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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following papers: "The Knowledge of Cats: Epistemological Foundations of Mathematics Education" (R.B. Davis) and "PME Algebra Research: A Working Perspective" (E. Filloy); "Some Misconceptions in Calculus: Anecdotes or the Tip of an Iceberg?" (M. Amit & S. Vinner); "Difficultes Cognitives et Didactiques dans la Construction de Relations entre Cadre Algebrique et Cadre Graphique" (M. Artigue); "Unbalance and Recovery: Categories Related to the Appropriation of a Basis of Meaning Pertaining to the Domain of Physical Thinking" (R. Cantoral); "On Difficulties with Diagrams: Theoretical Issues" (T. Dreyfus & T. Eisenberg); "The Two Faces of the Inverse Function: Prospective Teachers' Use of 'Undoing'" (R. Even); "Intuitive Processes, Mental Image, and Analytical and Graphic Representations of the Stationary State: A Case Study" (R.M. Farfan & F. Hitt); "The Role of Conceptual Entities in Learning Mathematical Concepts at the Undergraduate Level" (G. Harel & J. Kaput); "Mathematical Concept of Formation in the Individual" (L. Lindenskov); "Pupils' Interpretations of the Limit Concept: A Comparison Study between Greeks and English" (J. Mamona-Downs); "Infinity in Mathematics as a Scientific Subject for Cognitive Psychology" (R.N. Errazuriz); "Organizations Deductives et Demonstration" (L. Radford); "The Teaching Experiment 'Heuristic Mathematics Education'" (A. Van Streun); "The Understanding of Limit: Three Perspectives" (S.R. Williams); "Self Control in Analyzing Problem Solving Strategies" (G. Becker); "Influences of Teacher Cognitive/Conceptual Levels on Problem-Solving Instruction" (B.J. Dougherty); "Can Teachers Evaluate Problem Solving Ability?" (F.O. Flener & J. Reedy); "Teacher Conceptions about Problem Solving and Problem Solving Instruction" (D.A. Grouws, T.A. Good, & B.J. Dougherty); "Math Teachers and Gender Differences in Math Achievement, Math Participation and Attitudes Towards Math" (H. Kuyper & M.P.C. van der Werf); "Teaching Students to be Reflective: A Study of Two Grade Seven Classes" (F.F. Lester & D.L. Kroll); "Students' Affective Responses to Non-Routine Mathematical Problems: An Empirical Study" (D.B. McLeod, C. Craviotto, & M. Ortega); "Accommodating Curriculum Change in Mathematics: Teachers' Dilemmas" (R. Nolder); "Teachers' Characteristics and Attitudes as Mediating Variables in Computer-Based Mathematics Learning" (R. Noss, C. Hoyles, & R.

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Sutherland); "Teachers' Perceived Roles of the Computer in Mathematics Education" (J. Ponte); "Mathematics Process as Mathematics Content: A Course for Teachers" (D. Schifter); "Psychological/Philosophical Aspects of Mathematical Activity: Does Theory Influence Practice?" (R. Scott-Hodgetts & S. Lerman); "A Web of Beliefs: Learning to Teach in an Environment with Conflicting Messages" (R.G. Underhill). Includes 27 poster presentation abstracts and a listing of author addresses. (MKR)

International Group For the Psychology of Mathematics Education

PROCEEDINGS

Fourteenth PME Conference

With the North American Chapter
Twelfth PME-NA Conference

(July 15-20)

México 1990

VOLUME I

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Paul Cobb
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P R E F A C E

The fourteenth annual meeting of PME is the first to be held in a Latin-American country: Mexico. The diverse traditions of Mexican Culture provide a stimulating backdrop against which to continue conversations started in previous years. Conference participants can join these conversations in a number of different ways: research reports, poster presentations, discussion groups and working groups.

Two innovations made this year both reflect the view that PME serves primarily as a forum for researchers in mathematics education. First, there will be a discussion group that has as its focus the aims of PME. Clearly, tensions about aims and goals are to be expected in an international organization such as PME given the geographical, cultural and philosophical diversity of its members. One of PME's strengths is the common, democratic belief that such tensions should be the subject of debate and argumentation.

A second innovation concerns the inclusion of a plenary symposium in the program. In taking as their theme the responsibilities of researchers in mathematics education, the symposium participants will attempt to spark discussion of broader issues in mathematics education from a variety of perspectives. The program committee is grateful to Kath Hart for organizing the discussion group and to Alan Bishop for organizing the plenary symposium.

The major interest of PME members as indicated by the research reports continues to be the cognitive analysis of students' mathematical conceptions and learning. A glance at the contents pages of these proceedings reveals that these contributions address a wide variety of different conceptual domains and age levels. These do, however, appear to be some underrespresented areas of investigation, particularly measurement and statistical reasoning.

In addition to this cognitive emphasis, trends noted by organizers of PME 13 continue to develop. One concerns the effective and metacognitive aspects of students' mathematical experiences and their relationship to their cognitive development. A second concerns the conditions in which cognitive development occurs, with particular emphasis on the social setting as well as on the problems that students attempt to solve. A third concerns the growing attention being paid to students' mathematical activity in computer environments. More than in previous years, contributors to this area of research are developing theoretically grounded rationales for their construction of the environments. In addition, there appears to be a growing realization that the problem of accounting for students' learning while interacting with the computer is empirical in nature and requires careful, detailed analysis. The final trend concerns the growing interest in didactical issues and in teachers' pedagogical activity in the classroom. This line of work offers the possibility of developing theoretical frameworks and methodologies that acknowledge the mutual interdependence of teachers' and students' activities.

PME 14 1990
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HISTORY AND AIMS OF THE PME GROUP

At the Third International Congress on Mathematical Education ICME 3, Karlsruhe, 1976, Professor E. Fischbein of Tel Aviv University, Israel, instituted a studying group bringing together people working in the area of the psychology of mathematics education. PME is affiliated with the International Commission for Mathematical Instruction ICMI.

The major goals of the group are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Membership

1. Membership is open to persons involved in active research in furtherance of the group's aims, or professionally interested in the results of such research.

2. Membership is on an annual basis and depends on payment of the subscription for the current year January to December.

3. The subscription can be paid together with the conference fee.

The present officers of the PME group are:

President	Nicolás Balacheff (France)
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A Note on the Review Process.

The Program Committee received a total 152 research report proposals that encompassed a wide variety of theoretical and empirical approaches. Clearly the process of reviewing such a diverse collection of papers cannot be reduced to an algorithmic procedure. It is a process that copes with novelty and diversity by relying on situated wisdom and judgment. Nonetheless a few general remarks can be made.

Each proposal was sent to three colleagues for review with the request that comments be provided when considered appropriate and that these would be forwarded to the author(s). The review categories were:

A: Definitely accept
B: Accept with reservations
C: Accept as Poster
D: Reject

Some colleagues were unable to complete their reviews and others did not writ comments to explain their decisions. All comments received were sent to the authors when they were informed of the Program Committee's decision.

The Program Committee completed additional reviews to ensure that every paper received a minimum of two reviews. In 55 cases. there was clear agreement between two reviewers and additional reviews were not solicited. All other papers received a minimum of three reviews.

The Program Committee took the view that PME actively encourages participation and serves to stimulate intellectual dialogue. Consequently, there had to be clear evidence that a proposed research report was either inappropriate with respect to the goals of PME or contained inadequacies or inconsistencies before it could be rejected. Every proposal receiving review categories of B, B and C or better was automatically included in the Program. The most favorable reviews received by any of the rejected papers were B, B, D. Each of these cases was debated in detail by the Program Committee. The Program Committee completed additional reviews of all cases in which a unanimous decision could not be made on the basis of reviewers comments. These included all proposals that received ratings of A and D from different reviewers. Some papers therefore received five reviews. Eventually, 32 proposals were accepted as Posters, 111 proposals were accepted as Research Reports and the remaining 9 proposals were rejected.

A note on the Grouping of Research Report

The Program Committee followed a reflexive process to group the research reports. An initial organization scheme was derived from readings made when deciding whether to accepted particular research papers. We attempted to place all accepted reports into these groupings and became aware of limitations in the initial categorization scheme. We therefore revised this initial scheme and attempted a second grouping of papers. This process was repeated several times to yield the grouping used in these conference proceedings. The majority of papers focused on either *Mathematical learning*, *Mathematical teaching* or *Social interaction*. These were used as the three main categories for sorting papers. Several papers dealt instructional approaches in a theoretical way and it was not possible to justify their placement in any of these categories. They were therefore taken to constitute a fourth grouping that we have called *Didactical Analyses*.

Most of the reports that focused on mathematical learning investigated student's conceptualization of specific mathematics content and could reasonably be further organized in terms of these concepts, for example rational number or algebraic thinking and functions. Other papers that focused on the mathematics student were grouped together as Affect, Beliefs and Metacognition. A final category within the grouping of mathematical learning was formed of those papers which examined social-psychological issues of Social Interactions, Communication and Language.

The reports that focused on mathematics teaching were also fairly easy to separate into those that dealt with teachers beliefs or social-psychological factors and those that dealt with their mathematical understandings. Reports in this latter group are concerned with teachers' mathematical understandings as they relate to their pedagogical practice. Those papers which focused on teachers or prospective teachers as learners of mathematics *per se* were grouped with other reports on mathematics learning.

It will be noted that we have not used the traditional categories of problem solving or of instructional technology. Recent research has demonstrated that mathematical cognition is situated, and domain specific conceptualization plays a crucial role in successful mathematics problem solving. A report of a study that used whole number problems can be interpreted as an investigation of an important aspect of arithmetical competence. Such paper were therefore put with the mathematical content on which they drew. Similarly, papers that could have constituted a grouping on technological issues typically dealt with student learning of concepts and skills in computer environments. While technology is clearly an important theme for PME, the Program Committee took the view that this work and that which focuses on the development of specific mathematical concepts should mutually inform each other. Papers that dealt with learning in computer and non-computer environments were therefore grouped together according to the mathematical concepts of interest to facilitate dialogue among researchers. It would also have been possible to form a separate grouping of reports that emphasized the influence of social or cultural factors on the development of specific mathematical concepts. Again, while such investigations are central to the aims of PME, issues raised in these reports are relevant to colleagues who investigate learning across computer and non-computer environments and in relation to concept development or problem solving. Consequently, reports across these different perspectives were grouped together to facilitate intellectual interchange.

Thus the major categories used in grouping research reports related either to the mathematical content or to affective issues. Within these grouping, papers were further sorted into those that referred to student learning, to factors in teaching, to the use of technology, to issues of problem solving or to social and cultural aspects related to that content. Care was taken in finalizing the program that those primarily interested in, for example, the impact of technology would be able to select across the various categories to find presentations on their interests at any given time slot. Similar attention was given to the papers on problem solving, studies related more to teaching and those focussing more on learning. Any overlap between presentations were minimized by also taking the level of teaching or learning into account when allocating specific time slots.

The program committee wishes to emphasize that grouping used in these proceedings are in no way absolute. They are merely a social construction that seemed useful to organize the reported research activities. Conference participants are encouraged to read the abstracts when selecting sessions to attend.

Working Groups

Ratio and proportion	Kathleen Hart et al.
Geometry	Helen Mansfield et al.
Advanced mathematical thinking	Gontran Ervinck et al.
Psychology of in-service education of mathematics teachers: A research perspective	Barbara Jaworski et al.
Research on the psychology of mathematics teacher development	Robert Underhill, Carolyn Maher et al.
Social psychology of mathematics education	Alan Bishop et al.
Micromath research methodology	Nurit Zehavi et al.
Representations	Gerald Goldin et al.
Teachers and teacher educators as researchers in mathematics education	Steve Lerman, Rosalinde Scott-Hodgetts et al.

Discussion Groups

Learning mathematics and cultural context	Bernadette Denys
Philosophy of mathematics education	Paul Ernest
Theoretical and practical aspects of proof	Gila Hanna, Nicolas Balacheff, Daniel Alibert, Daniel Chazan, Uri Leron.
PME scientific orientation	Kathleen Hart
Discussion group on algebraic thinking	Lulu Healy, Romulo Lins, Teresa Rojano, Rosamund Sutherland, Sonia Ursini
Classroom research	Jan Van Den Brink

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PLENARY SYMPOSIUM

The responsibilities of the PME research community.

In this symposium the panel members will be challenged by the chair to specify their understanding and beliefs concerning the theme.

Questions such as the following will be debated:

- What is the main function of research in mathematics education?
- How should we choose what to research?
- Who are the best people to do research?
- Where should the research be carried out?
- To whom is a researcher accountable?
- How should researched knowledge be used?
- There will also be opportunities for questions and comments from the floor

The Knowledge of Cats: Epistemological Foundations of Mathematics Education

Robert B. Davis
Rutgers University
New Brunswick, NJ

In the recent past data from tests and questionnaires has been respected, but data from videotapes has tended to be questioned. Reasons are given why that is now changing.

Mathematics education is achieving a maturity that many of us never expected to see in our lifetimes. It is also witnessing some major changes in people's notions of what constitutes mathematics, what it means to know mathematics, and how people can best learn mathematics.

While all of this is of the greatest importance, I want to deal with another change that is probably less apparent, but may be of special significance for those of us who are concerned with research in mathematics education. Specifically, I want to consider what <u>kind</u> of knowledge one seeks when one studies the mathematical performance of human beings.

The true source of my remarks is to be found within mathematics education itself (see, for example, Davis, 1967; Davis, 1988a), but there is some measure of an external foundation, and a large part of this comes from the work of George Lakoff and Mark Johnson [Lakoff & Johnson, 1980; Lakoff, 1986; Johnson, 1987].

A Rapid Trip Through History. In the earlier years of this century, "knowledge" was thought to be very closely related to experience. Experience was greatly valued; by contrast, what was often referred to as "book learning" was usually not held in high regard. Thomas Edison once said that he did not want his son to go to college, and few presidents of the United States had been college graduates. At that point in history it seemed relatively easy to describe the kind of knowledge that one needed to acquire: one needed experience. Knowledge about the teaching and learning of mathematics was a matter of experience in teaching, learning, and using mathematics. This notion of "what constitutes knowledge" came to be challenged, probably because of the growing influence of universities.

In the middle of the century, and especially after World War II, a different view emerged, perhaps by optimistic analogy with physics: researchers pursued forms of abstract generalizations, educational equivalents of $F=ma$, or $e=mc^2$. One sought "objectivity", people spoke of "hard data" and "generalizability", and considerable attention was paid to the choices that students made among the options on multiple-choice tests. Let me warn listeners that I shall not attempt any serious history of the methods and accomplishments of this period, believing myself not to be competent for such an undertaking, in part because my own optimism in this direction was short-lived, and I quickly turned my attention to the work of Kurt Lewin and to others who saw the world much as he did. Lewin had a quite different notion of what constitutes "knowledge."

Piaget, of course, was also an early convert to a different methodology -- abandoning work on IQ tests, he announced that studying "who got the 'right' answer" was less important than understanding how subjects decided upon the various answers that they did give, which were often quite at variance with popular conventional (adult) wisdom. Everyone has noticed that Piaget was making a shift in methodology; people have usually failed to notice (indeed, Piaget himself failed to notice) that he was also shifting to a new kind of knowledge. He tried to cloak his new knowledge in conventional garments, by (for example) speaking of stages. He even tried to invoke group theory (rather as Kurt Lewin tried to invoke topology and vector geometry). But his real contribution was far deeper than this -- he gave us a new way of looking at human thought processes, and he gave us an entirely new set of expectations. I personally will never forget watching my young son confidently maintain that there were more pennies in a row where they were spread further apart. Without Piaget, I would never have thought to test such a ridiculous possibility.

So -- what is it that science (or serious research, whatever you may choose to call it) gives us? Sometimes it is, indeed, some kind of generalization that can be coded in some symbolic or abstract form: "chlorides tend to be soluble, sulfides tend not to be" or "the length of stretching is proportional to the applied force" or "$pV = nRT$" or "there are exactly 92 elements and can never be any others." [Of course, as this last generalization shows, some of these statements may subsequently turn out to be false; this is one of the most endearing aspects of science: truth is always temporary.]

Of course, as Herbert Simon (personal communication) points out, sometimes a "scientific theory" seems more like a piece of friendly advice. To use Simon's example: What is the "germ theory of disease"? Simon's answer: "The germ theory of disease is this: If you want to know why someone is ill, try looking for some kind of bug." Simon makes an important point, but please notice that what he has described does not sound in the least like the usual popular view of what "science" is!

This "germ theory of disease" is not really an abstract generalization at all. It is, in fact, a new way of thinking about some class of phenomena. This is also the kind of thing that Freud gave us (not entirely correctly, as we now know). Freud's contribution might best be described as "some new ways of thinking about why people behave the way that they do in certain kinds of situations," as when we interpret feelings by saying "I am angry at the boss because I am really very angry at my father." Here, too, the contribution or "result" is nothing like a specific, sharply defined, abstract generalization. It might better be described as a suggestion of one kind of thing that one ought, perhaps, to look for.

My purpose in these remarks is to consider in greater detail what it means to have a new conceptualization, a "new way of looking at certain phenomena." I want to argue that what it really means is to have a new collection of basic metaphors. And that, in turn, brings into question the idea of what a metaphor actually is.

We used to believe, Lakoff argues, that "metaphors" were certain kinds of things that we make use of in order to communicate with someone else. But in believing this we were mistaken. Metaphors are far more important than mere tools of communication -- they form a large part of the mental representations by means of which we think. Forget input-output operations; metaphors are essential to our own personal internal information processing.

Each of us has lived so closely with our personal collection of metaphors -- and our culture has so long relied upon its collection of commonly-shared metaphors -- that they have become nearly invisible to us. We have the idea of many things which, arguably, do not exist at all. Consider the case of street corner. We say "I'll meet you at the corner of Hollywood and Vine," and nobody notices how unreasonable this language is. There is no such thing as a "corner"! If a drug store is "on the corner", is the stationery store next to it also "on the corner"? How far back does the corner extend? How many people can stand "at the corner"? Can three people stand at the corner -- or is that too many? Perhaps only one person can stand "at the corner". Any others can only be near the corner. [This is an example from Lakoff & Johnson, but it called to my mind a recollection from my own childhood, when precisely this question bothered me: How far does "the corner" extend? Does it extend as far as the middle of the block? Can a whole drug store be "at the corner", or is it only the corner of the drugstore that can really be "at the corner"?] Yet we talk about "meeting someone at the corner of Hollywood and Vine" and never give a second thought to the thing (or should I say, "to the non-existent thing") that we have just referred to.

We say "The foundations of that theory are not clear." What is this? Does a "theory" have foundations? A building has a foundation, but can a theory also have a "foundation"? To try to put the matter briefly, hopefully without doing too much damage to the ideas of Lakoff and Johnson, one might say: We have grown up in a world where a building is built up, brick by brick, from the ground upward, until it stands there before us, awe-inspiring or beautiful or practical or ugly, and we are able to behold (and to make use of) this product of our (or someone's) construction.

Now, consider a "theory." The idea of someone thinking and testing and contradicting and concluding and generalizing and doing some more thinking and some more testing and some more contradicting and some more concluding -- how can we think about anything as elusive as that? Well, we can't. So we don't. But there is something that we can do. We can map all of this into a mental representation that we already possess. We can think of a theory as a building. We already know about buildings! [Of course, it is interesting that our mapping is not one-to-one onto; we do not usually speak of "the roof of the theory" or "the windows of the theory" or "the doors of the theory." We might say "That theory has gargoyles on it!" -- but then everyone would say that we were speaking metaphorically. As long as we stay within the boundaries of the commonly-accepted mapping of "theory" into "building", people do not recognize that we are speaking metaphorically at all, even though we are. Everybody talks about "the foundations of a theory." Indeed, everyone talks about "theory building," by analogy with putting one brick on top of another. This is so common that it has become invisible, and only by real effort can we even notice that we are doing it. But, Lakoff and Johnson would say, without it we would be unable to think.]

This is the kind of observation that you can test for yourself. You try to think about a "theory" without making any use of metaphors, without using ideas that were originally developed in your own mind in some quite different context, for some quite different purpose. [I just followed my own advice and tried it myself - I found myself thinking of "connections" -- one idea was connected to another idea. But isn't "connection" an idea that I knew about, quite early in life, and learned in a quite different context -- long before I was concerned with explicitly thinking about theories?]

Our ability to talk in this way -- more acurately, our ability to think in this way -- depends upon the collection of ideas that we have built up in our own minds. Following Lakoff and Johnson, I will usually speak of these ideas as metaphors, in order to emphasize that we are mapping new perceptions into previously-established mental representations (see, for example, Davis, 1984). Equally, a culture is in large part defined by the collection of metaphors that are shared in common by those who live within that culture. What Freud and Piaget and Pasteur gave us were major additions to our collection of basic metaphors; after that, because we had some new metaphors, we saw the world differently. We thought about the world differently.

The Main Claim of this Note: With those preliminaries completed, I can now state my main point. In recent years there has been a growing concern to give good descriptions of instances of a human being thinking about some mathematics problem (or in various other ways dealing with mathematical situations). One form of this interest is the growth of videotaped task-based interviews -- that is, videotapes that show someone working on a piece of mathematics, perhaps while an interviewer is also present. Another form consists of videotapes of actual classroom lessons, as in the extremely interesting tape "Double-Column Addition: A Teacher Uses Piaget's Theory" [Kamii, 1987]. While many of us have come to value highly such descriptions -- indeed, to see the collecting of such descriptions as one of the main present-day tasks of mathematics education -- there are others who prefer to ignore these descriptions as "merely anecdotal." Clearly I disagree with this assessment. It seems to me that those who disregard descriptions are seeing "science" mainly as a collection of some kind of abstract generalizations, whereas an equally important part of science -- perhaps a more important part -- is the collection of metaphors it gives us that allow us to think about the world in certain particular ways.

It is important to emphasize how invisible these metaphors or "basic ideas" really are. We have had them so long, and used them so often, and built upon them so successfully, that we have come to take them for granted. We can hardly imagine a human being trying to think without them. An example may help: Let us say that, not having most of these usual basic ideas, I do nonetheless have some sort of idea of "living thing" or "animal." I decide to measure the size of the animals in my neighborhood. What will I report?

Some very confusing data, you can be quite sure of that. I will report heights of 2 inches, and 14 inches, and 2.4 inches, and 20 inches, and 5 feet six inches, and eight feet, and 6 feet, and lots more. There may seem to be no pattern. But if I hold up, for a moment or two, on my busy activity of making all of these measurements, and instead try to _see_ some of these animals, I may develop a few basic ideas that will serve me well indeed: I may learn the idea of a _cat_, and the idea of a _mouse_, and the idea of a _dog_, and the idea of a _human being_, and the idea of a _horse_. Now, all of a sudden, my weird distribution of measured heights may begin to show a useful pattern, whereas before they did not. Measuring heights is one thing, but knowing what a cat is is something else entirely, and for many purposes it is more important to know the difference between a cat, a dog, a mouse, and a human being.

The Cats of Mathematics Education. All right. What are these different _kinds_ of things, these "cats" and "dogs" and "mice" of the world of mathematics education? Perhaps the main point is that most people don't know. Consider this example: Typical taxpayers in the United States read that "the math test scores have gone up this year" and they are delighted. They do not seem to know that it is possible that the schools are teaching far more directly to the tests, perhaps at a very great price in what the students are learning. (And notice the many ways that testing, and teaching to the test, may _reduce_ what a student will learn. Here are but a few (see especially Koretz, 1988): (i) If it isn't on the test, students won't try to learn it, and teachers may not dare to take the time to teach it; (ii) if it _is_ on the test, it may become so elevated in urgency that many students feel panic; this is undoubtedly a major source of "math anxiety", and may also be a source of gender differences in mathematics test scores [de Lange, 1987; Dienes, 1963]; (iii) that same elevation in urgency may cause teachers to teach it more directly, allowing the students less opportunity to assemble their own background knowledge, to consider alternatives, to ask questions, etc. [essentially, the "haste makes waste" phenomenon], so that their learning, while superficially adequate, may in fact be brittle and temporary, and may cover up deep and permanent misconceptions [Rosnick & Clement, 1980]; (iv) that testing itself takes up time that might have been devoted to other opportunities for learning [a New Jersey public school recently took stock and discovered that they were devoting 36 days each school year entirely to testing]; (v) the content topics that are on the test may not represent what students most need to learn; (vi) the kinds of behavior that students are asked to demonstrate on the tests may be mainly simple imitation and memorization, neglecting other behaviors that are of at least equal importance in long-term performance levels in mathematics [see, for example, Hall & Estey, to appear]; (vii) many students make little effort to deal with tests, and may merely put their heads down on their desks, or otherwise show clear evidence of non-participation and probably an overpowering sense of hopelessness; (viii) a heavy emphasis on tests may induce teachers to use short-cuts that, in the short run, produce higher test scores, but may actually give wrong ideas about mathematics [as when a teacher told a student that one could delete the final zero from 37.10 "because it was not held in by 'book-ends'", whereas one must not delete the zero from 37.01 "because the zero is held in by the 'book-ends' of the decimal point on the left, and the digit 1 on the right"].)

Anyone who was familiar with the many ways that teachers and students respond to various forms of testing would have many questions far deeper, and far more important, than "whether the test scores went up or down."

I do not mean to suggest that the important unknown elements -- the "cats" and "dogs" -- lie only in the area of testing or test-related matters. It would be a mistake to point to any single area of human mathematical behavior and to suggest that there is where the main zoological distinctions need to be worked out. Quite the contrary! Pick any area of mathematical behavior, and most of the ingredients are, at present, largely unexplored, unidentified, and even unimagined. How would a person go about solving the "three-switch" problem (Davis, 1985) -- or, for that matter, how would a person go about solving almost any problem? We have very little in the way of good descriptions of what someone actually did when they solved some particular problem (and much of what we do have is due to John Clement, whose work in this direction has set a world-class standard -- see, for example, Clement, 1982; Rosnick & Clement, 1980; Clement, 1988; Brown & Clement, in press -- or to Alan Schoenfeld [see Schoenfeld, to appear]). Getting this kind of data has not always been highly valued, because such descriptions have too often been seen as "just anecdotal" (although Clement and Schoenfeld themselves may have broken through this barrier, but most others have not). Well, all right, if you really don't want to know about dogs and cats. But you are going to have great difficulty in making sense out of those numbers you are carefully measuring for the height of "an animal". It really helps to be able to distinguish the mice from the horses, so you know more precisely what it is that you are measuring.

Lest I seem to overemphasize tests as the area of greatest interest, let me look at one other area. the specific treatment of highly specific content topics. This is one of the great disappointments in mathematics education. Good teachers repeatedly invent some extremely effective ways of presenting certain specific topics, but their methods are rarely passed on to other teachers, who are consequently left to invent their own methods, and the later methods are often inferior to those that were lost. Probably the specificity of the topics and the methods make transmission to others quite difficult. Also, the advantages and disadvantages are often impossible to predict in advance. We have all seen many examples. In my own case (and this is an area where we are almost compelled to use personal data, since so little shared data seems to be available in any public forum), I originally taught mathematical induction largely as I was taught it, and rather along the lines in most books. I used induction to establish De Moivre's theorem, or the formulas for the sum of the first n positive integers and for the first n squares:

P(n): $1 + 2 + 3 + \ldots + n = [n(n+1)]/2$ (eq. 1)

Q(n): $1^2 + 2^2 + 3^2 + \ldots + n^2 = [n(n+1)(2n+1)]/6$ (eq. 2)

If one watched students carefully, one saw that some made correct proofs. Many, however, wrote something that seemed to them like a proof -- indeed, it even <u>looked</u> like a proof, provided you didn't think about it too carefully. But it wasn't. In a correct proof, a key step involves getting from P(n) to P(n+1). One is supposed to do this by adding (n+1) to each side of equation (1), and then simplifying the right hand side of the resulting equation. From this one can legitimately conclude that the statement P(n) implies the statement P(n+1). By contrast, the incorrect "proof" that some students wrote involved getting from equation (1) to P(n+1) by merely replacing the variable n in P(n) by n+1. The resulting equation looked good -- but the method by which it was obtained had no legitimate logical justification.

All of this occurred at University High School, in Urbana, Illinois, a very special school for academically-gifted students. The distinction between the two methods of obtaining P(n+1) was so subtle that it gave trouble to many students, gifted though they undoubtedly were. Two colleagues, Pat McLoughlin and Elizabeth Jockusch, suggested that we <u>not</u> begin the topic of mathematical induction with <u>algebraic</u> examples such as these, but rather with examples that involve no algebra whatsoever. We did this, using problems such as:

A "checker board" type of playing board has 2^n squares along each side. One corner square is removed. Prove that the resulting board can be tiled exactly by using a three-square tile with the three squares arranged in the shape of an "L".

There are 2n points in a plane. Prove that it is possible to connect each point to one other point in such a way that the connecting line segments do not intersect. [This problem was suggested by Brian Greer, and was a welcome addition to our collection.]

When we changed our presentation in this way, always being careful to begin the topic of "proofs by mathematical induction" with non-algebraic examples, the incorrect process of merely substituting n+1 for n in P(n) was, of course, impossible, and students had no difficulty (other than the difficulty of inventing a suitable proof, of course). As often happens, once students had learned a correct notion of what was going on, when we did finally turn to "algebraic" theorems, the students were well able to deal with the "P(n) to P(n+1)" distinction that had, previously, turned out to be so troublesome.

Now in one sense this is a small detail. Of course it is. But precisely this kind of "small detail" can make the difference between a course in which students move quickly to a powerful command of mathematics, and one in which students are often confused, much time is wasted, and things don't seem to progress as we would wish.

It would seem that this kind of knowledge would be avidly sought after, and eagerly transmitted to colleagues. Few of us wish to be selfish, and to hoard whatever improved methods we may devise. Yet in fact very little of this kind of knowledge is ever shared with others, beyond perhaps colleagues in our own school. I suppose the reason is that mathematics education has never found any suitable way to report such a "result". It is rather as if we have come upon some new kind of cat, but have no means for describing it. Such small "details" probably cannot, and probably should not, be made the subject of a large-scale statistical study. There are far too many of them. We meet several, or perhaps dozens, every day. Dealing with them each in separate "scientific" studies would be unthinkable! Of course, this is a place where videotapes of actual classroom lessons may help; or one can hope that many of these ideas will be incorporated into better textbooks (but this rarely seems to happen, perhaps because prospective purchasers are not quick to see the subtle merit of the slightly different approaches). In any event, this is another kind of case where specific knowledge, not abstract generalizations, seems to be what is needed. I suppose every one of us who has taught mathematics has a large repertoire of "specific" methods of this sort. And few of us will be able to pass many of them on to anyone else!

This is a large loss. It is also a big mistake that academic life does not seem to value this kind of "knowledge". It was precisely this kind of knowledge about harmony and counterpoint and texture that let Beethoven compose his magnificent string quartets. He was lucky; the results could be widely shared, even though his actual methods of thinking about music have for the most part not been. We should no more dismiss someone as "a great teacher" -- a demeaning assessment in modern intellectual life -- than we should suppose that Beethoven was nothing more then a careful craftsman, with nothing of intellectual value to share. A "great teacher" has a very large store of valuable specific knowledge that is precious beyond measure.

Comparing Kinds of Knowledge. That brings us back to the question of "measuring". If the specific knowledge of what a "cat" is and what a "dog" is has been undervalued, is it possible that reports of measurements have often been overvalued? Clearly I would argue that they often have been, as in the reporting of test results, the meanings of which are usually not clear at all, and seem only rarely to be brought into question.

There is an intellectually trivial, but practically important, case that should be mentioned in passing: specifically, the use of seemingly careful quantitative measures, where what is being measured is almost completely undefined. As one example: Those of us who work in in-service teacher education are familiar with a few school administrations that seem to want to buy generic black-and-white labeled "teacher education in mathematics". These schools appear to have no concern as to what is taught or learned in the in-service course, nor how it is presented to their teachers. The in-service mathematics "content" might be the good use of calculators, or the advocacy of 37 (poor) ways to use calculators, or two dozen reasons for not using calculators, or how to teach tensor calculus to fourth graders, or how to conduct more efficient rote drill. The actual content seems not to matter. Yet precisely this sort of thing is sometimes "quantified." This is very much like buying "two quarts of something" without regard for what. To have any reasonable notion of what you are buying, you need to be able to specify it in far greater detail than that; indeed, in the case of in-service teacher education, you need to specify not only the content, but also the kind of experiences the teachers will have. Will they listen to lectures? Use Cuisenaire rods? Work with computers, hands-on? And what will be the purpose for the use of computers? Will the in-service teachers explore their own ways of trying to solve novel, unfamiliar problems? Or what? Will the atmosphere be challenging, or supportive, or what? Will the details of the concepts and the experiences have been carefully thought out beforehand by the teacher educators? Do the members of the (probably visiting) instructional team agree or disagree on these details? The "something" that you are buying as "teacher education" may perhaps be well enough defined if you say, for example, "two weeks of study in the First Course of the Marilyn Burns program," or perhaps "the initial course with the Mathematics Their Way group" [see, for example, Burns, 1984]. But anything much less specific than this is not really quantifiable, because you don't know what it is.

But the phenomena I want to discuss are deeper than this. Can "quantitative" knowledge mislead us in fundamental ways? Can there be important knowledge that is not, and cannot be, "quantitative"? I want to present arguments for the affirmative answer to both questions.

Are there Fundamental Reasons for Asking Whether Abstract Generalizations and Quantitative Knowledge about Learning Actually Exist?

Is "hard evidence" really evidence at all? To put this in perspective, let me ask some subsidiary questions: Should we try to use human intelligence, empathy, intuition, and all other available human resources to try to improve the teaching and learning of mathematics? Answer: Of course we should!

Is there such a thing as <u>knowledge</u> about the teaching and learning of mathematics? Answer: <u>Of course there is, and many good teachers have a lot of it</u>!

Does this knowledge take the form of abstract generalizations? [The answer here must necessarily be left as an exercise for the reader, but you might ask yourself this: What abstract proposition, if it were suddenly to come to be known to be true, would revolutionize the teaching and learning of mathematics?]

Perhaps more fundamentally, <u>do important educational phenomena replicate</u>? It has generally been assumed that they do. (There certainly are a great many unimportant educational phenomena that replicate very nicely, but when you are studying them you may mainly be studying the statistical properties of white noise.) Of course you can average over many cases. This will produce some numerical results. Unfortunately, that which is true for most people (or even true of <u>all</u> people) is nearly always trivial. (Remember John Maynard Keynes' basic economic law: "In the long run we'll all be dead.")

How Can This Be? A Practical Demonstration. How can there <u>be knowledge</u>, and yet have a situation where that knowledge is <u>not</u> in the form of abstract generalizations? Let me give, first, a practical demonstration suggesting that this might be so.

Example 1: At the time of the Cuban missile crisis, I was teaching the daughter of a TV newscaster who was involved in reporting the situation. She -- and more particularly some of her classmates -- were quite visibly upset by the crisis. So were the teachers, for that matter. Who would not be? If you knew much about what was going on at the time, you had to worry that this could be the start of World War III; and, given the presence of nuclear weapons in nearby locations, that might turn out to be a very short war indeed! One could easily imagine that democracy was about to be erased from the record of human history. I and other teachers believed that this had a very noticeable effect on the schoolwork of these students. I had never seen anything like it before, and I have seen nothing like it since. But a great many things will happen only once in a lifetime. That is to say, they may never happen again in any world that resembles the one we live in today. And they may never have happened before in any world that resembles the one we live in today.

Example 2. In the work of the Madison Project [see, for example, Davis 1988b; Lockard, 1967; Howson, Keitel, & Kilpatrick, 1981] it appeared that many of our experimental classes were greatly influenced by a few -- one, two, or three -- dominant students. The values of these students shaped the commonly-expressed values of most of the class. But these students tended to be very individualistic. Like John Kennedy and Jimmy Carter and Ronald Reagan and FDR, no two were alike. We never found the situation repeating itself. [Those who have viewed many of the Madison Project films, which show actual classroom lessons, have seen this phenomenon for themselves.]

Example 3. One of the most effective teachers I ever had was Professor Hans Mueller of the MIT Physics Department. Among his other features, he had a striking and delightful German accent [think, for example, of the Danish accent of Victor Borge]. I always felt sure that this helped hold students' attention, and certainly seemed to make physics a more "human" activity. But would anybody suggest that a foreign accent is routinely to be considered a desirable attribute for a college teacher? On the average I'm sure it is not.

Have I learned anything from all of these experiences? I would argue strongly that I have. What kind of thing that knowledge is is a matter I will return to shortly.

How Can This Be? Theoretical Considerations. Let me give three kinds of reasons for suspecting that "average" behavior may be mainly an illusion.

Reason 1. Consider the case of a random walk. We know that, if a random walk takes place along a line (a "one-dimensional random walk", as mathematicians say), then with probability 1, we will find ourselves in a small neighborhood of any point that you may select. In particular, if we have visited a neighborhood once, we know we will pass that way again. So, if the educational phenomena we care about should be represented by a one-dimensional random walk, there will be some reasonable kind of replicability.

The same thing will be true if, instead, the educational phenomena we care about should be represented by a two-dimensional random walk, by motion in the Euclidean plane (that is, there are two important variables). Here, too, each neighborhood will be revisited with probability 1.

But suppose that the educational phenomena that we care about may involve <u>three</u> variables, and so need to be represented by a three-dimensional random walk. No similar result applies. Pick a neighborhood of some point in three space, and you can no longer be confident that we will pass that way again.

Now -- how many variables are needed to describe the essential features of important educational phenomena; how many dimensions do you think are needed for a good representation of most of the educational phenomena that we really care about? Ten? Or a hundred? Or perhaps a billion? Or 10^{23}? Or how many? So, if we have a good geometric description of the space of possible situations, how likely are we ever to see <u>that</u> situation again? <u>It is next to certain that we never will</u>!

Reason 2. The educational phenomena that we really care about can not be tested independently. As someone has described it, "they must be tested as a corporate whole." Consider, for example, the case of Sharon Dugdale's computer-based lesson (on the University of Illinois's PLATO computer system) using a simulated pin ball machine. The pin ball machine simulation is attractive. Any one of us might enjoy playing it. But when Sharon first developed this lesson, she used it for the mathematical topic of <u>fractions</u>. Although its appearance was the same in both versions, the fractions game was <u>not</u> fun. It was too slow; few if any of us can solve problems involving fractions with the lightening-like speed that is required to make pin ball exciting. When -- keeping the format unchanged -- Sharon altered the mathematical content, from fractions problems to whole-number problems, the game became fast enough to be real _fun_. Now, was that a good format, or a poor one? It depended upon _just how_ it was employed! (The first college course in psychology that I ever took was subtitled: "It's not what you do, it's how you do it!") Consequently, the common idea of testing _parts_ or _single aspects_ of educational experiences or performances is usually untenable. It's really true: It's not what you do, it's how you do it! You have to test _that particular combination of things_. Testing one or two variables at a time, as is commonly done, is very often misleading. (And some people have wanted to test the proposition: "Is computer use in school really helpful?"!)

Reason 3. Thanks to James Gleick's book, everyone now knows about _chaos_ [Gleick, 1987]. It was not always so. Perhaps one should start the story in 1961, when Edward Lorenz's mathematical models for weather phenomena suddenly demonstrated an unanticipated aspect: Given the slightest, smallest change in initial conditions, the long-term consequences could be unbelievably great. [Gleick, pp. 16-18.] The phenomenon has been described by saying that the motions of the wings of a butterfly in Brisbane, Australia may cause a tornado in McCook, Nebraska. In a sense, a very small event may act as a "switch" to turn on powers far vaster than it itself, whose effects may be felt in remote times or places. Lorentz himself said: "... any physical system that behaved non-periodically would be unpredictable." [Gleick, p. 18.]

But some weird form of instability is not the only issue. There is also the question of _averages_. Given any finite sequence of numbers, you can, of course, compute an average -- but what you get is, in actuality, a number. That number may, or may no, have much meaning. If I have a random sequence of numbers, I can compute their average, but the next time I have a new random sequence, I may well compute an entirely different "average" for this new collection. There is no necessary relation between these two "averages."

In fact, there are many different kinds of sequences that may appear, whether we are studying meteorology (like Lorenz), or abstract mathematical systems (like Stephen Smale and James Yorke), or the population level of fish (like W. E. Ricker), or income distribution (Benoit Mandelbrot), or the price of cotton (Houthakker), or abstract populations (like Robert May), or epidemics of measles or polio. Gleick [p. 72] gives this sequence:

.4000, .8400, .4704, .8719, .3908, .8332, .4862, .8743, .3846, .8284, .4976, .8750, .3829, .8270, .4976, .8750, .3829, .8270, ...

and so on. You can compute the average of the first *n* of these numbers, but you may be hard-pressed to explain what that average actually means. Habit has made us all accustomed to believing that the average of a sequence of numbers has some sort of meaning, something close to the popular notion of "average." But in all of the fields mentioned above, there are now some famous examples that show that these "averages" may be little more than numbers obtained by a mathematical calculation, not easily related to real world phenomena. What these sequences of numbers are telling us is complicated, and I refer the interested reader to the relevant literature, perhaps starting with Gleick, 1987.

How Hard Is "Hard" Data? Perhaps the most remarkable aspect of the question of "hard data" is actually very simple. People, from newspaper readers to educational researchers, do believe in "hard" data. But think, for a moment, of what is actually involved. A group of 11th graders, say, come into a room to take a test on "mathematics". Two students put their heads down on the table and make no pretense of trying to answer the questions; one feels hopelessly defeated by school; the other does not, but has had little sleep for three days now, because he works nights. Another student, Bill, has memorized all of the things that the teacher has told him to memorize. He doesn't really understand it, and if you were to ask him what the role of axioms is in mathematics he would have no idea, since the teacher has not told him to memorize that, and it never occurred to him to think about why axioms were used so prominently in tenth-grade geometry. Another student has worried so much about whether she will remember all of the formulas and definitions that she is virtually unable to think about what day it is, what city she lives in, or what her name is. Alex has practiced solving type problems, and does them quickly, but also does not understand, and doesn't want to. Tom is the best student in the class when there is a really hard and novel problem that no one else can solve, but he is not at all diligent in doing homework or remembering definitions, and will get quite a few problems wrong because of large gaps in what he has attended to. Andy works hastily, and makes many errors. Jill has been absent, due to illness, and finds that many problems on the test are totally new and totally meaningless. They are also meaningless to Toby; he had planned to study hard for this test, but wrote down the wrong date on which the test was to occur; he had planned to use all of the coming weekend to study for it. Carolyn really understands mathematics, and has come into the test expecting that she will have time to derive any formulas that may be required; she will turn out to be mistaken in her estimate of the amount of time that she will have to work on each problem. [These are all like students I have actually observed; anyone who sees many classes of students could add some further types of likely behavior.] Now they take the test.

Every bit of the complexity of who the students are and what they are doing could be matched by complications in the design and selection of the material to be tested, and in the expectations of the kinds of behavior that the students should be able to demonstrate. Is "removing parentheses" actually a part of basic algebra? Must students know how to determine some of the properties of a <u>function</u> by scrutinizing a table or a graph? Should students know the abstract definition of a "function" as a set of ordered pairs? How much credit should be given for the solution of a novel problem, not expected by the students, that they can solve only by ingenuity or by the skillful use of heuristics? How much "partial credit" should be given for a correct method, if the work also contains a "minor" numerical error? Would your assessment be different if the resulting erroneous "answer" should have been seen to be clearly ridiculous?

But -- somehow -- this diverse group of human beings, with their very different patterns of behavior, do (mostly) manage to work on this not-at-all "objective" collection of problems, and to display behaviors that will be imperfectly reflected in what they write on their papers, and which will be somehow "evaluated" by essentially arbitrary weighting of what kinds of behaviors are most important, and what kinds of errors are most forgivable.

The result will be a collection of numbers that are said to constitute "hard data". How can anyone believe that hard data actually exists in the form of some abstract numbers obtained from such an operation?

If we had a large presence of video cameras in schools, regularly allowing parents and researchers and taxpayers to see the teaching, the behavior of the students, the conditions of the washrooms, and so on, would that constitute "hard data" -- harder then the test results, or less hard?

Then How Can Anyone Know Anything?

If, in fact, the kind of "data" that is commonly gathered and commonly used as the basis of studies of education does not necessarily give us a good description of reality, and if our "knowledge" is not to be caste in the form of abstract generalizations, then how can anyone know anything?

For one thing, it is possible that a human being can react to the visual image of other human beings ("body language"), to the nuances of the human voice, and to the apparent content of "ideas", in a way that is not well understood at present. Consider how messages are sometimes interpreted by machines, by trying to locate the message content through a process of matching against possible messages. This is, in effect, a search for the appropriate point in a space of possible meanings. One looks for the point that is "nearest" to the present incoming stimuli. This, then, is a question of what is the appropriate metric to determine the "distance" between different meanings. We know relatively little about the nature of this metric in human thought.

When we restrict ourselves to abstract generalizations to encode "knowledge", by implication we also commit ourselves to a set of assumptions about the kind of "metric" that humans use in trying to recognize meaning. These assumptions may well be in error. Human beings may do better than that.

Abstract generalizations are not the only possible means of recording or communicating knowledge. When education students discuss classroom behavior, they may seem to understand; yet if, at this point, the students have an opportunity to view some films of actual classroom behavior, they regularly show surprise, spontaneously exclaiming remarks such as: "Oh! Is <u>this</u> what you meant!" Clearly, the abstract descriptions have not conveyed the actual reality nearly as well as the videotapes do.

But perhaps there is a more fundamental way of looking at the situation. The videotapes seem able to show <u>possibilities</u> that the education students did not have in mind. Hence, by viewing the tapes, the students may be enlarging their basic collection of metaphors. Thus, in the sense of Lakoff, they can now think about classroom behavior (or mathematical problem solving behavior) in ways that were not previously possible for them. They now have some important new metaphors, and can see the world in a new way. This is a powerful kind of knowledge.

Giving this knowledge to students (or to other researchers) is a major part of the job of mathematics education.

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PME algebra reserch. A working perspective.

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INTRODUCTION

It goes without saying that any analysis that seeks to clarify given educational problems - and that is a prime mover in educational research - will have to make within the context of an Educational System. However, as a countermeasure in itself, as well, this study can do no more than attempt to change the conditions under which mathematics is taught in that System. Naturally, the problematic is affected and, ultimately, the methodology of the research. However, there is also the point, usually ignored, that the results are conditioned by the need of be usable, to be put to tests precisely where supposedly, it is wanted to cast light where changes in them will have to be taken into account in order to continue to move forward, to go more deeply into the facts discerned, to be able to formulate new hypotheses that have already duly consider the work done.

This point impels the issues to be closely linked to the very teaching process, that is, to the pedagogy of mathematics at least, in some of their aspects. It must be clarified, however, that all the foregoing does not imply discarding some-what theoretical problems and their logical and appropriate methods, but rather than these investigations are made within the context of much broader programs in which direct work with student and teachers is involved. And so, in this paper, it will be proposed that studies in which historical - critical analysis of the development on mathematical ideas takes on full meaning, are those in which, not only such analysis make possible, for example, the construction of learning sequences that reflect the achievements of theoretical research, but when the history of ideas is enriched by new hypotheses arising from the test of pedagogical sequences within educational systems themselves. Only in this case, would we be able to be presenting a study in both the field of education in mathematics and in the history of epistemology of mathematics.

EPISTEMOLOGICAL ANALYSIS.- BRIEF EXAMPLES.

History has once again recovered its proper dimension after having been relegated to being a pastime for mathematicians, although striking works [53] were produced, as well as general panoramas seen from a new viewpoint, [7] reaching even textbooks [18].

However, Boyer himself [8] had already offered us more profound efforts to capture other more intense moments which concerned the history of ideas. Numerous other titles could be mentioned here that illustrate this great return of history as an instrument of seeing the present. We can speak of epistomological studies that have had influence in mathematical education research. By instance, the works of Brunschvig [10], Piaget [47], and Galperin [36] have been the motive force of a new outlook with respect to elementary mathematical operations. Our ideas as to the rudimentary processes of mathematical - model construction have changed completely since the appearance of historical works on the Babylonians [46]. Our concepts on the birth of the theory of proportions, deduction, axiomatization have begun to take on subtle gradations on which were unaware before [52]. And, the didactics of mathematics also have begun to benefit from the re-encounter between history and epistemology through the history of ideas.

As was mentioned above, it is still necessary to make an analysis of teaching and learning problems in mathematics using this historical-critical method and, subsequently, to put the theoretical findings to the test in the educational systems for its experimentation and then to go back, on the basis of the practical results, to obtain a view of the problematic of the history of ideas that corresponds to the didactic results.

It would seem extremely important at this point to go back to history and analyze the works of the Middle Ages in this respect. We owe a great debt in that area to the historians (see[12], [13], [19], [45], [37], for example) since their recopilations, translations, and commentaries give us live material which are at hand for whoever approaches them with a fresh outlook" on the problematic of the teaching of algebra at the very moment when algebra is ready to admit the introduction of analytical ideas in geometry and, immediately after, the methods of the Calculus.

An example: The Appearance of the Arithmetical - Algebraic Language

Oneway - The epistemological analysis. Thus, to understand the jump from arithmetic to algebra and the appearance of arithmetical-algebraic language; one has shed light on the period immediately previous to the publication of the books of Bombelli and Vieta.

Vieta's "Analytic Art" shows the construction of an algebraic language where, besides being able to model problematical situations resolved in the languages used by Bombelli and Diophanthus, we can also find language to describe the synthesis and algebraic properties of operations presented in the oldest texts. There, however, they were only employed to resolve and be used problem by problem, while, in Vieta's case, there is the possibility of describing the synthesis (algebraic theorems) and the syntactic properties of the operations. All of these can be described on this language level and besides this, can be added to the body of knowledge of those who dominate this new language level.

In the following sections we shall describe language levels before the introduction to the language of Vieta's "The Analytic Art". We shall try to illustrate some differences between the Abbacus books and the "De Numeris Datis".

The Abbacus Books

According to the compilation work of Van Egmond [19], the Abbacus Books represent the most feasible means of assimilating oriental mathematics to Western European civilization. Thus, a new mathematics takes shape from the adaptation of Indo-Arabic mathematics to the particular problems of a society in real economic ascent (the Italy of the XVth and XVIth centuries).

This mathematics is ready and available to be applied in the so-called Abbacus Books, whose contents, essentially, embrace the presentation of the positional system of Indo-Arabic numbering, the four elemental arithmetic operations and resolved commercial problems. These problems involve four elemental operations, the use of the rule of three, simple and compound, of simple and compound interest, and of the solution of some simple algebraic equations. Some books include multiplication tables and equivalent monetary, weight and measurement systems.

The Datis

In contrast with the books of Abbacus, which were used as elemental algebra texts in intermediate education to be applied to commercial life, Jordanus de Nemore's 'De Numeris Datis' [39] was a text directed towards university students of the period, with the aim of presenting and teaching solutions to non-rutinary 'algebraic' problems. In effect, the Datis offers a treatment of quadratic equations, both simultaneous and proportional, which presupposes the handing of equivalent contents as those in the "liber algebre" of Alkhwarizmi [2] and the "Liber abaci" of Fibonacci [21]. In both texts they begin with some definitions and the development of the equation $x^2=bx$, $x^2=c$ and $bx^2=c$, arriving very rapidly at the equations $x^2 + bx = c$, $x^2 + c = bx$ and $bx + c = x^2$.

According to Hughes [39] , both the kind of problems in the Datis and the way of presenting these belongs to a more advanced stage than in the other two works (currently know as elemental algebra). From this point of view (says Hughes) the role played by "De Numeris Datis" in mathematical history is comparable to that of Euclid's Data [20], in the sense that the former is the first book of advanced algebra while the latter is the first book of advanced geometry and which supposes a good background in fundamental geometry (contained in "Los Elementos") in order to present the ambitious student with the proof and solutions of theorems and non-standard problems by the method of analysis.

The difference of stages, indicated by Hughes can be noted in a revision of some problems from the "De Numeris Datis" and comparing, on the one hand, the method of presentation with, for example, problems of the "Trattato Di Fio retti" [45]. In the Datis propositions one has to find numbers, of which some numerical relations are known but these are given by constants. That is, one states, for example, that the sum of three numbers is known ($x+ y +x = a$) in place of which the sum of three numbers is equal to a certain number, let's say 228 ($x +y + z = 228$, as appears in Abbacus). On the other hand, the sequence of solution of Datis problems shows explicitly, the reduction of the new problem to one previously resolved. This kind of sequence is not completely absent in the texts of Abbacus. That is to say there also appears in the Abbacus problems the repeated application of rules or algorithms when the procedure for solution has led a well-identified situation in which the afore mentioned application is feasible.

Nevertheless, the text of Abbacus do not display in an explicit way, the intention of reducing to situations, to previous found and resolved forms. In the Datis this does form part of the method of solution.

This could be attributed to the fact that expressions like $x+y+z = 8$ and $x + y + z = b$ with $8 = b$ are not fully identified as equivalent with respect to the procedures and strategies for solution. These, in Abbacus depend strongly on the specific properties of the number a (or b) and of its relations with the other numbers which intervene in the remaining equations of the system in question.

It could be said that the De Numeris Datis in this sense has reached a more evolved stage, given that, through identification of more general forms it allows grouping into large families, problems which could be solved in the same way. This is not meant to imply however that those strategies and skills necessary for the solution of the problems in the Datis have reached a higher or more evolved level of abstractions in the symbolism than those developed by the texts of Abbacus.

The Appearance of Arithmetical-Algebraic Language

The first thing jumps to mind when faced with problems and solutions like the ones in Abbacus books is that nobody now uses this language. Perhaps with the translation written in current algebraic symbolism they could take on the appearance of typical advanced problems in a modern text, but they differ from these in that the strategies employed to obtain a result do not obey our habits. Apart from this, we would not effect many of the operations and intermediate steps which are apparently necessary there. Today, if we see if from the point of view of structured adult language, the language of the books of Abbacus and of the De Numeris Datis are dead tongues. The translation of these tongues to a live language impresses by the novelty (to us) of actions which lead to the same results as ours. These, however, follow unimagined routes and for their presence problem by problem, book by book, show us skills bearing no relation to those we have developed and utilize with our algebraic language. What is more, we were never impelled to build or employ them in order to confront problems with our skills and arithmetic knowledge.

The other way around: Empirical Analysis.

The contrasting of these language levels can be found in [24], [25], [26], [27], where they also analyze diagnosis made through questionnaires and clinical interviews realized with students of 12-15 years old which cover areas similar to the development of the level of algebraic language which these students develop.

We next enumerate two epistemological theses which guided these investigations.

Is is almost obvious that when a new conceptual apparatus is constructed and this is imposed on us without being structurally sound, together with others older and strongly-rooted, the new skills tend to shade the old. Given the fragility with which, in this moment, the new procedures, the new resolution techniques, etc., can be used, we find that even problems which had been previously dominated become difficult to model in the new language in which the incipient conceptual apparatus expresses itself.

But it also occurs that the well-anchored intellectual structures tend to perpetuate themselves and oblige us to review situations which in the new language, when taking shape, could be resolved with simple and routinely operations.

The Synthesis

We recall that we began with the reading and interpretation of ancient texts (epistemological level) and we have jumped to the plane of psychological processes. We put forward the view that it is precisely this jump there and back that allow us to offer hypothesis based on the construction of the general knowledge, and convert them (by way of a metaphor), into hypotheses on the didactical aspects of mathematics. We will attempt therefore to relate that process at the level of the individuals, of the children which in the case that concerns us, where the arithmetical language will have to give way to the algebraic, which will be increasingly more pertinent, even for those situations which have always been modelled in arithmetic.

The construction of the new language which will have to come from the elemental arithmetical operations will (as already mentioned) need to operate new objets. These will represent not only numbers but also representations of these as individuals (as unknowns, for example) as an expression of the relations between number sets (the proportional variation, for example), as functions, etc. Algebraic language tends to be built on new objects whose operations will not be totally determined until the contours of the objects' new universe are well delineated and these will not be well defined until the new operations are structurally finished, in its semantic and syntactic aspects.

These steps in the construction of semantic and syntactic fields corresponds to a stratification in the actions which finally have to be identified in an operation of the new level of language.

As an example, we note the constant observed in children constructing intermediate stages that carry the possibility of solving problems which are able to be modelled by the equation $Ax + B = Cx$ and therefore to the elaboration of the necessary syntax for the solution of such equation. This has a correlation in the evolution in the books of Abbacus and the next steps as, for example, is expressed in Vieta's "Analytical Art".

EXPERIMENTAL WORK

Observation in Class. One of the simplest phenomena that come to light from the observation in class of permanence phenomena at a "reading level" with children who have just finished primary education (around 12 years old) its the one appearing when faced with questions of the following type:

Evolution scheme of the equation Ax = B

1.- $3 \times \square = 12$

2.- $3 \times \square = 672$

3.- $\bigcirc \xrightarrow{\times 3} 672$

4.- $3 \times X = 672$

5.- $3X = 672$

Between ages 10 and 12 it is easy *"to center"* some of the students on the "reading" of all questions in the same manner as in [2]: what number, when multiplied by 3, yields 672?

When analyzing answers given by children of those ages, besides ascertaining that such questions are perceived as different, since some of them can be answered and others not, we find that it is fairly easy, with students of a certain profile, to succeed in "centering" them on the use of the preferred arithmetical method, which is trial and error. One can even lead them to keep on using such a method for a long time, in spite of the fact that B numbers, in the equation Ax=B, keep getting larger and larger; eventually, this situation places them in a position where they no longer possess enough arithmetical abilities to be able to answer such a question without making some mistake.

Throughout the first year of secondary school (in the Mexican Educational System), most students come to prefer the method of dividing B by A to solve the equation $Ax = B$, which is the intended objective of mathematics programs in this cycle. However, the same phenomenon mentioned above reappears, with students who had already achieved a great amount of operating abilities for solving all equations of the first degree, when the context in which the equation $Ax = B$ stems from the analysis situation during the resolution of a word problem.

But even more strikingly, it also happens when the expression $Ax = B$ occurs -written by the same subject being observed- , and this symbols are not recognized as the expression of an equation which a few moments before could be operatively handled to arrive at a solution. The context in which the equation appears, even in its written form, causes the formerly acquired operability to be "forgotten", and the subject once more shows a preference for the arithmetical method of trial and error; in some cases, the difficulties reach such a point that no solving method can be put into play. A more careful description of what is happening here, in the latter case, shows that it is the interpretation of the x symbol what becomes crucial for the decoding of $Ax = B$; thus x is interpreted as "an unknown", and the subject does not know that to do, for - in his own words- we are dealing with something which is not known" (it should be borne in mind that we are at the time in teaching when we are trying to succeed in starting the students in the use of what he or she has learned about the solving of first-degree equations, such method to be applied to the solution of problems appearing in mathematics, physics, and chemistry classes, among others).

These observations can easily be done in the classroom, and it is possible to infer there, that these facts are linked to many others, which are instances of the intrinsic difficulties which the learning of algebra presents: the usual errors when working operatically with algebraic expressions; translation error when using algebra to solve problems written in the usual language; erroneous interpretations concerning the meaning of algebraic expressions (given the different contexts where they appear); difficulties to find any meaning; the impossibility of using algebra to solve usual problems, etc.

Experimental Observation

In order to observe these phenomena with greater precision it is necessary to have recourse to an experimental situation which permits to control some disturbing factors which are always present in the classroom, i.e, to possess observation mechanisms which permit a more thorough and accurate analysis; this, however, in such a way that the situation observed does not have to do only with the problems presented by the subject under observation; it is necessary that the components which teaching puts into play are also present.

Five years ago, at the **Centro Escolar Hermanos Revueltas** in Mexico City, an experimental design was mounted whereby the teaching of mathematics, throughout the six years of secondary education, could be controlled from the standpoint of the teaching objectives aimed at, and also possessing a control on the teaching strategies employed throughout the whole of the middle-basic education cycle. Furthermore, a laboratory for clinical observation was installed, where individual or group interviews can be performed, with an option to videotape them. Clinical interviews possess a structured format; yet, the interviewer can move freely between each one of the previously designed steps, allowing it to be the interviewed subject's line of thought the one that defines each of the subparts in the interview. Except in those cases where the subject has no problem at all to solve the proposed task, the interviewer intervenes by proposing new questionings that allow the subject to learn (through discovery) the task which he or she initially could not solve. The idea is to discover the difficulties posed by the learning of initial algebra, given the usual ways in which at present this subject is tried to be taught. These are clinical interviews where the observation focus is placed on the usual teaching methods, and on the peculiar ways (along with their typical obstructions and difficulties) that subjects present during learning.

On the basis of this infrastructure, the project Evolution of Symbolization in the Middle-Basic School Level Population has been developing, and as a part of it, the study, Acquisition of Algebraic Language has centered on the interrelationships between two comprehensive strategies for the design of learning sequences that cover long periods of time, for the middle-basic algebra curriculum. These sequences are:

a) The modelling of "more abstract" situations in "more concrete" languages, in order to develop syntactic abilities.

b) The production of codes for the development of problem-solving abilities. The use of syntactic abilities for the development of problem-solving strategies.

Broadly speaking, through a) it is intended to give meanings to new expressions and operations, by modelling them in more concrete situations and operations. Under b), (in such a way that problem resolution codes are generated) the idea is to give to the new expressions and operations senses that arise from the fact that certain abilities in the syntactic use of the new symbols can be counted on, as well as on their use in a "more abstract" language.

The Theoretical Framework

Leaving aside empirical observations such as the one described in the first paragraph of this Section, the theoretical guidelines of this project derive, essentially, from three components: an epistemological one based on the analysis of Middle Age and Renaissance texts (a description of it can be found in [28] and [30]; a second line comes from semiotics, which is intended to be a guide for the analysis of algebra, when the latter is considered as a language (see, Bibliography, in U. Echo's work [17]); and lastly, cognitive psychology, with its recent developments in the field of language acquisition and its relationships to a language pragmatics, has been an invaluable theoretical source (see for instance, Series [50] and [51]).

To develop in a precise way the theoretical model which we are using is not within our scope in this moment (a brief description will be found in a latter section). We will here limit ourselves to appeal to the reader's intuitive concepts concerning terms such as semantics, syntax, semantic charge, more concrete or more abstract language level, the reading level of a text, and so forth. We do this, in spite of the fact that one of the consequences of the interpretations derived from these studies is, precisely, that many of the usual errors which are committed when using new expressions stem from the subject's anticipatory mechanisms when he or she is decoding a situation that must be modelled in that language, and where the semantic charge produced by the subject's previous experience plays a decisive role in the trust that statements given to some of the proposed problems are valid in themselves, even when "<u>read</u>" within the perspective of a different theoretical frame-work, and that "<u>facts</u>" described, even when given other interpretations, possess and intrinsic interest.

In a series of articles describing the results of our project, OPERATING THE UNKNOWN AND CONCRETE TEACHING MODELS (Filloy/Rojano), we tried to approach various aspects on the interrelationship between semantic and the syntactic components of the problem, seen from the point of view of teaching strategies for types a) and b) which have been briefly described above. As the title suggests, these articles focus on type a) strategies, and on the moment in teaching when the aim is to teach how to operate the unknown that appears in first-degree equations. We did not approach there the analysis of what happens when a totally syntactic model is used as a teaching strategy, although we anticipated that also in this case phenomena possessing the same nature as the one described here for concrete models, become present. It will not be missed on the reader that the aspects of type b) strategies also appear here when describing the mechanisms that come into play at the time when abstraction processes of the operations unchain themselves. Nevertheless, the whole focus is placed on type a) teaching strategies, on their relationships with the appearance of usual syntactic errors, on their differences from one model to the other, and on the relationship they maintain with respect to the subject's previous attitudes, especially in terms of extreme positions between the clearly syntactic and the purely semantic tendencies displayed by the subjects. In this article, emphasis will be made on the abstraction processes of the situation posed, as well as on the operations involved.

The general description of the contents in that series of articles shows that there exists a dialectic between the syntactic and the semantic progresses, and that progress in one of these two components implies progress in the other one. The analysis is made from the point of view of the usual strategies in the teaching of algebra. The starting point is the belief that the "facts" reviewed are not taken into consideration by the present educational systems, and that the various misconceptions and errors in the use of algebraic properties that are intended to be taught for the first time, are left to the later rectifications that the students might be able to achieve spontaneously. In the rest of this section we will present a brief summary the contends of those papers:

First.- <u>The solving of equations and the transit from arithmetic to algebra.</u>

We present here the theoretical and empirical background which is relevant for the proposed problem and above all for the determination of the moment, in the development of the algebra curriculum where experimental observation will take place.

Second.- <u>Concrete Modelling in a transition moment.</u>

The moment of observation is described, from the point of view of previous teaching, and also the population from which subjects are taken to perform the case studies that conform the clinical part of the study. This population is classified according to their previous abilities and knowledge, and an argument is advanced on why, for the study we describe here, work is done with subjects in the class called "higher stratum".

Third.- <u>Abstraction processes of the operations, from the use of a concrete model to learn how to operate the unknown.</u>

The description here, initiates with the performance of the subjects observed, after an instruction phase aimed at the operation of the unknown, based on the modelling of the equations in "<u>concrete</u>" contexts. A brief description is made of empirical results obtained, in order to have some referents that permit us to make a description of the interaction processes between the semantic and the syntactic aspects that become present in the acquisition of the early elements of algebraic language.

Fourth.- <u>Algebraic semantics versus algebraic syntax.</u>

Confronted here are two canonic subject's attitudes in the learning and use of mathematics, which possess specific characteristics in the case described: the application of the same model to operate the unknown. Two contrasting cases have been selected: one totally learning towards a syntactic attitude, and the other one being purely semantic.

Fifth.- <u>Modelling and the Teaching of Algebra.</u>

An analysis is made of concrete modelling as a teaching strategy for algebra. It is observed that its strengths sometimes become weakness, when such modelling is framed, as is done there, in the context just described: the operation of the unknown and the generation of some of the usual syntactic errors.

The results obtained in that part of the work allow us to assert that the rectification of algebraic syntax errors and of the operational vicissitudes which become present in the middle of complex processes of resolution of problems or equations, which are generated during the learning of algebra, cannot be left to the spontaneity with which children make use of the first elements with which they are provided to make incursions into the terrain of algebra, because the paths marked by such spontaneous developments do not go in the direction of what algebra intends to achieve; this is precisely the reason why such a rectification is a task for education. Therefore, if we are trying to introduce certain algebraic notions by means of models (including the syntactic model) it would be convenient to bear in mind which are some of the main components of modelling.

Modelling has two fundamental components. One of them is translation, by means of which new objects and operations being introduced, and which appear in abstract situations, are endowed with meanings and senses in a more '<u>concrete</u>' context; in other words, through translation; such objects and operations are related to elements pertaining to a '<u>concrete</u>' situation; the latter is a state of things representing, in turn, another state of things in the more abstract situation (in the case of the geometric model, equality between areas or magnitudes corresponds to an equality between algebraic expressions); thus, starting from what is already known at the more '<u>concrete</u>' level, about the resolution of such situations, operations are introduced which, even if performed on the '<u>concrete</u>', are also intended to be done on the objects pertaining to the more abstract level; for this reason, a two-way translation becomes necessary from one context to the other, in order to make it possible to identify each operation in the more abstract level with the corresponding operation in the '<u>concrete</u>' model.

A second component of modelling is the separation of the new objects and operations from the more '<u>concrete</u>' meanings which were introduced, i.e., modelling also tries to do what Mt. (the case of syntactic tendency mentioned in section IV) attempts from the beginning, namely, to detach herself from the semantics of the '<u>concrete</u>' model, since, ultimately, what we seek to achieve is not to solve a situation which we already know can be solved, but to find the ways of solving more abstract situations by means of more abstract operations. This second component is one of the driving principles which guides the function of modelling towards the construction of an extra-model syntax.

These studies which we are reviewing here shows that mastery of the first components of modelling (translation) can waken or inhibit the second one; such is the case with subjects who, achieve a good handling of the '<u>concrete</u>' model, but who, because of this very fact, also develop a tendency to stay and to progress within that context. This fixation on the model runs against the other component, that of abstraction of the operations towards a syntactic level, which would presuppose a breaking away from the semantics of the '<u>concrete</u>' model.

What we are remarking on the interaction between the two basic components in cases of a syntactic tendency, obstructions are generated during the processes of abbreviation of the actions and while producing intermediate codes (intermediate between the algebraic concrete situation and the algebraic syntax level); these are obstructions to the processes of abstraction of the operations effected on the 'concrete' model, and they are due to a lack, at this transition period, of adequate ways to represent the results or the states to which operations lead. Once more, this is a shortcoming in the second component of the modelling action.

The obstructions just pointed out constitute a sort of essential insufficiency, in the sense that modelling (when left to the spontaneous development on the part of the child), upon being strengthened in one of its components, tends to hide precisely that which, essentially is attempted to be taught, namely, <u>new concepts and operations.</u>

This sort of dialectics between the processes belonging to the two modelling components should be taken into consideration by teaching, and an attempt ought to be made at a harmonious development of both types of processes, in such a way that they do not obstruct one another. From the analysis of the cases presented here, it is indeed clear that this is a task for education, given the fact that the second aspect of modelling: the breaking away from previous notions and operations, on which the introduction of new knowledge finds support- is a process consisting in the negation of parts of the model semantics; these partial negations take place during the transference of the use of the model, from one problem situation to another (in the case of the geometric model, this is a transference of its application from one equation mode to another); but when this generalization in the use of the model is left to a spontaneous development on the part of the child, the partial negation can happen in essential parts of the model (in the geometric model, the presence and the operation of the unknown are negated); for this reason, the intervention of teaching becomes necessary in the development of these processes of detachment from, and negation of the model, in order to guide them towards the construction of the new notions.

The transference of the problem situation, semantics versus algebraic syntax, to a level of actions in modelling, permits the closing of gaps between teaching and such a problem situation, since, through the analysis of this interaction at this new level, didactics phenomena come to light which point out as necessary the intervention of teaching at key moments of the processes which are unchained during the initial stages of algebraic language acquisition.

BRIEF DESCRIPTION OF THEORETICAL FRAMEWORK OF THE MEXICAN TEAM

The stability of the observed phenomena and the well-established replicability of the experimental designs that were used in our studies confronted us with the need to propose a theoretical component to deal with different types of algebra teaching models for the teaching-learning processes together with (2) models for the cognitive processes involved, both of which are related to (3) formal competence models to simulate the competent performance of an ideal user of elementary algebraic language.

It was necessary to concentrate on local theoretical models appropriate to specific phenomena, which were nevertheless able to take account of all of these components; we also proposed ad hoc experimental designs to throw light on the interrationships and oppositions arising during the development of all the processes relevant to each of these three components.

Mathematical sign Systems.

We needed a sufficiently broad concept of mathematical sign systems (henceforward referred to as MSS) and a notion of sign meaning that embraces both the formal mathematical meaning and the pragmatic one.

We also needed a concept of MSS that was efficient enough to deal with a theory of MSS-production in which we would be working with intermediate sign-systems used by the learner in the learning/teaching process, during which the subject would have to rectify the use of these intermediate MSS so that, at the end of the teaching process, the student becomes competent, which is the educational goal of any teaching model.

These local theoretical explanatory models have to deal with at least four types of sources of meaning (see Kaput J. [3]):

1. By transformation within a particular MSS without reference to another MSS.

2. By translations across MSS.

3. By translations between MSS and non-mathematical sign systems, such as natural language, visual images and the behavioural signal-systems used by the subjects during the learning/teaching processes that permit us to observe the learner's cognitive processes and on the basis of these psychological results propose new hypotheses for a "mathematical didactics" analysis of the teaching models involved in the experimental design of the local theoretical model under study.

4. With the consolidation, simplification, generalization and reification of actions, procedures and concepts of the intermediate MSS created during development of the teaching sequences proposed by the teaching model component of the theoretical local model under study, these intermediate MSS evolve into a new "more abstract" MSS in which there will be new actions, procedures and concepts that will have as referents all the relevant actions, procedures and concepts of the intermediate MSS for their use in new signification processes. If the goals of the teaching model are achieved, the new stage has a higher level of organization and represents a corresponding new stage in the cognitive development of the learner.

Wile the first three sources of sign-functors (translations, following Kaput's terminology) represent means of dealing with primitive expressions and means of combining them, the fourth represents means of abstraction, by which compound objects can be named and manipulated as units and afterwards used in signification processes to solve the new problem-solving situations proposed to the learner in the teaching sequences of the teaching model theoretical component. If, as is the case, we have to work with mathematical learning/teaching processes, there is no way to avoid having these means of abstraction as our main focus of observation and we need a theory of MSS-production in which an abstraction-functor relates the different intermediate MSS (used during the development of the teaching sequences) with the final more abstract MSS (the goal of the teaching model under study). Later on, a mathematical didactic analysis could interpret this psychological evidence to propose related hypotheses to be observed by its own methodological means.

There is a MSS (with its corresponding code) when there is a socially conventionalized possibility of generating sign-functions (through the use of a sign-functor), whether the domains of such functions are discrete units called mathematical signs or vast portions of discourse (which we will call mathematical texts) in which a mixed concatenation of signs is produced using signs coming from different sign-systems (including natural language ones and the learner's personal signal-systems mentioned above), provided that the functional correlation has been previously posited by a social convention, even in the cases in which it is ephemeral as in the case of the didactic signal-systems appearing during the intermediate steps of the teaching sequences of certain teaching models (balances, piles of rocks, spreadsheets, Logo environments, diagrams etc). There is, on the contrary, a communication process when the possibilities provided by a MSS are used in order physically to produce expressions for many practical purposes. These performing processes require signification processes, the rules of which (the discursive competence) have to be

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taken into account by the theoretical cognitive part of the production of mathematical signs only insofar as they have already been coded because, as we have already mentioned, we are interested in observations in which new knowledge is acquired by the user with the expansion of those intermediate mathematical signification systems to new ones which embody them.

Stratified MSS and teaching models

What we use in order to think mathematically and to communicate our thoughts to others is a collection of stratified MSS with interrelated codes that allow the production of texts the decoding of which will have to refer to several of those strata; the elaboration of the text will use actions, procedures and concepts whose properties are described in some of the strata.

Two texts T and T′, both produced with a set of stratified MSS L will be called transversal if the user cannot elaborate T as in the decoding of T′ - that is, if T is not reducible to T′ with the use of L. Usually what happens is that the learner can elaborate T and T′, but cannot recognize the two decodifications as a product of the use of the same actions, procedures and concepts of the different stratum of L.

If we now have another stratified MSS M in which T and T′ can be decoded and the elaboration of both can be described through the same actions, procedures and concepts in M, the meaning of which has as referents the actions, procedures and concepts used in the decoding of T and T′ in L, then we will say that M is a more abstract stratified MSS than L for T and T′.

To accomplish this, the actions, procedures and concepts; used in M have lost part of their semantic-pragmatic meaning" they are more abstract.

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This brief description of how we can define the abstraction-functor allows us to give a preliminary definition of a Teaching Model as a set of sequences of mathematical texts T_n, the elaboration and decodification of which by the learner enables him at the end to interpret all of the texts T_n in a more abstract MSS F, whose code makes it possible to physically produce the texts T_n as messages with a socially well-established mathematical code, as was presupposed by the educational goals of the Teaching Model.

The analysis of how these processes of decodification to the sequence of texts T_n are better accomplished by the learner to become a competent user of the MSS F (as described in the formal competent model component) is part of a mathematical didactics study of the Teaching Model, which will have to take into account the cognitive processes described through the cognitive model component of the local theoretical model under a PME study.

RESULTS OF RECENT RESEARCH INTO PROBLEMS OF LEARNING ALGEBRA USED AS THE CORE FOR IN-SERVICE COURSES IN THE TEACHING OF MATHEMATICS.

In Mexico the need to apply a theoretical approach to the problem of teaching mathematics began to become evident towards the middle of this century. From the outset, this new awareness attracted the attention of groups of mathematicians, educators, psychologists and epistemologists, giving and impetus to new study programmes at all levels of the educational system. This activity resulted in many areas of enquiry being thrown up which had not previously been studied and which posed awkward problems.

Changes in mathematics curricula made it essential for teachers to have knowledge that was in accordance with the new ideas on mathematics teaching. There was also a need for researchers in the field.

The mathematics teaching section of the IPN's Research and Advanced Studies Center (Centro de Investigación y de Estudios Avanzados) has been training people to master's and doctoral degree level in the field of Mathematics Teaching since its formation in 1975. Four years later, it started the National Programme to train Researchers in Mathematical Education for the state universities, with the Ministry of Education, through the Directorate General of Scientific Research and Academic Achievement, part of the Sub-Ministry of Education and Scientific Research (Subsecretaría de Educación e Investigación Científica, SESIC).

The training of research personnel has made it possible to set up Centers of Research and Teaching in Mathematics and academic units (known collectively as REGIONAL NODES) in a number of universities.

The National Mathematics Teacher Training and Further Education Programme (NMTTFEP) started in 1984 with the support of SESIC through the University Academic Units Network made up of 16 state universities and 10 regional Technology Centers.

The national Mathematics Teacher training and further education Programme (NMTTFEP)

Aims

Promote interaction between groups of experts and researchers on the one hand, and practicing teachers on the other, in order to:

- Propouse curriculum changes

- Produce new teaching materials

- Design or implement evaluation techniques to assess the performance of the school system.

- Provide external support to the teacher in the classroom.

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- Propouse alternative solutions to mathematics teaching and learning problems.
- Train educators with the capacity to develop a fresh approach in schools.

Curriculum Organization

The curriculum is divided into three complete phases, completion of each phase entitling the participant to receive a diploma. The phases are:

I. Training phase:

The course content in this phase is basically the same as that of the courses given by the teacher at the level at which he or she will be working. The aim is that the teacher should be fully acquainted with the mathematics he or she will be teaching. Subjects covered will be Algebra I, Trigonometry, Analytical Geometry, Euclidean Geometry, Differential Calculus, Equations and Matrices systems, Graphically expressed functions, Integral Calculus, Probability and Statistics.

II. Further mathematics training phase:

The aim of the courses in this phase is that the teacher should increase his mathematical knowledge beyond the level to which he teaches, in order to enrich his teaching work and, where appropriate, enable him to tailor courses and lessons to the place and conditions in which he is working. Nine subjects are dealt with in this phase: Set Theory, Modern Geometry, Mathematical Proof, Vectors and Spatial Analytical Geometry, Algebra II, Linear Algebra, Computing, Advanced Calculus and one of the following options: Computation and Numerical Methods or Probability and Statistics. The Advanced Calculus course also consists of two options: Introduction to Analysis and Vectorial Calculus.

III. Further teacher training (subject related):

The content of these courses provides the teacher-pupil with the basis necessary to have a better understanding of the mathematics teaching and learning process, and also to interact with colleagues as well as experts and researchers in the educational field in order to produce new technological resources and new knowledge to improve teaching. The subjects in this phase are Conceptual Development of the Calculus, Algebraic Conceptual Development, Basic History of Geometry, Didactic Theory of the Calculus, Didactic Aspects of Proof, Psychology of Mathematics Teaching, Errors in Algebraic Syntax, New Teaching Methods: Audiovisual workshop, the teaching of Algebra, the Computer in Mathematics teaching and Evaluation.

Research in Mathematics Teaching and Mathematics Teacher Training.

In the further teacher training phase two types of course may be distinguished:

A. Courses whose main purpose is to provide the teacher-pupil with elements of theory which will help them to increase their understanding of the mathematics teaching learning process.

B. Courses whose main purpose is to enable the teacher-pupil to intermingle with experts and researchers in the education field in order to do research to obtain new technological resources and new knowledge leading to the transformation of the country's education system.

Description of two courses for teachers based on recent research on the teaching of Algebra

Example of a Type A course

The "Teaching of Algebra" course was held between May and July of 1987 at the School of Mathematics at the Universidad Autónoma de Yucatán, as part of the further teacher training phase organized by the NMTTFEP. This was a "Type A" course.

Those taking the course are teachers of mathematics at upper middle and upper levels in the states of Yucatán and Campeche working in technical universities of government schools. The course was divided into twelve weekly sessions, each lasting approximately two hours, with the teachers participating actively through workshops and activities to encourage the exchange of experiences and ideas and thus further enriching the course.

The following topics were studied:

i. General survey of the problems of teaching algebra.

ii. Syntax errors.

iii. Solving methods and remedial teaching.

iv. Problem solving.

The plan of the course shows the activities and evaluations carried out in each session and the work done by teachers between sessions. The course was designed in such a way as to provide teachers, before the session, with material and related reading schemes which would provide a conceptual basis for the topic to be studied and would be helpful in the discussion and in reaching conclusions.

Workshops were also held in error classification, where teachers had the opportunity to exchange experiences and to comment on the research findings related to the teaching of algebra from other parts of the world, such as the studies by Matz and Booth carried out in the United Stated and England respectively.

THE NEED FOR NEW CURRICULAR MATERIALS FOR WORKING WITH TEACHERS OF MATHEMATICS

Our presentation has shown one way in which the results of research into the teaching of Mathematics are being used in Mexico to bring about change in the country's education system, namely, by means of courses within a teacher training program (NMTTFEP) based on the discussion and reproduction of certain recent experiments in the field of research, the case illustrated here being the teaching of algebra. Next, a brief presentation was made of the type of theoretical framework used by the course designers (E. Filloy, M. May, E. Peraza, T. Rojano and M. Trujillo) to carry out their own research work (see the Proceedings of the Psychology of Mathematics Education Group for the last five years, where a description of it can be found [4]).

This was done to contrast the theoretical framework with theoretical references used when working directly with teachers on the courses we have described, which are of a very different nature. These courses are based on the discussion of work carried out by other teams; their explicitly stated theoretical viewpoints are studied with the teachers taking the course, as are the implications of the way problems are stated, the design of the experiments, and the way they were set up, on which the manner in which the data was processed is included as evidence. A study is made, jointly with the teachers, of the results that appear to be most important for them in their day-to-day teaching. We have tried to make clear that, in our work with teachers, there is a need to develop special new curricular materials to introduce new problem areas being brought out by research in mathematics education all over the world, as well as new methods of understanding these problems, novel ways of setting up experiments, experimental technique, the use of new methods of processing information, etc. These courses start from the assumption that it is possible to discuss recent research results with practicing teachers in the Mexican education system and, therefore, to develop special materials incorporating theoretical insights derived from areas of knowledge very far removed from what the teacher has hitherto had access to (cognitive psychology, artificial intelligence, psycholinguistics, mathematical didactic theory , to give some examples); the description given here of the theoretical framework used by the Mexican team to design, interpret and correct its experiments is very closely linked with a conception of algebra as a MSS (as described above).

From these courses, not only can an accurate view of teachers' opinions on the problems being considered be obtained, but a new area of discussion can be proposed to them, where prejudices rooted in years and years of teaching cease to be evident, at least to start with. The whole conception of discussion with teachers (the real backbone of the courses) is based on stating "facts" which are not know to the teachers and only recently published. This enables teaching problems to be considered from angles which are completely new to them. Working together with researchers becomes a collective activity which is not hindered by prejudices formed as a result of past practice, but proceeds as an innovative effort by all concerned, namely, teachers and researchers. From the experimental results, the teams so formed put forward new curricular ideas to be used by all teachers in the Mexican educational system. Meanwhile, all the time new hipotheses are emerging, which will serve as the basis for further joint activities.

In conclusion, to use the very suggestive and plentiful theoretical and empirical research findigs all over the world, new curricular designs need to be developed to enable a branch of knowledge which aspires to be intricately interwoven with other fields derived from linguistics, psychology, history, epistemology,. etc., to be transposed into a language and practice which are as well-expressed as possible in terms of the everyday discourse used by teachers working in our existing educational systems.

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Advanced Mathematical Thinking

Some Misconceptions in Calculus - Anecdotes or the Tip of an Iceberg?

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Several recent papers have argued that many students who passed a university calculus course have poor conceptual knowledge of the course. This is a general claim which does not explain where the problem is. In order to be more specific, a detailed analysis of students ideas is needed. In this paper we analysed the answers of one student to two questions, posed to him in a questionnaire. It can be considered as a "one questionnaire case study". The analysis shows that in spite of the fact that some elements in the student's answers indicate a good conceptual understanding, there are other elements which indicate a very serious misconception. It seems that this misconception implicitly directs the student's line of thought when he tries to solve a non-routine conceptual problem. On the other hand, the correct ideas appear when the student is asked the routine questions - either conceptual or computational.

In recent years.it has been discussed that teaching and learning mathematics became procedural and not conceptual. Namely, both students and teachers put the emphasis on procedures and avoid concepts. A lot can be said about the causes of this phenomena. There are psychological causes, educational causes, social causes and even political causes. We cannot deal with all these here.

The aim of this paper is to draw more attention to the above phenomena in the domain of calculus. It is not enough to know about it in general, exactly as it is not enough for the ecologist to know in general that disposal of toxic materials in the river harm the fish. The more specific our knowledge about it is, the greater our chances are to change the situation. Since sophisticated calculators today can execute symbolic manipulations on functions and also draw their graphs, there is no point in teaching and learning how to do it unless conceptual understanding is involved. It is a waste of time and energy to train somebody to do something that a machine can easily do. If some aspects of calculus are important at all to somebody who does not major in mathematics these are the conceptual aspects. The concept of the <u>derivative</u> is especially important. If this concept is not well understood then its relation to velocity, rate of change, etc. cannot be understood in natural sciences and its relation to marginal value concepts cannot be understood in economics and business administration. There are already some studies which indicate how poorly calculus students perform on conceptual tasks. Orton (1983) reported that most errors students made when carrying out some tasks in differential calculus were the result of the failure to grasp conceptual principles which were essential to the solution. Tufte (1988) found that the success percentage in technical calculus items was between 73 and 92 whereas the success percentage in calculus conceptual items was between 7 and 22. Selden (1989) who administered a non-routine conceptual calculus test to students who passed a routine calculus test with grade C, reports that the highest score gained in this test was 35%. Vinner (1989) reports that in high school graduates, who had completed a calculus course and passed matriculation exams with grade not less than 80%, only 24% knew the geometric interpretation of the derivative and 7% knew the algebraic interpretation of the derivative (the derivative as a limit). Note that the last aspect is the one that really counts in natural science. In this paper, however, we are not interested in statistics. We are interested in the ideas students have about calculus concepts related to the derivative.

There are two common methods to reveal students' ideas: a questionnaire and an interview. The common belief is that an interview is a better instrument than the questionnaire. This is because many ambiguities can be resolved in an interview that cannot be resolved in a questionnaire. Also, some spontaneous reactions in an interview can be extremely illuminating, much more than the controlled or even inhibited reactions one can get in a questionnaire. This might be true in many cases but there are also many cases in which the situation is more delicate. Assume that a student makes an ambiguous statement in an interview and the interviewer wants to ask a question which is supposed to clarify this ambiguity. Of course, this must be done in such a way that the student will not change his mind as a result of the question posed to him. Practically, however, this might be impossible. There are situations in which any reconsideration of a given answer causes a critical analysis. This analysis wil lead to a clarification in a direction different from the one in the original answer. Everybody with minimal self awareness knows that very often he has vague ideas which he believes in, but the moment he formulates them in words or even listens to somebody else's formulation he realizes that these were faulty ideas. So, there are cases in which an interview will not lead to clear and unambiguous information but even to distorted information. Also the belief that in an interview we can obtain more spontaneous reactions is not necessarily true. It depends on the student and on the interviewer and on the relations between them formed before and during the interview.

Thus, in this study of calculus misconceptions, we decided to use questionnaires and to analyse them in a very detailed way. A satisfactory analysis should be a coherent (not necessarily consistent) interpretation of the student's answers. In order to form such an interpretation it was necessary sometimes to use speculations. This might be considered as negative by some people and our answer to such criticism raises two issues:

1. It is impossible to make progress without making some speculations. The speculations should be examined, of course, by experimental data, but this is a long process and it is not a one study project. We ourselves are planning to examine these speculations in the next stage of our research and we hope that other mathematics educators, specializing in calculus, will also examine them and either support or refute them.

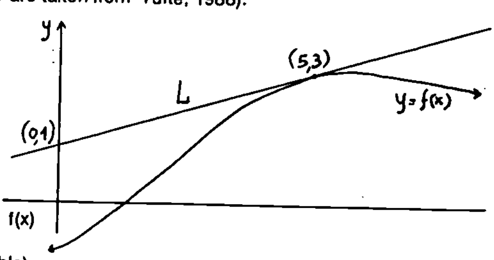
2. Athough we declare our hypotheses as speculations these are not at all detached from practice. They have strong support in our experience as calculus teachers. Many reactions of students in our calculus classes and office hours strongly support our Interpretations of the questionnaire. Some readers might consider this as anecdotal information. We believe that it is symptomatic and it is only the tip of an iceberg.

Method

<u>Two questions and One Student</u>

Out of a questionnaire that contained eleven questions we selected the following two questions to discuss here. (Questions 1A and 1B are taken from Tufte, 1988).

1. Line L is tangent to the graph of y = f(x) at point (5.3).
   A. Find f (5)
   B. Find f' (5)
   C. What is the value of the function f(x) at x = 5.08 (be as accurate as possible)



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2. A. What is a derivative? Define or explain as you wish.

B. What does it mean that the derivative of $f(x) = x^2$ is $2x$?

C. Using only a calculator, can you suggest a method to calculate an approximate value of the derivative of 4^x at $x = 2$?

Please explain and justify every step in your solution.

The student, whose answers will be analysed here, is a first year economics student at Ben-Gurion University who took a calculus course both in high school and at the university. The questionnaire was administered to him, as well as to another 130 students, at the end of the university calculus course. Because of confidentiality, we changed the student's name and called him Ron.

Results

Ron's answers and explanations are the following (this is a literal translation from Hebrew):

1A. The value of the function at 5 is 3. Explanation: (5.3) is the tangency point.

1B.

$$\frac{\Delta y}{\Delta x} = \frac{3-1}{5-0} = \frac{2}{5} = \mathrm{tg}\,\alpha$$

Explanation: The derivative at $x = 5$ is the slope of the tangent to the function $y = f(x)$ at this point.

1C.

$$y - y_0 = \frac{2}{5}(x - x_0)$$

$$y - 3 = \frac{2}{5}(x - 5)$$

$$y - 3 = \frac{2}{5}x - 2$$

$$y = \frac{2}{5}x + 1$$ This is the equation of the tangent at (5.3).

Now we will find the integral (a primitive function)

$$F(x) = \int \left(\frac{2}{5}x + 1\right) dx = \frac{(2/5)\,x^2}{2} + x + C = \frac{1}{5}x^2 + x$$

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x$$

$$f(5+0.08) = \frac{1}{5}\cdot 5^2 + 5 + \left(\frac{2}{5}\cdot 5 + 1\right)\cdot 0.08$$

Explanation: Using the slope of the tangent we found the tangent equation at (5.3). By means of this we found the integral (the primitive function) and by means of linear approximation (part of Taylor), first order approximation we found an approximation to the value of the function at x=5.08.

2A. The derivative is the slope of the tangent to the graph at a certain point. This is the derivative at a certain point. Generally speaking, it is the slope of the tangent to the graph (the tangent of the angle of the slope).

2B. It means that the slope of the tangent to the function x^2 is $2x$. For instance at $x = 2$, $y = 4$ the slope of the tangent is $y' = 2\cdot 2 = 4$.

2C. $$\lim_{h\to 0} \frac{4^{x+h} - 4^x}{h} = \lim_{h\to 0} \frac{4^x\ 4^h - 4^x}{h} = \lim_{h\to 0} \frac{16\cdot 4^h - 16}{h}$$

Explanation: $y = a^x$, $y' = a^x \ln a$, $y = 4^x$, $y' = 4^x \ln a$

Analysis

At first sight, if we ignore 1C which looks quite strange, Ron's answers are almost alright. There are some minor deficiencies: First of all, the explanations in 1A and in 2C are completely irrelevant. They have nothing to do with the answer. Most teachers tend to ignore this phenomenon as long as the answer is correct. When marking exams it is almost forgivable. Since we are not involved here with evaluation we would like to point out the phenomenon. Many students do not understand the nature of mathematical explanation. They point at a certain relation between their answer and another fact but <u>they fail to see that their answer does not follow from (or not implied by) that particular fact</u>. We will call this <u>an irrelevant explanation.</u> Secondly, there are some formulation deficiencies in Ron's answers. These are even more forgivable by teachers than the previous deficiency.

If you read 2A carefully you might be bothered by the use of the definite article: "the slope of the tangent to the graph at a certain point". This can be understood as if you choose a certain point and then define the derivative as the slope of the tangent to the graph at this point. Ron probably was aware of this interpretation and he bothers to tell us that it is not the correct one. This is by saying that "this is the derivative at a certain point". Namely, the point varies, it is not a fixed one. Thus, consciously, the idea is very clear and the answers to 2B and 2C support the impression that Ron's concept of the derivative is satisfactory. But what happened in 1C is an example of unconscious (or implicit) influence of ideas that when expressed explicitly are immediately recognized as wrong. We refer here to the above mentioned idea that the derivative is the slope of the tangent at a certain (fixed) point. When the concept of the derivative is explained to the student there is a typical drawing shown to him, similar to the drawing in question 1. Since there is a well known tendency in cognition that pictures replace the concepts, and in the picture only one tangent is drawn, this tangent might replace the derivative. The derivative will be considered as the equation of this tangent. This idea was expressed explicitly by several students on different occasions in classes, in office hours and in our questionnaire. Is it possible that this wrong idea, in spite of Ron's explicit intention, will direct his behavior in 1C? Note that if we ignore 1C, Ron would be considered as somebody who understood the concept of the derivative in a quite satisfactory way.

Let us now analyse Ron's answer to 1C. In this answer the required formula to calculate f(5.08) appears quite close to the end. This is the formula $f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x$ (the only mistake is a notational one ; it should be "≈" instead of "="). In 1A and 1B Ron calculated $f(x_0)$ and $f'(x_0)$, so he could use the above formula right away. Why didn't he substitute these in the formula and find an approximation for f (5.08)? Instead Ron calculated the equation of the tangent at (5,3) and got $y = \frac{2}{5}x + 1$. This equation is treated now as if it were the derivative of the given function. Therefore, in order to find the given function one should look for the integral of the above derivative. Thus Ron writes, using the common letter F to denote a primitive function:

$F(x) = \int (\frac{2}{5}x + 1)\,dx$. The answer $\frac{1}{5}x^2 + x + c$ brings up a certain difficulty, namely, the integration constant c. This is simply resolved by ignoring it (or substituting $c = 0$). Also this is a typical phenomenon: using arbitrary steps which facilitate the situation and make solution according to the original plan possible. At this point the notational conflict should be resolved. On one hand, Ron uses F(x) to denote the primitive function. The use of a capital letter is common in this context. On the other hand, the question speaks about f(x), therefore he returns to f (small letter). In this f, Ron substitutes 5 for x_0 and gets

$(1/5) \cdot 5^2 + 5$. In addition to this, $f'(x)$ is now $\frac{2}{5} \cdot x + 1$ and therefore $f'(5)$ is $\frac{2}{5} \cdot 5 + 1$, in conflict with the answer in 1B.

Thus, unintentionally, in spite of what was said in 2A, the derivative has become the equation of the tangent at a certain fixed point and the function itself is the integral of the equation of the tangent. The contradiction between 2A and 1C can be explained by compartmentalization. The fact that it happened this way can be explained by the existence in a suppressed form of the above misconception of the derivative. Apparently, Ron's line of thought was also directed by additional implicit ideas:

1. In order to evaluate the value of a function at a certain point one should know the function in terms of algebraic formula.
2. The above formula can be obtained if we have a formula for the derivative.

To these two principles a previous principle was added :

3. The formula of the derivative is the equation of the tangent (and if this is the case you must restrict yourself to one fixed point).

We hope that it is clear now that by an interview we would not have been able to clarify the conflict in Ron's thought. Explicitly, Ron knows the definition of the derivative. Any direct question about the derivative asked at the context of 1C will lead to an answer similar to 2A. Thus without the above speculation it Is impossible to explain what happened in 1C. As a part of an interview it might be considered as a meaningless accident caused by temporary confusion and not as evidence of a certain implicit common misconception. In an interview we would even be impressed by the technical terms used by the student: "Linear approximation" and "Taylor". Technical terms are always a trap for teachers and students know it very well. Technical terms are considered by teachers to be an indication of understanding. If one uses the right term can he have a wrong idea?

The above misconception can be explained even by a lingual analysis. The exact geometrical definition of the derivative is the following: The derivative of a function at a certain point is the slope of the tangent to the graph of the function at this point. It is quite hard to memorize and therefore some omission-transformations take place. The first one is harmless.

You say "the derivative" instead of "the derivative of a function". The second one is dangerous. You say "is the tangent" instead of "is the slope of the tangent". The third one is based on the convention that there is no need to distinguish between the function and its graph. Every mathematician can tell you from the context whether one refers to the function as an algebraic entity or to its graph, the geometrical entity. Thus, you say "tangent to the function" instead of "tangent to the graph of the function". The fourth omission is in the beginning of the definition. Instead of saying "the derivative at a certain point" you simply say "the derivative". Therefore, because of grammatical reasons when you reach the word "this" at the end of the definition you must use an indefinite article and make it "a point", or even better "a certain point", which echoes the expression omitted earlier. Hence, the final result of all the above changes is: <u>The derivative is the tangent to the function at a certain point.</u> This is a definition we got in many questionnaires and faced in many classes. Now, there are two possibilities: 1. The above formulation is used only to facilitate memorization and it serves the student to reconstruct the complete original definition. 2. The above formulation becomes the definition itself. In addition there is the above convention that when saying "tangent" you can refer either to the geometrical entity or to the algebraic entity - the equation of the tangent. Therefore, from these one can imply that the equation of the tangent at a certain point is the derivative. This "certain point" which already has the connotation of a fixed point, when associated with the prototype drawing of the derivative (something like we had in question 1) leads to the above misconception which has been found, as mentioned before, in many students. This misconception does not prevent students from passing, the university calculus course, sometimes even with good marks.

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DIFFICULTES COGNITIVES ET DIDACTIQUES DANS LA CONSTRUCTION DE RELATIONS ENTRE CADRE ALGEBRIQUE ET CADRE GRAPHIQUE

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Abstract : In this paper, starting from results obtained in a research on the teaching and learning of differential equations with beginners, we analyse cognitive and didactical difficulties linked to the building of relations between the algebraic setting and the graphical one, for the notion of function. This analyse tends to show that persistent difficulties are mainly concentrated in the interaction needed at level of proofs and that, at this level, cognitive difficulties are reinforced by didactical ones.
The presentation will be given in english.

I - INTRODUCTION

Les notions mathématiques fonctionnent généralement dans plusieurs cadres et une des caractéristiques de l'activité du mathématicien est le jeu qu'il opère entre ces différents cadres pour résoudre les problèmes qu'il se pose ou lui sont posés. D'un point de vue cognitif également, comme R.Douady l'a montré dans sa thèse (Douady, 1984), les déséquilibres entre connaissances et convictions issues de cadres différents sont des leviers sur lesquels le didacticien peut jouer efficacement, dans une perspective constructiviste de l'apprentissage. C'est dans cette perspective théorique que je me situerai dans cet article, à propos de la notion de fonction.

Depuis une dizaine d'années les recherches se sont multipliées à propos de cette notion, mettant en évidence les différents niveaux de conceptualisation qui marquent son apprentissage et les difficultés rencontrées par l'enseignement usuel, centré sur les aspects algébrique et ensembliste (Dreyfus, Vinner, 1982), (Dubinsky, 1989), (Sfard, 1989).
La plupart des chercheurs ont d'autre part élaboré et expérimenté des stratégies d'enseignement visant à surmonter ces difficultés. Un certain nombre de ces stratégies sont basées sur l'exploitation des possibilités offertes par l'outil informatique pour mettre en connexion étroite différents cadres de fonctionnement de la notion, en particulier les cadres *algébrique* - où la fonction intervient par l'intermédiaire d'une ou plusieurs formules - *numérique* - où elle intervient par l'intermédiaire de tableaux de valeurs numériques - et *graphique* - où elle intervient par l'intermédiaire d'une représentation graphique, et ceci que les chercheurs se réfèrent explicitement ou non à la théorie des jeux de cadres (Dreyfus, Eisenberg, 1987), (Guzman-Retamal, 1989).

Ces expériences dont les résultats sont le plus souvent mitigés ne peuvent manquer de susciter diverses questions :
- Quel rôle peut jouer l'établissement de relations entre les divers cadres de fonctionnement de cette notion dans sa conceptualisation ?
- Quelle est la nature exacte des difficultés rencontrées dans l'établissement de ces relations ? En particulier, quel est le poids respectif dans ces difficultés de la composante cognitive et de la composante didactique ? Comment ces composantes s'imbriquent-elles et pourquoi ?

Je me restreindrai ici à ce deuxième groupe de questions. Elles paraissent essentielles pour comprendre le fonctionnement de l'enseignement et analyser toute tentative d'intervention. Elles sont à rapprocher des questions que pose actuellement en didactique la théorie des obstacles épistémologiques (Artigue, 1988). L'analyse des travaux menés en ce domaine montre en effet que, le plus souvent, ce qui est identifié par les chercheurs comme obstacle épistémologique, par référence au développement historique des notions considérées, se retrouve étroitement imbriqué dans l'enseignement à des obstacles de nature didactique. L'enseignement usuel vit sur la fiction de la possibilité d'un apprentissage dans la continuité : le bon enseignant est celui qui permet à l'élève d'éviter les difficultés, qui prévient les erreurs, qui aplanit l'apprentissage pour en faire un processus graduel et sans ruptures. Ceci conduit, consciemment ou non, à des prises de décision didactique qui, dans leur volonté de contouner les ruptures inévitables, renforcent au contraire les obstacles épistémologiques par ce que l'on peut identifier comme des obstacles didactiques.

Ceci peut conduire à faire l'hypothèse de mécanismes analogues dans la construction des relations inter-cadres au niveau de l'enseignement. Est-ce-que la séparation des cadres, souvent présente dans l'enseignement usuel, est une réponse didactique à des difficultés cognitives réelles ? Si oui, ne les renforce-t-elle pas ? Si oui encore, est-ce-que les expériences menées basées sur l'établissement de relations inter-cadres ont réellement réussi à s'opposer à ces contraintes didactiques ou ne l'ont-elles fait qu'en surface ? En quoi ce phénomène pourrait-il expliquer certains des résultats obtenus ?

C'est au débat sur ces questions que je voudrais contribuer ici, en exploitant dans cette direction les résultats d'une recherche menée depuis trois ans sur l'enseignement des équations différentielles en première année d'Université, avec des étudiants d'orientation mathématiques/physique.

II - CADRE DE LA RECHERCHE ET METHODOLOGIE

Si la théorie des équations différentielles s'est mathématiquement développée dans plusieurs cadres, l'enseignement pour débutants se centre sur la résolution algébrique c'est à dire sur la résolution par l'intermédiaire de formules (formules explicites ou implicites, développements en série, expressions intégrales). La recherche menée avait pour objectif l'étude des possibilités, dès l'entrée à l'université, d'extension viable de l'enseignement à la résolution qualitative, c'est à dire à la caractérisation géométrique et topologique de l'ensemble des courbes compatibles en chacun de leurs points avec le champ de tangentes associé à l'équation (portrait de phase de l'équation).

Pour l'étude des questions posées dans l'introduction, cette recherche me semble présenter diverses caractéristiques intéressantes :

- la résolution géométrique des équations différentielles met en jeu de façon incontournable les relations entre cadre algébrique et cadre graphique : une équation (objet algébrique) étant donnée, cette équation se traduit dans le cadre graphique par un champ de tangentes et/ou un régionement du plan suivant la croissance ou décroissance des courbes cherchées. La résolution qualitative consiste ensuite en un va et vient permanent entre équation et tracés,

- dans le cadre de la recherche menée, cette interaction intervient, suivant les activités proposées aux étudiants, dans des registres et à des niveaux de difficultés très différents : ce ne sont pas les mêmes compétences qui sont nécessaires pour associer des tracés fournis et des équations, pour interpréter des tracés fournis, pour prévoir le portrait de phase d'une équation ou pour justifier des conjectures,
- dans l'expérimentation, l'enseignement des équations différentielles est préparé par un travail sur courbes et fonctions et l'on peut donc faire l'hypothèse que les difficultés identifiées dans la recherche sont réellement des difficultés résistantes dans la construction de relations inter-cadres,
- la recherche a donné lieu à trois expérimentations successives, prenant en compte les feed-back obtenus pour ajuster le processus d'enseignement d'une année sur l'autre, elle a concerné sur trois ans environ 300 étudiants de niveaux variés puisque la troisième année, il s'agissait des étudiants les plus faibles entrant à l'université, et une dizaine d'enseignants.

En ce qui concerne la méthodologie, il s'agit d'une recherche classique d'ingénierie didactique. C'est donc une recherche basée (après analyse des contraintes épistémologiques, cognitives et didactiques pèsant sur l'enseignement usuel dans ce domaine), sur la conception et l'expérimentation d'une séquence didactique jouant sur ce système de contraintes et la validation des hypothèses à l'origine de la conception s'effectue essentiellement par confrontation entre l'analyse a priori du processus d'enseignement construit et les données recueillies au cours ou à l'issue de l'expérimentation.
Dans cet article, je ne présenterai pas cet aspect du travail pour lequel le lecteur peut se reporter à (Artigue 1989). L'analyse des difficultés se fera en référence aux principaux registres d'interaction en jeu dans l'enseignement et, au niveau des données quantitatives, on s'appuiera sur les réponses à des questions représentatives extraites des évaluations menées chaque année à l'issue de l'enseignement.

III - INTERACTION DANS DES TACHES D'INTERPRETATION

Ces tâches sont présentes dans l'enseignement, dans les premières situations d'approche du qualitatif : situations dans lesquelles les étudiants ont à associer des tracés de champs puis des portraits de phase à des équations ainsi que dans les séances de travaux pratiques sur ordinateur où ils ont à produire les portraits de phase de diverses équations et les analyser.

Le bon fonctionnement des situations d'association, résolues chaque année en petits groupes, sans que l'enseignant ait à intervenir, témoigne de l'accessibilité des relations inter-cadres en jeu dans ces activités avec des étudiants de ce niveau : lier caractéristiques de l'équation et invariance des courbes solutions par des transformations géométriques simples, lier signe de la dérivée et sens de variation des solutions, lier zéros de la dérivée et pente horizontale, limite infinie et pente verticale, lire une pente et reconnaître des solutions particulières. Les séances de travaux pratiques montrent également que très vite la plupart des étudiants sont capables de prendre de la distance par rapport aux tracés fournis pour par exemple lire correctement comme asymptotiques des tracés qui visiblement se touchent et rejeter des tracés avec croisement dans le cas où de tels croisement sont théoriquement impossibles.

Les difficultés les plus résistantes repérées concernent dans ce registre l'étude des branches infinies. En effet, pour ces étudiants débutants, cette étude relève seulement de la recette algébrique : on cherche la limite du rapport f(x)/x, si cette limite existe, on cherche la limite du rapport f(x)/x - limf(x)/x .....

Cette conception est inadaptée à l'étude qualitative des équations différentielles où, la fonction étant inconnue, c'est en termes de limite de la dérivée que l'on aborde l'étude des branches infinies. Il y a donc nécessité d'enrichir en ce sens la conception initiale et de coordonner les deux points de vue. Ceci est d'autant plus délicat que la conception "dérivée" peut conduire à des regroupements des cas distincts de ceux de la recette initiale : asymptote verticale et branche parabolique de direction asymptotique verticale, par exemple. Ceci peut expliquer la résistance constatée de formulations contradictoires comme celle-ci :

"Il faut savoir si f(x) a une asymptote verticale ou une branche parabolique quand x tend vers l'infini."

Notons cependant que les erreurs de ce type, usuelles chaque année en début d'enseignement, regressent fortement quand la difficulté est prise en compte explicitement dans l'apprentissage, comme cela a été le cas ici à partir de la deuxième année.

IV - INTERACTION DANS DES TACHES DE PREVISION

Ce type de tâche (une équation différentielle étant donnée, prévoir son portrait de phase) est présent dans l'enseignement : deux séances de travaux dirigés au moins, un problème lui sont consacrés et il intervient dans l'évaluation finale. Dans ce registre également, la recherche ne met pas en évidence de difficultés fortement résistantes, au moins lorsque le tracé préliminaire à effectuer, pour régioner le plan suivant le signe de y' est d'un niveau de complexité raisonnable ou lorsque ce tracé, éventuellement plus complexe, est fourni. Les épreuves de l'évaluation finale montrent, même en temps limité, un niveau de réussite raisonnable et la baisse sévère du niveau des étudiants, la troisième année, n'altère pas les résultats obtenus les années précédentes.

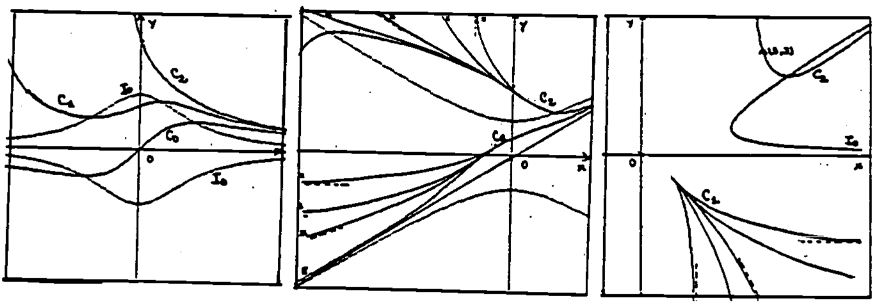
Les tableaux ci-après le mettent en évidence en présentant les résultats des tâches de régionement et de tracé demandées aux trois évaluations finales successives. Les équations concernées étaient respectivement :

1) $y'=(1/1+x^2)^2-y^2$, la première année, et l'on demandait de tracer sans justification les courbes-solutions C_0, C_1, C_2 passant par les points (0,0), (-2,1) et (0,2),

2) $y'=x^2+1-y^2$, la deuxième année, et l'on demandait, après avoir fait repérer la solution linéaire, de tracer des formes a priori possibles pour les solutions C_1 et C_2, issues respectivement des points (0,1/2) et (0,2), sans justification mais avec formulation des questions qui se posent à leur sujet,

3) $y'=y(x-y)-1$, la troisième année, et l'on demandait cette fois, de tracer des formes a priori possibles pour une solution C_1 passant par un point (x,y) vérifiant $x>0$ et $y<0$, puis de prévoir le tracé de la solution C_2 passant par le point (3,3).

Les tracés correspondant sont les suivants (dans les formes a priori possibles nous n'avons fait figurer raisonnablement à ce niveau que des types à dérivée monotone) :



Les résultats, donnés en pourcentages, pour des effectifs entre parenthèses, sont les suivants :

Année 86-87 Régionement Tracé C_0 Tracé C_1 Tracé C_2	Groupe I (29) 90 69 55 76	Groupe II (30) 83 -70 83 80	Groupe III (30) 80 57 50 63
Année 87-88 Régionement Tracé C_1 2 types au moins Tracé C_2	Groupe I. (23) 83 52 65 57	Groupe II (26) 96 77 73 73	Groupe III (32) 94 84 81 81
Année 88-89 Régionement Tracé C_0 3 Types au moins Tracé C_1	Groupe I (26) 88 88 85 80	Groupe II (31) 84 84 77 71	Groupe III (29) 86 86 62 69

Ce développement progressif mais sans obstacle majeur des compétences d'interaction nécessaires à la prévision, dans les cas simples du moins, est confirmée par les autres données issues de la recherche.

V - INTERACTION ET TACHES DE JUSTIFICATION

C'est en fait à ce niveau que se sont concentrées les difficultés rencontrées. Ceci est frappant si l'on se réfère encore une fois aux résultats des évaluations finales. Nous sélectionnerons trois tâches caractéristiques de ce registre : prouver qu'une solution coupe une courbe donnée - qu'elle ne peut couper une courbe donnée - prouver qu'elle admet une asymptote ou écarter la possibilité d'un tel type de branche infinie.

La première année, ces trois tâches sont présentes dans la justification demandée du tracé de C_1 sur l'intervalle $]-2,+\infty[$. Les résultats sont les suivants :

Intersection C_1, $I_0\,(x<0)$	10	30	23
Intersection C_1, $I_0\,(x>0)$	0	10	13
Non-intersection ensuite	3	10	13
C_1 asymptote à I_0	0	23	0

Ces résultats sont clairs. Ils opposent sans ambiguïté les compétences manifestées dans les autres registres à la faiblesse du registre justificatif. Si l'on étudie par exemple les justifications proposées par les étudiants pour les problèmes de croisement, on s'aperçoit qu'il y a deux grandes tendances : celle qui consiste à décrire simplement le tracé et celle qui consiste à produire une justification formulée dans le langage classique de l'analyse. On reconnaît alors dans les textes, malheureusement en vrac, les ingrédients des démonstrations analogues faites par les enseignants.

Des difficultés, d'origine cognitive et didactique, contribuent, me semble-t-il, de façon essentielle à ce phénomène. *Le passage du registre de la prévision à celui de la justification nécessite en effet un changement de point de vue.* Dans le premier cas, il s'agit de produire un tracé, le plus simple possible, respectant des contraintes imposées. Les règles du jeu sont les mêmes que lorsqu'il s'agit de résumer dans un graphe tous les renseignements obtenus sur une fonction, tâche classique pour les étudiants. Dans le second, le tracé produit doit à la fois être support du raisonnement et objet de doute : était-il le seul possible ? Contre quelles autres éventualités a-t-il été plus ou moins intuitivement choisi ? Ce renversement de point de vue n'a aucune raison d'être facile. La difficulté est renforcée par le fait que, dans l'enseignement usuel, il n'est nullement sollicité, le cadre graphique étant un sous-cadre utilisé pour la représentation, non pour la justification.

Une analyse plus fine de la séquence d'enseignement montre d'ailleurs que ce statut inférieur du cadre graphique n'a été entamé que superficiellement dans l'expérimentation réalisée : le cadre graphique y est omniprésent mais au niveau prévision - interprétation uniquement. Les notions de barrière, de zone... qui permettraient de le rendre opérationnel au niveau des justifications n'ont pas été introduites. Il y a semble-t-il plusieurs raisons à cette non introduction :

- la volonté, tout à fait légitime par ailleurs, de faire de l'enseignement qualitatif sur les équations différentielles une occasion privilégiée de faire fonctionner les outils fondamentaux de l'analyse élémentaire en cours d'apprentissage ,
- la force du rejet traditionnel par l'enseignement du cadre graphique comme cadre de justification. L'enseignant a du mal à s'opposer à ce rejet et, même s'il essaie, il se trouve confronté à des difficultés sérieuses : il lui faut négocier un contrat avec les étudiants à partir de rien. Pour une démonstration classique, ce contrat reste implicite et même si le

consensus évolue sans cesse, c'est sur la base d'un consensus de départ. Pour une argumentation graphique, toutes ces questions apparaissent au grand jour : quels sont les points qui méritent justification, quels sont ceux sur lesquels on peut, voire on doit, glisser ? A quels arguments a-t-on droit ? Comment rédiger ?

La seconde année, compte-tenu des analyses effectuées, diverses modifications ont été apportées. En particulier, le cadre graphique est devenu opérationnel au niveau justification par la définition des notions de zone, de champ rentrant et de champ sortant par rapport à une zone, l'énoncé de théorèmes permettant la manipulation de ces notions et la légitimation explicite de raisonnements formulés directement dans ce cadre. Si l'on se rapporte encore une fois à l'évaluation finale, les problèmes déjà cités étaient présents à travers les questions posées concernant la justification des comportements de C_1 et C_2 pour $x>0$. Les résultats obtenus sont les suivants :

Non intersection C_1, I_0	13	77	19
C_1 asymptote à I_0	39	62	44
Intersection C_2, I_0	13	54	13

Ils traduisent une progression évidente. Mais tout aussi frappante est l'importance des disparités entre les groupes. Les résultats des groupes II et III par exemple ne se différencient que dans ce registre et les différences sont particulièrement fortes dans la gestion des problèmes de croisement. Une analyse des procédures montre que, sur ce point, toutes les justifications correctes sauf 5 sont basées sur l'utilisation de zones et que les différents groupes n'ont pas construit le même rapport aux instruments de justification graphique introduits. Et même dans le groupe où visiblement ces instruments font partie à part entière des outils de légitimation (groupe II), il subsiste une différence entre les problèmes de non-croisement et ceux de croisement où les étudiants se laissent davantage piéger par l'évidence perceptive et regressent à des preuves du type : C_1 décroît et I_0 croît, donc elles se coupent.

En revanche, on ne note pas de différence inter-groupes, dans l'ensemble des données receuillies, au niveau des preuves ou rejets d'asymptotes : les étudiants ont massivement recours à une interaction entre cadres mettant en jeu l'énoncé suivant : si une fonction dérivable a une dérivée qui tend vers une limite non nulle à l'infini, elle tend elle-même vers l'infini. Cet énoncé, de forme classique, s'était révélé l'année précédente comme un théorème local du groupe II et expliquait les quelques réussites constatées dans ce groupe. Il est devenu un théorème "officiel" et ne pose visiblement pas les mêmes problèmes didactiques que les énoncés portant sur les zones et les barrières. Mais on voit aussi persister malgré l'insistance de l'enseignement sa version erronnée : si f(x) a une limite finie à l'infini, sa dérivée tend vers 0, claire manifestation de la difficulté à rejeter au niveau des généralisations nécessaires aux preuves le modèle monotone qui guide si efficacement les tracés au niveau des prévisions.

Ainsi donc les résultats obtenus dans l'évaluation finale, cohérents avec les autres données recueillies, tendent à montrer que dans un environnement didactique adapté, les relations entre cadre graphique et algébrique peuvent s'étendre des capacités d'interprétation et de prévision à des capacités de justification. Et l'on peut faire l'hypothèse que les différences constatées ici entre les groupes, au delà des différences de niveau sans doute réelles des élèves, témoignent aussi du degré avec lequel les enseignants impliqués dans l'expérience ont réussi à investir dans leur pratique ces outils non familiers et réussi à vaincre les réticences de l'enseignement usuel vis à vis du cadre graphique.

L'expérimentation de la troisième année, avec les étudiants faibles, tend à confirmer cette interprétation. Les résultats de l'évaluation finale concernant les problèmes d'asymptote, de croisement et non-croisement sont donnés ci-après (les enseignants des groupes I et II étant ceux des années précédentes, l'enseignant du groupe III étant un nouvel enseignant) :

Asymptote	54	48	28
Intersection	19	52	24
Non-intersection	46	45	10

VI - CONCLUSION

Les résultats obtenus, même s'ils restent locaux, tendent donc à confirmer l'hypothèse initiale d'une coexistence entre difficultés cognitives et difficultés didactiques dans la construction des relations entre cadre graphique et cadre algébrique à propos de la notion de fonction. Mais ils permettent aussi, me semble-t-il, de mieux comprendre l'imbrication de ces difficultés et d'évaluer l'influence de cette imbrication sur la résistance de difficultés que l'on pourrait hâtivement cataloguer de cognitives exclusivement. L'imbrication entre difficultés cognitives et didactiques n'est pas indépendante des registres dans lequel l'interaction est appelée à fonctionner. Elle se manifeste ici, de façon résistante, dans le registre de la justification traditionnellement inexistant dans le cadre graphique. Mais, dans ce registre, elle est suffisamment forte pour survivre à une expérience d'enseignement qui la prend explicitement en compte.

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UNBALANCE AND RECOVERY
CATEGORIES RELATED TO THE APPROPRIATION OF A BASIS OF MEANINGS PERTAINING TO THE DOMAIN OF PHYSICAL THINKING
CONSTRUCTING THE NOTION OF ANALYCITY (CASE STUDIES)

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In the development of a research aiming to extract the construction mechanisms of mathematical concepts and processes related to Calculus, when these are guided by the physical thinking of predicting the phenomena of continuous flux in nature, we studied those mechanisms (symbiotic and predator) operating the transition between the notions of Prediction and Analysis. We have adopted the clinical approach used in case studies, with teachers in the field of physico-mathematical sciences, pertaining to the Mexican Educational System.

§ 1. STATEMENT OF THE PROBLEM. Determining the variables.

Our research now reports results on a new stage, and it shows new approaches in the theoretical and methodological fields, as well as in the domain of didactics. In the theoretical realm, we started with an epistemological analysis of the Taylor's Series (TS) mathematical concept, focusing our attention on the transition mechanisms between these two notions of scientific contiguous domains: *prediction*, which belongs to physical sciences, and *analytical function*, peculiar to mathematics [Cantoral,1989a]. This permitted to recognise a basis of meanings for the TS, which places it as a cognitive instrument enabling to observe the evolution of flowing objects, and, consequentely, to predict their behavior. This meaning of physical nature, suffers a predator process of *Didactic Transposition* which conceals its primary meaning [Cantoral,1989] The new paradigm of Calculus assigns to the TS the meanings usually associated with a mathematical result, i.e. those that are nourished from the relationship with other mathematical objects. On the other hand, the models of inductive generalization and those of functional metamorphosis [Cantoral, 1989], are the ones prevaling in the present dav mathematical analysis.

Concerning the methodological level, we adopted the clinical approach used in case studies within a research atmosphere and controlled teaching, in which participating subjects were engineers and physico-mathematicians whose field of endeavor is teaching, in the domain of Engineering Sciences. This working system involves old didactic elements [Cantoral and Farfán, 1987], as well as the treatment of germinal ideas as recognized in the analysis of classic originals [Cantoral and Farfán, 1989].

Finally, in the didactic field, research points to the reconstruction of a Calculus didactics based on the teachers everyday intuitions and experiences, inspired bv a real phenomenical closeness, where the center of analvtical focus is the phenomenon, and not the concept.

§ 2. THE RESEARCH PROBLEM. An Anatomy.

The major problem in this research consisted in analvzing the construction processes of mathematical knowledge (the notion of *Analytical Function*) when these are guided by the thinking of nature's continuous flux phenomena. In connection with this, we studied the functional mechanisms which operate the relationship of a dialectic nature between the notion of *Prediction*, pertaining to the physical sciences and Engineering, and that of *Analysis*, which is peculiar to Mathematics.

§ 3. TEACHING AND ITS APPROACHES. Population and Contents.

Within the framework of a model of Mathematics Teachers Updating, among practicing teachers, we designed and implemented a research and teaching program whose main object of study is Calculus as taught to future engineers. This program started in December, 1987, and was attended by 49 teachers. As an everyday technique, we tried to approach the discourse to the research results. Mathematics in context was discussed, saturating the discussion with Newtonian mechanics aspects closely related to their professional experiences. The working dynamic, with ten-hour sessions every two weeks for almost two years, made it possible to work with these teachers without their having to abandon completely their teaching duties. At least two didactic approaches were discussed; one of them arose from the results of research concerning the reconstruction of school mathematical discourse, or from old strategies presenting variation together with the *Prediction Idea* (PI).

This didactic atmosphere considered an essential fact in the old days'didactics: PI makes of TS development the cognitive instrument *par excellence* in the observation of variation phenomena. It is used to state, solve, and interpret problems. We worked on the two basic models of discourse associated to TS: PI, and the Convergence Idea. To the conventional approach to Calculus, which makes of it an instrument of convergence $f(x) = P_n(x) + R$, where R is treated as a remainder or as an order of magnitude, we opposed another, which we recognized during our research, and which places TS as the Prediction instrument for things that flow continuously. Thus, when the initial state of the system in evolution is known through data such as x, $f(x)$, $f'(x)$, etc., the value assumed by $f(x+h)$ is announced, just with them!.

In order to develop the experiment, a sample of 4 teachers was chosen, seeking to make it representative of those who had constructed knowledge. The sample included 1 physico mathematician, and 3 engineers. They work for University Education Institutions, in Engineering areas, or for Medium-High Education, in technological areas, where, besides teaching subjects in those specialties, they also teach Mathematics or Physics.

§ 4. FROM PREDICTION TO ANALYSIS. Its productions.

We now present an analysis of productions in knowledge construction. This conservation of PI pointed to the fact that not only the action itself of Prediction had to be studied, but also that which permits to predict; the *Praediciere*, which makes itself manifest in the recognition of a basic information unit, by means of which it is possible to announce that which will happen. We classified PI according to the following scheme:

$$\textit{Prediction}\begin{cases}\textit{Long-term}\begin{cases}\textit{in discrete variation environments}\\ \textit{in continuous variation environments}\end{cases}\\ \textit{Short-term}\begin{cases}\textit{in discrete variation environments}\\ \textit{in continuous variation environments}\end{cases}\end{cases}$$

A comprehensive observation of solution provided by the teachers permite to recognize that when the following term is sought -and not the behavior *ad infinitum*- (short-term prediction) of a discrete variable describing the succesion a_1, a_2, a_3, ..., a fixation is produced on the local growth $a_1 \rightarrow a_2$, $a_2 \rightarrow a_3$, $a_3 \rightarrow a_4$, etc., and from this, by another *constantification* process, an attempt is made at recognizing the stable nature of the change process. The usual procedure is to observe the first diffrence $a_n - a_{n-1}$, the second difference $\left(a_{n-1} - a_n\right) - \left(a_n - a_{n-1}\right)$, and so on. In general, the regular behavior is sought, of that which is variable. This clearly determines primary approach strategies, which, due to the fact that they are functional mechanisms of a cognitive kind. they are preserved when passing from the variation in discrete environments to continuos contexts. In this sense, they are phenomenological principles inherent to the nature of the variation.

In this way of looking at the problem, a natural form is associated to the following array $a_n - a_{n-1} = \varphi(n)$, where $\varphi(n)$ is "easier" to study, as regards its variation, than a_n itself. Thus the following term a_{n+1}, depends on the preceding one a_n, and of something which is regular $\varphi(n)$, $a_n = a_{n-1} + \varphi(n)$. In this additive strategy predominated strongly their following results, and its presence temporarily precludes recognizing long-term prediction in discrete environments. Trough other questions, it was sought to determine prediction strategies when the additive recourse used so far was no longer available. In these, the task was to complete the linear sequence 1, 0, -1, 0, 1, ...; in three of the four answers, resort is made to a new strategy which does not use the preceding recourse, but an algorithm of a ciclic nature which links the last element with the first one. This is interesting, due to the fact that, in the absence of regularity patterns between succesive terms, the procedure becomes some sort of conservation principle by means of which it is sought that something *the ciclic remains constant*.

In questions suggesting long-term prediction, we tried to find out whither the strategy had been stable. Three of the four Professors recognize that the value of the last element in the array depends of the values of y_0 and of its successive differences y_0, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$, and $\Delta^5 y_0$. Thus initiates the recognizing process whereby a few initial values are sufficient to announce the final result.

This model, although permitting the successful solution of problems in this context. will become an obstacle for those in continuous variation. This scheme, which we call the *Praedicіere*, appears in the recognition of patterns which are only valid for the situations where it stems. Thev do not construct strategies that can be "inherited" by new situations. This situation becomes interesting because when passing from the prediction processes in discrete environment to those which are established in continuous environments. action strategies are inherited. Thus, a principle of conservation of the *Praedicіere* is operated.

As the kinematic representations of long-term prediction in continuous environments. they referred to the possibility of determining the position of a moving object at every instant, whenever its initial features of movement have been made explicit: thus we know its position and the manner in which it varies. Their answers find support in those stretegies belonging to discrete environments, and the studv of the wav in which it varies $\left(a_n - a_{n-1}\right)$ is decisive in the subsequent state of its evolution. The variation of position Δs with respect to time Δt is called the measure of the variation manner.

Finally, questions were asked about formulas describing position $s(t)$ at any moment. for this provided some information on the initial state. Here. a regular situation arises. as to performance so far obtained: three of our four teachers immediately attained long-term prediction in continuous variation environment. with a strong similarity with that obtained for discrite variation envirinments. In all cases. their answer was $s(t)=s_0+s'_0t+s''_0t^2/2!+\ldots$ On the other hand. there came to pass that in the absence of the telescopic predictor model for discrete variation. no general strategies inherent to the variation were constructed. but a search was initiated for formulas. or the task of completing tables was attacked.

§ 5. UNBALANCE AND RECOVERY.

It is in this sense, that the nature of movement phenomena. referred to continuos flux phenomena. given a tint to instruments and strategies which operate when approaching predictions problems. It is essential to describe *how* their evolution occurs. which means announcing what will happen with the behavior of flux: in other words. it is necessarv to *Predict* their development. For instance. the flow of water is induced by the presence of a difference in pressure at neighboring points $p(x+dx)-p(x)$. which. if zero. will indicate an equilibrium and. therefore. an absence of movement: naturally. if different from zero. it announces the presence of flow. which will have to occur in some preferential direction. Analogously. the prpagation of heat is determined by an effectivé difference of temperatures at neighboring points $T(x+dx)-T(x)$. The accumulation of heat in a body obeys to the action of the net difference in temperature variation at neighboring points. and this is expressed by $dT(x+dx)-dT(x)$.

The nature of flux phenomena underlines the need to study differences of the type $\varphi(A+dA)-\varphi(A)$. where φ can represent a wide variety of particular physical parameters. Thus. the fundamental difference becomes the cognitive instrument *par excellence*. and it participates of the nature of the phenomenon. Such a difference will be completely determined by the behavior of its variables at point A. i.e.. by means of the difference $\varphi(A+h)=\sum_{0}^{\infty}\varphi^{(i)}(A)h^{i}/i!$.

§ 6. GENERAL CONCLUSIONS.

In the productions analyzed, an evolution is perceived in the recognition mechanisms of the fundamental difference as the object of study of variation in discrete environments; such a difference is gathered and organized in a wider framework of "*mental homotaxis*" through a *reflection* process of one environment on another one. At that moment certain processes of analogy arrangement come into operation, enabling the embodyment of the former processes. A re-arrangement of such representation in continuous environments is not sufficient to achieve long-term prediction; it is necessary to construct praediction strategies which, propped up by the study of the *Praediciere*, succeed in being continued in long-term predictions. It is established as a *functional a priori* for this qualitative jump, the joining of the presence of the *Praediciere*, and the strategies linking the local fundamental difference to the comprehensive one, through a couple of basic principles: the *hereditary character* of the process, and its feasible *constantification*.

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ON DIFFICULTIES WITH DIAGRAMS: THEORETICAL ISSUES

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Although visual approaches are often advocated in mathematics, students do not readily adopt them. This paper puts forth cognitive and didactical considerations to explain this reluctance to visualize.

1. On the reluctance to process visually:

It is well documented in the literature that a vast majority of students are more inclined to process basic mathematical concepts in an analytical framework rather than in a visual one. The research of Dick (1988), Eisenberg and Dreyfus (1986), Monk (1988), Mundy (1987), and Vinner (1989) have established that this reluctance to use visual arguments exists and is wide-spread in student populations. The gravitation toward analytical representations affects all ability levels, even the mathematically precocious (Clements, 1984), and it doesn't seem to matter if the concepts were initially presented to the students in a visual framework or in an analytical one or in both; students are reluctant to visualize.

Students seem to consider the visual aspects of a concept as something peripheral to the concept itself. E.g., zeroes of a function f(x) are those values of x where f(x)=0, *and it just so happens* that at those values the graph of f(x) crosses the x-axis; a function f(x) is odd if f(-x) = -f(x), *and it just so happens* that its graph is symmetric with respect to the origin; functions f(x) and g(x) are inverses of each other if, for each x in their respective domains, g(f(x))=x and f(g(x))=x, *and it just so happens* that y=g(x) is the equation which represents the reflection of the graph of f(x) through the line y = x; this list could be extended ten-fold, but the conclusion is unavoidable; analytic descriptions of a property are preferred to visual descriptions of them.

More specifically,in the case of calculus Balomenos, Ferrini-Mundy, and Dick (1988) concluded:

> *Despite the calculus teacher's predilection for diagrams, our research indicates that students resist the use of geometric and spatial strategies in actually solving calculus problems* (p.196).

In another study Dick (1988) concluded:

> *There was no evidence of graphical interpretation of any kind .... even college students of relatively advanced mathematical training can be expected to ignore the use of their own graphs, even when these are produced immediately preceding a computational problem for which they could be used* ...(p2).

In other words, students do not know how to exploit diagrams they themselves draw in order to solve problems. The drawing is considered to be peripheral to the problem-itself, and this seems to be a general finding in mathematics instruction. Indeed, Sowell (1989) concluded from a meta-analysis of 60 studies that:

> *Instruction with pictures and diagrams did not appear to differ in effectiveness from instruction with symbols* (p. 499).

Why is this the case? Why is the reluctance to visualize basic mathematical concepts as wide-spread as it is? Authorities advocate the benefits of thinking of mathematical concepts in a visual way; e.g., Solomon Lefschetz (editor of the Annals of Mathematics) ... *saw mathematics not as logic but as pictures ... To be a scholar of mathematics you must be born with...the ability to visualize* ... (Halmos, 1987, p. 400), and to work in chaos theory ... *Graphic images are the key* (Gleick, 1987, p.38). According to Rival (1988, p.41): *Mathematicians are rediscovering the power of pictorial reasoning,* but why then is this interest in visualization not exploited by students? Some, like Polya (1945) and Sawyer (1964), have been proselytizing the use of visualization skills for many years. But a groundswell of advocates has never existed. Fischbein (1987) states: *What characterizes diagrammatic models is the fact that they represent intuitively the original reality via an intervening conceptual structure. Without a clear understanding of this intervening structure, with its laws and constraints, the diagram cannot deliver its message* (p.165). Fischbein seems to have put his finger on the problem; students often lack these intervening conceptual structures so that diagrams can deliver their messages, and there are reasons for this absence .

The studies by Goldenberg (1987), Hershkowitz (1989), Monk (1988), and Yerushalmi and Chazan (in press), are examples, from geometry and from analysis, where researchers have identified specific conceptual difficulties students incur when they need to use diagrams. But these are episodic in nature. In this paper, we attempt to identify some of the deeper common reasons underlying these difficulties. Our approach combines two points of view: a didactic one, based on work by Chevallard (1985), and a cognitive one, based on work by Larkin and Simon (1987).

2. **Why diagrams are difficult; a didactic reason:**

According to Chevallard (1985; see also Seeger, Steinbring and Straesser, 1989), knowledge undergoes a fundamental change when it turns from academic knowledge as known by mathematicians into instructional knowledge as taught in school; this change is called a "didactical transposition."

Academic knowledge is very intricate and contains many links and connections; these cannot be presented as a package since presentation is always sequential, one thing after the other. So the elements of knowledge have to be taken apart and ordered sequentially. The didactical transmission of knowledge implies the formation of a linear text, which structures the knowledge, giving it for instance a beginning and an end. As a consequence, links between concepts and procedures are omitted or destroyed; relationships, which are among the most important aspects of mathematical knowledge, have to be (re-)constructed painstakingly. In addition, in school, knowledge is necessarily taught separated from its context. These factors lead to a strong compartmentalization of knowledge: Mathematical knowledge is split from a "body of knowledge" into a large number of isolated "bits of knowledge. "

From Chevallard's didactical transposition, it follows rather directly that school knowledge is best represented sequentially, not diagrammatically. Because school mathematics is usually linearized and algorithmetized, it is so presented to students, and so preferred by them and so processed by them. An analytic presentation is basically sequential, and although it is possible to present intricate relationships analytically, this has to be done by taking them apart, quite as in the didactical transposition.

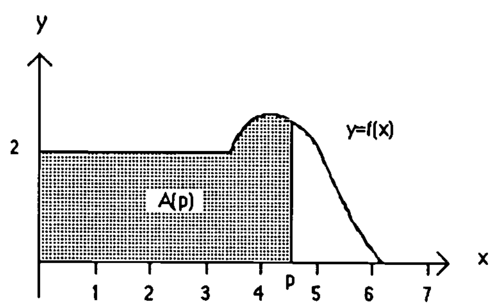
3. **Why diagrams are difficult; a cognitive reason:**

Larkin and Simon (1987) take an information processing point of view. They compare the accessibility of information needed to solve a problem when it is presented in a diagrammatic, versus a sentential form. (Accessibility can mean either ease of recognition or efficiency of search.) The distinguishing feature is that diagrammatic representations explicitly preserve topological (and geometric) relationships between components of the problem; sentential forms do not explicitly preserve these relationships. As a consequence, information may be more accessible in a diagrammatic representation than in a sentential one, even if the two representations contain precisely the same information. This strength of diagrammatic representations is achieved by "indexing information by its location in space"; that is, many elements in a diagram can be adjacent to each other, whereas in a sentence, any element is adjacent only to its two neighbors, the one preceding it and the one following it. In a good diagram, then, all information about a single element is grouped together.

Consider, for example, the following question from Monk (1988):

Fig. 1: A Pointwise and Across-Time Question

i) Pointwise: Determine the values of A(1) and A(3).
ii) Across-time: The point p moves from 4.5 to 6.0. Does the area A(p) increase or decrease?



In order to realize how much information is implicit in the diagram, the reader may try to give the data and formulate the question in a sentential format, avoiding a diagram. Then, imagine a student who needs to answer the question. As teachers, we naturally expect the question to be easier to answer if given in diagrammatic form. This is, however, only conditionally so. In fact, it is so only for students who have learned how to read and use diagrams of this kind. This diagram (as well as any other one) uses conventions, notations, generalizations, and abstractions without which the diagram is unintelligible. These start with the properties of the number line(s), the association between points in the plane and number pairs, the possibility of interpretation of a point in terms of a preimage-image pair of a function, the graph as the set of all points of the form (x,f(x)), and continue with properties of the function as represented by the graph such as continuity, area under the graph, the relationship between neighboring points on a graph (increase, decrease, concavity) etc. Some of this information will be needed for solving the problem, some will not. Therefore, even if all of the above elements are at the disposal of the student, it may be very difficult for him to quickly focus on the relevant information, disregarding the rest.

Larkin and Simon provide and analyze in much detail, examples from physics and geometry which exhibit the differences between diagrammatic and sentential representations. Although their main aim was the analysis opposing the two kinds of representations, they have also noted that many diagrams, and among them function graphs, do not describe actual spatial arrangements; therefore, they have inherent interpretations and conventions of the kind pointed out above; and consequently, they are useful only to those who know these interpretations and conventions and can thus develop thinking processes which exploit the advantages of the diagram. In summary, while to the knowledgeable diagrammatic representations are far superior to sentential ones for solving many problems in mathematics and physics, they may be completely unhelpful to the neophyte.

4. **Combining Chevallard, Larkin and Simon:**

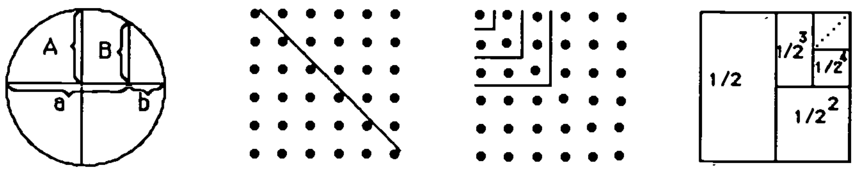
In brief, Larkin and Simon have shown that diagrams contain information, in particular, relational information, in highly concentrated, localized, strongly non-linear form. Chevallard has made the point that knowledge, as it enters school has to undergo a didactical transposition, one of whose main features is linearization. From this it follows logically, that it is natural to present school knowledge analytically, rather than diagrammatically; and it therefore should come as no great surprise that students prefer an analytical framework over a visual one.

An analytical presentation, being sequential, is simpler to absorb -- elements are presented one after the other, none are missed. Relationships between the elements may be lacking; if they are present they have to be introduced separately from the elements, tacked on to them. Diagrammatic representation is simultaneous, the elements and relationships between them are apparent at the same time, at the same location. They are therefore likely to be difficult to read, absorb, and interpret. Similar statements apply to thought processes: Visual processing is anything but linear, and as such, it represents a higher level of mental activity than analytic processing.

As an example, a rather arbitrary one, consider the following "proofs without words".

Fig. 2: Proofs Without Words

a) b) c) d)



$A = \frac{a+b}{2} > B = \sqrt{ab}$ $\sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i = n^2$ $\sum_{i=1}^{n}(2i-1) = n^2$ $\sum_{i=1}^{\infty} 1/2^i = 1$

Each diagram conveys the cognitive structures which have to be built in order to understand the proof. Focusing in on Fig 2c, we see how naturally the three separating lines structure the diagram in such a way that the equality of the two expressions in the equation becomes evident. This is due, to no small extent, to the properties inherent in the diagram itself. The diagram is structured so that the needed groupings and relationships become apparent by proper spatial relationships. Successive terms in the sum, for example, are represented by neighboring groups of points. The proof thus becomes a single unit, immediately understandable.

But how would one present such a proof to students? It is rather difficult to decide what to say first, where to start and how to get to the conclusion. Not only are many of the relationships likely to be lost on the way; it may take quite an effort for the students to get to the stage where they can see the entire argument from the diagram. And even if they do (or seem to), they may well turn and ask whether *the statement can now be proved mathematically*, i.e., analytically. It will probably be quite a bit easier for the teacher to rely on the fact that the sequence is arithmetic, to use the "well-known" formula for the sum of a finite arithmetic sequence, and to do the necessary algebraic manipulations. The students will no doubt accept that the statement is true, but their understanding of what the implications of the proof are, and where else the result could be used, will be considerably reduced. Vinner's (1989) results show clearly that an analytic formulation of a proof is preferred by the students, even if they seemingly could not make much sense out of it.

5. Conclusion:

This paper has documented that students are reluctant to think of mathematical concepts visually. It presented reasons as to why this reluctance to visualize is as widespread as it is. *Diagrams are useful only to those who know the appropriate computational processes for taking advantage of them* (Larkin and Simon 1988, p. 99). It seems as though visual processing is at a higher cognitive level than analytic processing. This hierarchical ordering of these two skills gives rise to a host of reasons why visual processing should be stressed in the curriculum at the expense of analytic processing. One of the foremost of these reasons is that obtaining the skill to think visually will automatically improve one's skill to think analytically; but the data seem to show that the opposite is not the case. Reading a diagram is a learned skill, it doesn't just happen by itself. To this point in time graph reading and thinking visually have been taken to be serendipitous outcomes of the curriculum. But these skills are too important to be left to chance.

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Department of Mathematics
Ben - Gurion University
Beer Sheva, Israel
January 26, 1990

Dear Teresa,

Enclosed please find one copy of our PME-14 paper. We hope that this gets to you in time, and that preparations are progressing as expected.

We are looking forward to the conference.

Sincerely,

Tommy Ted

Tommy Ted

THE TWO FACES OF THE INVERSE FUNCTION

PROSPECTIVE TEACHERS' USE OF "UNDOING"

Ruhama Even[1]

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This study investigates prospective secondary math teachers' knowledge and understanding of the inverse function. It draws on analyses of questionnaires and interviews with subjects from eight universities in the USA. The findings suggest that many prospective teachers, when solving problems, ignore or overlook the meaning of the inverse function as "undoing" what the function does. They also overgeneralize the idea of undoing. Their "naive conception" results in mathematical difficulties, such as not being able to distinguish between an exponential function and a power function, and claiming that log and root are the same thing.

Introduction

Functions opened new opportunities in mathematics. In addition to the typically algebraic operations of addition, subtraction, multiplication, division and raising to power, functions can also be composed and inverted. "The strength of the function concept is rooted in the new operations--composing and inverting functions--which create new possibilities" (Freudenthal, 1983).

The study reported here is part of a larger cross-institutional study of prospective secondary teachers' knowledge of functions (Even, 1989). This paper describes the prospective teachers' knowledge and understanding of inverse function. It concentrates on two different aspects of conceiving inverse function as "undoing". "Undoing" is an informal meaning of the inverse function which captures the essence of the definition. The importance of this informal meaning is also recognized by the National Council of Teachers of Mathematics who recommends that all students explore the concept of inverse function informally as a process of undoing the effect of applying a given function, while the precise definition of inverse function and composition of functions be reserved for college-intending students (Curriculum and Evaluation Standards for School Mathematics, 1989). The paper starts with a discussion of the use (or lack of use) that prospective teachers do with their informal knowledge and understanding of the meaning of inverse function as undoing. Then it describes problems with the inverse function as a result of dealing with it on an informal level of "undoing" only, with no relation to the mathematical notion of inverse function.

Method

Participants were 162 prospective secondary mathematics teachers in the last stage of their formal preservice preparation at eight midwestern universities in the USA. Data were gathered in two phases from November 1987 to April 1988. During the first phase, 152 prospective teachers completed an open-ended questionnaire. This questionnaire included non-standard mathematics problems addressing six interrelated aspects of function knowledge (Even, Lappan, & Fitzgerald, 1988). The questionnaire also asked respondents to appraise and comment on examples of students' work (each of which represented some misunderstanding or error related to functions). An additional ten prospective teachers completed the questionnaire during the second phase of data collection, and intensive interviews were conducted with the ten subjects in order to augment the analysis.

Meaning of Inverse Function as Undoing

When working on problems and answering questions that dealt with inverse function, many prospective teachers seemed to ignore or overlook the meaning of an inverse function as "undoing" what the function does. Instead, they used unnecessary calculations. For example, the participants were asked the following question:

****

Given $f(x) = 2x - 1$ *and* $f^{-1}(x) = \frac{x + 10}{2}$. *Find* $(f^{-1} \circ f)(512.5)$. *Explain.*

****

The following Table summarizes the ways in which the first phase subjects answered the question. The rows present the number of people who used each of the different methods. The columns describe the correctness and completeness of the use of the method.

	Correct	Incorrect	Not complete	No Answer	Total
Inverse property only	27	7	1	—	35
Inverse property & calculations	26	0	0	—	26
Calculations only	31	17	14	—	62
No answer	—	—	—	29	29
Total	84	24	15	29	152

From the Table we can see that about half of the participants who answered this question did not refer to the concept of inverse function and its "undoing" meaning at all in their answer even though by using the idea of "undoing", the answer to this question is immediate. These people just went ahead and attempted to calculate the answer. Less than half of the participants (53) based their answer on the correct "undoing" idea. One subject, for example, answered:

— " $(f^{-1} \cdot f)(512.5) = (512.5)$ When you put in a value and then put it in to the inverse function you'll get back the original value."

Although using the inverse property was sufficient, half of the participants who used this argument (26 out of 53) added calculations of some sort. There were several reasons for using unneeded calculations together with an explanation that was based on the meaning of inverse function. One reason was that the solver knew and was able to use the inverse property, but either felt uncomfortable not using all the given data (the specific functions at hand) or felt that stating a property (or a definition) was not enough to be considered as an explanation. One subject, Valerie, for example, wrote:

" $\frac{2(512.5) - 10 + 10}{2} = \frac{2(512.5)}{2} = 512.5$ All that has been done is taking the inverse of a function." Later she explained that she used the calculations as a way of explanation.

> ...So I didn't know if you wanted me to show it or just explain, so I just wrote it out...I wasn't sure what explanation you wanted.

This attitude points to a misunderstanding of what counts as an explanation in mathematics--a finding that fits with other research on prospective teachers' understanding of mathematics (Ball, in press; Even & Ball, 1989; Martin & Harel, 1989), and research on students' understanding of the validity of formal proofs in mathematics (e.g., Fischbein & Kedem, 1981).

Another reason for using the "undoing" idea together with calculations was that the solver did not consider the meaning of an inverse function until confronted with the original number: 512.5 as the result, realizing that the result of the execution of the calculation should have been known from the beginning. For example, Mike, who did not use the inverse property in his answer to the questionnaire, was asked to explain his work (all he did on his questionnaire was to give instructions of how to find $f(512.5)$ and then to plug that result into f^{-1} to find the answer, without really doing that).

> R: Ok. So what was the answer [the number]?
>
> M: I didn't figure out the answer.
>
> R: Can you figure it out right now?
>
> M: Ok. (Figuring answer.) $f^{-1}(512.5) = \frac{512.5 + 10}{2}$...
>
> And then you're going to do f of this, and I said this equals c. Take f of... (works on the calculations and gets back the number 512.5). I guess I should have known that you are going to get... since you get the same

number. I didn't realize it happens before (looks embareced).

R: Ok. And why are you supposed to get the same answer?

M: I just worked it out. (Laughs at himself.) Because you're just taking a function and then taking its inverse. So... I should have known that.

It is clear that Mike and others "knew" the inverse property. Still, they did not draw upon their conceptual knowledge but rather approached the problem by using an unnecessary procedural knowledge.

Looking at the inverse function as "undoing" what the function "does" is helpful in understanding the concept of inverse function. But limiting one's conception of inverse function to this "naive conception" only, results in mathematical difficulties. This is discussed in the next section.

Undoing as Naive Conception of Inverse Function

A power function (e.g., $f(x) = x^3$) and an exponential function (e.g., $f(x) = 3^x$) look similar. This similar appearance completely disappears when the inverses of the two functions are considered. Root (which is actually also a power function) is the inverse function of an odd power function (e.g., $f(x) = \sqrt[3]{x}$, since $\sqrt[3]{x^3}=x$), while log is the inverse function of an exponential function (e.g., $f(x) = \log_3 x$, since $\log_3 3^x=x$). An even power function does not have an inverse function since it is not a one-to-one function.

The following question deals with the relationships between these four functions.

****

A student said that there are 2 different inverse functions for the function $f(x) = 10^x$: One is the root function and the other is the log function: Is the student right? Explain.

****

The term "root function", which is the inverse function of a power function, is not used very often. So the participants had to decide about the meaning they wanted to attach to it. The most common description of the root function by the participants was the xth root of 10 or just the xth root (without specifying 'of what'). These subjects overgeneralized the idea of a root function, such as the square root: $f(x) = {}^{2}\sqrt{x}$, or, in general, $f(x) = {}^{n}\sqrt{x}$ (where n is a parameter), to an exponential function in the first place--$f(x) = {}^{x}\sqrt{10}$, and to an incorrect use of variables and parameters in the second place--$f(x) = {}^{x}\sqrt{\ }$. Both "function" descriptions meant: take the xth root of what you have (which was 10^x) but neither description was appropriate. One subject, Tracy, used ${}^{x}\sqrt{x}$ in the same manner, checking both the log and the "root" functions.

"$f^{-1}(x) = \log x$: $\log(10^x) = x\log 10 = x$ -- correct

$f^{-1}(x) = {}^{x}\sqrt{x}$: $(10^x)^{\frac{1}{x}} = 10$ -- incorrect."

Tracy used correctly the algorithm for checking whether a function is an inverse function but she did not really use her own definition of a root function. She composed the two functions and checked to see if she got the identity function $f(x)=x$ as the result. She explained why the root was not an inverse: "...you're not going to get x back out of it, so that's how I determined it." Tracy used her procedural knowledge of inverse functions and therefore correctly chose log as the inverse function of $f(x)=10^x$.

But the root function appealed to many of the participants. About one-third of the participants who answered the question (23 first phase subjects out of 63) used their naive conceptual knowledge of what an inverse function was. These people used the idea of "undoing" as their interpretation of inverse function. The xth root of 10 seemed to them to "undo" what 10^x does: In order to get 10^x, one starts with 10 and then raises it to the xth power. By taking the xth root of 10^x, one gets 10 back. One subject, Bob, composed log with $f(x)=10^x$ and got x, and then composed the xth root with 10^x and got 10. He then accepted both functions as the inverse function, even though the second time he got 10 instead of x. The "feeling" that an inverse function gives back what you started with (10 in our example, instead of x) lead many others to wrongly conclude that root was the inverse function of $f(x)=10^x$.

Accepting the root function as an inverse function because of its "undoing" appeal created a dissonance: many of the participants also remembered from previous study that log was the appropriate inverse function and inverse function was unique. To solve this uncomfortable situation, about one-third of those who answered stated that the log function and the root function were both inverse functions for the given function: $f(x)=10^x$, since they were the <u>same function</u>. Jenine, for example, wrote: "I believe that there is only one function. The root function and the log function are just two different ways of representing the same function." In her interview she added:

> ...Log is a power and that's what a root is...It's just a different way of expressing the same thing...

Jenine seems to think of root function, in this case, as the xth root of 10 ($\sqrt[x]{10}$) and of log as a root, or a power, a different way to describe powers. This wrong conception of log did not interfere with her ability to successfully solve regular log problems, as she recalled, since these problems usually require only procedural knowledge of logarithms.

Conclusion

Exponential and logarithmic functions as well as power (as a special case of polynomial) and root (power) functions are common as illustrations of theorems and properties in mathematics. They are used as specific cases to clarify general properties. Most of the prospective secondary math teachers who participated in the study did not seem to have a good understanding of them. They did not understand the difference between exponential and power functions and thought that taking the log and taking the root were the same thing. In such a case it is not clear how these functions can clarify theorems and properties. These functions are also an important part of the high school mathematics curriculum. The National Council of Teachers of Mathematics recommends that college-intending students develop a thorough understanding of specific functions including polynomial, exponential and logarithmic (Curriculum and Evaluation Standards for School Mathematics, 1989). Teachers, therefore, need a thorough understanding of these functions based on an understanding of inverse function. But, as this study shows, the participating prospective teachers seemed to have a fragile knowledge about these functions and inverse function.

Inverse function, as any other concept, cannot be understood in one simplistic way only. Understanding this sub-concept of the concept of function requires understanding the general meaning as well as the formal mathematical definition. Perceiving inverse function as "undoing" is powerful on one hand but is not enough for dealing with all aspects of the concept of inverse function on the other hand. This term is too vague and not precise. So, a solid understanding of the concept of inverse function cannot be limited to "undoing" only.

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INTUITIVE PROCESSES, MENTAL IMAGE, AND ANALYTICAL AND GRAPHIC REPRESENTATIONS OF THE STATIONARY STATE.
(A CASE STUDY)

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In the present work we examine the view held by Mathematics teachers (at University level) on the stationary state. The study was carried out during a process of research and controlled teaching. The stationary state was characterized by the intrinsic phenomenology of the concept arises from the need of determining the stationary state of heat flux. The observation method is that of case studies.

INTRODUCTION: Our study lies within the framework of a broad research project aimed at the interpretation of Fourier's work and its connection with the teaching of mathematics. In fact, it is our belief that the formalization process undergone by mathematics completely concealed heuristic ideas and processes of major importance for the acquisition of concepts such as the stament of physical problems aimed at the development of mathematical abilities.

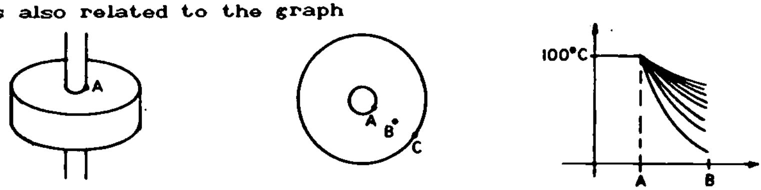
The wealth of heuristic processes in Fourier's work allows us to think of research alternatives that might be applied to the teaching of mathematics. In the first part of our research we must necessarily plunge into the study of the history of mathematics, especially for the period from late 18th century to early 19th century, in connection with Fourier's work [Farfán, 1989]. Another fundamental part is the study of processes developed by Mathematics teachers when faced with problems of a physical nature, showing a similarity with those approached by Fourier. A third component is that of analyzing the behavior of this same population, within a mathematical context.

In this paper, we will confine ourselves solely to the second part of our research; i.e., we will focus our attention to the analysis of processes developed by Mathematics teachers on the face of a problem on heat transmission.

CONCEPTS	SUMMARY OF QUESTIONS
GRAPHIC REPRESENTATION	Draw a graph for the function of temperature $T_{t_o}(x)$ for time t_o. Does its shape follow a known pattern?. What happens to the temperature at each point of the ring when t is "almost infinite"?. Plot the corresponding curve (temperature vs. radial position). How is this graph related with those you have drawn for various times ?.
STATIONARY STATE	Can temperature at A be 105°C ?. Can temperature at C be 100°C ?. What happens to temperature in the course of time ?. What happens to temperature at each point of the ring when time is "almost infinite"?
ANALYTICAL REPRESENTATION	Does your graph $T_{t_o}(x)$ follow a know pattern ?. Suggest a method to find its formula. Suppose you have the curve relating temperature T and position x for to,but you don't have the formula. Can an equation be found relating them?. Does it exist a formula representing each curve at each time t?.That is, Does it exist an analytical expression T(x,t) which establishes the functional dependencies ?.

From now on, we will call Model M1 the one which is related to the explanation and the figure proposed in the questionnaire, which is also related to the graph

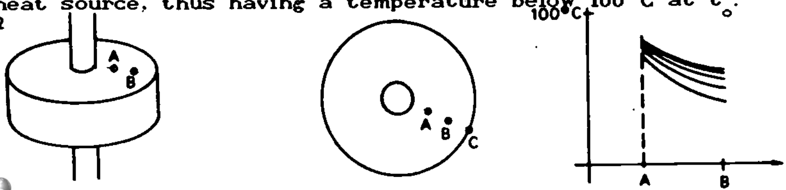
M1



In other words, in model M1 point A is next to the heat source, and it has a temperature of 100°C for t_o.

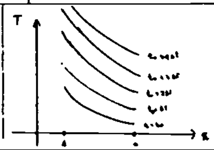
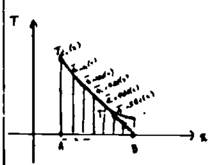
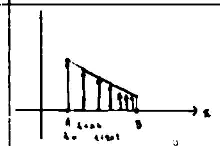
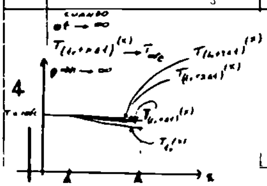
We will call model M2 the one related to the drawing below, our interpretation, which is also related on the right hand side. In this model M2, point *A* is a certain distance away from the heat source, thus having a temperature below 100°C at t_o.

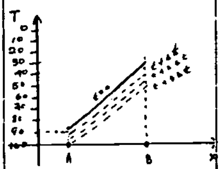
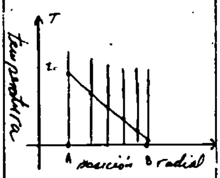
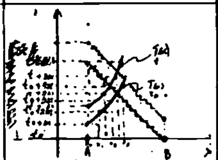
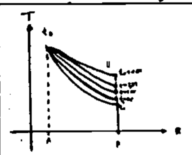
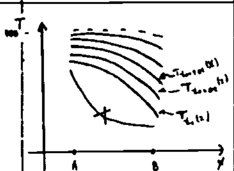
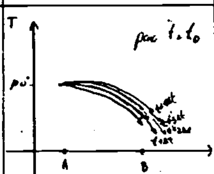
M2



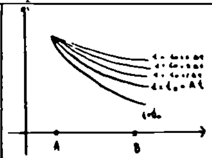
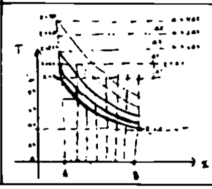
GRAPHIC REPRESENTATION AND THE CONCEPT OF STATIONARY STATE

In the table show below we can observe that 6 teachers drew a graph which is related to model M_1, and the rest drew one which relates to model M_2. Teachers 2 and 3 confused the Distance vs. Temperature graph with the one relating Time vs. Temperature. Teachers 6 and 7 confused the Distance vs. Temperature graph with the one linking Distance vs. Time. It was probably a mistake on the part of the researchers, not to have used a different notation for the Temperature axis (i.e. C could have been used, for instance, instead of T, to refer to Heat in degrees Centigrade).

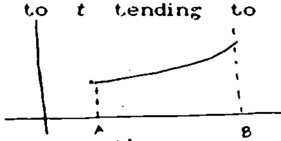
TEACHER No.	GRAPHIC REPRESENTATION	THE STATIONARY STATE CONCEPT.	ANALYTICAL REPRESENTATION FORMULA	METHOD EQUATION FOR t_0	METHOD FUNCTION 2 VARIABLES
1		Temperature is greater at A, and less at B		Applying successive derivatives, and Taylor method.	By Taylor's series, $T(x,t) = T(x_0,t_0) + \frac{\partial T(x_1,t_0)}{\partial x_0} h$
2		Temperature is a constant (it reaches an equilibrium); the coldest points receive more heat is reached.	Not possible ..distance and time vary at the same time. until equilibrium	It can be found, but it is almost impossible.	Yes, it does exist: a relationship can be established between Temperature-Distance, and Temperature vs Time
3		If we consider the rings to be infinite, T will become zero at $t \simeq \infty$	For heat transmission, you have a s. line	No. It may be possible, but using a thermo. for some.	Yes, it does exist; it involves partial derivatives.
4		A heat equilibrium is reached; i.e. there are no longer any temperature variations.	$T_{t_0}(x) = k e^x$	1 Regression. 2 Least squares 3 Differential equations	This could be done by means of a two variable Taylor Series.

TEACHER No.	GRAPHIC REPRESENTATION	THE STATIONARY STATE CONCEPT.	ANALYTICAL REPRESENTATION		
			FORMULA	METHOD EQUATION FOR t_o	METHOD FUNCTION 2 VARIABLES
5		The points closest to the source will remain in equilibrium; the more distant ones will be almost cold.	I could find the equation for the stright line $y-y_1=$ ...	By using the Taylor Series try to find the equation that conformed to the real thing.	I should thing so, for the only thing I see is that f moves vertically.
6		Temperature prsents a maximun, and from there on, it will decrease ... and it will tend to become standardized.	Yes, there are aproximative methods, or the Taylor Series.	I don't remember, but there does exist an equation.	There are always mathematical methods to conform...
7		...temperature is the same at every point, when t is almost infinite	Parabola $T(x)=x^2$	Yes by comparing with Taylor's Series.	Yes $T(x.t) = f(x)$ $f(x)+$ $f'(x)h+$... I believe it can be done by applying Taylor's Series
8		It increases at each point, but eventually an equilibrium is reached, because at the last point.	$y=\frac{1}{x}$	It would be the preceding formula: $T(x)=\frac{1}{x}$	This formula can be the expression for $T(x)$: $T(x+h)=$ $T(x)+T'(x)h+$ ...
9		The time will come when temperature at each point becomes constant.	Yes. One way is by least squares: $T_t(x)=a+b\cdot\sin cx$	By finding the min. of $S=\sum_{i=1}^{n}[T(x)_i-a-b\cdot\sin cx_i]^2$	$T(x,t)\simeq T_o(x)+a_1 T_{\Delta t}(x)+\ldots+a_n T_{n\Delta t}(x)$. $T(x,t)=\lim_{n\to\infty}$
10		It tends to become stationary, i.e. to remain constant, or it would reach the limit.	Yes, the curve resembles the "bachystocrone", and with Taylor.	Yes $T(x)=T(x)+T'(x)\Delta x+T''(x)\Delta x+$...	Yes, it does exist It is a Fourier series since every curve exponential sin and cos

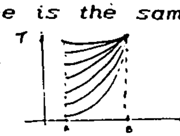
T COPY AVAILABLE

TEACHER No.	GRAPHIC REPRESENTATION	THE STATIONARY STATE CONCEPT	ANALYTICAL REPRESENTATION: FORMULA	METHOD EQUATION FOR t_o	METHOD FUNCTION 2 VARIABLES
11		When time is very large, temperatures for each point will have become stationary ...	It is an exponential curve: there exist approximative methods.	Same answer as above.	Such a formula does exist ... variation in time and position gives us a Differential Equation.
12		..because it can flow externally, temperature does not become almost infinite.... very great.	Yes, it resembles: $f(x)=\frac{a_o x^n+..}{b_o x^n+..}$	Yes. It can be found by means of a two-variable Taylor's Series.	There must be a formula ... by means of two-variable Taylor time and distance.

Teachers 3, 5, and 6, associated a straight line to the function, for t_o. Teaches 5 showed a surprising feature: he changed the origin of the vertical axis, thus reversing the scale: we interpret (from his graphs) that he is, indeed, thinking of a family of bounded functions. However, in the question related to t tending to infinity, he draws the following graph:



Teacher 7 reverses the curves and plots Distance vs. Time. From his drawings it can be deduced that temperature increases. Nevertheless, when explicitly asked about temperature as t tends to infinity, his answer is that "*temperature is the same at all points*", and he proposes the following graph:



i. e. temperature at A is less than temperature at B for every instant in time. Another contradiction is perceived in the answer given by Teacher 9, who wrote that when t grows sufficiently large a level of equilibrium is reached it the temperature of the solid body. Yet, in his graphs when t grows so does temperature, and he does not indicate any boundary.

Teacher 12 is in a better position than his colleagues, for he does not incur in a contradiction [Hitt, 1989]. He is consistent in his error when he points out (both in the graph and in his explanation) that temperature will become great as *t* tends to infinity.

Thus, we can be certain that some teachers (those that fall in contradictions) need to strengthen the interaction between the image conception they about the physical phenomenon, and their written interpretation.

ANALYTICAL REPRESENTATION

These teachers received strong instruction concerning the development of functions through the use of Taylor's Series. In fact, in a large number of answers (8) it is thought that a function can be associated, to the graph they found, using Taylor. They did not realize that they would have to be in possession of the function, either to approximate it or to develop it in its Taylor's Series.

Teacher 2 and 6 pointed out: *"it is almost impossible, but a formula exists"*, and *"There are always mathematical methods for approximating"*, respectively. These statements suggest that Mathematics is so powerful that no matter how difficult something might be, we can always construct it. [Vinner, 1983] points out something similar, concerning his students. He mentions, about the question: Is there a function that corresponds 1 to each positive number, corresponds -1 to each negative number, and corresponds 0 to 0? that *"functions (which are not algebraic) exist only if mathematicians officially recognize them (by giving them a name or denoting them by specific symbols). This view was expressed by answers like..*

No. as a matter of fact perhaps there is such a function but I do not know about it"

COMMENTS

There is little interaction between the intuitive ideas and mental images presented by these teachers in their first drawings and their representations of the same problem in a more general context. The need is seen to close the gap between these situations. Doing so is of fundamental importance, considering that some of the teachers were in such a contradictory situation, without their being aware of it.

Concerning the mathematization process of the physical phenomenon, we can infer that their general knowledge of Taylor's Series played a very strong part, in that such knowledge prevented them to properly associate their intuitive ideas about the behavior of the physical phenomenon, to formal mathematical ideas. In some cases, this teachers explicitly proposed a particular function, but when passing to the general context they did not take into account their own initial proposals. The specific case is ignored when passing to the general case.

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THE ROLE OF CONCEPTUAL ENTITIES IN LEARNING MATHEMATICAL CONCEPTS AT THE UNDERGRADUATE LEVEL

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In this paper we begin to examine the role of cognitive entities in the learning and use of certain ideas important to the undergraduate mathematics curriculum where the conception must act as a mental "object" in comprehension or reasoning processes. Depending on specifics of the situations, these include the idea of function, limit of functions of two or more variables, differential and integral operators, cosets in vector spaces and groups, among others. We attempt to distinguish between the general function of cognitive entities as means for overcoming natural working memory constraints, via chunking or encapsulation processes, and their more specific roles in certain types of mathematical thinking and certain concepts.

Introduction

We have all had the experience of thinking in terms of mental objects. For example, suppose one asks if a vector space V and its double dual V** are isomorphic. At one level, one is asking about the "objects" V and V**, and to begin describing an isomorphism, one may go on to describe a correspondence between respective vectors in the two spaces, which again, are treated mentally as objects, although they might be n-tuples or matrices, for example. The aim of this paper is to begin to discuss in somewhat more precise terms the processes of using and building such mental objects and their roles in helping us to understand ever more complex mathematical concepts. In a subsequent paper (Kaput & Harel, in press) we will extend our discussion to include the role of notations in the processes of forming and applying mental objects.

Greeno (1983) defines a conceptual entity as a cognitive object for which the mental system has procedures that can take that object as an argument, as an input. The cognitive process of forming a conceptual entity has been called "encapsulation" (Ayers, Davis, Dubinsky, and Lewin, 1988), or "entification." Ayers et al. view this process as an instance of "reflective abstraction" (Beth and Piaget, 1966), in which "a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower" (p. 247).

The construction of function as a conceptual entity is an example of the entification process (Thompson, 1985; Harel, 1985; Ayers et al., 1988). One level of understanding the concept of function is to think of a function as a process associating elements in a domain with elements in a range. This level of understanding may be sufficient to deal with certain situations, such as interpreting graphs of functions pointwise or solving for x in an equation of the form f(x)=b, but it would <u>not</u> be sufficient to deal meaningfully with situations which involve certain operators on fuctions, such as the integral and differential operators, as we will see later in this paper. For the latter situations, the three components of function--the rule, the domain, and the range--must be encapsulated into a single entity so that these operators can be considered as procedures that take functions as arguments. Incidentally, a formal definition of a function as a set of ordered pairs, a mathematical entity, does not appear to play a role in such situations - when would one conceive of a function as a set of ordered pairs in the context of applying a differential operator to that function?

The construction of conceptual entities embodies the "vertical" growth of mathematical knowledge (Kaput, 1987). For example, at lower levels the act of counting leads to (whole) numbers as objects, taking part-of leads to fraction numbers, functions as rules for transforming numbers become objects that can then be further operated upon, e.g., differentiated. It complements the kind of "horizontal" growth associated with the translation of mathematical ideas across representation systems and between non-mathematical situations and their mathematical models.

Greeno (1983) examined the role of conceptual entities in understanding and in problem solving. He conjectures four functions of representational knowledge that involve conceptual entities: forming analogies between domains, reasoning with general methods, providing computational efficiency, and facilitating planning. Greeno offered empirical findings that are consistent with his conjectures; these findings deal with elementary mathematics--geometry proofs and multidigit subtraction--as well as physics, puzzle problems, and binomial probability. He also suggests that instructional activities with concrete manipulatives can lead to an acquisition of representational knowledge that includes conceptual entities. Recent work by Dubinsky and Lewin (1986), Dubinsky (1986), and Ayers et al., (1988) demonstrates how computer activities in learning mathematical induction and composition of functions can facilitate the construction of these concepts as entities.

Roles of Conceptual Entities

In this section we will discuss the concepts of function, operators, vector-space, and limit in terms of the role that conceptual entities have for alleviating working memory or processing load when concepts involve multiple constituent elements, facilitating comprehension of complex concepts, and assisting with the focus of attention on appropriate structure in problem solving. The first two of these psychological necessities will be discussed below in turn. Space limitations prevent an examination of the third.

Working-memory load

One psychological justification for forming conceptual entities lay in their role in consolidating or chunking knowledge to compensate for the mind's limited processing capacity, especially with respect to working memory. To avoid loss of information during working memory processes, large units of information must be chunked into single units, or conceptual entities. Thus, thinking of a function as a process would require more working-memory space than if it is encoded as a single object. As a result, complex concepts that involve two or more functions would be difficult to retrieve, process, or store if the concept of function is viewed as a process. This is true for many concepts in advanced mathematics. Imagine, for example, the working-memory strain in dealing with the concept of the dual space of the nXn matrices space if none or only a few of the concepts, matrix, vector-space, functional, and field are conceived as total entities.

Comprehension: "Uniform" operators versus "point-wise" operators

Despite the heavy working-memory load involved in understanding the dual space of the nXn matrices space without most of its subconcepts being entities, it is still possible to make sense out of it, at least momentarily. In some situations, however, the justification for the formation of conceptual entities is more than just a matter of cognitive strain that results from a memory load. In such situations comprehension requires that certain concepts act mentally as objects due to an intrinsic characteristic of the construct involved. Examples of such situations include those which involve the integral or differential operator. These types of "uniform" operators <u>cannot</u> be understood unless the concept of function is conceived as a total entity. We distinguish these from other types of operators on functions which we could be termed "pointwise" operators, and for which there is no need to conceive functions as objects. Operations with functions--such as sum and composition--can be treated as "pointwise" operators; the cognitive process of understanding these operators involves the conception of a function as a process acting on elements of the domain. For example, in constructing the composition of two functions f and g, say fog, one must first perform the process g on an arbitrary element x of the domain, generating a result $g(x)$, and then performing the process of f on that result to obtain $f(g(x))$, all conceivable as acting on individual elements of the domain. These two separate operations are coordinated to produced a new process, which then it interiorized, and the resulted in a new process fog (Ayers et al., 1988; p. 247). Similarly, in constructing $f+g$, for every input x, the outputs, $f(x)$ and $g(x)$, are produced to construct the sum, $f(x) + g(x)$.

In "uniform" operators, in contrast, the point-by-point process is <u>inapplicable</u>. For example, to understand the meaning of $\int^t f(x)dx$ as a function, it is necessary to think of $\int^t$ as an operator that acts on the process $x \to f(x)$ <u>as a whole</u> to produce a new process $t \to \int^t f(x)dx$. <u>It is the awareness of acting on a process as a whole--not point-by-point--that constitutes the conceptions of that process as an object.</u>

Mathematically unsophisticated students attempt to interpret "uniform" operators as "pointwise" operators apparently because they cannot conceive of a function as an object. Consider the derivative operator. Our experience in the classroom suggests that many students understand that $f'(x)$ means: for the input x there is the output $f(x)$, and for that output we get the derivative $f'(x)$. This understanding is likely the conceptual base for many students' answer to the question, Find the derivative of the function, $f(x) = \begin{cases} \sin x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$,

to be:

$$f'(x) = \begin{cases} \cos x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Interpreting "uniform" operators as "pointwise" operators is quite common among students, not only with the concept of derivative. It likely has to with students' limited understanding of the concept of variable. Apparently, this misconception is inferred by students from the formal computations that are used to introduce algebra in high school, especially evaluation of expressions for single values of x and the solving of equations for a single unknown (Kaput & Sims-Knight, 1983) in which the true notion of variable is not employed because variables take on only single values. For them, a variable is "an unknown number."

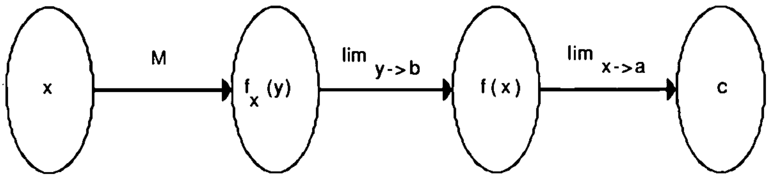
Comprehension: the case of object-valued operators

To begin with, the real-valued function $f(x, y)$ of two real variables can be interpreted, as a process, in two ways. The first is amenable to the process-conception of function: $f(x, y)$ is thought of as a process mapping points on the plane, (x, y), into points on the real line, $f(x, y)$; thus, students who possess the process-conception of function would likely have no difficulty dealing with this interpretation.

In a second interpretation, $f(x, y)$ is a process associating points on the real line, x, with functions, $f_x(y)$, which are themselves processes from the real line into itself. Like the "uniform" operator in which a function is thought of as an input-argument, in this interpretation a function is thought of as an output-argument. Cognitively, thinking of a function as an output is <u>not</u> different from thinking of it as an input, in the sense that in both cases a function is treated as a variable and as an instantiation of "something else" which is viewed as an entity. This "something else" is, in the "uniform" operator case, the noun, <u>input</u>; and in the second interpretation of f(x, y) is the noun, <u>output</u>.

This analysis, which has to be empirically substantiated, is supported by our informal observation while teaching undergraduate mathematics classes the concepts of double limit,

$\lim_{(x,y)\to(a,b)} f(x, y)$, and the iterated limit, $\lim_{x\to a}\lim_{y\to b} f(x, y)$. As some textbook authors indicated (e.g., Munroe, 1965; p. 108), we observed that while computationally the iterated limits is easier than the double limit, conceptually the iterated limit involves a more sophisticated idea, which causes difficulty for students. In stating and proving certain theorems on iterated limits (e.g., theorems concerning conditions on the equality between this limit and the double limit), the analysis of $\lim_{x\to a}\lim_{y\to b} f(x, y)$ as being a composition of the following three mappings (see Figure 1) is inevitable:



1. M: $x \to f_x(y)$, whose domain is a set of numbers and whose range is a set of functions;
2. $\lim_{y\to b}$: $f_x(y) \to f(x)$, whose domain and range are sets of functions;
3. $\lim_{x\to a}$: $f(x) \to c$, whose domain is a space of functions and its range is a set numbers.

Students responses and questions indicate a difficulty in dealing with aspects concerning the operator M, which, as indicated earlier, requires the object-conception of function. While the operator M must be understood as an object-valued operator, the other two operators, $\lim_{y\to b}$ and $\lim_{x\to a}$, can be viewed in two ways determining different levels of understanding the concept of iterated limit. In one way $\lim_{y\to b}$ and $\lim_{x\to a}$ are uniform operators acting on objects which happen to be functions. This level of understanding, although desirable, is not achieved by an average undergraduate student, who usually views these limits, and the concept of limit in general, as pointwise operators.

The limit of function is another example of a pointwise operator. To understand this complex concept, many clusters of knowledge about different concepts in mathematics are required. We will not attempt to analyze this knowledge in this paper; however, the process-conception of function is sufficient (and necessary) to understand this concept. It is so because $\lim_{x\to a} f(x)=L$ involves the <u>dependency</u> between the behavior of the numbers "near" a, x's, or inputs of f, and the behavior of their outputs, $f(x)$'s, "near" L.

Beside the iterated limit, the undergraduate mathematics curriculum is full of situations that involve object-valued operators. For example, those which concern parametric functions, such as $f(x)=ax+b$, $f(x)=\sin(ax)$, $f(x)=\log_a x$, etc., or parametric equations involving such functions. In these situations the correspondence between the parameters and the function, or the equation, constitutes an object-valued operator. In (Kaput, 1986; in preparation) we report on an extended study of secondary level students whose task was to determine an algebraic rule that fit a student-controllable set of numerical data. Their behavior allowed a clear and stable decomposition of the group of students into two sets, one of whom consistently used a pointwise pattern-matching process, mediated by natural language formulations of their proposed "rules," while the other searched for and applied a parametrically mediated formulation of their proposed rules. The latter, for example, would look for constant change in the dependent variable, identify this as the "m" in y=mx+b, and proceed from there. For them the process was a search for parameters. In effect, they were dealing with a space of functions (albeit a limited one), whereas the other group of students conceptualized the task as a pointwise pattern match.

Another common example, related to the vector space discussion above, involves the construction in abstract algebra of the quotient object associated with a "normal" subobject, e.g., in the case of groups. The cosets must be conceived as objects if they are to participate as elements of a group. However, the existence of a "representative element" for a coset, where the operation defined on cosets can be given in terms of an operation on their representatives, makes it possible to deal successfully with many aspects of the quotient group on a symbol manipulation level without treating the subsets of the group as objects, or even as subsets. Students' inadequate conceptions are revealed when one asks them to attempt to create a group using a non-normal subgroup's cosets - they often cannot understand why the subsets "fall apart" when they attempt to multiply them together as sets, or by using representatives.

Reflections

This paper is but the briefest introduction to a complex set of issues, most of which were avoided because we concentrated in getting some specific examples on the table for discussion. We likewise avoided significant examination of empirical findings that may lie behind some of the assertions made. However, we hasten to add that we know of relatively little empirical work done from the perspective of this paper beyond that which was cited. Much is needed. In subsequent papers we hope to clarify the roles that different notations may play in entity formation and application, as well as to attack perhaps an even more subtle matter - the difference between a well-formed schema and a cognitive entity.

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Mathematical Concept Formation In the Individual.

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The project desribed here discuss how conceptions such as Etnomathematics may inspire the study of learning advanced mathematic in industrialised countries. The project involves interviewing four pupils at work. The results give support to the framework of rationales of learning. It will be explicated how pupils' learning processes and learning results can be seen as organic results of the individual's view of teaching and learning mathematics and of the individual's specific wishes and needs in relation to the content and the structure of mathematics instruction in the school.

FIELD OF INTEREST.

The importance of conceptions such as Ethnomathematics, natural mathematics and Folk Mathematics is evident in analysing education in the Third World. In industrialised countries a number of curriculum planners and teachers are inspired by these ideas. Most of the attempts to utilise the ideas concern education on lower level and adult training in elementary mathematics.

In this project the field of interest is to analyse teaching and learning of advanced mathematics in industrialised countries. What part could conceptions such as etnomathematics play in the analysis ? What inspiration is produced by such conceptions and how should they be developed with special reference to advanced mathematics in industrialised countries ?

We have no investigations that help us identify what may correspond to etnomathematics when looking at advanced mathematics in industrialised countries. In this project the conception of everyday knowledge is used with the aim, firstly, of grasping not only everyday thinking tools corresponding or contrasting to those found in mathematics, but also knowledge about parts of the world that are in correspondence to or contrast with the way mathematics is applied to those parts. Secondly, the aim of introducing the conception of everyday knowledge is to stress the focus on what happens and what is comprehended outside organised instruction.

It is unquestionable that the relationship between everyday knowledge and mathematical conceptformation is of a qualitatively different nature depending on whether the subject under consideration is below or above what we could call the Arithmetic Border. As mentioned above, our concern is to draw attention to the levels above the Arithmetic Border, and we have chosen to investigate the level just above that border.

The project will be searching for

a - specific kinds of structures and contents in everyday knowledge in a country like Denmark, and for
b - specific features of the relation between everyday knowledge and mathematical conceptformation, which

1.- can either serve as butterfly-nets, i.e. nets for grasping mathematical concepts
2.- or will function as blockage for the development of mathematical concept-building.

In the former case everyday knowledge, respectively the relation between the two areas, may potentially have a productive function for the learning of mathematics. If it were possible to make explicit or describe in some way such everyday knowledge, that could be an element in developing teaching methods based on equal dignity-relations between the participants.
In the latter case everyday knowledge, respectively the relation between the two areas, could have a destructive function for the learning of mathematics. If again it were possible to make explicit or describe this type of everyday knowledge we might be able to formulate some of the psychological and cognitive reasons for some pupils learning-problems.

Investigating the relations between the two areas does not constitute a field of well-defined problems, but rather certain optics on the problem field of mathematics education.

This optics is across prevalent theoretical considerations like:
Rationales of learning,
Sociology of Youth,
Cognition,
Needs of qualifications,
Teaching,
Theory of transfer.

The optics is capable of throwing spotlight on problems of several sorts. Problems such as
-blockings
-learning difficulties
-sorting
-discrepancies between "the matter taught" and "the matter learnt"
-predictions of future qualification needs and demands
-poor transferability.

Simultaneously the optics carries the seed of new principles of how to structure cources in order to remedy some of the problems. Perhaps it might be possible to get to know more about how to
-get rid of blockings
-clarify sorting mechanisms
-indicate keys for particularly productive modes of understanding
-provide opportunity the acquisition of more future orientated qualifications.

PLOT AND DESIGN OF RESEARCH
Teaching and learning mathematics give birth to problems of high complexity. I have been searching for an avenue of investigation

capable of capturing this complexity without destroying or removing significant features - yet, it must be possible to restrain and to communicate the plot and the method.

I chose to approach **individual learning** of mathematics as it takes place **in ordinary instruction.** I chose to investigate individuals in the start of the gymnasium, because changing school may illuminate one's learning style and specific learning problems.

My endeavour has been two-sided:
a) It has been my wish to describe and analyse the individual's preknowledge from the former school, his/her attitudes to school and to the subject of mathematics, something I call his/her "field of attention", and the development of mathematical knowledge and skills. I wanted to describe and analyse processes of learning as well as products.

b) And I wished to develop a method, which I call "interviewing pupils at work".

This method consists of the researcher being together with each individual, after school without a fixed time-table (but max. two hours). During this being together several instruments are played:
- the researcher interviews the pupil about attitudes to education, school, subjects, specific classes, learning habits, learning methods
- the pupil thinks loud while working with textbook and exercises,
- the researcher questions about specific issues/topics,
- the researcher presents mathematical explanations and methods and listens to the pupil's reaction,
- the pupil asks questions,
- the pupil suggests which exercises to work through and choose which topics to discuss,
- spontaneous conversation.

The being together must run through a relatively long period in order to establish possibilities of recognisé of development and changes and in order to establish opportunity to become aware of and correct researcher's possible wrong perceptions of what is going on. In this case it was 4 months in autumn 1989. In addition I followed some of the mathematic lessons as an observer.

I chose four 16-17 years old from the same class. They are all above average in their age group as regards interest, knowledge and skills, as they follow the danish "mathematical gymnasium".

I wanted to choose four pupils differing in their ways of learning. I made the choice after letting all the pupils in the class answer a quetionnaire. I had constructed the questionnaire in order to obtain information about :
A. the level of support versus press from the family
B. pupils' experiences of success and failure as something stable and inner-grounded versus something unstable and grounded on external factors.
C. the individuals' own criteria for gaining understanding and knowledge.

RESULTS

My being together with the four pupils has produced a very rich fund of empirical knowledge concerning cognition in relation to mathematic as it takes place in the individual.
Some of the main results may be given in statementform as follows:

1: Firstly, you find a strong consistency throughout a person's view of mathematic, rationale of learning, way of working with the subject and the character of the concepts built by that person. The result provides a corroboration for the validity and usefullness of the theoretical frame-work Rationales of learning, and suggests a detailization.

2: Secondly, potentials in pupils' conscious, but non-stated needs and wishes towards teaching-content and teaching-style can be made public, and it is my thesis that these potentials may serve as a building stone for new manner of teaching learning.

3: Thirdly, the selection proces in mathematic instruction, which divide the pupils in those with succes and those without, functions partly through invisible mechanisms.

4: Fourthly, it is meaningsfull to see the learning of mathematics as a vehicle for stabilization or destabilization of self-confidence.

5: Fifthly, some of the basic difficulties of the pupils are grounded on specific features of the relations between everyday knowledge and mathematical concept formation.

Finally the results as a whole make it possible to formulate conjectures of how important features, such as examinations and exercises, could be changed in order to fulfil the idea of competence-constructing, democratic, equal-dignity mathematical education. The challenging problem then becomes how does an education following this guidelines relate to the future qualification needs.

RATIONALES OF LEARNING - The engine that creates the dynamics.

In what follows I shall present the first results from my investigation.

During the four months it became possible for me to see important elements of the pupils' rationales for learning breaking through every of their concerns towards working with mathematics. It is these observations that convinced me of the validity and the usefullness of the frame-work. The rationales of learning are so to speak the engine that creates the dynamics.

The four pupils represent four qualitatively different specifications and mixtures of Rationales. In the following the pupils are called Ann, Paul, Mary and Michael.

Ann is fascinated by two features of the mathematics:
-she **is fascinated by the way maths produces its** results. She likes the known staging with figures and solutions as steady members of a theatrical company, and she likes that assertions so obviously happen to be either true or false.
-she likes **the possibilities** offered by mathematics of getting **to know** certain things as **immediately** obvious. In learning foreign lanquages you just have to try to remember, she tells me, whereas in mathematics you sometimes "see the logic". This two-sided fascination constitutes her S-Rationale.

From the very beginning in the gymnasium it is obvious that Ann is shocked. The speed is too high for her, and the level too.She is not doing as well as she expected. The low level of her own ability becomes her main concern. She has to repress her wishes to see mathematics as "logical". She becomes thrown upon to concentrate on searching for important features to remember. In reading the textbook she looks for emphazised and framed-in elements, and otherwise she just "reads from the bottom to the end".
Her dominating wish "to do well" does not constitute any internal filter, so this wish does not afford her any help in her help-demanding position.

Using the definitions of S.Mellin-Olsen I suggest, that because Ann's S-rationale does not get any response from her new surroundings, and because of her learning problems, her I-rationale becomes dominant. Unfortunately it does not help her problems.

Paul is engaged in searching tasks that are **ready** to be accomplished, tasks that the institution "gymnasium" asks him to accomplish. In addition to searching for tasks, he is also looking for rules to be useful in solving the tasks. What it is all about is of minor importance. His interests in mathematics are born as mainly operational, not orientated towards understanding or criticism. His interest in Reality is just the same.

Paul is engaged in building upon his very rich fund of knowledge of the rules of the game and of the standard behaviour in school-math. He generalizes his fund of data to generate principles of the system, as for instance
"in most exercises it is appropiate just to use the most recently discussed frames and rules",
"exercises with per thousands are solved by dividing the smallest number with the biggest number",
"if I get a solution containing more than three decimals, then I have made a mistake",
"if the figures 120 and 15 occur in the same exercise, you should in most cases divide 120 by 15".

Pauls concepts are heavily affected by his interest in action. His concept of division of fraction by fraction, for instance, is governed by rules, formulated as manipulations of symbols.

Using the frames of S.M.O I shall suggest that no antagonism exists between the S- and the I-Rationales. They are rather identical.

Mary is engaged in **searching for new ways of** seeing offered by mathematics. Asked about which elements were the most fascinating in the first four months she mentions features that lie outside her previous knowledge and outside her previuos imagination of things mentally created.

She always reflects upon definitions. For instance her reflections on the definition of a function, which is an analogy to a feature in the everyday world : the prices of articles in a grocery, where every article has onen and only one price - while it is possible to find examples of prices given to more than one article. When I asked her why she is performing these reflections - whether the teacher, the textbook or I ask her to - she answers, that she cannot find of any other way to cope with the concepts. It is beyond questioning for her just to read, doing nothing else.

She dislikes exercises demanding several calculations, unless she can connect each step of the calculations with something she finds meaningful, but it is of no importance whether the meaning is inner- or extra-mathematical. She is, however, disappointed of the inferior power towards Reality offered by the newly learned mathematics. She thinks she learns too little about the Real World.

Mary is marked mainly by her S-Rationale which is constituted by two different interests: her interest in getting to know new ways of seeing and her interest in getting to know about the world outside the school. The former interest is partly satisfied in the course, the latter is not.

Michael is engaged in **searching for the meaning** of mathematics. His thoughts circulate around questions like : " Why this definition ? Why this concept ? Why equations ? Why proofs ?"
If he is finding no satisfactory answers, he is not able to go on working. Not every kind of answer satisfies him. Answers telling which type of exercises could be solved by the mathematical tools taught to him will not meet his interests and curiosity.
He has a feeling of mathematics as a fund of knowledge filled by meaning, and he loves to participate in lifting a corner of the veil covering that meaning. He assumes that all mathematicians have a clear sight of this meaning, and he feels deserted by the mathematical culture, which does not let him know it.

For him the most fascinating elements in the first four months of the course is the use of the new methods to problems partly in the courses of mathematics, partly in the courses of physics. He is creative himself in posing problems. He creates problems concerning control of the formulas towards reality, and he likes to "become aware of my own ability to check the scientific results", as he puts it.
Michael got his first meeting with proofs in the gymnasium. He really wants to know about this new aspect of mathematics, but he is bound to interprete the task according to his present view of mathematics and according to his present view of reason. His

interpretation of the proof of the Theorem of Pythagoras is whether or not a specific triangle is right. The way of teaching the proof and the way of formulating the task to be done by the pupils give Michael no help to grasp the new orientation of mathematics provided by the concept og proof.

His rationales of learning are primarily social, and are primarily built upon his interest in getting to know the meaning of mathematics and this is grown inside his everyday conception of what is meaningful and rational.

SUMMARY AND FINAL COMMENTS

The project desribed here gives support to the framework of rationales of learning. It demonstrates the existence of special wishes and needs in the individuals in relation to the content and the structure of mathematics instruction in the school. These wishes and needs dominate the individual's choice among the manifold of possible ways of coping with the school.
These wishes and needs also determine what kind of Activity the individual chooses to be engaged in, they determine how the individual interpretes the tasks presented to him/her, and they determine how the individual participates in Activities.
Each pupil seems to think that his or her way of interpretation and participipitation is natural and the only existent one.

The relative weight of S-rationales towards I-rationales was astonishing high.
The S-rationale can be detailed towards the different basic generating factors.
It is obviuos that the school only provides feeble correspondence to the S-rationales. The reasons for this are complex. Part of them consist in the fact that the school does not know about the importance of the rationales and does not give attention to the rationales of the individual pupils.

This project implies that the S-rationales are wowen together with the individuals' everyday knowledge about what mathematics and rationality really is.

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PUPILS' INTERPRETATIONS OF THE LIMIT CONCEPT; A COMPARISON STUDY BETWEEN GREEKS AND ENGLISH.

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Pupils from English and Greek schools at pre-university stage were asked two questions on the nature of limits on the real line. Their responses were examined for their main conceptual formations on the subject, and different trends were identified between the pupils of the two countries. The English tended to use infinitesimal reasoning, whereas most of the Greeks were adroit in using standard procedures (available to them but not to the English because of their pedagogical background). Thus the English have a psychology of the "continuum" that seems to be closer to the Leibniz–Cauchy model than to that of Weierstrass; the Greeks most accept the Weierstrass model but not without conflict sometimes with the "dynamic" approach, suggesting that the latter is closer to their intuition.

This research is extracted from a wider work [Mamona, 1987] which deals with students' interpretations of some concepts, especially that of limit, met in a first course of Real Analysis; (By the term of Real Analysis, we mean the classical development, essentially as G. H. Hardy would have understood the term, exemplified in his book [1908], and not the topological one based on set theory and on the notion of mapping of one topological space into another). In this paper, we are interested in how English and Greek pupils in their final year in school before entering university (typically 17 years old) think of limits and the real continuum. A comparison between the nationals is interesting in that the English have no formal instruction about limits on the real line, contrary to the Greek case. We find the English use "infinitesimals" which often confounds the completion of a limiting process, whereas the Greeks sometimes display difficulties in using formal symbolism and reasoning, suggesting that little insight is given by the strict definition. We aim to study these traits in more detail.

Because of the special place of the concept of limit in mathematics as being almost symbolic of the first cross-over from naive mathematics to rigour, it has attracted a fair amount of attention in educational research. Sierpinska [1985] focused her attention on the "epistemological obstacles" relative to the notion of limit. Tall and Vinner [1981] described "concept images" of limits which are approximations made by the subjects to the formal definition. The research discussed here overlaps with these and similar works; however, this paper stresses how important the background of a pupil is in influencing his/her approach.

METHOD

We gave 20 English and 20 Greek pupils at their final year of school before entering university a question sheet that contained the two questions 1 and 2 below, amongst other questions involving limits.

1) Is 0. 999...=1? and 2) What is the limit $\lim_{x \to 2} \frac{x^5-2^5}{x-2}$?

We picked out these questions for this paper because the responses to these were particularly rich. It should be said that for 1) and the Greeks, the question "Is $0.33\ldots = \frac{1}{3}$?" was included.

Our analysis of the responses is not quantitative; our sample is small and categorization is too clumsy to be effective in partitioning responses with only subtle differences. We simply try to identify the influences, prejudices and lines of thought suggested in the data, and we contend that the sources of each can be satisfactorily explained. From this it can be understood that each major conceptual formation may be called endemic to a population with a certain background, and although a particular subject may not use a particular approach in one question, he/she is still liable to use it in another. In this way, we think most of these phenomena are so widespread that any experienced teacher would recognize them straightaway just from his/her experience: their exact relative frequency is of little import. We conserve our time more to portray the problems revealed and go some way in discussing their causes.

RESULTS

(I)The First Question

For the English pupils:

"Is the statement $0.999\ldots = 1$ true or false? Give reasons for your answer."

For the Greek pupils:

"Are the statements $0.333\ldots = \frac{1}{3}$ and $0.999\ldots = 1$ true or false?"

Conceptual formations and prejudices identified

A. *A Decimal Expression for a Number Is Unique.*

A prejudice that can be expressed in very general terms; it is that any representation is faithful, or, in other words, if objects have different representation (especially in the same system) then they are different. So, in our case, $0.\dot{9}$ does not equal 1 simply because of their symbolic forms. It is difficult to find explicit evidence of this, but we feel as a subconscious influence it is quite widespread. The most suggestive responses are a couple that argue that a number given by an infinite decimal expansion somehow is different in character from a fraction, or is somehow less "concrete": ". . .in 0.333. . . the number 3 recurs an infinite number of times, so $0.\dot{3}$ cannot be the same as the fraction 1/3." or ". . . the number 1 and 1/3 have a definite value but 0.333 . . . and 0.999. . . cannot be concrete."

B. $0.\dot{9}$ *is an on-going sequential process.*

This is to say that $0.\dot{9}$ must be constructed by an unending process of adding a 9 to what you already have, starting with 0.9. This process is ruled by time; every step has a least interval of time for it to be performed. Key words to look for are "always," "never reaching" when seeking for evidence of this approach, e.g., "The statement is false as although in the limit it may be said that this is true 0.999. . . would never actually reach 1 but would always be a very small amount less than 1" or "The statement 0.999. . . = 1 is false. . . there will always be the .000 . . . 1 which has to be added to make it up to 1."

C. *Infinitesimal Reasoning*

The response placed in part *B* are relatively few; the remainder seems to perceive $0.\dot{9}$ as not needing construction, or that $0.\dot{9}$ is the final result of an infinite procedure. Any sequence used is completed and not ongoing. However, there are two major ways a sequence is regarded as completed; one is by limit (part *D* below) or by infinitesimal reasoning. The latter says that the difference $1 - 0.\underbrace{9\ldots9}_{k}$ becomes closer to 0 as k increases, and when the process is completed the difference becomes "infinitely small" but not 0.

Examples: "$0.\dot{9} = 1$ is false since the difference between the two is 1/infinity, which although infinitesimal is not zero", or

"The numbers 0.333. . . and 0.999. . . approach the numbers 1/3 and 1 "infinitely" near, but they cannot reach them."

D. *Explicit Usage of Limits*

Usual successful application of the rule to find the limit of a geometric series, once the equality $0.\dot{9}=\sum_{n=1}^{\infty}\frac{9}{10^n}$ was recognized. However, awkwardness was obviously evident in using notation, and no one attempted "ε - δ" reasoning.

Finally beyond the above four main ways of dealing with the limit in the discrete case, three other different approaches appeared which are worth mentioning:

1. Usage of the "algebra" of infinity as in the answer: "yes, it differs from 1 by $10^{-\infty}$ which is zero."
2. Approximations, i.e., the more practical-minded pupils accept the equality on the grounds of rounding up approximate values, in almost the scientific spirit of getting more and more accurate "measurements" consistent to a desired result. For example the answer: "$0.\dot{9} = 1$ is true since all calculations are rounded up".
3. Symbolic "juggling" where operations are conducted on infinite decimal expansions: for example, "let a = 0.999 . . . -->10a =9.999... --> 10a-a=9--> 9a = 9 --> a = 1, so 0.999... = 1"

   For this answer, (and others similar to it), we think that though resourceful and flawless, is however a bit contrived, it gives no indication of why the answer is achieved, there is a sense almost of accident, rather than logical inevitability.

We highlight now the differences in "tone" between the Greek and the English responses. For the majority of the English pupils $0.\dot{9} \neq 1$. The prevailing reasoning was of an infinitesimal character. The pupils expressed their intuitive feelings of how, in a dynamic procedure, the $0.\dot{9}$ will get as close as it can to 1. On the other hand the Greek pupils were about equally divided between those who did not accept the equality $0.\dot{9}=1$ usually because of the form of the actual numbers $0.\dot{9}$ and 1 (rationality-irrationality, $0.\dot{9}$ not quite accepted as a number), and those who gave a strict justification of 1 being the limit of $0.\dot{9}$ (sometimes impressive for their inventiveness and basically correct reasoning). Even amongst the latter there remains an uneasiness that $0.\dot{9}$ in itself can represent a limit, e.g. "In other words it would be better to write $\lim(0.333...) = \frac{1}{3}$. The same for lim 0.999...=1" ['better' than just writing 0.333...].

(II) The Second Question:

Find the $\lim_{x \to 2} \frac{x^5 - 2^5}{x - 2}$

For this question, there were just two main conceptual formations, one used almost exclusively by the English and the other by the Greeks. Because of this, our description from the start is in a format comparing the two groups. We give a representative answer from each group followed by a commentary on them. (We also give an extra answer from the Greeks which was particularly impressive).

Representative English Answer

α. $\lim_{x \to 2} \frac{x^5 - 2^5}{x - 2}$ *Let* $x = 2 - \delta x$ *then*

$$\frac{(2 - \delta x)^5 - 2^5}{- \delta x} = \frac{2^5 - 5.2^4\, \delta x + 10.2^3\, (\delta x)^2 + \ldots - 2^5}{- \delta x} =$$

$= 80 - 80\delta x + \ldots + (\delta x)^4$, *as* $\delta x \to 0$, *the function* $\to 80$

Greek Answers

Representative example:

β. $\lim_{x \to 2} \frac{(x^5 - 2^5)}{(x - 2)} = \lim_{x \to 2} \frac{[(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)]}{(x - 2)} =$

$= \lim_{x \to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

Substituting x by 2 is permitted here because the function is polynomial, so continuous.

Exceptional example:

γ. The $\lim_{x \to 2} \frac{x^5 - 2^5}{x - 2}$ is the derivative of the function

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$y = x^5 \quad at \quad x_0 = 2.$

We know that $y' = 5x^4$. *So the limit is* $5x_0^4 = 5.2^4 = 80.$

The English responses to the second question showed certain concept images more readily identified in another question where pupils were asked to express in words what is meant by saying: "$f(x) \to l$ as $x \to a$"". Their predominant answer was of the type: "...it means as x gets closer and closer to a, $f(x)$ approaches the value l". (Note that the English sixth form syllabi deal with limits of functions in an intuitive manner and only at a later stage a more formal definition may be given). As with the limit in discrete cases, the above answers show that pupils think about the limit of a function in a dynamic way. Expressions like "as x gets closer and closer to a, f(x) approaches the value l" do convey a feeling of motion and flow. The question "how close do you mean?" disturbs pupils who give either tautologous answers of the kind "as close as you can", or again infinitesimal arguments such as "x differs from a by an incredibly small amount". The point x does not seem to be the point which immediately precedes a, but they rather think of x as lying in an infinitesimal neighbourhood which is closer to a than the immediately preceding point. Let us be more explicit. At this stage the pupils are not sophisticated enough to reject the existence of a "previous" number on the grounds of the nature of the continuum, (as it is formed after the Weierstrassian revolution in the theory of Real numbers). This "previous" number is distinct from a and thus can be represented on the real line as a specific distinct point b, where the interval between a and b is infinitesimal yet nonetheless it exists. These responses suggest that pupils do naturally think of the real line as composed of points and their infinitesimal neighbourhoods in a naive way. So, one can say, that their concept of the continuum is closer to the Cauchy–Leibniz one than that of Weierstrass.

The Greeks, who have been exposed to the formal treatment, basically gave answers similar to the β above. As well as factorization and cancellation done within the lim sign, L'Hôpital's rule was also used in a few responses. For this population, questions like this are familiar in that they often appear in the entrance examinations. Our particular example β shows expertise and confidence in using standard knowledge. We asked a few subjects, after the test, who had used factorization and cancellation to justify their use of cancellation under the lim sign; the following kind of explanation is typical: " We may divide x–2 into x^5-2^5 whenever x is not 2; it's true then, however close x is to 2, and so we can do the same in the limit". As in question 1, nobody offered an ε–δ argument. We pick out the response γ because it was impressive that a pupil with so little experience can have already insight in fundamental notions of Analysis. There was just one answer of this type.

Finally answers of the type: $\lim\limits_{x \to 2} \frac{x^5 - 2^5}{x - 2} = \frac{2^5 - 2^5}{2 - 2} = \frac{0}{0} = ?\ 1\ ?\ 0\ ?\ \infty$

which were found in both groups, brought to the surface pupils' difficulties with handling zero. These difficulties really have nothing to do with the limit concept; zero is a bane for nearly every pupil some time in his (or her) mathematical career!

CONCLUSION

The question of when Analysis, as opposed to Calculus, should be first taught is one of great contention. In Greece, pupils meet Analysis in their last year of school, whereas in England students first meet it in their first year at university (and there is now a move there to delay it even further to their second year). The English view seems to be that the bulk of Calculus may be taught without reference to the first arguments, and first arguments when needed may be adequately explained by infinitesimal reasoning. The Greek approach seems more philosophic, that the first principles should be "properly" (i.e formally) explained as soon as Calculus is introduced. Our study suggests that the English are deprived of insight about the mainstream modern model (Weierstrass) of the real continuum, and think more in terms of the antiquated (though briefly revived in Non-Standard Analysis) Leibniz–Cauchy model, where the numbers on the real line have infinitesimal neighbourhoods. The Greeks, although they did not use for example the "ε– δ" definition and preferred to use standard procedures, did seem to be able to accept a limit as a mathematical object rather than a "dynamic" process. This presumably is influenced by their more formal background. However a few Greeks did show some conflicts between the dynamic and static approaches, suggesting that the first is more natural to their original intuition. We feel a gentler introduction may be devised which converts their intuition rather than trying to destroy it by abruptly giving the formal presentation; where in fact the old and new ideas clash.

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INFINITY IN MATHEMATICS AS A SCIENTIFIC SUBJECT FOR COGNITIVE PSYCHOLOGY

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Summary:

Cognitive Psychology has not studied enough the infinity, which has been an important concept for human knowledge. A brief bibliographical revision is presented. In general, litterature is orientated towards didactics of the concept of infinity and its educational implications, but not to a real comprehension of the understanding of this concept. It is suggested to consider the cognitive activity that infinity requires as an independent scientific object in order to develop a solid theoretical corpus that permits the creation of new concepts for a better discrimination of the phenomena that until now aren't well identificated. Besides, the study of the conception of infinity allows to investigate areas of cognitive activity not based on direct experience (because of our finite reality), offering us very little explored aspects of the pure mental activity.

Resumen:

La psicología cognitiva no ha estudiado lo suficiente un concepto tan medular en la historia del saber humano, como es el infinito. Se presenta una breve revisión bibliográfica. En general la escasa literatura al respecto está orientada a la didáctica del concepto de infinito en matemáticas y a sus consecuencias pedagógicas, y no a la comprensión misma del entendimiento del concepto. Se propone considerar a la actividad cognitiva que requiere el infinito en matemáticas como un objeto de estudio científico independiente con el fin de crear un marco teórico sólido para acuñar nuevos conceptos que permitan discriminar fenómenos que hasta hoy no son claramente identificados. Además el estudio de la concepción del infinito permite investigar dominios de la actividad mental no basados en experiencias directas (debido a nuestra realidad finita), haciéndonos acceder a terrenos muy poco explorados de la actividad puramente mental.

El problema

En relación con ciertos conceptos como espacio, tiempo, belleza, o bondad, ya el hombre de las civilizaciones más tempranas reconoció algunas cualidades especiales posibles como "inagotable", "interminable", "indefinido" e "infinito". En pueblos cuyas culturas y orígenes geográficos diferían, conceptos de ese tipo se enraizaron fuertemente en otros tan abstractos como místicos, manifestándose bajo otras formas como "eternidad", "perfección", o "dios".

Este artículo se centra en el más abstracto de ellos: en el concepto de infinito. A través de la historia de la humanidad, complejos dominios de la actividad mental humana como la filosofía, la religión, las ciencias y las matemáticas han dedicado grandes esfuerzos al estudio del infinito. Como dijera el eminente matemático alemán David Hilbert, "The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite..." (citado en The Open University, 1988). Este concepto siempre ha constituido un tema crucial para las distintas ramas del conocimiento humano, presentando mucha controversia y provocando los más variados sentimientos: impotencia, respeto e incluso miedo, entre otros.

La situación no es distinta en matemáticas. A través de su historia, desde las paradojas de Zenón hasta las discusiones actuales sobre los números transfinitos de Cantor, el infinito ha sido un concepto difícil de tratar. Aunque en muchas oportunidades ha sido abiertamente negado, o dejado a la voluntad de naturalezas divinas, este concepto ha estado presente en las mentes de los matemáticos, teniendo una evolución muy dinámica y enfrentando teóricos con posiciones radicalmente diferentes.

Si aceptamos que el infinito es un concepto importante para el conocimiento humano, y dado el desarrollo de la psicología en los últimos 40 años, se esperaría una rica y abundante literatura relacionada con la comprensión del entendimiento de tan trascendente concepto. Paradojalmente la situación es otra. Como Fischbein afirmó (Fischbein, Tirosh and Hess, 1979, p. 3)

> It is surprising that psychology has done so little in exploring the fascinating concept of infinity, whose importance for science, mathematics and philosophy is undeniable. Even Piaget, who is an ...infinite source of new ideas and new outlooks concerning a variety of fundamental scientific concepts, has made a very limited contribution in this direction.

En el presente artículo se limitará el concepto de infinito al área de las matemáticas, cuya consistencia, rigor formal y naturaleza abstracta pueden facilitar la aproximacón científica al fenómeno.

Si tomamos algunos de los cuestionamientos que se planteó Galileo, ¿hay más números naturales que pares?, u otros de Georg Cantor, ¿hay más números racionales que enteros positivos? o ¿hay tantos puntos en la superficie de un cuadrado como en uno de sus lados?, ciertamente como psicólogos cognitivos estamos en presencia de una temática interesante. Desde el punto de vista de la psicología cognitiva pueden identificarse dos grandes orientaciones en cuanto a la relación que existe entre el aparato cognitivo del individuo y la estructura teórica de las matemáticas. Para la primera, la estructura teórica matemática (los números, los cuadrados y los puntos, recién mencionados en las preguntas de Galileo y Cantor) es una entidad independiente del aparato cognitivo, preexistente a éste. La manera en que el aparato cognitivo aprehende esta estructura teórica es el objeto de estudio de este enfoque. La matemática *es*, tiene sus leyes, y lo que interesa es estudiar cómo el individuo las descubre y las aprende. Para la segunda orientación, por el contrario, todo concepto matemático es una creación del sistema cognitivo en su interrelación con el medio (naturaleza, sociedad, etc.). De esta manera, el objeto de estudio lo constituyen las características, necesidades y propiedades del aparato cognitivo que hacen posible la creación y la existencia de determinados conceptos matemáticos (p.e. los números, los cuadrados y los puntos).

Es este segundo enfoque el que motiva al autor. Desde un punto de vista psicológico deberíamos preguntarnos entonces ¿cuál es la necesidad real de construir un concepto como el infinito?, ¿cómo es que somos capaces de pensar en el infinito?, ¿porqué podemos concebir una noción como esa, crear un concepto cómo ese?, ¿cuáles son las condiciones que necesitamos para ser capaces de concebir el infinito?, ¿qué tipo de actividad cognitiva funciona cuando estamos pensando en el infinito?

Lo que se ha hecho

Al analizar la literatura relacionada con el infinito en matemáticas, aún cuando las corrientes teóricas sean variadas (teoría de la información, cognitivismo, psicología genética, etc.), se puede constatar que en general ella se orienta fuertemente hacia la educación y sus aplicaciones pedagógigas, por sobre un interés epistemológico o de ciencia básica pura.

Ha habido cierta tendencia a intentar dilucidar ciertos problemas respecto a la enseñanza del infinito en el dominio de los números y las dificultades de su aprendizaje (Tirosh, Fischbein & Dor, 1985; Evans & Gelman, 1982; Duval, 1983; Falk, Gassner, Ben-Zoor & Ben-Simon, 1986). Siempre ligados a la realidad escolar, algunos trabajos han tratado sobre las nociones de infinito que están a la base de ciertos conceptos en cálculo infinitesimal, como series (Davis, 1982), límites (Smith, 1959; Sierpinska, 1987) y continuidad (Tall & Vinner, 1981; Furinghetti & Paola, 1987).

Entre algunos estudios que intentan responder a necesidades menos aplicadas directamente a la educación, se puede citar un trabajo de R. Falk y otro de A. Sierpinska. En el primero se intenta estudiar la concepción que tienen los niños de la naturaleza del abismo entre cantidades finitas inmensas y la cantidad infinita más pequeña que se pueda concebir (Falk & Ben-Lavy, 1989). En el segundo, se pretende saber en qué condiciones las concepciones de infinito y de matemáticas de los estudiantes comienzan a funcionar como obstáculos epistemológicos para apender otras nociones (Sierpinska & Viwegier, 1989).

Por otro lado, un grupo de trabajos, ha centrado su interés en aspectos más fundamentales y básicos de la psicología cognitiva respecto al entendimiento del infinito. Entre los primeros esfuerzos realizados se puede citar un trabajo de A. Rey sobre las cantidades límites en el niño (Rey, 1944), y algunos trabajos de Piaget, que teniendo un interés claramente epistemológico, no profundizan lo suficiente como para hablar de un estudio del pensamiento y la cognición humana frente al concepto de infinito. Así, se puede mencionar sus estudios sobre la noción de punto y del continuo aparecido en sus trabajos sobre la génesis del número y la representación del espacio en el niño (Piaget & Inhelder, 1948). En su libro "Epistémologie Mathématique et Psychologie" publicado con el lógico E. Beth, se dedican ciertas reflexiones a la intuición del infinito, aunque si bien es cierto, éstas son escritas por Beth (Beth & Piaget, 1961). En cuanto dice relación al infinito y la naturaleza operatoria del número, él dedica algunas páginas a ese tema en "Introduction à l'épistémologie génétique. Tome I: la pensée mathématique" (Piaget, 1950). En general, en la fructífera y creativa obra de Piaget, quien por lo demás estuvo siempre cerca de la matemática y de conceptos formales, no se encuentran grandes aportes relativos al infinito. Al parecer tampoco ha habido aportes provenientes de autores neo-piagetanos.

Un primer estudio, sin aplicación educacional inmediata sobre la intuición del infinito en matemáticas a distintas edades fué desarollado por Fischbein y colaboradores (Fischbein et al, 1979). Uno de sus objetivos fue el de estudiar los aspectos contra-intuitivos de la naturaleza del infinito. Ellos concluyen que los esquemas lógicos están naturalmente adaptados a realidades finitas; que a partir de los 11 años se comienza a tener una cierta intuición del infinito, pero que a causa de su naturaleza contradictoria es muy sensible a los contextos conceptuales y figurales de las situaciones planteadas. Por otro lado ellos concluyen que en general la intuición del infinito no es afectada por el entrenamiento en matemáticas, la que influye solamente en la comprensión formal y superficial del concepto.

Después de esa interesante publicación, que ya tiene más de 10 años, Fischbein no ha seguido incursionando en el dominio del infinito (Fischbein, 1989); sí lo han hecho algunos de sus colaboradores, aunque con una orientación hacia la educación (Tirosh, Fischbein & Dor, 1985). Otros dos trabajos interesantes son los de Langford sobre el desarrollo de los conceptos de infinito y límites en matemáticas (Langford, 1974) y de Taback (Taback, 1975), quién estudió los conceptos asociados a correspondencia, punto límite, y vecindad asociadas a la noción de límite. Interesantes resultan las observaciones de Langford de niños de diferentes edades sobre las capacidades de concebir iteraciones indefinidas producidas mediante las 4 operaciones aritméticas fundamentales. Concluye que en condiciones favorables, hacia los 9 años el niño es capaz de concebir la iteración indefinida mediante la adición, la resta y la multiplicación, pero que mediante la división no lo logra sino hasta los 13 años.

Por último, un aporte interesante es el de Tall (Tall, 1980) que propone interpretar las intuiciones del infinito no en el sentido tradicional y contraintuitivo esquema de la cardinalidad, sino en el de los números de medidas infinitas. El piensa que el hecho de que la medida se muestre más cercana de la intuición se debería a que es una extensión natural de nuestros esquemas relativos a la noción inicial de punto. Aspecto que le parece fundamental al momento de estudiar las intuiciones en los niños debido a que ellos no tienen acceso a esquemas de matemática formal superior.

De estos primeros estudios se puede esbozar algunas ideas. La conceptualización del infinito es sensible a los contextos en los que se desarrolla la actividad cognitiva, por lo que es necesario indagar más profundamente en ellos. A la base de un primer entendimiento del infinito podría estar la operación mental de iteración indefinida de operaciones básicas realizadas con elementos simples de estimaciones cardinales o de medidas (quizás antes incluso que la consolidación de la noción de número entero positivo y de operación aritmética). Ello hace resaltar la importancia de ciertos elementos teóricos como el rol de la convergencia y la divergencia en el desarrollo del concepto de infinito, asociado a lo que vulgarmente se llama infinitos grandes y chicos.

Discusión

Es indudable que la literatura que existe en psicología cognitiva respecto al infinito en matemáticas es pobre comparado a la importancia que éste parece tener. Si bien es cierto existen ciertos esfuerzos por entregar elementos clarificadores de ese fascinante e intrigante mundo, ellos son esfuerzos aislados, discontinuos en el tiempo y carentes de lazos teóricos entre sí.

A nuestro parecer, además de la aperente importancia ya discutida del concepto de infinito, el estudio de la concepción del infinito permite investigar dominios de la actividad mental no basados en experiencias directas (debido a nuestra realidad finita), haciéndonos acceder a terrenos muy poco explorados de la actividad puramente mental. Vale decir, incursionando en el entendimiento del infinito se tiene acceso a un universo de actividad mental singular y cualitativamente diferente, en la medida que estamos en condiciones de explorar procesos cognitivos sin (o con escasos) remanentes experienciales empíricos.

Considerando las contribuciones hechas por los trabajos pertinentes, nos parece que para poder sobrepasar la frontera del conocimiento que se tiene hoy al respecto, se necesitan nuevas nociones que permitan discriminar mejor las distintas cualidades y sutilezas conceptuales. Hoy en día englobamos bajo el nombre de infinito, un gran número de conceptos afines que hipotéticamente deberían poner en funcionamiento procesos cognitivos muy diversos al ser evocados. Así, por ejemplo, podemos hablar de infinitos potenciales y actuales; grandes y pequeños; referidos a contextos tan diferentes como series, geometría euclideana, cardinalidad, límites; bajo concepciones que pueden ser dinámicas o estáticas, etc., y siempre hacer referencia al infinito (de hecho el presente artículo peca de la misma falta de precisión conceptual). Nos parece que ha llegado el momento de enriquecer nuestro vocabulario para poder continuar haciendo ciencia. Para ello es preciso identificar bien y definir el objeto de estudio.

Dado el gran número de conceptos aislados que se están presentando, parece oportuno comenzar a considerar seriamente en psicología cognitiva al infinito como un objeto científico independiente que permita desarrollar y acuñar nuevos conceptos al interior de un marco de referencia sólido y bien estructurado.

Finalmente, definiendo bien al infinito como objeto de estudio científico para la psicología cognitiva, parece interesante vislumbrar no solamente las aplicaciones que podría tener en el campo de la educación de los diferentes aspectos del infinito matemático y de la elaboración de curricula, sino además las relaciones teóricas que se pueden establecer con el estudio de la actividad cognitiva humana en el mundo de la informática (tan enraizado hoy en nuestra sociedad), mundo en el que el infinito no parece tener significación (Núñez Errázuriz, 1989).

La frase anteriormente citada del matemático D. Hilbert aparece como un llamado urgente, sobretodo su final: "... no other concept stands in greater need of clarification than that of the infinite...".

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ORGANISATIONS DEDUCTIVES ET DEMONSTRATION

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The following is a presentation of the results of an experimental study which was carried out with university students about the arrangement of the propositions of a demonstration.
The data obtained show a great diversity of arrangements. The roots of several difficulties which impede to arrive to a correct arrangement of a demonstration are detected.
The results show new elements to be considered in a demonstration teaching program.

Les recherches actuelles, menées dans le cadre de la Didactique des Mathématiques, font état d'un interêt croissant pour la compréhension des problemès liés à l'apprentissage de la démonstration.

Les travaux de Balacheff [B1], [B2], inspirés de celui de Lakatos, montrent, à travers une situation d'intéraction et communication, le rôle que jouent chez les élèves l'incertitude et l'évidence, dans une démonstration. Dans [R3], nous exhibons certaines règles, de type "social", qui commandent la rédaction des textes de démonstration. Ces règles obéissent, en particulier, au besoin du locuteur de faire admettre un resultat en se faisant comprendre par l'auditeur, et prennent en compte des éléments logiques et linguistiques. Duval et Egret [D1], dans une perspective cognitive, montrent une différence importante entre la structure de démonstration et celle du discours usuel en langue naturelle, mettant en évidence le rôle joué par ce qu'ils appellent l'Arc Transitif de Substitution, et proposent certains principes pour l'enseignement de la démonstration, dont celui de distinguer les tâches heuristiques et les tâches spécifiques de démonstration, principe qui avait été suggeré auparavant par Gaud et Guichard [G1].

En suivant cette distinction didactique entre tâches heuristiques et tâches de démonstration, et en nous arretant sur cette dernière, on peut distinguer un certain corpus d'éléments cognitifs de nature différente qui sont mis en oeuvre lors d'une tâche de démonstration: d'une part on a le plan des énoncés ou propositions; d'autre part on a le plan des règles (qui peuvent être de nature différente, v.gr. règles de substitution syntactique d'expressions, règles de transitivité, règles de causalité, règles de type logique); enfin, un troisième plan qui correspond aux productions, une production étant l'acte qui permet d'associer un nouvel énoncé (l'énoncé resultat ou final de la production) à d'autre(s) énoncé(s) à travers une règle. Souvent, les productions sont vues comme relevant du raisonnement déductif;. on parle alors de **déduction**. Cette interprétation -qui prend ses sources dans ce qu'on apelle el metapostulado logicista de la psicología cognoscitiva [R4] - suppose, ne serait-ce qu'implicitement, que les processus de pensée sont isomorphes aux calculs formels de la logique symbolique. Des résultats que nous avons trouvé précédemment [R1], [R2], permettent de voir que les productions n'ont pas forcément une signification logique.

Or, il existe une composante qui agit sur les plans précédents (propositions, règles, productions) et qui joue un rôle d'organisateur des productions. Cette composante cognitive -sur laquelle nous voulons nous arrêter dans ce travail- gère la suite des productions de façon à ce que l'organisation des énoncés qui en resulte devienne effectivement une démonstration.

Les actions didactiques qu'on retrouve dans la presque totalité des manuels exhibent en fait l'état final de l'organisation. On sait très bien que cette pratique de l'enseignement de la démonstration n'a pas eu le succès attendu. La question qui surgit maintenant est donc celle de connaître de plus près le fonctionnement de cette composante d'organisation dans une tâche strictement de démonstration.

L'EXPERIMENTATION

Pour aborder ce problème, nous avons mené une expérience avec 70 élèves de première année de l'Ecole d'Ingenieurs de l'Universidad de San Carlos de Guatemala.

La passation a eu lieu en mai 1989, à la fin du premier semestre. Il faut dire que la plupart de ces élèves ont été soumis à une éducation mathématique où la rencontre avec la démonstration s'est faite à travers les manuels, ou en classe en suivant le modèle d'"apprentissage par imitation".

Nous avons présenté aux élèves une épreuve écrite où se trouvaient les énoncés de deux théoremes. Ensuite, en bas de chaque énoncé, il y avait, dans le désordre, les propositions qui constituaient une démonstration du théoreme correspondant. On demandait aux élèves de reconstituer la démonstration.

Voici les deux énoncés:

a) Dans tout triangle rectangle, la longueur de l'hypoténuse est plus grande que celle de chacun des côtés.

b) Soit n un entier strictement positif. Soit $x_n = 1 + 1/n$. Alors pour tout $n > 2$, $2 - x_n > 2/5$

Voici maintenant la liste des propositions, dans l'ordre presenté. Nous avons ajouté ici une colonne qui comporte des expressions de réperage (Fn ou Gm), qui nous sera utile pour l'analyse des resultats. Cette colonne ne figurait donc pas lors de l'experimentation

Première démonstration:

F3 mais $b^2 + c^2 > b^2$

F4 donc $a^2 > b^2$

F7 donc $a^2 > c^2$

F2 D'après le Théoreme de Pithagore
on a: $a^2 = b^2 + c^2$

F5 C'est-à-dire $a > b$

F6 De la même façon $b^2 + c^2 > c^2$

F1 Soient a la longueur de l'hypoténuse
et b et c les longueurs des côtés

F8 C'est-à-dire $a > c$

Deuxième demonstration:

G3 $3/5 > 1/n$

G6 $2-(1+1/n) > 2/5$

G2 $n > 5/3$

G1 Soit n tel que $n > 2$

G5 $1 - 1/n > 2/5$

G7 $2 - x_n > 2/5$

G4 $1 - 2/5 > 1/n$

RESULTATS

LA PREMIERE DEMONSTRATION

L'étude des organisations effectuées par les élèves en vue de produire une démonstration montre que, pour le premier théorème, dix élèves arrivent à trouver une organisation déductive de démonstration.

ORGANISATIONS PARTIELLES :

Parmi les organisations proposées par les élèves il y en a une assez fréquente (39/60): il s'agit de productions comportant de petites chaînes qui n'arrivent pas à être integrées dans une organisation majeure. Il y a des cas où un élève produit une seule organisation, et d'autres où l'élève en produit deux organisations "non connexes":

Erick: F1, F2, F4, F5, F7, F8, F3, F6. On reconnaît dans [F4, F5] et [F7, F8] les organisations non connexes.

PRODUCTIONS SIMPLES ET NON SIMPLES :

Parmi les 60 élèves qui n'arrivent pas à trouver une organisation déductive de démonstration, 17 commencent en mettant les deux premiers énoncés (hypothèse et théoreme de Pythagore) sans pouvoir continuer avec l'organisation déductive. De plus, on voit qu'un individu qui arrive à placer les trois premières propositions avec succès i. e. F1, F2, F3, mène à bon terme la tâche d'organisation. Reconnaître la place de la troisième proposition (F3), demande la prise de conscience que cette proposition, qui relève de l'algèbre élémentaire, fonctionne comme hypothèse au même temps que le théoreme de Pythagore, pour arriver à la conclusion partielle F4. Cette proposition n'est donc pas entraînée par les précédentes, mais elle doit être inserée dans l'organisation pour obtenir une autre proposition. Une telle production, que nous appelerons production "non simple" (énoncé - énoncé auxiliaire - règle -énoncé), s'avère cognitivement plus difficile que les productions "simples" (énoncé- règle- énoncé). En effet, ces 17 élèves qui mettent en tête de leur organisation F1, F2, sans pouvoir continuer avec F1, F2, F3 ... présentent des organisations partielles "simples".

ORGANISATIONS SUPRA DEDUCTIVES:

Un autre type d'organisation rencontré est celui qui consiste à organiser les propositions en groupes, de tel sorte qu'un groupe "implique" celui qui le suit dans l'organisation. Ainsi, par exemple, nous trouvons la suite chez Edwin: F1, F2, F5, F8, F4, F7, F3, F6. Ici, [F5, F8] "implique" [F4, F7].

ENONCES QUI SONT DE PLUS:

L'impossibilité d'intégrer des énoncés dans une chaîne, amène 22 élèves à placer lesdits énoncés à la fin du texte. C'est très souvent le cas des énoncés F3 et F6 (cf. Erick et Edwin plus haut). Mais on rencontre aussi le cas de l'hypothèse. Voici cet exemple: Carlos: F2, F4, F5, F7, F8, F6, F3, F1.

ORGANISATIONS SANS CARACTERE DEDUCTIF :

Il ya des cas aussi où l'élève produit une séquence sans caractère déductif: Anabella: F2, F1, F8, F5, F3, F4, F7

LA DEUXIEME DEMONSTRATION:

Il n'y a qu'une seule démonstration correcte. Seulement 6 élèves finissent leur organisation avec la conclusion cherchée. Il y a 27 élèves qui partent de la conclusion.

ORGANISATIONS DEDUCTIVES INVERSES:

16 élèves produisent comme texte de démonstration une organisation déductive complète mais "inverse". C'est-à-dire qu'une fois l'hypothèse posée, les élèves partent de la conclusion et parviennent à obtenir une organisation déductive, épuisant la suite des propositions données. Voici un exemple: Angel: G1, G7, G6, G5, G4, G3, G2. Nous voyons donc que ce type d'organisation -que Pappus et Proclus situaient au rang de l'analyse, en contraposition de la synthèse [H1] , [P1]- est assez fréquent chez nos élèves, et possède un rang de démonstration.

Du point de vue déductif, l'organisation tient aussi bien dans un sens comme dans l'autre; seulement chaque sens ne prouve pas la même chose, et ce qui est relevant ici c'est que la bonne organisation du point de vue mathématique ne corréspond pas avec l'idée de la bonne organisation chez l'étudiant. L'organisation chez l'étudiant se voit guidée, dans ce cas-ci, par l'idée qu'il se fait des opérations arithmetico-algébriques. En effet, 1- 1/n est vu comme le resultat d'éffectuer 2 -(1 + 1/n), et non pas à l'inverse. Donc, on passe plutôt de G6 à G5, que de G5 à G6, bien que dans les deux cas la règle qui permet la production soit de même nature (règle de substitution d'expressions). Le rapport entre le nombre d'organisations correctes avec celui d'organisations inverses (1/16), permet de nous donner une idée de la difficulté à marcher dans le bon sens, ainsi que du degré de la complexité qu'il y a derrière l'écran de démonstration.

Enfin, il convient de remarquer que dans cette deuxième démonstration, toutes les productions étaient "simples"; de plus, presque la totalité des élèves produisent des organisations déductives partielles.

CONCLUSION

L'orsqu'on distingue, dans une approche didactique, les tâches spécifiques relévant de l'heuristique et celles de la démonstration, on est amené à s'interesser à la composante organisatrice des productions. Les problèmes auxquels nous avons confronté les élèves permettent déjà d'apprécier une grande diversité d'organisations visant un texte de démonstration: organisations sans caractère déductif, organisations partielles déductives, organisations déductives inverses et organisations déductives de démonstration. Nous avons pu voir que la mise en oeuvre d'une production devient plus difficile quand il s'agit d'une production "non simple" -i. e. quand il faut tenir compte des propositions auxiliaires- que dans le cas des productions "simples". En outre, *l'interprétation* que fait l'élève des énoncés devient vital dans l'organisation de ceux-ci: c'est justement l'interprétation de G5 comme resultat d'éffectuer les opérations qu'apparaissent en G6 qui amène les étudiants à placer G5 comme conséquence de G6, alors que c'est l'inverse qu'on attend dans une tâche de démonstration. C'est pourquoi nous affirmons que dans l'organisation le contenu ne peut pas être evacué, comme semblent le suggérer Duval et Egret [D1]. Le statut d'un énoncé est *aussi* fonction de son contenu, faute de quoi le traitement des énoncés deviendrait formel, ce qui n'est pas le cas.

Devant la diversité importante d'organisations rencontrées et des difficultés observées chez les élèves dans les tâches demandées, la question qui se pose est celle de déterminer les moyens qui pourraient éventuellement amener les étudiants à produire des organisations déductives de démonstration. Mais cela demande de mieux connaître la composante d'organisation, ainsi que sa relation avec les plans cognitifs signalés auparavant (propositions, règles, productions). On peut se demander dans quelle mesure les caracteristiques des productions (productions d'association d'énoncés sans caractère logique [R1], productions de type logique, etc) vont influencer la structure logique de cette composante d'organisation. On peut formuler la même question au sujet du rapport entre la nature des règles (règles de type pre-logique, règles de type logique, etc) et la composante d'organisation.

Les démarches sousjacentes à une tâche de démonstration restent encore mal connues. Et c'est, il nous semble, un problème auquel il faut faire attention, dans un cadre didactique de la démonstration.

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THE TEACHING EXPERIMENT 'HEURISTIC MATHEMATICS EDUCATION'

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Abstract

Learning to analyse problems and learning to use heuristics is the main focus of the research project 'Heuristic Mathematics Teaching', which is being carried out the last eight years in 4th year secondary education. In the summary of the thesis 'Heuristisch wiskunde-onderwijs' (Van Streun 1989), dealing with that project, we read the next conclusions.

The educational experiment showed that educational arrangements do influence better problem solving development. On a number of points one educational variant is more conducive to that development than another. Important aspects of a successful educational arrangement are a balanced variation of 'plain' and 'applied' problems, explicit attention for heuristic methods and priority of an heuristic exploration of the specific domain to the exact formulation of mathematical concepts and techniques.

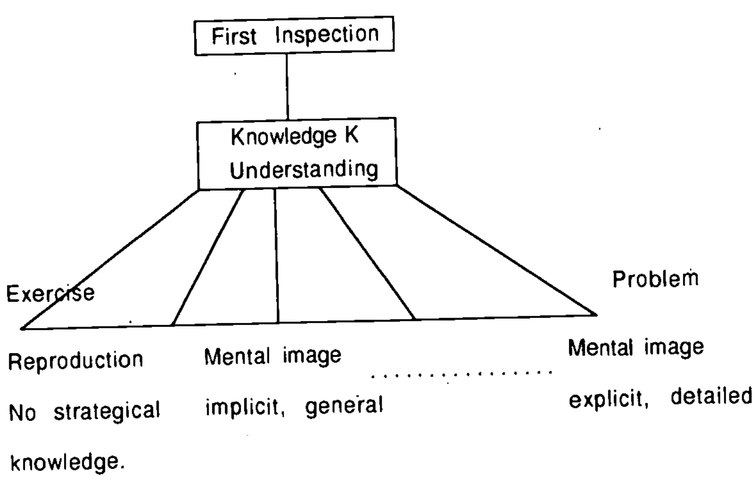
The theoretical framework

The knowledge required to solve problems has several different aspects (Van Streun, 1982a, 1982b, 1990). Knowledge which is specific to the subject can be subdivided into having a *command of the concepts* to be used and having a *cognitive schema* at one's disposal which makes it possible to relate the problem to a network of concepts, relations and experience in the memory. *Understanding* the problem can be regarded as forming an *internal mental representation* (which from now on will be referred to as the *mental image*) of the problem situation which includes all of the solver's ideas about the problem situation. Understanding a problem correctly can be described as having formed an adequate mental image of the problem situation, by means of which all the relevant components of the problem can be related to the knowledge the solver already has.

In addition to the knowledge required to understand the problem we also have *algorithmic* knowledge (the ability to carry out precisely defined methods for solving problems) and *strategical* knowledge (the ability to approach the problem by means of problem analysis and heuristic methods).

Analysing the verbal reports of solution processes of first-year students of mathematics (Van Streun, 1990) we managed to formulate several aspects of the relation between knowledge and heuristic methods, using the next figure.

Figure 1. Aspects of the solution process.



After an initial inspection of the formulation the solver tries to comprehend the problem situation and to link it to the knowledge he already has at his disposal in his long-term memory. This link sometimes leads to *recognition* and *reproduction* of the solution without conscious making use of strategical knowledge. The solver 'sees' the solution immediately. If the solver does not succeed in 'locating' the problem as a type of exercise then he can proceed to a problem analysis which can be either very general and implicit or explicit and detailed. That depends on the solver's individual approach to the problem and his perception of the situation. A good problem solver can switch backwards and forwards from a general approach to a more detailed one.

The experimental design

This is the report of an investigation of designing and teaching Mathematics Education, in which students learn how to make the most of their basic knowledge of mathematics in problem solving. The research project was started in 1980. During the first few years research literature about mathematical problem solving was being studied and 'think-aloud' protocols were being analysed (Van Streun 1982ab, 1990). This has resulted in formulating didactic starting points for designing Mathematics Education referred to, in which heuristic methods take an important place. In the course of the developing process in eight fourth grades vwo (in '82-'84) a complete course of instruction for mathematics was written, fitting into the new (HEWET)curriculum for applied mathematics.
On the basis of the didactic differences in presenting the same subject-matter among the heuristic course and two common courses hypotheses were formulated about the prospective differences in mathematical and applied problem solving. These hypotheses were tested in a competetive educational experiment in 21 forms in '84-'85. At this competetive expe-

riment an investigation was made in order to ascertain if the differently built-up textbooks for students cause different performances.

The three educational arrangements

The similarities and dissimilarities among the three educational variants HWO (the developed heuristic mathematics education), HEWET (developed in an official educational experiment) and WEDT (a traditional textbook) and the expectations based upon them about the differences in output, can be summarized schematically as follows:

Educational variants	HEWET	HWO	WEDT
subjects	equal	equal	equal
number of lessons	100	100	100
nature and number of problems	equal	equal	equal
arranging applied and 'plain' mathematical problems	continual variation	in phases	first 'plain' then applied
attention for heuristic methods	implicit	explicit	no
expliciting mathematical concepts and techniques	late/little	gradual/limited	rapid/frequent

The test results and the hypotheses

The test results can be put in one scheme together with the formulated

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hypotheses. Only the results printed in bold type have reference to statistical significant differences; the other inequalities point to non-significant differences.

The main project-hypothesis about increasing ability into problem solving by means of mathematical knowledge can be adapted.

	Result	Hypothesis
Ability of solving mathematical and applied problems	**HWO > HEWET > WEDT**	HWO > HEWET >WEDT

Special attention paid to problem solving does not derogate from the mastery of mathematical basic kowledge.

	Result	Hypothesis
Mastery of basic knowledge of concepts and techniques	**HWO > HEWET** HWO > WEDT > HEWET	WEDT > HWO > HEWET

We conclude that emphasizing mathematical basic concepts and techniques versus stressing heuristic methods does not necesseraly result in a corresponding difference in the frequency of the use of these problem solving procedures.

	Result	Hypothesis
Use of heuristic methods in solving mathematical and applied problems	**HEWET > HWO**	HWO > HEWET > WEDT
	HEWET > WEDT > HWO	
Use of basic knowledge of concepts and techniques in standard problems	**WEDT > HWO ≈ HEWET**	WEDT > HWO > HEWET

We observed that the nature of the problem to a high degree determines the solver's employing algorithmic or heuristic methods. Designing sets of problems which can be solved with the same basic knowledge but are at unequal distance from analogous standard problems rendered good services to the investigation of employing solving methods. This variable distance can be arranged objectively, for instance according to the number of necessary problem transformations and be fixed objectively per solver.
In the course of one scholastic year it appeared that some shifting in employing heuristic and algorithmic methods had already taken place. Students having acquired an increasing command of the subject area are going to apply algorithmic methods in a larger number of problems than less succesful students. Being more succesful in problem solving is attended by more frequently employing algorithmic methods.

Conclusions

The educational experiment showed that educational arrangement does influence better problem solving development. On a number of points one educational variant is more conducive to that development than another. It appears that a cognitive schema built up with 'plain' mathematical problems and applied problems in the end is inferior in applied problem solving to schemes built up with continual or phased variation in types of problems. Explicit attention for heuristic methods and gradual and limited formulating of mathematical concepts and techniques in mathematical education achieve a higher problem solving ability than implicit attention for heuristic methods and late and little formulating of mathematical concepts and techniques.

Our conclusions with regard to the problem solving processes are as follows. The nature of the problems presented and the distance to analogous standard problems determine to a high degree the choice of the solution method. During the process of the solver's mastering the subject-area his choice - the problems remaining equal - is shifting towards algorithmic methods. Heuristic methods have to be integrated in the cognitive scheme of mathematical concepts, techniques and applications.

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THE UNDERSTANDING OF LIMIT: THREE PERSPECTIVES

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Summary: In a case study involving college calculus students, eleven subjects attempted to deal with anomalous problems designed to alter their view of limits. Although most students eventually agreed that a limit could be reached and even surpassed, the dynamic view of limit was remarkably resilient. Three perspectives on the results are provided, dealing with knowledge of limit as a cognitive model, as an amorphous collection of phenomenological primitives, and as embedded in the concernful activity of the subjects.

Introduction, Theory, and Design

The notion of limit among calculus students has received increased attention in recent years, owing to its particular importance as a foundational concept in analysis and the rather persistently reported misconceptions which students have. These misconceptions have been repeatedly documented (Ervynck, 1981,; Tall, 1980; Tall & Vinner, 1981; Davis & Vinner, 1986) in various populations and teaching situations. The current study focused on three major confusions regarding limits: 1) confusion over whether a limit is actually reached, 2) confusion over whether a limit can be surpassed, and 3) confusion regarding the static character of a limit. These three areas correspond roughly to the three major epistemological obstacles which were overcome in the development of the modern day calculus (Grabiner, 1981; see Kaput, 1979 for a different view).

Various methods of viewing students' knowledge about limit have appeared in the literature, including Tall and Vinner's (1981) <u>concept image</u> to Cornu's (1983) <u>spontaneous models</u>.

This study took seriously the notion that students had fairly well structured models of limit, something like paradigm cases, upon which their performance was based, and that these models had to be altered in order to replace improper with proper conceptions. The study attempted, therefore, to document the nature of these models and to study the process whereby they changed.

In order to describe the process of cognitive change, the study employed a framework developed by Posner, Strike, Hewson, & Gertzog (1982). They liken the process of conceptual change in individuals to that of the scientific community and draw upon recent developments in the philosophy of science to gain insight into the process of conceptual change. They assert that in order for accommodation (or the radical reorganization of central concepts) to occur, three conditions must be met. There must be some sense of dissatisfaction with the existing conceptual framework; there must be alternate conceptions which are both intelligible and initially plausible; and the alternate conceptions must be seen as fruitful, useful, or valuable. Several factors are identified as having an impact on whether these conditions are met. These factors include the individual's epistemological and metaphysical commitments, the individual's knowledge of other fields, and the character of the anomalies which give rise to dissatisfaction with existing conceptions.

Nussbaum and Novick (1982) suggest a three-part instructional sequence designed to encourage students to make desired conceptual changes. They propose the use of an exposing event which encourages students to use and explore their own conceptions in an effort to understand the event. This is followed by a discrepant event which serves as an anomaly and produces cognitive conflict. It is hoped that this will lead the students to a state of dissatisfaction with current conceptions. A period of resolution follows in which the alternative conceptions are made plausible and intelligible to students, and in which students are encouraged to make the desired conceptual shift.

This basic sequence was followed with each of the eleven students in the study. Students were chosen from a second semester college calculus course based on their answers to a preliminary questionnaire. All students indicated having confusion about limits in at least one of the ways listed above. Following a session in which students were encouraged to describe their models of limit using repertory grid techniques (Mancuso and Shaw, 1988), students were presented with three sessions specifically designed to change their views of limit. During each of the sessions, students were asked to explain two opposing statements about limits and to choose the one most like their own view. Anomalous problems were then presented which were designed to favor a model of limit closer to the standard, formal definition, and the students were asked to work them. For example, determining the limit as x approaches 0 of the function $f(x) = x \sin(1/x)$ favors a model of limit in which limit is seen as surpassable. Finally, the students were asked to explain the anomalies from both viewpoints, in an effort to make both the anomalies and the alternatives intelligible.

Results

It is not possible to provide in a few pages the dialogues which underlie these results. However, the results can be summarized. The attempt at having students make their model of limit precise was for the most part disappointing. The number of constructs used in describing their models of limit were in general small, and they failed to cluster in any significant way. A possible reasons for this will be discussed below.

Data from the interview sessions, however, was more rewarding. It is clear, for example, that views of limit as not reachable or as a boundary are relatively easier to correct than the dynamic view of limit. Specifically, it is easy to convince students that a function can both assume the value of its limit point, and go beyond that value, because simple functions exist which easily serve as counterexamples. It is less easy to rid students of the belief that there is a process of taking a limit, (as distinct from the procedures for evaluating limits) in which the limiting value is never reached. It is very difficult to convince students that a dynamic view of limit is inappropriate (this result is not entirely surprising; see Kaput, 1979).

Three Interpretations

One possible interpretation of the failure to produce change in the dynamic notion of limit is that students' epistemological and metaphysical assumptions were not conducive to change. To be sure, students in the study were seen to have various views on the nature of mathematical truth, the value attached to non-mathematical aspects of limit models, idiosyncratic base metaphors for limit, and faith in a practical, generic model of limit, all of which were seen as potential inhibitors of conceptual change. For example, students were able to compartmentalize their knowledge so that counterexamples were seen as minor exceptions to a general rule which remained largely untouched. They also had extreme faith in the ability of inspection or graphing to portray all the necessary information to work a limit problem--the generic model mentioned above. In general, they valued their own models more because they saw them as simpler, and eminently practical--the models allowed them to work all the problems they needed to work.

A second interpretation would call into question the basic assumption made throughout this study that students' understanding of limit is structured in some definite way, akin to a theory or a model. diSessa (1988) argues that a student's intuitive understanding of physics, for example, "consists of a rather large number of fragments rather than one or even any small number of integrated structures one might call 'theories'" (p. 52). diSessa goes on to assert that these fragments are "phenomenological primitives," and can be understood as "simple abstractions from common experiences that are taken as relatively primitive in the sense that they generally need no explanation; they simply happen" (p. 52). The problem of conceptual change then becomes not one of attempting to shift from one model to another, but rather the building of a coherent model from largely unrelated units of prior knowledge. This would account for the failure of the repertory grid techniques to display any structure, and is in fact supported by the finding that students' knowledge of limits is highly case-specific and compartmentalized.

A final view of the data, one which I am coming to favor, is hermeneutical. It does not so much replace, as offer an alternative way of viewing, the claims of the other two perspectives. This perspective views the students as being engaged in concernful activity, in which they relate to the world in terms of their own goals and the pre-understanding they bring. Hermeneutics insists that such concerns are primary, and cannot be relegated to the background in favor of studying "cognitive processes." Thus it is entirely correct to say that epistemological and metaphysical beliefs played a role--in fact, as part of the students' overall world context, they played the major role in determining whether students would expand their horizon to include a new view of limit.

The data makes it clear that the primary concern for the students was passing their calculus class. Despite a careful attempt to focus on the subject matter in the sessions, a great deal of information was volunteered regarding the calculus class, its teacher, and its relation to school and life in general. Specifically, students reported that their task was to master the skills necessary to do calculus problems of the type they knew would be on a test. In this sense, the background was the message: my goal for them as a researcher was not their goal. It made no sense, except as a sort of language game that they were willing to play for me.

A second aspect of the hermeneutical model is the preeminence of the practical -- the mathematical experience gained by the student through years of doing mathematics as part of their concernful activity. Thus, as diSessa suggests, it was through the use of knowledge imbedded in the practice of doing mathematics -- phenomenological primitives of a sort, that students went about accomplishing their aims. The use of graphs, tricks, intuition, and educated guessing which has become part of their repertoire over the course of their years in mathematics class, were the tools brought to bear on problems. It is the use of mathematics as _ready to hand_, to use Heidegger's term. This was the idea I attempted to capture in the concept of a _generic model_ -- something which remained unarticulated (and perhaps unable to be articulated) but which was buried deeply in the practical experience of the students.

It may seem that this is an overly cynical view, or at best, that it states what is obvious, something we must move beyond in order to really understand the processes of learning limits. However, from a hermeneutical perspective, there is no learning, no understanding, separate from the context which the student perceives and the goals he or she brings to the task. The importance of students' engagement in the educational endeavor cannot be overstated.

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Affect, Beliefs and Metacognition

SELF-CONTROL IN ANALYZING PROBLEM SOLVING STRATEGIES

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Although many efforts have been made in the past to analyze human problem solving strategies, our knowledge about problem solving processes is restricted especially with respect to non-routine problems of a certain degree of complexity. Particularly, the role of cognitive strategies of different "levels" and the influence of situational components are nearly unaccessible to research by usual methods. The author used problems originally designated for being given to a students' problem solving working group to analyze detailed protocols of the own solution attempts. Two forms of protocols were tried out: 1. writing down all ideas concerning the solution itself, assessments of their productiveness and of expected success when making special attempts, impressions and emotions accompanying the solution process; 2. tape-recording when "thinking aloud" and endeavouring to verbalize all ideas as mentioned in 1. (Both forms turned out to be incomplete.) Additional thoughts and impressions recalled by retrospective were marked separately.
Evaluations of the protocols show unexpected findings and demonstrate that we usually have extremely simplified conceptions of problem solving processes, which are rather determined by our knowledge referring to fairly simple problem types.

Purpose and intentions

As mathematics educators we prefer challenging problems far more than routine tasks as materials to be given to students. The consequence is that our interest has to be focussed on investigating thinking as processes far more than as results, in order to obtain fitting ideas and a basis for describing and understanding these phenomena. The complexity of a problem corresponds to the refinement of methods to be used when analyzing solution attempts and the individual progress of a problem solution.
There are some well-known case studies on thinking processes, parts or special components of them in the field of mathematical topics. Professional mathematicians interested in the process of mathematical thinking have published examples for the finding process of special results representing profound theorems (Van der Waerden, 1968; Hadamard, 1954) by retrospective.

These publications trace the way from the question raised at the beginning to the main intermediate results, special cases, ideas for a generalization until the final result, and try to make obvious how these could generate. They mention the problem solvers' impression of the "sudden and immediate appearance of a solution" (Hadamard, 1954, p 8), or, quoting Henri Poincaré, of the "appearance of sudden illumination" (Hadamard, p 14). Gestalt psychologists, such as K. Duncker and M. Wertheimer, have analyzed probands' solutions of problems, and the key ideas leading to them, thus being concentrated on thinking processes; among the tasks serving as examples, we find mathematical problems, too. G. Polya's books contain valuable collections of mathematical examples, by which he tries to illustrate heuristical strategies obtained by a kind of subsequent analysis of problem solutions. His analysis of plausible reasoning represents a helpful attempt to express subjective procedures in objective rules and, thus, essentially contributes to a communicable language. In further elaborating Polya's ideas and related approaches, many authors have attempted to show how these can be used to initiate and support students search for problem solutions in mathematics education. As the most conspicuous result of these efforts, with respect to educational practice, can be regarded the elaboration of model discerning different phases on the whole structuring the problem solving process (f.i. Strunz, 1968, pp 22-239; Becker, 1980, pp 109-119). Developing heuristical strategies obviously is based on simple techniques, thumb rules, and elementary forms of heuristics, but since we do not want students to confine themselves to the latter, we urgently need knowledge on further developed and more generalized strategies.
One main reason for this shortage in our present knowledge is the difficulty to completely and objectively observe these processes and to describe thinking processes of a sufficiently high level by means of an elaborated language. Persons interested in solving mathematical problems usually are not trained in observing themselves when thinking, especially if both activities are carried out simultaneously; additionally their endeavour will be concentrated on finding solutions rather than on learning a specialized language allowing them to express their own thoughts. Very often students even do not understand what we want to know when asking them to describe how a problem solution was found; the answers we obtain will be "I did it in the same way as before", "I did it in that way, since I was sure it was correct", or the like. (Impressing examples of students' reactions to questions with this intention are to be found f.i. in Scholz, 1987.) Besides this, problem solving processes are nearly inaccessible to an "objective" observer, at least in essential parts. We cannot describe these internally progressing processes and phenomena by our usual language; we need to utilize comparisons, analogies, images, metaphors, and being concentrated on solving a problem calls for so much of our concentration that this process would suffer from directing our attention to observing ourselves. And the efforts to formulate our thoughts in any linear sequence of words seems to slow down the problem solving process itself and to ob-

struct endeavours to find productive ideas for the solution itself.
Van der Waerden describes this dilemma sharply as follows: "The psychology of finding has its particular difficulties. Most of us find it difficult to subsequently remember all which was passing trough our minds. It is even harder for us to give a report of our own short preparing reflections such that other individuals, too, are able to understand them. The short hints by which you talk to yourselves cannot be communicated to others without being put more precisely and explained , and putting our thoughts more precisely modifies them." (Van der Waerden, 1968, p 26; originally in German).
However, self-observation and self-control turn out to be the least dubious method to learn about our own more complex thinking processes. The most essential condition is an observer's interest both in the topics and in heuristical strategies. Many reservations have been formulated against self-observation as scientific method, by good reasons, but up to now no alternative method is known which could make us discover so many important details of problem solving processes, at least in cases of sufficiently complicated problem types.

Problem examples

The problems chosen for this purpose are taken from different collections destinated to be presented to students as additional offers going beyond the usual demands of school tasks. Problems for which a solution was not to be found immediately or in fairly short time by the author, ware taken as topics for the sketched purpose. (It cannot be excluded that at the very beginning of looking for a solution any blockage prevented immediatly finding a result and continued effecting the later progress of the problem solution; but in the present context this was regarded as helpful under aspects of the purpose of this study.) So, the problems used here can be characterized by a well-defined goal and a vastly extended repertory of well-defined means; the difficulty is rather determined by subjective estimation and depends also on situational conditions. Too simple problems would scarcely allow to distinguish them from routine tasks, whereas too difficult ones can be expected to claim so much concentration that observation of problem solving behaviour could seriously be impaired.
Some outcames of the study will be illustrated by findings from the following example:
"M is the midpoint of a chord AB of a circle. Any other chord CD is drawn through M. Tangents drawn at C and D meet $\overleftrightarrow{AB}$ in P and Q respectively. Prove: CP = QD, and PA = QB." (National Council 1966, p. 5, problem nr. 67)
Most of the examples are taken from geometry; problems belonging to this topic area seem to suitable especially because of the condition concerning the means. Geometry problems seem to represent best the conditions formulated above, especially with respect to the means.

Method and data collection

The data were collected in two ways.

1. During the own attempts to find the solution of a problem, all consciously registered thoughts, associations, impressions were written down on the work sheets together with the solution attempts and their persuences, as completely as possible. Against usual practice the figures were not corrected or repeatedly accomplished but newly designed; this procedure makes recognizable the lines of reasoning more easily. Finally, a readible form was drawn up, and additional thoughts, which could be remembered shortly after the experiment sessions, were added (with special markings).

2. Thinking aloud was tape-recorded when solving the problems, and it was tried to formulate all conscious solution steps, including remarks as mentioned under 1., later the tape-recordings were transcribed.

In a "complete" version of the protocols all utterances have been recorded. An abbreviated version contains only the "essential" ideas (f.i. remarks of emotional character and assessments referring to the estimated productiveness omitted).

Aspects and criteria for analyzing the protocols

The abbreviated protocol versions claim to contain the "essential" ideas of the solution process. A serious question when drawing up the elaboration of these protocols was the "extension" or the originally supposed range of one "essential" idea. It is obvious to condense f.i. the repeated transformation of any obtained formula by routine techniques in only one step, even if it takes several lines on the work sheet. But f.i. the new attempt to transform a formula, already in cases where it is done with a little modification in substituting any partial formula, a second run to do the same transformation without a previously committed error, a new attempt using other combinations of parts occurring in a figure, or the expressed purpose to control the correctness, are considered as a new idea. The short touch of an impulse to try out any proof part even if it was forgotten later or not put into effect by any reason, is regarded as a separate step. These "smallest elements" shall be named steps; shorter or smaller parts (such as the single lines of a transformation of a formula) are not registered as independent steps. Mostly, a step consists in using a well-defined technique (solving an equation or an inequality, searching for a formula in order to substitute a term in any longer formula, and so on).

These reflections suggest a distinction between a strategy and a step, not as much in terms of time consumed to carry out a step, but of the range of a solution element. It is hard to adhere to this distinction throughout the the whole solution process, since it rather turned out and can be seen as one result of the experiment, that repeatedly, an idea which at first came only as a rather superficial hint or an "aperçu" which was not taken too

seriously at the very moment of originating , later "extended" to a farther-reaching strategy with more or less restricted substrategies. So, the abbreviated protocol versions contain solution elements of two different "levels" defined by their respective range, even if the original protocols or a subsequent analyse occasionally would admit to identify "intermediary" ranges.
The single solution steps have been classified by an easily identifiable denotation, such as solving an inequality, comparison of angle measures, theorems about central and inscribed angles, and assigned to a superordinated strategy. Some of them are additionally characterized by the "place" of such an element in the solution process. Demonstration problems contain steps (f.i. properties of a figure) which are equivalent to the givens, and those equivalent to the goal property, steps which are immediate inferences from the givens, steps following from the latter, steps immediately allowing to draw the final conclusion, and steps preceding the latter. Corresponding "places" can be attributed to single steps in construction problems.
These categories are included into the analysis, provided that any single step, irrespective of whether or not correct, suggests such an identification.
Finally, a systematization of the figures on the sheets has been attempted. It is based on the strategies recognizable in these figures.
Elaboration of the protocols, especially classifying the single steps into the superordinated categories, depend on knowledge or remembrance of the original thoughts, which means that they can hardly be elaborated by an objective observer.
As an example, a part of an abbreviated protocol will be presented.
It refers to the problem quoted above; two main strategies dominate the process of its solution: finding a chain of triangles, for which certain proportions formed by lengths of line segments can be stated, the product of which by cancelling was expected to lead to the result 1, and some kind of extending the figure, successively different tangents being concerned. The work sheets number 57 to 72 show a mixture or a combination of both strategies (which only occurred here), a phase determined by extending figures with tangents strategy, and the last phase with the triangle chain strategy before finally solving the problem by the further, the single steps subordinated to the strategies, most of them characterized as expected to precede to any proof step immediately allowing to draw the final consequence, and the respective figure types with modifications.

Some results and critical remarks

The outcomes of the investigation contribute to our understanding of problem solving processes in some aspects which are hardly accessible to other methods of investigation. We can take the recorded problem solving processes as typical for a type of problems with non-routine character, determined, as mentioned above, by a vastly extended repertory of well-defined means.
The most remarkable result is that problem solving processes are composed

of different phases of distinctive course, more or less of increasing length, in which search for a solution is directed by one main strategy, whereas the partial steps in the framework of the strategy may show great variety.
The problem solving process starts with a series of rapidly changing attempts to try out different and heterogeneous solution ideas; among these may well be a successful one, which finally even turns out to lead to the solution, but without being recognized in its role during this very initial phase. In the course of the solution process, one or two, sometimes three of these solution ideas increasingly form as farther reaching strategies and successively prevail against others; but in case of failing success the superordinated strategy is abandoned (but later may be taken up again), and then an initial sequence with rapidly changing steps leading up to a longer period of stabilizing new strategy marks the beginning of the next unit under the predominance of a strategy. There are not yet enough examples to definitely bear out this structure of problem solving processes. It turned out that it is not easy to find suitable problems in sufficient number for this purpose.
Another quite astonishing result is the fact that situational conditions have unexpectedly high influence on the selection and assessment of the range of a strategy. Preference given to a strategy arises from familiarity with it by any context of working (a strategy being used shortly before the experiment is preferred, and so on), by any "nice" result (even if this may be wrong) or a seemingly smoothly flowing technique.
In the added example the successful strategy presumably was not recognized by an incorrect figure suggesting wrong properties, an unseccessful strategy favoured by a seemingly good result (as if standing shortly before the end, but counterfeited by a computation error).
Not surprising is an effect which may cast some light on the method of "thinking aloud". By far not all associations or ideas coming to conscience at any time of the problem solving process actually are uttered or at least sketchily formulated. It could be observed that several associations came into conscience apparently simultaneously; one of them, representing the stronger impulse, may suppress the others, which sometimes may come up later on again by recalling any partial sequence of the problem solving process.
The attention claimed by the constraint to grasp all elements coming to conscience seems to seriously interfere with the problem solving process itself; otherwise certain unusual errors would not to be explained. Obviously, in order to compensate both the attention directed to the utterance or writing activity, felt innervations or movements, and the considerable insecurity to be registered, many repetitions of the same solution attempt are carried out. The verbal formulations registered by tape show fairly bad style and many repetitions, sounding (in the tape-recordings) as if uttered without concentration, seem to be directed to catch weak remembrances. The urge to notice all conscious thoughts is far away from usual problem solving situations and can be assumed as affecting the results. And so is the condition to work on the prob-

lems only in experimental situations, especially refraining from thinking of the problems during the rest of the time (which actually was not to be kept up; f.i. during a bus drive the author suddenly caught himself thinking of the problem and after finding an advertising circular lying on a seat, used it to write down at least the thoughts which had come up immediately before). May-be the interferences by consciously controlling ourselves are comparable to conditions impairing problem solvers by any other reason. Under these aspects the susceptibility of problem solving processes to any interference in general is confirmed by the experiment, too.

The urge only to work under rigid self-control conditions turned out to be annoying, sometimes even galling or demotivating (as can be seen from several remarks on the work sheets). The endeavour not to lose any impression or idea lead to the feeling which can be described by the metaphor of tripping oneself up.

Solution attempts carried out only under the condition to add remarks later show far less associations and other accompanying elements. Most of the latter must have been forgotten, until the second reading of the written lines on the work sheets starts. So, retrospective comments are obviously incomplete and unsuffient, may-be even unreliable in their content.

It can be ascertained from the self-control experiment that our heuristical strategies together with "superordinated" impulses and control instances represent an undeterminably complicated system of dispositions and tendencies, and it can be assumed that only a small part of them actually become conscious. Only occasionally and partially we have conscious experience of competitions between different strategies offered by our cognitive system, a kind of uneasiness towards certain strategies, an impression of how near we are to the goal, and so on. A typical situation where the latter is to be felt occurs towards the end of the problem solving process, when we are irrevocably sure that we have found the solution, without having elaborated the solution idea and before having done all necessary steps in detail; this feeling even does not prevent us from committing errors, which then are experienced as non-essential and not detracting us from our certitude. This feeling of standing shortly before the goal is accompanied by an impression which can be described as: "it" works in ourselves).

No result was discovered with respect to the appearance of the final solution idea. F.i. during reading in a book on psychology of mathematical thinking (namely Strunz, 1968) especially the chapter about problem solving a typically extremely short idea suddenly flashed into the author's mind, to try out a special attempt to solve one of the problems in question; and the accompanying impression was: I shall try out the idea later on, not because I am convinced of its productiveness, but only because of keeping to the experimental conditions.

These findings are in line with results reported by many psychologists (for details, see Hadamard, 1954). This view finds support by an author representing quite another approach: namely by S. Freud's statement "that the unconscious 'search-readiness' is far more likely to lead to success than the consciously directed attention" (Freud, 1941, p 178, originally in German). Finally, the findings cast some light to the frequently used method of thinking aloud. This method has to be seen more critically, at least with respect to problems demanding a high amount of concentration. Although the methodical weakness of non-separating subject and object is the most serious objection against the reported experiment, this experiment shows that thinking aloud only partially reveals consciously experienced thoughts and even these not always reliably.

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INFLUENCES OF TEACHER COGNITIVE/CONCEPTUAL LEVELS ON PROBLEM-SOLVING INSTRUCTION

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The purpose of this study was to investigate cognitive levels of eleven intermediate grade teachers and their relationships with teacher mathematics conceptions, teacher problem-solving conceptions, and problem-solving instructional practices. In a case study approach, teachers were studied via written assessment (the Harvey's This I Believe Test), individual interviews, and four problem-solving lesson observations. Strong relationships were found supporting the theory that cognitive structures are related to instructional practices and conceptions about mathematics and problem solving.

Introduction

Considerable attention has been given to teacher conceptions about subject matter and its teaching and the way in which these conceptions filter into the instructional process (Cooney, 1985; Bush, Lamb, & Alsina, 1987; Thompson, 1982; McGalliard, 1988). How teachers act on these conceptions may be a result of the cognitive organization of their knowledge and beliefs. Individual cognitive configurations may impose a hierarchy upon factors encountered in the classroom, focusing more attention on certain factors than on others. The general cognitive premise that emphasizes the construction and accommodation of knowledge as it relates to instructional decisions is the basis of this research.

Underlying psychological tenets are general with respect to overall cognitive levels that help to characterize how an individual views his environment and impinging stimuli. Structural characteristics of each cognitive level should be valid in specific contexts, such as instructional situations involving the teaching of mathematical problem solving, and thus be descriptive of overt acts (teaching practices) representative of particular levels. Hence, pairing conceptions of mathematics and cognitions should provide a theoretical framework to explain instructional practices used in problem-solving instruction.

The theoretical framework used to determine teacher cognitive levels is a set of psychological attributes similar to those found in Harvey, Hunt, and Schroder's scheme (Harvey, Hunt, & Schroder, 1961). Their scheme presents a hierarchic model of four broad systems descriptive of ways an individual "establishes and maintains ties with the surrounding world" (Harvey et al., 1961, p. 11). Each of the levels or systems is described as follows.

The four cognitive systems lie on a concrete-abstract continuum. The most rigid of the systems, A, is identified as concrete formalism. It is heavily rule governed with strong reliance on a single Authority. Acceptance of new ideas is not specifically related to personal experiences, but rather to the position of the Authority. An individual identified with this system expects relatively uniform performance from all with little regard or expectations of individual differences. Judgments and evaluations are polar, either right or wrong.

System B, social pluralism, emphasizes social utility and acceptance. While there is still some reliance on a single Authority, an individual may look to others for guidance. Acceptance of new ideas is based on their perceived social usefulness. Adequacy of others' performance is dependent on how well it conforms to existing social structures. Personal performance is based on the likelihood of social acceptance and judgment and evaluations may change from one social context to another but with the same regard for correct-incorrect.

Individual of level C, integrated pluralism, realize multiple viewpoints or authorities exist but tend to rely on a single Authority. Acceptance of new ideas is contextually dependent and may still be socially related. Adequacy of others' performance may change from one context to another. Personal performance is judged with respect to the given situation and judgments and evaluations are contextual bound with little recognition that they differ from one context to another.

System D, abstract constructivism, is the most abstract or flexible of the four systems. It is characterized by autonomous thinking with respect for others' opinions and views. Multiple views are weighed before adopting a new idea or integrating into existing structures. Individual differences are expected and tolerated and personal performance is flexibly assessed. Multiple choices are explored before any judgments or evaluations are made.

Teacher conceptions about mathematics, specific topics, and instruction should relate significantly to the overall cognitive level. Because knowledge is organized relative to the features to which one attends, resulting structures should reflect attributes compatible with, and representative of, the cognitive level.

Thus, cognitive structures are responsible for an individual's general behavior and their influence can be felt in specific contexts that require active involvement as in the act of teaching. The extent to which these structures encroach on instructional decision-making processes has not been fully explored in specific subject matter areas such as mathematical problem solving. With renewed emphasis on altering teaching practices related to problem solving, more insight is needed regarding the influences on teacher decisions about those suggested practices and resulting implementations. The overall significance of this study lies in its potential to link instructional practices to teacher cognitive levels and thus deepen our understanding of this fundamental relationship.

Methodology

The voluntary sample includes eleven intermediate grade teachers (4th, 5th, and 6th grades) from four school districts in the Midwest. To gain an understanding of teacher cognitive level, the This I Believe Test (TIB) (Harvey, 1989), a written, open-ended paragraph instrument, was administered. Completed assessments were sent to Dr. O. J. Harvey for evaluation and results were not revealed to the investigator until all observations and interview analyses were completed.

Individual teacher interviews were used to ascertain teacher conceptions of mathematics, problem solving, and instructional aspects of both. Seventeen interview protocol questions were developed from those used in other studies (Grouws, Good, & Dougherty, in progress; Brown, personal communication, November, 1988) and piloted with preservice and inservice teachers not affiliated with the study.

An observation coding instrument was used to record field notes and quantitative data obtained during classroom observations. These observations were made in an effort to observe and record instructional practices in a naturalistic setting and to capture consistencies between practices and cognitive levels, as well as among mathematics and problem-solving conceptions, and practices. Classroom variables to be observed included, but were not limited to: amount of time spent on lesson development, types of problems selected for examples during development, teaching techniques used for problem-solving instruction, teacher use and types of questioning, teacher modeling, lesson format, and so on.

To minimize the effects of different mathematics content, four problem-solving lesson outlines were given to teachers in an effort to have them focus on one particular content strand with specific objectives. Each outline included lesson objectives (short-term) and suggestions for example problems but gave no instructional recommendations, scripts or other aids. The problems were structured so that teachers had their choice among those with an algorithmic solution (usually computation), practical application, or creative solution involving higher-thinking skills but teachers were not explicitly told which type each problem was. Four heuristics were used as objectives and included: (1) guess-and-test, (2) work backwards, (3) make a diagram, and (4) make a table and/or find a pattern.

Results

To obtain the best possible evaluation of teacher responses on the This I Believe Test, Dr. O. J. Harvey scored the test with regard to concrete or abstract levels as well as other specific dimensions that included evaluativeness (tendency to make evaluative judgments), richness-complexity (depth of thought expressed), and openness (willingness to seriously consider, or accept, a position contrary to his/her own view on a central issue).

Eight teachers are considered to have a dominant system A, indicating the most concrete of the four levels, and, in progression toward abstractness, each of the other three systems (B, C, and D) are identified with one teacher each (see Table 1). This distribution is not surprising since other studies have revealed few elementary teachers in the abstract domain.

Table 1

This I Believe Test Results

Teacher ID No.	Dominant System	Secondary System	Evaluative-ness	Richness-Complexity	Openness
0104	A		4	2	1
0106	A		5	1	1
0204	A		3	4	3
0206	A	B	3	5	3
03015	A		4	2	2
03061	A	C	3	3	3
05051	A		3	2	3
05061	A	B	3	3	3
0105	B	C	5	3	2
05042	C	A	2	3	4
05041	D		2	5	5

Note: 1 indicates low strength, 7 high strength

Responses to specific questions from the interview protocol indicate how the teacher views mathematics as a discipline. Four categories are used for response classification: (1) Mathematics is a set of rules and procedures, (2) Mathematics is a tool for everyday life, (3) Mathematics is an application of logical thinking and/or step-by-step methods, and (4) Mathematics is experiential and not a static body of knowledge. Considering a concrete-abstract continuum for mathematics conceptions, it appears that the first three conceptions are at a more concrete level than the fourth classification. By comparing conception response with functioning level, one can see that there are apparent relationships (see Table 2).

Table 2

Mathematics Conceptions Groups Compared to Cognitive Level

Teacher ID no.	Cognitive level	Mathematics conception
0104	A	1
0106	A	1
0204	A	1
0206	A	2
03051	A	1
03061	A	3
05051	A	1
05061	A	2
0105	B	2
05042	C	3
05041	D	4

Note: 1. Mathematics is a set of rules and procedures, 2. Mathematics is a tool for everyday life, 3. Mathematics is an application of logical thinking and/or step-by-step methods, and 4. Mathematics is experiential and not a static body of knowledge.

Using the classification scheme developed by Grouws, Good, and Dougherty (in progress), teacher responses regarding their definition of problem solving are placed in one of four categories: (1) Problem solving is word problems, (2) Problem solving is finding solutions to problems, (3) Problem solving is solving practical problems, and (4) Problem solving is solving thinking problems. Table 3 shows comparisons of teacher conceptions with their cognitive level.

Table 3

Problem-Solving Conceptions, Mathematics Conceptions, and Cognitive Levels

Teacher ID no.	Cognitive level	Mathematics conception	Problem-Solving conception
0104	A	1	1
0106	A	1	2
0204	A	1	2
0206	A	2	3
03051	A	1	3
03061	A	3	2
05051	A	1	1
05061	A	2	3
0105	B	2	3
05042	C	3	3
05041	D	4	4

Note: Problem-solving conception 1: Problem solving is word problems, 2: Problem solving is finding solutions, 3: Problem solving is solving practical problems, 4: Problem solving is solving thinking problems.

Few consistent and direct relationships between problem-solving conceptions and mathematics conceptions are found. However, if concrete-abstract conceptions are examined, it is evident that the larger grouping of more concrete mathematics conceptions (mathematics is a set of rules and procedures, mathematics is a tool for everyday life, and mathematics is applications of logical thinking and/or step-by-step methods) corresponds to the concrete grouping of problem-solving conceptions (problem solving is word problems, problem solving is finding solutions, and problem solving is solving practical problems). All teachers describing mathematics in a concrete way defined problem solving in the same manner, and conversely. The same is true of the abstract aspect.

During the entire mathematics period, classroom observation data include the amount of time spent on lesson development. Cognitive levels A, B, C, and D have mean lesson development times of 18.8, 20.5, 30.0 and 38.8 minutes, respectively and corresponding standard deviations of 7.6, 12.3, 7.2, and 9.3. There is a noticeable increase in mean times as the level of abstractness increases. As these means are monotone increasing, they suggest practical differences in development time.

Observations demonstrate strong consistencies between cognitive level qualities and instructional practices specific to mathematical problem solving. In particular, individuals associated with the cognitive level A use teacher-directed lessons, dogmatically adhering to lesson objectives. Teaching practices succumb to an inability to effectively deal with ambiguity that could occur if student input is predominate or if no prescription is given for finding problem solutions. Consequently, an approach similar to an algorithmic or mechanistic style dominated problem-solving activities and instruction. That is, little deviation from predetermined solution methods is observed and students are expected to conform to teacher modeling as if following some particular rule. Evaluation of student responses is dichotomous, typically correct-incorrect, and is based on procedural aspects, again related to a teacher need for structure. Teachers demonstrate an inability to restructure mathematical content or to present multiple representations in the event of student difficulty.

Cognitive level B attributes and instructional practices also indicate consistencies. Specifically, the social utility of problem solving is alluded to in each lesson and chosen example and practice problems are perceived by the teacher as useful in student daily life. There is less tendency to mandate solution processes but nevertheless, objective strategies are generally used. Often, teacher comments reflect the importance placed on student self-worth and their role in the classroom society as a possible result of the social context and structure influence.

Cognitive level C practices are systematic instruction. Instructional practices tend to be easily influenced by others' opinions, in particular, another fourth grade teacher, or materials supplied by that teacher. Problem-solving instruction was considered different than other mathematics instruction in that logical thinking was needed to be successful. Hence, different instructional contexts do imply different teaching approaches.

Abstract level D and its associated instructional practices show robust consistencies. Since individual differences are respected and tolerated, the teacher highly regarded student opinions and encouraged them to be creative in their thinking. Concomitantly, teacher reactions to student discussion reflected an appreciation of divergent thoughts and of individual differences. The autonomy and flexibility of level D's thinking processes are evident in the teacher's desire for students to assume responsibility for contributing meaningful explanations and presentations to the lesson.

Conclusions

Conceptions alone, as Thompson points out (1982), are not simply related to instructional practices. Nevertheless, this study found relatively high consistencies among cognitive levels, conceptions, and instructional practices. Although caution should be exercised in making conclusive statements due to the complexity of the subject matter as well as to the small sample size, the findings supported the original hypotheses. That is, teacher cognitive levels are related to their conceptions about mathematics and instruction on specific topics. Consequently, instructional acts portray cognitive level qualities in quite a distinct manner.

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CAN TEACHERS EVALUATE PROBLEM SOLVING ABILITY?

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Hypothetical students' solutions to mathematical problems were sent to 2200 teachers in the Chicago area, with 446 responses. The purpose was to examine the consistency among teachers in evaluating solutions, and to investigate their reaction to solutions having considerable insight. There was considerable variation among teachers in grading the solutions, and less than 25% of the teachers appeared to give credit for creative solutions of the problems.

Related Background Information

Problem solving has been the focal point of mathematics education for more than a decade. Research has followed three separate, but not independent, paths; psychological, curricular and pedagogical. The psychological aspects of problem solving has been centered primarily on characteristics of the problem solver, examining traits like ability levels or cognitive development (e.g. Confrey, Dienes, Krutetskii, Schoenfeld) The curricular focus has been on the nature of problems appropriate for the schools, and the research has been broad (NCTM Standards, or the University of Chicago's School Mathematics Project) and generally has been influenced by external factors such as international studies or reports from national organizations and federal agencies.

The study reported here is in the area of pedagogical research, for which there has been research with respect to problem solving. Much of the focus in this category has been on the so-called "effective teaching" research (Grouws et al, Ducharme and Kluender), and some of the findings may actually be counterproductive when teaching problem solving. Let me explain.

It would seem reasonable to assume that the concept of problem solving, as it is used in mathematics education, is clearly understood, but it is not. Not only is there a lack of agreement as to what the expression, "problem solving" means, many judiciously avoid defining it. In the 1980 NCTM Yearbook on problem solving, none of the 22 articles contained a specific definition of the term. The closest was in a chapter written by Kantowski; she briefly states what she considers to be a "problem."

> A problem is a situation for which the individual who confronts it has no algorithm that will guarantee a solution. That person s relevant knowledge must be put together in a new way to solve the problem. (p. 195)

The paradox is that if the purpose of <u>teaching</u> is to have students acquire the knowledge to solve problems, can teachers confront students with content for which for which they "have no algorithm that will guarantee a solution." Effective teaching research reports that clarity of presentation is highly correlated with student achievement, and teachers assume their students must be equipped with all the necessary concepts and skills needed to solve what problems are posed. They believe they should <u>clearly present</u> the "algorithms that will guarantee solutions" of the problems.

Teachers in the United States, almost uniquely among teachers throughout the developed world have the dual--albeit sometimes contradictory--responsibility of both teaching and assessing learning. They are expected to "teach" a specified body of knowledge, then determine if the students "learned." If students have not learned, who is at fault? Was the material not presented clearly, or did the students lack ability? It is not easy to determine the cause, and it is even more difficult when measuring problem solving achievement.

If solving a problem is taking relevant knowledge and "put(ting it) together in a new way," then how does a teacher determine a student's ability to do so? If a student cannot solve such problems, does the student have a reasonable complaint by saying, "You never taught us how to do this type of problem."? Teachers often respond by not testing for problem solving. Problems on tests are only those which were discussed in the class. Furthermore, insightful solutions may be dismissed as incorrect. The methods of solution <u>must</u> be those that were taught: creative solutions are not accepted.

The motivation for this study stems from an exercise given in a graduate level test construction course. The students, who were also teachers, were shown hypothetical solutions to two algebraic problems and were asked to evaluate the solutions. The solutions of one student, "Allen," were assumed to be complete, methodical solutions. Three of the students had various types of errors in their solutions. One student, "Betty", had the correct answers, but the solutions implied a high degree of insight, not using procedures which are normally "taught." There was considerable inconsistency among the graduate students in grading the solutions, but the grading of Betty's solutions was highly enlightening. Her solutions were given a full range from "0" through "10" by the graduate students. One teacher who gave 0's said that it was obvious the student did not use the methods <u>which were taught</u>, and should not be given credit for <u>merely finding the answer</u>. Apparently, for this teacher, creative problem solving is not a trait to be measured on a test. How pervasive is such an attitude among teachers? The exercise led to the investigation of a larger sample of secondary mathematics teachers.

Can teachers recognize solutions which show "reasoning and creative thinking" instead of taught procedures, on a test how much credit will they give to such solutions, and is there consistency among teachers in evaluating students' solutions to problems? Generally, there were two research hypotheses tested in this project:

<u>Hypothesis 1</u>: When grading student solutions to mathematical problems, secondary teachers will recognize and give credit for insightful, creative solutions.

<u>Hypothesis 2</u>: Teachers will use a conceptually consistent methodology when evaluating students solutions to mathematical problems.

<u>Methodology</u>:

The solutions to two mathematical problems from the five hypothetical students were sent to approximately 2200 secondary mathematics teachers in the Chicago area. The teachers were asked to to "grade" the solutions using a 0 - 10 point scale. The following are examples of solutions from three students for one of the problems.

1. Shu had a train ticket to Kalamazoo. He noticed the number on his ticket was two-digit number, and that the sum of the digits is 18. He said, "if five times the first digit is added to six times the second digit, the result is equal to the original number." What is the original number.

The three students' solutions were given as follows:

<u>Allen</u>

$$
\begin{aligned}
x + y &= 18 \\
5x + 6y &= 10x + y \\
-5x + 5y &= 0 \quad \text{or,} \\
-5x &= -5y \quad \text{or,} \\
x &= y \\
x + x &= 18 \\
2x &= 18 \\
x = 9 &\text{ and } y = 9 \\
&99
\end{aligned}
$$

<u>Betty</u>

Obviously!!

99

<u>Diane</u>

$$
\begin{aligned}
x + y &= 18 \\
5x + 6y &= x + y \\
4x + 5y &= 0 \\
-4x - 4y &= -72 \\
\text{adding,} \quad y &= -72 \\
4x + 5(-72) &= 0 \\
4x &= 360 \\
x &= 90 \\
90 - 18 &= 72
\end{aligned}
$$

The teachers' grades were recorded to examine variation among the teachers, and their evaluations were classified into four categories according to the implied procedures used in grading.

I. <u>Absolutes</u> These are responses from teachers who grade the solutions as fully correct or incorrect.

II. <u>Methodology Centered</u> These are responses from teachers who appear to weigh the credit according to the degree to which the "taught" algebraic methods are shown.

III. <u>Problem Solving Centered</u> These are responses from teachers who recognize and give the credit for both knowledge and insight demonstrated by the solutions.

IV. <u>Unclear Evaluators</u> Responses for which there is no apparent pattern.

After the responses were organized and classified, a random sample of the teachers from each of the categories were selected for follow-up telephone interviews.

<u>Results</u>

A summary of the teacher response is shown in Table 1. After the tabulation of responses, they were classified according to the criteria which is given following Table 1.

Table 1: Summary of Teacher Responses

Student: Score	Allen #1	Allen #2	Betty #1	Betty #2	Chuck #1	Chuck #2	Diane #1	Diane #2	Ernie #1	Ernie #2
0	0	1	93	73	73	294	45	44	21	21
1	1	2	62	32	16	43	18	20	2	1
2	1	1	90	58	83	66	100	89	12	13
3	2	1	18	25	70	19	88	55	29	14
4	0	1	17	26	60	19	70	64	65	25
5	3	3	60	49	64	19	65	45	106	78
6	6	5	4	10	30	2	28	27	96	58
7	2	4	7	19	26	0	18	17	59	75
8	32	23	9	26	19	0	14	24	50	132
9	38	24	4	7	0	1	0	6	5	26
10	361	381	82	121	5	5	0	55	1	3
Total	446	446	446	446	446	446	446	446	446	446
Mean	9.6	9.6	3.8	5.0	3.3	1.1	3.4	4.3	5.3	6/2
Range	9	10	10	10	10	10	8	10	10	10
SD	1.07	1.44	3.80	3.87	2.34	1.81	1.98	3.03	1.95	2.18

I. <u>Absolutes</u> There was no apparent partial credit. Solutions were scored as 0, 5 or 10 depending on whether the answer was correct regardless of the method of solution. Forty five of the respondents (10%) were placed in this category.

II. <u>Methodology Centered</u> Solutions appeared to be given credit according to algebraic methods shown in the solution. The answer sheets ranked the students on the basis of the algebraic information included in the solution. 191 of the respondents (43%) were placed in this category.

III. <u>Problem Solving Centered</u> These responses appeared to give credit to the solutions which were solved with insight and creative thought as well as with algebraic methods. 100 of the respondents (22%) were placed in this category.

IV. <u>Unclear Evaluators</u> Those for which no pattern was apparent. 110 of the respondents (25%) were placed in this category.

Telephone interviews were then conducted: and although the interviews were generally informal five specific question were asked. (1) Do you present problems in class for which the method for solving the problem has not been? (2) How often do you try to evaluate students' problem solving abilities? (3) Approximately what percentage of your tests involve solving problems which were not presented in class? (4) Briefly, how do you divide up the points when grading solutions to word problems? (5) Would you give students credit for solving a problem through reasoning and creative thinking or would you expect them to use the procedures taught in class?" (If the teacher said "No" he was then asked if he had a particular reason for not giving credit?)

Three "Absolutes" were called for interviews. All three said that they rarely present problems in class for which the methods for solving have not been taught, they rarely test for problem solving ability, but they would give credit for creative thinking if the solution was valid and the work was shown.

Nineteen "Methodology Centered" respondents were called. There was considerable variation in the degree to which they claimed they covered problem solving or tested for it. They claimed to be explicit when grading solutions, giving points for specific tasks such as defining the variables, determining the equations and correctly solving the problem. They also said that they would give credit for creative reasoning, provided the equations and solution was reported.

Ten "Problem Solving Centered" teachers were contacted. Nine said they presented problems in class which had not been previously introduced in class. A few said they did so with the stipulation that the problem be related to the objectives. Nine said they evaluate problem solving ability at least once a week, but there was considerable variance in the percentage of a test given to problem solving. Some said they would give credit for creative solutions, but that they did want to know how the students arrive at their answers.

Thirty of the "Unclear Evaluators" were called in follow-up interviews, and it was difficult to determine any response patterns to the interview questions. The one pattern which did emerge was that, like the "Absolutes," these teachers did not give much emphasis to problem solving either in teaching or testing. It may be an invalid conclusion, but it is possible that many of the teachers in this category did not have a clear understanding of problem solving, and therefore had difficulty evaluating students' solutions.

Discussion

Statistical analysis of the data does not appear to be necessary to be demonstrate the serious problem which exists when teachers are asked to evaluate students solutions to mathematical problems. For almost every solution there was a range of 10 points. This was true for even the solutions that were completely correct (Allen) or completely (Diane). Why did some of the teachers give very low scores to Allen's solutions? Was it because the digits were represented by "x" and "y" instead of the traditional "t" and "u"?

What happens if students such as Betty use insight to solve the problem? Can teachers not recognize it, or worse yet do they penalize the student? More teachers (93) gave her no credit than gave her full credit (82). About 78% of the responding teachers either had difficulty grading problem solving or only gave credit for methods which were taught in class. Only 22% appeared to be measuring problem solving ability. If these data be reasonably inferred to the general mathematics teaching population, it seems to imply that problem solving as proposed by the mathematics education community is not going to be easily accepted by the current teaching force.

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TEACHER CONCEPTIONS ABOUT PROBLEM SOLVING AND PROBLEM-SOLVING INSTRUCTION

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Twenty-five junior high teachers were interviewed to determine their conceptions about problem solving and its instruction. Teacher responses were grouped by common characteristics of their descriptions of problem solving and a classification framework consisting of four categories emerged. These include: (1) Problem solving is word problems, (2) Problem solving is solving problems, (3) Problem solving is solving practical problems, and (4) Problem solving is solving thinking problems. While some relationships were evident between conception and reported instructional practices, other aspects of instruction were heavily influenced by external factors such as textbooks, district expectations, and standardized testing and were similar across all teacher responses.

Introduction

A crucial dimension that impacts teacher decision-making is teacher conceptions of mathematics and mathematics instruction. Thompson (1984) indicates that conceptions of the subject matter and its teaching influence teaching actions. She points out that "teachers develop patterns of behavior that are characteristic of their instructional practice. In some cases, these patterns may be manifestations of consciously held notions, beliefs, and preferences that act as 'driving forces' in shaping the teacher's behavior" (p. 105). It is clear that without a better understanding of teacher conceptions and the role they play in the decision-making process, little progress can be made in improving the quality of school mathematics programs. Because the subject matter of mathematics is comprised of many strands, it is hypothesized that teachers will view each strand differently, yet with some characteristics inherent in their own views about mathematics. Therefore, the purpose of this research study is to deepen our understanding of teacher conceptions specifically within the mathematical problem-solving instruction domain.

Methodology

Twenty-five teachers drawn from eight junior high schools in a large midwestern school district comprised the sample. The volunteer sample represented over 80 percent of the junior high mathematics teachers in the district. Together they taught 119 classes composed of more than 2500 students. The SES level of the schools in the district ranged from lower-middle to upper-middle class.

Using a pilot-tested protocol, teachers were individually interviewed for approximately 50-55 minutes concerning their beliefs and teaching practices, with special attention to problem solving. At the beginning of the interview teachers were reminded that although problem solving was important there was not a consensus about what it means or how it should be taught. They were asked to be candid in their responses and reminded that all data collected were confidential. Discussions were audiotaped and later transcribed.

Transcribed interviews were analyzed to identify patterns of responses and to detect relationships among the responses. In some cases responses could be classified using a simple yes-no system, while in other cases, responses were classified using a multi-categorical system. In situations where there was a possibility of coder reliability problems, consistency of classification checks were done using multiple coders.

Results

Discussion of results will focus on identified dimensions important to instruction. These include: (1) problem-solving conceptions, (2) lessons goals and related instructional methods, (3) format of problem-solving lessons, (4) time allotted to problem-solving instruction, and (5) student affective factors. In some instances, direct relationships among these aspects are clear; other times, responses across all teachers are similar and show no specificity to any particular group.

Problem-Solving Conceptions

To help determine the sample teachers' definition of problem solving, they were asked to state in their own words how they would define the term. Careful assessment of their responses showed they clustered into four distinct categories. Many of their responses clearly focused on types of problems while others centered on features of the problem-solving process. The four conceptualizations identified were: (1) Problem solving is word problems; (2) Problem solving is finding the solutions to problems; (3) Problem solving is solving practical problems; and (4) Problem solving is solving thinking problems. The first three definitions focus on the nature of a problem and its computational aspects while the last one is primarily concerned with processes involved in finding a solution.

About one-fourth of the teachers (n=6) felt problem solving could be defined as word problems. The mode of presentation of the problem situation was the determining factor--it must be stated in words. It was often mentioned that these problems could be solved by applying computation or by transforming the problem into an equation and solving. None of these teachers mentioned problem or solution complexity as part of their discussion of what constitutes problem solving and, in fact, most of the problems teachers mentioned as examples were from textbooks. Analyzing a strategy game (e.g., Nim), physically demonstrating the possibility of a spatial arrangement (e.g., Polyominos), finding a pattern, and so on, were clearly not a part of their conception of problem solving. The level of thinking required in tasks was not a consideration.

The largest group of teacher responses (n=10) emphasized problem solving as the solving of problems. Some mentioned that students did not have to be doing word problems (the criteria for the previous conception group) to be involved problem-solving tasks. Instead, any time students found an answer to a mathematical problem, they were doing problem solving. Processes of finding solutions were interpreted as problem solving, but process had a distinct connotation for these teachers. The emphasis was clearly on step-by-step adherence to predetermined guidelines. Many teachers mentioned a four-step approach that students must follow to successfully solve problems: (1) read the problem, (2) determine what the problem is asking, (3) solve the problem, and (4) check the work. Each response indicated that the third step involved computations or setting up equations.

The third category (n=3) emphasized a different problem feature: contextual situation. These teachers consistently discussed solving problems of a practical nature. Example problems consisted of what teachers perceived to be real-life situations, but the solution process of those problems were applied computations. Teacher responses indicated a belief that students should solve problems like these to be better able to transfer their learning and understanding to situations encountered outside the classroom, such as at work or home, where they must function without the aid of the teacher. Problem focus, however, was very narrow, involving checkbook, discount, and purchase tasks.

The remaining teachers (n=6) suggested that problem solving is solving thinking problems. The incorporation of *ideas* into the solution process was a primary focus in responses of this category. Problems mentioned by these teachers required the use of something new and different, a novel approach that had not been practiced by students. The idea of nonroutine problems was frequently mentioned and most example problems required a high level of thinking. Teachers expressed a desire that students use creative solution techniques and find multiple solutions to problems.

Lesson Goals and Related Instructional Methods

Primary goals in problem-solving lessons were reported across all teacher responses as those that teach students how to solve problems, use thinking skills to become more sensitized to reasonable answers, and to develop logical reasoning skills. No specificity to problem-solving conceptions could be detected in teacher reports of specific problem-solving lesson goals.

To attain these goals, many from each conceptualization reported using the four- or five-step general approach to problem solving usually advocated by most textbook series (ie., (1) read the problem, (2) decide what operation to use, (3) solve the problem, and (4) check your answer) and became the backbone of their instruction. Teachers were adamant about having students read the problem as many times as necessary to glean insight into what the problem was asking.

The next step of the process description varied from teacher to teacher but they all seemed to mean the same thing: find what the problem asks. The algebra teachers tended to operationalize this as defining a variable whereas the general mathematics teachers thought of it as deciding what operation to use. All of the teachers in the fourth problem-solving conceptualization (problem solving is solving thinking problems) mentioned they would encourage students to use a strategy such as make a list, table, or chart; find a simpler related problem; or draw a picture. (Interestingly, some teachers commented that guess-and-test should not be used because it was not an acceptable mathematical method.)

The third stage had basically the same meaning for all teachers: solve the problem and get a correct answer. It was in the fourth step that some differences among teachers were noted. The majority of teachers indicated they had students go back over computations to find errors, but a few, including all teachers in the fourth conceptualization, said they made students take a closer look at the question the problem posed to see if their answer made sense or actually answered the question. None of the teachers, however, suggested problem extensions or generalizations as part of the fourth step.

Mention of a key-words strategy was scattered throughout some responses. Several teachers felt reading skills were so low that keyword lists were a necessary component of successful problem solving. The lists they compiled included words such as altogether which was to be taken to mean add, left to mean subtract, and so on and were posted on blackboards for students to refer to as they solved problems. The fourth definition category "problem solving is solving thinking problems" did not have any teachers indicating they used the keyword approach while the majority of teachers in the other categories relied on it as a method to improve problem-solving skills.

Some teachers expressed frustration in not being able to help students understand how to solve problems better. They felt that students either know how to proceed or not. If not, then teachers were at a loss as to what to do to help them approach a problem. They did not know what kinds of questions to ask or what hints to give.

All teachers were concerned about getting students to the point where they could solve problems and get a correct answer. They found this difficult due to their perceived inability to motivate or direct students and also to perceived student deficiencies in content and interest.

Format of Problem-Solving Lessons

There was considerable similarity in teacher descriptions of a typical problem-solving lesson. Instruction usually began as a teacher-directed activity with the teacher modeling the problem-solving process in the hope students would emulate those behaviors when confronted with problem situations on their own. For most teachers, this meant that the *exact* solution procedure they would like to see is presented at the chalkboard along with guidelines, such as show all your work, begin by defining a variable, and so on. After modeling, teachers presented problems to students and provided time for them to work individually with little intervention or guidance, unless it was student initiated. None of the teachers provided explanations of how they draw these experiences together for students, either during that specific lesson or in subsequent lessons. One can assume that this was not part of a typical lesson.

Controlling factors of a lesson (factors that guide decisions about the flow of the lesson) influence teachers as they determine exactly what is to be presented. In this sample, those factors tended to be directly related to teacher needs or to external factors not associated with students. Teachers reported that they drove the lesson in the direction they felt was most appropriate, *or*, alternately, they closely followed the textbook presentation. In neither case were student needs nor responses used to determine lesson flow and direction.

During the modeling portion of lesson, textbook problems provided examples for most teachers. The teacher's text edition often gave similar problems to what would be assigned for independent seatwork or homework and, thus, they felt comfortable and justified in using them. Often, problems from the homework set were used as examples so students could specifically see how to pattern their work on other problems. (Or, teachers would provide clues on how to solve all the homework problems in what they called an attempt to reduce student anxiety levels.)

Time Allotted to Problem-Solving Instruction

Time considerations were common to a variety of responses. Teachers often complained there was not enough time during the class period to do problem-solving activities. It was clear that these teachers felt problem solving was a distinct topic and was not integrated within other strands. In light of this view, it was surprising that, with an emphasis on needing more time, teachers did not indicate they had revamped the structure of their class time (ie. decreased seatwork time, graded homework in a different manner, and so on) to allow for a longer development portion of the lesson during problem-solving instruction nor to accommodate separate problem-solving activities as warm-ups or supplements to other lessons.

There was also concern expressed about the role of problem-solving instruction during the course of the entire school year. These teachers felt some topics in the curriculum were pushed by the school district and were primarily computational. Also, with the emphasis the district placed on standardized test results, teachers felt students needed to review topics prior to the test administration. Thus, the amount of time left for what teachers considered important (e.g., problem solving) was minimal. It was quite clear that in reality problem solving is far down the list when instructional time is allocated. No teachers were found who did problem-solving work first and then searched for time to squeeze other things into the curriculum or integrated problem solving and other topics.

Student Affective Factors

Teachers expressed the view that students are frustrated by problem-solving tasks and, as a result, have lowered self-confidence. When this occurred, teachers felt that teaching became a much more difficult task because students presented more management problems such as being off-task and were harder to motivate. To give their students more confidence to attack problem-solving activities, teachers preferred modeling problem solutions so that students would have guidelines to follow as they attempted problems on their own.

Teachers generally did not expect students to perform well on problem-solving tasks. Lowered expectation levels could prove detrimental because students become aware that feigning incapability usually resulted in an increase in teacher assistance or a decrease in the number of homework problems. And, teachers reported such an increase in student requests for help did occur during problem-solving lessons. They attributed it to low student success rates and self-confidence.

Summary

We are beginning to better understand teacher beliefs and conceptualizations about problem solving. In particular, it now seems clear that problem solving has varied meanings and these may differentially influence many aspects of the problem-solving instructional process. Specifically, teacher responses were captured into four conception groups: (1)Problem solving is word problems, (2) Problem solving is solving problems, (3) Problem solving is solving practical problems, and (4) Problem solving is solving thinking problems. Relationships between conceptions and particular instructional practices in some instances are clear. For example, the use of problems stated in words, particularly of the variety found in most textbooks, provided the impetus for problem-solving lessons for those teachers that believe problem solving is word problems. Teachers that view problem solving as the solving of practical problems tend to use real-life situations as the motivation for their lessons. Further, their goals include the desire to help students become independent and functional in problem-solving contexts outside the classroom. Similar relationships exist between other conceptions and problem-solving instruction.

Many factors were reported as influences on problem-solving instruction; some are teacher-controlled and others external to the immediate classroom environment. Based on our data, the textbook is the external factor that most heavily influences classroom practices. Other influences mentioned in our interviews include: classroom management considerations, perceived student ability levels, standardized tests, and, of course, teacher conceptions of problem solving.

With the identification of variability in conceptions about problem solving, areas where problem-solving instructional practices seem to differ, and important external factors, informed naturalistic studies involving observations can now focus on links between problem-solving instruction and teacher beliefs. The relationships among these factors are no doubt complex. In fact, we now know that we must carefully describe what is meant when a teacher gives critical importance to problem solving and its instruction in her classroom. Similarly, we must probe the tradeoffs that occur between conceptions about problem-solving instruction and powerful external factors such as textbooks and standardized testing. Although the relationships between conceptions and practice are not simple, it is essential to arrive at an understanding of them if we are to understand and improve problem solving instruction in mathematics.

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MATH TEACHERS AND GENDER DIFFERENCES IN MATH ACHIEVEMENT, MATH PARTICIPATION AND ATTITUDES TOWARDS MATH

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ABSTRACT

The influence of math teachers on gender differences in achievement, attitudes and participation was investigated by means of questionnaires and observations. It is concluded that there are small differences in achievement and large differences in attitudes and participation, but that these differences cannot be attributed to the math teachers.

INTRODUCTION

It has been found that Dutch math teachers in secondary education have different perceptions and expectations of girls and boys on math relevant dimensions (Jungbluth, 1982; Van der Werf et al., 1984).

It has been suggested that because of these differences math teachers treat girls and boys differently during their lessons, which in turn helps to create the large gender differences in math participation, as soon as math is no longer compulsory. A research project* was undertaken in order to test these suggestions. This paper focuses on the influence of math teachers on gender differences in achievement and participation and in attitudes towards math on the one hand, and teacher student interactions on the other hand. If it appears that different teachers do have an effect on the gender differences, i.e. if the TeacherxGender interaction is significant, this effect may be explained by different teacher behaviors towards girls and boys. This is tested with multi-level analysis (Aitkin & Longford, 1986).

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METHOD

The research was conducted at the three levels of Dutch general formative secondary education, MAVO (Low Level: LL), HAVO (Medium Level: ML) and VWO (High Level: HL).

Data collection took place during the spring of 1986 (*Take 1*) and 1987 (*Take 2*). At *Take 1* students in the grade *before* and the grade of the choice of examination subjects were investigated. At *Take 2* these students, who were now respectively in the grade of and the grade *after* the choice, were investigated again, and many new ones. The longitudinal aspect is not relevant for this paper and is therefore neglected. More than 5800 students and their teachers participated.

Variables of interest are: 1) teacher perceptions of girls and boys in general, 2) teacher expectations of individual students, 3) frequencies of teacher - student interactions, 4) marks on math, 5) intended and actual math participation, 6) attitudes towards math, and 7) perceptions of the teachers' behaviors during math lessons. These variables are explained together with the results in the next section.

RESULTS

First the teacher perceptions and expectations are discussed shortly and the teacher - student interactions in some detail. Next the gender differences on the four groups of student variables are presented. Finally the Teacher effect on the student variables is reported.

<u>Teacher perceptions and expectations</u>

1. It appears that *tidiness* is attributed more to girls than to boys by three quarters of the teachers; *industriousness* is also seen as more typical for girls and *disturbing order* as more typical for boys. No difference between the sexes is indicated on *unattentiveness, taking initiative* and *studiousness*.

2. For each student the teachers were asked three questions.

It appears that more boys than girls are expected to choose math and that more boys than girls will be advised to choose math. On the last question *Do you think this student could do better on math than appearing from her/his achievements?* (4-point scale) the overall difference is 0.2 in favor of the boys.

Teacher - student interactions

3. These interactions were measured by means of observations. The purpose of these observations was two-fold: to investigate whether girls and boys are treated differently by their math teachers and to construct indices of gender specific treatment - for each teacher. Both teacher behaviors and student behaviors were observed.

In general, both groups of behaviors are either self initiated (spontaneous) or reactive. The relevant spontaneous teacher behaviors are 1) giving a turn to an individual student, 2) giving help. The relevant reactive teacher behaviors are 1) giving feedback, 2) giving help (when requested), 3) (not) answering a question, 4) (not) permitting a student who raises hand to say something, 5) making a disciplinary remark. Turn giving was divided in five subcategories, feedback giving in seven. All teacher behaviors directed towards the class as a whole are not relevant in this context.

The spontaneous student behaviors are 1) raising hand, 2) raising hand after a question of the teacher to the class, 3) answering directly after such a question, 4) making a statement or question, 5) asking help, 6) asking feedback. The last three behaviors may or may not be preceded by raising hand and being permitted. The reactive student behaviors are 1) answering when having the turn, 2) not answering when having the turn.

Each class was observed during three lessons. We report the observation data on the highest level of aggregation, i.e. over lessons, classes, levels and takes, but separately (of course) for girls and boys. The figures given represent hundred times the relative frequency for girls and for boys to receive or to show a specific behavior during one lesson - a probability-like measure. Below are reported the behavior categories in which the gender difference is at least 3.

The teachers gave more turns to girls than to boys (38 vs. 35) and continued the same turn more often with girls than with boys (32 vs. 29). Consequently the girls more often answered on a turn (67 vs. 62). The boys more often answered spontaneously when the teacher asked a question to the class (19 vs. 15) - which may explain why the girls got more turns. The boys more often received positive feedback (68 vs. 65).

The girls more often raised hand spontaneously (31 vs. 27) and more often were permitted (27 vs. 23); (thereafter) they asked more often for help (17 vs. 13), which they also did more often spontaneously, without hand-raising (22 vs. 18). Consequently, the girls more often received help after request (34 vs. 27). The teachers also gave more help to the girls unsollicited (19 vs. 16). Finally, the boys more often made a statement spontaneously (14 vs. 11), and the teacher more often aimed disciplinary remarks at boys than at girls (23 vs. 20).

In our opinion these differences are not very large. Moreover, at lower levels of aggregation they are not completely consistent. The general pattern is that the teachers not actively aggrieve the girls; the contrary is more plausible. For the rest, the gender differences in teacher behavior appear to be caused largely by different behaviors of girls and boys themselves.

Gender differences

4. Dutch achievement marks range from 1 (very low) to 10 (excellent). The marks at the last two school reports were provided by the teachers. At the average the boys obtained 0.2 higher marks than girls, which is a rather small difference, taking into account the possible range and the standard deviation. In the separate groups the difference varies from -0.1 to 0.6.

5. The intended math participation was measured on a five-point-scale (*not; may be not; may be not, may be yes; may be yes, yes*). The actual math participation (dichotomous) could be obtained only from the students who were at *Take 2* in the grade *after* the choice. At HL either *applied* math and/or *pure* math can be chosen.

In all groups a higher percentage boys than girls indicated to choose math certainly. The difference ranges from 20% until 35%, with one exception: at HL the difference is much smaller for *applied* math. Taking into account the students indicating to choose math probably, the difference becomes in some cases more, and in some cases less pronounced.

At LL in the grade after the choice 52% of the girls and 83% of the boys had chosen math. At ML these percentages are 48% and 76%, at HL 31% and 60% for *pure* math and 59% and 67% for *applied* math.

6. On the basis of factor analysis four attitude scales were constructed. The first scale (7 items, a=.86) was labeled '*difficulty* of math', the second (5 items, a=.86) '*pleasure* in

math', the third (3 items, a=.62) 'math as a *male domain*', and the fourth (2 items, a=.69) '*usefulness* of math'. The pattern of the gender differences on these scales is very consistent over the groups. Girls significantly perceived math as more difficult and less useful, and (surprisingly) less as a male domain than boys did. On *pleasure* there is tendency that girls have lower means (indicating less pleasure) than boys. However, in the grade after the choice, i.e. among the students who had chosen math indeed, the girls' means tended to be higher than the boys.

7. On the basis of factor-analysis three scales for perceived teacher behaviors were constructed: *accessibility* (7 items, a=.86), *gender specific behavior* (5 items, a=.70) and *relevance transfer* (making clear the relevance of math; 4 items, a=.76). Only small gender differences occur on these scales. On the first there is no difference whatsoever, on the second there is a tendency that girls perceived less gender specific behaviors than boys did, and on the third girls tended to perceive less relevance transfer than boys.

The influence of the teacher

The results reported above are only pertaining to differences between girls and boys, i.e. the Gender main effect. The analyses reported in this section focus on the influence of the teachers, i.e. the Teacher main effect and - more importantly - the TeacherxGender interaction.

Two-way analyses of variance were performed on the three scales for perceived teacher behaviors in nine separate groups (the relevant combinations of Level, Take and Grade).

On *accessibility* the Teacher main effect was significant ($p<.01$) in all nine cases, the Gender main effect was not significant in any case, and the interaction was significant in two cases. The averaged percentage of variance accounted for (VAF) by Teacher is 44%, by Gender 0%, and by the interaction 3%.

On *gender specific behavior* Teacher was significant again in all nine cases, Gender was significant in five cases, and the interaction in one case. The averaged percentages VAF are 26%, 2% and 3%.

On *relevance transfer* Teacher was significant again in all nine cases, Gender in one case, and the interaction was not significant in any case. The averaged percentages VAF are 30%, 0% and 2%.

It can be concluded that there are large differences in the way individual math teachers are perceived by their students. The (non)significance of Gender corresponds with the remarks made at the end of (7) above. The finding that the interaction is not significant - with a few exceptions - means that it hardly occurs that specific teachers are perceived quite differently by girls than by boys.

In order to investigate the influence of the math teachers on gender differences in achievement, participation and attitudes multi-level analysis was performed (Aitkin & Longford, 1986). In the present case the analyses of variance reported above are analogous to the first level, in which students are the unit of analysis. At the second level, in which teachers are the unit of analysis, teacher variables are related to the teacher parameters resulting from the first level. The teacher variables used are gender and eight categories of observed teacher behaviors - for this purpose transformed into indices ranging from - 1.00 (only aimed at boys) to + 1.00 (only aimed at girls). In order to obtain a reasonable amount of statistical power for the second level of analysis LL, ML and HL* were taken together.

In accordance with the primary research question (i.e. the influence of teachers on gender differences) not the teacher parameters itselves were analysed, but the parameters of the TeacherxGender interaction. The dependent variables were math achievement, intended and actual participation*, the four attitudes towards math, and (for comparison) the three scales of perceived teacher behavior.

Analyses were performed with VARCL (Longford, 1988). The test statistic for a certain effect is the difference in *deviance* between two fitted models - in the present case the models with and without the Teacher Gender parameters. The distribution of this difference is (asymptotically) chi-squared, in the present case with two degrees of freedom. The critical value ($p<.01$) is 9.2. The results of the first level analyses are given in table 1.

Although the difference in deviance is significant in a number of cases, most differences are small, taking into account the large numbers of students involved.

In the four cases in which the difference is above 20.0 the second level of analysis was performed. On *male domain* none of the teacher variables appeared to be related to the interaction parameters in either case. At further inspection, the significant result is due to a

* At HL *pure* math was taken.

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systematic difference between LL, ML and HL: in both cases the gender difference is smaller at HL than at LL and ML. Because teachers are nested within Level this explains the significant result.

Table 1: Test of the Teacher Gender interaction: differences in deviance

	Take 1; grade before choice	Take 1; grade of choice	Take 2; grade of choice
math achievement	2.2	2.0	10.9*
intended particip.	9.6*	4.7	11.7*
actual particip.	-	0.0	-
difficulty	4.3	0.0	1.2
pleasure	3.6	2.5	13.5*
male domain	27.4*	14.6*	39.8*
usefulness	4.5	9.0	0.3
accessibility	12.2*	6.9	22.3*
gender spec.beh.	11.5*	5.5	27.4*
relevance transf.	5.9	6.7	2.8

* Significant at 1%

On *accessibility* the teacher's gender is significantly related to the interaction parameters. It appears that female math teachers are perceived to be more accessible by girls in comparison with the three other gender-gender combinations, which do not differ. This accounts for 15% of the variance in the interaction parameters.
On *gender specific behavior* again the teacher's gender is significantly related to the interaction parameters. It appears that male teachers are perceived to behave more gender specificly by girls, whereas female teachers are perceived to be more gender specific by boys. This relation accounts for 8% of the variance.

CONCLUSIONS

The finding that Dutch math teachers have different perceptions and expectations of girls and boys is replicated. Girls in general are perceived as more tidy and industrious, boys as more troublesome. Boys are expected to have more math capacity in reserve.
The observations show some difference in teacher behaviors towards girls and boys. However, these differences can be attributed largely to gender differences in student behaviors. Moreover, it is not true that girls are treated less favorably than boys, in the contrary.

The gender difference in math achievement is small, but large in attitudes towards math (especially on *difficulty* and *usefulness*) and in math participation. There are also large differences in the way individual math teachers are perceived (especially on *accessibility*), but there are only small differences between girls and boys in this respect, both overall and per teacher (the interaction).

Finally, and possibly most importantly, the differences in achievement, attitudes and participation cannot be attributed to (characteristics of) individual math teachers. There is some evidence, however, that the gender of the teacher influences the perception by girls and boys of their teachers' behaviors in a way that might be labeled 'own sex-favoritism'. The observed teacher behaviors do not influence this perception.

In our opinion the results fit nicely into a general pattern, which can be verbalized as follows: *the gender differences in math are not the teachers' fault*.

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TEACHING STUDENTS TO BE REFLECTIVE: A STUDY OF TWO GRADE SEVEN CLASSES[1]

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This paper reports on a study of the effects of mathematics problem-solving instruction designed to increase students' cognitive self-awareness and ability to monitor and regulate their cognitive performance. The instruction, which also included many of the features of previous research on problem-solving instruction, took place over a period of 14 weeks and involved the teacher in three distinct, but related roles: external monitor, facilitator of students' metacognitive development, and model of a good problem solver. The paper provides an overview of pre-instruction to post-instruction changes in students' problem-solving performance and a brief description of five general observations about the efficacy of problem-solving instruction designed to increase students' reflectiveness.

Background

For generations, mathematics teachers have voiced concern about the inability of their students to solve any but the most routine verbal problems, despite the fact that they seem to have mastered all the requisite computational skills and algorithmic procedures. Until recently, researchers have been content to attribute problem-solving difficulties almost exclusively to cognitive aspects of performance. However, there has been growing sentiment for the notion that a much broader conception is needed of what mathematical problem solving involves and what factors influence performance.

The rather elusive construct referred to as metacognition is among the factors that are currently considered to be closely linked to problem solving. Briefly, metacognition refers to the knowledge and control individuals have of their own cognitive functioning: that is, what they know about their own cognitive performance and how they regulate their own cognitive actions during the performance of some task. Metacognitive knowledge about mathematical performance includes knowing about one's strengths, weaknesses, and processes, together with an awareness of one's repertoire of tactics and strategies and how these can enhance performance. Knowledge or beliefs about mathematics that can affect performance are also considered metacognitive in nature. The control and regulation aspect of metacognition has to do with the decisions individuals make concerning when, why, and how they should explore a problem, plan a course of action, monitor their own actions, and evaluate their own progress, plans, actions, and results. This self-regulation is influenced by the individual's metacognitive

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knowledge (see Garofalo & Lester [1985] and Schoenfeld [1987] for more extensive discussions of metacognition and its relationship to mathematical behavior).

The research discussed in this paper is a continuation of our earlier work involving the role of metacognition in young children's mathematical problem solving and is an extension of the research reported at the PME XI meeting in Montreal (Garofalo, Kroll & Lester, 1987). Our most recent research project has had two main goals: (1) to investigate the influence of metacognition on the cognitive processes students use during mathematical problem solving, and (2) to study the effectiveness of instruction designed to increase students' cognitive self-awareness and ability to monitor and evaluate their own cognitive performance. A detailed discussion of the goals of this project is given in the final report of the project (Lester, Garofalo, & Kroll, 1989). In this paper we consider only the second of these goals: in particular, the question of how students can be taught to be more reflective about their problem-solving behaviors.

The Research Plan

In recent years there has been much research conducted on various approaches to mathematical problem-solving instruction. Detailed reviews of this research can be found in Kilpatrick (1985) and Lester (1985), and extended discussions of problem-solving instruction can be found in Charles and Silver (1989) and Schoenfeld (1985). One observation is common to all these reviews and discussions: namely, that none of the approaches has been shown to be substantially superior to the others. Furthermore, in reviewing the literature we found no evidence of systematic attempts to design instruction in mathematical problem solving that would emphasize the development of metacognitive skills in the context of learning regular mathematics content and that would expose students to a wide variety of problem types over a prolonged period of time. Consequently, we decided to conduct an exploratory study to investigate both the relative effectiveness of various teacher roles in promoting metacognitive behavior in students and the potential value of instruction involving a wide range of types of problem-solving activities.

Description of the Instructional Component of the Study

The instructional approach we used came about as a result of experience gained during several previous studies. In the mid-1970s, the Mathematical Problem Solving Project (MPSP) at Indiana University (Stengel, LeBlanc, Jacobson, & Lester, 1977) created a problem-solving teaching approach which was later refined by Charles and Lester (1982). The approach was characterized by the inclusion of: (1) practice in the use of strategies (strategy training), (2) instruction concerning the value and significance of strategies (awareness training), and (3) instruction concerning the monitoring and control of strategies (self-regulation training). Perhaps the most important features of the teaching approach are that it identifies rather specifically a set of ten "teaching actions" to guide the teacher during classroom problem solving lessons (see Charles & Lester, 1984, for a description of the teaching actions) and that it includes attention to monitoring and control strategies. In a study designed to investigate the potential effectiveness of the teaching approach, Charles and Lester (1984) found significant growth in students' problem-solving abilities with respect to comprehension, planning and execution strategies. From these findings, we became convinced that training in the use of a collection of skills and heuristics is much more effective when accompanied by attention to affective and metacognitive aspects of problem solving. In the current study, we decided to add an even more explicit focus on metacognition by having the teacher model strategic behavior and vocalize metacognitive thinking and decision making as he attempted to solve problems in front of the class. The notion of having the teacher serve as a model of a metacognitively-aware problem solver stemmed from Schoenfeld's (1983) recommendation that teachers should attempt to model good problem solving for their students.

The instruction was presented by Frank Lester to one regular-level and one advanced-level class of seventh grade students about three days per week for a period of 14 weeks (each class met for a period of 45-minutes per day). (Diana Kroll served as research associate and classroom observer.) In addition to instruction in problem-solving strategies, the instruction included three teacher roles focused on developing students' reflectiveness: the teacher as an external monitor, the teacher as facilitator of students' metacognitive awareness, and the teacher as model of a metacognitively-aware problem solver.

The role of teacher as an external monitor involved the teacher in: directing whole-class discussion about problems that were to be solved; observing, questioning and guiding students as they worked either individually or in small groups to solve each problem; and, leading a whole-class discussion about students' solution efforts.

The teacher as facilitator role involved the teacher in: asking questions and devising assignments that required students to analyze their mathematical performance; pointing out aspects of mathematics and mathematical activity that have bearing on performance; and helping students build a repertoire of heuristics and control strategies, along with knowledge of their usefulness. One way in which we attempted to direct students to reflect on their own cognition was to have them complete self-inventory sheets on which they listed their own strengths and weaknesses in doing mathematics. Another activity was to ask students to write short statements about their thinking during their solution attempts immediately after solving a problem.

The third role, teacher as model, required the teacher to demonstrate regulatory decisions and actions explicitly while solving problems for students in the classroom. The intent was to give students the opportunity to observe the monitoring strategies used by an "expert" as he solved a problem that he had never solved before. In addition, the teacher directed a discussion with the class about their observations of his behavior.

Data Collection

Written Pre- and Post-tests. Prior to the beginning of the instruction, written problem-solving tests were administered to all students in the two classes. A posttest, parallel to the pretest in terms of problem structure and difficulty, was administered to all students within a week after the end of the instruction. The problems on both tests were chosen to include some routine problems like those commonly encountered in school, as well as some non-routine, "process," problems like those considered during the instruction. The intent was to include some problems which students could not solve simply by means of the direct application of one or more arithmetic operations, problems that would require students to engage in strategic decisions and regulatory behaviors. The tests included a one-step, a two-step, and three process problems. Each problem was scored with respect to the degree of understanding and planfulness apparent, as well as with regard to the correctness of the answer.

Pre- and Post-instruction Interviews. Pre-instruction and post-instruction interviews were conducted with a subset of the students from each of the two classes. In these interviews the students were videotaped as they solved several problems (either individually or in pairs), and were questioned about their work and their thinking. No findings from the interviews are reported here because this paper focuses on the instruction, rather than the interview, phase of the project. However, observations resulting from these interviews are summarized in the project's final report.

Observations of Instruction. Instructional sessions were video- and audio-taped for both classes. Also, in order to pick up conversations between individual students or small groups of students and the instructor, the instructor wore a lavaliere microphone attached to an audio-cassette tape recorder which was worn on his belt. The tapes were a primary source of data on the effectiveness of the instruction.

A standard practice followed on almost all occasions was for the observer (who also operated the video camera) to debrief the instructor shortly after a session ended. That is, the observer and instructor discussed how the session had gone, what had gone well (or not so well), and what might be done as a follow-up activity on subsequent days. On occasion the observer called the instructor's attention to something that he may not have noticed (e.g., a group of students who had not been attentive) or suggested an idea for modifying an activity. In addition to the observer, the regular teacher sat in on about half of every class session. She never made comments or intervened during a lesson, but she did make several valuable suggestions to the instructor afterwards.

Results

Results of two types of analysis are presented in this section: (1) pre-instruction to post-instruction changes in students' problem-solving performance, and (2) observations of instruction.

Pre-Instruction to Post-Instruction Changes in Students' Performance

Both the regular class and the advanced class showed an overall mean gain in total score from pretest to posttest and the amount of gain was about the same for the two classes. Of 10 possible points on each test, the regular class mean went from 4.7 to 6.0 and the advanced class mean went from 6.7 to 8.4. However, these gains were not statistically significant. One interesting result is that the pretest mean of the advanced class exceeded the posttest mean of the regular class. Further, four students in the advanced class scored a perfect 10 on the pretest (only two scored 10 on the posttest), whereas no student in the regular class scored 10 on either the pre- or the posttest. The presence in the advanced class of several perfect scores, and the fact that only two students in this class scored lower than 8 on the posttest may indicate a ceiling effect on these tests for the advanced class. In other words, the advanced class students may have learned more about problem solving than their scores indicate. On the other hand, the tests did not seem to be too easy for the regular class. Assuming that the tests did measure problem-solving ability, it appears that the instruction was moderately successful. Also, a closer look at the students' written test results, together with an analysis of their class work and homework performance, suggests that the instruction was most effective with average ability students.

Both the pretest and the posttest also contained a series of four multiple choice self-inventory items accompanying each of the problems to be solved. These items provided information about certain aspects of the students' metacognitive awareness. Specifically, the items involved students' assessment of problem difficulty, confidence in the correctness of their solutions, familiarity with the types of problems, and interest in solving problem like the ones under consideration. No significant changes were detected from before to after instruction in any of the four areas, nor was any significant correlation found between students' problem-solving scores and any of the self-inventory areas.

Observations of Instruction

Since the instruction was exploratory in nature, we decided to attempt to describe it as completely as possible. Our approach was to prepare written accounts of the instruction from the point-of-view of three persons: the problem-solving instructor (FKL), the observer (DLK), and the regular mathematics teacher. These accounts are recorded in the final report of the project (Lester, Garofalo & Kroll, 1989). There was general agreement about the instruction on five points, each of which is stated below.

Observation 1: Control processes and awareness of cognitive processes develop concurrently with the development of an understanding of mathematics concepts. Thus, attempts to make students more reflective about their problem-solving should take place in the context of regular mathematics instruction. In this study, it was important for the instructor to be willing and able to deal with questions about mathematics content (e.g., how to find percentages) and about problem-solving skills (e.g., how to organize a table) at the same time that he was facilitating instruction designed to increase students' reflectiveness.

Observation 2: Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner, on a regular basis, and over a prolonged period of time. Furthermore, the teacher must play a prominent role not only in organizing instruction, but also in directing class discussions and aiding students in learning how to be reflective about their thinking (But, see *Observation 4*).

Observation 3: In order for students to view being reflective as important, it is necessary to use evaluation techniques that reward such behavior. That is, great care should be taken to insure that what is evaluated is consistent with what is intended to be learned.

Observation 4: The specific relationship between teacher roles and student growth as problem solvers remains an open question. In particular, the roles of teacher as facilitator and teacher as model need much more attention, and student expectations about the role of the teacher must be considered. For example, attempts to have the teacher model monitoring behaviors while solving an unfamiliar problem in front of the class were less successful than expected. The teacher found it difficult to maintain his role as problem solver (lapsing frequently into a teacher-like explaining role rather than a problem-solver-like modeling role). And the students indicated uneasiness because they expect a teacher to demonstrate the *right* way to solve a problem, not to stand in front of the class looking confused about a problem and making false moves.

Observation 5: Willingness to be reflective about one's problem solving is closely linked to one's attitudes and beliefs. We observed that students' attitudes and beliefs about self, mathematics, and problem solving frequently played a dominant role in their problem-solving behavior. It was often just as important to ensure that the students were motivated, engaged, and confident about trying to solve a problem as to ensure that they possessed sufficient mathematics knowledge or adequate monitoring skills.

Discussion

The relationship between problem solving and metacognition have been of interest to us for several years. Despite this long-term involvement we believe that we have only just begun to scratch the surface of what there is to know. At present, what we know about the role of metacognition and other noncognitive factors in mathematical problem solving is based more on our own experiences as teachers and learners of mathematics than on the results of carefully and systematically conducted research. Additional insights into the effectiveness of the instruction might be gained from further analyses of the data that have already been gathered. Moreover, in a future study we intend to undertake much more thorough scrutiny of various facets of problem-solving instruction.

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STUDENTS' AFFECTIVE RESPONSES TO NON-ROUTINE MATHEMATICAL PROBLEMS: AN EMPIRICAL STUDY

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Solving non-routine problems inevitably involves overcoming blockages and interruptions. Mandler's theory of emotion suggests that such blockages will result in relatively intense emotional responses. Protocols from seven university students provide support for Mandler's theory. Students reported both positive and negative emotional states while solving problems; reports of frustration were the most common response.

Research on mathematical problem solving has concentrated mainly on the cognitive processes of problem solvers. More recently researchers have turned their attention to the role of affect in student performance on non-routine mathematical problems. The purpose of this study was to investigate how affective factors can help or hinder the performance of young adults on problem-solving tasks.

The role of affect in problem solving has been identified as an underrepresented theme in research on this topic (Silver, 1985). Although affective factors have received more attention recently, (e.g., McLeod & Adams, 1989), we still have very little data on the affective states of students (especially more intense affective responses) as they solve non-routine problems. The data gathered for this study focus particularly on the emotional reactions of problem solvers.

Conceptual Framework

Research on the affective domain has usually concentrated on attitudes and beliefs, the less intense kinds of affect that can be assessed through the use of questionnaires (Reyes, 1984). Problem solvers, however, often exhibit more intense emotions. They report feelings of frustration (or elation) as they struggle with (or triumph over) mathematical problems (Mason, Burton, & Stacey, 1982). Experts as well as novices report rather intense emotional responses to mathematical problems (McLeod, Metzger, & Craviotto, 1989).

Mandler (1984) has developed a general theory of emotion that provides a strong conceptual framework for research on affect in this context. For a description of how the theory can be applied to the teaching and learning of mathematical problem solving, see Mandler (1989) and McLeod (1988). A brief summary of the theory will be presented here.

Mandler's view is that most affective factors arise out of the emotional responses to the interruption of plans or planned behavior. In Mandler's terms, plans arise from the activation of a schema. The schema produces an action sequence, and if the anticipated sequence of actions cannot be completed, the blockage or discrepancy is followed by a physiological response. This physiological arousal may be experienced as an increase in heartbeat or in muscle tension. The arousal serves as the mechanism for redirecting the individual's attention, and has obvious survival value which presumably may have had some role to play in its evolutionary development. At the same time the arousal occurs, the individual attempts to evaluate the meaning of this unexpected or otherwise troublesome blockage. The interpretation of the interruption might classify it as a pleasant surprise, an unpleasant irritation, or perhaps a major catastrophe. The cognitive evaluation of the interruption provides the meaning to the arousal.

In mathematics education, problems are usually defined as those tasks where some sort of blockage or interruption occurs. The student either does not have a routine way of solving the problem, or the routine solutions that the student attempts all fail. As a result, the kind of problem solving that is attempted by mathematics students results in just the kind of interruption that Mandler has analyzed in his theory. In this study students' performance was analyzed to see how they reacted to interruptions and blockages while solving mathematical problems.

Design and Procedures

Seven subjects (four females and three males) were chosen from among volunteers who were enrolled in two mathematics courses for non-majors at a large state university. One course was intended for business majors (four subjects); the other was a content course for prospective elementary school teachers (three subjects).

Each student was asked to think aloud while solving problems during a one-hour interview. Interviews were videotaped. The students worked on an assigned non-routine problem until they obtained a solution or decided to quit. After the students finished a problem, they were asked to describe their feelings during the problem-solving episode. The interviewer followed up their responses with specific questions about important points during the problem-solving process, particularly points at which they had run into difficulty or experienced some success. The interviewer also asked about particularly positive or negative feelings that the students had experienced. At the end of the interview, each student was asked to draw a graph that showed his/her emotions during the problem-solving episode.

Five different non-routine problems were used. Although the problems are well known, the students were generally not familiar with them. In one problem ("the handshake problem") students were asked to find the number of handshakes if there are eight people at a party and they all shake hands. Another problem ("chickens and pigs") involved finding the number of chickens and pigs in a barnyard if the farmer says that there are 60 eyes and 86 feet. A third problem ("seven gates") asked how many apples a man gathered in an orchard if he had to give a guard at the first gate half of the apples plus one more, and then had to give half of the remaining apples plus one more to a guard at each of the next six gates, finally leaving the orchard with just one apple.

Five students worked on three or four problems; one student tried two, and the remaining student attempted all five. Three students solved two problems, and three students were not able to solve any problems. The remaining student solved one problem, yielding seven correct solutions out of 25 problems.

Results

The analysis of the data followed the factors listed in McLeod (1988). Students were generally able to describe their emotions in reasonable detail. They reported mainly frustration and happiness as their emotional responses to problem solving, along with occasional references to other emotions like panic and satisfaction. The students drew graphs that indicated rather wide swings between positive and negative emotions, and suggested that the negative emotions were particularly intense. In the chickens and pigs problem, for example, a student drew a graph that went up when she realized that there were 30 animals altogether. The graph turned down, however, when she divided 86 by 30; she later said that she was just "playing with the numbers." She went on to try to solve the problem through trial and error, and made good progress. She would have been successful except for a computational error that resulted in feelings of frustration and the comment "I hate word problems." Her graph showed the rise and fall of her hopes and emotions. Her frustration with the computational error was quite intense; it seemed that her feelings of frustration used so much of her working memory capacity that she was unable to find her error in computation.

Figure 1 shows graphs drawn by the four students who attempted the "seven gates" problem. The first student reported feeling confused at the beginning of his attempt to solve the problem, saying that he "didn't even know where to start." Later he developed a suitable plan to find a solution by working backwards. Even though an error resulted in an incorrect answer at first, he was able to correct the mistake and solve the problem in about five minutes, generating the positive feelings shown at the end of his graph. The second student worked on the problem for about eight minutes, using mainly trial-and-error methods, before quitting and reporting feelings of frustration. The third student started out using algebraic methods to solve the problem. The computations got complicated very quickly, but the student felt that she was making progress toward a solution, which resulted in some positive feelings. Unfortunately, the resulting solution was a negative number of apples, and her emotional response changed from moving in a positive (upward) direction to negative. She stopped working after about 15 minutes. In her words, "I wasn't going to get it; I was frustrated." The fourth student spent about 11 minutes working on the problem. This student used algebraic methods and reported that feelings of frustration were quite consistent, although there were two more positive interludes when some progress was being made on the problem.

Asking students to graph their emotional reactions to a problem

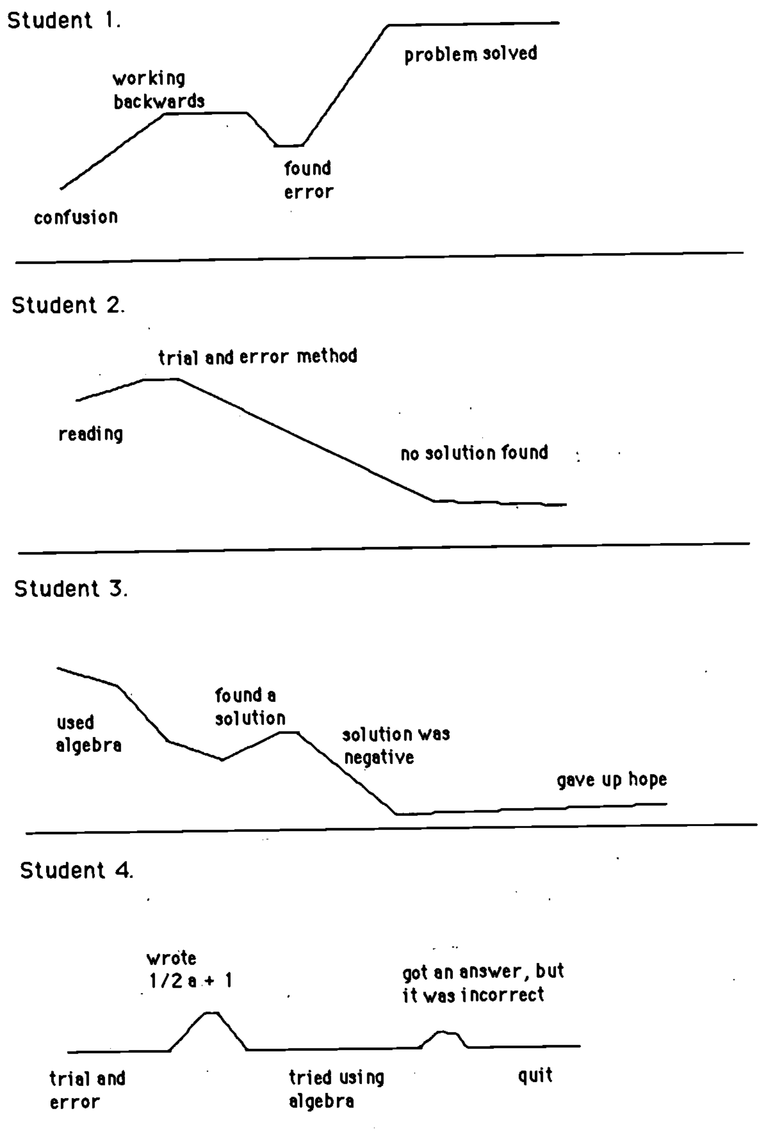

Figure 1. Student graphs of their affective responses to the "seven gates" problem.

appeared to be a useful technique. It gave students a way to describe variations in their affective responses at different stages of the problem-solving process, allowing them to show changes from positive to negative, as well as the differing levels of intensity of their emotional responses.

Conclusions

Students exhibit substantial swings in their emotional responses to problem solving. These swings occur quite regularly even among students who report that they have a negative attitude toward mathematical problems. The emotions that occur during problem solving appear to be relatively independent of traditional attitude constructs.

The appearance of these emotional swings corresponds in general to the interruptions and discrepancies that occur as a natural part of solving non-routine mathematical problems. Thus the data provide support for Mandler's (1984) theory of emotion and its application to research on mathematical problem solving.

In an earlier study (McLeod, Metzger, & Craviotto, 1989), experts and novices exhibited similar kinds of emotional reactions to problem-solving tasks. Experts, however, were better able to control their emotional reactions than novices. Students need help so that they remember the satisfactions, not just the frustrations, of problem solving. They also need to remember that feelings of frustration are a natural part of solving non-routine problems. A repertoire of heuristics can help students control their emotional responses. Further research along these lines should provide more information on how to help students use their problem-solving resources more effectively.

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ACCOMMODATING CURRICULUM CHANGE IN MATHEMATICS:

TEACHERS' DILEMMAS

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ABSTRACT

It is generally acknowledged that the social context within which teachers work imposes limitations upon what is possible both in terms of classroom practice and curriculum innovation. This paper examines factors which mathematics teachers in two schoools perceived as significant in influencing decisions they made relating to innovative practice. It describes dilemmas they had to resolve and points to the consequences for teachers in terms of feelings of competence and confidence associated with these dilemmas.

INTRODUCTION

Surveys of classroom practice suggest that in spite of recommendations for the inclusion of more practical work, investigations and applications in the mathematics curriculum at all levels of schooling, the pattern of teacher exposition and pupil practice has continued to dominate. The social context within which teachers work has been identified (see for example Popkewitz, 1988) as exerting a powerful influence over the process of schooling in general, and hence over the way in which curriculum reform is implemented. However, relatively little exists by way of research, particularly with regard to mathematics teaching, which details the major constraints which teachers perceive to be limiting their practice and the manner in which these constraints might restrict innovation. Research carried out by Desforges and Cockburn (1987) suggests that the mismatch they found between teachers' aspirations, which echoed those of 'experts' in mathematics education, and their everyday practice, resulted from 'the constraining factors of the classroom' (p.155) of which current approaches to enhancing mathematics teaching do not take account. Other research, for example that of Cobb, Yackel and Wood (1988) and the Low Attainers in Mathematics Project, LAMP, (DES, 1987), offers examples of teachers who were able to change their classroom practice despite the constraints within which they were operating. Neither of these studies set out to examine in detail the consequences for teachers of implementing change within the context of these constraints. Such an examination formed a part of my research and some preliminary findings are reported in this paper.

THE RESEARCH

The aim of the research has been to examine the ways in which mathematics teachers in two secondary schools have responded to the changes in classroom practice demanded of them in the course of curriculum change. The fieldwork for this study was carried out in the period September 1985 to July 1988, a time of substantial changes in school mathematics curricula. Recommendations regarding the teaching of mathematics contained in the Cockcroft Report (DES, 1982) had filtered through to some schools, a new public examination at age sixteen plus was being introduced and proposals for a National Curriculum in mathematics were emerging.

The two departments in my study were participating in a local curriculum development initiative (Nolder & Tytherleigh, 1990) which sought to support schools in devising curriculum innovations in mathematics for the intake year which were appropriate to their own individual circumstances. In particular schools involved in the project were aiming to adopt an investigative approach to the teaching and learning of mathematics.

At this time I worked as a mathematics curriculum support teacher in schools involved with the project, including the two departments which participated in my study. As such, I have had the dual role of support teacher/researcher.

RESEARCH STRATEGY

This study falls into the category of 'interpretivist research' (Eisenhart, 1988, p.103). It seeks to understand teacher behaviour by observing teachers in their natural settings and by eliciting from them the meanings they attach to actions and events. It begins from the standpoint that this is best achieved by using qualitative research methods. Participant observation has been my research strategy and a variety of data has been collected and analysed including field notes, documentation and audiotapes and transcripts of interviews.

From this analysis a network which represents a set of interrelated concepts associated with professional change and the dynamic relationships between these concepts has been developed. This will be described in a later paper. Within this paper I focus on one aspect of the network, that which relates to factors teachers perceived as constraining their practice.

THE TEACHERS

Lack of space limits the detail in which teachers in the study may be described. They varied in age from mid-twenties to late forties, were experienced teachers, and were regarded as competent practitioners whose students obtained good results in public examinations. The style of teaching in the two schools prior to involvement in the curriculum innovation may be described as 'formal' (Herscovics & Bergeron, 1984). Teachers viewed teaching largely in terms of 'getting things across' and their teaching focused on the effective transmission of knowledge via clear explanations. Progress was measured by 'coverage' of the mathematics syllabus. Teachers' major sources of feedback on their performance were results from public examinations, school tests and examinations, and students' responses to their lessons. The issue of motivation for change is a complex one and beyond the scope of this paper, but all the teachers in this study were motivated to some extent to change their practice by the imminent changes to public examinations, in particular the inclusion of teacher-assessed practical and investigative work.

CONSTRAINTS

Teachers in the research were subject to all the 'normal' constraints of a mathematics classroom and the everyday exigencies of teaching (see Jaworski, 1989). The focus here is upon constraints which were of particular significance as teachers set about implementing what they described as 'radical' curriculum change.

The main constraints teachers referred to were time, parental expectations and public examinations, and these interacted to exert substantial pressure on teachers. The latter two constraints, which were also identified within the LAMP study, reflect teachers' concerns with respect to accountability. A further constraint upon innovation was the influence of teachers' residual ideologies (Kirk, 1988) of traditional mathematics teaching with regard to the teaching/learning process and to criteria for successful teaching. All these constraints were associated with a set of dilemmas, discussed below, which teachers had to try resolve in order to accommodate the curriculum innovations.

ACCOUNTABILITY

At a time of declining numbers in the secondary school population, schools were competing for students and were increasingly aware that they were being judged by parents and other 'outsiders' on the basis of the curricula they offered and the public examination results their students obtained. Teachers exhibited anticipatory anxiety in relation to innovative practice on two counts. Firstly, it was felt that parents' views of what constitutes 'proper maths' might be at odds with new approaches.

> Many parents still expected their sons to have a 'sound' mathematics education and I'm sure the idea of playing with bits of coloured paper and sellotape instead of doing 'proper' maths like they had to, would have horrified some of our parents.
> (Rik, essay, 22/10/86)

Secondly, there was a possibility that examination results might deteriorate as a consequence of experimentation with 'untried' methods.

> I think the lack of confidence is knowing the effect it will have elsewhere. So, for example, if I change my style and do it this way, supposing my results are not so good, what will be thought of me?
> (Nan, interview, 13/7/88)

Teachers had to decide whether to keep to tried and tested teaching methods of which parents approved or whether to experiment with unproven methods which were vulnerable to parental complaints. In either case there was the risk that examination results could deteriorate, either because old techniques were inadequate in the new assessment context or because teachers were less skilled with newer techniques.

TIME

Time was a constraint on teachers' practice in the sense that it was limited, in terms of the length of the school day, the amount of contact time with students, and the amounts of their own time teachers could devote to preparation and assessment. Teachers perceived the new teaching approaches as demanding more time in the classroom as well as for preparation and assessment. This created time 'balancing' dilemmas for teachers. Teachers had to to make decisions as to how to share their preparation and assessment time among classes and how to allocate lesson time between 'transmission' and 'discovery' in order to 'get through the syllabus'.

> Lee said that it had taken a month to complete the unit and 'inside I was screaming "I'm never going to get this done!"' She felt there was a dichotomy between wanting to work in the new way and 'Are we going to get through the syllabus?'
> (Field notes, 23/10/86)

> Some more comments made by Von about the time and effort being expended on the Second Years at the expense of A Level.
> (Field notes, 9/10/86)

> ... we set out by reducing our Second Year syllabus a lot at the end of the year. We wrote out a beautiful new syllabus and when you actually come to read through it, there are huge chunks which we haven't done. We spend more time on investigation.
> (Nel, curriculum review meeting, 2/6/87)

RESIDUAL IDEOLOGIES

An underlying assumption of the curriculum development project with which teachers were involved was that the processes of changing established practice and modifying beliefs associated with that practice go hand in hand). Consequently teachers experienced some tension due the initial mismatch between their residual ideologies and the ideas about learning underpinning the curriculum innovations.

> ... this is a completely different way (of teaching) and half the time you're not opening your mouth and doing any teaching, you're just asking questions. It's hard for us to accept that they are going to learn maths from thin air almost and that we're not going to, you know, that because we haven't put the

> pearls in, they won't learn anything, because we didn't, as I say, put the pearls in.
> (Lee, interview, 12/11/87)

Related to this issue was teachers' traditional practice of assessing progress by the mathematical content they had 'covered' and the results of class tests, techniques which were frustrated by the process-focused approach of the new curricula and the potential within it for students to have differing mathematical experiences.

> ... everyone enjoyed in a sense what they were doing though they found it very difficult to measure achievement in terms of the pupils, how much they were really taking in and how much it was staying there because so much of it was finding out rather than being given to. So there was less formal assessment being possible so you felt that you were assessing very much more by feel than by actually looking at marks ...
> (Koo, interview, 11/7/88)

The problems teachers experienced in relation to residual ideologies exacerbated the 'transmission' versus 'discovery' dilemma referred to earlier.

DISCUSSION

The innovations carried out in the two schools involved in this study were regarded as 'successful' by the local Adviser for Mathematics in that they resulted in what he regarded as substantial curriculum change. Such an evaluation, however, fails to take into account the consequences for teachers of their involvement in curriculum change.

In this paper some dilemmas teachers experienced when making decisions relating to innovative practice have been described. A good deal of uncertainty was associated with these dilemmas which in turn affected teachers' perceptions of their own competence and confidence. The data are peppered with such words as 'worry', 'anxious', 'risk', 'doubt', 'apprehension', 'pressure', 'nervous', 'depressed', 'discomfort', reflecting teachers' feelings at this time. Teachers conceptualised change as 'a struggle', as 'more work', and as 'a compromise'. Preliminary analysis suggests that the motivation for change, which for many teachers was assessment-led, played an important part in sustaining change in the face of difficulties teachers experienced, as did the support teachers received from colleagues within the project and members of the advisory service. The issues of confidence, competence, motivation to change and support for change are considered in detail within my research and will form the basis of future papers.

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Teachers' characteristics and attitudes as mediating variables in computer-based mathematics learning

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A 30-day INSET course on using the computer in the Mathematics classroom took place during 1986/87 (Course 1) and 1987/88 (Course 2). Our research aims were twofold: i. to map the views and attitudes which the participants held initially about mathematics, mathematics teaching and computers; and ii. to describe and analyse the reciprocal interactions between participants' views and attitudes, their activities on the course and what they said about their practice. The data was analysed on a three distinct levels: in this paper, some overall conclusions of the cross-sectional analysis are presented.

Outline theoretical framework

In the last ten or so years, considerable attention has been paid to the processes of children's mathematical learning in computational environments, and the extent to which such learning may take place independently of explicit teaching. We have tried to broaden the idea of computational environments, or microworlds, beyond the merely technical, and to suggest that the teacher has a fundamental role to play in such settings (see Hoyles and Noss 1987). As a recognition of this, we have turned our attention to teachers, and it is the results of this research which we outline below.

The context of the research was the development, implementation and evaluation of a programme of in-service teacher education concerned with the use of generic computer applications (Logo, spreadsheets, databases) within the secondary school mathematics curriculum[1]. Course 1 took place during 1986/87 and course 2 during 1987/88 . The number of teachers was 13 in the first year, and 7 in the second: the majority were in positions of some responsibility within their mathematics departments. The course was substantial in terms of contact time: 30 days spread throughout the year in fortnightly sessions and three 3-day-blocks. The rationale for course implementation was based upon the need to:

• develop a reciprocal relationship between teachers' *personal and professional* skills;

[1] These we refer to as the 'Microworlds Courses', part of the Microworlds Project (1986-89) funded by the Economic and Social Research Council.

• encourage participants to view their own learning critically, and to reflect on it as a paradigm for thinking about pupils' learning;

• emphasise the importance of focussing the computer, the software and pedagogical issues as interrelated elements.

Our research aim was to map out some of the ways in which the teacher-participants on the microworlds courses thought and felt about employing the computer in their mathematics teaching, how their interactions with the computer influenced (and were influenced by) their pedagogical approach, and how they integrated the computer into their classroom practice.

Our conceptualisation of the classroom is as a setting where teachers and students mutually produce mathematical meanings from their practices: and thus we reject the idea that the teachers' role is merely to transmit mathematical knowledge. In considering the introduction of the computer, we reject a view which sees it as a technical fix, a technological solution to a well-defined problem. This kind of technological determinism ignores, among other things, that the computer has the potential to overturn many of the assumptions about what children can and cannot do, the 'hierarchies' of understanding that have been painstakingly drawn up, and the 'readiness' of pupils to understand this or that mathematical concept.[1] Second, and more fundamentally, if we regard the computer as merely a high-technology means of delivering pre-specified curricular objectives, we guarantee our inability to investigate the extent to which the computer's presence actually perturbs teachers' thinking, curricular attitudes and classroom practice. In effect, we rule out the possibility that the computer can bring anything fundamentally new to the pedagogical situation: yet this is precisely the object of our enquiry.

Thus our starting point is to reject the view that the computer is 'an innovation' which can be grafted onto practice. As we pointed out above, centring attention on the innovation itself deflects consideration away from the complex issues involved in integrating new ideas into the thinking and practice of those responsible for its 'implementation'. As far as computers are concerned, we want to reassert the importance of viewing the computer, the specific software, the accompanying pedagogy (which is not uniquely determined by the software, or by the intentions of its designers), and the classroom setting as forming an organic whole — perhaps an innovation in a broad sense, but not in the reified sense of something which can be 'applied' piecemeal to a teaching situation. This has an important corollary for our

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[1] We cannot discuss this literature in detail here: see for example Hoyles and Sutherland (1989), Hoyles and Noss (in press).

research methodology: a method of investigating teachers' roles which starts from pre-specified cognitive objectives for their students, treats as unproblematic the ways in which the computer's presence may influence, not just the means by which learning takes place, but the very nature of what is taught.

Teachers have beliefs and attitudes which underpin their ways of reading and acting upon innovations. If such beliefs and attitudes are viewed as an 'obstacle' to the transmission of 'good practice', one strategy might be to make the implementation of the innovation so routine, so 'teacher proof' that such beliefs might be sidestepped or at least suppressed. We recently attended a seminar which reported on an innovation which is based on teachers reading scripts 'like an actor': beliefs about mathematics, about mathematics teaching, about computers as a cultural innovation, will always crucially determine what teachers say and do, whatever script is presented.

Some methodological considerations

Essentially, our priorities for the research were as follows:

1. to map the views and attitudes which the teachers held initially about mathematics, mathematics teaching, and computers;

2. to describe and analyse the reciprocal interactions between teachers' views and attitudes, their activities on the course and what they said about their practice. (We did visit all the teachers in their classrooms, and followed three in some detail. Nevertheless, the constraints of the study did not allow us to investigate classroom practice as a central element of the research).

Thus our concerns went far beyond looking for 'treatment' effects of the course (this would be doomed in any case since the notion of a 'control group' simply does not make sense within our framework). Of course, we were interested to see which aspects of which beliefs and attitudes were changing, and to see if we could at least hypothesise as to the reasons behind them. But we were at least as interested in what the teachers actually did on the course; and we were particularly concerned to see how teachers existing views and attitudes influenced these activities.

Data was collected from the following sources:

- Interviews at the beginning, mid-term and end of the courses
- Examination of project work by teachers
- Examination of participants' case-studies of pupils
- Data collected from observation notes of participants' activities on the course

• Classroom observations and follow-up data

• Post-course questionnaires distributed to the teachers

In what follows, we are able to present only the most general of our conclusions. However, we believe that it is helpful to outline the ways in which the data was analysed and presented, even though the interested reader will have to go elsewhere to find it (see Hoyles, Noss and Sutherland 1990).The methodology we developed involved three levels: *caricatures, case studies and cross sectional analysis.*

The first level was to develop *caricatures* of the course participants[1]. The caricatures do not represent real people: they are a synthesis of the views, attitudes and practices of a set of individual course participants which have been developed to do what caricatures do best — to focus attention on the significant points, perhaps to exaggerate them (at least by a relative de-emphasis on other facets), and to allow a loss of fine-grained detail in order to highlight variants and invariants across subsets of the data. We have chosen to label them as caricatures (rather than, say, as 'ideal types[2]'), since they were developed in the course of analysing the data, rather than as a priori theoretical constructs.

All of the data reported within a caricature is, of course, completely true to the original data sources. Nevertheless, by themselves there is a level of richness which is missing, not least because we did not have sufficient time to follow all the teachers into their classrooms and observe their practice. There were, however, three teachers whom we were able to observe in detail in the classroom, and for these we developed detailed *case studies* . This second level allowed us to address a range of issues based on direct observation of practice as well as 'hearsay', and to consider in depth the ways in which the three individuals structured their course experiences. The third level of analysis involved the development of a *cross-sectional analysis,* which attempted to transcend individual cases.

Cuts in data such as this are always problematic. But by considering the data from three distinct perspectives, we can at least be explicit about the sensitivity of the cuts, and highlight points at which we are unable to fit individuals into our neat classification. In what follows, we only report from the perspective of the second level cross-sectional analysis, but we should stress

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[1] Again the reader is referred to Hoyles, Noss and Sutherland 1990 for details of how these caricatures were developed.

[2] As conceived by Weber, an ideal type is constructed by abstracting from elements which, although present in reality, are not present in this ideal form. We have resisted using the term 'ideal type' in part because of the connotations of the word 'ideal' when applied to a group of people, and worse, when abstracted into one (non-existent) individual.

that this was developed from both the caricatures and case studies, and thus represents some attempt at synthesising the findings from a diversity of perspectives.

Some conclusions of the cross-sectional analysis

Teacher projection. We observed an almost universal tendency for participants to project their own preferences and attitudes onto their pupils — their own feelings were expressed as insights about their pupils. As examples, those teachers who displayed initial anxiety in using the computer suggested that their pupils would be similarly anxious, and proposed pedagogical implications which flowed from this; teachers who favoured a particular style of interaction (for example, an approach which was more strongly directed than that adopted on the course) tended to argue that their pupils would benefit from a similar approach. Thus there was a dialectical relationship between teachers' own attitudes and those they attributed to their pupils. We do not suggest that this process was uniquely attributable to the computer setting: only that the course highlighted this relationship, by encouraging reflection on participants' and pupils' learning.

Motivations for approaching computer-based mathematical learning. We found that participants approached the courses with a range of motivations. A key classification in terms of understanding their course activities and shifts in attitude, was that between *proactive* and *reactive* views of the computer as an innovation. In the former category, we identified teachers who held a more or less articulated position on mathematics and its teaching and who were seeking mechanisms by which to instantiate their 'programme'. At the other extreme, we identified a number of teachers whose practices were being directed towards change — these teachers' were primarily motivated by the need to accommodate and react to curricular or organisational pressures (such as new National Assessment procedures).

This classification enables us to be rather more specific about the mechanisms by which the course may have influenced the participants. For the proactive participants, a key role of the course was that it *legitimated* the kinds of approaches and theories which they held — at least on a theoretical level — and allowed them the opportunity to operationalise their ideas. For the reactive teachers, the course offered a mechanism by which they could *implement* the approach which they were being encouraged to adopt.

This proactive/reactive distinction also related to the question of mathematical content. In general, the proactive teachers tended to view the computer as a medium for extending the range of mathematical activities which they could offer their pupils, while the reactive teachers were concerned to develop criteria which measured the computational activities against the yardstick of, for example, texts or curricula and thus were interested in identifying facets of the work which could be incorporated within their existing curricular priorities. There was similarly some evidence that proactive and reactive participants tended to focus towards personal and professional prioritisation respectively during the course activities. Although these categories are useful for a first crude classification, we identified a more subtle underlying issue: the extent to which a participant was able to integrate the computer into his or her mathematical pedagogy (theoretically and/or practically) appeared more related to the *direction in which a participant's thinking was already developing and with his or her commitment to change*, rather than the style of teaching approach, view of mathematical activity, or rationale for attending the course.

The role of mathematics. Although a central focus of the course was to encourage the integration of computational and mathematical activities, there were interesting interactions with various characteristics, primarily with the participants' view of mathematics itself. As an example, for those whose position was essentially 'ethnomathematical' (i.e. who viewed mathematics as 'everywhere') the computer work posed an opportunity to extend the ambit of ethnomathematical practice to encompass the computer activities themselves. For those who took a much more curriculum-focussed view of mathematics there was a tendency for the computer to be seen as simply a vehicle with which to introduce curricular content.

Changes in pedagogy and intervention strategies. We classify in two main ways the extent to which participants reevaluated their pedagogical approaches. We do not mean to imply that all participants did so, or that those who did, did so to an equal extent. But one cut across the data which stands out is that between those who came to reevaluate aspects of their intervention strategies in essentially *quantitative* terms, and those who came to see a need for some *qualitatitive* reevaluation. In the former category, it appears that this exclusively involved those who saw the need to intervene *less*. In general, these participants tended to be in the reactive category (although the converse was by no means true). The ethnomathematical teachers in contrast, initially tended to adopt an abstentionist position with respect to intervention for the computational activities in ways similar to those they adopted in traditional mathematical 'investigations'. However, they showed evidence of *qualitative* changes in their pedagogy — at least in the computer setting — in that they tended to be prepared to see in computational work the need for novel intervention strategies. This applied particularly to the proactive teachers. We note that qualitative shifts of this kind were *not* related to mathematical qualifications or experience, or to the relative prioritisation of the personal and professional.

Resonance. We tried to gain a picture of the extent to which the course resonated with the thinking of individual teachers, the ways in which the activities impinged on their views and attitudes. The course failed to resonate initially with those participants whose own learning (and teaching) styles did not mesh with those adopted on the course. Some of the participants undoubtedly assumed that the course would teach them 'how to do it'. However, the views of all but one teacher in this category changed as the course unfolded, in relation to the extent to which the individual participants realised the mathematical power of the computational approach for his/herself rather than pedagogically — i.e. the personal dimension was again critical.

Transition to the classroom

Still at the level of cross-sectional generalisations, we end with a brief overview of some of the issues determining the extent to which course participants integrated the ideas of the course into their classroom practice.

Implementation. The first point that emerged was that planned and careful organisation was a prerequisite for the integration of the computer into the classroom. That is not to say that the converse did not operate (i.e. that those who became committed to such integration found ways to organise their classrooms), but it was very evident that — at least in the computer-impoverished setting of most of the participants' classrooms — routine access to the machines (on the part of both teachers and pupils) was and remains a necessary if not sufficient condition for classroom implementation.

It is evident from our follow-up interviews and from the post-course questionnaires, that almost all the teachers cited technical difficulties and access problems as major obstacles in using the computer in the classroom: classroom implementation appears to be unrelated to any questions of commitment or pedagogical strategies. It is simply the case that mathematics departments have low priority in access to computers, and in some cases, had even been forced by the school organisation to hand over what limited machines they possessed to other curriculum areas. Of the nine teachers who reported continued (and extended) use of the computer in their classrooms, six would be classified as proactive, and all of them were committed to change at some level. Moreover, in every case, these teachers were using the major applications which had formed the backbone of the course (notably Logo and spreadsheets), whereas some of the other participants reported subsequently that their main activities were restricted to topic-specific software.

A further critical barrier to continuing computer use was lack of support from other members of staff, heads of department, and heads of school. There is a need for a critical mass of teachers committed to using the computer for mathematical purposes within any one school.

Dissemination. The success of any course dissemination relates to the personal status of the individual within the department and his/her relationship with other staff and advisors. Additionally although all the teachers received formal support from their LEAs, we found many cases where such support stopped short of developing the teachers' contribution beyond that of his or her own classroom. However, our evaluation showed that dissemination was largely restricted to those who had 'reached' the second phase as referred to above. In itself, this is an unsurprising finding, in that the motivation for dissemination among teachers of mathematics would be likely to rest on the computer's role in 'aiding' the process of teaching and learning mathematics. Even this apparently banal finding is interesting: a significant number of participants joined the course believing that the computer formed a potential area of study *unrelated* to mathematics.

Two years after the completion of the course the picture of dissemination is reasonably positive. At least four in-service courses, modelled on the Microworlds Course, are now in progress.

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TEACHERS' PERCEIVED ROLES OF THE COMPUTER IN MATHEMATICS EDUCATION

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The reform movements concerned with innovative pedagogical approaches and the possibilities offered by information technologies rise new problems to inservice programs. These must give careful consideration to their pedagogical and cultural frame and to its inner dynamics. This study focus in the conceptions and attitudes of teachers involved in such a program regarding the educational role of the computer.

Automatic information handling media acquired a prominent role in many fields of our society. They are essential in research, design, control, management, and communication. One finds examples of changes fostered by these technologies in all domains of economical, social, and cultural life. The development of the ability to use critically and efficiently these media is becoming an important educational objective.

The computer is a particularly significant tool in mathematics, allowing to work simultaneously with different representations of data and yielding the automatization of the execution of repetitive tasks. The computer brings with it new concepts and problems, enabling the extension of the range of questions and strategies that the students can deal with.

World wide economic competition pressures school systems for educational reform. Attention is being paid to the development of student "basic competencies" and professionally oriented school programs. But there is also a generalized concern with the present inefficacy of the educational systems to promote in most students higher literacy competencies (see Romberg, 1988). Mathematics is one of the subjects that most contributes to the failure, frustration, and social unadjustment of many students.

Therefore, it is not surprising that, in mathematics education, the major strand of the current reforms concerns not the updating of the content (as was the case in the sixties), but the establishment of new goals and methodological approaches. Problem solving, project work, embedding mathematics in real word contexts, stressing the student' role in the learning process, interest the possibilities offered by the new information technologies have been orientations behind most recent research and development efforts (APM, 1988; ICMI, 1987; NCTM, 1989).

Research on inservice work with teachers aimed at the introduction of these innovative ideas in schools is thus required. But, to be successful, the inservice framework must be consistent with the sort of pedagogy that is advocated for schools. The development of new conceptions, attitudes, and competencies should not be viewed as a mere process of "training" but as a multifaced process of "teacher development".

The inservice program in which this study is based stands on the assumption that thinking on how to use the computer in their classrooms and in other school settings, can be a good starting point for teachers to reflect in a global manner on their own practice. Although the computer may be introduced with little or no change in teachers' conceptions and teaching methods, their interest in making a sensible use of this instrument and their disposition to learn new things, assume new classroom roles, and establish new teacher/student relationships creates a stimulating environment for general educational reflection.

This program is carried as part of the National Project MINERVA, aimed at the introduction of computers in Portuguese schools. Our group is connected to 27 schools, of which 23 at middle and secondary level. In these schools it is constituted an interdisciplinary coordinating team, with 3 to 5 teachers. Depending on the school, mathematics teachers may or may not integrate it. This team is encouraged to organize activities to disseminate the use of new information technologies, to promote the development of disciplinary and interdisciplinary activities and projects, and to support other teachers that intend to use the computer in their classrooms. These activities are proposed to foster a new structure and atmosphere influencing the teachers' professional role (Romberg, 1988).

Different inservice opportunities are offered in this program, targeted to teachers in a variety of situations. For example, there are shorter courses focused in a single powerful piece of software, like LOGO or spreadsheets, intended for "beginners", and longer ones centered in one school topic, like mathematics or language, intended for teachers having already some experience. There are also more extensive courses for members of the school coordinating teams and the members of the Project group. Most of these courses have flexible organization schemes, alternating formal sessions, sometimes in concentrated periods, with work in the schools. For the teachers, all the activities carried within the Project are considered as part of the inservice program, including the local support directly given to them, the participation in school projects, and the meetings with teachers from other schools.

The inservice program was designed with two essential elements: (a) its cultural and pedagogical frame, based in the innovative potential of the new information technologies and in the concept of project work (Monteiro & Ponte, 1987), and (b) its dynamics, considered at three levels: personal involvement, group processes and the role of the program team. This study focus in the conceptions and attitudes of mathematics teachers, concerning how they view the computer and its role in mathematics education.

Theoretical Background

Reforms aimed at the promotion of new pedagogical approaches or the introduction of new technologies in schools are examples of attempts to educational change. One must be aware that the most critical aspect for the success of any intended process of change in large organizations concerns the role of the people involved (Huberman, 1973; Knupfer, 1989-90).

People can change in various respects. For example, Lewin (1948) distinguished as possible aspects of personal change: (a) change in cognitive structure (like learning new knowledge); (b) change in motivation (such as learning to like or dislike something); (c) change in ideology or in fundamental beliefs; and (d) change in behavior (like control of body muscles).

The cultural and pedagogical frame is an essential aspect of the inservice program. Teachers have their well established systems of ideas and beliefs about themselves, about the subject they teach, about their profession and about their practice (Jones, 1988). An intended process of change necessarily carries with it an underlying rationale. The specification to the teachers of this cultural and pedagogical rationale is essential to introduce new information and conceptual elements that challenge the closed circle of their conceptions and values, their "certainties" (on what works) and their "impossibilities" (in doing anything different). The assumption is that it is much more likely to begin a successful process of questioning these conceptions bringing in new perspectives from the outside, than searching contradictions and weaknesses inside the teachers' conceptual frameworks.

Furthermore, this cultural and pedagogical frame ought to be clearly stated to the teachers if they are to play the role of subjects in the process. Teachers should have the option of adhering or not, the possibility of accepting or not the new views and proposals. The ultimate decision to change is theirs, and they must be provided with all the relevant information to make it conscientiously.

In fact, the personal involvement of the teachers is a fundamental condition of personal change. This involvement should yield them to levels of progressively more autonomy regarding the program team (Can rio, 1989).

To foster the involvement of the teachers, the program must take into account their interests, objectives and experience. For them, a very important part of the process of assuming their own process of learning and professional development depends also on becoming confident in defining and solving their own problems (Easen, 1985).

The analysis of needs of the participants has been pointed as the key element in the design of an innovation. However, the identification of needs is a complex task. Teacher training institutions may have more or less defined views about teachers' training needs, but the teachers themselves may consider them irrelevant or unacceptable. Regarding teachers as true professionals is quite contradictory with giving somebody else the role of stating what they need. But the articulation by the teachers themselves of their needs may also be very difficult. They may not be used to this process of self analysis and may not be aware of what possibilities are available and what are their implications. This analysis may only be possible as a result of effective professional development, and not as the beginning point of the process (Easen, 1987). In this program, the analysis of needs is considered as an essential task, but to be carried on an interactive way by participants and trainers (Can rio, 1989).

The dynamics of group processes is also a fundamental element of the inservice program. It is quite difficult to surmount all the difficulties surrounding innovations in isolation. To resist to constant criticism, to draw in the experiences of the others, to have reflection partners, teachers find a strong support from their pears involved in the same process. Furthermore, group dynamics, appropriately designed may be an important factor in the change process. As Lewin (1951) as shown, so far as the values of the group remain the same, the individual will resist change, and that so much as he or she will be required to deviate from the norms of the group. If the norm itself will change, the resistance caused by the relationship between the individual and the group is eliminated.

The role of the program team is essential in this process. It has the responsibility of creating the working framework, constructing the necessary materials, make the general proposals, introduce the cultural and pedagogical framework. The team is seen with an affirmative role of creating the appropriate environment to foster the program objectives.

Like all adults, teachers try to protect their self-image as far as possible (Rogers, 1977). Many teachers see programs offered by training institutions as oriented by systems of pedagogical beliefs non-congruent with their own. It is not surprising that they adopt in such cases a defensive attitude. They do not examine the suggestions and proposals that are presented with an open mind, but as instances of a foreign and threatening point of view, that should be distrusted. It is therefore an important task to establish a climate of confidence and a good relationship with the teachers. This may be achieved by working together in an open way, emphasizing the idea of sharing. The conceptions of the team members are not to be hidden neither to be imposed upon the teachers.

The interest for the new ideas and approaches develops naturally in a stimulating environment with its own challenges. In this respect, the basis for learning is regarded as being the same for children and adults: strong motivation, great amount of activity, reflective looking back and conceptualization.

Results

For this study, data was collected from questionnaires specifically distributed to the teachers involved in different kinds of inservice work. The questionnaires were given at the beginning and at the end of the first set of formal program sessions. The responses were analyzed in the light of the reports and discussions with the team in charge of each course. (Later in the program, teachers will be interviewed--their projects and school activities will then be detailed discussed.)

This study was mostly concerned with the professional profile of the participants, their reasons for registering in the course, their intentions regarding the use of computers in their schools, and their perspectives of the impact of the computer in mathematics education and in education in general.

Participants. The study included 30 mathematics teachers: 11 were on a course in LOGO.GEOMETRY (a program for problem solving in Euclidean geometry), 4 on a course on LOGO (which also included teachers from other subjects and primary school teachers), and 15 were on a course on using computers in mathematics education, in which previous experience was required. The teachers on the two first courses will be called the "beginner's group" and the teachers in the third course the "disciplinary group".

All the teachers in the beginner's group work in secondary schools, with an average teaching experience of 12.7 years. In this group, 11 teachers were female and 2 were male.

In the disciplinary group, 6 teachers come from secondary schools and 8 from middle schools. One was a middle school teacher now teaching at secondary level. She was teaching for 12 years. The years of experience were 12 for the secondary school teachers and 19.9 for the other middle school teachers. All of the teachers in this group were females.

Combining both groups, 80% of the teachers have more than 10 years of experience. This shows that it is not the younger teachers who mostly come to this program.

Reasons for coming to the course. One may get involved in inservice work of this kind because of a general interest on what is being proposed regarding the use of computers in education, or because one wants to have an active role in his/her school, where computers are already being used. Of course one may just want to learn more about the actual use of computers. Teachers could indicate one or more of these reasons or give any other response.

The intention of making actual use of the computers was high in both groups (Table 1), with some teachers indicating the two reasons. However, this intention can refer to its use in the classroom, in club activities, in interdisciplinary projects, in other school activities. The higher rate of responses for the general interest in the different uses of computers from the more experienced teachers may indicate that they do not feel already quite confident in that respect.

Table 1

Reasons for coming into the inservice program

	Beginners' Group	Disciplinary Group
Immediate intention of using computers	10	8
General interest for the use of computers in education	5	10

In the disciplinary group 7 teachers indicate that would like to know more software and 12 indicate an interest in analyzing other possible uses of the computer. In this group several teachers show a clear concern with the classroom, others refer to the club, others to both, but none speaks in terms of general school activities. It becomes quite obvious that the teachers are essentially concerned with the teaching of their subject.

Intended activity after the course. What sorts of activities these teachers intend to do in their schools? After the first formal part of the program, are they planning immediate use, or are they still reluctant or undecided?

In the disciplinary group, 14 teachers reported intention of immediate use. In the beginners' group, 9 indicated willingness for immediate use and 6 showed some reluctance. From these, some indicated that they would not have enough conditions (meaning lack of physical resources--time, space, equipment), others that they did not had given enough thought to it, and finally others felt that they would need more preparation.

The responses also made clear that many middle school teachers think in terms of club activities--8 refer to it. That is not the case with secondary school teachers, who mostly are concerned with classroom activities--only 1 refers to the club setting.

Perceived educational roles of the computer. It is important to know what teachers see as the major role of the computer in education. Is it an instrument for individualized support to students? An auxiliary means to create new learning dynamics in the classroom? A resource to the realization of interdisciplinary activities and projects? Will it be essentially used in computer related subjects?

The responses are summarized in Table 2. Again teachers could give more than one response. We may conclude that the dominant concern of the teachers in both groups is the creation of new dynamics in their classrooms.

Table 2

Perceived educational roles of the computer

	Beginners' Group	Disciplinary Group
Individualized support	1	5
Classroom dynamics	12	15
Resource for projects	6	9
Computer related topics	-	2

A significant number of teachers indicated interdisciplinary activities and projects as an important role, although not as their first choice. This appears to result from the stress of that concept in the inservice program, but should be noted that it is far from being at the center of the teachers' concerns.

Conclusion

Teachers coming to the program have generally a considerable teaching experience, most of them maintaining a stable appointment to their schools. Showing a general interest for the applications of computers in education, they are specially concerned with its role in the teaching of their discipline.

These teachers indicate a major attention to the uses of the computers in classrooms. Although our project emphasizes the possible role of alternative working spaces in the schools, the concept of club as a significant learning environment is only noted in middle school teachers.

Most teachers come to the inservice program motivated to learn how to use the computer in their school. They reveal an intention of immediate use following the fist set of formal sessions of the program. Some major ideas presented in the program (such as interdisciplinary projects and school involvement) appear not to be rejected, but are not present in their main concerns.

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MATHEMATICS PROCESS AS MATHEMATICS CONTENT: A COURSE FOR TEACHERS*

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A major obstacle to the transformation of the mathematics classroom into an environment which produces mathematical understanding is that most teachers have not learned to think mathematically. This paper describes an experimental mathematics course for inservice teachers in which the notion of "mathematics content" as the familiar sequence of curricular topics is reconceived as "mathematics process": at once the active construction of some mathematical concepts--e.g. fractions, exponents--and reflection on both cognitive and affective aspects of that activity. The work of the course is organized around experiences of mathematical exploration, selected readings, and, perhaps most importantly, journal keeping. Teachers' learnings are illustrated by excerpts from their journals.

Introduction

While responses to the current crisis in mathematics education in the United States have been varied, one increasingly influential trend proposes that the mathematics classroom be reconceived as a problem-solving environment. In such a classroom, organized around students working collaboratively, debating ideas and approaches among themselves, the development of generalized problem-solving skills would be more highly valued than the memorization of algorithms or their rote application to particular problems. Correlatively, the role of the teacher would now be to stimulate students to construct their own understandings of mathematical concepts, to guide them in that process, and so to help them know their powers as mathematical thinkers.

One considerable obstacle to widespread implementation of such a reconceived mathematics pedagogy is that most teachers simply do not have an understanding of mathematics sufficient to allow them to promote exploration and debate in their classrooms. Themselves the products of traditional mathematics education, these teachers doubt their own abilities to think mathematically, viewing mathematics as no more than a collection of facts, definitions, and rule-governed procedures. Now while it is clear that such teachers need more extensive mathematics training, the sorts of courses generally offered at colleges and universities, either in mathematics or in education departments, will not solve this problem. For, while lectures on calculus or mathematics methods may be valuable for other reasons, they do not focus in the right way on the needs of adults who have had limited experience in, and a restricted view of, mathematics. Rather, what is needed are mathematics courses whose primary--and inseparable--goals are to help teachers learn to reason mathematically, to lead

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them to question and broaden their understanding of what mathematics is, and, by so doing, to validate them as mathematical thinkers.

In this paper I describe the structure and content of a course guided by these goals. Teachers' learnings are illustrated by excerpts from their course journals.

Course Structure

During the spring semester of 1989, the SummerMath for Teachers Program at Mount Holyoke College offered an experimental course for mathematics teachers. A total of forty-nine teachers were enrolled in two sections, one mostly of teachers of grades K-4, the other mostly of teachers of grades 5-8, but with a few high school teachers as well. About half the teachers had previously been involved in the program. Each class met weekly for one 3-hour period.

The course had three major components: mathematical explorations, reading assignments, and journal-keeping. Each session began by offering teachers the opportunity to bring up any thoughts or questions about the previous class, the math homework, or the reading assignments. Discussion usually lasted half an hour to an hour, leaving the balance of class time for mathematics explorations. The format for such explorations involved working from an activity sheet in small groups and then sharing discoveries and questions with the whole class. Homework included further questions related to the mathematical explorations, a reading assignment of one or two articles, and writing in one's journal.

Because teachers were concerned about being better prepared to teach their own classes, the mathematics topics chosen for exploration were usually selected from those in the elementary and middle school curriculum: whole number operations, integers, fractions, decimals, exponents, functions, area and perimeter, and properties of geometric figures. Often, however, class discussion led to such other topics as limits and non-Euclidean geometry. But the choice of particular mathematics topics was, in the end, of secondary importance. They were the means through which the primary goals of the course were pursued.

The reading assignments, distributed thoughout the course on a weekly basis, addressed constructivist mathematics, mathematical misconceptions, affective aspects of mathematics learning, metacognitive processes, and instructional approaches to particular topics. The papers were chosen to help teachers interpret their own experiences in class and to enable them to translate those experiences into their own teaching.

Journals were used as a means of reflection as well as a vehicle for dialogue. Teachers wrote about what they learned and what they found interesting, the ideas and concepts with which they were currently struggling, events in their own classes, and their personal reactions to course activities. The instructor collected the journals once a month and responded to them in writing before returning them one week later.

Developing Mathematical Reasoning Powers

Giving Meaning to Symbols. When one works "abstractly," with symbols say, one is likely to forget--if one ever knew it--that these derive their meaning from conceptual structures ultimately rooted in--"abstracted from"--experience. For example, the words and symbols that designate division ('÷,' 'divided by,' 'quotient,' 'remainder') represent conceptual structures ultimately derived from e.g. the experience of sharing (distributing objects into equal-sized groups and deciding what to do with what is left over). In traditional mathematics education, however, students from first grade on are generally taught a variety of formal expressions and operations which are never connected to their informal mathematical knowledge. As a result, students are often unable to use such expressions or operations in contexts other than those of the mathematics text book or classroom.

The teachers in the course were themselves subject to this problem. Adept with the algorithms for the basic mathematical operations, they needed to attach meaning to already familiar formalisms. To this end, the first lessons were organized around explorations of the properties of the number system: teachers were asked to explore the commutativity and non-commutativity of addition, subtraction, multiplication; and division and they looked for patterns, considered special cases, and illustrated each discovery with manipulatives, diagrams, or word problems. And where these activities stimulated explorations of such topics as negative numbers and the meanings of 'inverse' and 'reciprocal,' major emphasis was given to the concrete representations of patterns.

Such activities led teachers to a growing awareness of the possibility of attaching meaning to familiar symbols and operations. As one teacher who was grappling with the shallowness of her own understanding wrote in her journal:

> If I keep playing, the inter-relationships between the operations will become more and more tangible for me. I can say--subtraction is the reverse of addition--but that's a

limited way of looking at that operation. Besides, I guess I don't even know what "is the reverse of" means--AND what the implications may be in various situations.

The lesson sequence continued with similar explorations of the associative and distributive properties, again emphasizing concrete representations. Having completed a homework assignment on the distributive property, teachers reported that it held in the following cases: a. (4+2)x12 and 4x12+2x12; b. (4+2)/12 and 4/12+2/12; c. 12x(4+2) and 12x4+12x2. But not in d. 12/(4+2) and 12/4+12/2. Yet, their feeling about this finding can best be described as mystified. Why should the pattern hold in cases a-c, but not in d? Teachers were looking at arrays of symbols without attaching meaning to them.

As the class mulled this over, one teacher, thinking about previous lessons, suggested that they make up word problems for each expression. For c, the class suggested, "There were 4 boys and 2 girls, and each child had 12 candy bars. How many candy bars were there altogether?" They were satisfied that both expressions fit the word problem. And for d they suggested: "There were 4 boys and 2 girls who had 12 candy bars to share among themselves. How many did each child get?" Now they saw that that fit the first expression. For the second expression they began by analogy, "There were 12 candy bars to share among 4 boys and another twelve to share among 2 girls..." Suddenly, there were several gasps and "oh's" in the room. "It's a different situation!" The concrete context gave meaning to the symbols, meaning that offered grounding, access, and a sense of ownership over the ideas. One teacher described her experience: "Seeing the division example as a word problem was boggling. Suddenly the 'why won't it work' appeared so clear."

Exploring Mathematical Themes and Questions. As teachers attached meaning to familiar symbols, they came to see mathematics as a web of logical connection. The rules governing the basic mathematical operations were not arbitrary, need not simply be accepted, but could be demonstrated through exploring this web. The teachers could make these discoveries themselves, communicate them, and so corroborate one another's findings.

Yet, the development of mathematical systems is not, in itself, completely determined by logic. Of course, the particular designations '7,' '+,' or '=' are conventional, and so is the choice of 10 as the base of our number system, rather than 8 or 12. But while some conventions seem totally arbitrary, others have powerful systemic ramifications. The instructor frequently looked for opportunities in class discussion to point out the role of choice, agreement, and theoretical coherence.

For example, the activity sheets for the exploration of exponents were designed so that as students worked from the definition of 'whole number exponent,' they derived the product, quotient, and power rules. Once these rules had been established, students were challenged to give meaning to such expressions as '2^0' and '3^{-2}.' From a journal:

> The exponent work was interesting. What's even more fascinating is the notion that mathematicians have these "agreed on" rules. I've tried to imagine any other discipline that requires a similar function. The theories of the social sciences are different. One either accepts a theory, develops it further, or rejects a theory and maybe develops another.
>
> So $2^5/2^5=2^0=1$ because we need to fit the "subtract exponents" rule. But what I sense is that 2^0 is a symbol--and that the agreed on rules are for language's sake. No, it's more that that. For language's sake, scientists agree on xyz as the name of a newly discovered element. There's no need to fit the agreement into an existing schema.

And as this teacher reflected further on the dialectic between logical determination and convention, she concluded:

> So 2^4 means 2 multiplied by iteslf 4 times, 2x2x2x2. That's agreed on. Therefore, 2^0 means 2 multiplied by itself 0 times. No No No. This time the zero represents prior manipulations of the exponents. $2^x/2^x=2^{x-x}$ So we're changing the meaning of exponent to make a rule work. But then we say 2^0 has no meaning anyway--so we'll give it a meaning to fit our rule.
>
> I want to come back to this someday. I think it's very convenient and very logical and very clever. I need to observe if there is a related process in any other discipline. I can't think of any right now, but I haven't been thinking along this vein before. Not ever.

Other mathematical themes and questions that were explored in the course included: the uses and limits of physical models in the development of mathematical ideas; the need to continually extend one's understanding of basic operations as one begins to operate with new kinds of numbers--for example, if one understands multiplication as repeated addition, how does one interpret '-3 x -2' or '1/3 x 1/2?'; and the variety of meanings behind simple formalisms--for example, '15-9=6' might be interpreted as "take-away," "comparison," "missing addend," "unknown part," etc.

Reflecting on One's Own Thinking. If the primary goal of the course was to enable teachers to become mathematical thinkers, in the instructor's view that would not be so much a matter of providing opportunities for them to work on particular mathematics topics, as it would be of providing them the occasion to simultaneously step back from mathematical content in order to reflect on their mathematical process. (The most important means through which this was to be accomplished was the journal each participant was required to keep.) As an example of this process, teachers began to see that "understanding" wasn't just "on" or "off," "yes" or "no", but that there are different levels or stages of understanding.

> There's still much more to realize about dividing fractions. I know that I need to work more to solidify what I have just "discovered" before moving on to the gray areas. I think this is the big news for me. I don't have to get it all right now. I'm not only learning math concepts, but I'm becoming more aware of myself as a learner. Was this planned?

Another teachernoted the relation between acquiring new bits of mathematical understanding and developing a framework that would give these bits new solidity and significance:

> Back to fraction reflections--I think my thoughts are so sketchy becase the fractional thinking hasn't settled into any cognitive slots yet. They're still in the making. I know it's not enought to say I have a better "sense" of fractions....A framework is developing....

But in the process of self-reflection, cognitive and affective issues were registered as inextricably related to one another. This teacher wrote about the feelings she associated with deepening understanding:

> The idea that new knowledge is often "compartmentalized so that it does not interfere with existing concepts" (Hiebert and Lefevre, 1986) has been explored in this course. The excitement and empowerment of making connections with previously learned material and higher-order concepts unleashes this binding tendency to remain with surface characteristics of a newly-learned or surfacely-learned concept.

In this journal excerpt, the excitement and empowerment of new understanding are emphasized, but when one's investment in that "binding tendency," that "compartmentalization," to which this teacher refers, is threatened by change, the experience is often one of anxiety, frustration, or anger. Many teachers realized for the first time that such "negative" emotions are part of the process and that avoiding them actually short circuits the learning:

> The complexities of math are still baffling to me and I certainly didn't expect to have all the tangles unraveled in one short course. But I have learned that little bites of understanding are possible and, for me, the best way to approach mathematics. I'm not nearly as frustrated by my lack of conceptual understanding of math's big ideas.
>
> That's not to say I'm not frustrated when a new math topic is presented! Goldin's (1988) article about affective learning sets really helped me to see my own learning process. I no longer go directly from frustration to anxiety to fear/depression. I can stop and pick up some tools I've learned to use in this course.

By articulating their own internal experience as mathematics thinkers, teachers learned that puzzlement, fuzziness, and frustration--indicators that had previously signalled the end or failure of learning--are part of the process. They also came to know the satisfaction, excitement, and pleasure associated with understanding.

The Nature of Mathematics

The process of becoming a mathematical thinker necessarily involves a changed conception of the nature of mathematics. The teachers' initial sense of what it meant to do mathematics had been drawn from their experiences in the traditional mathematics classes they had taken from grade school on. In general, this involved learning, in Pavlovian fashion, procedures which would, if correctly followed, lead them to the right answers to examples or problems whose form one came to recognize:

> I was always very successful with math during high school and college--a straight A student. I am perplexed about how I could have done so well, truly understanding so little. I realize now that it was possible; I had mastered the mechanics, not the concepts. And it wasn't like faking it because I didn't realize that anything was missing.

By contrast, this course provided teachers with experiences involving open-ended explorations designed to develop conceptual understanding. Near the end of the course a journal assignment asked teachers to write about what they now thought mathematics was.

> I see mathematics as a combination of structure and creativity, the number system and algorithms providing the structure. The creativity comes in reaching beyond the algorithms to search for the how? and why?

Another teacher wrote:

> Mathematics is an infinite structure with countless connections for people to make. It shouldn't be structured (as it so often is) so that students (and teachers) believe there is only one way to get to the "right answer." Also, in mathematics and the development of mathematical concepts, half the excitement, enjoyment, the learning is "getting there." The trip of exploring, manipulating, and connecting new and old ideas is the most important part of math; not the finished puzzle or right answer.

Personal Relationship to Mathematics

As teachers came to recognize their own abilities as mathematical thinkers, and as their views of the nature of mathematics changed, many of them expressed a new sense of personal power over mathematical ideas:

> I guess even on this simplistic level I find myself--dare I say it?--thinking along mathematical lines. I used to quickly shut down if any notion appeared to be connected to "complicated" math ideas. I know what it is. Confidence. Some of the mystery is lifting.

Another teacher wrote of how her increased confidence had freed her to own her mathematical powers:

> Because of the confidence and new perspectives towards problem solving this course has given me, I was able to follow (albeit gingerly) a line of thought that I never would have attempted to attend to before--and my reward was a personal immediate

> experience in which I was conscious of the living power of mathematics thought--not someone else's account, but mine!

Finally in this last excerpt a teacher described how he came to find his own place in the tradition of mathematical thinkers:

> I've come to see mathematics more as a commitment to and respect for knowledge and understanding. The most memorable moments of the class were those in which we shared a fascination with mathematical questions, elegant proofs, or paradoxes. At such times, I felt we shared not only ideas but basic values--a love of inquiry for its own sake, an appreciation of human reason, a respect for the intellectual history of humankind. This insight ties the study and practice of mathematics in a much more tangible way than before, to some fundamental moral and ethical drives that I bring from other areas of my life. It somehow makes me feel that I have the right to participate in mathematics.

Conclusion

In this paper I have described a mathematics course for teachers whose major goal was to help participants become mathematical thinkers. The idea for the course actually came from teachers themselves. For several years, participants in SummerMath for Teachers had been asking for a mathematics course. They were aware that theirmathematical knowledge was too superficial to allow them to teach as they now believed they should, but they rejected those courses already offered at local colleges and universities. "We need a math course taught the way you're teaching us to teach."

And it must be emphasized that this <u>was</u> a mathematics, not a methods, course. But as the goal of the course was to enable its participants to become mathematical thinkers, the notion of "mathematics content" was reconceived as at once the active construction of mathematical concepts and reflection on that activity. The type of thinking that teachers applied to fractions and exponents was qualitatively different from that required to memorize procedures for, say, finding derivatives or integrals.

Furthermore, as the role of the teacher in the type of mathematics classroom described here is a considerable departure from the way teachers have been teaching, such courses must attend to teachers' affective relationships to their subject matter: it is no easier for math teachers than it is for anyone else to make profound changes in the central activity of their lives.

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PSYCHOLOGICAL/PHILOSOPHICAL ASPECTS OF MATHEMATICAL ACTIVITY: DOES THEORY INFLUENCE PRACTICE?

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At successive meetings of the British Society for Research in Learning Mathematics in 1988/89, we initiated discussion about the nature of 'radical constructivism' as a theory of knowledge and its possible implications first for the working mathematician, and then for mathematics education. We proposed that the radical statement of constructivism does not deny the existence of the real world, but makes it similar to an undecidable statement. Recognising that attitudes to the nature of mathematics affect working mathematicians, including teachers, we summarise the discussion here, quote from some interviews, and propose that this position potentially empowers one to engage mathematically with the world around.

Introduction

In this paper, we summarise and extend the main points of the discussions held at BSRLM, and reported briefly in the proceedings [Scott-Hodgetts and Lerman 1989], and we also quote from some interviews with mathematicians and mathematics educators, in which they were asked to talk about their views of mathematical knowledge, and how it affects their work. The connections between teachers' perceptions of mathematics and their teaching styles has formed the theme of some of our earlier work [e.g. Scott-Hodgetts 1987, Lerman 1983, Lerman 1986], and one of us has written and published on constructivism and mathematics also [Lerman 1989].

Our starting point for an examination of the implications of radical constructivism for the practice of mathematical activity is the, in our view, mistaken interpretation of Kilpatrick [1987], which was more clearly stated in his presentation than his published paper, that there is an ontological commitment to the non-existence of the real world implicit in radical constructivism; he appears to pursue this with the suggestion that since this is patently absurd, one can dispense with the radical idea completely. However, in our view the strong statement, namely:

"Coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower." [Kilpatrick 1987]

precisely and deliberately does not make this commitment. Rather it "places ontological questions (within the context of human thought) in a similar position to that of undecidable statements (within the context of mathematical logic)" [Scott-Hodgetts 1988 in Scott-Hodgetts and Lerman 1989]. Consistent with this position is the assertion, "It (radical constructivism) intends to be no more and no less than *one* viable model for thinking about the cognitive operations and results which, collectively, we call 'knowledge'" [Von Glasersfeld 1985].

In our view, Kilpatrick is quite right when he says ".. radical constructivists claim that we need to abandon our search for objective truth" [Kilpatrick 1987] but where he sees this as unsatisfactory for us as mathematics educators, we would tend to see this as challenging and empowering. As Bloor writes "Are believers in a flat earth the only ones amongst us with the right to operate with the distinction between 'up' and 'down'?" [Bloor 1982].

The Nature of Mathematical Activity

Where practising mathematicians have explicity concerned themselves in an in-depth study of the nature of mathematical truths, their discussions have often shown an emotional involvement which might appear surprising if one has the image that philosophical activity, as well as mathematical activity, is concerned with the gradual discovery or development of objective truths.

Consider the following extracts from the correspondence between Frege & Hilbert [Frege 1980]:

"I call axioms propositions that are true but are not proved because our knowledge of them flows from a source very different from the logical source, a source which might be called spatial intuition. From the truth of the axioms it follows that they do not contradict one another. There is no need for a further proof." [Frege to Hilbert 27.12.1899].

"I found it very interesting to read this very sentence in your letter, for as long as I have been thinking, writing and lecturing on these things, I have been saying the exact reverse: if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence." [Hilbert to Frege 29.12.1899]

These statements are strongly held and defended by the writers, with feelings of frustration, for instance, being expressed:

"There is widespread confusion with regard to definitions mathematics... it seems to me that complete anarchy and subjective caprice now prevail." [Frege to Hilbert 27.12.1899].

An image more consistent with the demonstrated emotional commitment is the seeing of a alternative constructions of the nature of mathematics as the result of different and competing *perspectives* upon mathematical activity. The notion that we might regard accounts of the nature of mathematical truth in this light is perhaps supported by the analysis of Benacerraf (1973):

"It is my contention that two quite distinct kinds of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous semantical theory in which the semantics for the propositions of mathematics parellel the semantic for the rest of the language and (2) the concern that the account of mathematical truth mesh with a reasonable epistemology. It will be my general thesis that almost all accounts of the concept of mathematical truth can be identified with serving one or another of these masters *at the expense* of the other."

In the context of the discussion we wish to evoke, the details of Benacerraf's arguments are not as important as the justification of his own view that different philosophers have focussed on different aspects of mathematics, and have built upon that narrower perspective theories which purport to account for all of mathematics. For example, he points out:

"The difference is that its proponents, although realists in their analysis of mathematical language, part ways with the platonists by construing the mathematical universe as consisting exclusively of mathematically unorthodox objects: Mathematics for them is limited to metamathematics, and that to syntax."

An essential difference between the competing theories refered to by Benacerraf and the radical constructivist thesis is the explicit assertion within the latter that it makes no claims to be the "right" position, but merely to be one model for thinking about [mathematical] knowledge, to stand alongside the alternative positions - for example the formalist one expressed by Hilbert and the platonist/logicist one purported by Frege. Then, just like mathematical modellers, we are free to make use of the competing models in whatever ways seem appropriate to our needs. We would claim that the criteria for choice in both cases are similar - the degree of resonance with previous experience and the extent to which a particular model seems to 'fit' our current observations: whilst some aspects of this decision making process might be held to be objective, it is clear that others are subjective - a point to which we will return later.

Practising Mathematicians

Wittgenstein once said that mathematics was nothing more than the contents of the notebooks of mathematicians. The relationship between philosophical theory of mathematics and the actual day-to-day activity of mathematicians has, however, been largely ignored. In examining the potential applicability of the radical constructivist model to this area, we felt an essential starting point was the consideration of what mathematicians "think they are about" when engaged in mathematical activity. We therefore decided to compile five case studies of mathematicians working in a variety of areas; this work is ongoing, but the starting point was simply to ask them to talk about their views on the nature of mathematics in relation to their own practice. There is not room here to discuss all of the responses comprehensively, but the following quotes indicate the variety of focus and belief:

"I see mathematics as a combination of concepts and intuition. The conceptual side is basic to the extension of mathematics, with intuition central to the development of theorem-proving"

"At the start of a new branch of mathematics there is often an application driving the development of the mathematics. The symbols and axioms are therefore developed with the intention that they should provide a model of something in the real world. This modelling capacity is not a characteristic of the mathematics but is a reflection of the human power to associate properties of the model with real events."

"The mathematical model used by statisticians of different schools would be symbolically identical. But this is superficial since different humans are associating the symbols with different meanings."

"Because doing mathematics is a reflex action - almost subconscious - you need to be relaxed to do it well... I don't know why you can just look at a result and it's obvious how to prove it... it's as if you just pull things out of a hat."

"I had a traditional operational research view of mathematics as being, or providing, the 'rationality' of the decision making paradigm . . . more recent paradigms have devalued the use of mathematics as providing a complete picture of decision making and view its use as describing and structuring essentially 'messy' problems."

What we are attempting to do now is to look at the appropriateness of analysing the responses we have within a rational constructivist framework, and we are finding the explanatory powers very powerful in relation to other theoretical positions. The following extract, taken from the response of a (former) set theorist forms a good basis for the illustration of this point:

Mathematics - the everyday solving of mathematical problems - seems to me to involve the creation and manipulation of mathematical objects, and the study and elaboration of their, sometimes hidden, properties. This seems to involve both the act of creation and that of revelation, each in a very real sense; there is no feeling of taking part in an elaborate psychological game: the interest is real and sometimes passionate. There can be a definite "I want to know the truth of this" feeling.

However, I believe that this feeling of discovery does not bind me to the reality of the mathematical objects involved; rather, I feel that in the process of mathematical activity, we postulate the existence (or non-existence) of one or more mathematical objects with a given set of properties. Having done so, and regardless of the nature of this "constructed" object, the process of discovery involves the unravelling of hidden structures inherent in the initial definition (postulation).

Common to these activities are both standard modes of reasoning, which are stable across large portions of the mathematical community at a given time, and general principles which we perhaps take to be more fundamental (although all of our mathematical truths are at the same level of *logical* truth!), for example certain properties of the natural numbers. Having studied, and cared about, whether certain generalisations of the Continuum Hypothesis are "true", i.e. are provable within a particular set theory, I would be happier to abandon my beliefs about such issues, than to similarly jettison my beliefs regarding such "truths" as " 1 + 1 = 2 " in the domain of the natural numbers. The further we get from the objects of immediate perception (I can see two things, even though "two" may not be such a thing), the easier it becomes to believe that properties of these mathematical entities may be this or that. Perhaps this merely underlines my own lack of ability to visualise more complex ideas, and thereby fill them with more "reality".

When challenged, A admitted that he did indeed have an instinctive belief which could be expressed in Orwellian terms, i.e. that all mathematical truths are true, but some truths are more true than others! Refering back to the distinction highlighted by Benacerraf, A's position was that having a homogeneous semantical theory would be regarded as sufficient to account for truth across mathematics, including those truths relating to the basic concepts of number theory, but that what *actually* happened for *him* was a switch in focus, from the formal to the intuitive, when dealing with these latter concepts. In *these* areas, his view is much more in line with Frege's first assertion, or perhaps with that of the empiricist, Kitcher, when he says:

"We might consider arithmetic to be true not in virtue of what *we can do* to the world, but rather of what *the world* will let us do *to it*" [Kitcher, 1984].

In fact, A went on to talk about a qualitative difference for him between concepts which had an embodiment capable of perception *in an instant* (i.e. the 'twoness' embodied in two tables), and those which would need an operation to be performed before they could be verified. This difference held even when the operation needed was as simple as that of counting.

Clearly A's beliefs as described above do not fall neatly within the established schools of thought concerning the philosophy of mathematics, and therefore to use one of the standard theoretical frameworks in order to explain them could not provide an adequate analysis. We could, of course, just dismiss the subject as being confused in his views, but before doing so we should take account of two factors:

(1) A sees his current position as unproblematic, in the sense of being consistent with his mathematical experiences to date - he does not perceive a need to strive for a 'better' explanation

(2) A is a successful mathematician - one whose results have been valued by the mathematical community as evidenced, for example, by the award of a D.Phil. in Mathematical Logic.

We would claim that radical constructivism can explain A's current viewpoint, and also validate the 'confusion' which has certainly not impeded, and may well have contributed to, A's mathematical achievements. Nobody would seriously argue that *this* mathematician has, in any consistent sense, discovered "an independent, pre-existing world outside the mind of the knower". Certainly he himself does not claim that, although *some* mathematical objects or concepts do seem to him to be "very real" whilst the physical existence of others seems less likely (even where their 'truth' can be proved mathematically).

What A has been engaged in, without doubt, is what is described as "an adaptive process that organises one's experiential world". In doing this he has brought to bear different models of the nature of mathematics, picking and chosing in order best to 'fit' his particular experiences at different times. The strong radical constructivist statement precisely 'fits' the phenomena which we observe here; also it 'allows' the ontological commitment which A clearly has in relation to *some* mathematical truths whilst at the same time explaining the lack of consistency in this area. Clearly an implication of adopting the radical constructive stance is that any ontological commitment must be regarded as an act of faith rather than the result of logical deduction. We would argue that this certainly provides an adequate explanation of A's position. As we are speaking from a radical constructivist position (at this point in time) we would not dream of claiming that it was *THE* explanation.

Mathematics educators

In a similar series of interviews, mathematics educators including teachers, researchers and lecturers were asked to talk about their views of mathematics and its relation to their practice. Again the following quotes illustrate the variety of ideas:

"Mathematical concepts and knowledge have always been there, it may just have taken a long time for them to be discovered. So mathematical knowledge is certain. This provides your security as a teacher. You, the teacher, know the theorems in geometry, for example, and so the problems arise in putting them across, not in the knowledge itself. Children may develop their own methods and understanding, but provided they can see that it works for themselves and can show me they understand, that's OK. I don't expect them to repeat back what I gave to them.

"The thing I really like about investigations, especially ones that I haven't done before - actually even the ones I have done before, because the kids always come up with something new - is that for that period of time, it feels like you are creating mathematics. You may find some new mathematical description of wallpaper patterns or butterflies wings, and even if someone somewhere has done something on that, you don't know anything about it, and its new for everyone".

"Sure it's a bit disturbing at first doing this kind of work, you always sort of wonder whether you'll know enough maths to cope with whatever the kids are coming out with, but of course you do, because you know what to do and that's the most important bit. For instance, if through looking at a number of cases you generate a formula or something, you know that it has to be tested, on further specific cases, and then if it's a really sophisticated bit of work, to try and deduce the result, which was suggested by the data".

"I haven't really thought about what mathematics as such is, I didn't do a full maths degree. I suppose it has the image of heavy, powerful and in many cases ancient knowledge, but it's growing and changing all the time, and people are doing that. I mean I don't understand chaos theory for instance, I saw something about it on television, but whereas you can think that algebra or geometry is somehow part of the way the world is, no-one can suggest that chaos theory is, or if you do claim that, it's pretty far-fetched. I should hope that the mathematician always has the excitement about creating something, rather than discovering something that has always been there".

The following is an extract from a lengthy interview with C:

"I don't have a clearly worked out philosophy of mathematics education, but as I am attracted by the aesthetic side of maths rather than the practical, I tend to see it as a game, or as patterns, games in the sense of creative play. This comes over in my teaching in that I encourage students to follow their own interests, not aiming for some right answer, and to see that what they do has value in itself. There are rules but you can change them.

I have seen teachers who, as a result of the imposed introduction of group work after being used to individualised work, don't know how to talk to them, or how to make an input, and indeed pupils resent their interventions! I agree with the idea that children construct their own knowledge, although the way we present things affects children and what they make of it. This attitude is more insecure than the traditional "This is the right way to do it" but it is more challenging and interesting, there's diversity.

When I'm doing maths it's real to me, although it depends what one means by 'real'. Some children are motivated by aesthetics. some by applications in the real world, and the teacher is there to make things concrete with real examples. You have to accept where they are at, and find the common ground."

There are strong elements of formalism here, with an emphasis on the aesthetic and mathematics as games or patterns. At the same time, the influence of her successful teaching strategies filters in, through the notion of changing the rules, thus making the play creative; through seeing challenge in diversity and insecurity, and through focussing on the effects of presentation by the teacher. Mathematics is real, although for C an aesthetic reality is quite adequate, whilst recognising that concrete applications provide reality for others.

Here too we would claim that the radical constructivist perspective provides a powerful explanatory framework of C's views. There is a strong 'fit' between her views and her practices. The theories that describe and justify her practice becoming absorbed into her image of the nature of mathematics. The apparent inconsistencies form a satisfactory and homogeneous rationale for C's teaching, in which as the mathematician A, above, she has achieved considerable 'objective' recognition and success. This rationale 'fits' the strong radical constructivist statement, but again, we would not claim that it was *THE* explanation.

Whether adopting a radical constructive stance would have made Frege and Hilbert at least more tolerant of each others views is something we certainly *can not* know. Hopefully those who *now* adopt that stance *are* tolerant.

I guess we would say that if you have an alternative model which fits as well, have faith and use it - tentatively! After all, that's the way you'd use mathematical models, isn't it?

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A WEB OF BELIEFS: LEARNING TO TEACH IN AN ENVIRONMENT WITH CONFLICTING MESSAGES[1]

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Abstract

Prospective teachers function in pre-service environments in which there are frequently several voices representing beliefs about the nature of mathematics and the nature of learning and teaching mathematics. This paper presents certain aspects of the first of several environments which are being studied within the context of a broader project (see footnote). The beliefs which are expressed through the voice(s) of the schools are of great interest in attempting to sort out the complex web of influences during the novice, pre-service period.

Introduction

Images. Images. Images. The construction of mathematical knowledge as a synthesis task of the knower was elaborated by Kant (Werkmeister, 1980; Hintikka, 1974), and Dewey (1938) was probably the strongest 20th century proponent of the social construction of knowledge. It was Piaget (e.g., Flavell, 1963 and Piaget, 1954) who sought to synthesize these two aspects into a genetic epistemology which accounted for both the personal and social aspects of knowledge construction. As these perspectives continue to unfold at the end of the 20th century, we find many mathematics educators, cognitive psychologists, and anthropologists attempting to understand the processes of social construction and enculturation into the mathematical aspects of a given community (e.g., Bishop, 1985; Carraher and Schliemann, 1985; Lave, 1985).

The project discussed in this paper is supported by the National Science Foundation under grant No. MDR 8652476. All opinions expressed are, of course, solely those of the author. Project Staff: PIs: Hilda Borko, University of Maryland; Cathy Brown and Bob Underhill, Virginia Tech; Project Director: Doug Jones; Graduate Research Associate: Pat Agard; Consultant: Margaret Eisenhart.

More recently, considerable interest has been expressed in the construction of professional knowledge by teachers as they enter into socialization and change processes (e.g., Cobb, Yackel, and Wood, 1988; Underhill, 1986) and research, methodology and examination of research traditions bearing on socialization and enculturation processes have come to the fore (e.g., Eisenhart, 1988, and Desforges and Cockburn, 1987). Considerable research interest has been expressed concerning teacher's beliefs; see Underhill (1988) for a summary.

This paper is an introductory exploration of a complex set of data. In our research, we have followed eight senior-level college students through three seven-week school placements in one small metropolitan school division. We are documenting the process of learning-to-teach mathematics. In order to study influences, we have two major data sets which we call the voice of the school and the voice of the university. By following our student teachers through three-student teaching placements and then (four of the eight) through their first year of teaching, we will capture the interactions of school, university and novice teachers over a two year period. We are developing case studies which reflect a thorough and careful exploration of similarities and differences across those case studies.

In the following presentation of simulated first-person cases, the beliefs of school personnel from the central staff, building, and classroom are presented. The Associate Superintendent has a math background and has been an administrator for more than 10 of his 30 professional years. The Math Supervisor has taught high school math for eight years and has been in her present position for five years. The principal taught grades six and seven for several years, was an assistant principal for two years and is in his second year of principalship. The classroom teacher has taught several elementary grades for 24 years and has been at this school since it was built in 1970.

Views at the School Division Level

The Associate Superintendent

About Math. I believe that the curriculum is pretty much dictated by state adopted textbooks, and state literacy passport tests. My own view of mathematics is captured in non-routine problem solving, generalizability of concepts and structures, applications and in fostering alternative solution strategies. I want kids to see algebra as generalized arithmetic and vice versa.

About Teaching and Learning. I believe that teaching should focus on understanding and that this is best achieved by using manipulatives, focusing on applications, and encouraging divergent thinking as in fostering alternative solution strategies. I value teacher innovation, and we encourage it through mini-grants for teachers to try things out. We are constantly pushing for math enrichment in classrooms. I recognize that our Staff Development Office pushes the Madeline Hunter model pretty hard, but I'm not much of a fan of that approach myself. I want teachers to get away from the strict use of basals: the supervisor reorders chapters to fit more closely with my belief that intermittent contact with topics should be in a matter of days or weeks rather than months.

About the Schools. The division's single greatest resource is its math supervisor. However, teachers have MUCH flexibility in terms of print material, resources, time allocated and so on. The central office mainly serves to encourage, support, assist. It does not require. It hopes, and it helps.

The Math Supervisor

About Mathematics. My own view of mathematics is captured in interesting mathematical activities whether they be through emphases on usefulness (applications), or modeling experiences, or interesting problem-solving experiences. I believe you can move on to more challenging conceptual ideas even if you haven't mastered, say, all of the 100 whole number multiplication facts. Mathematics is very open-ended. My view of mathematics is more fully captured in my views of teaching and learning.

About Teaching and Learning. I encourage instruction which is "highly manipulative and open-ended." I also encourage teachers to be creative in their math instruction and to focus on problem solving. I agree with the Associate Superintendent when he says, there is "no wrong way," teachers should "examine what's going on in the mind of the child." I think there should be very little emphasis on rote procedures: teachers should be open and flexible and listen to the children. There should be lots of peer interaction, verbalization and use of concrete models. Our teachers have fraction bars, decimal squares, measuring instruments, geometry models and other aids. At my office, there are "all kinds of materials that can be checked out: calculators, Miras, pattern blocks and so on." The math teaching is "not as creative as I'd like to see it," but it's "pretty typical of school systems in general," nothing outstanding. I think math contests and other competitive activities are useful, but I also promote the use of cooperative learning as in "groups of four." Students have many needs, they "should be grouped and regrouped. ...Whole group is not appropriate 100% of the time."

About this Particular School. There isn't much grouping and regrouping there in the regular classes. They use the textbook series' instructional management system (IMS) some for diagnosis and regrouping, but not nearly as much as some schools in our division. Most of the teaching in the regular classrooms follows the curriculum guide and textbook; it is mostly characterized by routine strategies, and no one there is particularly interested in manipulatives. They sort of do what's expected. There is little evidence of using special equipment or resources in the regular program. The program in that school for the gifted 5th and 6th graders is different in about every respect from what is going on in the regular classrooms.

Views at the School Level

The Principal

About Math. I believe math is more than rote rules for calculating. I think there are concepts to be learned. In primary grades these are learned through the use of manipulatives, but as children get older conceptual learning depends on applications and integration of content as our school division curriculum guide suggests in learning decimals and fractions together and in learning area and multiplication together. Problem solving requiring higher order thinking is also important.

About Teaching and Learning. I encourage teachers to order materials by placing catalogs in their boxes, but "predominantly I believe most of the teachers are using the Silver Burdett basal" and its IMS. The teachers decide what other printed materials and resources they wish to use. Our math priorities are based on the IMS pre- and post-test results. I believe in acceleration for advanced students, but grade-level teachers decide how to do it. We follow the curriculum guide which rearranges topics in the basal. Teachers have no flexibility in content, only sequence. At grade 6 we have 50-60 minutes for math. Usually it's about 30 minutes of directed teaching and 20-25 minutes for old and new homework. We follow the school division's Madeline Hunter effective instruction model. At the 6th grade level, the main resources are for geometry, especially large chalkboard protractors and such. All such materials are kept in the classrooms of those who request their purchase. I would like to see less pupil stress and repetition in math classes and more practical, real world situations, more use of calculators, more focus on conceptual and "higher level problem solving. ...We have concentrated too much. . .on rote practice."

About This School. We are in 3 buildings. One for 5 and 6 one for 2, 3, 4 and one for gifted, K and the library. Teachers decide how to group learners by grade level, and I okay their ideas. Grades 4-6 have team teaching. In grade 6, one teacher teaches both classes of science and one teacher teaches both social studies. I require teachers to work together. I support alternative approaches by allowing them to decide their grouping or teaming arrangements.

About This Teacher. He is quite traditional. He puts a problem on the board, goes over it, and has his students read and work problems. He does use quite a few visuals and charts and occasionally he uses the overhead projector (OHP). he also has Cuisenaire Rods and geoboards for his students which he uses occasionally. Since he likes science, he uses science applications in his math teaching. He gives lots of individual help and works with small groups when students need remediation.

Views at the Classroom Level

The Teacher

About Math. The math I teach is determined by the State Standards of Learning and the basal pre- and post-tests; that's mostly skills and concepts. I like geometry, so I add a special unit to my curriculum. Getting the steps right is really important in math, and accuracy. For example, it is really important to remember to invert when dividing fractions and to know how to count up the number of places to move the decimal in decimal multiplication. I like science, so I try to focus on applying math. In much of math, "This is how it's done!"

About Teaching and Learning. Students need compassionate teachers, and they need teachers who will help them remember important information (like the multiplication facts) and mathematical processes. Teachers must help students learn the steps in mathematical activities like inverting in division and placement of the decimal in multiplication. Careful and detailed explanations are important, and students need plenty of practice.

About This School. The principal focuses a lot of attention on basal and standardized (SRA) test scores. He expects us to raise those scores. The expectations are very high. There isn't much support for innovation or using alternative approaches here. There are also no special pieces of equipment or resources for teaching math. We are expected to team at the 6th grade level. I teach my own math and both sixth grade science classes. The other teacher teaches both of the sixth grade social studies classes.

About My Own Classroom. I have two reading groups and 3 math groups. Three of the math students are ahead of the rest of the class; they are not gifted, though. I give them independent work. I have some resources which I share with other teachers. I use lots of OHP transparencies, and I sometimes use sound filmstrips for students who are having trouble. I teach about 45-60 minutes each day, and I basically follow the school division's version of the Madeline Hunter model, using it to help my students learn how to do math. I occasionally use supplementary print materials, but basically I follow the basal and curriculum guide. I care very much about the students' self-esteem.

Conclusions

The mathematical conceptions appear to be watered down considerably as one moves from the highly specialized mathematics leaders at the school division level to the principal to the classroom teacher. Central level staff seem to have well-defined conceptions of what they want mathematics teaching and learning to be, but, at the same time, they seem to have fairly realistic images of its actual classroom practices. In the research project, we are especially interested in following these voices through three schools in which the participants have upper-level, self-contained and departmentalized placements. In studying these over a period of time, we hope to document the mosaic of influences and the clarity with which these voices and those of university personnel are articulated (or NOT articulated!) in the actions and statements of novices.

The presentation will allow for considerable discussion of this web of beliefs and will focus further than this brief paper has been able to do on a second school and a second placement. Certain tentative implications will be drawn based on work with two participants in two placements.

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POSTER TITLE Students' Performances in Clinical Interviews about Fractions

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This research was carried out by eight university teachers of UFRJ and four secondary teachers of the team of Projeto Fundão, from 1986 to 1988. Its objective was to verif y the efficiency of a didactic proposal on teaching fractions to students of 5th grade of the 1st degree (± 10 years old) and of 1st grade of 2nd degree prospective teachers course (± 16 years old). Some questions have been shown to be important during the analysis of the written tests, applyed to 131 students, before and after the teaching using the proposal. To make them more explicit, four students of 2nd degree were interviewed. These interviews have shown students' mental processes and difficulties that deserve special atention. To start the discussion about each one of the eleven items chosen,which envolve the concept of fractions of continuous and discrete sets, equivalent fractions, and order of fractions, the interviewer presented a task to the student. The most relevant points observed will be presented in the poster.

POSTER TITLE: A COGNITIVE APPROACH TO INSTRUCTION FOR PROSPECTIVE TEACHERS

PRESENTER: Nadine Bezuk, Judith Sowder, and Larry Sowder
INSTITUTION: San Diego State University

Cognitive science research on intentional learning and motivated comprehension was combined with cognitively based research on rational numbers and probability and statistics in designing a mathematics course for prospective elementary teachers. Much of the research on rational number learning (c.f., Post, Harel, Behr, & Lesh, 1988) and probability (c.f., Shaughnessy, in press) has reported similar conclusions: many students are developing only rote, procedural knowledge and possess deep-rooted and serious misconceptions. Two sources in particular offered assistance in formulating instructional guidelines based on cognitive research: work on intentional learning (Scardamalia, Bereiter, McLean, Swallow, & Woodruff, 1989) and work on motivated comprehension (Hatano & Inagaki, 1987).

The major goal of this course was to lead students to a better understanding of rational numbers and stochastics, and to examine the limitations of their prior understandings and make necessary changes. Instructional techniques included making knowledge-constuction activities overt, maintaining attention to cognitive goals, using cooperative learning groups, and organizing lessons around problems selected to induce cognitive incongruity. Alternative types of evaluation were utilized, including nonroutine tasks and students' written reflections of their learning.

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THE ROLE OF IMAGERY IN MATHEMATICAL REASONING

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The relationship between spatial visualization and mathematics competence has been widely debated. While strong arguments have been put forth for the critical role of imagery in mathematical reasoning, this view has been challenged by others. Since meaningful mathematical activity deals with relationships, it is likely that dynamic imagery plays an important role in mathematical meaning making. The purpose of this study was to examine the role of imagery in mathematical reasoning. Previous research (Brown and Wheatley, 1989) using clinical interviews, showed that students who had high scores on a test of mental rotations (Wheatley Spatial Ability Test, 1978) were making sense of mathematics as evidenced by solutions to nonroutine mathematics tasks. In contrast, students scoring low on the WSAT had not constructed meaning for many mathematical relationships even though they were judged successful in school mathematics.

For the present study, a group administered paper-and-pencil test of mathematical problem solving and concepts was constructed for grade five students. The twenty-eight item test included nonroutine problems and questions on numeration, measurement, and number operations. This test along with the WSAT was administered to four classes of grade five pupils in two public elementary schools. One school (School One) had a high percentage of minority students and students from low socioeconomic homes while the other school (School Two) was judged by the state to be exemplary although not high SES. At School One the correlations between the WSAT and the mathematics test were relatively high ($r = .65$ and $r = .52$) while at School Two the correlations were markedly lower. Analysis of gender and race data indicated that nearly half of those scoring high on the WSAT were white males while half of the students scoring low were black females.

The nature of the relationships between imagery and mathematics reasoning was probed in individual interviews. Three students from each of the eight cells (rotation x gender x race) were selected for further study. Individual interviews were conducted with these persons to determine the use of imagery in completing spatial tasks and solving nonroutine problems. All interviews were video recorded for subsequent analysis. An effort was made to construct a viable explanation of the children's mathematical reasoning and use of imagery. These explanations will be presented and related to the test profiles.

MATEMATICA Y REALIDAD

PROPUESTA DE UNA DIDACTICA INTEGRADORA DE LA MATEMATICA CON EJERCICIOS DE COMPUTACION

YOLANDA CAMPOS CAMPOS — SEP - DGENAM - CAM.DF
ELOISA BERISTAIN MARQUEZ — Mg Graw Hill de México
CESAR PEREZ CORDOVA — Libro Electrónico S.A

RESUMEN

This paper presents the theorical reference on which a proposal of the learning integration with realia it can be based. We present a scheme of the didactic strategy, and in one of its phases we foresee the performance of computing exercises and games. The examplification of the afore mentioned deals with "Geometric Transformations", a topic in the seventh unit of the second grade of the Mexican secundaria goverment syllabus.

POSTER TITLE: PIAGETIAN AND VAN HIELE THEORIES: THE SEARCH FOR FUNCTORIALITY

PRESENTER: Livia P. Denis, Ph.D.

INSTITUTION: State University of New York at Albany (SUNY-A)

The purpose of this paper is to examine the relationship between stages of cognitive development (Piaget, Grize & Vinh-Bang, 1977) and the van Hiele theory (van Hiele, 1957, 1984; van Hiele-Geldof, 1957, 1984) of mathematics learning from a category-theoretic (Arbib & Manes, 1975; MacLane, 1971) perspective. Hoffer (1983) pointed out the importance of devoting attention to the testing of the van Hiele phases of learning as functors between each van Hiele level category, as well as the need for testing the existence of the functions in each van Hiele category. This paper moves in that direction. Neo-Piagetian theories of cognitive development (Halford, 1989; Davidson, 1988) were examined to develop a model to explain the relationship between Piagetian theory of cognitive development and the van Hieles' theory of mathematics learning. The proposed theoretical model serves two functions. Firstly, it explains the relationship between the two theories under study. Secondly, it helps in the clarification of the formulation made that the two theories belong to two different research programs (Orton, 1987). Students do not in general acquire formal-operational abilities as early as was originally thought (Farmer et al., 1982; Flavell, 1977; Shayer & Adey, 1981). Copeland (1984) indicated that logical processes, such as mathematics, must be based on the psychological structures available to the child. It is postulated in this paper that the attainment of van Hiele level seems not only to imply the existence of a knowledge domain and a forgetful functor (Davidson, 1988), but it also suggests the existence of a functor adjoint to the forgetful functor. Existence of this functor may be evidenced by student's explanations given about their reasons for actions and solutions to given geometric problems at each van Hiele level (see Fuys, Geddes & Tischler, 1988).

POSTER TITLE : Tendencies of learning thinking styles and effect of mathematics learning.

PRESENTER : Dr. Hamdy A. El-Faramawy.

INSTITUTION : Dept. of Psychology - Menoufia University, Egypt.

ABSTRACT

The styles of learning * thinking indicate tendency of person to depend on one hemisphere rather than the other. Therefore, individual tends to use one side of brain, left / right, or both (which called integrated style) in his mental processes and behaviour. The present study however, aims mainly to answer the two following questions :

1 - Are there differencies in tendency of learning - thinking styles between maths. students and history students.

2 - Is there any effect of maths. learning on the dominant style among maths. sample.

Samples are chosen from Maths. and history students in college of education - Egypt.

Learning - thinking styles test of Torrance was used. Results indicate that :

* Maths. students are tending more to use right side of brain rather than history. students.

* There is significant difference of learning - thinking styles between first-grade maths. students and fourth grade maths. Student for the favour of fourth grade sample. Other findings are implicated.

SOCIAL CONSTRUCTIVISM AS A PHILOSOPHY OF MATHEMATICS: RADICAL CONSTRUCTIVISM REHABILITATED?

Paul Ernest
University of Exeter

This paper argues that the traditional absolutist philosophies of mathematics as a body of certain truth are defunct, and need to be replaced by a conceptual change view of mathematics (Confrey, 1981). This recognizes that mathematics is fallible, like any other field of knowledge, and the creation of human beings. Such views are increasingly widespread, and are reflected in the work of Lakatos (1976, 1978), Davis and Hersh (1980), Kitcher (1983), and Tymoczko (1986).

Social constructivism, a novel approach to the philosophy of mathematics which fits into this new tradition, is introduced. It starts from the two principles of radical constructivism (Glasersfeld, 1989). However, it adds further assumptions, to avoid the pitfall of solipsism (Goldin, 1989). These are the assumption of the **existence** of the physical and social worlds (without assuming that humans have any **certain knowledge** of them). Central to the social world is the phenomenon of human language. Building on the work of Wittgenstein (1956) and Bloor (1976) it is argued that just as language is a social construction, so too is mathematics. The result is a philosophical analogue of Restivo's (1988) sociological account of mathematics as a social construction. Only a brief sketch of social constructivism is provided. (For a full account, see Ernest, in press, The Philosophy of Mathematics Education, Falmer). However, if the theory is accepted tentatively, it is possible to indicate how it addresses the problem of accounting both for the apparent objectivity and the utility of mathematics. It also suggests how some of the criticism directed at radical constructivism (Goldin, 1989; Kilpatrick, 1987) can be overcome. Finally, the implications for education are considered briefly.

POSTER TITLE	Students' Preference for Numbers of Variables in Translating Algebraic Sentences Involving Two Quantities
PRESENTER	Aparna B. Ganguli
INSTITUTION	General College, University of Minnesota

Seven hundred sixty three college students enrolled in three levels (elementary, intermediate algebra part 1, intermediate algebra part 2) of remedial algebra classes were given seven sentences to translate algebraically. All the sentences involved two unknown quantities. The sentences were :

1. The sum of two numbers is 139. The smaller number is x. What is the larger number ? 2. The sum of two numbers is 35. Write an algebraic expression.

3. One number is three more than another number and their sum is fifty three. Write an algebraic expression. 4. $4,500 is invested, part at 8% and the rest at 10% simple interest. If x is the amount of money invested at 8%, what is the amount of money invested at 10%? 5. One number is four less than the other number. Find the number in terms of one variable. 6. There are seventy five coins consisting of nickels and dimes. If the number of nickels is x, find the number of dimes. 7. The sum of two integers is 11 and their difference is 35. Write an equation to describe this and then solve to find both the numbers.

Analysis of the written solutions indicated that many students preferred to use two variables in algebraic translation when two quantities were involved. Most of the students attempted to translate the sentences directly. Whenever the word ' two numbers' appeared, students used two symbols. The symbols ranged from different letters to number signs such as '#' or simply a blank space such as ' - '

Sixteen students, at least five from each level, were interviewed individually and videotaped. During the 15 to 20 minutes of videotaped interviews, the students were asked to think aloud while answering questions similar to the written questions. Transcripts of the videotapes and the students' written responses provided data for this study. The results indicated that the students experienced considerable difficulty in translating these simple sentences algebraically. Only 30% (n= 763) students wrote 139 - x as the correct answer for question number 1. Many chose to write only y as the final answer.

The major obstacle in students' thought processes was that unless the first unknown was found in terms of a concrete number, another unknown could not be expressed in terms of the first unknown. It was relatively easier to name another symbol for the second unknown and write the equation using two symbols. The concept of relations between two quantities in terms of one variable seemed to be extremely abstract for all levels of remedial mathematics students.

The analysis of the videotaped responses revealed that when two quantities were involved the more concrete task was to write an equation by using two symbols rather than writing an algebraic expression. Even the most successful students followed only the method of syntax and failed to internalize the concept of variables.

PERCEIVED DIFFICULTY OF PROBABILITY/STATISTICS CONCEPTS

WILLIAM E. GEESLIN
UNIVERSITY OF NEW HAMPSHIRE

College students frequently view a probability and statistics course with fear and are reluctant to enroll in such a course unless it is required. At the same time some authors have "lowered the level" of their textbooks by deleting various topics from their text or by labeling certain topics as "optional." The most common deletions have been from the areas of combinatorics, conditional probability, and probability models/distributions. At the University of New Hampshire we have two one-semester introductory probability and statistics courses: one with a calculus prerequisite taken primarily by mathematics, computer science, and engineering majors; the other requiring only high school algebra taken primarily by liberal arts and health sciences majors. While teaching both versions of these courses over a four semester period, the author has collected concept-ranking data concerning which concepts were viewed as easiest and which were viewed as most difficult by the students. In addition students responded in writing to questions concerning how probability and statistics differed from other mathematics courses. Normal achievement data was collected for the purposes of assigning course grades as well. Of interest is that non-mathematics students appeared to do as well on test problems as mathematics/science majors (on problems that were covered in both courses, i.e., non-calculus questions and questions that did not involve special distributions such as the Weibull taught only in the "calculus" section). Bayes rule when taught using tree diagrams was perceived as an easy to learn concept (contrary to texts which delete this topic). Combinations and permutations were viewed as easy to learn concepts by many students. Students felt that probability was more difficult than statistics, although approximately 20 percent disagreed with this. In general students did not list many differences between probability and other mathematics courses that they had taken. Data on relative perceived difficulty on main concepts will be available.

Proposal for Poster Session: PME 1990

INTRODUCING TEACHERS TO MISCONCEPTIONS IN SECONDARY SCHOOL MATHEMATICS

Anna O. Graeber
University of Maryland

Secondary school mathematics teachers regularly face students with conceptual and computational misconceptions in mathematics. Yet there are few readily available resources to help them understand the sources or consequences of their students' difficulties. This is true despite the fact that during the past decade, research has begun to produce an impressive data base of common conceptual errors and intuitive misunderstandings pertinent to topics in secondary school mathematics.

A National Science Foundation funded project is preparing materials for college and university instructors and their secondary mathematics preservice and inservice teachers that will give them ready access to these research findings. Materials have been designed to provide instructional materials and a data bank of selected references on (1) examples of common misconceptions and the related performance errors, (2) diagnostic tools and procedures, and (3) instructional methods and references to resource materials. The materials have been tested in a variety of settings, including methods courses for preservice teachers in secondary mathematics as well as graduate level courses for mathematics teachers.

The poster presentation will include a display of selected project products, the handbook for college instructors, and examples from the bibliograhic data base that is being compiled using Hypercard software. The project was begun in January 1988 and is funded through December 1990. Thus, a vast majority of the project materials are well developed at this time.

Among the project products is a set of short articles on misconceptions ["Multiplication makes bigger, division makes smaller; The equality symbol as operator; Frame of Reference; The probability heuristic of representativeness; Graph as Picture; The variable reversal error; Systematic Errors: Fractions; Systematic Errors: Decimals; Misbeliefs about mathematics]. These articles are offered as examples of misconceptions that could be categorized as overgeneralizations, overspecializations, mistranslations, and limited conceptions.

Approaches and theories that researchers have offered to help teachers assist pupils overcome or control the influences of misconceptions are also included in the materials. An outline of Swan's Conflict teaching approach (Swan, 1986), Driver's general structure for lesson schemes (Driver, 1987), and Fischbein's (1987) didactical implications about the role of intuitions are presented and related to Flavell's (1977) theory of the steps to equilibrium .

The data base currently includes about approximately 900 entries. These can be accessed by author, mathematical topic that is the subject of the misconception, or key words in title. Each of the entries is abstracted.

POSTER TITLE The Conception of Inner Form:Nature and Role in Learning Mathematics

PRESENTER Keiko ITO

INSTITUTION Graduate Student, University of Tsukuba, Japan

"Mathematical form" has sometimes played a fundamental role of extending the definition of mathematical entity and directing the evolution of mathematics. For instance, mathematical form that is shown as $(a^m)^n = a^{mn}$ enable us to extend the exponents from whole number to fraction and define the power $a^{q/p}$ Such form is one of the characteristic powers inherent in mathematics.

In teaching and learning mathematics, teachers hope that pupils also develop their own mathematical knowledge based on form. However, it is considerably difficult for pupils to do that. Most pupils cannot appreciate both why and how to define such entity and consequently learn them by rote. Thereafter, they don't come to be able to make the best use of the definition.

Why are pupils not able to learn mathematics based on form? When considering its reason, we cannot ignore the sign of their learning mathematics by their own basis.

Let's see an example. The way of computation of division by fraction is taught to pupils based on proportion form underlying the computation of division by whole number. However, when pupils are asked to compute $p \div (q/r)$ (p,q,r: whole number)before the instruction, most of them compute it as follows: $p \div (q/r) = q / (p \times r)$. One of pupils say, 'I did so because I had learned to compute $(q/r) \div p = q / (r \times p)$ '. His reason indicates that he didn't compute it at random but invented the computation by his own basis. That is, in the division of fraction by whole number, whole number needs to be put on denominator. He extended the definition to the division of whole number by fraction and defined it.

In this way, pupils learn mathematics on their own way without ignoring mathematical form completely. The author calls pupils own basis inner form . Inner means that it is considered in pupils. Inner form is expected to clarify the relevance of pupils own basis to mathematical form for the purpose of developing mathematical knowledge in pupil.

In this poster, the author tries to clarify the nature and role of inner form in learning mathematics. In order to do that, an idea of mathematical symbol system (J.J.Kaput(1986)) is used. A mathematical symbol system is a symbol scheme S together with a field fo reference F where a mathematical structure is associated, and a systematic rule of correspondence c between them, perhaps, but not necessarily, bidirectional. A symbol system will be denoted by an ordered triple $\mathbf{S} = (S, F, c)$.

Using this idea, mathematical form is represented on the left side of Fig.1. While inner form is represented on the right side of Fig.1.

(S . F . c)	(S . F . c)
(S . F' . c')	(S' . F . c')
(S' . F' . c")	(S' . F' . c")

Fig.1 Mathematical Form and Inner Form

In the poster, Fig.1 is illustrated concretely with examples of pupil's performances of operations of fraction and so on.

Poster Title: Using Metaphors for Reflection in Teacher Education
Presenter: Elizabeth H. Jakubowski
Institution: Florida State University

Personal epistemologies influence teachers' conceptualizations of their roles and associated beliefs. When teachers teach they do what makes sense to them in the circumstances. Our research has indicated that the sense making process is associated with an understanding of the roles which are of greatest salience to the classroom. Teachers appear to make sense of salient roles in terms of images and metaphors in which are embedded belief sets and epistemologies. The image and/or metaphor serves as an organizer of belief sets. Metaphors and belief sets associated with major roles such as management, facilitating learning, and assessment influence the way teachers plan and implement the curriculum.

When teachers adopted the metaphor of teacher as learner, changes in educational practices were evident. Changes could be observed in their role conceptualizations, beliefs and then in classroom practices. As teachers accepted the metaphor and role of teacher as learner, they were able to resolve the conflict of always having to have the answer or always being the expert in either mathematics or science. Thus, they were willing to consider and learn from new ideas which might facilitate children's learning. Associated with the reconceptualization of roles were different beliefs about learning and teaching. Decisions were made to change learning environments thus moving their educational practices from technical interests towards more practical interests.

Adoption of the role of learner/researcher enabled teachers to ask questions about what was happening in their classes. Consequently, teachers were interested and alert to finding out what worked and what did not. Teachers were reminded that they should expect some things not to work on some occasions for some students. Their role as researchers was to identify what was happening, work out why, and plan changes to enhance the quality of the learning environment. The raising of questions brought teachers to a new level of awareness regarding what their students were doing and the effectiveness of their strategies. Being a researcher stimulated reflection in and on practice. Raising questions, seeking answers, reflecting on alternative answers and making changes resulted in shifts in beliefs about learning and teaching.

MATHEMATICAL FEATURES OF DYSLEXIA/SPECIFIC LEARNING DIFFICULTY

by

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Summary

About 60% of dyslexics have problems with mathematics as well as language. The most influential factors accounting for their poor mathematical achievement relate to their lack of "efficiency" in using central, cognitive processing strategies like verbal labelling, abstraction, generalisation and short-term memory. This results in difficulty interpreting mathematical symbols, understanding the structure of the number system, appreciating commonalities amongst units of measurement and money and much more. These students' styles of learning are illustrated and explanations offered in terms of psychological and educational models.

POSTER TITLE IMAGES OF AN ACHIEVABLE TECHNOLOGICAL FUTURE

PRESENTER JAMES J. KAPUT

INSTITUTION SOUTHEASTERN MASSACHUSETTS UNIVERSITY

This "poster" takes the form of a pair of computer-animated videos. One depicts a variety of mathematics learning software exemplifying linkable representations and consistent interface across topics and grade levels. The target grade levels of this software range from 1 - 8.

The second video depicts a dynamic interactive environment for learning elementary graphical calculus in the context of simulated driving of vehicles. Here the student can generate graphs of velocity and/or distance traveled vs time in "real time" while driving the simulated vehicle.

Each of these "draft" videos concentrates on the software itself, rather than on tasks and contexts for its use. These will be discussed in accompanying written materials.

Funding for the development of these videos has been provided by the National Center for Research in Mathematical Sciences Education and Apple Computer, Inc.

Mathematical Lessons Via Problem Solving
For Prospective Elementary Teachers[1]

Ioanna Masingila and Vânia Santos
Indiana University

Teachers must have the experience of constructing their own mathematical knowledge for the content they will teach. The construction of their own knowledge can also facilitate the changing of their attitudes and beliefs about mathematics. The beliefs a future elementary teacher has about what mathematics is and what it means to know and do mathematics are driving forces in that prospective teacher's learning, applying, and teaching of mathematical ideas (e.g., Carpenter, 1989).

At Indiana University a new mathematics content course, using a problem solving and cooperative learning approach, has been designed to provide prospective elementary teachers the opportunity to develop and construct their own mathematical knowledge. The philosophy behind this course is to actively involve the students in "doing mathematics" and thinking throughout this process--making and testing conjectures, and convincing themselves, their small group and the whole class. An important aspect of the course is the process in which the students engage while exploring certain key, unifying mathematical ideas (e.g., place value, decomposition of numbers, equivalence, congruence, similarity, measurement). This is in agreement with Schroeder and Lester's (1989) ideas that foundational to developing mathematical understanding is to be able to (a) relate a given mathematical idea to a variety of contexts, (b) relate a given problem to a greater number of mathematical ideas implicit in it, and (c) construct relationships among the various mathematical ideas embedded in a problem.

The concept of decomposing numbers is a foundational mathematical idea present in the elementary curriculum. In order to have future elementary teachers confident and familiar with the relevant aspects involved in this key idea, they should explore the idea from a variety of perspectives. A traditional approach to number theory concepts would begin with a lesson which introduces the concept of decomposing numbers into factors by giving definitions and examples of prime and composite numbers, presenting an algorithmic method of finding the prime factorization of whole numbers, and assigning practice exercises which focus on computational aspects. Furthermore, in the traditional approach students usually work, and are assessed, individually and are generally not encouraged to articulate their reasoning and understanding of the concepts in either verbal or written form.

In contrast, the activity we designed to introduce this concept and lead into a further exploration of number theory, used cooperative learning and a problem-solving approach, and was part of eight hours of classroom activities on number theory. Prior to the first activity the students completed, as homework, problems involving key concepts of number theory. The first activity began with the students making individual concept maps about factors. They were then given a challenging problem, the Locker Problem, which had ideas embedded in it of divisibility, factors, primes and composites, and the categorization of numbers based on their number of factors. Following activities provided opportunities for the students to explore and deepen their understanding of number theory ideas and, at the same time, clarify misconceptions that were evident in their pre-activity homework and concept maps. After a period of three weeks, to allow for maturation and reflection, the students made a second concept map about factors. Then they wrote a brief reflective paper after examining their pre- and post-instruction concept maps. The diagnostic homework, concepts maps, and reflective paper were instrumental in providing us with information about their knowledge, thoughts, and beliefs before and after the number theory activities. We observed that these enabled the students to: (1) become more aware of their own knowledge, (2) identify what they still needed to learn, and (3) recognize the difficulties involved in this topic. This indicated that the prospective teachers were not only using their cognitive knowledge, but were also starting to develop their metacognitive knowledge.

Some examples of comments from the reflective papers are as follows:

"I didn't really think factors were very important because I had always learned about them separately from the rest of mathematics. Now I see that factors are vital to many operations and problems. While working with factors, I was also amazed at the patterns that emerged. For example, I never realized that numbers with three factors are squares of prime numbers. I also understand why this is true now."

"My concept map of factors is certainly more complex this time than the first one--and I know its because some of the things we did with factors in this class, while not necessarily new, related in a different way to what I've done before. Looking for ways to characterize the number of factors, finding the patterns--those things were new for me."

Additional examples of reflections and concept maps will be presented during the poster session.

[1]The research reported in this paper was supported by National Science Foundation Grant No. NSF TEI8751478 to the Indiana University Mathematics Education Development Center. All opinions and conclusions presented are the sole responsibility of the authors and do not necessarily indicate the views of NSF.

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EXPLORACIONES SOBRE EL RAZONAMIENTO EN MATEMATICAS.

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ABSTRACT.

The mathematical reasoning is an important theme for mathematics education. Although, it does not reflect in school mathematics curriculum and research. There are several approache s but a lot of questions are without satisfactory answer yet. The purpose of this presentatation is to show different approaches to study mathematical reasoning and present some results of studies related with types of reasoning in mathematical textbooks and understanding of logical thinking. Many forms of mathematical proofs have found which consider induction and analogy as deduction. The context is an interesting and important theme for research. And the students have many problems with basic elements of logic.

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MATHEMATICAL CONCEPTS AS TOOLS: Learning about angles in LOGO programming

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University of California, Berkeley

In an investigation of 7th and 10th graders using LOGO graphics, students manifested significant conceptual problems in choosing inputs for rotational primitives (Carraher & Meira, 1989). The notion of angle emerged as an essential "tool" for the adequate use of LOGO rotational primitives. This raises the issue of two contrasting learning environments: (1) Computational microworlds in which mathematical concepts function as instruments for reaching one's goals (such as rotating the turtle in LOGO); and (2) Standard mathematics lessons in which concepts are viewed as objects of study in themselves. Vergnaud (1984) has emphasized the need for building a theory of learning in which concepts and competences are solutions to specific problems that people face. He is concerned with the task of describing learning environments that support use of procedures, representations and concepts as solutions to problematic situations. According to this perspective, LOGO users would come to understand angle as they work to build their programming projects. The goals of this study were: (1) To investigate the effect of varying amounts of LOGO programming experience on the students' previous knowledge of angles; and (2) To explore aspects of the interaction between LOGO-based learning of angles (a tool for reaching goals) and school-based learning of angles (an object of study).

The sample consisted of 46 7th grade and 38 10th grade students. The 10th graders formed the "expert" group, for they had received more school-instruction on angles than the 7th graders. Subjects in each grade received three levels of programming training in LOGO: Zero (control group), 15 and 30 hours. The training procedure was open-ended, based on student-initiated projects. Students received no direct instruction on angles during the training sessions. After training was completed, all groups solved a written test about angles with 33 questions involving comparing, estimating and drawing angles, supplements and congruency. Students were then assigned an "angle knowledge score" based on Guttman's scale. The group averages were (%): (1) 7th graders (0; 15; 30h)- 38.8; 50.0; 62.6; (2) 10th graders (0; 15; 30h)- 75.0; 68.7; 91.7.

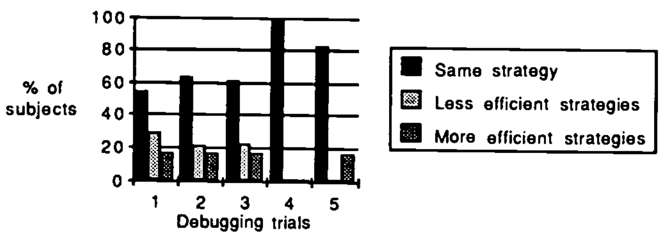
Performance on the test was strongly associated with the amount of training in LOGO. Both 7th and 10th graders with 30 hours of training scored significantly better than their classmates with no training at all ($p < .03$, Mann-Whitney's test). However, 10th graders from the 15-hour group scored consistently worse than their classmates with no training in LOGO. The results suggest that: (1) Initially, at the 15-hour level, experience in LOGO interfered with the 10th graders' existing standard knowledge of geometry (interference not observed among 7th graders given their non-expertise in the subject); (2) Then, at the 30-hour level, experience with LOGO enabled both transcendence of existing interferences and significant improvement on the angle test, for expert and non-expert groups when compared with their classmates. The study lends support to Vergnaud's functionalist perspective of knowledge construction, with the caveat that there must be significant experience in the domain for the appropriate level of meaningfulness to emerge. It suggests the value of school teaching in which the target knowledge appears as the solution, as the tool that students can use to cope with challenging and meaningful problems.

CONFLICTS IN COMPUTER PROGRAMMING: Do empirical contradictions affect problem solving?

Luciano L. Meira
University of California, Berkeley

Psychology and cognitive science have long argued on the role of empirical contradictions in promoting cognitive development and learning (Piaget, 1980; Newell, 1988). In science and mathematics education, the notion of "cognitive conflict" has given theoretical substance to widespread instructional approaches oriented to help students overcoming faulty or "misconceived" knowledge (Novak, Ed., 1987). Computer programming appears to constitute a privileged context for the study of conflict. A program generated in LOGO graphics, for example, can be considered an explicit statement of the programmer's believes about how to obtain geometric figures in that environment. If the program does not succeed, an empirical contradiction should arise that can make the user to experience conflict. Children's strategies for choosing rotation inputs in LOGO drawing were described in Carraher & Meira (1989). Three strategies were identified and a hierarchy suggested in which the strategies were ordered according mathematical efficiency and sophistication . This study investigated the role of empirical contradictions in the restructuring of those strategies.

Thirty two children (7th and 10th graders, aged 13 to 16) were interviewed after 15 or 30 hours of practice in LOGO. The task was to design a program for generating a cross formed by four equilateral triangles. Subjects worked in "LOGO editing mode" and had a maximum of five chances to debug and test their programs. After each debugging trial, the children were asked to justify their choices of angles in order to obtain a classification of strategies according to the hierarchy mentioned above. A conflict-event was considered to have occurred each time the subject tested out a complete program with unsatisfactory results. The chart shows the percentage of subjects who used the same strategy, changed to a less efficient strategy, or changed to a more efficient strategy after each debugging trial.



Averaging across all trials, 72% of the reactions consisted of using the same strategy after debugging. Of the subjects who never changed strategy along trials, 79% used the least efficient strategy in the hierarchy (again averaging across all trials). The results support the hypothesis that conflict based on empirical contradictions is not a sufficient condition for progress in problem solving, even when negation of a clearly stated "theory" is involved (Balacheff, 1986).

THE SHIFT OF EXPLANATIONS FOR THE VALIDITY OF CONJECTURE: FROM EXPLANATIONS RELIED ON ACTUAL ACTION TO GENERIC EXAMPLE

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In Japan, the guidance of proof geometry begins from 8th grade. In this guidance, students are intended to understand the significance of deductive explanations as assurance of the generality of properties and relations which plane figures have, together with cultivating a better understanding for plane figures. But a research precedent have already showed the following:in Japan, half of the students, even if they are 9th grade, incline to regard the explanation by actual measurement and/or manipulation, besides the explanation by deduction, as explanation enough for assuring the truth of a statement.

How students can detach their explanation for the validity of conjecture from actual actions(for example, actual measurement and/or manipulation) as the method by which they convince the validity of conjecture?. This problem is a fundamental point of view for my research. Accordingly, it is indispensable for my research to set up the level of explanation by actual actions and that of explanation being apart from them.

Now, my research problem is that how students shift from explanations relied on actual actions to generic example. Actual actions necessarily need concrete cases or materials. Then the former explanations is generated with using these cases or materials. On the contrary, generic example is not apart from actual actions completely, but is a explanation which never refer only to the specificity in itself. Rather, students see the generality of conjecture in generic example. In this sense, generic example is of great interest for consideration, because it is placed on the middle between the two levels of explanation. Then considering the shift from explanations by actual actions to generic example is the first step for a fundamental point of view for my research.

In the experiment, after presenting tasks which require to make conjectures and the explanation for assuring the validity of conjectures, students try to make them under a condition that they must use the concrete case or materials. Due to this condition, actual actions can appear, and then the process of the shift from the explanation by actual actions to generic example can be observed. After this observation, the interview to students is carried out for confirming that what is generic example for him in this process.

This process is analyzed by the following point of view.

1:What the object of validity is ?.
2:What the method for assuring the validity of the object is ?.
3:What explanation is used as generic example ?.

Then I discuss about the shift of explanation in this process on the basis of the relations between the object and the method of validity.

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POSTER TITLE Calculus as a bridge between intuitions and reality

PRESENTER Ricardo Nemirovsky

INSTITUTION TERC, Cambridge MA.

This poster session will describe a new project called Measuring and Modeling. This research project of Technical Education Research Centers (TERC), will explore ways to help students bridge the gap between formal calculus and the intuitions about real-life situations that can be modeled with calculus. The project, funded for 21/2 years by the National Science Foundation, will conduct a series of teaching experiments to explore learning situations that combine intuitive physics and mathematical modeling.

In the Measuring and Modeling project high school students will work in pairs, conducting experiments on real physical processes that they can measure, modify, and explore in the laboratory. Such events — for example, moving objects, flowing water, or heating substances — typify simple systems which change over time. The students will use probes to measure the physical quantities generated during the experiment. They will then try to emulate the observed behavior creating a mathematical model of the event. To create a model of the process they have observed, students will use calculus concepts like rate of change and level of accumulation. The project will study how students transfer models across various problem types and whether the transfer helps students to understand the mathematics underlying many different types of phenomena. Preliminary results will be reported.

USE OF VAN HIELE PHASES BETWEEN LEVELS IN PRIMARY SCHOOL GEOMETRY.

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Final year students in special method mathematics courses for the primary school appear to experience difficulty in making the links between psychological theories such as van Hiele's, and actual practice in their mathematics classrooms. Analysis of van Hiele's levels and the phases between them suggests several practical activities suitable for primary school pupils who are growing from the basic level, recognition of shapes, to the next one, analysis of properties of these shapes. Suggestions are also given for activities facilitating growth to the next level, ordering of properties.

POSTER TITLE The relationship between environmental and cognitive factors and performance in Mathematics of Indian pupils in the junior secondary phase

PRESENTER Anirud Rambaran & Tinus van Rooy

INSTITUTION University of South Africa

SUMMARY

Both inheritance and environment are important factors in the development of the child. Whilst innate ability affects development, the development of that innate ability is also influenced by environmental factors. Experience facilitates neural development and neural development facilitates higher levels of learning. Hence the type of environment the learner is exposed to has important implications for the learning of mathematics. This research attempted to investigate the influence of environment on the cognitive abilities of the pupils and hence on their performance in mathematics.

POSTER TITLE Cognitive Strategies and social Classes: a comparative study of working and middle class english children.

PRESENTER ANTONIO ROAZZI

INSTITUTION Universidade Federal de Pernambuco (Mestrado em Psicologia)

Seventy two English children between 6 and 8 years of age from different SES groups were tested on a task aimed at discovering the strategies used for solving a cognitive problem. The experiment was designed to investigate whether: (1) the use of different strategies (figurative versus operational) in solving cognitive problems may depend on experience and (2) whether these strategies are influenced by social class.

The task consisted of judging the number of sweets contained inside a non-transparent box by comparing its weight with other boxes using a balance-scale. These two comparison boxes were presented in two different conditions Visual and Number. In the Visual condition the two comparison boxes were transparent, and the number of sweets inside was visible. In the Number condition, the two boxes were not transparent but the subject could know the quantity of sweets because the number of sweets in the box was written on the lid. Half of the visual problems and half of the number problems had only one solution - Task 1 (e.g. target box contains 5 sweets where comparison boxes contain, respectively, 4 and 6 sweets), while the other half had two possible solutions - Task 2 (e.g. target box contains 5 sweets where comparison boxes contain, respectively, 3 and 6 sweets).

The results indicates a superior performance of middle class children on the Number condition. In the Visual condition middle class children outperformed working class children only on Task 1. In Visual Task 2 no significant differences were found. Taking into account the type of explanations given by the subjects the results were interpreted in terms of cognitive strategies. It is hypothesised that low SES children make relatively more uses of figurative strategies, and middle-class children of operational strategies.

J U E G O S   M A T E M A T I C O S

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CENTRO DE ESTUDIOS TECNOLOGICOS No. 1
"WALTER CROSS BUCHANAN

JUEGOS MATEMATICOS, nació hace aproximadamente 7 años como un intento de ayudar a integrar a nuestros alumnos (mayores de 18 años, con 3 o más años de haber abandonado las aulas, trabajando actualmente, con familia, etc.) al proceso de enseñanza-aprendizaje en el área de matemáticas.

Las prácticas de JUEGOS MATEMATICOS tienen una estructura tal, que mediante el manejo de materiales concretos y siguiendo una serie de instrucciones sencillas, los alumnos JUEGAN con los conceptos matemáticos, los "palpan", los "sienten" y como consecuencia se observa un buen rendimiento y un mayor interés por las matemáticas en la mayoria de ellos.

Actualmente contamos con un promedio de 15 prácticas por curso en el nivel de bachillerato, siendo algunas de ellas adaptaciones de juegos y entretenimientos conocidos, permitiendo su adecuación y aplicación en primarias y secundarias.

En el *poster presentation* se pretende dar a conocer algunas de las prácticas, propiciando que los participantes las realicen, para posteriormente llevar a cabo discusiones utilizando técnicas grupales.

PROSPECTIVE MATHEMATICS TEACHERS' CONCEPTION OF FUNCTION: THE REPRESENTATIONAL SIDE OF THE COIN

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Abstract

This paper describes a study aimed at providing prospective teachers with an opportunity to enrich their subject matter and pedagogical knowledge of the concept of function. First, we describe a learning module which leads to the exploration of the Parallel Axes Representation (PAR) -- an unconventional graphical representation of functions. Then, we describe its implementation in a course for prospective teachers. Our initial analysis illustrates that working with PAR enriched prospective teachers' mathematical knowledge and helped them develop a more critical approach towards the use of representations in instruction.

Development of some aspects of mathematical thinking in an Analytic Geometry Course.

María Trigueros
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A course in Analytic Geometry at college level is being used to teach some aspects of mathematical thinking: abstraction, logical inference and problem solving as part of the course itself. The course has been taught during three semesters and the study is still in progress. The analysis of evaluation questionaires, interviews with the students, a follow up of some of them in calculus and algebra courses taken simultaneously and comparison with other groups suggest that the students improve in the thinking aspects already mentioned although they feel that the course is difficult and generates anxiety.

COMPUTER GRAPHICS FOR THE ADQUISITION OF FUNCTION CONCEPTS

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ABSTRACT

It is assumed that the process of reflective abstraction is the key to the cognitive construction of logico-mathematical concepts.

The four types of reflective abstraction necessary to construct the function concepts -generalization, interiorization, encapsulation and coordination- can be enhanced by intuitive meanings of the mathematical ideas. These intuitive meanings may be developed by an inductive approach whereby the experimental phase will be done with a graphics program for microcomputers by means of a graphics environment which could be described as a "generic organiser"

Results of field experiences show that computer graphics may indeed be useful for the construction of certain function concepts.

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