16 COURSE |
| IMPOSSIBLE | 8,12 | 46% | 63% | 100% | 50% | 71% | 93% |
| CERTAIN | 2, 13 | 23% | 6% | 29% | 0% | 29% | 50% |
| COMPLEMENT | 9,18,20 (9, 16) | 31% | 25% | 79% | 40% | 20% | 56% |
| EXCHANGE | 15,16,17 | 31% | 25% | 64% | 50% | 14% | 79% |
| EXCHANGE | 19,20 | ----- | 0% | 14% | ----- | 7% | 7% |
| EXCHANGE-COND | 21,22 (19,20) | 8% | 13% | 57% | 0% | 14% | 64% |
| CAUSAL-COND | 23,24 (21,22) | 23% | 44% | 93% | 60% | 50% | 100% |
| DIAGNOSTIC COND | 22 (20) | 23% | 25% | 71% | 10% | 43% | 64% |

TABLE 3. Comparison of the success in some of the objectives.

There is a great difference in the results of questions concerning impossible events and those referring to certain events. Questions 2 and 13 asked about certain events. Question 2 posed a qualitative approach, whereas Question 13 was

13. What is the chance of getting a sum between 1 and 13?
a) 0............[ ]
b) 1............[ ]
c) 13...........[ ]
d) None of these..[ ]

This question was expected to be difficult for younger pupils and those without an introduction to Probability, mainly because of the assignation of number 1 to the certain event might not seem natural to them. Nevertheless, the results from pupils who had been taught Probability would seem to suggest that the difficulty is rather the recognition of the certain event.

For the older pupils, questions 9, 18 and 20 concerning the complementary events of those introduced in questions 8, 17, and 19, respectively, were better answered than these last. In particular, question 19, which posed an event defined as

...getting only one 6 from any of the two dice (bags)...,

was as difficult for pupils who had done a course (14% correct answers with Form A and 21% with Form B) as for those who had not (19% correct answers with Form A and 14% with Form B).

Questions requiring the idea of exchangeability (permutation between the elementary events composing the event under consideration) were difficult for children without an introduction to Probability.

Unlike the younger pupils, the children aged 14-16 showed an ordered, gradual improvement in performance, which referred to conditional probability, that is, questions 21, 22, 23 and 24 (corresponding to 19, 20, 21 and 22 for the younger children).

Finally, questions appealing to causal reasoning were better answered than those requiring diagnostic reasoning by all the population.

## FURTHER RESEARCH

Although there appears to be a better performance of children (mainly the younger) with the questionnaire using the urn model than the context of dice, the difference was not significant, despite the length of the context with urns, which could be a reason for a poorer performance. Nevertheless, there is still the doubt of whether they can transfer (or rather repeat) a correct answer given to a question posed in one of the contexts to a similar question posed in the other context.

In spite of the success shown by children previously taught Probability, the review of the consistency of the answers given to groups of questions concerning the same notions appears to reveal that even after an introductory course, elementary notions of Probability, such as certain and compound events, are difficult to handle.

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RESEARCH PAPER FOR PRESENTATION AT P.M.E. 14 - MEXICO

AUTHOR: ROBERT PEARD

TITLE: GAMBLING AND ETHNOMATHEMATICS IN AUSTRALIA

## Summary

*The phenomenon of gambling is widespread throughout Australian society. Thus we have an identifiable subgroup of the population for whom gambling, particularly on horse-racing, constitutes a form of "ethnomathematics". Children from this group bring probabilistic knowledge with them to the school environment. This study researches what this knowledge is, how it is constructed and used, and the implications of this in the school system.*

## Introduction and Rationale

This study will involve research in two broad, independent, yet interrelated fields of mathematics education. The first of these is what may be loosely defined as "ethnomathematics" within Australian culture. This part of the study will involve the exploration of culturally-based mathematical knowledge in probabalistic and related concepts of a segment of the society for whom the phenomenon of gambling is inherent in their culture.

The second field employs the ideas of "constructivism" in the learning of mathematics. In this section, the research will explore how this knowledge is used to construct mathematical procedures and concepts. In addition, the relationship between these constructs and present classroom instruction in the topics will be researched in order to determine how such knowledge may be meaningfully incorporated into classroom practices.

If we view Australian society as a changing, developing multicultural mixture any ethnomathematical concepts will of necessity be confined to various sub-groups. In this study the term "ethnomathematics" will refer to the inherent

Those who reasoned in this manner effectively constructed their own "common denominator" algorithm and employed this to questions of the type of number 1. Others reasoned:

> "3 gives 4 or 7 gives 9", so 6 gives 8, 7-6 =1, 1 gives 8/6, so 7 gives 8 + 8/6 = 9 2/6 which is greater than 9 so 4:3 is better".

It is interesting to note that although those using this procedure sometimes made errors - both computational and procedural - none was consistently incorrect and none used the incorrect "additive" algorithm reported to be common amongst children 12-16 by researchers such as Hart (1984) "erroneous reasoning .. referred to as the incorrect addition strategy" (p.4).

Such reasoning here would go

> "3 gives 4 or 7 gives 9, if 3 gives 4, 7 gives 4 + 4 = 8 which is less than 9"

Non-gamblers employed a variety of techniques including the finding of a common denominator and conversion to decimals for the first type of question and were generally unable to attempt the second.

(3) The concept of equal likelihood of occurrences.

Sample Questions;

1. When a single die is rolled are all numbers 1 to 6 equally likely? Is, say, a "six" harder to get than any other?

(4) The concept of independence.

Sample questions

1. When a fair coin is tossed it is just as likely to land heads as tails. If a fair coin is tossed three times and lands heads each time, is it still just as likely to be heads as tails on the next (fourth) toss?

(a) 2:9 = 10 : x
(b) 4:7 = 5 : y

2. (a) If $10 is bet at odds of 9:2, how much can be won?

(b) If $5 is bet at odds of 7:4, how much can be won?

Analysis: The "gamblers" consistently employed the same algorithm in all questions, in nearly all cases successfully. Typical reasoning was:

"2 gives 9, 10 is 5 x 2, so 10 gives 5 x 9 = 45"

"4 results in 7

4 is one less than 5

This additional 1 gives $\frac{1}{4}$ x 7 = 7/4

So 5 results in 7 and 7/4 = 8 3/4"

"Non-gamblers" were generally unable to attempt the second question but answered the first using more traditional school-taught algorithms, though not always correctly. It would appear that in this situation the gamblers do construct their own algorithm and apply it in traditional non-gambling situations. Further research will be conducted in this area.

(2) Equivalence of fractions and comparison of "odds".

Sample Questions

1. Which is the larger, 3/5 or 5/8? Why?

2. Which are the better "odds" 4:3 or 9:7? Why?

Analysis: Again, the gamblers constructed their own algorithm, though not as consistently as in the first instance. Some reasoned:

"3 gives 4 or 7 gives 9

So 7 x 3 gives 4 x 7 or 3 x 7 gives 3 x 9

21 gives 28 or 21 gives 27

28 is greater than 27, 4:3 is better".

This research will therefore employ the use of a number of case studies as the dominant research methodology.

Research Questions

This research is currently in progress. Twenty case studies are being conducted with upper secondary pupils. Ten of these come from a social background in which gambling practices are commonly accepted. The most prevalent of these practices is betting on horseracing and trotting, though gambling on card games is also evident. The other ten pupils are from a similar socio-economic background but are unfamiliar with such practices.

The research questions include:

- What mathematical knowledge (both concepts and processes) do pupils from a gambling background bring to school with them?
- How is this knowledge acquired?
- Does this knowledge transfer to use in classroom situations to perform traditional, related mathematics?
- To what extent do these procedures parallel or differ from those employed by non-gamblers and traditional classroom practices.

Students from both groups were each asked a number of questions in a clinical interview situation. The full report will contain the questions, responses, analysis and implications of the results to the date of the presentation but some preliminary results from five areas of study are presented here for discussion.

**Results and Implications to Date**

**(1) Algorithms employed in calculations involving proportions.**

Sample questions:

1. Complete the following proportions:

Clements (1988) "it needs to be remembered that often in Australia there are unique factors influencing how children learn mathematics" (p. 5).

Secondly, that the phenomenon is inherently mathematical in nature. Bishop (1988a) notes that developing ideas about chance and prediction are important mathematical activities (p. 106) and that gambling games are part of modern western society at present (p. 112). Investigations in our mathematical culture include experimental probabilities (p. 117). D'Ambrosio (1985b) has stressed the need for incorporation of ethnomathematics into the curriculum in order to avoid the "psychological blockade" that is so common in mathematics.

The interrelation between ethnomatics and constructivism results from the use of the learners' experiences which are culturally determined, to construct mathematical concepts. As Davis (1989) notes:

> It is now far from a new idea that mathematical ideas and concepts are actively constructed by individual children and older people alike (p. 32).

This research will explore how probabilistic ideas and concepts are actively constructed by individuals in, as Davis says as "intelligent responses to their environment" (p.32).

Leder (1989) refers to "the growing adoption by contemporary mathematics educators of constructivist perspectives" (p. 2).

Higginson (1989) refers to constructivism as a conception of knowing and learning with its emphasis on the active involvement of the learner (p. 11).

Harris (1989) argues that:

> The mathematical meanings acquired in the classroom are the personal constructs of each individual learner and that they are strongly influenced by past experiences and the social context of the school (p. 81).

mathematical ideas related to the probabilistic concepts in gambling that some of the population possess and that might be reasonably expected to be brought to school by the children of this segment. This working definition is in keeping with that employed by other researchers in the field. Beth Graham (1988) in researching the ethnomatics of Aboriginal children of Australia uses the term to refer to "the mathematical understandings that the Aboriginal children bring to the educational encounter ... the mathematical relationships inherent in their own culture". (p. 121). Gerdes (1988) uses the same definition to describe the intuitive mathematics of the native culture in a post-colonial society. Carraher (1985) in Brazil uses the term to refer to the knowledge of "the everyday use of mathematics by working youngsters in commercial transactions" (p. 21). D'Ambrosio (1985a) first coined the term to refer to "mathematics which is practiced amongst identifiable cultural groups" (p. 45) and it is in this context that the term is employed in this study.

The rationale for the selection of the phenomenon of gambling relies on two major factors:

Firstly, that the phenomenon is widespread within the culture and is related to the culture in a unique way. Award winning Australian author Peter Cary (1987) in refering to the social history of Australia commented "It was as if the colony were founded on gambling" (p.263). In the Australian Newspaper (26/3/89) Phillip Adams refers to Australia as "the gold-medal country of gambling" (p.42). Statistical evidence showing the monies bet on legal gambling - TABs and casinos per capita of population supports this, as does the observation of social phenomenon such as "Melbourne Cup Day".

Thus it may be that in researching the concepts employed in gambling the study will add to factors unique to our culture. The need to do this is supported by

2. If it is equally likely for a new-born child to be a boy or a girl. Which sequence is more likely when a family has four children.

(a) BBBG, (b) BGBG, (c) both equally likely

Shaughnessy (1981) refers to the use of "representativeness" in this estimation (p.91). This study will explore the comparative use of "representativeness" amongst the "gamblers" and "non-gamblers".

(5) Combinations and Permutations in Probability Estimations.

Sample Questions

1. A student has eight books at school and decides to take two home. In how many ways can this be done?

2. There are eight horses in a race. To win "the double" you must select the first two (i.e. first and second but not necessarily in the right order). How many selections can be made?

3. A committee is to be selected from ten students. Which of the following committee sizes would result in more possible committees:

(a) 8, (b) 2, (c) no difference between 2 and 8.

Shaughnessy (1981) refers to the use of "availability" in making this type of estimation (p.93). Again, this study will compare the use of this by "gamblers" and "non-gamblers".

The results of the questions of item (3) to (5) are presently incomplete. These and other data are to be presented for discussion and reaction.

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# A Mathematization Project in Class as a Collective Higher Order Learning Process

Hans-Georg Steiner

The Institute for the Didactics of Mathematics (IDM)

University of Bielefeld, F R Germany

*Various positive classroom experiences with a mathematization program related to mathematical modelling of situations and problems in the social-political domain of decision making by voting (voting bodies) provide us with a learning context for students in grades 11 – 12 and an educational setting in which social cognition and social learning, metacognition, learning about learning, communication about communication play a significant role and can be made a matter of in depth didactical investigations. It is considered important that, in the context and setting specified, these factors are not seen independent from the content and its epistemological structures. Rather they are viewed as being profoundly connected with the constitutive interrelation between theoretical concepts, applications, knowledge development and social interactions in the broad field of mathematics-related activities in science, education and practice, understood as a socio-historical reality. The paper is concerned with interpreting the observed classroom processes from these epistemological and socialcognitive points of view and identifies further research questions.*

## 1. The Learning Context

The learning context is a *developing* and *expanding* one and grows out of discussions in class about *situations* and *problems* related to the role and functioning of bodies in various domains of society that reach decisions by voting (Steiner 1986a, 1988). The students immediately provide a number of examples and find more of them – and more specific information – by searching in the library or talking to people inside and outside of school: the Federal House of Representatives, a city council, a jury, an examination board, the Security Council of the United Nations, a stockholders' meeting, their own class when e. g. electing a class-speaker etc. Problematic situations and special concerns among the students come up when trying to describe such voting bodies in general terms like the given vote distributions ($v$) on the set of voters ($V$) and the majority quotient ($q$), or when interpreting particular situations like the position of a chairperson in case of a tie, the very unusual regulations for the Security Council etc.

Stimulating points and questions of debate and personal involvement in the beginning phase are e. g.: Meaning of and reasons for unequal vote distributions; can there meaningfully be a member with 0 votes?, meaning of and reason for majority quotients like 2/3, 3/4, i. e. different from 1/2 (simple majority) or 1 (unanimity); does the generalization $1/2 \leq q \leq 1$ make sense?, can we describe the decisive role of a chairperson in case of a tie by giving him or her a bigger number of votes?, can the number of votes be meaningfully used as a measure of power?, why does the Security Council $SC = F \cup T$, consisting of the Big Five $F$ and the Small Ten $T$, have such strange by-laws, saying that in non – procedural matters a proposal is carried if at least 9 of the 15 members, including all big five, vote for it?, can we replace these unusual regulations, which do not apply any vote distribution and majority quotient, by assigning a certain vote distribution to the members and determing an appropriate majority quotient?, what power do the Big Five and the Small Ten have?, can we explain or define power positions like veto-power, powerlessness, dictatorship?

Dealing with these questions and problems in the collective learning process goes together with the development of an *enriched theoretical framework* starting out from *empirical conceptualizations*, based on concrete voting sets, vote distributions and majority quotients and the definition of the (normal) voting body to be a triple $(V, v, q)$ with $v$ being a mapping from $V$ into the set $\mathbb{N}$ of natural numbers, including 0, different from the constant 0-mapping, and $q$ being a rational number with $1/2 \leq q \leq 1$. The understanding of $V$, $v$, $q$ as *variables* is the first step to creating a *theoretical model* which can be put in a *dynamic relation* to concrete cases and related problems.

The next step consists in the introduction of concepts like winning-, losing-, and blocking coalitions, first described as *derived concepts* with respect to normal voting bodies $(V, v, q)$ by saying that a subset $C$ of $V$ is called a winning coalition, if it is strong enough to carry any proposal, i. e. if $v(C) > q\ v(V)$ in case of $q = 1/2$ and $v(C) \geq q\ v(V)$ in case of $q > 1/2$ (and for the losing- and blocking-coalition correspondingly).

By flexibly using the term *winning coalition* also with respect to non-normal voting bodies like *SC*, by e. g. saying that the above identified subsets with 9 elements are the minimal winning coalitions in *SC*, the concept *winning coalition* is also made a variable for a possible *richer theory* and a *larger domain of applications*. In this *anticipating way* the concept is creatively used by the students, e. g. to define concepts like "$x$ is a *powerless member*" by means of "$x$ does not belong to any minimal winning coalition" or "$x$ has *veto-power*" by means of "$x$ belongs to all minimal winning coalitions". The need to further elaborating the theory and giving it firm foundations appears in connection with the problem whether all voting bodies of the *SC*-type can always be *weighted*, as was found by the students to be the case for *SC* itself (by e. g. putting $v(f) = 7$, $v(t) = 1$ for all $f \in F$ and $t \in T$, and $q = 13/15$). This leads to an *axiomatic definition* of a voting body $(V, W)$ *of the Security Council type* in which $W$ is a non-empty set of subsets of $V$, called winning coalitions, and in which the concept of winning coalition $W$ is *implicitly* defined. A big breakthrough and an *expansion of the context* is then made by proving that there really exist voting bodies of the *SC*-type (from 5 elements up) which *cannot be weighted*. Thus the new theory turns out to be the more general one and gives reason for further applications and for total reorganization of the developed body of knowledge by embedding the old knowledge into the new conceptual framework.

We can only mention here that in the course of further explorations some situations appear which seem to be covered by the theory developed thus far but actually create a kind of *paradox* and *crisis* which lead to an even more extended theory that we called the theory of $\alpha$-, $\beta$-voting bodies (Steiner 1969b). We should also mention that the core of the learning context described can be extended into various directions which have been pursued by specials groups of students on their own: different theories of power, relations between non-measurable voting bodies and finite geometries, relations to the theory of games (Steiner 1976b, 1986a, 1986b, 1988).

*2. The Setting, Goals, Activities, and Processes*

The classroom experiences were made over several years in gradses 11 and 12 of West-German high schools (Gymnasia and comprehensive schools), and in grades 9 and 10 within the gifted student program of the Comprehensive School Mathematics Program (CSMP, at Carbondale Ill.) and the ongoing project Mathematics Education for Gifted Students in Secondary Schools (MEGSSS at Ft. Lauderdale, Fl.) (see also Steiner 1966, 1969a, 1969b, 1976a). This present report is referring to *specially arranged settings* in West-German classrooms in which the mathematization program was made a matter of a *project-type teaching-learning activity* lasting for about 3 weeks with a total number of about 15 classroom-meetings of 45 minutes each. The social organization of students' work changed between all class discussions, group work and individual work. Each project was taught by one teacher (including the author) and observed by another; the explorative teaching was done by the author. The analysis and interpretations are based on observations, related notes, papers produced by students, a kind of individual diary kept by the students and two kinds of essays written by the students at home at the end of the whole process when the students had to choose between a systematic deductive presentation of the knowledge developed in the project and a genectic description of the actual processes in the course of creating the theory.

As for the *goals*, the project was explored and designed to give students a special opportunity to experience how in a collective activity a mathematical model related to a relevant problem domain originally situated outside of mathematics can be developed. They should learn by doing, that such a model is not to be taken as ready-made mathematics but can be created in a process of mathematization in which the students themselves can actively *play different roles*. First, they are *people concerned* with the situations and problems in the political domain of voting and decision making, with related values, interests and expectations they may hold themselves and which via social interaction and communication should go into criteria for acceptability and adequacy of the model, thus representing an important (the external) aspect of the social dimension of mathematics. Second, they are having the *role of mathematizing mathematicians*, i. e. of *experts*, as which all of them should be accepted and respected, though at the same being learners, yet somehow sharing this with searching mathematicians. Especially in this role they have to take into account the concerns related to the problem domain and be in communication with others. But they should also experience that the more internal mathematical problems of the model construction are often matters of *social interactions and negotiations among mathematicians*, which represents another (the internal) aspect of the social dimension of mathematics (see Steiner 1988).

As an example of the second aspect experienced in class the following phenomenon can count which happened in almost all relatizations of the project: When trying to define the *concept of dictator* within the conceptual framework of the not yet fully elaborated theory, usually several suggestions are made, among which the following three almost always appear: (1) a dictator is making all other people powerless, (2) a dictator is by himself a winning coalition, (3) a dictator belongs to all minimal winning coalitions. Reactions are: that only one of the definitions can be "true" which causes debates about the *nature of definition*: not being true or false, but useful, adequate etc.; that the suggested definitions may be logically equivalent, which would be an indication of adequacy since different experts have different intuitive ideas but are logically saying the same. The first suggestion can be interpreted as expressing a kind of pitiful attitude towards what a dictator does to other people, the second one as representing a kind of egoistic position. The students then get very much involved in trying to prove or disprove logical equivalences, and in doing so, proof becomes a *matter of social interest and communication* and is no longer something the teacher is giving to them as an order. They find that (1) and (2) are indeed logically equivalent and that (3) is actually saying something different, which they identify as veto-power. When standardizing the definitions, again arguments are being exchanged and the preference of (2) is usually based on the agreement about its simplicity in directly referring to the fundamental concept of winning coalitions, whereas (1) is rejected as using powerlessness as a more complex derived concept. The equivalence of (1) and (2) is then turned into a theorem saying that $x$ is a dictator if and only if $x$ makes all other members powerless. This again is a matter of social acceptance in relation to building up a collectively owned theory.

Social interactions also play an important role in *coping with epistemological obstacles* (see Sierpinska 1989). A profound obstacle is coming up for the students in connection with the implicit definition of winning coalitions within the axiomatic definition of voting bodies of the *SC*-type because of the *circularity* involved and the contrast to the concrete empirical meaning given to this concept in normal voting bodies. The broad bases laid in the *total learning context* together with intensive *social interaction and communication* among the students in which many aspects and views of the problem are expressed, contributes to a dynamic attitude and a flexible relation to possible applications which is the adequate way to handle and develop theoretical concepts (see Jahnke 1978, Steiner 1990).

The content of the project does not belong to the obligatory normal school topics and is not meant to have this status, in particular it is not thought of as learning material to be spread in bits and pieces over longer periods of time. Its strength and potential lies in the concentration during a limited project time on a dynamically coherent and surveyable context which is rich in different kinds of activities, interactions and reflexions and is put by the students themselves into a dynamic relation to their previous learning experiences in mathematics and to other mathematics-related contexts they may meet in the future. Because of this intended biographic role, the project is purposefully placed at grade level 11 or 12.

In this way the project creates a kind of distance to the normal mathematical classroom which causes reflexions and transfer. The role of definitions, proofs, theorems, problems, applications etc. is comparatively discussed by the students, sometimes in a spontaneous way and at a local level: definition, theorems, proofs in geometry, applications of algebra and calculus etc.

In describing the overall project work at the end in two different ways, the *systematic reorganization* of all knowledge gained from an axiomatic point of view on one hand, and the *genetic reconstruction* of the developmental processes and events on the other hand, the students become aware of several *styles* to talk about and to present mathematics. They are trying, now at a more global level, to characterize how other parts of school mathematics have been taught to them and they are wondering why these parts have not been developed in the same genetic and inductive way as experienced in the project. They want to know how mathematics is taught at the university and they begin to make differentiations between research and learning, between different learning styles, they bring in aspects of time and time economy etc.

Already during the project it could be observed that most of the participating students deeply changed their *attitude* towards mathematics in a positive direction. This particularly holds for *girls*, a matter which deserves special attention and further analysis. It has also been found in later conversation and *interaction* with students who participated in the project that they had very much internalized their project experiences and used them as guiding references for their appreciation of mathematics. Interestingly several of them declared that becoming mathematics teachers might be the best way to learn more about and to professionally enjoy unexpected aspects and dimensions of mathematics they had experienced in the project.

## 3. *Theoretical Concepts, Applications, Social Interactions, Knowledge Development, and Higher Order Learning*

In order to design and execute more specified research in relation to teaching and learning in a context and setting as has been indicated with respect to the mathematization project, it seems important to have a sufficiently developed *theoretical background.* The author sees substantial and relevant components for this background in socio-historical and epistemological studies in mathematics and the empirical sciences (Sneed 1971, Jahnke 1978, Steiner 1989, 1990) on one hand and in the presently very dynamically growing research characterized by terms like "Zone of Proximal Development" (Vygotsky 1978), "Construction Zone" (Newman et al.), "Learning by Expanding" (Engeström 1987) on the other hand, both components being essentially interrelated. The interrelation consists in the analysis of the mutual interdependence between theoretical concepts, applications, social interactions and communications, knowledge development, and higher order learning (see also Seeger 1990).

Engeström (1987) and Newman et al. (1989) are both referring to the so-called *"learning paradox"*, formulated by Fodor (1980), as a challenge to cognitivists, as follows: "There literally isn't a thing as the notion of learning a conceptual system richer than the one that one already has" (p. 149). They both are criticizing Bereiter's (1985) interpretation of the paradox as basically being a problem to understand how a learner can *internalize* more complex cognitive structures located in the culture while not knowing how internalization actually takes place, which means dismissing Vygotsky's cultural-historical position as a solution.

Newman et al. (1989) are responding: "Internalization need not be the construction process which creates the more powerful structures. We are pointing to the social interaction in the zone of proximal development as the more central locus for constructive activity in the Vygotskian framework" (p. 68). Engeström (1987) who is reacting from his concept of *learning by expansion as transcending given contexts*, based on Leont'ev's (1978) *activity theory* (see also Steiner 1987), linked with Bateson's (1972) complex hierarchy of learning processes, is also considering the general *problem of how the new is generated from the old* as well as Davydov's *searching paradox*. He points out that "the new is not generated from the old but from the *living movement* leading away from the old" (p. 164).

Jahnke (1978) who is concerned with the problem how *justification and development of knowledge*, especially in mathematics, are related, is referring to the *paradox of the proof* and the *dilemma of the theorist* which have a structure similar to that of the learning paradox and can also be related to the problem of the relation between reflective and simple abstraction in Piaget's genetic epistemology or the dual-control-problem in artificial intelligence research (see also Otte 1980). From his profound historical and epistemological investigations applying Sneed's (1971) clarification of the nature of *empirical theories* and *theoretical concepts* as well as Churchman's (1968) *systems approach* and *philosophy of the maximal loop*, he makes clear that the kernel of the problem lies in the *dialectic relation between sign and signified* which turns out not only to be the crucial point in *relating theory and applications* from a developmental point of view but also to be deeply connected with *social contexts and communication* as indispensable components of understanding the problem. With respect to the dynamically inseparable connections between justification and application, Jahnke comes to the conclusion that "justification (evidence) is, so to say, placed into the future. The more general, more extended, more developed theory is founding and justifying the less general theory" (pp. 108–109).

Engeström (1987), Newman et al. (1988) and others have designed research methodologies to study learning in the construction zone and learning by expansion. At the IDM in Bielefeld research on classroom interactions (Bauersfeld et al. 1988) and on the epistemology of school mathematics (Steinbring 1984) are now being put into closer relation (see Seeger 1990). It seems important to do more detailed empirical research based on these theoretical and methodological developments particularly in contexts and settings with respect to existing observations as have been sketched in this paper.

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**ALICE ALSTON**
Rutgers University
Center for Maths,
Science & Comp. Educ.
New Brunswick, New Jersey 08903
**USA**

**MIRIAM AMIT**
Technion-Israel Inst.of Technology
Dept. of Ed. in Sc. & Technology
Technion, Haifa 32000
**ISRAEL**

**ABRAHAM ARCAVI**
Science Teaching Weizmann
Institute, Rehovot 76100
**ISRAEL**

**MICHELE ARTIGUE**
Irem Paris VII 2 Place Jussieu
Paris 75005
**FRANCE**

**ALFONSO AVILA**
Unidad de Apoyo Didáctico
Sría.de Educ.Cultura y Bienestar
Social del Estado de México.
**MEXICO**

**CARMEN BATANERO**
Escuela Univ. Profesorado Egb.
Campus de Cartuja
Granada 18071
**SPAIN**

**LUCIANA BAZZINI**
Università di Pavia
Strada Nuova 65
Pavia 27100
**ITALY**

**CANDICE BEATTYS**
Rutgers University
Center for Maths,
Science & Comp. Educ.
New Brunswick, New Jersey 08903
**USA**

**GERHARD BECKER**
Modersohnweg 25 D-28
Bremen 33, 2800
**GERMANY**

**MERLYN BEHR**
Mathematics Department
Northern Illinois University
Dekalb, ILL 60115
**USA**

**ALAN BELL**
Shell Centre Math. Educ.
Univ. Nottingham
Nottingham NG7 2RD
**ENGLAND**

**JACQUES C. BERGERON**
Université de Montreal Fac. des
Sciences de I'Education C.P.6128
Succ "A" Montréal, P.Q. H3C 3J7
**CANADA**

**KATHRYN BERTILSON**
Elem/Sec Education,
Washington State University
Pullman, WA 99164-2122
**USA**

**NADINE BEZUK**
Center for Research Math. &
Science Educ.
San Diego State University
San Diego, California 92182-0315
**USA**

**VICTORIA BILL**
Learning Research & Develop.Center
University of Pittsburgh
**USA**

**DERRICK BIRKS**
Schell Centre Math. Educ.
Univ. Nottingham
Nottingham NG7 2RD
**ENGLAND**

**ALAN J. BISHOP**
Cambridge Univ. Dept.of Education
Cambridge CB1 1QA
**ENGLAND**

**OLE BJORKQVIST**
Faculty of Education,
Abo Akademi Box 311
Vasa 65101
**FINLAND**

**JOAN BLISS**
552 King's Road
London SW10 OUA
**ENGLAND**

**PAOLO BOERO**
Univ. Genova Via L.B. Alberti 4
Genova 16132
**ITALY**

**ANDRE BOILEAU**
Department of Maths. & Inform.
Univ. du Québéc à Montréal
CP 8888, Montréal Québéc H3C 3P8
**CANADA**

**LESLEY R. BOOTH**
James Cook University
School of Education
Townsville, Queensland 4811
**AUSTRALIA**

**GILLIAN BOULTON-LEWIS**
Brisbane CAE
School of Early Childhood Studies
130 Victoria Park Rd Kelvin Grove
Kelvin Grove, Queensland
**AUSTRALIA**

**CHRISTINA BOYD**
Phillip Institute of Technology
Melbourne
**AUSTRALIA**

**YOLANDA CAMPOS CAMPOS**
Calle D No. 16 Mz. 13
Col. Educación,
México, D.F. 4400
**MEXICO**

**RICARDO CANTORAL**
Nicolás San Juan 1421
Col. Del Valle
México 12, D.F. 03100
**MEXICO**

**FEIYU CAO**
Curr. & Teaching Math. Research
Ins 55 Shatan, Hou Jie
Beijing
**CHINA**

**DAVID CARRAHER**
Fygenho Poeta 66-202
Recife 50 000
**BRAZIL**

**TEREZINHA CARRAHER**
R. Mendes Martins 112 Varzea
Recife Pernambuco 50741
**BRASIL**

**IVANA CHIARUGI**
Univ. Genova Via L.B. Alberti 4
Genova 16132
**ITALY**

**VICTOR CIFARELLI**
Univ. of California
Office of Academic Support, B-036
La Jolla California 92093
**USA**

**McKENZIE A. CLEMENTS**
School of Educ. Deakin Univ.
Geelong Victoria 3217
**AUSTRALIA**

**ANIBAL CORTES**
Lab. Psydee 46 Rue St. Jacques
Paris 75005
**FRANCE**

**MARTIN COOPER**
Po Box 1 Kensington School
Educ. Univ. New South Wales
Kensington 2033 NSW
**AUSTRALIA**

**CATHLEEN CRAVIOTTO**
Elem/Sec. Education
Washington State University
Pullman WA 99164-2122
**USA**

**REGINA DAMM**
4 Rue Charles Appell.
Univ. Louis Pasteur. Irem
Strasbourg 67000
**FRANCE**

**GARY DAVIS**
School of Math. & Inform.
Sciences. La Trobe University
Bundoora Melbourne 3083
**AUSTRALIA**

**ROBERT B. DAVIS**
Rutgers University
Center for Maths, Science &
Comp. Educ.
New Brunswick, New Jersey 08903
**USA**

**LUCIANO DE LEMOS MEIRA**
U.C.Berkeley-Tolman H. 4533
Berkeley, California
**USA**

**LIVIA P. DENIS**
2016 State Street, Schenectady,
N Y Schenectady, NJ 12304
**USA**

**BERNADETTE DENYS**
5/7 Rue de la Comete
Paris 75007
**FRANCE**

**ANALUCIA DIAS SCHLIEMANN**
Rua Aroa Lins de Andrade 275
Jaboatao Pernambuco 54320
**BRAZIL**

**JUAN DIAZ GODINO**
Escuela Univ. Profesorado Egb.
Campus de Cartuja
Granada 18071
**SPAIN**

**BARBARA DOUGHERTY**
Curric. Research & Developm.Group.
1776 University Av. CM 130
Honolulu, Hi 96822
**USA**

**JEAN-LUC DORIER**
Equipe de Didact.des Maths.
38 402 St. Martin D'Heres
Grenoble Cedex
**FRANCE**

**TOMMY DREYFUS**
Science Teaching Weizmann
Institute, Rehovot 76100
**ISRAEL**

**LAURIE EDWARDS**
115 Miller Hall DQ-12
Seattle Washington 98195
**USA**

**THEODORE EISENBERG**
Math Dept Ben Gurrion Univ.
Beer Sheva 84120
**ISRAEL**

**HAMDY ALY EL FARAMAWY**
College of Educ. Menoufia Univ.
Shebin El Koom
**EGYPT**

**NERIDA ELLERTON**
Sch Educ. Deakin Univ.
Geelong Victoria 3217
**AUSTRALIA**

**HIDEYO EMORI**
2-28-3 Horinouchi Omiya-Shi
Saitama-Ken 330
**JAPAN**

**PAUL ERNEST**
36 Belmont Road
Exeter Devon EX1 2HG
**ENGLAND**

**A. ESTEPA CASTRO**
Escuela Univ. Profesorado Egb.
Campus de Cartuja
Granada 18071
**SPAIN**

**RUHAMA EVEN**
Science Teaching Dept.
Weizmann Instit. of Science
Rehovot 76100
**ISRAEL**

**ROSA MARIA FARFAN**
Nicolás San Juan 1421,
Col. Del Valle,
México, D.F. 03100
**MEXICO**

**PIER LUIGI FERRARI**
Dipartimento di Matematica
Via L. B. Alberti 4
Genova 16132
**ITALY**

**FREDERICK FLENER**
Northeastern Illinois Univ.
5500 N St Louis
Chicago ILL 60625
**USA**

**FULVIA FURINGHETTI**
Via All'Opera Pia 9 N 11
Genova 16145
**ITALY**

**GRAZIA FRACASSINA**
Univ. Genova Via L.B. Alberti 4
Genova 16132
**ITALY**

**AURORA GALLARDO**
Dakota 379, Col. Nápoles
México, D.F. 03910,
**MEXICO**

**APARNA GANGULI**
Div. of Science, Business & Math
Univ. of Minnesota
Minneapolis, Minnesota 55455
**USA**

**MAURICE GARANÇON**
Department of Maths. & Inform.
Univ. du Québéc à Montréal
CP 8888, Montréal Québéc H3C 3P8
**CANADA**

**FRANCISCO GARCIA**
Unidad de Apoyo Didáctico
Sría.de Educ.Cultura y Bienestar
Social del Estado de México.
**MEXICO**

**WILLIAM E. GEESLIN**
18 Thompson Line
Durham New Hampshire 3824
**USA**

**JOAQUIN GIMENEZ**
Gral. Mitre 216 Pral 2
Barcelona E 08006
**SPAIN**

**GERALD A GOLDIN**
Rutgers Univ. SERC Bldg Room 239
Buch Campus
New Brunswick NJ 8903
**USA**

**TOMAS A. GOOD**
University of Missouri
Columbia
**USA**

**ANNA GRAEBER**
Dept. of C&I 2311 Benjamin Bldg,
UMPC College Park
College Park Maryland 20742
**USA**

**DOUGLAS GROUWS**
University of Missouri 212
Townsend, Columbia Missouri 65211
**USA**

**DOMINIQUE GUIN**
Irem de Strasbourg 10 Rue
du General Zimmer
Strasbourg 67084
**FRANCE**

**ANGEL GUTIERREZ**
Dept. Didact. Matem.
Univ. Valencia
Alcalde Reig 8, Valencia 46006
**SPAIN**

**ISMENIA GUZMAN**
Irem de Strasbourg 10 Rue
du General Zimmer
Strasbourg 67084
**FRANCE**

**RINA HADASS**
Univ. of Haifa, Oranim Shool
of Educ.
Kibbutz Movement
Tivon 36910
**ISRAEL**

**GRAEME S. HALFORD**
Brisbane CAE
School of Early Childhood Studies
130 Victoria Park Rd Kelvin Grove
Queensland
Kelvin Grove
**AUSTRALIA**

**CHRIS HANCOCK**
TERC 2067 Massachussets Avenue
Cambridge, Massachussetts 02138
**USA**

**GUERSHON HAREL**
Dept of C&I School of Educ.
Purdue Univ.
West Lafayette N 47907
**USA**

**NIRA HATIVA**
School of Educ. Tel Aviv Univ.
Tel-Aviv 69972
**ISRAEL**

**LULU HEALY**
Institute of Educ. Univ. of London
20 Bedford Way
London WC1H OAL
**ENGLAND**

**NICOLAS HERSCOVICS**
Concordia University
7141 Sherbrooke St. West
Montréal Québéc H4B 1R6
**CANADA**

**RINA HERSHKOWITZ**
Science Teaching Weizmann
Institute, Rehovot 76100
**ISRAEL**

**SARA HERSHKOVITZ**
Center of Educational Technology
Tel Aviv
**ISRAEL**

**FERNANDO HITT**
Dakota 428,
Col. Nápoles,
México, D.F.
**MEXICO**

**CELIA HOYLES**
Institute of Educ.Univ.of London
20 Bedford Way
London WC1H OAL
**ENGLAND**

**PIET HUMAN**
Stellenbosch University
Stellenbosch 7600
**SOUTH AFRICA**

**KATHRYN IRWIN**
Educ.Dept.Univ. of Auckland
Auckland 649
**NEW ZEALAND**

**KEIKO ITO**
1-8 Mitswa-Sou 3-18-10
Amakubo Tsukuba-Shi
Ibaraki-Ken 305
**JAPAN**

**ADELA JAIME**
Dept. Didact. Matem.
Univ. Valencia
Alcalde Reig 8, Valencia 46006
**SPAIN**

**ELIZABETH JAKUBOWSKI**
Florida State University
221 MCH Tallahassee
Tallahassee, Florida 32306
**USA**

**BARBARA JAWORSKI**
School of Education
University of Birmingham
Po Box 363
Birmingham B15 2TT
**ENGLAND**

**LYNN JOFFE**
Joffe Consultancy Services
19 Haynes Close
South SL3 8NA
**ENGLAND**

**ROCHELLE KAPLAN**
William Patterson College
388 Edgewood Ave
Teaneck New Jersey 7666
**USA**

**NELLY KAVAFIAN**
Lab. Psydee 46 Rue St. Jacques
Paris 75005
**FRANCE**

**KONRAD KRAINER**
Universitat Klagenfurt IFF
**AUSTRIA**

**TAPIO KERANTO**
Visamaentie 15
Hameenlinna 13100
**FINLAND**

**LENA LICON KHISTY**
Elem/Sec Education,
Washington State University
Pullman, WA 99164-2122
**USA**

**CAROLYN KIERAN**
Department of Maths. & Inform.
Univ. du Québéc à Montréal
CP 8888, Montréal Québéc H3C 3P8
**CANADA**

**DAVID KIRSHNER**
Louisiana State University
Dept.Curriculum & Instruction
Baton Rouge LA 70803-4728
**USA**

**GOTZ KRUMMHEUER**
Inst. for Didaktik of Math.
Karl-Glockner-Str 21 C
Giessen 6300
**WEST GERMANY**

**HANS KUYPER**
Rion/Postbus 1286/ 9701 BG
Groningen 9701
**NETHERLANDS**

**DIANA LAMBDIN KROLL**
309 Educ. Bldg, Indiana Univ.
Bloomington Indiana 47405
**USA**

**HILDA LEA**
Univ. of Botswana, Private BAG 22
Gabarone
**AFRICA**

**TAMARA LEMERISE**
Dept. of Psychology
Univ. du Québéc à Montréal
CP 8888, Montréal Québéc H3C 3P8
**CANADA**

**STEPHEN LERMAN**
Roughdown End
Roughdown Villas Rd
Boxmoor
Hemel Hempstead
Herts HP3 OA
**ENGLAND**

**FRANK LESTER**
309 Educ. Bldg, Indiana Univ.
Bloomington Indiana 47405
**USA**

**FOU-LAI LIN**
3-6F Alley 15 Lane 147
Sec. 3 Hsin-Yi Rd
Taipei
**TAIWAN, REP. OF.CHINA**

**LENA LINDENSKOV**
Roskilde Univ. Center IMFUFA
Roskilde 4000
**DENMARK**

**LIORA LINCHEVSKI**
Hebrew Univ. School Education
Mount Scoups
Jerusalem 96-664
**ISRAEL**

**ROMULO LINS**
Schell Center for Math. Educ.
University Park
Nottingham NG7 2RD
**ENGLAND**

**JANE-JANE LO**
152-15 Bliss Dr. Tallahassee
Tallahassee Fl. 32310
**USA**

**IVI MACHMANDROV**
Center of Educational Technology
Tel Aviv
**ISRAEL**

**VERONICA P. DE MAGALHAES**
Maestrado en Psicología
Universidade Fed. Pernambuco
**BRAZIL**

**CAROLYN A MAHER**
Rutgers University
Center for Maths,
Science & Comp. Educ.
New Brunswick, New Jersey 08903
**USA**

**JOANNA MAMONA**
LRDC, University of Pittsburgh
3939 O/Hara Street
Pittsburgh PA 15260
**USA**

**ZVIA MARKOVITZ**
Dept. of Mathematical
San Diego State University
San Diego, CA 92182
**USA**

**DOUGLAS McLEOD**
Elem/Sec. Education
Washington State University
Pullman WA 99164-2122
**USA**

**JOOST MEIJER**
Grote Bickersstraat 72
1013 KS Amsterdam
**THE NETHERLANDS**

**ANA MESQUITA**
R. Maria Brown 7-8C
Lisboa 1500
**PORTUGAL**

**MIKIO MIYAZAKI**
3175-1 Kojyou Ryugasaki-Shi
Ibaraki-Ken 305
**JAPAN**

**CANDIA MORGAN**
10 Holly Road, Hounslow
MIDDX TW3 IVN
**ENGLAND**

**JUDIT MOSCHKOVICH**
829 A Castro State
San Francisco California 94114
**USA**

**JUDITH MOUSLEY**
School of Education,
Deakin University
Deakin 3217
**AUSTRALIA**

**NITSA MOVSHOVITZ HADAR**
Dept.of Educ.in Science & Tech.
Technion Haifa 32000
**ISARAEL**

**SWAPNA MUKHOPADHYAY**
Univ. of Pittsburgh Learn
Research Develop Center
Pittsburgh
**USA**

**HANLIE MURRAY**
Stellenbosch University
Stellenbosch 7600
**SOUTH AFRICA**

**NICOLE NANTAIS**
Faculte D'Education
Univ.de Sherbrooke 2500
Sherbrooke Québéc J1K 2R1
**CANADA**

**LILIAN NASSER**
Rua Xavier Da Silveira 34-902
Rio de Janeiro 22061
**BRAZIL**

**RICARDO NEMIROVSKY**
TERC.2067 Massachussetts Av.
Cambridge MA. Massachussetts 2140
**USA**

**NOBUHIKO NOHDA**
Takezono 2-809-1
Tsukuba Ibaraki 305
**JAPAN**

**RITA NOLDER**
Dept. Educ. Loughborough Univ.
Loughborough Leics LE11 3TU
**ENGLAND**

**RICHARD NOSS**
Univ. London Inst. Education
20 Bedford Way
London WC1H OAL
**ENGLAND**

**RAFAEL NUÑEZ ERRAZURIZ**
Univ. de Fribourg Suisse
Inst. Psych. Route de Fougeres
Fribourg Suisse CH 1701
**SWITZERLAND**

**ANA MARIA OJEDA SALAZAR**
Nicolás San Juan 1421,
Col. Del Valle,
México, D.F. 03100,
**MEXICO**

**JOHN OLIVE**
Univ.of Georgia Dept.Math.Educ.
105 Aderhold Hall
Athens GA 30602
**USA**

**ALWYN IVO OLIVIER**
Dept. of Didactics
Stellenbosch Univ.
Stellenbosch 7600
**SOUTH AFRICA**

**ALTON OLSON**
5215 142 State
Edmonton Alberta, T6H 4B6
**CANADA**

**MICHELE ORTEGA**
Elem/Sec. Education
Washington State University
Pullman WA 99164-2122
**USA**

**ROBERT PEARD**
Dept. of Math. BCAE
130 Victoria Park Rd.
Kelvin Grove Queensland Q 4059
**AUSTRALIA**

**MARIE-JANNE PERRIN-GLORIAN**
6 Rue Einstein
Antony 92160
**FRANCE**

**ANGELA PESCI**
Via Loatti 12
Pavia 27100
**ITALY**

**DAVID JOHN PIMM**
Centre for Math. Educ.
The Open University Walton Hall
Milton Keynes
**ENGLAND**

**JOAO PEDRO PONTE**
Clagáde Do Galvao 21
Lisboa 1400
**PORTUGAL**

**NORMA CHRIST PRESMEG**
22 Glenlorne, 31 Vause Rd.
Durban 4001
**SOUTH AFRICA**

**LUIS RADFORD**
50 Calle 16-66 Zona 12
Guatemala 1012
**GUATEMALA**

**TIKVA RAGER**
School of Educ., Tel Aviv Univ.
Tel-Aviv 69978
**ISRAEL**

**JULIA REEDY**
Saint Victor, High School
Arlington Heights, Illinois
**USA**

**LAUREN RESNICK**
824 LRDC Univ.of Pittsburgh
3939 O'Hara Street, PA 15260
**USA**

**ANTONIO ROAZZI**
Edf.Vila das Pedras Bda.Viagem
Recife Pernambuco 51020
**BRAZIL**

**TERESA ROJANO**
Nicolás San Juan 1421
Col. Del Valle,
México, D.F. 03100
**MEXICO**

**GUILLERMO RUBIO**
Nicolás San Juan 1421
Col. Del Valle,
México, D.F. 03100
**MEXICO**

**ADALIRA SAENZ-LUDLOW**
Univ. of Georgia Dept.Math.Educ.
105 Aderhold Hall
Athens GA 30602
**USA**

**HARALABAMBOS SAKONIDIS**
552 King's Road
London SW10 OUA
**ENGLAND**

**VANIA MARIA SANTOS**
Indiana Univ. School of Educ.
Math. Education Room 309
Bloomington 47405
**USA**

**LEONA SCHAUBLE**
824 LRDC Univ.of Pittsburgh
3939 O'Hara Street, PA 15260
**USA**

**DEBORAH SCHIFTER**
Summer Math. for Teachers
Mount Holyoke College
South Hadley MA 1075
**USA**

**JOY FRANCES SCOTT**
35 Success Crescent, Manning
W.A. Australia 6152
**AUSTRALIA**

**ROSALINDE SCOTT-HUDGETTS**
Dept.of Math.South Bank Polytechn
Borough Road
London SE1 OAA
**ENGLAND**

**FALK SEEGER**
Institute for Didactics of Math.
University of Bielefeld
Postfach 8640, 4800 Bielefeld 1
**GERMANY**

**KENNETH SHAW**
4750 Collegiate Drive,
Florida State University
Panama City Florida 32405
**USA**

**MARTIN A. SIMON**
Penn. State University,
176 Chambers Univ. Park PA 16802
**USA**

**ADELE C. SMITH**
Florida State University
**USA**

**JUDITH & LARRY SOWDER**
Dept. of Mathematical Sciences
San Diego State University
San Diego, CA 92182
**USA**

**RUTH STAVY**
School of Educ., Tel Aviv Univ.
Tel-Aviv 69978
**ISRAEL**

**LESLIE P. STEFFE**
Univ. of Georgia Dept.Math.Educ.
105 Aderhold Hall
Athens GA 30602
**USA**

**HANS GEORG STEINER**
Institute for Didactics of Math.
University of Bielefeld
Postfach 8640, 4800 Bielefeld
**GERMANY**

**TUULA STRANG**
Kamarineidonkuja 4 B
Espoo 2940
**FINLAND**

**ROSAMUND SUTHERLAND**
Inst. Educ. Univ. London
20 Bedford Way
London WC1H OAL
**ENGLAND**

**ALBA G. THOMPSON**
Dept.of Math.Illinois State Univ
Normal Illinois 61761
**USA**

**PATRICK THOMPSON**
Dept.of Math.Illinois State Univ
Normal Illinois 61761
**USA**

**CORNELIA TIERNEY**
60 Buckingham Street
Cambridge MA 2138
**USA**

**LUCIA TINOCO**
Praca Radial Sul 25/002
Botafogo Rio de Janeiro 22260
**BRAZIL**

**DINA TIROSH**
School of Educ., Tel Aviv Univ.
Tel Aviv, 69978
**ISRAEL**

**MARIA TRIGUEROS**
ITAM Rio Hondo No. 1
San Angel México, D.F., 01000
**MEXICO**

**BOB UNDERHILL**
Memorial Hall, Virginia Tech.
Blacksburg VA 24061-0313
**USA**

**SONIA URSINI LEGOVICH**
Augusto Rodin 395-17
Col. Insurgentes Mixcoac
México, D.F. 03810
**MEXICO**

**JAN VAN DEN BRINK**
Research Group OW & OC
Tiberdreef 4 Utrecht 3561 GG
**THE NETHERLANDS**

**M.P.C. VAN DER WERF**
Institute of Educational Research
RION University of Groningen
**THE NETHERLANDS**

**TINUS PETRUS VAN ROOY**
Dept.of Didactics
Univ. of South Africa
Po Box 392
Pretoria 1
**SOUTH AFRICA**

**ANNO VAN STREUN**
Boelens Straat 10 9203 RM
Drachten 9203
**THE NETHERLANDS**

**GERARD VERGNAUD**
Lab. Psydee 46 Rue St. Jacques
Paris 75005
**FRANCE**

**SHLOMO VINNER**
Israel Science Teach. Centre
Hebrew University
Jerusalem
**ISRAEL**

**ELFRIEDE WENZELBURGER**
Parque del Virrey 4
Parques de Coyoacán
México, D. F. 4890
**MEXICO**

**GRAYSON H. WHEATLEY**
Florida State University
**USA**

**STEVEN WILLIAMS**
Dept.of Math. Washington St.Univ.
Pullman Washington 99163-2930
**USA**

**TERRY WOOD**
Dept. of Educ. Purdue Univ.
West Lafayette Indiana 47906
**USA**

**ERNA YACKEL**
Dept. Math&SC Calumet Campus
Purdue University
Hammond Indiana 46323
**USA**