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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following research papers: "Some Aspects of the Construction of the Geometrical Conception of the Phenomenon of the Sun's Shadow" (P. Boero, R. Garuti, E. Lemut, T. Gazzolo, & C. Llado); "Towards the Design of a Standard Test for the Assessment of the Student's Reasoning in Geometry" (A. Gutierrez & A. Jaime); "Investigating the Factors Which Influence the Child's Conception of Angle" (S. Magina); "Children's Construction Process of the Concepts of Basic Quadrilaterals in Japan" (T. Nakahara); "Spherical Geometry for Prospective Middle School Mathematics Teachers" (E.D. Obando, E.M. Jakubowski, G.H. Wheatley, & R.A. Sanchez); "Spatial Patterning: A Pilot Study of Pattern Formation and Generalization" (M.L. Taplin & M.E. Robertson); "Students' Images of Decimal Fractions" (K.C. Irwin); "Preference for Visual Methods: An International Study" (N.C. Presmeg & C. Bergsten); "Visualization as a Relation of Images" (A. Solano & N.C. Presmeg); "Cognitive Processing Styles, Student Talk and Mathematical Meaning" (N. Hall); "Listening Better and Questioning Better: A Case Study" (C.A. Maher, A.M. Martin, & R.S. Pantozzi); "Classroom Communication: Investigating Relationships between Language, Subjectivity and Classroom Organization" (J. Mousley & P. Sullivan); "Mathematical Discourse: Insights into Children's Use of Language in Algebra" (H. Sakonidis & J. Bliss); "Teaching Realistic Mathematical Modeling in the Elementary School: A Teaching Experiment with Fifth-Graders" (L. Verschaffel & E. De Corte); "Seven Dimensions of Learning: A Tool for the Analysis of Mathematical Activity in the Classroom" (S. Goodchild); "Teaching Mathematical Thinking Skills to Accelerate Cognitive Development" (H. Tanner & S. Jones); "Towards Statements and Proofs in Elementary Arithmetic: An Exploratory Study about the Role of Teachers and the Behavior of Students" (P. Boero, G. Chiappini, R. Garuti, & A. Sibilla); "Proving to Explain" (D.A. Reid); "Beyond the Computational Algorithm: Students' Understanding of the Arithmetic Average Concept" (J. Cai); "Learning Probability Through Building Computational Models" (U. Wilensky); "Algebra as a Problem Solving Tool: One Unknown or Several Unknowns?" (N. Bednarz, L. Radford, & B. Janvier); "Negotiating Conjectures Within a Modeling Approach to Understanding Vector Quantities" (H.M. Doerr); "Preferred Problem Solving Style and Its Effect on Problem Solving in an Adult Small Group Mathematical Problem Solving Environment" (V. Parsons & S. Lerman); "Proportional Reasoning by Honduran Tobacco Rollers with Little or No Schooling" (S.M. Fisher & J.T. Sowder); "Can Young Children Learn How to

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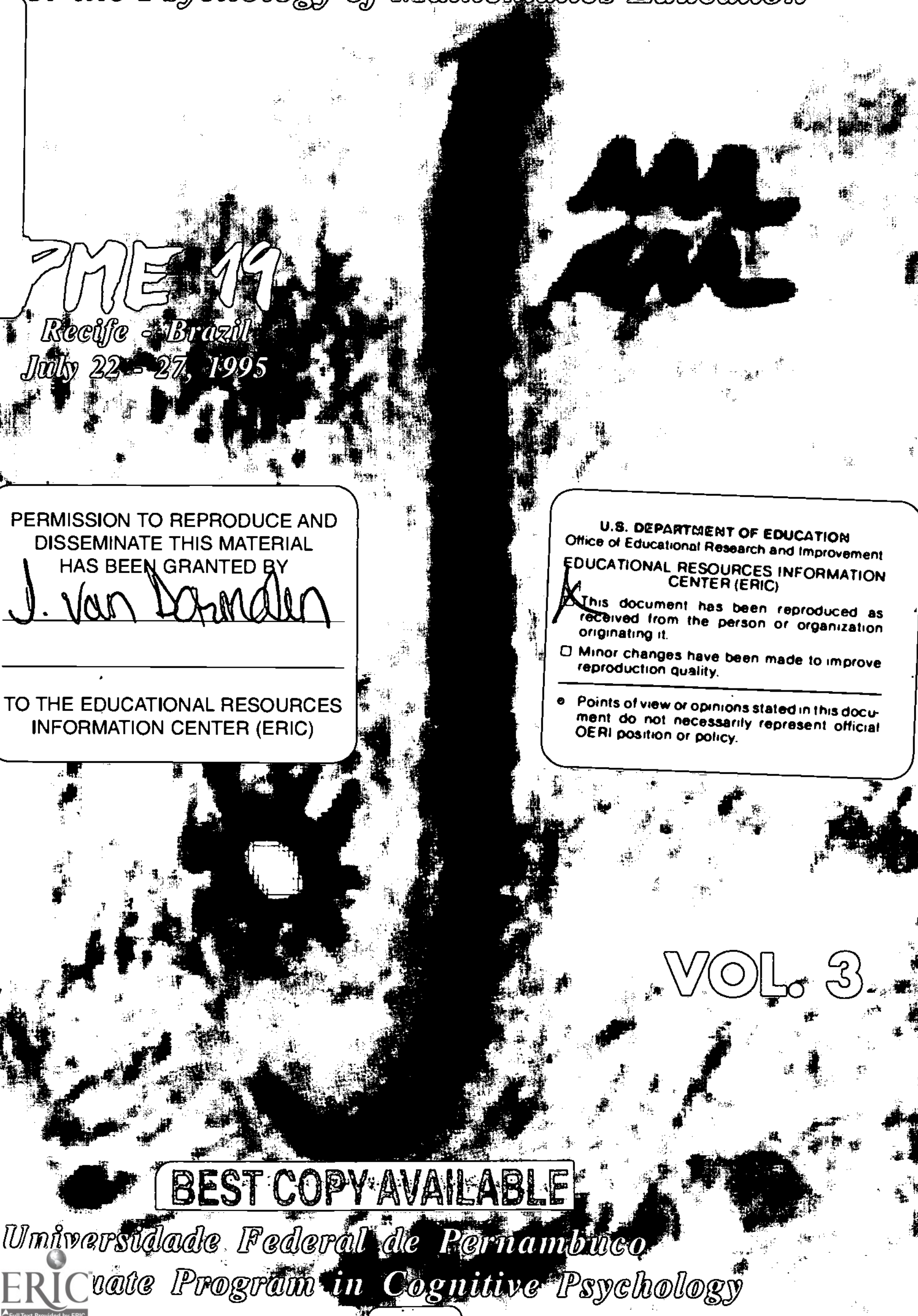

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The cover painting reflects the integration of the PME Community with the warmth of the tropical sun and beaches of our home town and host site of PME 19: RECIFE.

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RESEARCH REPORTS

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Some aspects of the construction of the geometrical conception of the phenomenon of the Sun's shadows

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a Giovanni Prodi

The persistence of "naive" conceptions relative to many natural phenomena in subjects that have learnt in school a "scientific" interpretation for them, and their difficulty in using school-learnt mathematical models to interpret non-trivial situations raise interesting issues for psychological and educational research. This report analyses some aspects relative to the passage to a geometrical conception of the phenomenon of the Sun's shadows from the "naive" non-geometrical conceptions that most 9/11 year-old students have of this phenomenon.

1. Introduction

Mathematics plays an important role (in the history of culture and the intellectual maturation of the individuals) for the construction of "scientific" conceptions of phenomena pertaining to a variety of fields (from astronomy to genetics). "Scientific" interpretations based on mathematical models learnt at school, however, appear fragile, and "naive" conceptions not only persist in common culture, but resurface also in cultured people when, in difficult problem situations, mathematical models are for some reason not serviceable. This happens in particular for the case of the geometrical modelisation of the Sun's shadows (Boero, 1985). This is a phenomenon which lends itself particularly well for the study of this subject. Many children manifest, in fact, deeply radicated "naive" (non-geometrical) conceptions of this phenomenon even at a relatively advanced age (9 - 12 years). At this age the elementary geometrical modelization of the phenomenon is accessible at school as it requires elementary mathematical tools. On the other hand, the phenomenon has been widely used for dozens of years now in different countries in renovating mathematics teaching to introduce different geometrical concepts and motivate geometric activities, and this offers a wide base of experiences for further investigation (Barra & Castelnuovo, 1976; Berté, 1985; Lanciano, 1990; Lladò, 1984; Trompler, 1983).

The investigation dealt with in this report concerns the transition from non-geometric naive conceptions to a "geometric conception" (i.e. a conception based on geometric aspects) of the phenomenon of the Sun's shadows. In particular, this report wants to highlight some "variables" on which the teacher may act to help the construction of the geometric conception. It is our hypothesis that this construction requires the mastery of different geometric aspects of the phenomenon, which are independent of each other from a cognitive point of view and not spontaneously acquired.

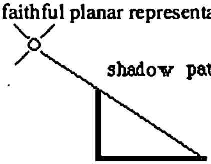
The mastery of the geometric aspects considered in this report, albeit necessary, does not appear however sufficient to build at school a stable and deep geometric conception of the phenomenon (see point 5.).

For what concerns the methods and the types of results obtained (both positive and negative), we believe that the investigation dealt with in this report may be extended to the issue of the construction of "scientific" conceptions relative to other phenomena.

2.Methods

In the elementary geometric modelization of the phenomenon of the Sun's shadows, we may discern the following aspects, forming three facets of the same model:

a) The "shadow pattern" representing a sunray touching the extremity of the object and identifying the border of the shadow. In particular, if the object is a nail or a thin stick, the "shadow pattern" is a faithful planar representation of a tri-dimensional reality (see below).



shadow pattern

representation of the "shadow space"
(in the case of a stick)

b) The "shadow space", that determines the visible shadow as its section with the surface on which the shadow is projected. The shadow space is tri-dimensional in nature, and not directly visible in its entirety. It is not easily representable by drawing, except in the case of the shadow of a nail or a thin stick (surface enclosed by the "shadow pattern" - See above).

c) The relations between the length of the shadow and the angular height of the sun, and between the height of the object and the length of the projected shadow. These are relations concerning the metric aspect of the phenomenon and that, at an elementary level, may be taken into consideration in qualitative terms (for instance: *"if the sun is high, the shadow is short"*) or in quantitative terms (in particular through the constant ratio between the heights of the objects and the lengths of the projected shadows).

Our investigation dealt with these three geometric aspects of the phenomenon of the Sun's shadows. We tried to verify the following hypotheses, resulting from past observations of the behaviour of students and cultured adults in various problem situations relative to the Sun's shadows and from a previous analysis of the tools necessary to successfully face them:

* An operative mastery of each of these three subjects is not spontaneously reached (requiring therefore the mediating action of the teacher).
* An operative mastery of one aspect may be reached without necessarily requiring or implying mastery in the others (cognitive independence).
* None of the three aspects is by itself sufficient to face problem situations that require either a geometric interpretation or a qualitative or quantitative prediction about the phenomenon of the Sun's shadows.

In order to verify these three hypotheses, we carried out systematic observations on classes experimenting the projects of the Genoa Group and the Sabadell Group for the integrated teaching of mathematics and sciences in primary and comprehensive school. In these projects the phenomenon of the Sun's shadows is made object of extensive and in-depth didactic activities (that have extended for some teachers well over fifteen years: see Belcastro, 1981; Lladò, 1984; Boero, 1985). We have, in particular, compared the behaviours of the students in grades IV, V, VI, VII where their didactic routes dealt in depth, at the beginning, for a sufficiently long period, with <u>only one</u> of the geometric aspects of the phenomenon mentioned above. In these classes, problem situations, suggested by previous difficulties met by the teachers and suitable to highlight the effects of the didactic choices made, were planned, managed and observed. Some of these problem situations had also been presented to cultured adults (primary school teachers following in-service training activities with us.

The observation was carried out by collecting texts and drawings by the students and the adults and through discussions and interviews, aiming at determining the nature of the students' non-geometric naive conceptions and clarify the role of the previously identified geometric aspects in the transition to the geometric conception of the phenomenon of the Sun's shadows.

3.General review of the results of the observations carried out.

We will consider six problem situation and for each of them we will indicate the results of the relative observations.

3.1: *"Did you notice that on sunny days your body makes a shadow on the ground? Do you think that the shadow is longer at 9 in the morning or at 12 noon? Why?".* This question was presented to grade IV students at the beginning of the work on the Sun's shadows(410 written answers were collected) and to VI grade students within an initial diagnostic questionnaire (over 4000 written answers collected from 1980 on).Results: For percentages of students reaching 59% at age 9 and 47% at age 11, the Sun's shadow is longer at 11 with explicit motivations of the type: *"because the Sun is stronger", "because the Sun hits more".* Further interviews (beginning with the request to *"explain your answer"*) on smaller samples show that, in the majority of cases, this motivation corresponds to a conception of the shadow as an "appendix of the object" (therefore belonging to the object), whose length is controlled by the strength of the Sun. A typical statement is: *"I answered at 11 because at 11 the Sun is stronger, and so (....) my body makes a longer shadow because the Sun hits more and gets a longer shadow out of me"* (see also Boero, 1985).

In order to ascertain if the "shadow pattern" may be spontaneously acquired, after the initial questionnaire we asked to eight grade VI classes: *"Explain in a drawing how shadows are made".* In these classes, even most of grade VI children who had correctly answered the question above that the *"shadow is longer at 9"* were not able to supply a graphic interpretation in terms of "shadow pattern". The fact that the acquisition of the "shadow pattern" is not spontaneous seems confirmed by other results also. Most of grade VI classes, after the experimental verification of the initial hypotheses were asked: *"Explain with a drawing why, when the Sun is high, the shadow is short and when the Sun is low, the shadow is long".* Less than 15% of the children are able to produce a drawing where the "shadow pattern" is used to explain the dependence of the shadow's length to the Sun's height.

3.2. *"Facing away from the Sun, we walk toward a wall. At one point we see that our shadows begins to go up the wall. How do you explain this?"* This question is asked after observing the phenomenon. An exploratory investigation, carried out with individual interviews in three grade IV and two grade VI classes at the beginning of the activities on the shadows, indicates that, for many 9-11 year old children, the shadows goes up *"because it cannot squeeze against the bottom of the wall"*, for others, it goes up *"because it has its own length"*, for others again *"because if it finds a* *wall, it tries to look like the person"* (all in all, over half of the children manifests this type of conceptions). Virtually none of the children refers to the "shadow space" or to the "shadow pattern". Even the answers to this question bring out the conception of the shadow as an appendix of the object that projects it, as well as the non-spontaneity of the conception of the shadow seen on a surface as a section of the "shadow space". The same question, presented to V grade students that had already carried out an extended activity on the shadow space only, gives very different results (more than half the students refer to the *"shadow space meeting the wall"* in their answers). Lower (about 40%) are the percentages of grade VI students that, after activities on the "shadow pattern" only are able to give a correct interpretation of the phenomenon (mostly everybody refers to the "shadow pattern"; virtually no child refers to the "shadow space").

3.3: The children look at the shadow of a long factory shed, while they are standing completely in the shadow at about ten metres from the shadow's edge. They are asked: *"Where and how must you walk to see the Sun move like a cat from left to right on the roof of the shed?"* The same question was asked to primary school teachers during in-service training activities. The problem situation is difficult; we have observed non-negligible percentages of success, with exhaustive motivations of the correct answer (between 25 and 40%, according to the age) only if the question was preceded by extended activities on the "shadow pattern". We also noticed that the activities on the "shadow space" by themselves allow an exhaustively motivated answer only to the first part of the question *("where do you need to walk")* but not to the second part *("how do you need to walk")*. The difficulty seems to lay in the transition from the correct collocation in the shadow space to the identification of the direction of the movement.

3.4: The children observe the shadow of a factory shed, with the Sun low, while they are in the sun at about ten metres from the shadow. They were asked: *"Where do you need to move to remain all in the shadow?"*. We asked this question both to IV and VI grade students that had not yet carried out activities on the shadows and to IV, V and VI grade students after extended activities on the shadows. In both cases, if the activities did not deal with the shadow space, most of the children (over 70%) suppose that *"to be all in the shadow, it is enough that I enter in the shadow with my feet"*. Similar results are obtained with the question: *"What will happen if Mary walks on the shadow of the shed, at less than a metre from its edge? Will she be all in the shadow or not?"* The percentages of correct answers pass from less than 10% to over 60% after performing activities on the shadow space.

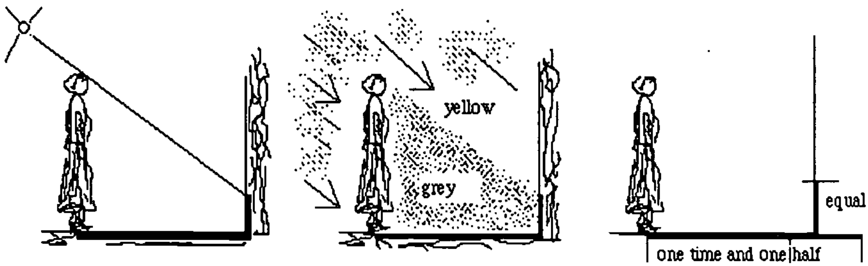
3.5: *"Two boards with two nails of identical length are placed one in the yard and the other on the terraced roof of the school. How are the 'fans' of the shadows recorded at the same times of the same day?"*. The question is asked to grade V students after carrying out extended activities on the shadows (in particular, after extended activities on the "shadow pattern" but without activities on the shadow space, or vice-versa). Over 60% of the students that worked mostly on the "shadow pattern" answered that the 'fans' are different (with longer shadows on the terrace) and the exhaustively motivated correct answers are less than 15%. Most of the students use the "shadow pattern" in a stereotyped manner, without realising that the distance of the Sun is enormously greater than the difference of level of the two measuring points of the shadows (for further details, see Scali, 1994). The percentage of exhaustively motivated correct answers increases instead to about 35% in the case of extended prior activities on the shadow space. The light investing the two points of observation is often represented with one single "light beam".

3.6: Determination of the height of objects that may not to be reached for direct measurement, using the length of their shadow: In Garuti & Boero (1992), a teaching experiment is described during which the students go from an initial prevalence of additional-type reasoning to multiplicative-type reasoning (thanks to the teacher-led discussion and verification of the strategies as they are produced by the students and to the reference made to the "shadow pattern"). The investigation proves, in our opinion, that the relation of direct proportionality between height of the objects and length of the projected shadow is not spontaneously acquired, and that its acquisition may make use of the "shadow pattern". In a subsequent teaching experiment in other two grade VI classes, it was also noticed, however, that the students may reach that proportional reasoning also via other ways, without using the "shadow pattern".

In general, for what concerns aspects a), b) and c) considered at point 2, the observations seem to indicate that:

* Mastery of each of the aspects is spontaneously acquired only by a very small percentage of students (see the situation described at 3.1 for a), the situation described at 3.4 for b) and the situation described at 3.6 for c)).
* Mastery of each of the aspects may be acquired independently from the others, without effects on the spontaneous acquisition of the others (see situations 3.2., 3.3., 3.6.).
* Each of the aspects is necessary to successfully approach some of the problem situations that have been considered: In particular, a) appears necessary for the situation described at 3.3; b) for the situation described at 3.4 and, obviously, c) for the situation described at 3.6.

A fact that appears clear from the observation we made is that, even if each of the three considered geometric aspects may be the object of a separate activity, does not require the others and does not produce the acquisition of the others, it may be useful to weave the work on the three aspects together (for time reasons, if nothing else) thus trying to recompose the unity of the "geometric vision" of the phenomenon. As a matter of fact, the qualitative relationship between the height of the Sun and the length of the shadow may be used to construct the modelization of the Sun's rays with straight lines (using the graphical products of some students: see 3.1.), the mastery of the shadow space may be linked to its sections with planes containing the Sun ("shadow pattern"), and the "shadow pattern" may turn out to be quite useful to arrive to a geometric model of the proportional relation between the heights of the objects and the lengths of their projected shadows. Nevertheless this does not seem to necessarily produce an integration of the three aspects: the drawings reproduced below (situation 3.2.) are frequent also in classes where the activities on the three aspects have been performed together in the way described here.



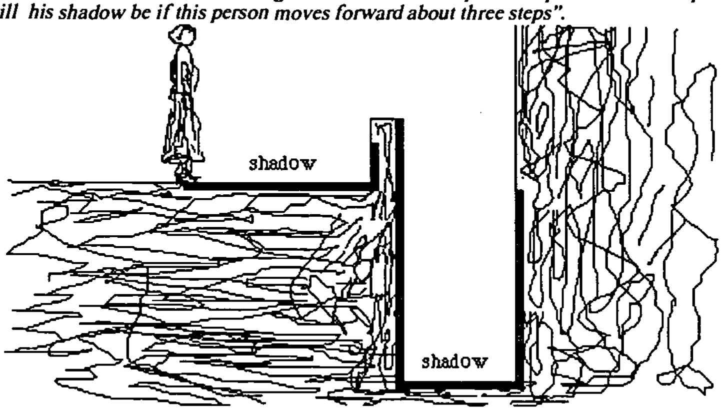
Although facets of a same geometric model, the three aspects considered seem to correspond to different ways of seeing the phenomenon of the Sun's shadow. Also taking into account some interviews we made, they seem to correspond, respectively, to the idea of a dynamic relationship between the position of the Sun and the edge of the shadow, to the idea of shadow as a "lack of light" (like in the second drawing), and to the idea of quantitative regularity (obvious in the third drawing).

4. Behaviour of students and adults in a difficult problem situation.

We considered two variations of a same problem situation:

(A) Symbolic situation: The following drawing is reproduced on a sheet of paper, with the caption:

"The drawing represents in section a situation of shadows made by the Sun. A person is coming close to the low wall and is represented with his shadow. On the other side of the low wall is a deep hollow space and then another high wall. The drawing shows the area of the hollow space that remains in the shadow". The assignment is: *"Draw the person represented in the picture and where will his shadow be if this person moves forward about three steps".*



(B) Real situation: In a situation similar to that schematised above, with the Sun behind, the teacher moves slowly towards the low wall and asks the students to predict what will happen to the shadow. Mostly everybody will say that the shadow will move up the low wall. The teacher starts moving again. When the shadow is at about two thirds of the height of the low wall, he stops and asks the students to write what will happen to the shadow if he should move closer to the wall. Once the answers have been collected he asks those who did not do it to illustrate their answer with a drawing.

These two situations have been presented to V, VI and VII grade classes and to primary school teachers during in-service training activities after some months of work on the shadows (observation of the phenomenon of the shadows, various types of verbal and graphic description of the shadows of people and objects at different times of the same day and in different days of the year) or after short activities of observation of shadows. With different percentages according to the classes, the age of the students and the activities carried out previously, the following types of predictions have been recorded (together with other non-pertinent answers and non-answers):

PR1) *"The shadow of the body appears on the wall in front, above the shadow of the low wall"*

PR2) *"The shadow ends up in the darkness of the hollow space" or "The shadow is not long enough to go up the wall in front"*

The following table summarises the distribution of the percentages of these answers in relation to the situation (A or B), the type of previous activity on the phenomenon of shadows (on the "shadow space", or on the "shadow pattern", or on all geometric aspects, or some introductory observations only), the type of subject being interviewed (grade V and VI students or primary school teachers). All percentages refer to a group of at least 40 people (for the students, at least two classes).

	STUDENTS				ADULTS			
	situation A		situation B		situation A		situation B	
	PR1	PR2	PR1	PR2	PR1	PR2	PR1	PR2
only shadow space	14%	41%	9%	50%	18%	31%	16%	37%
only shadow pattern	46%	27%	31%	38%	57%	11%	48%	20%
all aspects	45%	24%	32%	36%	55%	9%	49%	22%
only some observations	11%	58%	8%	64%	17%	33%	15%	42%

From the table we may gather the following indications:

- Generally speaking, the results in situation A are better than those in situation B. This is particularly true for the students and the adults that have carried out extended activities on the "shadow pattern". The "shadow pattern" is not spontaneously used (by those not familiar with it) even in situation A, which would appear to suggest an easy geometric construction of the solution.
- The "shadow pattern" appears to be necessary but not sufficient to successfully approach the two problem situations. As a matter of fact, only those groups with an extended working experience with the "shadow pattern" display significant percentages of PR1 predictions. Even in these groups, however, many are not able to successfully approach the two problem situations (note in particular that only in the adult group and only in the situation A, there is a success percentage above 50%).
- A previous performance of activities on the "shadow space" and/or the proportionality between height of objects and length of projected shadows does not seem to affect the percentage of success;
- The analysis of the percentages of PR2 answers (confirmed by the analysis of some interviews gathered among those formulating this prediction) confirms the fact that non geometric "naive" conceptions persist even after extended activities on the geometric aspects of the phenomenon of the shadows and shows that these conceptions surface again even at an adult age.

5. Discussion

In consideration of the behaviours described at points 3.3 and 5, and in particular the fact that even the majority of those students that had carried out extended activities on all geometric aspects of the phenomenon of the shadows dealt with in this report, do not succeed in passing these tests, we may ask ourselves what further factors are involved in the transition to a geometric conception of the shadow capable to approach tests similar to those described here.

Generally speaking, it seem that one of the factors is the quality of the activities carried out by the subjects of the investigation, and especially the degree of interiorisation of spatial relations. These relations remain for many people (even adults) confined to the space of the representation on a sheet of paper and are not connected to the space of the phenomenon nor to the imagination of the phenomenon. In general they are thus unable to use autonomously the representation on paper to correctly approach the problem (see, as well as 3.1, also B) of 4.). In particular, the connection between the "shadow pattern" and the phenomenon appears to be very tenuous (see Situation 3.5.), and this may be interpreted with the fact that the "shadow pattern" is really a reduction to the microspace of the paper of a situation placed in the macrospace (see also Berthelot & Salin, 1992; Scali, 1994). It appears that the arguing activities relative to the phenomenon of the Sun's shadows also are important for the interiorisation of spatial relations, as well as for the development of skills for the deliberate control and direction of thought processes (necessary in particular to perform the connection between the "shadow pattern" and the phenomenon). As a matter of fact, some good results have been obtained in activities as demanding as those described above in those classes where the teachers had requested accurate verbal descriptions of different aspects of the phenomenon of the Sun's shadows, accurate comparisons of hypotheses, and class discussions to validate the hypotheses produced (anticipating the results of experimental verifications).

It would appear opportune to perform further in-depth investigations on these subjects so as to obtain a more complete and organised picture of which variable intervene (beside those of a geometric type considered in this report) in the transition to a coherent and stable geometric conception of the phenomenon of the Sun's shadows.

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TOWARDS THE DESIGN OF A STANDARD TEST FOR THE ASSESSMENT OF THE STUDENTS' REASONING IN GEOMETRY

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Abstract. *In previous publications, we outlined a theoretical framework for the design of tests to evaluate students' Van Hiele level of thinking and for the assignment of Van Hiele levels to the students. Based on this framework, we present here a test aimed to assess students in Primary and Secondary Schools. The subject area of the test are polygons and other related concepts. The test is integrated by open-ended super-items, each one of them having several related questions dealing with the same problem. In this paper we describe the items, analyze the structure of the test, and present the results of the administration to a sample of 309 primary and secondary students.*

Introduction.

A constant in the research on the Van Hiele Model of Geometric Reasoning over the years is the expressed need of an assessment instrument fitting the usual requirements of reliability and validity, and also fitting the requirement of easy administration in a short time to big samples. Unfortunately, the third requirement seems to be incompatible with the previous ones. Clinical interview is considered the most valid and reliable technique, but it can hardly be used with medium sized samples. Written tests do not have this inconvenience, but it is usually harder to verify their reliability and validity, since written answers are poorer than oral answers. Some attempts have been done in the direction of building written tests to assess the Van Hiele levels but, unfortunately, they have not been successful enough (Mayberry, 1981; Usiskin, 1982; Senk, 1983; and Crowley, 1989).

We are working in a research program, a part of which is presented in this paper, whose main objective is to build a written test with a structure as close as possible to semi-structured clinical interviews. The research is divided into three related parts:

- Definition of a model for the evaluation of tests and assignment of Van Hiele levels (Gutiérrez, Jaime, Fortuny, 1991 and Gutiérrez et al., 1991): The continuity of the Van Hiele levels has been showed by lots of students' answers. Then we proposed a method to assign to students a degree of acquisition of each level, mirroring the reality that most students use a level of thinking or another depending on the task they are solving.

- Identification of a framework for the design of tests to evaluate students' levels of thinking (Jaime, Gutiérrez 1994): The reasoning of each level is characterised by several key processes or abilities, so a balanced and valid test should assess everyone of them. On the other side, most tasks can be answered from several levels of thinking, so students' level of thinking is determined by their answers, not by the statements of the tasks. Then, researchers should consider the range of possible levels of answer to each item of a test.

Furthermore, this helps to obtain a reasonable amount of items assessing each Van Hiele level without the inconvenience of a long test taking too much time to be answered.

- Design of a test to assess the Van Hiele level of reasoning of students in Primary, Secondary, and University. In this paper we present results of this part of the research. Namely, we describe and analyze a test made of open-ended items based on polygons and other related concepts, and present the results of the administration of this test to students from 6th grade of Primary to the last grade of Secondary. Previous versions of this test were also administered to university students (preservice teachers).

To conclude this introduction, just to mention that we consider Van Hiele levels 1 (recognition), 2 (analysis), 3 (classification), and 4 (formal deduction).

Description of the Test.

Each Van Hiele level is integrated by several key processes of thinking so, for a student to attain a certain level, the student should show mastery in all the processes characteristic of this level. These processes are (Jaime, Gutiérrez, 1994):

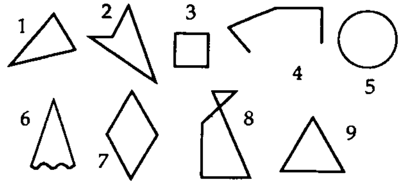
- **Identification** of the family a geometric object belongs to.
- Work with the **Definition** of a concept, in two ways: To *use* a known definition, and to *state* a definition for a class of geometric objects.
- **Classification** of geometrical objects into different families.
- **Proof** in some way of a property or statement.

The table below summarises the key processes characteristic of each Van Hiele level 1 to 4. The "---" mark means that this process is not a part of the reasoning of the level. Then, any test designed to assess the Van Hiele levels of thinking should have items evaluating every process of each level.

	Identification	Definition	Classification	Proof
Level 1	√	√ (State)	√	---
Level 2	√	√ (Use & State)	√	√
Level 3	---	√ (Use & State)	√	√
Level 4	---	√ (Use & State)	---	√

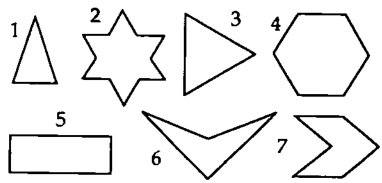
For several years, we have piloted and improved a set of paper and pencil items. Usually, pilot trials of the items were followed by clinical interviews of some students, to check the reliability of their written answers. The final result in the subject area of polygons is the test we describe below. It is integrated by 8 open-ended items, each one of them having several questions. When evaluating a student's answers, we do not consider each answer independently, but the answers to all the questions in the same item or section. In this way, we can have a clearer picture of the student's way of reasoning, and we can judge inconsistencies, contradictions, etc. among different answers. These are the items in the test (due to the limited space, the text of some items is shortened):

Item 1. - Write a **P** on the polygons, write an **N** on the non-polygons, write a **T** on the triangles, and write a **Q** on the quadrilaterals. If necessary, you may write several letters on each figure.



- Write the numbers of the figures which <u>are not polygons</u> and explain, for each of them, why it is not a polygon.
- The same questions for figures which <u>are triangles</u>, and figures which <u>are quadrilaterals</u>.
- Is figure 8 a polygon? Why? - Is figure 2 a triangle? Why?

Item 2. - Write an **R** on the regular polygons, an **I** on those that are irregular, a **V** on those that are concave, and an **X** on those that are convex. If necessary, you may write several letters on each figure.



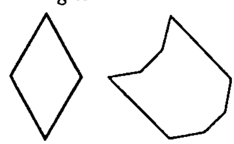
- For polygons 2, 4, 5, and 7, explain your choice of letters or why you did not write any letter.

Item 3. A) - Write all the important properties which are <u>shared</u> by *squares* and *rhombi*.
- Write all the important properties which are <u>true</u> for *squares* but <u>not</u> for *rhombi*.
- Write all the important properties which are <u>true</u> for *rhombi* but <u>not</u> for *squares*.

B) - The same questions as in A) for *equilateral triangles* and *acute triangles*.

Item 4. A) - You can see a shape in figure -a- (a rhombus). Make a list of all the properties that you find for this shape (you can draw to explain the properties).

B) - The same question for shape in figure -b-.



- a -   - b -

Item 5.1. - Recall that a *diagonal* of a polygon is a segment that joins two non adjacent vertices of the polygon. How many diagonals does an n-sided polygon have? Give a proof for your answer.

Item 5.2. - Complete the three following statements (you can draw if you want):

In a 5-sided polygon, the number of diagonals which can be drawn from each vertex is ..... and the total number of diagonals is .....

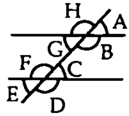
In a 6-sided polygon, the number of diagonals which can be drawn from each vertex is ..... and the total number of diagonals is .....

In an n-sided polygon, the number of diagonals which can be drawn from each vertex is ..... Justify your answer.

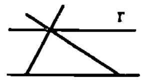
- Using your answers above, tell how many diagonals an n-sided polygon has. Prove your answer.

Item 6.1. - Prove that the sum of the angles of <u>any</u> acute triangle is 180°.

Item 6.2. - Recall that, if you have two parallel straight lines cut by another straight line: All the acute angles marked in the figure (A, G, C, E) are equal. All the obtuse angles marked in the figure (B, H, D, F) are equal.



Taking into account the figure on the right (line r is parallel to the base of the triangle) and the properties mentioned above, prove that the sum of the angles of <u>any</u> acute triangle is 180°.



Item 6.3. - Here is a complete proof that the sum of the angles of <u>any</u> *acute* triangle is 180°. Read it and try to understand it.

- The sum that we are supposed to calculate is $M + R + T$ (figure 1).
- Construct a parallel to the base of the triangle through the opposite vertex R (figure 1). Extending a side, we have two parallel lines cut by a transverse, so $T = t$ (figure 2).
- Extending the other side we have two parallel lines cut by a transverse, so $M = m$ (figure 3).
- Therefore, $M + R + T = m + R + t = 180°$, as the latter three angles form a straight angle (figure 4).

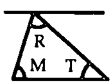


Figure 1.

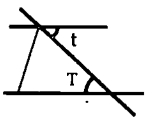


Figure 2.

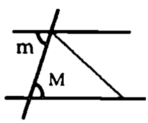


Figure 3.

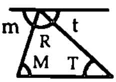


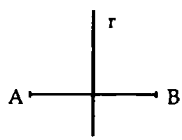
Figure 4.

- You have seen above a proof that the sum of the angles of an acute triangle is 180°. Is it true that the sum of the angles of a *right* triangle is 180°? Prove your answer.

- Tell how much is the sum of the angles of an *obtuse* triangle: <u>Exactly</u> 180°, <u>more than</u> 180°, or <u>less than</u> 180°. Prove your answer.

Item 7. A) - Prove that the two diagonals of <u>any</u> rectangle have the same length.

B) - Recall that the *perpendicular bisector* of a segment is the line perpendicular to that segment that cuts it through its midpoint (line *r* is the perpendicular bisector of segment AB). Prove that any point of the perpendicular bisector of a segment is equidistant from the endpoints of the segment.



Item 8. - Usually a *parallelogram* is defined as a quadrilateral having two pairs of parallel sides.

Could a *parallelogram* also be defined as a quadrilateral in which the sum of any two consecutive angles is 180°? Justify your answer: If your answer is affirmative, prove that both definitions are equivalent. If your answer is negative, draw some example.

Usually all the questions of an item are presented in the same sheet. However, each section in items 5 and 6 is presented in a different sheet, and students are not allowed to "go back" to correct or complete answers in previous sheets after they have moved to the second or third section. In both items, the first section just states a property to be proved; then, the second and third sections provide some hints to help students to understand and complete the proof. In this manner, we allow more able students to complete the proofs on their own, and less able students to work on the problems with some help and to produce some kind of answer, in a way similar to the procedure used in clinical interviews, were the researcher, when necessary, guides the student with a hint, comment, question, etc.

The table below summarizes the key processes evaluated in each item and the possible Van Hiele levels of students' answers. It supports the validity and reliability of the test, since i) every process is considered at least by an item, and ii) for each level of thinking, there are several items that can be answered by students in that level. In Spain, students in upper Primary and Secondary are likely acquiring thinking level 2 or 3; for this reason we have included in the test a high number of items assessing these levels.

	Van Hiele levels					**Definition**			
Item	1	2	3	4	**Identif.**	Use	State	**Classif.**	**Proof**
1	•	•			•				
2	•	•			•				
3	•	•	•			•		•	
4	•	•			•	•			
5.1, 5.2		•	•	•		•			•
6.1		•	•	•					•
6.2, 6.3		•	•						•
7		•	•	•		•			•
8			•	•		•	•		•

This test was administered to students in upper Primary (grades 6 to 8) and Secondary (grades 1 to 4) (ages from 11 to 18). To optimize the administration, we did not present the eight items to all the students, but we took into account the particular characteristics of students in different grades: Since primary school students were most likely reasoning in levels 1 or 2, we reduced the number of items evaluating levels 3 and 4 in their test. In the same way, we reduced the number of items evaluating levels 1 or 2 in the test for upper secondary school students. On the other side, the mathematical content of an item may increase its difficulty in certain grades, so we avoided items whose contents likely still had not been studied in some of the grades 6 to 8. Then, we administered three different sub-tests:

A) The test for students in grades 6, 7, and 8 had items: 1, 3, 4, 6, 7.

B) The test for students in grades 9 and 10 has items: 1, 2, 3, 5, 6.

C) The test for students in grades 11 and 12 has items: 1, 3, 5, 6, 8.

All the test have five item, three of them being the same items, to guarantee the possibility of comparison of results, and the other two items selected depending on the expected students' level of thinking and their knowledge of geometric facts. Tests A and B do not assess the process of statement of definitions. Although it may be considered as a weakness of these tests, in pilot trials, we noticed that questions asking to compare definitions or to build a definition (i.e. a list of necessary and sufficient properties) from a list of given properties were meaningless for most students in those grades. Then, we decided to exclude this kind of questions, to have a shorter and more efficient test.

Results of the Administration of the Tests.

The test was administered to 309 students. The table on the right shows the number of students in each grade. The answers were codified according to the method of levels and types of answers defined in Gutiérrez, Jaime, Fortuny (1991), and a vector with 4 percentages was assigned to each student, showing the student's acquisition of Van Hiele levels 1 to 4. Both researchers made independent assignations of level and type to each answer, then both assignations were compared, and the disagreements were analyzed. Some times this analysis resulted in an improvement of the marking criteria, and a new marking of some answers if necessary.

Grade	Students
6th Primary	34
7th Primary	62
8th Primary	83
1st Secondary	35
2nd Secondary	36
3th Secondary	28
4th Secondary	31

In order to make more meaningful the results of the evaluation of the tests, the numeric scale of percentages of acquisition of a Van Hiele level can be translated into a qualitative scale of degrees of acquisition of the level, as follows:

[No acquisit.] Low acquisit. [Intermed. acq.] High acquisit. [Complete acq.]

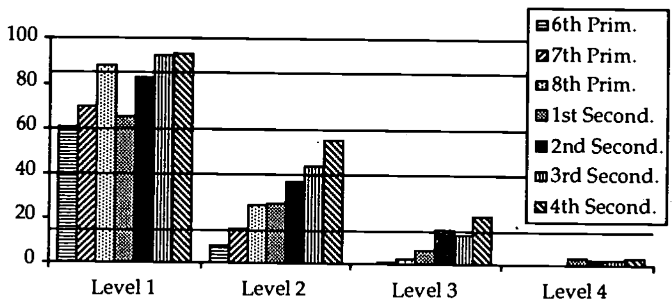
0% 15% 40% 60% 85% 100%

The vectors of the degrees of acquisition of the four levels provide information about the behaviour of every individual student. Analysing those data, we have identified several profiles, that correspond to different styles and qualities of reasoning. The table in the next page presents the most significant of such profiles and the percentages of students having each profile (for instance, in CHI≤L, C stands for complete acquisition of level 1, H for high acquisition of level 2, I for intermediate acquisition of level 3, and ≤L for low or null acquisition of level 4).

Profiles	6th	7th	8th	1st	2nd	3rd	4th
C C I,H ≤I	0	0	1,2	0	5,6	3,6	12,9
C H I ≤L	0	0	0	0	5,6	3,6	9,7
C H ≤L N	0	4,8	4,8	0	0	25,0	12,9
C I ≤L N	0	3,2	10,8	11,4	5,6	7,1	25,8
C ≤L ≤L N	26,5	43,6	59,0	17,1	44,4	42,9	22,6
H ≤I ≤L N	20,6	9,7	12,1	28,6	26,8	0	0
I N N N	29,4	8,1	7,2	2,9	0	14,3	0
L ≤L N N	20,6	25,8	2,4	34,3	2,8	0	0
N N N N	2,9	4,9	1,2	0	0	0	0

An evolution in the students' kind of reasoning along the grades can be observed: The higher the course, the bigger the number of students showing better profiles of reasoning, with the exception of 1st and 2nd grades of Secondary.

It is also interesting to analyze the relationship between the results of students in different grades. Next chart shows the means of the acquisition of each Van Hiele level by the students in each grade, so it provides with a global picture of the differences from students in different grades. Some conclusions can be drawn from both table and chart:



- Most students had a high or complete acquisition of level 1. There is a progress in the acquisition of this level along the Primary grades, and also along the Secondary grades. However, it is noticeable the reduction of the acquisition of this level in the first grade of Secondary. A reason for such reduction may be that some students in this grade had not enough time to complete the test, since 8 students (23% of the group) did not answer the last item in the test (item #3), and half of them neither answered the previous item (#2). Both items assess levels 1 and 2. The influence of this problem in the results of level 1 is rather important since only three items evaluate this level, but its influence in the results of level 2 is smaller because it is assessed by all the items in the test.

- Students in 7th grade of Primary and 1st, and 2nd grades of Secondary had not completely acquired the first level, although they showed at the same time a low acquisition of level 2. In the same way, students in the upper grades of Secondary showed an intermediate acquisition of level 2 and also a certain acquisition of level 3. One of the main characteristics of the Van Hiele Model is the hierarchy of the levels (a student is supposed to begin the acquisition of a level only after s/he has completed the acquisition of the previous level), but the reality of the teaching of mathematics is that often students are being taught in the higher level of reasoning, and teachers force them to answer according to that level. The result is that students are not allowed to complete the acquisition of the lower level but, sometimes, they acquire practice, although only a few, in the higher level. However, the phenomenon of "reduction of level" has to be taken into consideration (Van Hiele, 1986).

- Only 17 students in the sample had an intermediate or better acquisition of level 3, and only 7 students showed a low or intermediate acquisition of level 4. So, most Spanish students leave the Secondary School having a low or null acquisition or level 3, i.e. almost completely unable to make any kind of mathematical deductive reasoning (neither formal nor informal). These poor results may be a consequence of the usual way of teaching Mathematics in Secondary School, where many teachers emphasizess formal proofs (level 4) when, as we see in the previous graph, students are reasoning only in level 1 or 2.

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INVESTIGATING THE FACTORS WHICH INFLUENCE THE CHILD'S CONCEPTION OF ANGLE

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This paper summarises the main results of a PhD thesis which aimed to investigate the factors which influence the child's conception of angle by analysing children's responses in a variety of situations and under different conditions. Theories from psychology and mathematics education were interwoven to form a basis for designing the study and the interpretation of the results.

This study was guided by constructivism and authors who follow this position. In this way, from Piaget (Piaget et al 1968, Fourth 1969, 1977) I embraced the perspective in which knowledge can be understood from two viewpoints: as describing things -- figurative knowledge -- which initially arises from imitation, starting the symbol formation, and as operating on a thing -- operative knowledge -- which is concerned with the transformation of reality states. It involves a logical thought. The 'figurative' aspects of symbolic acquisition and their usage, including language, are subordinated to the child's 'operative' aspect of knowledge.

From Vygotsky's theory (1962) I borrowed two main ideas. The first is the zone of Proximal development, which allows children to reach higher stage (level) with the help of 'others'. The second is his distinction between spontaneous and scientific concepts and how both are elements of the same process, i.e., of the concept formation. They are continually influenced by each other. The spontaneous concept arises from the child's everyday life experience, whilst the scientific concept is usually acquired at school, with the help of the teacher.

Vergnaud (1987, oral communication) makes a similar distinction between spontaneous (called 'ordinary') and scientific concepts, which illuminate the understanding of the results of this research. He argues that ordinary concept has much to do with a person's level of competence. This competence is shown by the operational invariants which emerge from schemes acquired from a child's interaction with the situation. Thus the operational invariants will constitute theorems-in-action as well as the theorem-in-conception. He emphasises that ordinary and scientific concepts can co-exist in harmony, depending on the situation in which each concept, or combination of concepts, might be applied. Thus, it is essential to confront a child with problem solving which puts him/her in a position of understanding the meaning of the concept .

Another important idea which was also helpful for the interpretation of my findings comes from Nunes (1992, 1993). She argues that the representational systems influence the functional organisation of the children's activities However, these systems may not be able to influence the functional organisation without the support of particular cultural sign systems . This means that the same children may perform differently when carrying out the same function supported by different systems.

Finally, from Van Hiele (1986) I embraced the argument that a context involves many different symbols, and any given symbol is not restricted to one context only. For Van Hiele, the starting-point of symbols is an image, in which the properties and relations are projected. Through learning, the symbol loses its peculiarity of image and achieves a verbal significance. Therefore, the symbol becomes more useful for operations involving thinking.

The design of the research was built up based on the main issues of the above theories, from which three fundamental questions arose:

1) Considering Piaget's and Vygostsky's developmental perspectives, considering also Vygotsky and Vergnaud's ideas that firstly, a concept emerges spontaneously and thus is transformed into a scientific one and, still having in mind Van Hiele's considerations about the learning process, I consider how the angle is understood by a child spontaneously, and to what extent this understanding varies with age and schooling?

2) From Van Hiele' statement about the importance of presenting a content inserted in a paper situation plus Nunes's consideration about the influence of different representational systems over the functional organisation of people activity, my question is: does a child have a different perception of an angle in different situations?

If so, and thinking in terms of children' semiotic function, a finally ask:

3) How do different situations give meaning to the child's understanding of an angle?

THE STUDY

The research was carried out in Recife, a city situated in the north-east of Brazil. The sample was comprised fifty four students divided on the basis of school levels into nine groups of six, with the youngest group consisting of 6 year-old pre-school children and the oldest 14 year-olds in the last class of elementary school. In Brazil, school is divided into elementary school (from first to the fourth class) and middle school (from fifth to eighth class). In the elementary school children have only one teacher who is responsible for all central subjects in the curriculum including mathematics.

In contrast to the earlier years, children in the middle school have different teachers for different subjects. In Brazil the teaching of geometry occurs differently from the elementary to the middle school. As far as the curriculum relevant to this study is concerned, children have some contact with the topic of angle in the elementary school -- although this is largely confined to 'playing' with shapes. More analytical activity including angular measurement is not introduced until the middle school. For this reason, the sample was divided into two groups: Group 1, composed of 6-10 year-old children from early elementary school, and Group 2 which was made up of 11-14 year-old children from middle school.

The study included 92 activities, i.e., children were asked 92 times to give an answer in three separate sections. The activities, which were the central point of the research, were elaborated according to six interwoven sets of variables, as described and defined, in summary , in the next figure:

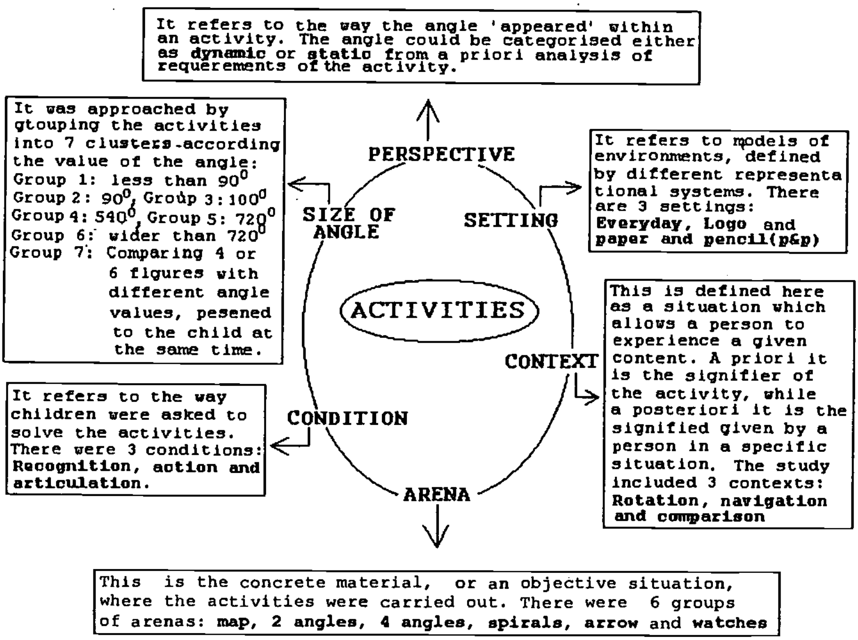

Figure 1: The Universe of the Study

I now shall briefly give some examples of how activities were carried out:

Example 1: *from the map arena*

Recognition: Did you turn 90° at any point along route **A**? Where?
Did you turn 90° at any point along route **B**? Where?
Articulation: Explaining your answer

In this example, the activities, beside the recognition and articulation conditions as well as group 2 of the size of the angle, were using the dynamic perspective, for all 3 settings, experienced the navigation context.

Example 2: *from the watch arena*

Action: Where would the minute hand be half an hour later?
Articulation: Explain how did you come to your answer

This example shows an activity related to: group 3 of the size of angle variable, action and articulation conditions, inside of everyday and Logo settings, experiencing the rotation context, in a dynamic perspective.

Example 3: *from the 4 angles arena*

Recognition: Which of the angle is the largest value?
Which of the angle is the smallest value?
Articulation: Explain why you came to your answer

This example shows an activity related to group 7 of the size of angle variable, recognition and articulation conditions, inside of paper and pencil (p&p) and Logo settings, experiencing the comparison context, in a static perspective.

RESULTS

Next Tables below show the number of incorrect answers given by children in both Group 1 (from 6 to 10 year-olds) and Group 2 (from 11 to 14 year-olds) over the 92 activities. The tables will take into account: contexts (Table 1), settings (Table 2), arenas (Table 3), conditions (Table 4), and size of angle (Table 5)

% OF INCORRECT RESPONSES 3 CONTEXTS				
	Ages	NAVIGATION	ROTATION	COMPARISON
ELEMENTARY SCHOOL	6	89.39	76.67	80.30
	7	81.06	64.6	70.71
	8	77.27	30	64.65
	9	81.06	28	62.12
	10	74.24	19.44	55.55
	Average	80.60	43.74	66.67
MIDDLE SCHOOL	11	82.61	20	49.48
	12	81.16	16	44.79
	13	66.67	6.67	23.44
	14	45.65	0	14.06
	Average	71.27	10.67	32.94

Table 1: The result of children's performances in the 3 context

	Ages	1 EVERYDAY	2 PAPER & PENCIL	3 LOGO
% OF INCORRECT RESPONSES 3 SETTINGS				
ELEMENTARY SCHOOL	6	75.22	90.38	71.84
	7	68.02	75	67.24
	8	47.75	69.87	49.42
	9	54.05	68.59	43.1
	10	46.4	51.28	40.23
	Average	58.29	71.02	54.37
MIDDLE SCHOOL	11	48.15	60.90	40.23
	12	43.52	51.92	31.61
	13	32.41	42.95	21.84
	14	21.76	31.41	8.04
	Average	36.46	46.79	25.43

Table 2: The result of children's performances in the 3 settings

	Ages	1 RECOGNITION	2 ACTION
% OF INCORRECT RESPONSES 2 CONDITIONS			
ELEMENTARY SCHOOL	6	81.42	72.58
	7	73.77	61.83
	8	65.57	32.79
	9	66.12	32.26
	10	57.92	22.04
	Total	68.96	44.3
MIDDLE SCHOOL	11	61.94	24.73
	12	53.61	19.98
	13	43.33	10.21
	14	29.72	1.61
	Total	47.15	14.13

Table 3: The result of children's performances in the 2 conditions

	Ages	1 MAP	2 WATCH	3 2 ANGLES	4 4 ANGLES	5 ARROW	6 SPIRAL
% OF INCORRECT RESPONSES 6 GROUPS OF ARENAS							
ELEMENTARY SCHOOL	6	78.28	74.44	78.43	100	79.41	72.22
	7	72.73	57.78	66.67	95.83	73.53	63.89
	8	65.66	27.78	60.78	95.83	41.18	63.89
	9	67.68	34.44	63.72	95.83	35.29	36.11
	10	59.59	21.11	58.82	70.83	25.49	33.33
	Total	68.79	43.11	65.68	91.66	50.98	53.89
MIDDLE SCHOOL	11	73.44	22.22	47.06	66.67	28.43	30.55
	12	61.98	6.67	43.14	70.83	27.45	38.89
	13	59.37	4.44	29.41	37.50	12.74	13.89
	14	44.27	0	11.76	25	0.98	16.67
	Total	59.76	8.33	34.25	50	17.4	25.02

Table 4: The result of children's performances in the 6 arenas

		% OF INCORRECT RESPONSES 7 CLUSTERS						
	Ages	1 < 90	2 90	3 180	4 540	5 720	6 > 720	7 4 e 6 ANGLES
ELEMENTARY SCHOOL	6	85.08	85.9	75	79.17	75	72.22	94.4
	7	70.17	83.33	62.12	79.17	58.33	63.89	100
	8	68.42	75.44	37.12	45.83	8.33	63.89	80.55
	9	69.30	79.82	39.39	20.83	0	36.11	88.89
	10	62.28	72.81	29.54	20.83	4.17	33.33	69.44
	Total	71.05	79.30	48.63	49.17	29.17	53.89	86.67
MIDDLE SCHOOL	11	64.03	65.79	38.64	29.17	8.33	27.78	66.67
	12	57.90	64.49	29.54	25	12.5	38.89	55.55
	13	46.49	54.38	20.45	.33	0	13.89	30.55
	14	20.17	28.07	17.42	0	0	16.67	16.67
	Total	47.15	53.18	26.51	15.62	5.21	24.31	46.36

Table 5: The result of children's performances according to the size of the angle

DISCUSSION

Taking into account the six variables of the study, the discussion will be placed in terms of issues:

Development: The results pointed to the presence of the developmental factor as influencing the children's performances. A progressive increase, in the averages of correct answers, was noted as the child matured (from 6 to 14), although this difference was less in the performances of 8 to 12 year old children. However, these differences were not very marked and age did not necessarily lead to improvement, as shown the difference between the results of 10 and 11 year-old children. I shall take two points into consideration. Firstly, 10 and 11 age-group children were classified in the same sub-group. Secondly, the difference in favour of the 10 year-old children was small. Therefore I do not consider that this divergence refutes the developmental factor. Instead I understand that this difference shows that there were other factors influencing children's performances.

School: I came up to state that school was one of the responsible for the children's performances based on the evidence that middle school children were able to solve at least half of the activities, whilst among the elementary school children this only occurred with the 10 year-old children. On the other hand, the children's results were not wholly consistent. In fact, younger age-group children presented better performances than older ones depending on which value of angle was involved in the activity. For instance, 12 year-old children made more mistakes than the 9 year-olds in those activities which involved angles of 720^0 and wider (clusters 5 and 6 respectively).

Angle Perspective: The quantitative findings evidenced a diminution, from 8 to 14 year-old children, with regards to the number of the correct solutions in activities inserted in the dynamic perspective of angle[5]. Moreover, the 12, 13 and, above all, the 14 year-old age-groups, as well as 6 and 7 year-old children, showed better performances in the static activities. Why middle school children were better in this perspective while the elementary school children were better in the dynamic perspective: did the middle school children have difficulty in perceiving the movement of figures? In fact, when the qualitative data are taken into account we note that children between 8 and 14 referred quite often to the turns of the figures, mainly when the activities in the everyday and Logo settings. The same was not noted among 6 and 7 year-old children, who referred to this category very little in comparison to the older age-groups. Moreover these younger age-groups continued to used the static reference as much as the older children. This is an evidence that for younger children were concerned with the figure itself as it was in a given moment, i.e., before or after it had moved -- children were only using the figurative aspect of knowledge.

Setting: The majority (and in some age-groups, all) children presented their best performances in the Logo setting. This result was true for all the age-groups, independent of school level. My first explanation for this result does not relate to any of the theories mentioned, but to the children's motivation to play with Logo, since the computer had never been used before by these children.

Context: The great majority of the children presented their best performances when the activity was part of the rotation context. In contrast, a large number of children could not solve activities in the navigation context. The first fact to be taken into account from this result is the mathematical properties involved in the contexts: whilst rotation involves turning around the same point (same axis), navigation presupposes translations and rotations and rotations occur in different axis. Therefore, in the mathematical sense, the context of navigation is more complex than rotation, since rotation is one of the steps involved in navigation, i.e., children need to know (or at least, to carry out) rotation in tasks involving navigation, but the contrary is not true. From the psychological perspective it is also possible to note differences between the two contexts which were probably influencing the children's experience.

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5 - I am not afirming that 8 year old children solved more dynamic questions correctly than the 14 year old children. Rather, it refers to a comparison, within groups, between the average of correct responses taking into account the dynamic and static perspectives.

My last reflection concerns the dynamic and static perspectives. The finding showed that the older children, the more they referred to the dynamic perspective. It was also showed that the p&p setting is the hardest one for children of all ages. P&p was the setting which, unlike the other two, basically explored static activities. This happened because I was interested in exploring the relationship between p&p and school, as I did between the spontaneous concept and the everyday setting, and also between dynamic and the Logo setting. However, thinking only in terms of dynamic and static perspectives, I noted that arenas could be better balanced intra and inter settings. In this way the everyday setting should involve other static arenas besides the stick game, the one in which children did not make association with their daily life. And the p&p setting should involve dynamic arenas other than the arrow in which children can make associations with their everyday life. For future research I would propose a design which included, in each setting, a balanced number of arenas, in terms of dynamic and static as well as in terms of cultural and non-cultural meaning.

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CHILDREN'S CONSTRUCTION PROCESS OF THE CONCEPTS OF BASIC QUADRILATERALS IN JAPAN

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(ABSTRACT)

The objective of this study is to investigate and clarify the process of children's construction of the concepts of basic quadrilaterals. We conducted three kinds of research tests and discuss the results from three points of view.

Section 1 discusses the existence of common cognitive paths among basic quadrilaterals. In section 2 we attempt to analyze the process of cognitive development with mutual relationships among basic quadrilaterals. Section 3 examines if actual children's thoughts of basic quadrilaterals follow the thought levels proposed by van Hiele.

0. INTRODUCTION

This paper attempts to investigate the process of children's construction of the concepts of basic quadrilaterals, or specifically trapezoid, parallelogram, and rhombus, from the following three points of view.

P1. Vinner's common cognitive path

P2. Recognition of mutual relation among quadrilaterals

P3. Van Hiele's theory of thought levels

P1 is based on the idea that there may be a concept construction path common to most children in mutually related concepts A and B. This idea was proposed by Vinner et al. (1980). Section 1 discusses the existence of common cognitive paths among basic quadrilaterals.

P2 is an attempt to analyze the process of cognitive development with mutual relationships among the basic quadrilaterals. In Japan, how to teach this topic significantly changes each time the mathematics programs by the Ministry of Education undergo revisions, which indicates there are a lot of need to be explicated on the development process and difficult points. Section 2 discusses these subjects.

P3 is associated with the concept construction process over the relatively longer term. The theory of thought levels proposed by van Hiele is widely known today, and studies which have found some positive results supporting the theory are being carried out at present (NCTM, 1988). With such a situation in mind, Section 3 examines if actual children's thoughts of basic quadrilaterals in Japan follow the thought levels by van Hiele.

Problems and subjects of this research are as shown below:

Problems for research: Research Problem Set I , II and III

(Appendix ①, ② and ③ attached at the end of this paper)

Table 1. Subjects of the research

School		Elementary School			Secondary School	
Grade (Age)		4 t h (9~10)	5 t h (10~11)	6 t h (11~12)	7 t h (12~13)	8 t h (13~14)
Number		1 0 6	9 7	1 1 2	1 0 6	1 0 1

1 . COMMON COGNITIVE PATH

Teaching of the basic quadrilaterals in Japan at present is mostly carried out in the fourth grade of elementary school. This section analyzes and discusses the results from the tests of the research problem sets I and II conducted with fourth graders and fifth graders. The research problem set I relates to the extension of geometrical figure concepts, while that of set II to its connotation. The results from the tests were processed as shown below.

With the problem set I , subjects who answered correctly five problems or more out of seven have been assumed to have nearly-perfect understanding of the extension of concept for the geometrical figure. Totalling the number of these subjects for each of the geometrical figures has resulted in the next order: parallelogram → rhombus→ trapezoid. For that reason, common cognitive paths from parallelogram to rhombus, and from rhombus to trapezoid have been anticipated. Accordingly, two-by-two contingency tables have been developed based the distinction between nearly-perfect or not, and χ^2 tests have been conducted on those. One of the tables is as shown in Table 2.

Table 2. Two-by-two contingency table of rhombus-trapezoid (Set I)

(4th graders)		Trapezoid ○	Trapezoid ×	Total
Rhombus	○	48 (45.3%)	32 (30.2%)	80 (75.5%)
	×	3 (2.8)	23 (21.7)	26 (24.5)
		51 (48.1)	55 (51.9)	106 (100)

○: Subjects with nearly-perfect understanding
× : The others

$\chi^2 = 16.6 \quad p < 0.01$
(Yates'correction)

Regarding the problem set II, on the other hand, a subject has been judged to be nearly-perfect with the connotation of the concept, when he (or she) answered correctly five problems or more out of six for each of the figures. Totalling again the number of nearly-perfect subjects with each of the geometrical figures has similarly resulted in the order of : parallelogram, rhombus, and trapezoid. Since the same cognitive paths from parallelogram to rhombus, and from rhombus to trapezoid have been anticipated, we have developed two-by-two contingency tables between them for χ^2 tests. The results are shown in Table 3 and 4.

Table 3. Results of common cognitive paths relating to extension

	Parallelogram to Rhombus	Rhombus to Trapezoid
4th Graders	Exist ($p<0.01$)	Exist ($p<0.01$)
5th Graders	Exist ($p<0.05$)	Exist ($p<0.07$)

Table 4. Results of common cognitive paths relating to connotation

	Parallelogram to Rhombus	Rhombus to Trapezoid
4th Graders	Exist ($p<0.01$)	Exist ($p<0.01$)
5th Graders	Exist ($p<0.05$)	Exist ($p<0.08$)

These results signify that the common cognitive paths for the basic quadrilaterals are from parallelogram to rhombus, and rhombus to trapezoid, both with connotation and extension. Those paths are considered to indicate the process and paths of constructing the concepts. Therefore, teaching these concepts should be more effective when made in order of parallelogram, rhombus and trapezoid. It is noteworthy that the order is not a mathematical transition from the general to the special nor the special to the general.

2. RECOGNITION OF MUTUAL RELATION AMONG QUADRILATERALS

Researching problems for this view point are primarily the problem set III. The results focusing on the rate of correct answers are shown in Table 5.

Table 5. Results of problem set III

	Rectangle & Quadrilateral					Triangle & Isosceles Tri.					Parallelogram & Trapezoid					Parallelogram & Rhombus				
Q. / Grade	a* A	b E	c I'	d O	To.	a A	b E	c* I'	d* O	To.	a* A	b E	c I'	d O	To.	a A	b E	c* I'	d* O	To.
4th N=106	% 72	% 79	% 42	% 55	% 62	% 55	% 66	% 64	% 65	% 63	% 16	% 33	% 58	% 29	% 34	% 62	% 48	% 48	% 62	% 55
6th N=112	87	88	67	75	79	83	71	80	83	79	20	41	60	44	41	74	64	70	69	70
8th N=101	91	91	79	86	87	86	79	87	85	84	41	58	76	63	60	73	83	79	76	78

From the above results, the following conclutions may be drawn, wherein the following discussions triangles and quadrilaterals are called general figures, and isosceles triangles, parallelogram etc. that are derived by adding restrictions are called special figures.

i) Mutual relationship between a general figure and a special figures as between a triangle and an isosceles triangle or between a quadrilateral and a rectangle is basically recognized by 65 % ~70 % of fourth graders and by 85 % ~90 % of eighth graders. These high rates of correct answers indicate the ease of recognizing the mutual relationship between a general and a special figure as compared with other relations.

ii)Mutual relationship between a parallelogram and a rhombus is basically recognized by approximately 55 % of fourth graders, approximately 70 % of sixth graders, and by 75 %~80 % of eighth graders.Those numbers show that recognizing mutual relationship between special figures is more difficult than between general and special figures.

iii)Only approximately 20 % of sixth graders, and approximately 40 % of eighth graders gave correct answers to the questions on the mutual relationship between parallelogram and trapezoid.That indicates that recognizing the mutual relationship between parallelogram and trapezoid is the most difficult relationship for children.

iv)The significant difference in the rate of recognition of the mutual relationship between parallelogram and rhombus and that of parallelogram and trapezoid indicate a difference in the construction of concepts between rhombus and trapezoid. Results from the research problem sets I and III indicate that many children have an image of a footstool about trapezoids, and they understand trapezoids as having only a pair of parallel sides. The former is a prototype phenomenon (Hershkowitz et al.,1990, pp.82-83) induced by typical trapezoid figures presented in early stages of teaching. The latter, on the other hand, comes from the difficulty in the definition of trapezoid involving the use of a logical term "at least", or from ambiguity due to avoidance of that difficult term. Results from the problem set III also show that once a child understands the definition in such an exclusive manner, it is difficult for him (or her) to correct the exclusive understanding even after learning the logically correct definition afterward.

v) The recognition of the mutual relationship between quadrilaterals reflects common cognitive paths from parallelogram to rhombus,and rhombus to trapezoid.
From all of the above, the following are derived :

(a) Recognition of mutual relationship between figures varies in level of difficulty according to that of the object figure.

(b) With quadrilaterals, mutual relationships are recognized and constructed at first between a general figure and a special figure, then between special figures, and lastly between difficult-to-define special figures.

(c) In teaching trapezoids, some improvements should be made with pictures and definition to be presented in the early teaching stage.

3. THEORY OF THOUGHT LEVELS

Today, a variety of research problem sets have been developed to identify the thought levels proposed by van Hiele. By the standards suggested by Mayberry (1983, pp. 60-61), the research problem set Ⅰ, Ⅱ, and Ⅲ are judged to be those to determine if the thought level 0, 1, 2 are reached respectively. Therefore, using those problem sets, this section discusses the thought levels on quadrilaterals.

Results from the research have been processed as shown below: with the problem set Ⅰ, those subjects who gave two or less wrong answers have been assumed to have nearly-perfect understanding, while, with the sets Ⅱ and Ⅲ, those with only one or no wrong answer have been assumed to be the same. On that assumption, research have been carried out to determine if the subjects are nearly-perfect with respect to each of the problem sets Ⅰ, Ⅱ, and Ⅲ, and to summarize the results. Table 6 shows the results with fourth graders and eighth graders.

Table 6. Results of analysis for thought levels

○ : Subjects with nearly-perfect understanding   × : The others

Thought Level	Problem Set Ⅰ	Ⅱ	Ⅲ	4th Grader (N=106) Paral.	Rhom.	Trape.	8th Graders (N=101) Paral.	Rhom.	Trape.
Level 2 ···	○	○	○	23%	14%	1%	69%	62%	36%
Level 1 ···	○	○	×	33	33	31	22	19	31
Level 0 ···	○	×	×	23	18	13	4	4	4
	×	×	×	8	13	32	1	2	10
	Sub-total			86	78	77	96	88	80
	○	×	○	6	10	3	2	5	3
	×	○	×	5	2	14	0	1	12
	×	○	○	2	2	1	1	3	4
	×	×	○	2	8	5	1	3	1

The following examines these results from two view points presented by Mayberry (1983, p.58): The first is an inversion phenomenon, and the second the consensus of levels between concepts. In order for thought levels to represent development levels, there must not be cases of inversion phenomena where, for example, a subject can solve level-2 problems, but still cannot solve level-1 problems. To examine the phenomena, a sub-total line has been inserted. Patterns above the sub-total line include no examples of inversion phenomena, while those below it represent subjects showing the phenomena. The subtotals represents the rate of the numbers of subjects showing no inversion phenomena.

The tables indicates that 77 % or more of children show no inversion phenomena with each of the geometrical figures for each class of school graders. No firm criterion, or minimum proportion of subjects without inversion phenomena has been established yet to determined if the thought levels represent the development levels. The above results, therefore, show that, by setting the criterion to 77 %, the thought levels can represent the development levels with each of figures and for each class of school graders. For that reason, one of the conclusions of this paper is that the concepts of basic quadrilaterals develop in accordance with thought levels presented by van Hiele.

The second view point, the consensus of levels between concepts, represents a condition, for the purpose of this paper, where a child is on the same level with respect to parallelogram, rhombus, and trapezoid. In the above table, the numbers of children at each of levels vary significantly by geometrical figures. For example, the numbers of children at the level of " ○○○ ", or level 2, vary significantly with respect to parallelogram, rhombus, and trapezoid. Judging from these facts, the consensus is not observed.

Those results indicates that the thought levels of the same child will vary according to different geometrical figures. Therefore, it should be understood that a child's thought level depends on concepts.

The results, however, do not necessarily negate generality in the development of thought levels. We think that it is possible that although the development level itself is at the level 2, the level with a paticular concept still remain lower because the subjects had yet learned it or learned it insufficiently.

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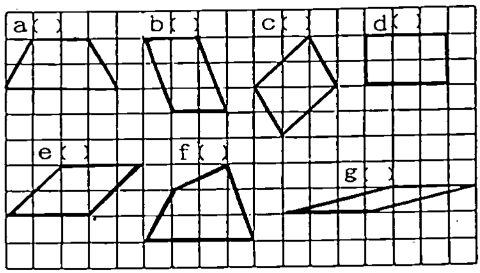
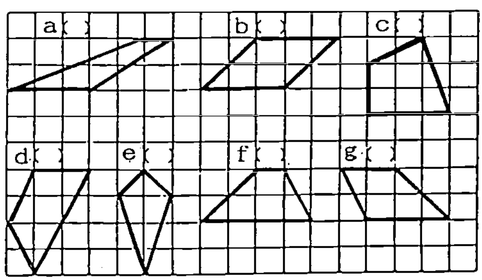
Appendix① RESEARCH PROBLEM SET I

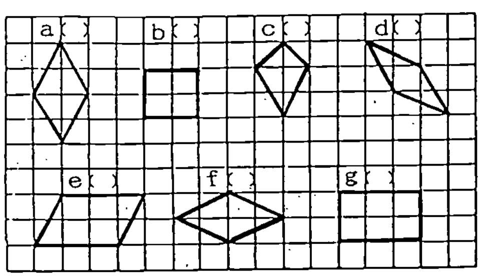
Q 1. Among the quadrilaterals shown in the figure below, enter a circle (○) in the parentheses ( ) if the figure belongs to the class parallelograms, and a cross(×) if not. If you are not sure, enter a triangle(△).



Q 2. Among the quadrilaterals shown in the figure below, enter a circle (○) in the parentheses ( ) if the figure belongs to the class trapezoids, and a cross(×) if not. If you are not sure, enter a triangle(△).



Q 3. Among the quadrilaterals shown in the figure below, enter a circle (○) in the parentheses ( ) if the figure belongs to the class rhombuses, and a cross(×) if not. If you are not sure, enter a triangle(△).



Appendix② RESEARCH PROBLEM SET II

Read the following sentences a. through f. in Q 1 ～ Q 3 and circle the letter if it is right, or cross if not.

Q 1. (Properties of a parallelogram)
  a. Both pairs of opposite sides of a parallelogram are same length.
  b. Both pairs of opposite sides of a parallelogram are parallel.
  c. Adjacent sides of a parallelogram are same length.
  d. Opposite angles of a parallelogram are the same.
  e. Adjacent angles of a parallelogram are the same.
  f. Some parallelograms have four sides of different lengths.

Q 2. (Properties of a trapezoid)
  a. Both pairs of opposite sides of a trapezoids are same length.
  b. One pairs of opposite sides of a trapezoids are parallel.
  c. Adjacent sides of a trapezoids are same length.
  d. Opposite angles of a trapezoids are the same.
  e. Adjacent angles of a trapezoids are the same.
  f. Some parallelograms have four sides of different lengths.

Q 3. (Properties of a rhombus)
  a. ～f. : The same kind of questions in Q 1

Appendix③ RESEARCH PROBLEM SET III

Read the following sentences a. through d. in Q 1 ～ Q 4 and circle the letter if it is right, or cross if not.

Q 1. (Rectangles and Quadrilaterals)
  a. All rectangles belong to quadrilaterals.
  b. No rectangle belongs to quadrilaterals.
  c. Only selected rectangles belong to quadrilaterals.
  d. Some rectangles do not belong to quadrilaterals.

Q 2. (Triangles and Isosceles Triangles) The same kind of questions in Q 1.
Q 3. (Parallelograms and Trapezoids) The same kind of questions in Q 1.

Q 4. (Parallelograms and Rhombuses)
  a. All parallelograms belong to rhombuses.
  b. No parallelogram belongs to rhombuses.
  c. Only selected parallelograms belong to rhombuses.
  d. Some parallelograms do not belong to rhombuses.

SPHERICAL GEOMETRY FOR PROSPECTIVE MIDDLE SCHOOL MATHEMATICS TEACHERS

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The purpose of this study was to investigate the viability was the implementation of a spherical geometry unit in a university geometry course designed for prospective middle school teachers. Using qualitative data collected from students enrolled in this course, instructor reflections, and participant observations, it was found that the proposed unit promoted novel and rich discussions during its development that allowed prospective middle school teachers to construct alternate views of mathematics learning environments.

Preamble

Currently, there have been many calls (National Council of Teachers of Mathematics [NCTM], 1989; NCTM, 1991; National Research Council [NRC], 1989) to implement changes in content and pedagogy in elementary and secondary school mathematics. These suggestions propose a shift from traditional practices to more student-centered activities.

As a result for these calls, the Mathematics Education Program at Florida State University is currently carrying out a National Science Foundation funded research project entitled "Development of Effective Mathematics Learning Environments and Tasks for Prospective Middle Grade Teachers." The main aim of this project is to plan and implement courses in mathematics and mathematics learning and teaching for prospective middle mathematics teachers. Providing opportunities for prospective teachers to construct discipline specific pedagogical knowledge is a priority of this project so that prospective middle school teachers will construct and adequate meaning of mathematical concepts as central to middle school mathematics (Jakubowski, Wheatley, & Erhlich, 1993).

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Course Development

An experimental course, "Elements of Geometry" has been developed as part of the activities in this project and it will be used as the focus of this report. This course was designed to provide opportunities to construct geometric patterns and relationships. At the same time these issues will be examined from a learning and pedagogical perspectives. Among the content of this course, one and perhaps the heart of this course, is a unit of non-Euclidean Geometry.

Students entering the geometry course do so with a rather uniform set of beliefs about what mathematics is. This set of beliefs, largely resulting from many years of mathematics courses, is well represented by the belief that mathematical activity is nothing but "applying" a fixed procedure to get the answer to a task that already existed or was developed by someone else. Contrastingly, one of the main goals in this course is for students to construct for themselves mathematical patterns and relationships. Thus tasks and learning environments have been created so that they will provide potential learning opportunities for the participants. With this goal in mind, the proposed primary instructional strategy of this course is problem centered- learning (Wheatley, 1991) and assessment procedures have been developed accordingly.

In an attempt to help students understand the role of geometry in the real world, a set of spherical geometry activities have been developed and field tested. For instance, Sullivan (1969) argues that concepts developed in mathematics program will be helpful in map reading because the study of coordinates could help students in learning about latitude and longitude. Congleton & Broome (1980) describe a geometry module designed for use at the high school level. This module included topics such as spherical geometry, the coordinate system used to describe points on the earth' surface, parallel and meridian sailing, and a solution of right spherical triangle problems. Van Den Brick (1993), in his workbook "Mecca," that deals mainly with spherical geometry on the globe, explains the subtleties inherent in designing a new topic (i.e., spherical geometry) and students' experiences when activities were conducted at school. Lénárt (1993) reports an experiment involving 400 students from middle school to college level, in Budapest, in teaching elements of spherical geometry, contrasted with concepts of plane geometry. He stressed the reflections and remarks of students and teachers involved in the experiment. Indeed, he claimed that the experience was successful and appropriate for students, including middle school students. Casey (1994) described a classroom activity where students were introduced to the curvature of surfaces using a wide variety of objects. He concluded that projects of this nature are appropriate to introduce students to fascinating geometrical phenomena exhibited by surfaces.

Overall, the above reports emphasize the viability of introducing students to spherical geometry activities. In fact, such activities rich potential learning opportunities for students and thus it seems worthwhile to introduce prospective middle school teachers to topics of spherical geometry which in turn will help them to become better prepared for teaching middle school students.

The Research

This research project is part of a larger study. Data reported in this paper was collected during the Fall Semester 1994 from one of the sections of the undergraduate course "Elements of Geometry" at Florida State University [the section used to collect the data for this report was an experimental course of the research project]. The main goal of this report was to investigate the success of the spherical geometry unit in a university geometry course designed for prospective middle school teachers. This research was conducted under a constructivist framework (von Glasersfeld, 1987). The assumptions guiding this research were essentially taken from an interpretive perspective (Erickson, 1986). Several techniques were used to collect data - classroom and participant observations, informal interviews, audio tapes, various students documents [of their solutions to some geometric tasks of the spherical unit], learning portfolios, and journals. Data collected were analyzed on a continuous basis throughout the semester and more in depth during the implementation of the spherical unit. This analysis provided rich descriptions of relationships between the students' world of geometry and the mathematics that emerged from the engagement of students in specific activities and/or during discussion time. Triangulation of data (Lincoln & Guba, 1985; Patton, 1990) was used to support assertions and assure their viability.

Most of the students in this course had little or no experience with practices informed by constructivist views. The instructional strategies included mathematical problem solving in small groups as well as whole class discussion. All participants were encouraged to share their views with the class to provide additional opportunities for students to test and/or reconstruct the viability of their mathematical constructions. Mathematical procedures and formulas were not emphasized in this course. In fact, the emphasis was placed on students constructions of conceptual or relational, rather than instrumental understanding (Skemp, 1978).

The spherical geometry unit was implemented over a period of five weeks with two sessions per week of an average of 75 minutes per session. When the unit was first introduced, students were given a set of tasks designed so that they have the potential of being problematic, this was guided by the main rationale in this course being that mathematics was to be viewed as a personal activity with opportunities for each person to construct their own mathematics (Wheatley, 1991). The class was divided into small groups of three members each. Each group was provided with a globe of the earth (the sphere) and a beach ball that they could use as a model of the earth. Students were provided with additional materials such as markers, string, scissors, and tape. The first task given to the groups was to draw on the beach ball the equator and the Greenwich meridian, longitude lines, and latitude lines. Although it seemed to be a simple task, students were actively engaged and the task itself generated rich opportunities for clarification of the concepts and characteristics of longitude and latitude. In addition, negotiation of conventional mathematical terms of longitude and latitude was ensured. There was no specification of strategies to be used in solving the tasks. In fact, students were encouraged to work in collaboration and become autonomous in their negotiation of social norms. They were encouraged to devise their own methods to solve the tasks and to share their findings during whole class discussion.

Subsequent sessions dealt with concepts of great circle, "straight" line, spherical triangle, and quadrilateral, among others. In the following section of this paper we will illustrate some of the findings using descriptions of students' experiences and reactions during the implementation of this unit.

Findings

For the purpose of this paper, the findings reported will illustrate the views of students that are representative of students' elaborated constructions that emerged from the spherical geometry unit.

One student commented the following, "it amazes me that everyone in this class is at least twenty years old and yet we are figuring out how to calculate longitude and latitude." Simple events developed in the classroom such as drawing longitude and latitude lines, identifying places such as Tallahassee on the globe and locating those places on the model of the earth (using the beach ball) certainly helped students to construct a deeper understanding of the concepts involved. Several students from this class shared the view that activities of this nature are extremely beneficial to the whole class, in the sense that the activities challenge them to integrate what they know whether it was learned from a formal setting or learned from their lived experiences. At the same time, they were given the opportunity to construct or reconstruct the world in which they live. Another student said, "this is the first time that I have thought about relating map reading to mathematics," while a different student added, " I think that with more exposure to these activities, I am steadily defining within myself several characteristics of what mathematics is." These are few of the comments that students made when they began to solve the tasks of this unit. Overall students expressed both surprise and satisfaction of what they have learned by the end of the spherical unit.

All students from this class "understood" the concept of a triangle, however when they where asked to define a triangle on the sphere, most of them had no alternatives at the beginning (other than the traditional definition of a triangle). After a long process of negotiation in their small groups, students came to a consensus that a spherical triangle is formed by the intersection of three great circles. Some students in their effort to make sense of a triangle on the sphere cut pieces of strings and taped them on the sphere. Later on they took out the constructed triangle and laid it down on a flat surface. To their surprise they found out that the sum of the angles of the spherical triangle was greater than 180 degrees. They could hardly believe that a triangle on the sphere could actually have that characteristic. In fact, one group characterized spherical triangles in the following manner: "Spherical triangles are those which are formed by the intersection of three great circles (as opposed to the intersection of three line segments on a flat surface), each angle of the triangle is formed by the intersection of two great circles, a triangle can have one, two, or even three right angles, and the sum of the internal angles is greater than 180.°" A different group made sense of an equilateral triangle by relating it to problem (2) in appendix A, where the distances traveled defined perfectly an equilateral triangle since they were asked to travel from the North Pole 25% of the way around the globe and this direction was given three times and every time they were asked to turn 90° (see appendix A).

Another relevant concept that was discussed was the negotiation of a quadrilateral on a spherical surface. Students' characterizations of these quadrilaterals were as follows: "A spherical quadrilateral is constructed by the intersection of four great circles at four different points (that represents the vertices of the quadrilateral);" "A spherical quadrilateral uses curved lines (as opposed to straight lines in Euclidean geometry) and the sum of its internal angles is greater than 360;° " "A quadrilateral on a spherical surface with the length sides equal was not a square, but a parallelogram;" "A quadrilateral on a spherical surface with all angles equal to 90° was not a square, but a rectangle." The previous conceptualizations that students constructed are flowed, however they indicate ways in which students made sense of spherical issues.

Since the assessment for this class relied heavily on a learning portfolio, students had to reflect on what they learned with each unit. This event lead some students to develop activities for middle school students about spherical geometry issues. For example, one student created a problem-solving task where students needed to work in small teams to develop a plan for a vacation. This activity involved some spherical geometry concepts and so students would have the need to resolve similar issues in order to develop their plan.

The data analysis suggests that spherical geometry units are appealing and have a great potential for learning geometry in an innovative manner.

Conclusions

The spherical geometry unit provided opportunities for prospective middle school teachers to make sense of geometric concepts in an interesting and captivating way. Overall, the activities implemented during the development of this unit promoted active engagement of students enrolled in this course. They valued the fact that they were given the autonomy not only to make sense of the mathematical concepts entangled but also they were given freedom to make sense of mathematical concepts and to develop their own methods to deal with particular tasks. Indeed, during discussion time students even commented on the usefulness and appropriateness of these tasks for middle school students.

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APPENDIX A

Sample Tasks of Spherical Geometry Unit

1. Given that the circumference of the earth is 25,000 miles,
   a. Estimate the distance from New York City to San Francisco.
   b. Estimate the distance from Tallahassee to Moscow.
   c. Estimate all points that are exactly 3000 miles from Boston.

2. Begin at the North Pole. Travel 25% of the way around the globe, turn 90° to the right, travel another 25% of the way around the globe, turn 90° to the right again, and finally travel 25% around the globe. Where are you?

3. Begin at (lat, lon) = (85, 0), heading due West. Travel in a straight line 25% around the globe. Determine your final heading.

4. Investigate the relationship between the side lenghts and the angles for the following (on a sphere):
   a. Equilateral triangles.
   b. Isosceles, right triangles.
   c. Regular polygons having four sides.

SPATIAL PATTERNING: A PILOT STUDY OF PATTERN FORMATION AND GENERALISATION

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The purpose of this paper is to report the outcomes of a pilot study of a larger project designed to explore the types and levels of cognitive functioning underlying the conceptual development of spatial patterns and relationships. It focuses specifically on pattern formation and generalisation and links these to the SOLO Taxonomy (Biggs and Collis, 1982, 1991). Observation of students' responses to three tasks suggests a unistructural-multistructural-relational cycle in their attempts to recognise generalisations from their representations of the patterns. The study also reflects a preference for students to model or draw external representations of the pattern as a basis for making generalisations.

Expressing generality from patterns is a notion fundamental to the development of mathematical concepts. It is, for example, one of the three subheadings in the algebra section of the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) and is also an important component of the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989). The National Statement on Mathematics for Australian Schools recommends that children "work with a variety of numerical and spatial patterns, and find ways of expressing the generality inherent in them....leading children to recognise that different descriptions can fit the same spatial arrangements" (p.191). It is important to explore the most effective ways of implementing these ideas in the classroom at all levels. The focus of the larger project of which this study is a part, on the conceptual development of spatial patterns, was chosen because of the acknowledged importance of spatial thinking in its own right as well as its powerful contribution to mathematical thinking in general (Lean & Clements, 1981, Bishop, 1983, Australian Education Council, 1991).

Several writers (Bishop, 1983, Presmeg, 1992, Australian Education Council, 1991, Thomas and Mulligan, 1994) have acknowledged the importance of encouraging students to use visual processing in order to succeed at mathematical tasks, and this is particularly true of spatial patterning. However, there is evidence that some children have difficulties with visual processing (Bishop, 1983) and that there is a need to understand more about how it can be developed. Kosslyn (1983) contributes to this understanding by defining four stages of image processing: generating an image, inspecting an image to answer questions about it; transforming and operating on an image; and maintaining an image in the service of other mental operations. This is reflected in the National Statement on Mathematics for Australian Schools, which claims that, to be able to represent a pattern internally, children first need to be able to see it, then find ways to express it verbally.

The pilot study reported here was concerned with finding out more about the first of Kosslyn's stages, that is how children go about generating an image in order to be able to see the pattern. This is a critical step. In fact, Resnick and Ford (1981) suggest that "the important intellectual work is over once a representation has been developed" (p.220). In particular, this part of the study was concerned with the kinds of external representations children may need to create in order to be able to transfer to a mental representation. To establish whether some forms of representation lead to more successful outcomes than others, required an understanding of the contributions of previous research regarding children's approaches to processing mathematical information.

There is evidence that successful mathematicians do not necessarily all use the same modes for processing information (Krutetskii, 1976, Shama and Dreyfus, 1994). The modes they use can include verbal-logical and visual-pictorial (Krutetskii, 1976), physical/kinaesthetic, ikonic or notational forms, or various combinations of these (Gardner, 1983, Thomas and Mulligan, 1994). While Mayer & Sims (1994) found that some students do not need visual prompts because they can generate their own representations, others have reported the manipulation of materials (Owens, 1994), drawing diagrams (Resnick and Ford, 1981), or a combination of these (Bishop, 1983) to be important in establishing internal representations and extracting meanings. Krutetskii (1976) claimed that students can be equally successful at mathematics with different correlations between visual-pictorial and verbal-logical components. Watson, Collis and Campbell (1994) comment on the need for all of these forms to be used to support instruction in the early high school years.

In spite of this knowledge, there is evidence of a mis-match between students' preferred methods of processing information and the way in which the information is presented to them (Resnick, 1992). Resnick suggests that this may be due to failure to encourage children to build on to their already established, intuitive ideas about mathematics. The contributions of Biggs and Collis (1991) and others (for example, Campbell, Watson and Collis, 1992, Collis, Watson and Campbell, 1993, Watson, Campbell and Collis, 1993, Watson et al., 1994) explore this notion of multimodal functioning, particularly in relation to the ikonic and concrete symbolic modes of the SOLO Taxonomy (Biggs and Collis, 1982).

If we are to find out more about how children process information about spatial patterns, it is important to investigate these individual differences in the use of information processing (Presmeg, 1992). Two questions arise within the context of this project.

(i) Is there an observable progression in children's ability to recognise generalisations from their representations of spatial patterns?

(ii) What kinds of external representations do children make of patterns, and what intuitive or concrete symbolic processes do they apply to these representations, in order to form generalisations?

In exploring these questions, we seek to investigate the interaction between development of children's external processing of generalisations from spatial patterns and the development of intuitive and concrete symbolic thinking in relation to this. The most suitable model for doing this is the SOLO Taxonomy (Biggs and Collis, 1982, 1991), because this has been adapted "in response to recent structuralist evidence identifying a multiplicity of intelligences" (Campbell et al., 1992, p.279).

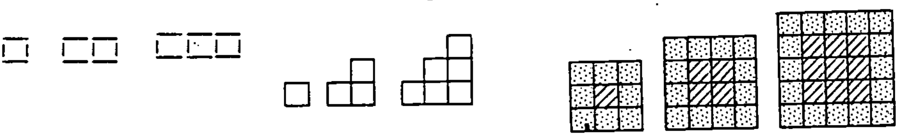
Like Watson et al. (1994), the focus in this study was on school-aged children. As a consequence it concentrated mainly on two of the four modes of functioning identified by Biggs and Collis: the ikonic and concrete symbolic modes. It will also explore the existence of a unistructural-multistructural-relational cycle within these modes (Campbell et al., 1992).

Tasks

Three tasks were selected in which students were asked to express generalisations from patterns. These tasks were chosen for the following reasons: they are spatial in nature, they are suitable for representation in different formats, namely physical/kinaesthetic, visual-pictorial, and verbal-logical representation, and they are typical of patterning tasks recommended in documents such as the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) and Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989). The three patterns are represented, in pictorial form, in Figure 1.

Figure 1: Tasks

Task 1: The Match Pattern   Task 2: The Step Pattern   Task 3: The Path Pattern



To give students the opportunity to respond in either an ikonic or a concrete symbolic mode, or a combination of both, and to represent the patterns in the most suitable way for this, the tasks were presented in three different formats. These formats are consistent with the multiple modes of processing suggested by other researchers and described earlier in this paper:

(i) concrete modelling, in which a representation of the pattern was made from blocks or other materials, to cater for the kinaesthetic learners,
(ii) diagram, to cater for the visual-pictorial learners,
(iii) word description, presented both verbally and in writing, to cater for the verbal-logical learners.

In order to ensure that the format in which the experimenter presented the task was not likely to influence the student's form of representing it, the tasks were presented in a cyclic rotation of formats, as shown in Table 1. To ensure an even distribution of the three modes across the first, second and third tasks respectively, the sequence in which the tasks were presented was held constant.

Table 1: Formats in which tasks were presented

Tasks			
Match	Step	Path	
concrete	diagram	verbal	Students 1, 4, 7, 10 etc.
diagram	verbal	concrete	Students 2, 5, 8, 11 etc.
verbal	concrete	diagram	Students 3, 6, 9, 12 etc.

Procedure

The sample consisted of 40 Year 7 students, in their first year of secondary school and aged 12 and 13 years. There were equal numbers of males and females. The students were selected randomly from several classes from a population described by their teachers as being of average ability. Data were collected in individual clinical interviews, each of approximately twenty to thirty minutes duration. Observation and Teachback (Pask 1976) strategies were used to monitor the students' responses and interviews were tape-recorded for later analysis. The students were given the tasks one at a time. Students were shown the patterns and asked to identify the fifth, tenth and 100th terms in the sequence (Orton and Orton,1994). They were given a selection of materials, including matches, blocks, squared paper and blank paper, and told that they could represent any steps of the patterns in whatever way they chose. They were then asked to describe the pattern and a generalisation for "any term". It has been reported elsewhere (Robertson and Taplin, 1994) that the most frequently chosen format of representation was concrete modelling. This was irrespective of the format in which the task was presented. Robertson and Taplin (1994) also reported that the main reason given by the students for this preference was that it gave a physical picture of the pattern that was quick and easy to construct.

Results

Table 2 summarises the formats in which the students represented the match task, and their attempts at forming generalisations. Because the study reported here was a pilot investigation, the consequent small number of students in each cell necessitates that the analyses be descriptive. The match task has been chosen for discussion here because the patterns represented in Table 2 are very similar to those formed from students' responses to the other two tasks. Details of responses to these latter tasks are available from the authors.

Responses of two of the children, who were unable to generate correct rules for the pattern, indicated only a pre-structural understanding. One child, for example, could only describe the tenth term of the sequence as "a long straight line". Three children counted the number of squares, which matched directly to the step number, and ignored the number of matches in the pattern. Ten children relied on constructing the sequence to the tenth term and directly counting the matches. Four others chose to count on in multiples of four, since they assumed that each square had four sides. None of the children who used these strategies chose to continue with this "counting on" method to the hundredth term and, understandably, none could predict a rule for "any" term. The "counting on" group of strategies has been classified as unistructural because each of the variations uses only one relevant aspect of the mode, namely the number of blocks or matches they could see.

A more sophisticated strategy was to look for patterns. The simplest of these, most likely based on the children's previous experiences, was to construct the fifth step and then use multiples of it. The justification behind this was that, since 10 is 5 doubled, the tenth term must be double the fifth term. Four children used this strategy. The second strategy in this category was slightly more sophisticated. Seven children recognised that the first square contained four matches and that each subsequent one required only an additional three. These children were, however, only able to reach the tenth or hundredth terms by adding on from the first term. Although there is a clear progression in the sophistication of the responses in this category, they have both been classified as multistructural because several disjoint aspects are processed in some kind of sequence.

Ten students were able to form generalisations, although nobody was able to express them in algebraic terms. It is interesting to note that these were not all correct, despite the fact that the children were demonstrating quite a sophisticated level of reasoning. Two explanations came close to predicting general terms. One of these, *"The amount of how many squares, times 3 and add 4"* actually predicted the (N+1)th term rather than the Nth. Another explanation, *"There are four ends to a square so you have to times it by the number for the step and then take away that number"*, gave the Nth term + 1. An example of a more accurate explanation was, *"Take 1 off the number and times that by 3 than add the 4 at the beginning"*. The eight students who used this latter type of strategy were not only able to predict the Nth term correctly, they were also able to test their predictions by calculating specific examples. These responses have all been classified as relational because they reflect an integrated understanding of the relationships between the different aspects of the task.

While this analysis deals only with one phase in the process of generalising from patterns, namely pattern recognition, it is clear that there is a progression in sophistication of the students' responses. This progression is summarised in Table 2. The majority of students used either "counting on" or "looking for patterns" strategies and were unable to obtain successful generalisations using these. Table 2 also summarises the numbers of students who represented the pattern in various ways. For example, at the unistructural level, seven students needed to make a physical model of, and four students needed to draw a diagram of, at least the fifth step before they were able to recognise and work from a pattern. At this level, four students gave a verbal-logical response rather than making any external representation of the task. Even as the responses became more sophisticated, at least some students still chose to model the sequence in order to explore the rules, rather than to use diagrams or verbal descriptions.

Table 2: Summary of students' responses to match task

Type of Response	SOLO Classification		No. Students	Way in which Ss represented task		
				Model	Diagram	Verbal
no particular system	pre-structural	i	2	2	0	0
counting on:	unistructural	i	15	7	4	4
one-to-one matching: "every number you say there has to be that number [of squares] in the group" *counting matches in representation (could not do100th)* *counting on by 4s (ie 5th=5x4 10th=10x4) or 3s*		ii	2	0	1	1
looking for patterns which are logical to Ss, possibly based on previous experiences: *based on multiples of 5th step (ie 5th has 16, 10th has 16x2,100th has 10thx10)* *starting at 4 and adding 3 for each step but adding from start each time (i.e. 4+3+3+3...)*	multistructural	i	11	5	3	3
recognition of a generalisation (not expressed algebraically):	relational	i	2	2	0	0
3N+1 *3N+4* *4N-N* *3(N-1)+4*		ii	8*	6	1	1

i Ss who responded at this level and had incorrect generalisation
ii Ss who responded at this level and had correct generalisation
* two of these Ss could successfully predict any step number, but would not articulate the rule

Discussion

In response to the research questions outlined earlier in this paper, the following observations can be made. The results can only be interpreted for the small grade 7 sample and the three spatial patterning tasks used in this study. There does, however, seem to be an observable progression in the children's ability to recognise generalisations from their representations of spatial patterns, which fits the SOLO model. This sequence can be summarised as follows:

Unistructural: counting on, mostly from external representations of the pattern; counting in either ones or a multiple of some number suggested by the pattern.
Multistructural: recognition of patterns and use of these as a basis for finding specific terms in the pattern; starting from first term of pattern each time in order to calculate a given term.

Relational: recognition of patterns and use of these to predict any given term directly, without needing to start from first term; articulation of generalisation, but not in algebraic terms.

The unistructural type of response was efficient for calculating the fifth and tenth terms of the patterns, but students either lost patience or made arithmetic errors when trying to calculate bigger terms, such as the hundredth. It was not possible for the students operating at this level to make generalisations about the patterns. Responses at the multistructural level offered more efficient systems for calculating bigger terms, but most students were still unable to generalise using these approaches. The relational level responses allowed a more efficient system for generalising. However, several of the children responding at this level gave incorrect formulae and did not seem to have systems for checking the validity of these formulae.

The above suggests a unistructural-multistructural-relational cycle (Campbell et al., 1992) at this particular grade level. Responses at the unistructural and multistructural levels are consistent with Biggs and Collis' (1982, 1991) description of the ikonic mode, with some concrete symbolic support. The students drew on some concrete symbolic experiences with counting and patterning, but used intuitive strategies to try to make this previous knowledge fit the patterning tasks they were given. At the relational level there is some suggestion of transition to concrete symbolic mode. This warrants further investigation with older students to explore when and how the transition develops with this particular type of task.

At all levels of the unistructural-multistructural-relational cycle, more than two-thirds of the students chose to either model or draw at least the fifth step of the pattern before they moved to working from an internal representation. The design of this study did not allow for distinguishing between the functional level at which they chose to represent the task and the optimum level at which they were capable of representing it (Lambom and Fischer, 1988), Watson, Collis, Callingham and Moritz, 1994). Nevertheless, it could support the idea that teachers should be encouraging modelling and drawing (Campbell et al., 1992) as an important step towards efficient mental processing of the information. This question warrants further investigation with a larger sample. It also suggests the need to continue the study with older students to explore if, when and how they "transcend the concreteness of an image or diagram" (Collis et al., 1993, p.119). It is also important to explore the links between external and internal representations. Some data regarding this were collected in this study and their analysis will be the focus of another paper.

In addition, the findings of this study suggest some further implications for future research. One of these is the need to investigate the link between students' representations of spatial patterning in these structured algebraic tasks and problem solving in "real" space, such as the interpretation of maps, graphs and charts (Bishop, 1983). Another question that arises is the need to consider "non-mathematical variables, such as student motivation, work habits, teaching, and language performance which could contribute significantly to mathematical performance" (Lean & Clements, 1981, p.296).

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STUDENTS' IMAGES OF DECIMAL FRACTIONS

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This study explored the images held by 36 students of 10, 11, and 12 years which related to decimal fractions. Some of these images demonstrated an awareness that numerical quantity could be continuous, and that decimal fractions represented small quantities. Other images indicated that the students saw whole numbers as discrete, with nothing coming between 0 and 1, and decimal fractions as 'just a number' without a quantitative referent. Those students whose images helped them to generalize from a quantitative understanding of 0.1 to 0.01 were those who described decimal fractions as proportions, those who saw them as rational fractions, and those who used words that described small portions.

Students have more difficulty in understanding decimal fractions than might be expected from the logic of this part of the number system. The logic behind this notation is a downward extension of the place-value system of whole numbers. In this system a number in each place is one tenth the size of the the same number in the place to the right. Students' poor understanding of this system has been demonstrated by the errors that they make in ordering decimal fractions (e.g. Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989) and by their predictions of the result of operations that include decimal fractions (e.g. Brown, 1981; Wearne & Hiebert, 1988).

These errors indicate that students often deal with decimal fractions as though they were whole numbers, or that they deal solely with symbolic features of the notation rather than relating the decimal fractions to appropriate quantity. Put another way, many students lack number sense for decimal fractions.

If we are to improve the chances of students developing number sense for decimal fractions, it is important to know what concepts they bring to understanding this aspect of number so that teachers can help them construct understanding on this base. Relevant to the study reported here is the informal or intuitive knowledge that students bring to understanding of fractions as explored by Clements and Del Campo (1990) and Mack (1993).. One way of exploring students' understanding of a mathematical field is through asking about the images that they have. Following on from the work done by Presmeg in this field (e.g.1986) Reynolds & Wheatley (1992) claim that for mathematics to be meaningful to children it needs to be based on appropriate images.

The descriptions of images discussed in this paper were provided by students who took part in a larger study of understanding of decimal fractions. The students came from schools in multicultural, lower economic areas of Auckland. An earlier study had shown that students from these areas made less progress in understanding decimal fractions than did students from schools in more affluent areas of the city where residents were predominantly of European descent (Britt, Irwin, Ellis, & Ritchie 1993).

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Method

Thirty-six students were interviewed individually on several aspects of their understanding of decimal fractions. The students came from two multicutural schools in lower income areas, where students of European descent were in the minority. Past records showed that the students from these schools made slower than average progress in mathematics. Three boys and three girls were interviewed from each school at each of the ages of 10, 11, and 12 years (mean ages 10 years 5 months, 11 years 5 months and 12 years 7 months). The ethnic background of the children was Pacific Islander (Samoan, Niuean, Tongan, Cook Island Maori, and New Caledonian) - 56%, New Zealand Maori - 22%, European - 19%, and Indian - 3%. This was similar to the ethnic makeup of the school populations. All but one of these students had had the majority of their schooling in New Zealand and were fluent in English. The principals of the two schools said that the sample of children interviewed was representative of all students in their schools in mathematical achievement.

In the part of the interview reported here students were asked to shut their eyes, think about what came between zero and one, and tell the interviewer what picture they saw. This question was asked in an attempt to see if they could picture the quantity represented by number as continuous rather than discrete, and to see if they had any sense of decimal fractions as coming between zero and one. Alternatively their responses could have shown that the students referred to only symbolic features of these numbers, seeing them as symbols without referents. A similar procedure was used for discovering their visualization of "zero point one" and "zero point zero one". Both of these questions were asked to see if students had an appropriate quantitative concept of these decimal fractions or if they thought of them as either whole numbers or as symbols that did not have referents. After their first response to each of these questions, students were asked if they could think of any other picture.

Two other tasks which produced evidence of students' images are also referred to in brief. One was a task in which students were presented with a piece of paper with a large square on it, told that this represented a field, and asked to show how much one person would get if the field were divided among 10, 100 and 1000 people. Grids were provided to help them with this division if wanted. Responses to this task were compared to responses given to a different set of students who were asked to divide up a rectangular cake without the aid of grids. The other task which yielded some understanding of students' images of decimal fractions was one in which they were asked to find out, on a calculator, what 0.01 needed to be multiplied by to get the answer 1.

Results

Visualization of What Comes Between Zero and One

Student's responses to this request for visualization were categorized as shown in Table 1. Examples of the types of response in each category are given in the text.

Category of response	Age 10	Age 11	Age 12
• Number line	3	5	5
• Plausible representations other than number lines	2	0	0
• A symbol which could be put between 0 and 1	3	5	6
• Nothing comes between 0 & 1	4	2	1

Table 1. Number of students giving a response in each category when asked to think of what came between 0 and 1.

The first two categories indicate some quantitative understanding of either numbers or space coming between zero and one or that zero and one were part of a continuous scale. An interesting result here was that the proportion of students in each age group giving an answer in these categories (5 out of 12) did not differ for ages 10, 11 and 12. What did change across age was the way in which they described what, for this purpose, were misconceptions. Students aged 10 were more likely to say that nothing came between zero and one, appearing to be sure that these represented discrete steps, while older students were more likely to give an answer which related to symbolic or syntactic features of numerical representation which could be said to ignore both their discrete and continuous nature.

The one difference between students from the two different schools appeared on this task. All of the 12-year-old students who offered a number-line representation came from the same school, while students from the other school predominantly thought of a symbol as coming between zero and one. Although this could be a random effect, it could also have been the result of teaching. The students visualizing number lines came from three different classes in the same school, but were their teachers did some of their planning together.

Most of the responses scored as number lines were given in terms of fractions. The simplest of these was that 1/2 came between 0 and 1. The next level of complexity was to add 1/4 to this number line before 1/2. Of the 12-year-olds who drew number lines, three used decimal divisions, one used fractions, and one mixed fractions and decimals incorrectly, first writing

1/4 1/2 3/4

and then writing:

0 0.1 1/4 0.01 1/2 0.001 3/4 1

This confusion of how decimal fractions fit into the number line was also demonstrated by an 11-year-old who drew the following number line:

0, 1ths, 10ths, 100ths, 1000ths, 1

The 12-year-olds who drew number lines with decimal fractions all displayed an understanding of the placement of decimals fractions with a different number of decimal places. Examples were:

a) 0
0.1
0.2
0.3
0.4
0.5 < 0.42
0.5
0.6
0.7
0.8
0.9
1

b) 0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1
^
.01
.001

Figure 1: Number lines drawn by two 12-year-old students that indicated understanding of the placement of decimal fractions of different sizes

The following excerpts are from the transcripts of the two students who gave non-number line responses that indicated an understanding of something coming between zero and one:

Interviewer Next shut your eyes tight and I want you to think about everything that comes between zero and one. Can you get a picture of what comes between zero and one?
M Yes.
I Tell me what your picture is like.
M A baby that's not quite one, not newly born, it's about three months old.
I That's good, could you draw that in any way or write it down?
M Yip.
I Any other pictures in your mind?
M Some lollies that are only sixty five cents, not quite one dollar and they're not free.

I. Let's do something different. Shut your eyes and think of everything that comes between zero and one. See if you can get a picture. What comes between zero and one? Can you tell me?
H Um, I think it's um centimeters. There's a zero and then there's a, there's a zero and then there's a centimeter
I You write it down.
H See, there's a zeros here and then you start centimeters here and then it's one, oh,
[draws 0 cm 1 2 3 4 5 6 7 8]
I Ok, where do you see it like that?
H On a ruler

Of the students who drew symbols as coming between 0 and 1, three used operation signs (e.g. 0+1), and the rest said that the decimal point came between 0 and 1 (0.1). It was not possible to get an additional response from students who gave 0.1 as their visualization. That was all that they saw. There was a possibility that the response 'a decimal point' was influenced by the design of the interview, as all students had previously been asked what they knew about decimals and had been asked to put decimals of different magnitude in order by size.

The belief that nothing came between 0 and 1 was also apparent in the answers of several students in another portion of the interview when they were trying to guess what 0.01 needed to be multiplied by on the calculator to get the answer 1. These students said that of their various guesses, 0 was closer to 1 than was .01, .0001, or other small decimal fractions.

Visualization of Zero Point One

Students were next asked what picture they could get for zero point one, and their responses were categorized as shown in Table 2. Visualization of a quantity represented by 0.1 did increase with age. Answers that indicated some understanding of the quantity for 0.1 were given by two of the 10-years olds, six of 11-year-olds, and seven of the 12-year-old students. There was no difference between the schools on this task.

Category of response	Age 10	Age 11	Age 12
• Verbal indication of quantity e.g."a small bit"	0	0	4
• Fraction equivalent	1	6	2
• One part in 10	1	0	1
• Just a number / same as 1	6	1	1
• Don't know / no picture	4	3	3
• Other, apparently not accurate	0	2	1

Table 2. Number of children giving different categories of responses when asked to visualize 0.1.

The first three categories in this table indicated a sense of the quantity represented by 0.1. While four of the 12-year-olds gave the concept of quantity in verbal terms, such as "just a small bit", the 11-year-old students were more likely to give or draw the fraction equivalent. Two 11-year-old students drew pie diagrams and shaded in one of ten portions, but used the word "half" in their explanation to mean a piece. The 10-year-old student whose response was categorized as one part in ten said that his mother had taught him, and explained in detail how a chocolate bar could be divided amongst 10 people and 0.1 would be his bit. The portion of students who said that 0.1 was just a number decreased as the age groups increased. Those who did not have a quantitative concept for 0.1 had all been dealing with decimal notation in the earlier part of the interview and had been introduced to it in at least one context at school.

Visualization of Zero Point Zero One

The same categories used in Table 2 were appropriate for classifying students' responses to this item. Indicating what 0.01 meant was more difficult than giving a quantity for 0.1. The proportion of students of 10, 11 and 12 giving quantitative responses for 0.01 were 1 of 12, 6 of 12 and 3 of 12 respectively.

Category of response	Age 10	Age 11	Age 12
• Verbal indication of quantity e.g."a very small bit"	0	0	2
• Fraction equivalent, spoken or drawn	0	6	1
• One part in 100	1	0	0
• Somewhere between 0 & 1	1	0	1
• Just a number /same as 100	2	2	4
• Don't know / no picture	8	3	3
• Other, apparently not accurate	0	1	1

Table 3. Number of children giving different categories of response when asked to visualize 0.01.

It was surprising that more 11-year olds than 12-year-olds gave quantitative responses on this task. It appeared that the 12-year-olds had less confidence that their understanding of fractions was relevant to understanding a decimal in the hundredths. Several who had given a reasonable response for 0.1 indicated that they had no idea what 0.01 meant.

There were three groups of students who were able to generalize from their statement of what 0.1 meant to a meaning for 0.01. One was the 10-year-old taught by his mother who went from his proportional explanation of one part in 10 to an explanation one part in 100. Another was the group of 11-year-old students who explained decimals in terms of their fractional equivalents. The third group were 12-year-olds who intensified their verbal description of size, for example saying that 0.1 was a small amount, and 0.01 was "more smaller" (the student who was not fluent in English).

One student presented an interesting pair of responses for representations for 0.1 and 0.01, as shown below. While these responses were considered to indicate a surface understanding of these numbers rather than a quantitative one, they gave insight into the sense that he was trying to make of place value.

I *Up here we got an answer called zero point one, what does that mean to you? Can you get a picture for that?...*
G *That's the middle of the two, it goes one, two, three, four, five, six, seven, eight and like that, the other one that goes like one two, three, four, five, six, seven, eight,*
I *So write that down for me, so I can see what you mean.*
G (writes starting from the right of the decimal point:)
4 3 2 1 0 . 0 1 2 3 4
I *What about zero point zero one, what does that mean?*
G *The zero is the middle to zero and tens. So it would go, zero, it would go ten, twenty, thirty, forty, fifty, sixty that way and it would go ten, twenty, thirty, forty, fifty, sixty that way.*
K *Write that down. Write it over here.*
G (writes) 5040302010.1020304050

Other Representations

In a previous series of interviews, similar students of the ages of 8, 10, 12, and 14 had been asked to show how they would divide a rectangular cake so that 10 people could have the same amount each, and how one of those pieces could be further divided to provide enough for 10, making 100 pieces in all. All students of 10 and older could do this task. However, many of them took a considerable amount of time to do it because they started out by showing a rectangle cut in half, then halved again and again, giving eight pieces. They saw that this did not give the right number of pieces, and continued to work on their drawings until they finally found a way of making 10 pieces. Their initial attempts at division by halving suggest that this was their preferred method of division and initially interfered with their ability to make divisions by 10 as required for a number sense of decimal fractions. The 36 students in the later series of interviews were given a task that was thought to be similar but turned out not to be. In this task the students were asked to show how much of a field one person would get if it was divided among 10, 100, and 1000 people and offered a variety of grids to help with the task. They were less successful on this task than similar students had been on the freehand task. Two types of responses indicated their images of the result of division by ten, a concept closely allied to decimal fractions. One response shown by several students was representing a piece one tenth the size of the previous division by halving the last division. The division by 1000 was drawn as 1/2 the size of the division by 100 by 17%, 42% and 33% of the 10, 11 and 12-year-old students. Another observed response was that students preferred to draw all divisions as of similar shapes, either all long strips or all squares, although the simplest method using the grids provided would have been to use a strip for 10ths, a square for 100ths, and a strip for 1000ths. This would be the 2-dimensional representation closest to the representation in Dienes blocks. The fact that many students did not chose this representation is interesting in the light of the fact that all would have had extensive experience with Dienes blocks, but did not chose this method of representation for their own drawings.

Discussion

No teachers in multicultural classrooms share the culture of all their students. Teachers must make a special effort to understand the concepts that their students come with if they are to help them construct a quantitative understanding of decimal fractions, a point emphasized by Mack (1993). Teaching without reference to these concepts is likely to lead to some of the misconceptions that concern mathematics educators. The fact that fewer of the 12-year-old students than the 11-year-old students in the this study had a quantitative understanding of hundredths may have been the result of emphasizing the procedures rather than the meaning in teaching students to operate with decimals. Teachers with whom these results were discussed were not surprised to learn that some students did not think that anything came between zero and one or did not give a quantitative description of decimal fractions. However, few teachers had introduced this topic through exploring continuous quantity.

Advisers for this project had expected students to have a better understanding of decimal fractions written in place-value notation than they had of rational fractions, because of recent changes in the curriculum and the widespread use of calculators. This was not the case. Understanding of fractional division by powers of 2 was more widespread than understanding of divisions by 10. This repeated division by 2 is related to early concepts of fractions described by Clements & Del Campo (1990) and Hart (1981), and is reinforced by folding paper to show fractional parts. With this group of children it was also reinforced by a tendency to call parts of any size "a half". This was more common in the previous series of interviews than in the data reported here. This phenomenon is familiar to parents of young children who talk about "the bigger half". It is apparent in mature people of Pacific Island descent who use the word half, or its transliteration "afa", to mean a part of any size. This linguistic influence can mislead teachers and interviewers, but in the data reported here needs to be separated from the ability to draw an object divided into 10 equal portions, even if one of these portions is called a half.

These students showed some evidence that it was harder for them for understand hundredths than tenths. A similar result was found by Brown (1981) and Britt, Irwin, Ellis & Ritchie (1993). This may be the effect of teaching which emphasises discrete divisions into tenths rather than teaching place value for decimal fractions as an extension of whole number place value. This potential difficulty was overcome by those students who had a concept for the size of tenths which they could generalize to hundredths. The three types of successful generalization given in this report suggest concepts which a teacher could exploit to help their students build a meaningful understanding of decimal fractions.

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PREFERENCE FOR VISUAL METHODS: AN INTERNATIONAL STUDY

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There have been assertions in recent literature on visualization that students are reluctant to visualize when doing mathematics, particularly at the high school and college levels. While there is research evidence that in the classes of 'nonvisual' teachers even 'visual' students will suppress their preferred visual cognitive modes in favor of nonvisual methods used by their teachers, our data show that it is simplistic to claim that students are reluctant to visualize. In the present international study, the same instrument for measuring preference for visual methods in solving nonroutine mathematical problems was administered to students in three countries, South Africa, Sweden and the United States. Some results are analyzed here.

Are students reluctant to visualize when they do mathematics? This claim was made by Eisenberg and Dreyfus (1991), and in 1994 Eisenberg wrote, "A vast majority of students do not like thinking in terms of pictures - and their dislike is well documneted in the literature" (p. 110). He went on to cite calculus studies by Mundy, by Dick and by Vinner in which students showed "a definite bias toward an algebraic approach even when it was more difficult than the visual one" (p. 111). He also described Clements' study of Terence Tao, a mathematically precocious Australian who preferred not to visualize in doing mathematics, and used this study as evidence that "the tendency to avoid visualization exists even in the mathematically precocious" (ibid.).

We suggest that the studies cited by Eisenberg admit of other interpretations than the one he has given. While it is certainly the case that students of all preferences may avoid visualization when this has not been encouraged as 'good mathematical thinking' in the classroom (Presmeg, 1985), recent research by Wheatley and Brown (1994) shows convincingly that far from being reluctant to visualize, many students use their visualizations as a tool for meaning-making in mathematics. Presmeg's (1985) study suggests that much of the visualizing done by students is of a private nature: their imagery may not be apparent in written protocols.

Further, the study by Clements of Terence Tao's mathematical cognition is quite consistent with Krutetskii's (1976) model grounded in extensive case studies of many mathematically gifted students. Krutetskii described representatives of each of his categories or types of mathematical giftedness, which were based on students' ability to use, and preference for, visual methods. Students who have the ability to use visual methods may prefer not to, and the various combinations led to Krutetskii's classification into "analytic" and "geometric" categories, and two subtypes of "harmonic" in which verbal-logic and visual-pictorial components are in equilibrium (Krutetskii, 1976, chapter 16). Like Sonia L. ("abstract-harmonic") but unlike Volodya L. ("pictorial-harmonic"), Terence Tao would probably have been placed in the "harmonic" category based on his abilities - or in the "analytic" category which evidences "a weak development of the visual-pictorial component" (Krutetskii, 1976, p. 318). However, Krutetskii also found no difficulty in identifying gifted students with the strong visual-spatial abilities and preferences characteristic of his "geometric" category.

Further, in the research reported here we describe evidence from three countries that individual students vary greatly in their preferences for visualization in mathematics. Certainly, visualization may be downplayed or devalued in certain classrooms or systems - leading some students to believe that visualization is not mathematics - but apparently there is no dearth of visualizers 'out there'. Given the opportunity, these individuals *prefer* to visualize in mathematics.

The Mathematical Processing Instrument

Complete details of the development of the visuality preference instrument we used in our research are given in Presmeg (1985), where definitions of terms are also provided. Suffice it to say here that we take *a visual image* to be a mental construct depicting visual or spatial information, and *visualization* to be the process of constructing or using visual images, with or without diagrams, figures or graphics.

The Mathematical Processing Instrument (MPI) measures preference for visualization, rather than ability, because of the strong research evidence that students who are able to use visual methods may or may not prefer to do so (e.g., Krutetskii, 1976). The MPI was designed for use with grade 11 students and their mathematics teachers (in a system with 12 grades). A previous instrument by the same name, designed by Suwarsono (1982) for grade 7 Australian students, was not suitable for use with mathematics teachers, although 13 of his more difficult problems were retained. Distilled from more than 300 nonroutine problems, the remaining 11 problems were drawn from a problem bank of more than 100 problems each of which could be solved by visual and by nonvisual methods. No diagram was present in any of the problems, since this might pre-empt visual methods. As in Suwarsono's instrument, a test and questionnaire were used. After students or teachers had completed the problems in the sections designed for them (A and B for students, B and C for teachers), they were asked to select a solution from the three to six given for each problem, which was "close" to the solution they had used, or to check a box entitled "none of these" and to describe their method. After three fieldtests involving several hundred students and teachers, most "new" solutions were catered for. According to their individually preferred methods of solution, students and teachers each solved 24 problems, as set out in the following table.

	Number of problems	Designed for	Level of difficulty
Section A	6	students	easy
Section B	12	students & teachers	intermediate
Section C	6	teachers	difficult

Nonparametric statistics were used to establish construct validity and reliability in the initial three fieldtests,(in Cambridge, England, and in Durban, South Africa) and construct validity in Sweden and the U.S.A. Data from 342 South African students yielded a Spearman split-half reliability coefficient, adjusted by the Spearman-Brown formula, of 0.827. For construct validity, based on interviews with approximately 20% of the students in each country, randomly chosen, Spearman rank-order correlation coefficients between interview and questionnaire scores were as follows:

	South Africa	Sweden	U.S.A.
Spearman's r	0.67	0.71	0.71
Total N	342	106	74

Scoring of the MPI was done by assigning 2 points for a visual solution, 0 for a nonvisual solution, and 1 point (occasionally needed) if the student checked "none of these" or if the item was omitted. Thus for a total of 18 problems, the possible Mathematical Visuality (MV) score was 36.

Sample Items from the MPI

For each of the 24 problems in the test, from three to six different solutions were given, worked out in detail, in the questionnaire. The following are sample items from the test, one from each section. (The number of solutions in the questionnaire for that item is given in parentheses.)

A-2 (6 solutions). Altogether there are eight tables in a house. Some of them have four legs and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs?

B-10 (4 solutions). If you place a cheese on a pan of a scale and three quarters of a cheese and a three quarter kilogram weight on the other, the pans balance. How much does a cheese weigh?

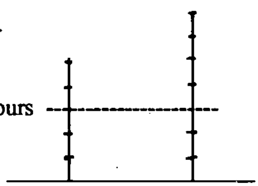
C-5 (6 solutions). Two candles have different lengths and thicknesses. The long one can burn three and a half hours, the short one five hours. After burning for two hours, the candles are equal in length. What was the ratio of the short candle's height to the long candle's height originally?

<u>Typical nonvisual and visual solutions to problem C-5</u>

Nonvisual. I reasoned from the data given. After two hours, fraction of tall candle used up was four-sevenths; thus three-sevenths remained. At this time, fraction of short candle used up was two-fifths; thus three-fifths remained. But these heights were equal. Thus three-sevenths of the length of the tall candle equals three-fifths of the length of the short candle. Thus the required ratio is 5 : 7.

Visual.

After two hours

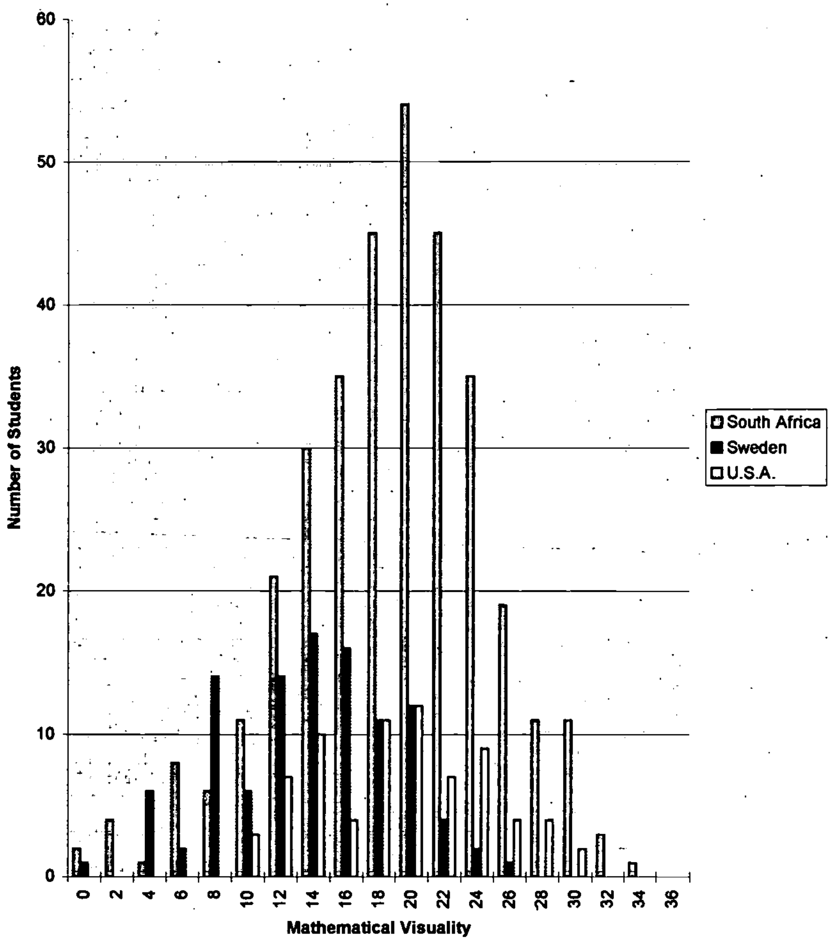


It can be seen from the diagram that the required ratio is 5 : 7.

Data from Three Countries

As an economical way of displaying our data, we include the following chart which shows the frequency distributions of scores from the MPI in the three countries. In Sweden a Swedish translation was used. All students were in grade 11 or equivalent at the time of the administration of the instrument.

FREQUENCY DISTRIBUTION OF MATHEMATICAL VISUALITY SCORES



Comparison and Discussion

While there are certainly visualizers amongst our mathematics students in all three countries, i.e., students who are *not* reluctant to visualize, there are some intriguing patterns in the international comparison which call for qualitative research to explore issues such as those raised by Eisenberg. The South African study of visualization was qualitative and involved in-depth interviews with students and teachers, and observation in mathematics classrooms over an extended period. This component was missing (or rather, has not yet been carried out) in the other two countries.

Of a possible mathematical visuality (MV) score of 36, the median scores of students in South Africa, Sweden and the U.S.A. were respectively 18, 14, and 20. In the South African data, there was no significant difference between median scores of boys and girls (20 and 18 respectively). However, the Swedish boys and girls had median scores of 12 and 16 respectively, reversing the literature prediction of higher visuality scores for boys, which we found in the American data in which boys and girls had median scores of 22 and 16 respectively - a significant difference according to the median test ($\chi^2 = 4.849$, df = 1, signif. $P<0.025$). The median scores are summarized as follows. (Numbers of students are in parentheses.)

Medians	South Africa	Sweden	U.S.A.
Boys	20(217)	12(66)	22(40)
Girls	18(125)	16(40)	16(33)
All	18(342)	14(106)	20(73)

Without qualitative research to suggest possible explanations for these international differences, we can but guess that the much lower median MV score for the Swedish students may have something to do with the fact that they are enrolled at a Science and Technology school, while the South African and American schools were of general intake. Qualitative research in Swedish schools could shed light on this issue. In the South African project, the visuality instrument (MPI) was effective for its purpose of choosing students and teachers for the study, but some of the richness of individual visualization, and student-teacher interaction in this regard, was fathomed in the qualitative part of the research.

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VISUALIZATION AS A RELATION OF IMAGES

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The purpose of this research was to construct explanations of students' learning dynamics that involve the use of imagery. Participants of this research are undergraduate mathematics education majors. This paper focuses on how participants used images and how they used relationships amongst these images (defined in this paper as visualization) while making sense of new geometrical tasks.

Background of the Study

In an effort to become more knowledgeable about how to build on the strengths of students' existing imagery and spatial visualization, for the ultimate purpose of improving both research and instruction in this area, this research was designed. The value of this research is based in the need to learn about specific cognitive constructions that students make at different levels while they are learning geometry. The participants in this study were a group of students enrolled in an informal geometry course for undergraduate mathematics education majors.

The purpose of the research is to construct explanations of situations where students are engaged in geometrical activities that were specifically designed by one of the researchers to facilitate the use of relation of images. However, participating students could approach the tasks in a manner consistent with their own sense making, whether this involved visualization or not.

Significant research has been reported in the area of geometry, and, more specifically, in the area of visualization and imagery (Shepard, 1978; Bishop, 1980b; Presmeg, 1986a; Presmeg, 1986b, Brown and Wheatley, 1989; and Wheatley, Brown, and Solano, 1994), however, the present study focuses on students' processes of sense making during engagement in geometrical tasks.

Theoretical Framework

Current emphasis in mathematics education is being placed more and more on student experience, analytical thinking, and creativity (National Council of Teachers of Mathematics [NCTM], 1989; National Research Council [NRC], 1989; Everybody Counts, 1989). Many of these recommendations can best be understood from a constructivist perspective. And so, in the conduct of this research a constructivist framework (von Glasersfeld, 1987) was used. From this perspective, mathematics should be viewed not as a body of knowledge but as a construction of knowledge.

Various constructivist proponents embrace the belief that knowledge construction is a personal event that results from the interaction of the individual with objects or phenomena occurring in a social context (von Glasersfel, 1989; Wheatley, 1991). This framework helped the researchers to make decisions about the nature of activities used in data collection, how those activities were conducted, and the nature of explanations constructed during data interpretation.

Another important part of the framework was a set of definitions, negotiated between the investigators, that are defined as follows,

Image: Image was defined as a mental construction of an object created by the mind through the use of one or more senses, where the mind plays an active role (i.e., rotating, translating, and transforming the image). Viewing an image in this manner, the mind is the protagonist, playing the principal role. Indeed when we say "picture in the mind," we may not think of the mind as having an active role for the uses of images as mental symbols of reality that can be used to solve a problem. Rather we may think of "picture in the mind" as a "fixed image" placed in the mind, nothing more - a view of imagery which we wished to avoid.

Visualization: Although visualization is often referred to as the ability to mentally rotate, manipulate, slide, and transform an object (Shepard, 1978), in this research, visualization was defined as the relationship among images. We claim that the process of establishing these relationships is what we would call logical thinking. In other words, in order to visualize there is a need to create many images (i.e., more than a prototype image (Lakoff, 1987)) to construct relationships that will facilitate visualization.and reasoning.

Imagery: Imagery was defined as a collection of one of more images. The power of imagery is that it may result in visualization that would help students create links which facilitate meaning-making in learning geometry, as our data illustrate.

Methodology and Procedures

The emphasis of the research is on how students create imagery and how they use it in the process of making sense of geometrical situations, thus a qualitative research methodology is viable, which precisely highlights the process rather than isolated events or happenings. In fact, the qualitative emphasis on process has been particularly beneficial in educational research (Bogdan and Biklen, 1992), which supports our choice of methodology.

The assumptions guiding this research were essentially taken from an interpretive perspective (Erickson, 1986). Various techniques were used to collect data. The main strategies included participant observations, and video recordings of participant's building of three dimensional solids, and of formal and informal interviews. Two groups were given voice in this research - participant students and the researchers. Four students from the course MAE 4816 Elements of Geometry were selected with their willingness to participate in the research. The main criteria for selection of participants were based on a preliminary analysis of the scores of the instruments that were administered at the beginning of the semester to this class. These instruments were: the Mathematical Processing Instrument [MPI] developed by Presmeg in 1985, The Wheatley Spatial Ability Test [WSAT], and the Visualization of Rotations Test [VRT] designed by Guay in 1976. At the beginning of the semester these instruments were proctored by the researchers, one of whom was the instructor of the class.

Each participant student was interviewed individually four times throughout the semester with each session video recorded. In order to gain information about students' process of sense making while engaged in geometrical situations, two instruments were given to the students during the interviews. Students were asked to talk aloud as they were solving the tasks. In addition, both the researchers and the participant had the opportunity to interact and ask for either clarifications or explanations during the interview, as needed. These two instruments, Dynamic Imagery Instrument A and Dynamic Imagery Instrument B were developed by Solano in 1994, as part of the research activities. These instruments were designed with the intention of providing rich opportunities for the use of imagery and visualization.

Findings

In this paper, only a limited set of explanations will be described. Marc, one of the participants, will be the focus of the findings reported, but when necessary, one other participant's elaborations will be provided to support or shed light on the findings reported. Reported findings represent only a small part of the analysis and interpretation, thus in this paper only data using the first instrument (Dynamic Imagery Instrument A) are used as evidence to support our interpretations..

For the following explanations, the reader will need to refer to Instrument A (see Appendix A). When Marc attempted to solve the first task (on Instrument A) he saw the problem and immediately responded, "it is one half of the square because these are two [talking about the two shaded regions] halves of the middle triangle" that he described as an isosceles triangle. To Marc this isosceles triangle was formed by the two black triangles since E is the midpoint of side AB, therefore Δ AED and Δ BEC are congruent and when put together we get Δ DEC. A different participant, Anna, made sense of the same task in a different manner but she evidenced once more that the use of imagery and the relationships constructed among these images were not only of high quality but also they made possible her success in solving the task. Her rationale was " if I draw an altitude for Δ DEC, then each of the triangles Δ DAE and Δ CBE are one half of half of the square, thus the shaded region must be ½ of the area of the square."

The main basis for the explanations that Marc constructed about the tasks 2, 3, 4, 5, 6, and 7, was an image of a diamond (that he introduced to help him making sense of the tasks). He used subsequent images along with this one plus relationships amongst these images to make sense of new tasks. The role of imagery in this case was a dynamic one. In addition, his visualization was powerful during the process of making sense of a new task. Following are a few examples to support this assertion.

In the second task Marc identified a shape he called "a diamond," and said, "it is also one half, since we have a diamond in the middle, another diamond if we put together the half to the left and right sides of the square, so we have two unshaded diamonds plus two more formed if we put together the shaded triangles with sides AE with DG and EB with GC." It was an interesting construction that Marc made, his use of imagery was very dynamic and the transformations he made were quite sophisticated. He added, " since I have four congruent diamonds, two shaded and two unshaded, therefore the area of the shaded ones is ½ of the area of the square." Marc used his knowledge from task number 2 to inform his construction on task number 3 , thus his answer to that task was ¼ of the area of the square. When he saw task 4, he immediately turned the page sidewise, since he already had seen the previous picture from which he knew that a diamond had an area of ¼ of the area of the square. Similarly, he made sense of tasks 5, 6, and 7.

Contrasting with Marc's approach to task 2, 3, 4, 5, 6, and 7, Anna approached the same tasks using her knowledge from task 1. The relationships of images continued being the focus for further constructions [and so her constructions were somehow knotted one into the next]. For instance, she transformed the square given in task 2 into four congruent squares and thus she came up with a representation of four squares similar to the one in task 1. The shaded regions were the opposite of the ones in task 1; however that did not interfere with her sense making of task 2. In fact, she said, "in each square I have one half of a fourth [meaning that each little square she constructed in task 2 has an area of ¼ of the total square] shaded, therefore my overall area in the whole square (of task 2) is 4/8, which is one half of the area of the big square." Similar explanations were provided by Anna in tasks 3 through 12. Marc and Anna had used rich images and relationships among these images to make sense of geometrical situations, however the images used by each of them were different in nature. But that event did not stop them from being successful in their solutions.

A different process was followed by Marc while making sense of task number 8. The first action he took was to draw a line parallel to the base of the square (DC) through the point F, and he said, " this is more than one fourth and less than one half" [referring to half of the square]. He explained, "because if you take this triangle [pointing at the one shaded above the line he draw] and slide it down to the Δ FCG you won't have half." He was quiet for few seconds and proceeded, "but this is one fourth" [meaning the right lower corner]. After that he realized that the other triangle shaded [the one next to the shaded square (lower right corner)] was actually one half of the fourth (1/8) so, he said, "it is 3/8 of the total area of the square." His approach to solving this problem proved to be a rich opportunity for Marc to use and to relate images.

Somehow surprising was the following finding. Marc had previously constructed [in problems 2 through 7] that a diamond has an area of one fourth. Also in problem 8 he learned that each corner (formed by triangles similar to Δ FCG) has an area of 1/8, however in problem 16 he was not able to use visualization as "relation of images," thus the task became more perplexing. Instead of visualization, he chose an algebraic approach which eventually misled him, so instead of getting ¼ his answer was 7/18. In fact, he even looked proud when explaining his algebraic thinking process.to get an answer, even though he had used dynamic transformations of imagery previously with great success. We also found that Marc was more relaxed and confident of his solutions when relying on visualization.

A rather interesting observation of the researchers was that Anna's consistency of elaborating on the previous tasks to help her make sense of the following one was very useful for her when she reached task 16. The same can not be said of Marc.

Conclusion

The analysis conducted in this research suggests that imagery and visualization as defined and described in this paper play a vital role in the processes used in solving mathematical problems. Individuals can profit greatly if they are given opportunities to form meanings that assist them in making sense of their mathematics. Based on our evidence we conclude that visualization is an important component of this meaning-making.

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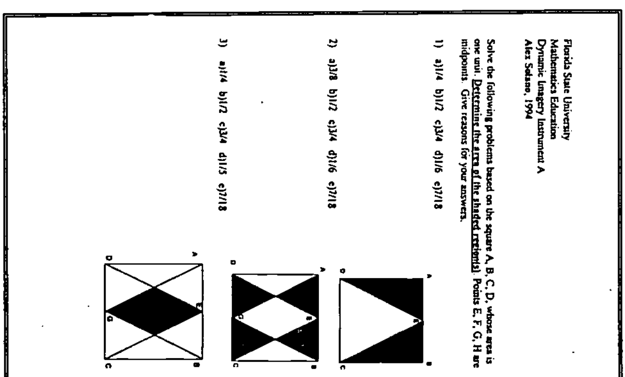
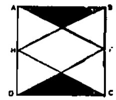
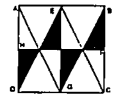
Appendix A



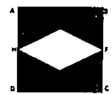
4) a)'/4 b)1/2 c)3/5 d)1/6 e)7/18



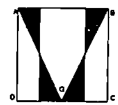
5) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18



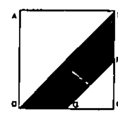
6) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18



7) a)1/4 b)1/5 c)3/4 d)1/6 e)7/18

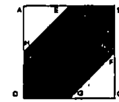


8) a)1/4 b)1/2 c)1/6 d)3/8 e)7/18

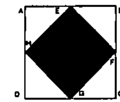


2

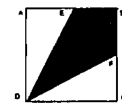
9) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18



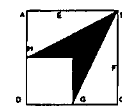
10) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18



11) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18

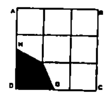


12) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18

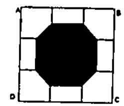


3

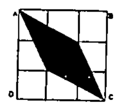
13) a)1/5 b)1/2 c)3/4 d)1/6 e)2/9
All small squares are congruent.



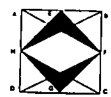
14) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18
All small squares are congruent, vertices of the octogon are on the midpoints of small squares.



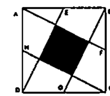
15) a)1/4 b)1/2 c)1/3 d)1/6 e)7/18
All small squares are congruent



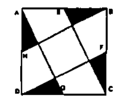
16) a)1/4 b)1/2 c)3/4 d)1/6 e)7/18



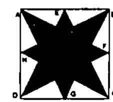
17) a)1/4 b)1/2 c)2/3 d)1/6 e)1/5



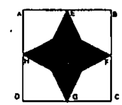
18) a)1/4 b)1/5 c)3/4 d)1/6 e)1/2



19) a)1/4 b)1/2 c)3/4 d)1/6 e)3/5



20) a)1/3 b)1/2 c)3/4 d)1/6 e)1/4



COGNITIVE PROCESSING STYLES, STUDENT TALK AND MATHEMATICAL MEANING

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This study investigated the impact of simultaneous and successive brain functioning processes on Year 4 students' abilities to complete subtraction problems using multibased arithmetic blocks and through written algorithms. This paper discusses the meanings these students appeared to make as they completed these subtraction activities, and as they talked about these algorithms and materials.

Brain Functioning

Simultaneous and successive processing refer to two methods of processing data within the brain. The Lurian (1966a, 1966b, 1973) model of cognitive processing is a model of brain-action, an information processing model, one concerned with the processes of analysing data in the brain rather than with making inferences about intelligence, developed by Luria from his clinical case study work with brain damaged clients. Luria (1966a, 1966b) regarded simultaneous processing as involving a synthesis of separate elements into a coherent whole, where all aspects of the situation are taken into account. For example, perception requires simultaneous processing, as does reading and our place value number system. Identifying meaningful relationships between a number of concepts and applying these concepts together in a problem solving situation also requires simultaneous processing. Simultaneous processing occurs then in situations where multi-attributed data are involved either from external stimuli or from memory retrieval where earlier patterns of relationships are recalled.

Luria (1966a, 1966b) posited successive processing as temporal, sequential and dependent, involving the analysis of data in sequence, of necessity temporally organised, and where data are linked serially and cannot be considered together in the one instant. The recognition of a musical tune, with notes identified one after the other, is an instance of successive processing, as are finger tapping, speaking and writing. Automatised routines and other cognitive activities not requiring introspection, for example the arrangement of words in a sentence, or the solving of an arithmetic algorithm, are based on successive processing (Kirby & Robinson, 1987).

Measuring simultaneous and successive processing

Reliable measures of simultaneous and successive processing have been widely described and applied (Das, 1988; Das, Kirby & Jarman, 1975; Kirby & Das, 1977; Kirby & Robinson, 1987; Solan, 1987). Examples of measuring instruments include the Number Span test where participants listen to a series of from three to ten numbers, then write them in order. The Letter Span and Word Span tests are also based on listening then writing. The Shapes test requires participants to select any number of five components to make up a given shape, the Paper Folding test involves predicting what a piece of paper with holes punched in it, will look like unfolded. The Matrix A test requires participants to copy shapes based on a 3 by 3 array of dots where each shape in turn is shown for five seconds.

This model of simultaneous and successive processing has been used in a number of investigations of cognitive processing, strategy development and teaching applications (Cummins and Das, 1978; Das, Cummins, Kirby and Jarman, 1979; Das, Naglieri & Kirby, 1994; Elliott, 1990; Molloy & Das, 1979; Kirby & Robinson, 1987). Classroom investigations have shown children as young as four have simultaneous and successive processing skills (Angus, 1985; Elliott, 1990). The impact of simultaneous and successive processing on aspects of language development, especially reading, has been a common area for research (Kirby, 1992; Leong & Sheh, 1982). Other studies have shown the model to have relevance in the teaching of school-level literacy and mathematics across grade and age (Hunt & Fitzgerald, 1979; Molloy & Das, 1980; Merritt & McCallum, 1984). Yet others have given more emphasis to the role of simultaneous and successive processing in mathematical learning (Merritt & McCallum, 1984; Molloy & Das, 1980). Das (1988) claims successive processing to be essential for reading achievement, but that in order to move to more advanced levels of reading, simultaneous processing is required. He reported too, that simultaneous processing was a good predictor of mathematics achievement and that successive processing was unrelated to mathematical achievement. In particular, Das et al. (1979), and Kirby and Das (1977) argue that neither simultaneous nor successive processing by itself is sufficient for high achievement in school.

The present work seeks to study the relationship between elementary students' cognitive processing styles, the way they solve subtraction algorithms, both in written form and through the use of multibased arithmetic blocks (MABs), and the way they talk about, and the meanings they appear to give to, these written and physical actions.

Method

The study reported here involved working with four Year 4 classes as they were taught subtraction algorithms through the use of MABs, written algorithms and word problems. A series of six tests[1] were administered to the students in class groups. A principal components analysis was conducted on these scores, and using a two factor analysis, the scores on the number, word and letter tests were added to give a score for successive processing, with the sum of the scores on the shapes, paper folding and Matrix tests giving a score for simultaneous processing. The median of these scores was used to categorise all students as high or low, so four groups were formed: students who were high simultaneous and high successive processors, students who were high simultaneous and low successive processors, students who were low simultaneous and high successive processors, and students who were low simultaneous and low successive processors. At the completion of the period of instruction, students were interviewed as they attempted to answer subtraction questions using written algorithms and MABs. These interviews were videotaped and later transcribed.

[1] The number, word and letter span tests, and the shapes, paper folding and Matrix A tests described earlier.

Results and discussion

High simultaneous and high successive processors

The text below shows Kae, a high simultaneous and a high successive cognitive processor, completing the written algorithm 653-472 to obtain the correct answer, 181.

R: What about that one?
S: Three take away two is one. Five take away seven you can't do so you trade, five, fifteen. Fifteen take away seven is eight and five take away four is one.
R: Good. Why did you have to trade?
S: Because five is less than seven.
R: Now when you traded you crossed that out. So why did you cross that out?
S: So I could put the five at the top and put the ten from there and make fifteen.
R: So why is that five and not four or three or two or something different?
S: Because the next number down from six is five.

The transcript indicates Kae has high procedural knowledge. She described the process of completing the algorithm briefly and accurately, including the need to trade and decompose. I take this to be a reflection of her high successive processing ability. She also gives reasons for these steps, and explains why she does a particular thing and not anything else. In particular, not only does she identify mathematical properties (*five take away seven you can't do so you trade*), but she recognises relationships (*because five is less than seven*), and establishes a logical explanation through the use of *because* and *so* in various combinations. That is, she recognises mathematical relationships, and in her interview provides extended explanations on various components of the algorithm, without requiring continual prompting. I take all this to mean she has a good understanding, a relational understanding, of the algorithm sequence, and the interrelatedness of all its components, and that these qualities are possible because of her high simultaneous processing capability.

When asked to complete 547-169 with MABs, she did this quickly and correctly, her procedures appear to have been automatised, which I interpret as another instance of high successive processing. Kae appeared very able to describe her working processes, and provided extended explanations. For example, she first described what to do *Take one of these away*, then gave it its technical term *trade*, then unasked provided a reason for this *there's only forty there*. This high language proficiency, and the giving of explanations beyond the immediate answer to the question asked, seems a feature of high simultaneous processors. Kae easily identified each step in the MAB process, she recognised the need to trade, and described the process. She identified facts or skills, she recognised relationships, and she elaborated on these: all signs of high level simultaneous and successive processing. When she repeated this second question using pen and paper only, she had no hesitation in predicting that her answer was the same as when she used MAB materials.

Of course not all high simultaneous and high successive processors showed these capabilities. For example, Lin is able to give only procedural reasons in explanations. She does not give reasons involving mathematical relationships. In explaining her method for 547-169 Lin claimed that you could not do 7-9, *Because if you try and take nine from seven you can't do it.* And she continued in this vein of calling up rules *But if you do it then you come up with the wrong answer*, rather than providing a mathematically based explanation. At the same time Lin described the correct procedure, and apart from a simple number bond error would have obtained the correct answer to 547-169. Even though there were exceptions of this kind, in general, students who are both high simultaneous and high successive processors appeared to have highly automatised procedures, to be able to explain their actions by referring to mathematical relationships, able to explain why as well as how, and to be able to link the use of MAB materials to written algorithms.

<u>High simultaneous and low successive processors</u>

Eve is a high simultaneous and low successive processors, here she is using MABs to calculate 746-382 (answer 364). Eve appears able to use the materials reasonably well, in particular, after completing the calculation for the units place value, she has no hesitation in exchanging 1 hundred for 10 tens. But she does appear to have some procedural difficulties. For example, she completes the subtraction with the MABs, but does not recognise she has the answer and has to be coaxed by the interviewer.

R: Can you tell me what you have got in front of you there now?	746-382
S: Seven hundred and forty-six.	746 in MAB
R: Good, what do you have to do now?	Identification
S: Take away three of these and eight of them and two of these.	Identifies the subtraction correctly
R: Off you go and do that.	
R: Hold on, just before you do that, what have you got there now?	Using MABs
S: I've got fourteen there.	14 tens
R: You've got fourteen. How did you get fourteen there?	
S: Well, I took away one of these flats and I got ten longs and now I'm taking away eight of them.	Identification and explanation Writes 360
R: So you go ahead and do that, all right.	
R: You read out your answer for me.	
S: Three hundred........	Uncertainty
R: Which is your answer?	
S: I haven't done it yet.	Not identifying step she is up to, or all MABs she has
R: No, you are right, it's just that that is your answer.	
R: How many units have you got there?	
S: No units.	Not identifying
R: Are you sure?	
S: Four units.	Identification Writes 364

These procedural difficulties continue when Eve is asked to repeat the calculation using a written algorithm. She immediately takes the smaller digit from the larger, suggesting that she has not automatised the correct procedure. A plausible interpretation of this and of the above text suggests that Eve's low level successive processing capacity has an impact on her ability to remember and reproduce the procedures necessary to solve algorithms, and has hindered her in automatising such procedures. Yet her ability to use the MABs relatively well, and to provide elaborate descriptions of what her actions on the materials, suggest that her high simultaneous processing capabilities assist her here.

From this and other interviews it appears that high simultaneous processors who are also low successive processors will generally be quite able when using materials, and will often give elaborate and correct descriptions and explanations as to what they are doing. But their low successive processing may lead to mistakes in the procedural aspect of what they do. For example, even though they give the correct description for using MAB materials, they may not actually put that description into practice, and without materials they may perform written procedures incorrectly.

Low simultaneous and high successive processors

Here is the case of Anna, a low simultaneous and high successive processor, who is attempting the written algorithm 547-169, and obtains the incorrect answer 398. She begins well by taking one of the four tens to the units column, making the 4 into 3, and the 7 units into 17. Or as she says *You go over to the four and you take a ten off there and put it with the seven and call it seventeen.* She mentions the terms *regrouping* and *renaming*, but does not identify *trading*. When asked for a reason for this action she was silent.

S: Because seven is a smaller number than nine.
R: So you can't take nine from seven so what do you do then?
S: You go over to the four and you take a ten off there and put it with the seven and call it seventeen.
R: Now what is that called when you do that?
S: Regrouping, renaming.
R: There is another word too. Can you think of another name? You said regrouping didn't you?
S: Yes.
R: And you said renaming as well. That's good. So now why did you (pause) You had five here (5 hundreds) and you wrote four (above the 5), so why did you write the four there?
S: Because (pause).
R: You are right. Now I'm just asking you the reason. There was five there and you write a four here and why did you do that?
S: Because the four you can't take away from six.
R: So you had a six here and a four here. What about this three?
S: That's for the seven.
R: So what subtraction are you doing in this column here? Are you doing four subtract six or three subtract six?
S: None. I'm doing fourteen.

The researcher reassures her, *you are right*, she says *the four you can't take away from six*, which has the digits reversed but identified the correct place value column. She knows that in someway the 3, renamed from 4 tens, is connected to the seven units, and says *that's for the seven.* The next question gives her a hint, *Are you doing four subtract six or three subtract six?* She then says *I'm doing fourteen*, and reasserts this in the next question. She then subtracts 6 from 14 and writes 9 as her answer. Here the student has the procedure generally correct, she has traded and decomposed correctly, and has the correct digits in the appropriate positions. She is able to explain the steps in the procedure *Because I had to take a ten off the five and put it with the four to make it fourteen*, but it is a procedural explanation not one calling on mathematical relationships. Her first error is subtracting from 14 instead of 13, and then she makes a mistake in the subtraction facts. On the whole, Anna seems reasonably able in performing the subtraction algorithm, but she seems to have little understanding of why her actions are appropriate. In the next section of the interview, she repeated 547-169, but using MABs this time. Anna recognised that 7-9 in the units position creates a difficulty, she knew the 7 has to become 17, so she goes to the bank for an additional 10 units. But she does not trade. This is repeated later when she recognises the difficulty with 4-6, and goes to the bank for 10 tens. She eventually gives the incorrect answer 488. There are two problems for Anna here. Firstly, she has a procedure but it is incorrect, and secondly she appears to have no understanding of either what trading is or of its proceduralisation.

Anna has considerable ability with written algorithms and frequently completes written algorithms correctly, including three digit subtractions with trading. The 14-6 instead of 13-6 is likely to have been a chance error or nervousness at being interviewed, rather than a regular strategy. That is, in spite of the algorithm errors in the interview, Anna seems to use her high successive processing skills to enable her to correctly answer subtraction algorithms. But Anna appears to have little knowledge about the use of MAB materials - she has not automatised the procedure, and is unable to use the materials effectively. I take this to be a case of low simultaneous processing being insufficient to interpret the use of MAB. That is, I am claiming that low simultaneous processors will have difficulty in using MABs effectively, and in establishing links between actions on MAB materials and a written algorithm intended to correspond to these materials and actions. Data from other low simultaneous and high successive processors appear to support this interpretation.

Low simultaneous and low successive processors

Students in this category appear to be the least able school mathematicians. For example, Andy frequently subtracted the smaller digit from the larger, simply disregarding place value. Within adjacent algorithms he would answer one using a correct procedure, and in the next revert to his smaller digit from the larger strategy. And in the problem solving questions, after he had written the algorithm corresponding to the word problem, rather than solve the algorithm, he would literally use a tally where one stroke was used to represent each number in the minuend, and the appropriate number were then crossed off. For example, in the problem where a milk truck delivers to 96 houses, but has only been to 59, he calculated how many remained by writing ninety six strokes and crossing off fifty nine of these. His class workbook showed that his algorithms were correct, and that trading has been used, at least according to marks on the page. Here he is completing the written algorithm 547-169 and obtaining the correct answer (378), during the interview.

R: Eventually when you crossed out the four (4 tens) you had a three here didn't you? So what was the next subtraction you had to do?
S: Three take away six you can't do so I crossed off the five, put a four and put a one next to it and take away six equals seven.
R: Now when you cross five off why do you put four? Why don't you put three or two or one or something like that?
(pause)
S: Because the four goes there and whatever you put a one next to it, whatever you crossed off.
R: What is that called, sometimes people use a special name for it? You cross the five, you write four and then you move one over there. What is that called?
S: Trading.

In the above text, it seems that Andy has a good understanding of the procedure. He recognises the need to trade, carries out the correct written actions and obtains the correct answer. However, when asked to explain why the 5 hundreds became 4 hundreds, and not some other number, there was a delay before he responded, and his explanation was *(b)ecause the four goes there*. This is not an explanation, but reference to a rule or statement which is used to support itself, so the argument is circular. It almost certainly indicates that even though Andy was able to recall and identify the correct instance of *trade*, he had little meaning of the concept beyond the written action. Indeed his class workbook indicates he is likely to make many procedural errors in subtraction algorithms. When he was asked to repeat the question with MABs he used them as counters ignoring place value, so he used 5 units to represent 500, 4 to represent 40 and 7 to represent 7. This incident also suggests that he had little knowledge of the materials and how they related to algorithms, and that he had little conceptualisation of what the materials or his actions on them meant. The MAB and actions on them seem to make no sense to him whatever.

Other low simultaneous and low successive processors also appeared to have declarative knowledge, they could say what had to be done but could not apply it. In particular, they made procedural errors on a regular basis, and their apparent inability to automatise the procedure is consistent with low successive processing. And their low simultaneous processing meant they did not link the materials to the written algorithm, making it problematic for them to understand why algorithms were structured and completed the way they were. Their language was generally narrow and unlinked, with few elaborations and with little reference to mathematical relationships.

Conclusion

In analysing these and other transcripts, there appears to be a pattern of mathematical skills and understandings that reflects the cognitive processing style of learners. In particular, high simultaneous and high successive processors appear likely to have automatised procedures, to have insights into mathematical relationships, and to understand both how to complete a procedure and why this is the case. They also appear likely to link actions on materials with written algorithms, and to have a good chance of recognising and correcting errors. High simultaneous and low successive processors seem to have insights into mathematical relationships, and can link actions on materials with written algorithms, but may have procedural weaknesses. That is, they have a good chance of understanding why, but are less effective when it comes to how. Low simultaneous and high successive processors are likely to have automatised procedures, but this ability to perform is not accompanied by knowledge of why. They appear unable to recognise mathematical relationships, and so are unable to link actions on materials to written algorithms. Low simultaneous and low successive processors are likely to be inconsistent in their completion of algorithms and are unlikely to understand the structure of an algorithm. They have declarative knowledge, but appear unable to proceduralise it, and are unlikely to have insights into mathematical relationships. They seem to interpret actions with materials and written algorithms as two unrelated sets of activities.

Research into simultaneous and successive processing in classrooms has largely been confined to language and arithmetic activities, so there is little data to indicate if these constructs are consistent across all school mathematics, and there is even less one can say about other school curriculum areas. From a practical classroom perspective this study suggests that some students will have difficulty linking their actions on manipulative materials to a written representation of these materials and actions. Teachers will have to be particularly insightful and sensitive in encouraging students to explore materials, in establishing procedures with both materials and algorithms, and in discussing various representations, the links within procedures, and the relationships between different representation systems. From a research perspective there is need for further studies to investigate the impact of various levels of simultaneous and successive processing on mathematical learning.

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LISTENING BETTER AND QUESTIONING BETTER: A CASE STUDY

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This paper describes the changes in the mathematical behavior of a twelve-year old student, Jeff, who has been observed as part of a longitudinal study of how children build mathematical ideas. Specifically, we examine his listening to and questioning of other students in small groups and the evolution of his mathematical behavior during his exposure to constructivist classroom settings where teacher/researcher questions are guided by student thinking. Over a span of seven years, changes in Jeff's manner of working with other students have been traced showing movement from focusing on reaching a correct answer to becoming more attentive to the mathematical ideas of others. Certain conditions contributing to the movement towards student led mathematical discourse in the classroom will be discussed.

> They never told us the answer! Yeah, you never tell us the answer. No, like, I remember, in third grade, the thing with the box, with the corner cut out, and I always wanted to find out how many black marbles were in there and how many yellow marbles were in there, but you never opened the box!
>
> - Jeff, grade 7, March 18, 1994

Certain features are typically associated with the teacher centered classroom. Vinner (1994) describes student behavior and how it can focus on pleasing the teacher. Once students find the answer that elicits a positive response from the teacher, they will try to repeat that answer in situations that appear mathematically similar. What appears to be meaningful mathematical dialogue may only be what Vinner calls "pseudo-conceptual" behavior. The teacher, motivated by pressures other than mathematical understanding, may accept the students' answers as evidence of comprehension, and move on with the lesson. The direction of mathematical activity in such situations lies mainly in the hands of the instructor. The class "moves along" as students respond, but this motion does not necessarily indicate mathematical growth on the part of the students. The students may simply be responding to cues instead of building mathematical ideas which they can use when the teacher cues are not present.

In the teacher-centered classroom, the instructor serves as the social and mathematical authority of the classroom group. Jaworski (1994) describes situations where teachers are conscious of this position of power and use their awareness to create a different kind of classroom -- one where *shared* meanings are created. The teacher overtly attempts to avoid imposing her meaning upon the students, opening the way for students to construct their representations of mathematical knowledge. This complex process involves constant decisions upon the part of the teacher as she interacts with students. In many situations, any statement on the part of the teacher may interfere with the student's mathematical development. In any case, the "culture" created in the classroom determines the nature of the interactions. Students will invent shared meanings if they believe that their meanings have value.

This "culture" has been described by many, including Davis (1989). Gooya (1994) describes the difficulties inherent in forming this culture. The teacher must play a "special role" in creating the environment -- long hours of reflection and lessons where only a few problems are covered may be necessary to change the nature of mathematical "authority." Elements of this change have been detailed (e.g., Pantozzi, 1994; Jaworski, 1994), where the teachers alter the nature of their questions with the aim of creating a problem-solving culture centered upon student thinking.

During a longitudinal study of children's thinking, we have observed changes in students' questions of each other when a group of students grows accustomed to the lack of specific teacher prompts. The teacher/researcher instead models open-ended questions that are based upon the students' thinking, forming a distinct classroom culture. Over time, students begin to question each other in the same manner, probing each other's thinking with questions instead of requesting information or specific answers. The teacher/researcher's role recedes as the students drive the focus of inquiry and use their work as the starting point for additional exploration. (See Martino & Maher, 1994.) Student-to-student questioning moves the activity forward; the students move into a position of mathematical authority since they do not look for the researchers to hand down knowledge. This case study focuses upon one student, Jeff, and the changes he makes in his listening to and questioning of other students. Examples of Jeff's interaction with other children in group problem-solving situations illustrate the transitions that accompany exposure to a constructivist classroom setting.

Background

This case study of one student arises from a longitudinal study investigating how children build up mathematical ideas as they are engaged in problem tasks with other students. The study takes place in a working class school district in the United States that has been the site of a teacher development project in mathematics over the last 10 years. Outside of their daily classroom schedule, students in this district participate in problem-solving sessions where they are given opportunities to do mathematics in an open environment, working in groups for extended periods of time. Researchers develop problem situations for the students, but the direction of the activity is heavily influenced by the students' questions and interest. The classroom situation is designed so that children are free to build solutions, discuss their ideas, and negotiate their conflicts. No "lesson" as such is taught by the researchers. One set of students has been studied continuously, as a group, for the past 7 years. This group has remained largely intact over this time, allowing the students to grow in familiarity with their classmates. Their development as a group and as individuals has been traced by researchers. Data on each student has been continuously collected over this time; one student was selected for this paper.

Methods and Procedures

In studying how children develop mathematical ideas, problem situations are developed by the teacher/researchers. The problem is presented to a class of students and they are videotaped as they discuss the problem in groups. Teacher/researchers observe the students as they interact, question the students after they have begun working towards a solution, and facilitate discussion. Their questions are based upon the students thinking, not upon a predetermined course of action. Thus a session may extend to several hours or extend to a number of days or months. As each session is videotaped by two or more cameras, researchers follow the students' discussions and note elements of the students' mathematical activity. After the videotapes are transcribed, they are examined for students' growth in mathematical thinking, and are used to develop future classroom sessions. Student work, the field notes, the transcript, and analysis by researchers and graduate students comprise a video portfolio of the development of the group and of each student.

Data Source

This research was motivated by data collected over the course of a longitudinal study that is now in its seventh year.[1] Classroom sessions and interviews with students over these 7 years were videotaped, transcribed, and analyzed by researchers and graduate students. The classroom sessions consist of the team's analysis of student discussions; of students working in groups on a problem task and recording their findings; of students sharing solutions and/or questions with the class; and of researchers interacting with the groups of students. Five vignettes focus specifically on Jeff in the second, third, fourth, fifth and seventh grades, respectively, and of his interactions with group members and researchers. These represent samples of data focusing on Jeff doing mathematics over the course of the entire longitudinal study.[2]

Theoretical Orientation

We have found that under these conditions described above, profound changes can occur in the location of mathematical authority in the classroom. For example, interactions and questions between students are altered and classroom settings where students guide mathematical discourse without direct teacher instruction are evident.

The shift from teacher authority to student authority over mathematical discourse transpires over an extended period. For the shift to occur, certain elements must be present on a consistent basis. Teacher/researchers model the kind of questioning that is based upon students' constructions. Over time, students begin to develop similar requests for justifications of solutions by asking: "Can

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2 Videotapes are available by writing to Carolyn A. Maher, Rutgers University, Graduate School of Education, New Brunswick, NJ 08903, USA.

you convince me?" or, "Tell me how you got that answer." At first, questions such as these are role imitations; gradually, however, they evolve as a real expectation of the culture that is being created.

Initially, students may develop the mathematical ability to explain their reasoning to each other, but may lack the "social graces" necessary to lead the mathematical discourse. (See Wilkinson & Martino, 1993.) Students also seem to pass through stages (see table 1, below) where they are not able to listen to another student's explanation while they are constructing one themselves. As a result, group disagreements, which can lead to productive problem-solving activity, can also result in a particular student dominating discussion and silencing opposition on the way to arriving at the correct answer. Constructive group discussion and shared leadership typically predict problem-solving success, but the absence of such elements does not constitute a "dead end" for problem solving. We conjecture that some students may first revert to modeling the central authority of the teacher-centered classroom before they embrace the culture of shared responsibility that the researchers have modeled over time. In addition, constructing mathematics and developing social graces simultaneously may be difficult for children.

The changes manifest themselves most clearly in the kind of questions students ask each other, the nature of the responses they receive, and the type of interaction between students in groups.

Table 1 summarizes the behaviors we have observed in these three areas.

One-way dialogue -- Students answer teacher/researchers' questions, make reports of findings, request and receive information from each other. New ideas depend largely on teacher/researcher questions.
Exchange of information – Students ask each other to perform tasks, ask questions of teacher/researchers. Input from other students is considered as students construct ideas.
One-way explanations – Students explain reasoning to each other and to researchers, but may not attend to cognitive needs of listener. Students reorganize ideas based on teacher/researcher questions.
Collaboration – Students attempt to convince others of their reasoning, and may use each other's comments to evaluate their explanations. Students begin to construct ideas without teacher/researcher questions.
Mutual awareness – Students make arguments and counter arguments, and ask questions of each other that take into account each other's previous statements – dialogue is interactive. Teacher/researcher questions often unnecessary.

Each questioning behavior listed builds upon the previous behavior, but the appearance of each is not necessarily sequential. Some or even all the behaviors may occur during one classroom session.

When students develop mutual awareness of each other's arguments, much of the teacher questions that cause students to reorganize their thoughts are asked collectively by the students themselves. Cobb (1994) suggests that shared authority is critical to mathematical learning; we have found that while students may not ask questions of each other "politely" at first, they can develop the skill over time. Part of these changes may be developmental; the children may grow more mature. However, maturity alone may not create the ability to lead the *mathematical* discussion; exposure to the culture of questioning and listening must also occur. At that point, student questions can drive the mathematical activity in the classroom.

Results

Limitations in space prevent a full description of the events. However, representative episodes are reported that characterize Jeff's behavior over the interval under study. These are presented as vignettes and are selected from classroom problem-solving activities during grades two, three, four, five, and seven.

Vignette 1. Grade 2. October 23, 1989: Jeff directs other students.

Jeff is observed working in a small group on a story problem and directing another student how to solve it.

Mike: *1,2,3,4,5,6,7.*
Jeff: *Now add 5* [He moves 5 stones toward Mike.]
Mike: [counting the 5 stones] *1,2,3,4,5.*
Jeff: *Now count 'em.*
Mike: *1,2,3,4,5,6,7,8,9,10,11,12.*

Jeff then directed the other group members to record the same solution.

Jeff: *Whatcha get for the number of pencils?* [points to Mike]
Mike: *Seven*
Jeff: *Whatcha get for the number of pencils?* [points to Aaron]
Aaron: *Seven*
Jeff: *What did you get?* [points to Michelle]
Michelle: *Seven*

Vignette 2. Grade 3. December 6, 1990: Jeff gives a give one-way explanation.

The students have been given a "mystery box" containing ten marbles, some of which are yellow, some of which are black. The students are unable to open the box, but can sample the contents by shaking the box and looking at small hole in the corner of the box 20 times. Jeff's solution differs from that of his partner's:

Jeff: *13* [yellow] *and 7.* [black] [The result of their sample.]*That means we have 7 yellow and 3 black.*
Jamie: *We don't have 3....7...*
Jeff: *That would be the answer though. Believe me. That would be right, though. We think it's seven yellow...*
Jamie: *Yellow... (grabs pen) I'm putting it in!*
Jeff: *Seven yellow and .. what is that?* [Jamie tries to write a different answer.]
Jamie: *Yellow?*
Jeff: *Yeah, seven yellow marbles... and three...*
Jamie: *No, we don't have seven here!* [She points at the 13 yellow marbles they sampled.]
Jeff: *Jamie, common sense, look, 1,2,3,4,5,6,7, count that and it's seven, if there's 13 here, see this would be all seven, and this would be a three. That would be seven and three.*
Jamie: *OK. OK. I think there's...*
Jeff: *That's what we found.*

A few minutes later, Jamie proceeded to erase Jeff's answer and change it to her own. She explained her solution to a researcher while Jeff turned his head away.

Vignette 3. Grade 4. February 6. 1992: Jeff chooses the creative role for himself.

As Jeff and his partner, Michelle, build all possible towers five cubes tall when they have available cubes of two different colors, Jeff assigns himself as the role of creating new towers and his partner the role of making an "opposite" [colors reversed for the same position] for each of his designs.

Jeff: *I'll make them and you make the opposite. This is easy...make the opposite.*
Michelle: *Why do I have to do it?*
Jeff: *Because I'm making the new ones. I can't do both.*

Vignette 4. Grade 5. April 2. 1993: Jeff collaborates with other students.

In Grade 5, we see Jeff ask questions that focus on the solutions of other students.

A group of students is trying to determine how many different combinations of 4 toppings could be placed on a pizza. The researcher asks a question about their method. Jeff elaborates the researcher's question.

Researcher: *I don't see... how you are going to consider all the possibilities?*
Stephanie: *Yeah, well no, because in this one nothing's going to be mixed, but in this one, something is mixed.* [Two toppings mixed together on a pizza.]
Jeff: *How come in this one, nothing can't be mixed?*
Stephanie: *Nothing is mixed because this is half a pizza.*
Ankur: *Yeah. Half.*
Jeff: *So why can't you just make this like, um,* [topping 1], [topping 2] *say...*
Researcher: *That's my question.*

Vignette 5. Grade 7. March 18-21.1994: Jeff listens to and questions other students.

Over time, we have observed Jeff modify his interactions with other students, his questioning of other students, and his interaction with the researchers. His experience deserves attention since he has focused on his personal search for mathematical meaning, frequently disregarding the ideas of others. Observed changes in his questioning that reveal greater interest in the ideas of other students are indicated in the March 1994 episode.

Students were given two 6 sided dice and were asked to invent a "fair game" based on the rolling of the dice. This required that they determine (a) which sums could be rolled, and (b) the number of ways each of the sums could occur.

On March 21, Jeff expressed belief that he had solved the problem. In his interaction with the other members of his group, he refers to their work from March 18. Jeff reaffirms his desire to find the correct answer over others' objections.

Student 1: *Should we explain this to somebody?*
Jeff: *We'll explain it when it goes up.*
Student 2: *Yeah.*
Jeff: *What's two?* [Which rolls result in a sum of 2?]
Student 1: *Oh, let me guess who's going to explain it, hmmmm?*
Jeff: *All of us.*
Student 1: *Hmmmm. He* [Jeff] *doesn't let anybody explain.*
Jeff: *You were just standing there, you weren't doing anything, what do you want me to do, just sit there and not do anything?*

As the session continues, Jeff realizes that his group has not considered all possible combinations of dice that result in the sums from 2 through 12. His group has not included (1,2) and (2,1) as distinct rolls of the dice. Jeff, listening to another student, realizes the omission.

Jeff: *Unfortunately, he* [referring to Student 4, below] *makes somewhat sense because actually you do have two chances of hitting it.* [Rolling a sum of 3.]
Student 2: *What?*
Jeff: *See look, because if you roll, if this die might show a one, and this die might show a two, but next time you roll, it might be the other way around.*
Student 3: *Look Jeff.*
Jeff: *And that makes it two chances to hit that, even though it the same number, it's two separate things on two different die.* [1 on one die, 2 on the other **and** vice versa]
Student 3: *Therefore there's more of a chance. Therefore there's two different ways, therefore there are...*
Jeff: *And that that likes blows up our plan..*
Student 3: *Therefore there are two ways to get three.*
Jeff: *And that just screws up everything we just did, worked on for about the last hour.* [He smiles, perhaps recognizing why the earlier group solution was faulty]

Jeff reconciled his misconception by paying attention to the idea of another student who suggested that the outcomes (2,1) and (1,2) were distinct. Jeff then considered whether the outcomes (1,1) and (1,1) were also distinct. He turned to a member of his group (Student 2) to share with her the other student's reasoning:

Jeff: *Well, then if, couldn't two come up twice then?* [That would be (1,1) and (1,1).]
Student 4: *No because Jeff, one on one die and one on the other die is still the same thing.*
Jeff: *Yeah, even if you just... OK...*
Student 4: *Yeah, there's still one in one and ...*
Jeff: *Yeah, if you do switch, yeah, because it seems like even if you do switch, it will still be like the same thing.* [The roll (1,1) will be unique.]
Student 2: *Yeah, but that's the same thing as that...* [Why are there not two distinct (1,1) rolls?]
Jeff: *No but this, look, on this one you have two and one, but you actually have to move the die to hit one and two, but on this is doesn't matter, you can just, do you know what I'm trying to say here?*
Student 2: *Sort of.*
Jeff: *That's good.*

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Conclusions and Implications

The story of Jeff and his growing independence in mathematical thinking suggests that students can and do indeed listen to the thinking of other students, reflect on their own and on other student ideas, and share them with others. Instruction, guided by close attention to student thinking, can make possible the creation of a "culture" where students play a large role in determining the course of mathematical investigation. The development of this culture may require considerable time, effort, and patience as teachers and their students work to redefine earlier traditional roles. The study of Jeff suggests that the effort may well be justified. It is interesting that Jeff recalls the marble problem after four years! What might have contributed to Jeff's changes was the creation of a classroom culture that left open to students discussion of ideas. Reliance upon reaching a "correct answer" made possible by teacher directed questioning may appear beneficial when students produce correct responses. However, declines in students' ability to think and reason critically belies this notion. Students who answer teacher questions correctly may either be responding to cues, or may already have developed the idea independent of teacher intervention. They may not be able to think and question based upon their *own* ideas. Students can learn to listen and question better, and lead mathematical discussion rather than follow it, through their exposure to a classroom culture that values student questions.

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CLASSROOM COMMUNICATION: INVESTIGATING RELATIONSHIPS BETWEEN LANGUAGE, SUBJECTIVITY AND CLASSROOM ORGANISATION

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This is a report of research in progress. A survey aimed at identifying features of quality teaching, followed by analysis of lessons which were aimed at demonstrating those features in a multi-media resource, led to a research focus on one of those features: Communication. It was noted that numbers of interactions were quite uneven, and that while some teachers did little to remediate this situation, others attempted to call on all students. When teacher educators watched a videotaped mathematics lesson, their written critiques regarding communication were surprisingly contradictory. This paper outlines some issues raised by this diversity.

Multiple perspectives

Over the past three years, we have been making a multi-media program and researching its use[1]. This CD-based resource for learning about teaching mathematics includes a variety of media that provide information about different aspects of the same phenomena, such as videos of lessons, transcripts of the verbal communication, films of pre- and post-lesson interviews with the teachers, teachers' notes, readings pertaining to the styles of teaching used and specific features of the classroom climate, graphic representations of the classroom interaction, and other components. Our research is focusing primarily on how undergraduate students and teachers use the program to observe and analyse classroom interactions.

One of the major areas of concentration in the resource, and the focus of this paper, is Communication. This was identified as an important component of teaching in the first stages of our research project, when planning for filming the lessons commenced with a literature search on aspects of good teaching as well as the implementation of a survey aimed at identifying common perceptions of the features of quality mathematics teaching (see Sullivan & Mousley, 1993, and Mousley & Sullivan, 1994).

There were six major groups of components identified: *Building understanding, Communicating, Engaging, Problem solving, Nurturing* and *Organising for learning.* These should not be thought of as independent factors — each is both dependent on and constitutive of the others. For instance, responses to the questionnaire demonstrated that communication is vital in organising for learning and engaging students as well as nurturing their all-round development and mathematical problem-

1 The research project "Features of Quality Teaching" is funded by the Australian Research Council, the Australian Catholic University, and Deakin University.

solving. Most importantly, communication was thought to be essential for building mathematical understandings, just as the other factors were. Thus the results of the survey could be represented as in Figure 1.

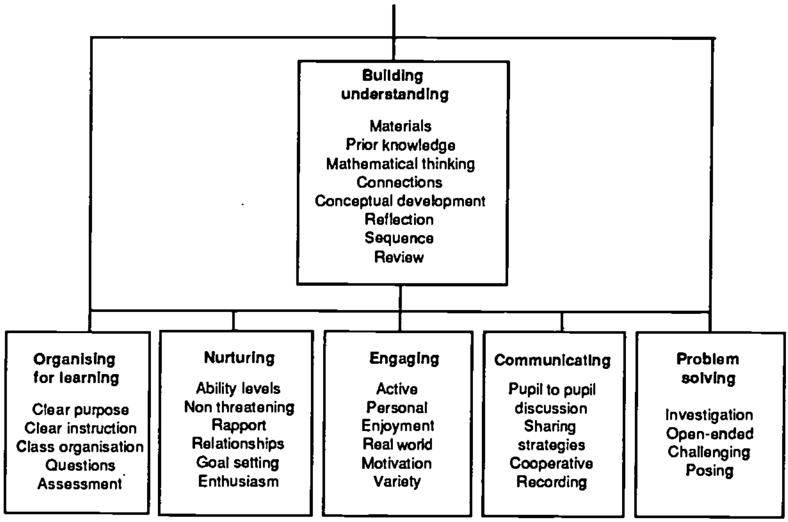

Figure 1. Interrelationship between some features of quality mathematics teaching.

During the development of the multimedia resource, thinking about how to encourage users to make logical connections between different representations of particular episodes provided opportunities for us to discuss and investigate a number of issues about communication in mathematics classrooms. For instance, when we graphed the number of interactions of each classroom participant for one lesson, and discussed possible reasons for variations in the results, further characteristics of the discourse were brought into question. These included, for instance, the total time of interactions between teachers and individual students, the types of interactions (such as a student posing a question or responding to a teacher's question), the qualities of those interactions, how such qualities contributed to the development of students' mathematical understandings, and whether other factors such as gender had been brought into play.

This led to decision-making about how we could provide for users wishing to explore such questions about the classroom communication using a variety of media, without our actually directing users to research topics that were essentially "ours".

Communication

Attention to language factors in mathematics education has long been recognised as an important focus for the improvement of mathematics pedagogy (see, for instance, Emori, 1993; MacGregor, 1994; Mousley & Marks, 1991; Thomas, 1994). However, there is a recent surge of new psycho-socio-linguistics interest with the recognition by post-structuralist theorists of the force of discursive practices and the ways in which people are positioned through those practices. Kenway, 1992, for instance, notes that school and non-school discourses, in concert and in contest, construct students', teachers', principals' and parents' identities in multiple and shifting, yet patterned, relationships of dominance and subordination.

Similarly, in practical contexts, teachers are now being urged to attend to individuals' explanations and questions, to take language-based "constructivist" approaches to the teaching of mathematics, and to involve their students in one-to-one, small group and whole class discussions. *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) epitomises this change, with claims like,

> The process of developing and building up mathematical knowledge through describing, questioning, arguing, predicting and justifying almost always requires a sharing of ideas. The productive sharing ideas depends on the clarity with which one can express oneself. Mathematical communication skills are needed in order to understand, assess and convey ideas and arguments which involve mathematical concepts or are presented in mathematical forms. (p. 13)

Such summarative recommendations of policy documents, however, do not hint at the complexities of the phenomena, or at the dilemmas that these present to teachers as they control classroom discourses and thus position themselves and students within the social arena. Two incidents during the preparation of this section of the multi-media program further raised our awareness of the complex and often contradictory nature of classroom discourse.

"George has a lot to say". When we were constructing the graph of the number of interactions for each child, we noted that one student had had a lot more to say than any other student, in each of the two lessons for his class. On asking the program to play all of his contributions one after the other, we found that George raised his hand frequently and that the teacher and other students regularly called on him to make opening remarks on a topic. His comments and questions, however, while serving the purposes of getting conversations started and keeping them moving, were relatively low-level so he had not made as valuable a contribution to the learning as had some other students with much lower numbers of interactions.

"But Jenny <u>was</u> involved". A group of graduate students trialing the program noted that one of the teachers had directed a question to a student who had played no previous part in the class discussion. The girl had appeared flustered and had given a very hesitant answer. Later questioning by the teacher in a one-to-one situation revealed that the child had had a good understanding of the concept and problem being discussed.

A third incident led to further research on this topic. After some of the lessons had be filmed, but before they were committed to CD-ROM disc, we were keen to validate the existence of each of the above components on the videotapes. We asked 24 teacher educators, many of who supervise student teachers regularly, to watch the tapes. Half were asked to write an unstructured critique, using any format they wished on a blank sheet of paper. The other observers recorded their critiques on a structured instrument which was basically a sheet divided into six sections, one for each of the six components above. These respondents were asked to rate the teaching for each component on a linear scale, then to write an unstructured comment on that component. In effect this forced their comments into the six components.

Through this exercise, we sought to determine whether the reports of the observable features of the videos were consistent with our impressions of the components of teaching presented; whether the six components are useful as a way of organising critiques of a lesson; and whether structured or open format is more informative for presenting critiques. The qualitative analysis program NUD·IST (Richards & Richards, 1990) was used in the analysis of the written critiques.

With regard to communication, there were surprisingly diverse and contradictory results (see Mousley, Sullivan, & Gervasoni, 1994). For instance, in a particular lesson the teacher did not seek to distribute questions and other interactions evenly, but rather allowed students the freedom to contribute publicly as they wished. There was a distinct division in the way that people commented on teacher-to-student communication. Some suggested that the teacher had focused only on some pupils and could be more inclusive by questioning all students. On the other hand, others commented on her nurturing manner of allowing students to join the dialogue when they felt they had something positive to contribute. Comments included the following:

- Mainly Emily, etc. were nurtured, but most others were left out of the class, so the problems of understanding by the majority of the class were not addressed. The range of abilities was not catered for all.
- I was also pleased you encouraged all children to participate.
- By the half way mark of the class seven children had been asked questions or had asked questions. Note how a few students seem to dominate. Do not rely on volunteers only.... Be aware of who is being involved and who is passive.
- The teacher directed learning sufficiently but enabled children to contribute voluntarily according to their potential.
- The teacher restricted communication etc. with her star pupils only, so what learning was achieved by the rest of the class is unknown. Only a handful of children talking ... M., L., E. and O., were about the only children really engaged.

Natural Communication

In non-institutional social situations, the frequency of people's contributions to discussions and their total time allocation are not controlled. We rarely feel worried about imbalances or attempt to redress unequal participation in everyday conversations. People place themselves within conversations according to a number of factors, such as their confidence in the situation, their knowledge of the subject matter, the contributions they wish to make (or to reserve), and the roles that they wish to play within the group. It is generally accepted that people can be fully involved and play an integral role in discussions by just listening. However, mathematics lessons are not natural social situations. Traditional patterns of control of communication have developed in schools just as they have in other social institutions.

Dominant models of classroom interaction have been brought into question with the recognition that discursive practices, in establishing both the terms of the pedagogy and the parameters of classroom action, produce what it means to be a subject, or to be subjected, within these practices (Walkerdine, 1989). Thus questions of whether communication should be controlled in traditional ways, or whether more natural patterns should be encouraged are raised (see, for instance, Lemke, 1990).

Clearly, there would be competing perspectives in such matters. Making sure that all children contribute would have some potential advantages, such as teaching all children the social skill of working together to solve problems and share mathematical understandings. Teachers can also judge students' understanding by their responses and hence adapt further teaching. As articulation of ideas helps to clarify them, expecting students to contribute to a discussion may also lead to improved learning. Encouragement to contribute may also result in a growth of confidence after successful attempts are made, leading to more willingness to contribute in future. Thus if teachers do not require oral participation, they could perhaps be thought to be denying some students potential opportunities for cognitive and social growth. Tobin (1984), suggests that teachers also use particular target students to direct the lesson flow.

However, the notions that there are only some acceptable indicators of participation and that teachers should control discourse arise from a didactic model of education where teachers set learning objectives in terms of measurable forms of predictable performance — and then structure, control and evaluate classroom activity in terms of these objectives. Some other models of education that are beginning to impact on mathematics pedagogy position teachers not at the centre of activity, but as facilitators of a variety of learning processes. Thus teachers do not dominate class and small-group discussions and the above functions of communication become the responsibility of all participants in classroom discourses (including the teacher). Using more natural patterns of interaction, students communicate with each other without regular deference to the teacher. Both teachers and students invite others to participate, but do not control participation. Thus students are not put on the spot through discursive imposition, are not positioned as "class" rather than "individual", and can take a more equal role in using language to establishing and maintain the aims of the social group.

While such models of education do not deny that curriculum content reaches students through the agency of teachers, they do require flexible patterns of communication within a context of new social relationships and practices in classrooms. Exploring the issue of oral participation within competing views of education demonstrated in the videotaped lessons has thus become a focus of our attention. This is not just a matter of counting numbers of interactions, examining ways that quieter students (or girls or members of minority groups) are encouraged to communicate more, or interviewing teachers about their theoretical stances. It requires probing examination of how and why learning processes are limited in particular ways. As Walkerdine (1985) notes,

> ... the issue of silence and speaking is not a simple matter of presence or absence, or of suppression versus enabling ... what is important is not simply whether one is or is not allowed to speak, since speaking is about saying something. In this sense, what can be spoken, how, and in what circumstances is important. It tells us not only about its obverse, what is left out, but also directs attention to how particular forms of language, supporting particular notions of truth, come to be produced. This provides a framework for examining how speaking and silence and the production of language itself become objects of regulation. (p. 205)

Thus the questions that are being raised are about how linguistic patterns in different classrooms (or with different activities) shape, constrain, or facilitate social relations between teachers with students, for students with each other, and of students with curriculum content and tools. They are also about which parties are talking (and when) for what purposes, and who is designing communication channels (and how, as well as with what effects). They are also about substantive messages about learning (as opposed to schooling) and about the nature of "doing mathematics" that are imparted during classroom discourses.

The lessons filmed do not demonstrate dichotomous pictures of individual teachers consistently controlling discussions in some classrooms compared with students free to participate as they see fit in other classrooms. These stances appear as points on a continuum with the teachers identifying speakers at times for specific purposes, then moving to positions of little control once those purposes are achieved, usually by creating windows for classroom participation in such a way that any student can refuse to take part in a discussion. At times, teachers control activity but not discourse, using invitations such as "Now let us hear a solution method from someone who hasn't already talked", or strategies such as students being asked to develop individual ideas then to share these in pairs before an open class conversation is held. Talk in these situations is not restrictively linear, does not put students into situations that they find threatening, and is not teacher-controlled in a limiting sense.

Conclusion

Weedon (1987, P. 108) claims that discourses are more than ways of thinking and producing meaning, in that the ways in which discourse constitutes the minds and bodies of individuals is always part of a wider network of power relations, often with institutional bases. Our research is continuing, with a focus on power relations which both shape and are shaped by the discursive fields of institutionalised education. Problematic aspects are being raised for discussion at conferences (e.g. Sullivan & Mousley, 1994), and teachers who participated in the filming are contributing to this dialogue (e.g. Mousley, Sullivan, & Gervasoni, 1994). With further reading, and with analysis of different media available for the examination of the classroom interaction, how various aspects of the communication position students is being examined.

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MATHEMATICAL DISCOURSE: INSIGHTS INTO CHILDREN'S USE OF LANGUAGE IN ALGEBRA

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ABSTRACT: The language used in the context of school mathematics has a number of special features and pupils have to come to terms with these features in order to gain access to the mathematical knowledge. This paper looks at the way children use the language in the context of three algebraic tasks and adopts the concepts of "register" and "cohesion" from Halliday's socio-semantic approach to analyse the language of their answers. The results show that to create written expressions to discuss abstract algebraic entities and relationships is not an easy task for the pupils.

HALLIDAY'S SOCIOLINGUSTIC APPROACH TO DISCOURSE

In order to learn, children must use what they already know so as to give meaning to what the teacher presents to them. Language helps to make available to reflection the processes by which they relate new knowledge to old. But this possibility depends also on the social relationships, the communication system between teacher and pupils, that is, the classroom discourse. Thus, the study of the classroom discourse is the study of language used in a specific social setting.

To understand the meaning of the social dimension of language, one has to think of language as the means of interacting with other people and constructing shared meanings, i.e. language as a means of communication. Human communication is not only a matter of generating sentences (linguistic competence) in the abstract, but also of being able to use and understand language in particular contexts (communicative competence). Halliday (1973) argues that language is a "meaning potential", a set of options in meanings. His theory makes two basic assumptions (Halliday, 1978):

(a) the linguistic system consists of three strata: *semantic* (meaning), *grammar* (the lexicogrammatical sytem, or wordings) and *phonological* (the sound);

(b) the semantic system involves three functional components: *ideational* (expresses the speaker's experience of the internal and external world), *interpersonal* (expresses social and personal relationships) and *textual* (makes the language "operationally relevant" in a context).

The textual component is of great interest because it is this component which defines the text-forming resources of the linguistic system. Halliday sees text as any written or spoken passage which forms a unified whole. Text is a semantic concept, it can be considered as a 'kind of super-sentence, something that is larger than a sentence but of the same nature' (1978, p.135). Sentences are in fact realisations of text rather than constituting the text itself. This attributes to the text an important role in gaining insight into the language used and thus the meanings exchanged in various social or other contexts. In particular, Halliday believes that there is a special value in analysing children's texts because young children tend to display their environmental links more directly, using less metaphorical mediation.

"Cohesion" is an important concept when considering text. Halliday notes that cohesion is the set of possibilities that exist within the language for making text hang together: the potential that the speaker or writer has at his disposal. Cohesion can be expressed either through grammar (grammatical cohesion) or through the vocabulary (lexical cohesion). So, for example, the phenomena of substitution, ellipsis and conjunction are grammatical types of cohesion, whereas those of synonym, collocation and others are lexical types of cohesion.

Halliday examines contexts of situations in which language is used and the ways in which one type of situation may differ from another. He uses the notion of "register" to show how linguistic situations differ from one another. Three variables need to be considered always, what is actually taking place (field) , who is taking part (tenor) , and what part language is playing (mode). Together these three variables provide the "register", since they determine the range within which meanings are selected and the forms which are used for their expression. The concept of register is a powerful one because it shows that the language we speak or write varies according to the type of situation. Halliday shows how situations determine language:

Situational elements	**determining**	**Semantic components**
Field (type of social action)		experiential, or ideational
tenor (role relationship)		interpersonal
mode (symbolic organisation)		textual

From the above discussion, it is clear that Halliday's theory offers a way of looking at pupils' written answers from a socio-semantic perspective i.e., in a functional way which considers the use of language as a choice of meanings, as an activity which has links with features of the situation in which it is used.

MATHEMATICAL DISCOURSE AND THE LANGUAGE OF THE SUBJECT: THE RELEVANT RESEARCH

Children in school need to learn how to use language in the context of the various subjects, in order to be able to cope with the demands of the curriculum. In the context of mathematics, this means that they need to learn how to use language in order to create, control and express their own mathematical meanings, but also to make sense of the mathematics of the others. It has been often suggested that the failure of so many children in mathematics is closely related to their limited access to the language of the subject.

In an earlier paper (Sakonidis and Bliss, 1991), we argued that this limited access is due to the fact that, although the language is used in the subject context, it shares with the rest of the English language the same basic grammatical and phonological elements, this is not the case for the semantic aspects. There are meanings special to mathematics that are not met in the everyday context and with which the learner has to come to terms. We believe that the term "mathematical register" in Halliday's (1978) sense, i.e. the set of meanings that belong to the natural language used in mathematics and that a language must express if it is used for mathematical purposes, signals successfully the language differences that are subject specific.

Research so far has mainly considered the lexical rather than the semantic or the syntactic components of the mathematical register. Three types of words used in the context of the school mathematics language have been identified: technical words (Twords), words common to mathematics and the everyday language (CEM); and ordinary words used in the context of mathematics but not carrying a mathematical meaning (NCEM). The relevant research (for example, Dickson, Brown and Gibson, [1984], Shuard and Rothery, [1984], Pimm, [1987]) indicates that Twords and CEM words present particular difficulty for pupils, but for different reasons. The former have a high level of technicality and the latter an everyday meaning which is often confused with that of the mathematical one. NCEM words can also be problematic because they often carry subtle, highly advanced and abstract ideas (for example, Reed [1984], Spanos [1988], Pimm [1984]).

From the above considerations, it becomes apparent that a study which examines children's language in the context of a mathematics topic could provide some useful insights into the way in which language functions in the context of that topic.

THE STUDY

The study which is presented here is part of a bigger piece of research which looks at pupils' ideas about algebra through their written language. The choice of the written language was made on the basis that it increases the possibility of a more thoughtful answer, whereas the choice of algebra was based on the assumption that there are very few means - other than linguistic ones - of expressing the abstract algebraic ideas. 394 pupils between the ages of 13 and 16 participated in the study: 155 year 9 (90 boys and 65 girls), 153 year 10 (73 boys and 80 girls) and 86 year 11 (44 boys and 42 girls). The subjects were taken from four urban schools: one boys, two girls and one mixed and they all had at least one year of formal teaching of algebra. The schools were banded for mathematics and a top and a middle group were taken from each school in years 9 and 10 and a top group only in year 11.

In order to examine the language of pupils, we decided to use Halliday's concepts of "register" and "cohesion", but not in quite the same complete way as Halliday. With "register", the concept of "mode" does not enter into this analysis since the responses are determined by a methodology that uses a written questionnaire. We use that aspect of "field" which is to do with the content of the lexical component, and those aspects of "tenor" which are to do with personal relationships, and with personal stance. Lastly we make use of part of the concept of "cohesion" by examining the connectives used by pupils. We now give more detail of those terms we are using.

Field: The category of "lexis" was formulated to describe word usage, that is:

(i) Technical (mathematical) words (Twords) : These are technical words, with a special mathematical meaning, found primarily in the context of mathematics, e.g. "equation", "numerator", "fraction".

(ii) Words common to everyday language and mathematics, but with different meanings in the two contexts (CEM lexis). These are words found both in everyday language and in a mathematical discourse, but with a rather different meaning in each, e.g."substitute" (as in an algebraic expression),

"re-arrange" (the formula), "simplify" (the algebraic expression). It is important to notice that most of these words do not have mathematical meanings in themselves, i.e. they are mathematically meaningless when isolated from the expression within which they are embedded.

Tenor: The following aspects of tenor were used:

(i) Personal relationships : The use of the personal and impersonal references. This category describes how the respondents define roles in the mathematical discourse, that is, whether they engage themselves or other human beings in the response (personal reference) or they attribute to the mathematical entities the main role (impersonal reference). Thus, the personal reference concerns the use of the pronouns "I", "we", "you", "us", whereas the impersonal reference concerns the use of the pronouns "it" and "they" (in a non human sense) and also the demonstratives "this" and "that".

(ii) Personal stance: The use of modal types/verbs . Here the category provides either the degree of certainty or uncertainty the pupil has about the ideas s/he is using, or the expression of the existence of alternatives/possibilities. The degree of certainty is expressed through the use of modal verbs such as "could", "might", "may" and so on.

Cohesion: There are three main kinds of conjunctive expressions examined in relation to this concept: *additive* ("and", "or", "also"), *adversitive* ("but", "yet", "although", "nevertheless"), *causal* ("because", "so", "therefore").

QUESTIONS GIVEN TO PUPILS

We now look at three of the five questions given to the 13-16 year old pupils in the form of a written questionnaire. We have chosen to analyse for this paper the three following questions since they generated more written text than the other two.

Question one: This question focuses on the area of a rectangle, and presents to the pupils the formula in three different forms: (i) area=width x length (mixture of words and symbols); (ii) A= a x b (symbols); and (iii) a x b (incomplete). Children are asked to choose, in part 1, the most helpful expression and, in part 2, the least helpful, each time giving reasons for their choice.

Question two: Pupils are given a problem where the relationship between two variables (C and p) is expressed through the formula C= p+2. They are presented with an answer given by a fictitious child and they are told that it is wrong. Pupils are asked to imagine that they are to help this pupil and to write down how they would explain what is wrong with the answer.

Question three: This question has two parts. In each part, pupils are given one linguistic expression used in an algebraic context and are asked to give "another set of instructions", which explains more clearly what the given expression entails. The two instructions are: Part 1 "remove the brackets" from the expression 3(x+1)-2x; Part 2 "substitute" s=2 and t=3 in the formula F=st.

In table 1 we give the percentages for the types of words used, Twords and CEM, by pupils in all three years for the above three questions. In Table 2 we give the percentages for the use of personal/impersonal pronouns and modal words (would, could) by pupils in all three years for the above three questions. We have not given a table for cohesion because the categories were not mutually exclusive and so discuss this for each question.

Table 1: Lexis/field within Register

	Twords			CEM		
	Y9	Y10	Y11	Y9	Y10	Y11
Question 1	%	%	%	%	%	%
Part 1	35	43	42	33	43	40
Part 2	31	33	28	38	45	47
Question 2	76	90	79	45	50	57
Question 3						
Part 1	77	90	66	28	61	38
Part 2	50	71	44	36	82	42

Table 2: Tenor within Register

	Personal relation*			Personal stance		
	Y9	Y10	Y11	Y9	Y10	Y11
Question 1	%	%	%	%	%	%
Part 1	41/55	37/63	29/57	5	12	5
Part 2	51/40	51/45	37/52	18	28	26
Question 2	14/64	13/67	20/72	41	40	44
Question 3						
Part 1	11/1.	21/3.	8/12.	4% over 3 yrs		
Part 2	10/2.	10/5.	12/6.	6% over 3 yrs		

* First use of personal then impersonal pronouns.

We shall now consider each of the three questions, in turn, in terms of the aspects of register detailed in the above tables.

Question 1

Field: The use of technical words is not very high in either of the two parts, being a little more frequent in part 1 than in part 2 for the older children. There are seven mathematical words used over the three years: "formula", "length", "width", "multiply", "rectangle", "equation", "number". Turning to CEM words, in part 1, the use is similar to that of TWords. In part 2, CEM words are used a little more frequently than TWords. There are four CEM words used across the three years, with relatively high and similar frequencies from year to year: "letter", "area", "stand for", "equal". Among them, two are algebraic CEM words (letter, stand for).

Tenor: The use of the impersonal reference in part 1 is higher than that of the personal reference across all years. In part 2, younger pupils (13/14 years), however, use personal reference more frequently than the older ones, this trend switching in year 10, with a greater use of the impersonal. The use of modal words is very infrequent in part 1 and a little more frequent in part 2, particularly with the older pupils.

Cohesion: Additive and adversitive connectives were used infrequently by all pupils. However, causal connectives, "because", "so", "therefore", were used in a very similar way and very frequently by year 9 and 10 pupils, for example for year 9 pupils , part 1: 54% and part 2: 64%. Year 11 pupils' use of connectives however falls because they seem to list criteria for choice rather than reasons for choice.

Examples of the aspects of register and cohesion in children's writing are as follows (we italicise the different aspects) :

"The most helpful answer is Jim's (area=width x length) *because it* isn't using *letters* which I sometimes get confused with. *It* simply gives *us* what to find, WIDTH, LENGTH (capitals in the response) without the use of A, B, A and the rest" (year 10).

"The most helpful answer is Jim's *because* the relationship is explained concisely due to the fact that each *variable* is named in the *equation*" (year 11).

Summarising for question one, the use of Twords does not exceed about 43% and CEM words are used as much as, if not a little more, than Twords. The use of reference is high, with the personal reference appearing less often over the years. There are very few pupils who use modal words, whereas the use of connectives is fairly high, particularly in the early years.

Question Two

Field: For all pupils, the use of technical words is very frequent, as high as 90% for year 10 pupils. Six Twords occur in all three years: "number", "add", "formula", "equation", "sum", "negative". Among them, the word "number " and then the words "add" and "formula" are fairly frequently used in all years. There are a number of additional Twords, e.g. "positive", "variable", appearing over the years, but very infrequently (maximum 3%). The use of words CEM is less frequent than Twords, increasing a little with age. There are five CEM words occurring across the years, namely "equal", "represent", "letter", "value", "stand for", only one of which is frequently used in all three years, that is, "equal" (about 33%). A considerable range of other CEM words occur in each year, but very infrequently (maximum 2%).

Tenor: On the whole, impersonal reference is used frequently increasing to 72% with year 11 pupils. Personal reference is used infrequently (maximum by year 11 pupils of 20%). The use of modal verbs is fairly high, its frequency being similar over the years (maximum 44%).

Cohesion: There is a use of all connectives, with causal words being the most frequent connectives (90% for year 10 pupils). Adversitive words are the next most frequently used type of connective words but never exceeding 28% and this for year 9 pupils.

Some examples of the pupils' responses are:

"*Your* way is wrong because *you could* have p+2=C and C *would* be the same *value*. The way I *would* do it is that C is a bigger *number because* if you had p=C the *values would* be the same but *if you* have p+2=C *you* know that p needs another *value* to equal C" (year 9)

"The explanation is wrong *because* just *because* the 'C' is on the left of the *'formula'* doesn't mean *that's* the bigger, it is C *because its* (it's) a total of an *addition* 'p+2', so the total has to be bigger than an individual *letter*" (year 10).

In summary, Twords are very frequent, whereas CEM words are frequently used but not as frequently as Twords. For both types of words, there is very little change over the years. The use of the impersonal reference is very frequent, unlike that of the personal pronoun where use is low. Finally, about 40% of the pupils use modal words, and up to 90% of them use causal connectives.

Question 3

Field: On the whole, the use of technical words has a fairly wide range (between 44% and 90%), being a little higher in part 1 (from 66% to 90%) than in part 2 (from 44 to 71%). There are five Twords used across the years in both parts : "multiply", "sum", "number", "formula", "equation". The words "multiply" and "sum" in part 1, and the words "number" and "formula" in part 2 are the most frequently used words. The use of CEM words is overall lower than the use of technical words in both parts , with the exception of year 10 pupils in both parts.

Tenor: In both parts, the use of either of the two references, that is, personal or impersonal, is very infrequent.

Cohesion: Overall, causal words, such as "because", "so", etc., are the only type of connectives used at all frequently (average about 20% in both parts).

Examples of answers to this question are:

"*Brackets* are often put in to help *you* in *equations* and not get you mixed up. *So* when *you* take the *brackets* away *you* usually have *simplified* the *equation* by using the *large number* outside, on the left. *So you* end up with a more easier task" (year 10, part 1).

"*Substitute can* be taken to mean exchange the *numbers* given for the *letters* in the *equation*. Take the *letters* out of the *equation* and put the *numbers* in" (year 11, part 2).

Summarising, on the whole, technical words appear more frequently than CEM words for years 9 and 11, but year 10 both types of word are used very frequently and similarly in both parts. The use of reference is very low, and the use of causal connectives is also fairly low.

DISCUSSION AND CONCLUSIONS

The written responses of the pupils to the three questions require, among other things, the ability to co-ordinate ideas and accommodate other points of view, components which entail considerable complexity. In addition, two other features need to be considered: the abstraction of the algebraic ideas and the fact that writing itself is not usually included among mathematical and, in particular, algebraic activities in school. It may thus be expected that written accounts could prove to be demanding tasks to perform.

On the whole, although the use of technical vocabulary is considerable in questions 2 and 3, its content is very restricted and similar across the years. This suggests that independent of age and algebraic experience, pupils use only a limited number of technical words in their responses. The use of CEM words is less than that of technical words. Older pupils, particularly year 10, use them slightly more frequently than younger ones. Overall, there seems to be an important common core of technical and CEM vocabulary used in all years. These two vocabularies are as follows:

Technical core vocabulary: "formula", "number", "add", "multiply", "sum", "length", "width".

CEM core vocabulary: "equal", "letter", "stand for", "represent", "value", "area", "expression", "replace".

The above seems to indicate that pupils have a facility with technical words which refer to operations but not to variables. When using the CEM words, there are one or two words that hint at the notion of variable. This possibly shows that pupils are attempting to understand it but do not yet have either a complete grasp of it or the appropriate means to express it.

The frequent use of impersonal reference (question 1 and particularly 2) could be mainly as a consequence of attempting to avoid using an algebraic term, consciously or unconsciously. Thus instead of saying "the variable should", they argue "it should", that is, in an anaphoric manner. The use of the personal pronoun is, on the whole, very low, except when the task is of a pragmatic nature; that is, the child is asked to judge certain ideas, without having to explain or instruct but rather needs to express his/her own opinion (question 1).

Modal words are infrequent and occur most often when the main concern of the response (and the task) is the idea of variable and the values for which it stands. This could be explained by a tendency of the pupils to again avoid dealing with the generality and abstraction of the algebraic ideas, either because they do not grasp them or because they cannot express them.

It appears that not only the nature of the tasks but also the writing task set by the different questions affect the way in which pupils use connectives. When the question and the writing task provide sufficient structures to indicate a need for the use of these words, then it becomes clear that they are present in the children's vocabulary and can be used well (questions 1 and 2). It is possible that to write spontaneously about algebra without some indication or hint of a definite structure might be to make a big cognitive demand on the pupils.

Aspects of Halliday's concepts of "register" and "cohesion" help us unravel some of the intricacies of the pupils' written texts, highlighting what for them is important in the mathematical discourse, the picture for the teacher possibly being quite different. Thus while for the adult the use of the impersonal pronoun could be seen as a sophistication, for the pupils such a use might be a helpful means of avoidance of difficulties.

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Teaching realistic mathematical modeling in the elementary school. A teaching experiment with fifth graders

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Abstract

Recent research has convincingly documented elementary school children's tendency to neglect commonsense knowledge and realistic considerations during mathematical modeling of word problems in school arithmetic. The present paper describes the design and the results of an exploratory teaching experiment in which a group of 11-12-year olds followed a course wherein word problems are conceived as exercises in mathematical modeling, focussing on the assumptions and appropriateness of the model underlying any proposed solution.

Theoretical background and research questions

Arithmetic word problems constitute an important part of the mathematics program at the elementary school. The most important reason for using this type of problem in schools is to train pupils in applying their formal mathematical knowledge and skills in real-worldlike situations. However, for several years it has been argued that the practice of word problems in school mathematics does not develop in pupils a tendency to include commonsense knowledge and realistic considerations in their solution processes. Rather than functioning as realistic contexts that elicit in pupils the use of their knowledge and experience about the real world, school arithmetic word problems have become artificial, puzzle-like tasks that are perceived as being separate from reality (Nesher, 1980).

Recent studies by Greer (1993) and by Verschaffel, De Corte and Lasure (1994) have yielded strong empirical evidence for this argument. In these studies large groups of pupils (11-13-years olds) were confronted with a set of word problems, half of which were standard items (S-items) that can be solved unambiguously by applying the most obvious arithmetic operation(s) with the given numbers (e.g., "Steve has bought 4 strings of 2 meters each. How many strings of 1 meter can he cut out of these strings?"), while the other half were problematic items (P-items) for which the appropriate mathematical model is less obvious and indisputable, at least if one seriously takes into account the realities of the context evoked by the problem statement (e.g., "Steve has bought 4 planks of 2.5 meters each. How many planks of 1 meter can he saw out of these planks?"). An analysis of the pupils' reactions to the P-items yielded an alarmingly small number of realistic responses or comments based on realistic considerations (e.g., responding the above-mentioned P-item with "8 planks" instead of "10 planks", because in real life one can only saw 2 planks of 1 meter out of a plank of 2.5 meter).

These well-documented undesirable learning outcomes are generally attributed to the following major characteristics of the current instructional practice: (1) the impoverished and stereotyped diet of standard word problems which can always be unambiguously modeled and solved through the most obvious arithmetic operation(s) with the numbers given in the problem, and (2) the fact that instruction is focused at teaching pupils to solve these problems by identifying and execute the correct arithmetic operation, rather than taking a different perspective whereby word problems are conceived as exercises in realistic modeling, focussing on the proper consideration of the assumptions and the appropriateness of the model underlying any proposed solution (Greer, 1993; Verschaffel et al., 1994). However, so far there is hardly any empirical evidence supporting these claims. In the present study an attempt is made to verify the hypothesis that through appropriate instruction one can develop in primary school children a disposition toward realistic modeling of word problems.

Research design and data analysis

Three classes from the same school participated in the experiment: one experimental (E) class of 19 fifth-grade children, and two control classes (C1 and C2) of 18 and 17 sixth-grade children, respectively.

The pupils from the E-class participated in an experimental program on realistic modeling (during the hours allocated for mathematics). The program consisted of five learning units of about 2 1/2 hours each, spread over a period of about 2 1/2 weeks. While the teacher was intensively involved in the preparation of the experimental program and the tests, the actual teaching was done by the first author. During the experiment, the pupils from the two control classes followed the regular mathematics curriculum.

The three groups were given the same pretest, which consisted of 10 P-items and five buffer items (S-items). The 10 P-items were grouped in five pairs: one item in each pair was similar to an item from one session of the experimental program both in terms of the context evoked by the problem statement and in terms of the underlying mathematical modeling difficulty (no-transfer item), while in the other problem the same underlying mathematical modeling difficulty had to be handled in a different problem context (= transfer item) (see Table 1).

With respect to each problem, pupils were not only asked to write their answer, but also to mention how they arrived at their answer and/or to write down possible difficulties or worries experienced during the solution of the problem.

Table 1. The ten P-items from the pretest

1A*	1180 supporters must be bused to the the soccer stadium. Each bus can hold 48 supporters. How many buses are needed?
1B	228 tourists want to enjoy a panoramic view from the top of a high building. In the building there is only one elevator. The maximum capacity of the elevator is 24 persons. How many times must the elevator ascend to get all tourists on the top of the building?
2A	At the end of the school year, 66 school children try to obtain their swimming diploma. To get this diploma one has to succeed in two tests: swimming 100 meter breaststroke in 2 minutes and treading water during one minute. 13 children do not succeed in the first test and 11 fail on the second one. How many children get their diploma?
2B	Carl and Georges are classmates. Carl has 9 friends he wants to invite for his birthday party, and Georges 12. Because Carl and Georges have the same birthday, they decide to give a joint party. They invite all their friends. All friends are present. How many friends are there at the party?
3A	Some time ago the school organized a farewell party for its principal. He was the school's principal from January 1 1959 until December 31 1993. How many years was he the principal of that school?
3B	This year the annual rock festival Torhout/Werchter was held for the 15th time. In what year was this festival held for the first time?
4A	Sven's best time to swim 50 meters breaststroke is 54 seconds. How long will it take him to swim 200 meters breaststroke?
4B	This flask is being filled from a tap at a constant rate. If the depth of the water is 4 cm after 10 seconds, how deep will it be after 30 seconds? (This problem was accompanied by a picture of a cone-shaped flask)

5A A man wants to have a rope long enough to stretch between two poles 12 meters apart, but he has only pieces of rope 1,5 meters long. How many of these pieces would he need to tie together to stretch between the poles?

5B Steve has bought 4 planks of 2.5 meters each. How many planks of 1 meter can he saw out of these planks?

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* The second problem of each pair is the transfer item.

At the end of the experimental course a parallel version of the pretest was administered in all three classes as a posttest. However, in one of the control classes - namely C1 - this posttest was preceded by an introduction of 15 minutes in which the pupils' attention was drawn to the fact that routine solutions for word problems are sometimes inappropriate when considered in terms of realistic constraints; a few examples of such inappropriate routine solutions were given, and pupils were warned that the test contained several items for which such routine solutions are inappropriate.

One month after the posttest, the pupils from the E-group received a retention test consisting again of 10 P-items and five buffer items (S-items) as part of a normal mathematics lesson. Half of the P-items were parallel versions of the transfer items from the posttest; the other half involved problem contexts and underlying mathematical modeling difficulties that were even more remote from those encountered during the experimental program (e.g., "Bruce and Alice go to the same school. Bruce lives at a distance of 17 kilometer from the school and Alice at 8 kilometers. How far do Bruce and Alice live from each other?" and "What will be the temperature of water in a container if you pour 1 liter of water at 80° and 1 liter of water at 40° into it?").

To evaluate the effects of the experimental program, an analysis of variance was performed with group (E versus C1 versus C2), time (pretest versus posttest) and problem type (no-transfer items versus transfer items) as the independent variables and the proportion of "realistic reactions" (RR) on the P-items as the dependent variable. The term RR refers to responses that result directly from the effective use of real-world knowledge about the context elicited by the problem statement (for example, the RR for the "planks"-problem mentioned in the first section, is "8"). However, a pupil's reaction was also considered as a RR, when a non-realistic answer was accompanied with a comment or consideration about the problematic assumptions of the mathematical model underlying the proposed routine solution (for example, when the "planks"-problem was answered with "10", but followed by a comment such as "Steve will have a hard time putting together the remaining pieces of 0.5 meter"). Significant main and interaction effects were further analyzed by means of Tukey's test.

Based on video-recordings of the lessons and pupil notes, detailed qualitative analyses of the problem-solving and the discourse processes in the experimental class, were also performed. Due to place restrictions, the findings of this analysis cannot be reported here.

The experimental course

This section describes and illustrates the major characteristics of the experimental program.

First, the impoverished diet of standard word problems offered in traditional mathematics classrooms was replaced by more authentic problem situations especially designed to stimulate pupils to pay attention at the complexities involved in realistic mathematical modeling and at distinguishing between realistic and stereotyped solutions of mathematical applications. Each learning unit focused on one prototypical problematic topic of realistic modeling. The topic of the first unit was: making appropriate use of real-world knowledge and realistic considerations when interpreting the outcome of a division problem involving a remainder. The opening problem involved a story about a regiment of 300 soldiers doing several typical military activities. Each part of that story was accompanied with a question which always asked for the same arithmetic operation (namely 300 : 8 = .) but required each time a different answer (respectively, "38", "37", "37,5" and "37 remainder 4"). The theme of the second learning unit was the union or separation of two sets with joint elements. The opening problem was about a boy who had already a given number of comic strips of "Suske and Wiske" and gets a package of second-hand albums of "Suske and Wiske" from his older cousins (who have lost their interest in these comics). The pupils had to determine the number of missing albums of "Suske and Wiske" in the boy's collection after getting this present (given the total amount of albums in this series). The third unit focused on problem situations wherein it is not immediately clear whether the result of adding or subtracting two given numbers yields the appropriate answer or the answer + 1 or - 1. In the opening problem, pupils were given the number of the first and the last ticket sold at the cash desk of a swimming pool on a particular day, and they had to decide how many tickets were sold that day.

The fourth unit dealt with the principle of proportionality and, more particularly, how to discriminate between cases where solutions based on direct proportional reasoning are or are not appropriate. The starting problem was about a young athlete who's best time on the 100 meters was given, and pupils were asked to predict the boy's best time on the 400 meters. The fifth and last learning unit started with a problem of a boy who wanted to make a swing and who had to decide about the amount of rope needed for fastening the swing at a branch of an big old tree at a height of 5 meter. In that session, pupils experienced that in many application problems one has to take into account several relevant elements that are not explicitly nor immediately "given" in the problem statement but that belong to one's commonsense knowledge base.

Second, the teaching methods used in the experimental program differed considerably from traditional mathematics classroom practice. Each session started with an opening problem (see above) that was solved in mixed-ability groups of 3-4 pupils. Within each group, the pupils had to solve the problem individually first (on a A4 sheet), and to agree upon a common response afterwards (on one common A3 sheet). In addition, the pupils were asked to answer a series of reflective questions such as "What difficulties did you encounter when solving this problem?", "On what points did you disagree?" or "What did you learn from solving this problem?". This group assignment was followed by a whole-class discussion in which the answers, the problem-solving steps and the additional comments of the different groups were collected, compared and evaluated, thereby focussing on the proper consideration of the assumptions and appropriateness of the mathematical models underlying the distinct proposed solutions. Then each group was given new worksheets containing a set of four problems, two with and two without the same underlying modeling difficulty as the opening problem. This group assignment was again followed by a whole-class discussion. Finally, each pupil was individually administered - either at a different time during school hours or as homework - one problem that involved once again the topical modeling difficulty, and the pupils' reactions to this individual assignment were also listed and discussed afterwards during a whole-class discussion.

The third major characteristic relates to a more subtle aspect of the experimental teaching environment, namely the establishment of a new classroom culture in line with the mathematical modeling perspective outlined above. This was attempted by the following activities: demonstrating and explaining valuable problem-solving strategies to the whole class (such as making a drawing or a diagram of the problem situation; thinking of a similar problem situation with easier numbers; applying informal, context-dependent solution procedures, etc.); giving appropriate hints and feedback during the group and individual assignments; spending a lot of time to listening to pupils' explanations and justifications of their own solutions; explicitly negotiating new social norms about what counts as a good problem, a good solution procedure, or a good response (e.g., "adults sometimes count on their fingers too"); and, discussing the role of the teacher and the students in a maths class (e.g., "Don't expect me to tell you what answer is the correct one").

Results

The analysis of variance revealed a significant ($p < .0001$) "group X time" interaction effect. During the pretest pupils from all three groups demonstrated a strong overall tendency to exclude real-world knowledge and realistic considerations from their problem solutions. The fifth graders from the E-class produced somewhat less RR on the 10 P-items of the pretest than the sixth graders from the C1- and the C2-classes - the percentages were 7 %, 20 % and 18 %, respectively -, but these differences were not significant. However, there was a significant increase in the number of RR from pretest to posttest for the E-group: from 7 % RR on the pretest to 51 % RR on the posttest. To the contrary, in the two control classes the progress in the amount of RR from pretest to posttest was non-significant, namely from 20 % to 34 % for C1 and from 18 % to 23 % for C2. The relatively small and non-significant increase in the number of RR in the C1-class as compared to the E-class, indicates that merely telling and illustrating that routine solutions for word problems are not always appropriate, is certainly not enough to transform pupils from mindless and stereotyped task performers into critical and realistic problem solvers. Such a transformation requires intensive training.

Furthermore, the lack of a significant "group X time X problem type" interaction indicates that the increase in the number of RR in the E-class from pretest to posttest, cannot be considered as a task-specific training effect. Indeed, while the increase in the percentage of RR in the E-group from pretest to posttest was larger for the five items that were similar to those from the experimental course (from 9 % to 60 %) than for the five transfer items (from 6 % to 41 %), the increase was significant (at the 5 % level) for both kinds of problem.

Additional evidence in favour of the experimental course is provided by the positive results of the E-class on the retention test. The percentage of RR for the 10 problems of the retention test (i.e., 41 %) was exactly the same as on the five transfer items from the immediate posttest. We remind that this retention test was administered as part of a normal mathematics lesson, and that it contained not only parallel versions of the transfer items from the immediate posttest, but also several P-items which were even more dissimilar from the training items. Interestingly, these latter P-items from the retention test elicited also percentages of RR that were much higher than those observed in equivalent groups of pupils who solved the same P-problems without special training in realistic mathematical modeling (Greer, 1993; Verschaffel et al., 1994)

Discussion

Although the positive results reported above are jeopardized by some methodological weaknesses of the present study (such as the small size of the experimental and control groups, the relatively short duration of the instructional treatment, and the absence of a retention test in the two control classes), they nevertheless provide good and promising support for the hypothesis that it is possible to develop in pupils a disposition towards realistic mathematical modeling, by immersing them into a classroom culture whereby word problems are conceived as exercises in realistic mathematical modeling focussing on proper consideration of the assumptions and appropriateness of the model underlying any proposed solution.

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Seven Dimensions of Learning - A Tool for the Analysis of Mathematical Activity in the Classroom

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Seven aspects of students' learning experience are tentatively identified as sufficient for providing a-coherent social constructivist account of goal directed activity. An example is then given illustrating the application of these in the analysis of an exploratory conversation with a student engaged in mathematical activity.

Unstructured conversations with students were held during their normal mathematics lessons with the specific intention of exploring the goals towards which they were working. These goals, defined at three 'levels', are recognised as *rationale*, *purpose* and *interpretation* (Goodchild 1994b). Analysis of the conversations, using Q.S.R. NUD.IST (1994) as a software tool for indexing, retrieving and structuring revealed evidence that four further 'dimensions' should be considered if a *complete* account of the students' goal directed learning activity was to be attempted. These additional dimensions are *awareness*, *conception*, *affect* and *content;* students' positions in each of these seven dimensions vary in dialectical relationship with each other. The shift from 'level' to 'dimension' is not accidental as dimension does not carry the same hierarchical meaning as level and the notion of hierarchy seems to be inappropriate in the context of 'dialectical relationships'. The seven dimensions are outlined below, together with a brief explanation which fixes them into the existing framework of learning theory. The application of these dimensions in the analysis of researcher - student conversations is then illustrated.

The epistemological basis of the research is that of social constructivism (Ernest 1991) and focuses upon students' goals in classroom activity and, it is hoped, will provide an empirical basis for demonstrating the coherence and sufficiency of social constructivism as a framework for explaining an important aspect of students' learning activity. Lerman (1994) has argued that social constructivism attempts to paste together inconsistent theories of constructivist and cultural psychologies and thus creates an incoherent account of cognition. The seven dimensions outlined here implicate both theoretical accounts and the coherence is established in the 'learning subject' rather than a synthesis of ontological and epistemological theories.

The choice of the first three dimensions; *rationale*, *purpose* and *interpretation*; were influenced by Lave's account of cognition in social practice (Lave 1988) and activity theory (Leont'ev 1979, Mellin-Olsen 1987). The initial interpretation of data was based on these but the need for further dimensions began to emerge from the very early stages of the analysis (Goodchild 1993, 1994a,b). A brief explanation of the dimensions is now given.

Rationale: Mellin-Olsen (1981) argues that successful learning requires students to hold a rationale for learning and he indicates two are possible, an S- rationale where students engage in learning activity because the content of the activity is perceived as being useful for its own sake; and an I- rationale where students engage in learning activity because it will enable them to achieve some other possibly unrelated goal, such as achieving success in examinations and thus gaining access to desired employment. It is apparent that these two rationales do not account for all observations of students engaging in classroom activity, some appear to work for no other reason than *this is what we do in mathematics lessons*. I call this a P- rationale ('P' for Practice), strictly it is not a rationale for learning, it is a rationale for a particular form of behaviour, in the sense that Bruner (1990) writes about behaving 'Post-Office'.

Purpose: As El'konin (1961) observed, there is a significant difference between everyday activity and educational activity in that in the former the subject aims to change the object of the activity whereas in the latter the subject aims to change him/herself. It is clear that this distinction is not always obvious to students who reveal that the purpose of their activity is the production of answers or solutions to problems rather than learning.

Interpretation: Learning is seen to be about changes which take place within the learner, and cognitively, those changes are due to the constructions of the learner, possibly through processes of assimilation and accommodation as proposed by Piaget. Constructivists agree that reflection is fundamental to the process of learning. Learners need to reflect both on their own thinking and on the object of their thinking, so to signal that reflection may be in both directions the word *interpretation* is preferred. There is also the possibility that students may be so engrossed in the production function that no reflection takes place at all and in this case the interpretation is labelled 'blind' (Goodchild 1993) following the use of the word 'blind' by Christiansen and Walther (1986) when they discuss this type of response to classroom activity.

Awareness: Students do not approach any activity with completely open minds and they will perceive the activity in the context of their own belief and value system, relating this to the object of their study, and practices in teaching and learning. Partly this includes metacognition (Schoenfeld 1987) and attempts to develop students' metacognitive processes have been made with some success (Bell et al 1994). In the present context *awareness* also includes students' beliefs about the nature of mathematics (Goodchild 1994a) and what constitutes 'proper' behaviour (by teacher and students) in the classroom.

Conception: Following the argument of Confrey (1991) the word misconception is avoided because each student makes her/his own conception as seems right from their own experiences of a particular concept. Their conception may *fit* with their experience but not *match* with that conventionally held, thus a distinction is made between the *conventional* conception and the *student's* own conception if it is seen to differ from convention. It is also necessary to distinguish between *syntactic* conceptions where students are able to follow the form of a process without establishing the meaning and *semantic* conceptions (Skemp 1982) where the object or process is meaningful, here again both syntactic and semantic conceptions may match or merely fit with convention.

Affect: much has been written about the power of affect in cognitive processes (e.g. McLeod and Adams 1989). Every student approaches mathematical activity with a previous history of success and failure, of pleasure and pain, of comfort and stress. Thus the affective dimension is of significance. Affect here is used as an all embracing term to include the whole range of attitudinal responses from cold to hot emotional reactions.

Content: superficially this is defined by the scheme of work, the teacher and the text book, in the event *content* is (continually being) redefined by the student through dialectical relationships with the other dimensions.

Methodology

One year ten (14 to 15 years old) class of 'intermediate achievement' was selected on the basis of practical considerations such as ease of access, familiarity with school, a class teacher willing to tolerate the researcher's presence, and a working atmosphere in the class which would facilitate the tape recording of conversations between researcher and student. Except on a very small number of occasions no demands were placed upon the teacher or deviations from the scheme of work were made to facilitate the research, thus as far as possible the class maintained its normal routine and conversations with students were held relating to their routine work. About 150 substantive conversations were held during the course of 85 lessons attended throughout the school year, the majority of those lessons missed were at the beginning of the year as the teacher established a working relationship with the class.

Of the various methodological problems with research of this nature perhaps the most significant is that introduced because the researcher is the instrument of both data collection and analysis. It is possible for the research to reveal no more than his own belief system thus control over the collection and interpretation of data weakens any claim of objectivity. One means of addressing this problem is to open up the ongoing analysis and interpretation of data to scrutiny and criticism, this is one of the aims of this report.

The conversation used below is chosen specifically because the 'seven dimensions' are not immediately obvious and are clearly the result of interpretation.

Example

The conversation takes place during a lesson in which the class is revising for an end of module test; the scheme of work is fragmented into modules of five or six weeks; each module consisting of a variety of topics; a test assessing performance in these topics is set at the end of each module. Student SJ initiates the conversation by claiming that she does not understand the first question on the revision sheet.

[numbers refer to 'text units' rather than lines].

1. I - . . . you can read it to me for a start
2. SJ - A garden centre sells fertiliser, the cost being, what's that?
3. I - Come on you can read that
4. SJ - Proportional is that how you say it?
5. I - Yeah that's right
6. SJ - Is that how you say it?
7. I - You just said it
8. SJ - Proportion-al
9. I - Proportional
10. SJ - Oh yeah, proportional to the amount bought four kilograms of fertiliser cost seven pounds fifty what is the cost of twenty kilograms?
11. I - And what don't you understand there?
12. SJ - How to work it out, like how did they get four grams cost seven pound fifty?
13. I - Well didn't they look in the catalogue and say "Oh, four kilograms cost seven pound fifty"
14. SJ - Forgot how you work it out
15. I - Hmm?
16. SJ - Forgot how you work it out
17. I - But you haven't got to work out what four kilograms they're telling you that.
18. SJ - Yeah but how did they?
19. I - Well you
20. SJ - Cause I got, how did they do that, so I can work out twenty?
21. I - Um, if you go to a shop, how do you know how much things cost?
22. SJ - Doesn't say
23. I - Do you ask the person behind the counter, "How did you work out the cost of that?"
24. SJ - Oh no, he just says it
25. I - Just says it. So, in this garden centre it just says
26. SJ - Four grams
27. I - Four kilograms costs seven pound fifty
28. SJ - So how do I work out twenty then? Do I do twenty times four?
29. I - Not quite
30. SJ - Share by?
31. I - Why would you do share by?
32. SJ - Cause, um, cause I need to know, cause I need twenty (I - Yes) kilograms instead of four, so if I share it, I work out instead of doing one, no instead of working out the four (I - Yes) I work out twenty (I - Yes) and then that'll be the answer
33. I - Can you think of another way of asking that same question?
34. SJ - Er, no
35. I - Look suppose you have worked out that for your garden you need twenty kilograms, (SJ - Hmm) you go to the garden centre and you find they sell bags, and each bag holds four kilograms, how many bags are you going to have to buy to get as much as you want?
36. SJ - Four
37. I - How do you work out four?
38. SJ - Timesed by itself
39. I - Why did you times it by itself?
40. SJ - Don't know, don't know how to work it out
41. I - Right, think you do . . you just need to be thinking a bit about it. You need to buy twenty kilograms, you need to buy twenty, but you look on the shelves and they've just got the bags and they hold four kilograms in each bag, (SJ - Yeah) how many bags are you going to have to buy?
42. SJ - Five
43. I - Why five?
44. SJ - Five, five fours are twenty
45. I - Five fours are twenty. Right, how did you work out that that was what you'd got to do?
46. SJ - Cause I divided twenty by four
47. I - You divided the twenty by four. Does that make sense?
48. SJ - Yeah
49. I - So if you've got to buy five bags, how much is it going to cost you?

50. SJ - Five times seven pound fifty
51. I - Right, which is?
52. SJ - [Uses calculator] Thirty seven pounds fifty

Asked to read the question aloud it is clear that she has a problem with the word 'proportional' but this is quickly overcome and she is stuck again as she believes she lacks some information or skill, she does not know how the price £7.50 for four kilograms is worked out. In trying to enable her to solve the problem it is recast into a more detailed narrative and broken down into two steps, 'how many bags are you going to have to buy?' And 'how much is it going to cost you?' To these questions SJ is able to give correct answers which she is able to explain meaningfully. SJ claims that this makes sense to her and her answers seem to bear this out. The conversation moves on to consider her feelings about the problem and its relevance to her everyday life.

Content: Superficially the content of the task is proportionality but this concept becomes obscured in the problem solving process. The problem is set in a real world context, its solution requires a knowledge of when division and multiplication are appropriate operations, as a calculator is available skill in these operations is not required nor any knowledge of number bonds. Proportionality was studied towards the beginning of the module, four to five weeks previously, four lessons being spent on the topic (there are three lessons per week), SJ had been present at all of these lessons. Although ostensibly set in the real world SJ's conception of the problem appears to situate it elsewhere.

Conception: SJ appears to situate the problem as a classroom task rather than in the given real world context. Evidence for this is apparent, firstly, [text units 1-10] where SJ has difficulty with the word proportional, this takes the problem out of her everyday experience. Secondly, SJ looks for the way to do it [text unit 12] 'how did they (get) four grams (sic) cost seven pound fifty?' [Text unit 20] 'how did they do that, so I can work out twenty?' And [text unit 62] 'I don't know where they've got seven fifty for four grams'. Compare these statements with her solution when the problem is rephrased [text units 35 and 41] and SJ is able to construct a solution which appears to be meaningful to her, based upon conceptions which appear to match those conventionally held [text units 42 - 52]. In both cases it is argued that the conceptions are situated in a context: SJ's conception - as a classroom problem; and its redefinition - as a real world problem, the change in context enables SJ to reach a solution. Further we note that in both cases SJ's conception is semantic in that she is relating to the meaning of the problem rather than a syntactic process of calculations; this is not to say however that she would not have been content with a syntactic solution if this had been made available to her.

It may be claimed that the support given to her [text units 17 - 41] changes the nature of the content and this is possible but in changing the nature of the content there is a response/reaction in the conception which SJ makes and that is part of the point being made here.

Awareness: SJ reveals differing states of awareness during the conversation. In the first instance her awareness is of not being able to do the problem, this is what she 'feels' arising from her conception. This uncertainty is revealed when asked 'what don't you understand here?' She responds 'how to work it out,' and later she asks 'so how do I work out twenty then?' [text units 11 and 12, 28]. She is also aware that there must be some mathematical process - knowledge of which she does not possess, [text units 16 - 20]. Later as the problem is restated her awareness changes to that of understanding [text units 53 - 58 and 65 - 68]. When questioned she also reveals her awareness of the value of the activity - none or very little [text units 77 - 116].

Awareness is important because she starts off aware that her knowledge is incomplete in the respect of this problem, without this she may not set herself a goal to learn. However she is aware that there is little value in this type of activity anyway so she is unlikely to set herself a learning goal in any case. Further, implicit in the conversation is her awareness of the nature of mathematics, that it is in some way 'special,' that there are special techniques and skills which must be applied to problems in the classroom and she does not possess all those necessary in this case. The language of the problem and her physical presence in a mathematics class seems to mystify the problem for her.

53. I - Thirty seven pounds fifty? Does that make sense?
54. SJ - Yeah, [text units 55, 56 omitted]
57. I - Do you think question one is a difficult question?
58. SJ - No, not really
59. I - OK, what was causing you the problem at the beginning?
60. SJ - 'Cause four's far away from twenty that's why I didn't understand it really
61. I - I haven't really understood what you're saying to me
62. SJ - 'Cause seven fifty's are four (I - Yeah) and I didn't know how to work twenty out, because, I don't know where they've got seven fifty for four grams
63. I - Right. So, if someone was, if you were to come along and and, help someone with that question, what would you say to them so that they could understand what the question was about?
64. SJ - Er, the same way you explain it, saying that I just don't know. How many twenties in four and then saying then you've got to times the answer by that there and that gives you the answer
65. I - But suppose I say to you now, well how do they work out four kilograms cost seven pounds fifty?
66. SJ - Is it four times something is seven fifty so to work out that they do seven fifty divided by four and you get the answer for working that one there out
67. I - And that will be what one is?
68. SJ - Yeah
69. I - Yeah OK. But is that important to know that?
70. SJ - No
71. I - What is important in this question?
72. SJ - Four
73. I - And what else?
74. SJ - Seven fifty
75. I - Yeah, anything else that's important in the question?
76. SJ - Um,
77. I - Do you think it's important to know this type of work?
78. SJ -
79. I - You don't
80. SJ - No
81. I - You can't think of where you might use it
82. SJ - No

[text units 83 to 119 have been omitted due to limitation of space]

Interpretation: SJ's awareness and conception indicates that she has engaged in some reflective activity but her reflection appears to cease as she constructs a conception which situates the problem in a classroom context. In the initial stages of the conversation SJ appears to employ a strategy of guessing thus revealing no reflection at all [text units 28 - 40]. Her guesses appear to be wild stabs at producing responses making use of any numbers which are at hand; her interpretation is 'blind'. Her concern after the initial problem with the word 'proportional' is not with understanding what 'proportional' means but with obtaining a method to produce an answer.

Purpose: SJ's willingness to guess at answers and her lack of awareness of any value in the activity reveal that she perceives the purpose of her work as being the production of answers within the classroom. Her productive intent is also apparent from the language used 'how to work it out' [e.g. text unit 12, 19, 20 etc.] which is quite common in a classroom where emphasis is placed upon answers rather than understanding the processes leading to the answers.

Rationale: SJ is not aware of any value in the work for her, she is aware that this is revision for a test and so it may be argued that she has an I-rationale for learning but in that she appears to be content with a 'blind', guessing interpretation it seems unlikely that she actually possesses any rationale for learning. However, SJ does engage in the work, of course it might be said - 'so would anyone with the attention she is getting' but in fact that does not necessarily follow, some students in the study did occasionally reject the opportunity to talk about what they were doing because, for a variety of reasons, they were not engaging with the task set. Further SJ's normal behaviour is to engage with the tasks set irrespective of whether she was concurrently held in directed conversation. Without the possession of a rationale for learning it is argued that she works as a result of the possession of a basic rationale for engaging in classroom activity, this is what I call the P-rationale.

Affect: SJ's readiness to engage in 'blind' guess work and her apparent reluctance to reflect upon the task may be an affective reaction related to her awareness of uncertainty in this situation. SJ is subject to the pressure arising from the conversation to produce answers and caught between her awareness that there is a mathematical process and that she does not know what it is. Mandler (1989) proposes a theory which accounts for anxiety arising from a 'blockage' in resolving a problem. SJ meets a blockage at the outset, with the word proportional, this is exacerbated as she loses sight of the real world context. Mandler goes on to suggest that the brain has limited capacity for concurrent processing and because the aroused anxiety occupies much of the brains conscious capacity little is available for rational reflective thought on the problem, SJ's response appears to fit with this theory as she apparently seeks her only possible route of escape which is to offer a blind guess.

It is accepted that much of the above is little more than conjecture but it is offered to demonstrate how these seven dimensions may provide a structure for the analysis of this type of conversation and possibly the interpretation of the context in which learners set their goals in classroom activity. The explanation reveals how SJ's initial conception of the problem (content) may have provoked both her awareness and elicited an affective response which influenced her interpretation, rationale and purpose. Provided the interpretation is not 'blind' in the sense suggested above then her reflection on the task is likely to impact upon her conception and so on, thus demonstrating how these 'dimensions' relate dialectically. Further it illustrates how the student is subjectified by the context of the activity at the same time as she is making her own idiosyncratic constructions of the problem.

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TEACHING MATHEMATICAL THINKING SKILLS TO ACCELERATE COGNITIVE DEVELOPMENT

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Abstract: *This paper describes a quasi-experiment in which 314 students aged between 11 and 13 followed a mathematical thinking skills course and were compared with matched control groups using pre-tests, post-tests, delayed tests and structured interviews. Assessment instruments were devised to assess students' levels of cognitive development, and their ability to use strategic and metacognitive skills. Statistical data were supported by participant observations. Intervention students performed significantly better than control students in both cognitive and metacognitive post-tests. Cognitive skills had not been taught directly by the course and transfer is claimed. Accelerated performance exhibited by intervention groups was maintained in delayed testing. The teaching of metacognitive processes resulted in accelerated and sustained cognitive development.*

Introduction: The Mathematical Thinking Skills Project (funded by the Welsh Office and the University of Wales 1993/4) aimed to develop and evaluate a thinking skills course to accelerate students' cognitive development in mathematics. The course was based on activities developed in the Practical Applications of Mathematics Project (Tanner & Jones, 1993, 1994). The National Mathematics Curriculum in England and Wales requires pupils to hypothesise and test, to generalise, and to prove their conclusions. The project aimed to accelerate the development of such formal modes of thought by enhancing key metacognitive skills such as planning, monitoring and evaluating. This paper focuses on a quasi-experiment to compare the performance of control students with those who had followed the course.

Mathematical thinking skills: The conception of thinking used in this paper is that of thinking as sense-making (McGuinness 1993). We do not intend to attempt to itemise mathematical thinking skills here for, as Lipman (1983 p.3) has observed "the list is endless because it consists of nothing less than an inventory of the intellectual powers of mankind." However, Coles (1993) has identified three dimensions: skills, dispositions and attitudes; which are generic to any discussion of the teaching of thinking. In mathematics, a student would know how to perform a procedure, when and why it should be used, and gain a certain satisfaction from using these skills.

From a Piagetian viewpoint, adolescence marks the onset of formal thought - the ability to argue from a hypothesis and to view reality as a reflection of theoretical possibilities. Formal thought has been described as a systematic way of thinking; a generalized orientation towards problem-solving with an improvement in the student's ability to organize and structure the elements of a problem (Sutherland 1992). However, these key aspects of problem-solving are metacognitive rather than conceptual in nature. It can be argued, therefore, that formal thought is underpinned by the development of metacognitive skills.

Accelerating cognitive development: Recent research suggests that cognitive development can be accelerated (eg: Shayer and Adey, 1992; Elawar, 1992). A key feature of these studies has been their deliberate enhancement of metacognitive abilities. Indeed, metacognition has been identified by McGuinness (1993) as a primary tool for conceptual development.

Several researchers have argued for the explicit teaching of thinking strategies to improve learning (see Christensen 1991 for a review). However, Christensen found that children who had been explicitly taught learning strategies failed to use them as efficiently or as appropriately as those children who had invented strategies for themselves.

We assumed a socio-constructivist epistemology, accepting that mathematics is actively constructed by students rather than transmitted by teachers but that construction takes place in a social context. Students were given opportunities to validate their constructions against those of others though discussion. Teaching approaches were intended to develop students' metacognitive skills and, by so doing, encourage them to construct and evaluate their own strategies.

The course targeted metacognitive rather than cognitive skills. It was expected therefore that "close transfer" would be achieved and that the metacognitive skills of students in intervention classes would be enhanced. It was assumed that students would apply their newly acquired thinking skills to any mathematics which they met subsequently thus learning it in a qualitatively different way. "Transfer at a distance" into the cognitive domain was not expected to be immediate, therefore, but as new topics were met. Thinking skills pay for themselves not so much during the week in which they are acquired but during the years that follow (Perkins 1987).

Methodology: An action research network of six secondary schools was established, drawing students from a variety of social and ethnic backgrounds. The schools developed and trialled teaching strategies and materials, supported by members of the project team. The sample was not random due to the degree of commitment demanded from the teachers involved and consequent difficulties of self selection. It may best be described as an opportunity sample approximating to a stratified sample of English-medium schools in Wales.

The action research paradigm was chosen due to the novelty of some of the activities proposed. Two teachers from each school, who were to be involved in teaching intervention lessons, attended an initial one day induction course to familiarise them with the theoretical underpinning to the project and the outcomes of previous work, in particular, effective teaching strategies. They then attempted to integrate these approaches into their own teaching styles.

Intervention lessons were led by normal class teachers rather than outside "experts". The advantages of this approach in terms of realism, pupil-teacher relationships and teacher development are clear. The approach carries the disadvantage, however, that the experiences of the intervention classes were not standardised. Regular participant observation by the university research team was necessary to record the nature of the interventions made. These observations revealed that the extent to which teachers were able to adopt the approach was variable. In one case at least, the attempt to marry contrasting styles resulted in confusion. In another case a traditional outlook overcame the novelty of the materials and a completely didactic approach was employed. Purely quantitative approaches often fail to see the realities of classroom interaction. Qualitative data adds some necessary illumination.

Two matched pairs of classes were identified in each school to act as control and intervention groups. One pair was in year seven and one pair in year eight. Matched classes were either of mixed ability or parallel sets in every case.

Written test papers were designed to assess pupils' cognitive and metacognitive development. The sections of the test designed to assess cognitive ability were based on a neo-Piagetian structure and items were classified as identifying one of four stages of development, which were referred to as: early concrete, late concrete, early formal and late formal. Items were placed in the context of four content domains: Number, Algebra, Shape and Space, and Probability and Statistics. Items emphasised comprehension rather than recall. Classification took account of the anticipated memory requirements, National Curriculum assessment, and the results of large scale studies such as the Concepts in Secondary Mathematics and Science Project, (Hart, 1981).

The metacognitive skills of question posing, planning, evaluating and reflecting were assessed through a section in the written paper entitled "Planning and doing an experiment". Metacognitive skills of self knowledge were also assessed by asking students to predict the number of questions they would get correct before and after each section. In addition to the written papers, the metacognitive skills of a sample of 48 pupils were assessed through one-to-one structured interviews. These were conducted whilst the pupil planned and carried out an investigation into the mathematical relationships inherent in a practical task.

The pilot course and intervention teaching lasted for approximately five months. Regular network meetings were held at which experiences were exchanged, strategies discussed and new activities devised and refined. Post-testing occurred at the end of the course. Delayed testing occurred four months later. The attitudes of the control and intervention pupils to mathematics were monitored over the duration of the project using a questionnaire which had been trialled and evaluated in another project (Hendley, Stables, Parkinson, & Tanner, 1995).

The Thinking Skills Course: There were two strands to the course:

- the development of a structured series of cognitive challenges to stimulate the progressive evolution of key skills in the areas of strategy, logic and communication;
- the use and development of teaching techniques which would encourage the maturation of the metacognitive skills of planning, monitoring and evaluation.

Underpinning both strands was a continual emphasis on the need to explain rather than describe, to hypothesise and test, and to justify and prove. Activities were structured to encourage the development of a small number of strategies. Teachers selected from groups of activities which were responsive to a range of strategies including, for example: identification of variables; systematic working; coping with real data - estimating, averaging.

Strategies were not addressed separately - skill in comparing and selecting strategies was required. Each group of activities was responsive to a small number of target strategies and a student who had attempted an activity from each group would have encountered a wide range of strategies. The activities in the course did not address directly the questions used in the test of cognitive ability. We were not "teaching to the test" but were hoping to establish "transfer".

Metacognitive skills were not taught through the content of the materials but through the teaching approaches used (see Tanner & Jones 1995), which tried to develop skills of planning, monitoring and self evaluation and, by so doing, encourage students to construct and evaluate their own strategies through discussion and debate. Teachers encouraged students to think and plan for themselves and discuss their work, but they were not afraid to intervene to guide discovery.

Results: The main hypotheses to be tested through the quasi-experiment were as follows:

H1. Pupils following the course would have their mathematical development accelerated and would improve their scores in the post-test more than the control groups.

H2. The metacognitive skills of the intervention classes, as measured by the metacognitive section of the post-test would be accelerated.

H3. Accelerated cognitive development, as measured by the cognitive sections of the post-test would be observed in classes where metacognitive skills were taught.

H4. Accelerated performance exhibited by intervention groups would be maintained in delayed testing.

In each case the null hypothesis was that there would be no significant difference between the intervention and control groups. The results indicated that the null hypothesis could be rejected at at least the 5% level in each case. The thinking skills course can thus claim success in each of its main aims.

Pre-tests: The assessment paper was trialled with 60 pupils from a school not involved in the project. Analysis indicated that the test was reliable (Cronbach's alpha = .86), and internally consistent for cognitive and metacognitive abilities. Correlations between assessments of cognitive and metacognitive ability made through interview and written paper confirmed that metacognitive and cognitive abilities were very closely linked ($p < .001$).

T-tests on the pre-test data showed no significant differences at the 5% level of significance between control and intervention groups for scores on the attitude questionnaire, the test, or its cognitive and metacognitive sections.

Post-tests: Covariate analysis of the overall test results using pre-test scores as covariates showed a significant difference in favour of the intervention groups at the 0.1% (0.001) level. Null hypothesis one could therefore be rejected. Mathematical development was accelerated.

Analysis of metacognitive skills showed improved performance by intervention classes and little change in control groups. These differences were significant at the 0.1% (0.001) level (table 1). Null hypothesis two could be rejected. Metacognitive development was accelerated.

Hypothesis three contended that cognitive acceleration would take place when metacognitive skills had been learned. Qualitative data collected during school visits indicated that the extent to which teachers were able to adopt the required teaching approaches was variable. In three cases it was clear that the required approach was not employed and metacognitive skills were not taught. Data from these schools was therefore rejected.

When these three classes and their associated control groups were removed, analysis of the nine remaining control and intervention pairs revealed accelerated cognitive development for the intervention groups which was significant at the 5% (0.05) level (table 1). Null hypothesis three may therefore be rejected and cognitive acceleration claimed.

Attitudes remained remarkably stable (tables 1 & 2). There was no significant difference in attitude between the groups at the 5% level at any assessment point. The similarities in attitude score suggest that there was little Hawthorne effect at work.

Following the analysis of the post-tests, teachers were invited to comment on the results:

"Sue": I definitely think it has helped their thinking skills. I said at the beginning that if you could convince me you could convince anybody because I was completely against it but now, I definitely can see the worth of it.

In the new classes formed for the new academic year some of the teachers now had students from both intervention and control groups. They were convinced that there was a marked difference between such students:

"Doreen": Well, the content that they were taught by us last term was exactly the same, both classes have done the exactly same work. But looking at the work this term, the intervention class metacognitively, planning and evaluating and that, the intervention class are, no doubt at all, far better. I have had much better work in from that half of the class - I've got the best of both classes now in the top set in year 9 from the intervention and control groups in year 8.

"Sue": Test and homework results this year so far are better from the students from last term's intervention class. They seem to be able to think more clearly.

An improvement in algebraic skills was noted in both the ethnographic and statistical data. Teachers reported a greater willingness on the part of intervention pupils to generalise with letters:

"Doreen": In investigations they have been far more adventurous in trying to use algebra but they were taught formulas in exactly the same way as the other class.

Such comments corroborate the statistical findings.

Delayed tests: The graphs of test scores for the valid schools (figures 1 to 4) show how the gap which opened up between intervention and control classes was sustained after the end of the course. Intervention students continued to progress in parallel with control students but at a higher level. Covariate analysis of the delayed test results using the pre-test scores as covariates (table 2) showed a significant difference between control and intervention classes at the 0.1% (0.001) level for the test overall and the metacognitive sections, and at the 5% (0.05) level for the cognitive sections. Null hypothesis four may therefore be rejected. A sustained improvement in mathematical performance is claimed. The improvement was sustained in both metacognitive and cognitive aspects.

Table: 1 Post-test v Pretest - Covariate Analysis (Valid Classes)

VALID CLASSES		N	Mean		SD		Prob	I > C ?
			Pre	Post	Pre	Post		
TEST	I	214	17.4	22.9	7.3	8.8		
	C	203	16.8	19.3	7.0	7.1	.000	Yes
METACOG	I	214	4.4	7.0	3.4	4.1		
	C	204	4.3	4.5	3.3	3.4	.000	Yes
COGNITIVE	I	215	13.1	16.0	4.6	5.3		
	C	203	12.5	14.8	4.5	4.4	.014	Yes
ATTITUDE	I	161	3.5	3.5	0.5	0.5		
	C	161	3.5	3.5	0.5	0.5	.097	ns

Table: 2 Delayed test v Pre-test - Covariate Analysis (Valid Classes)

VALID CLASSES		N	Mean		SD		Prob	I > C ?
			Pre	Delay	Pre	Delay		
TEST	I	210	17.2	24.2	7.2	8.8		
	C	213	16.9	20.9	7.1	7.2	.000	Yes
METACOG	I	210	4.3	7.6	3.4	4.4		
	C	214	4.3	5.2	3.4	3.6	.000	Yes
COGNITIVE	I	211	12.9	16.6	4.5	5.2		
	C	213	12.6	15.7	4.5	4.5	.030	Yes
ATTITUDE	I	158	3.5	3.5	0.5	0.5		
	C	168	3.5	3.5	0.5	0.5	.742	ns

Figure: 1

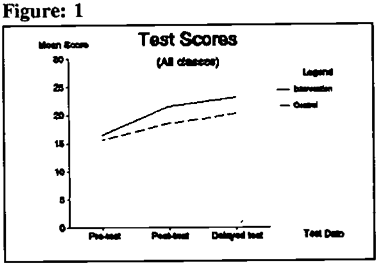


Figure: 2

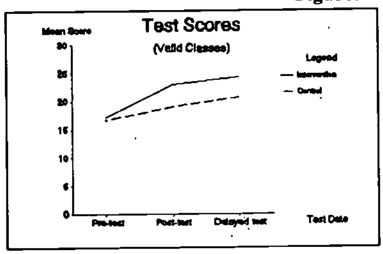


Figure: 3

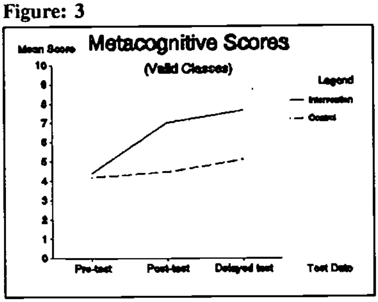
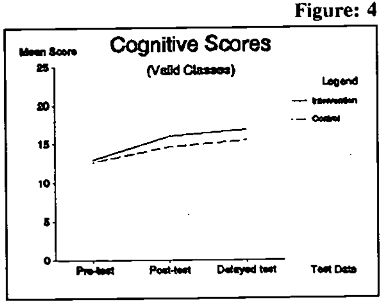


Figure: 4



Conclusion The improvement in metacognitive abilities of the intervention pupils was not unexpected as these skills had been targeted. The cognitive sections of the test, however, had not been taught directly. Improvement in these sections of the test may be explained by transfer of learning - by the application of improved modes of thinking to new mathematical contexts.

The sustained improvement of the intervention classes suggests that meaningful learning had taken place and provides a justification for the teaching of mathematical thinking skills. The course was very short and should be regarded as a pilot rather than a complete programme. Most of the intervention work was completed within twelve weeks. It is all the more surprising therefore that such clear and positive results have been achieved. Over a longer timescale it is probable that even greater acceleration could have been realized.

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Towards Statements and Proofs in Elementary Arithmetic:
An Exploratory Study about the Role of Teachers and the Behaviour of Students

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This report deals with the analysis of the behaviour of grade VI/VII students whilst constructively approaching, in a suitable educational context, statements and proofs of elementary arithmetic theorems. In particular, the report deals in depth with the issues of the teacher as a mediator of the most relevant characteristics of statements and proofs and the transition from the statements produced by the students to the relative proofs.

1.Introduction

The issue of the approach to mathematical theorems is dealt with in papers dealing especially with geometry, and above all, concerning the proof of theorems (see Balacheff, 1987; Hanna & Winchester, 1990; Hanna & Jahnke, 1993). Proving geometry theorems prevails also in high school students' work (see Moore, 1994).

In Boero & Garuti (1994), an analysis had been performed on how grade VI / VII students may realise, in a convenient educational context, a constructive approach to geometry statements. That report indicated some issues to be dealt with more in depth, concerning:

- The role of cultural mediation that may/must be performed by the teacher.
- How to implement the approach of the students to the proof of the statements that they themselves have formulated.

This report refers to a study having the following objectives:

- Analyze the behaviour of VI/VII grade students when approaching statements in elementary arithmetic (this is a field that has not been widely considered in the literature dealing with the approach to theorems).
- Deal more thoroughly with the issue of the role of cultural mediation performed by the teacher.
- Deal with the issue relative to the transition towards the proof in the arithmetic field, with special reference to two possible formulations of the statements produced by the students ("relational" and "procedural").
- Discuss the issue of the introduction of algebraic formalism as a "calculation technique" for the proving process.

Similarly to the approach to geometry theorems, historical and epistemological analysis helped us to identify some distinctive characteristics of arithmetic theorems and their proofs apt to be taken as reference points in the investigation of the cognitive behaviours of the students. This agrees with Vygotskij's theoretical framework: *"A suitably organised teaching-learning process results in mental development... Each scholastic subject has a specific relation with the child's development course, ... this takes directly to a review of the issue ... of the significance of each single subject from the point of view of the overall mental development" (Vygotskij, 1978).*

2. Historical and Epistemological Analysis

As for the historical analysis, it is known that even before the IV Century BC, the Greek mathematicians/philosophers discovered and proved many arithmetic properties. Concerning this period, Szabò (1961) underlines how the proofs relative to the properties of natural numbers form the first historical example of "μάθημα" (i.e. "science", "research"). In Euclid's *Elements* (Heath, 1956) we find arithmetic theorems that, from an epistemological point of view and as we'll see with some examples, show already all significant features of modern statements (conditionality, generality, relational or procedural formulation, etc.). The proofs instead are discursive and use a geometrical segment model to express "general numbers" (the algebraic language is not yet available).

Like geometry's, also arithmetic's statements are conditional, that is, expressed with the formulation: *"if.... then..."*. In Euclid's *Elements* we find statements with an explicit conditional formulation: *"If two numbers be prime to any number, their product also will be prime to the same"* (Tome VII, prop.24) together with others where the conditional form is implied: *"Any prime number is prime to any number which it does not measure"* (Tome VII, prop. 29).

As for the generality of the statements, the arithmetic field allows us to formulate significative statements with different degrees of generality. For instance, concerning the set of prime numbers, we may formulate statements relative to properties of the set itself (as for instance: *"Prime numbers are more than any assigned multitude of prime numbers"*, Tome IX, prop. 20) or properties relative to a generic element of the set (as for instance tome VII, prop. 29 mentioned above).

Arithmetic statements may be expressed either in a procedural or a relational manner, i.e. highlighting the procedure that leads to the result to be validated by the proof, or the relation, or property, that depends on the hypothesis being formulated and that must be proven. In Euclid's *Elements* we find statements expressed in a relational form (such as: Tome IX, prop.20 mentioned above) as well as statements expressed in a procedural form (Tome IX, prop.22: *"If as many odd numbers as we please be added together, and their multitude be even, the whole will be even"*).

Nowadays for many arithmetic theorems we use algebraic formalism in its functions of "generalisation - synthesis" (to express the statement) and of "transformation" (to prove the theorem by means of an algebraic "calculation").

For what concerns the introduction of algebraic formalism in arithmetic, the need to express the general resolutive methods of arithmetic problems with a suitable formalism, syntactically different from common language and from Euclid's segments model, is manifested in the work of Diofanto (IV Century AD). One thousand year had to pass, however, before Viete put together a formalism adequate to meet this need. Still from an historical and epistemological point of view, concerning the use of algebraic formalism, we can see how the statements and proofs relative to arithmetic theorems have formed, from the end of the past century on, the preferred field for logicians, mathematicians and artificial intelligence researchers to try and reduce the proofs to calculations, exploring thus the issue of the "truth" in mathematics and developing also automatic proving programs.

Concerning the significance of arithmetic statements, in the history of the theory of numbers we may trace back different significance criteria, often cohexistent at the same time: intellectual challenge (e.g.: the "Fermat's theorem"), relative to the difficulty of the proof; knowledge of the deep structure of the set of natural numbers (e.g.: infinity of the set of prime numbers); paradigmacity and importance for the construction of more general algebraic structures (e.g.: the theorem: "*given two natural numbers* a *and* b, *with* a>b, *there exist two natural numbers* q *and* r *such that* a=bq+r" produces in Algebra an axiom in the introduction of Euclidean rings). Let us note that, according to these last two criteria, a wide knowledge of arithmetic (if not of an even larger mathematical domain) is required to evaluate the significance of an arithmetic statement.

These historical and epistemological considerations have been useful for us, as seen in the next paragraphs, to focus the issue of the role of the teacher, set up the teaching experiment and analyze the behaviour of the students.

3.The teacher as a mediator

When approaching arithmetic statements and proof, there exist at first a wide gap between the knowledge of the teacher and the knowledge of the students. In particular, the teacher possesses knowledge and experiences unknown to the students in the areas of:

- Linguistic formulation of the statements and their characteristics of generality and conditionality.
- Significance (which, as we have seen at point 2, requires a reference to a mathematical culture which only the teacher has).
- Meaning of the proof in mathematics and the modalities and techniques for its attainment.
- Algebraic formalism (as an effective instrument to express statements and prove theorems).

The knowledge of the students may be made closer to the knowledge of the teacher:

- In part through constructive activity required of the students, therefore through an indirect mediation on the part of the teacher implemented with the choice of suitable tasks promoting the generation by the students of significant and useable products for classroom work.
- In part through classroom work on the students' products, through an indirect mediation on the part of the teacher, implemented in privileging some of the students' products and in gradually bringing out the relevant characteristics which "must" be possessed by their mathematical products (such as generality and conditionality of the statement as well as the logic consequentiality of the proofs).
- In part through a direct mediation on the part of the teacher such as the comparison with the official "texts" of arithmetics, the introduction of effective formalisms (such as the algebraic formalism), etc.

All these forms of mediation must take into account the cognitive requirements of the students, manifested through their productions and the cultural and cognitive meaning that these products have.

4.Planning of the Teaching Experiment

The educational context where our teaching experiment was located was that of the Genoa Group Project for the Comprehensive School. Relevant to the study reported in this paper are:

- The practise of written verbal reporting on the part of the students, concerning both the resolution of the problems and the relative reasoning and reflections.
- The development of competencies concerning arguing, producing hypotheses, etc. in extra-

mathematical "experience fields" (Boero, 1989).
- The creation of a classroom environment where the coherence of arguments, the quality of the processes producing hypotheses and the quality of the resolutive reasoning are very much valued.

The teaching experiment involved two classes in 1992/93 and two classes in 1994/95. Let's now go on to the description of the assignments (for further details, see Sibilla, 1994):

By means of the individual assignment: *"Suppose you have a certain set of numbers. Apply the transformation '+1' to all the elements of the set. What are the effects of the transformation?"* and in general, by means of assignments of the type *"what happens if...?"* referred to a set of numbers selected by the students, followed by the comparison between the "effects" that had been identified, the experiment aimed to create, in the numbers "field of experience" (by now sufficiently familiar to an 11-year old student), an initial awareness of the fact that there exist "unvarying" properties for the change of the given set of numbers being considered. The students, in other words, were to be led to identify and express in a conditional format, properties having characteristics of generality. Another purpose was also to raise the issue of the justification, through reasoning, of properties which do not appear immediately true (until their proper proof).

By means of the individual assignment: *"You have a given set. What transformation do you have to apply to the set so that the transformed set is only formed by even numbers? (Help yourself with tables if you want)"* and the assignment: *"You have a given set. What transformation do you have to apply to the set so that the transformed set is only formed by odd numbers?"* and by means of the subsequent discussion, the experiment wanted to "force", via the identification and the expression of a variable, the process of algebraic formalisation.

By means of the assignment *"What happens if you add together two consecutive odd numbers? Is there a regularity? And if so, why?"* and the following discussion, the experiment wanted to stimulate an experience of exploration of numerical facts possibly leading to the identification of various properties: general because they do not vary with the particular pair of selected consecutive odd numbers, and (implicitly) conditional because they would depend on the conditions (odd and consecutive) of the numbers.

Other assignments, such as: *"what happens if you add two even consecutive numbers?"* and *"what happens if you add three odd consecutive numbers?"* were used to evidence the conditional (as well as general) character of the statements relative to the properties of natural numbers, as well as develop a dexterity with algebraic formalism as a tool to explore and prove arithmetic theorems.

5. Analysis of the behaviour of students in the initial stage

5.1. Production and comparison of statements in a class

The assignment: *"Suppose you have a certain set of numbers. Apply the transformation '+1' to all the elements of the set. What are the effects of the transformation?"* produced in the four classes a large variety of answer texts, due to its character, purposely "unleading".

Some of the texts that were produced appear to be rather superficial, of little consequence from a mathematical point of view, and not very general in character: for instance: *"If I have the set 2, 3, 4, it* *becomes the set 3, 4, 5"*; other texts contain statements which, although not very general, appear to be quite significant: *"If the set contains numbers ending with 9, the transformation +1 transforms them in numbers ending with 0 and they have an extra digit".* Other statements that have been produced are more general: *"The set is transformed in another set with the same number of elements".*

Vis a vis with these products by the students, the teacher must select those which are more suitable, trying to bring out (or mediate) those aspects which are important from a cultural point of view (see point 3.).In a VII grade class, for instance, the following statements were compared:

a) *If I have 3, 4, 5, 6, 7, their sum is 25. When I add 1 to the numbers, their sum is 30.*

b) *After adding 1 there are both even and odd numbers.*

c) *By adding 1, if it is even it becomes odd and if odd it becomes even*

d) *An even number added to an odd number becomes odd, an odd number added to an odd number becomes even.*

The character of generality of the statements may be negotiated, at least in part, with the students. In practise, in this class, at this stage of the teaching experiment, the negotiation took place through the comparison of statements presenting common elements and asking the students to establish what happened to some properties that they had identified if the reference set was changed, and to compare statements which (like c) and d)) had common elements.

Through a teacher-led discussion, the students were able to make significant observations concerning the character of generality of the statements, in particular discovering that the first is valid only for that specific set. The second statement created some perplexity in the way it is formulated: *"It almost appears that you can get odd or even numbers from any set of numbers. It does not talk of the initial situation. If the beginning set is formed by even numbers only, the statement is no longer true".* The third and fourth statements were instead considered general (they are valid for all sets on which the "+1" transformation is carried out, but express a different degree of generality: *"In the fourth, one is considered as odd, but it could also be 3 or 5").* Many students observed also that the fourth statement goes beyond the level of generality requested by the assignment.

The conditional character of the statements, that was to be considered at several stages during the course of the teaching experiment, appeared in this class during the discussion of the statements above. In particular, the teacher led the class to the discovery that in the third statement the conditionality is explicit, while in the fourth, it is implied.

In this class, the problem of the significance had been introduced by comparing the statement a) with the other three: For the students, at this stage of the work, the triviality of the statement was clear (*"in a) it is like saying that all the numbers increase of one unit"*) without going further than this level of reflection. In our opinion, the capacity of the students to autonomously express a judgement relative to the significance may be the result only of an extended activity aiming to bring out as "significant" those properties which are not immediately apparent (intellectual challenge) and/or that contribute to a deeper insight in the numeric field.

This part of the teaching experiment confirms (in the field of arithmetic) the hypothesis of *feasibility of the objective to get grade VII students constructively involved in approaching statements* *of theorems in an adequate educational context, strongly depending on the role of the teacher*, stated in Boero & Garuti (1994) for geometry theorems.

5.2 Procedural and relational statements: Approach to the proof in two classes.

In two grade VII classes (39 students), with the same teacher, after the first comparisons, the attention veered on the following property, offered by a girl: *"If I add 1 the divisor changes. For instance, 365 is divisible by 5, 366 is not".*

Rather than a statement, this is the observation of a property that, even in its rather poor formulation, the teacher was able to judge as being very significant, not being immediately apparent and suitable to illustrate some structural characteristics of the set of the natural numbers.

The property was then submitted for discussion to the two classes. The students seemed little convinced that it was true and, moreover, they were perplexed concerning its formulation. At this point the teacher asked the students to ask questions in order to formulate the statement more precisely. These were the questions of the students:

- *"Is it true for all numbers? How can you be sure?"*
- *"Do all divisors change or only some?"*
- *"This is not true for 1, because 1 is the divisor of all numbers, but what about the other divisors?"*
- *"If the divisors change, do the multiples change also?"*

As it may be seen by the questions, the problem of the greater precision in the formulation of the statement was intimately weaved with the problem of the verification of its validity, even if gradually, through numerical examples, they started to realise that the property might be true.

Later the students were asked to rewrite the statement. In substance two types of formulations emerged (considered by the students as being equivalent: *"They say the same property")*:

I) A number and the number immediately after have no common divisors except for the number 1 (relational statement)

II) If you add 1 to a number, all its divisors change, except 1 (procedural statement)

At this point, it was interesting to determine how much the formulation of the statement could condition the proving process, and in what measure the proving process could be autonomously managed by the students in a situation where the students were strongly motivated to prove the property which they had discovered.

For this reason, the students were asked to: *"try and verify if the property expressed in the two statements is true and why".*

All the attempts made by the students of the two classes seem influenced by the formulation of the statement that they considered.

Most of the students (32) refer to the relational statement: they find all the divisors of a number and its next and verify that there are no common divisors, except for 1.

This procedure does not help them in the justification of the statement (none of them was able to attain to a real proof), but helps them in substance to get an opinion on its validity. Different behaviours, however, were observed amongst the students proceeding in this way:

- Some try on "large" number, looking for an empirical verification of the validity of the statement.

- Most of the students, in trying to reach a general justification try to proceed for "classes" of numbers, even if the justifications are only partial or wrong (as evidenced by the discussion following): *"Between a prime number and the number following it there may not be common divisors since a prime number has as divisors only 1, which we have excluded at the beginning, and itself, which changes from number to number"* (it was not hard for the students to realise that this statement is useless if one of the two numbers being considered is not prime, such as in the case of 14 and 15). *"A number may not have common divisors with the next number because one is odd and the other is even":* In this case also it was not difficult for the students to understand, through samples (such as 30 and 33) that the justification is not valid.

Some of the 7 students referring to the procedural statement consider the divisors of a number and try to establish if the same divisors are also the divisors of the following number, realising that the unit, added in the transformation, constitutes the remainder of the division. This is the way that a student comes to the following proof: *"This statement is true because between a number and the number immediately after, you add 1, so that the divisor of the first number is not right for the other because the 1 that was added forms a remainder. (...) For instance 15:3=5 and 16:3=5 remainder 1."*

Another proof that was produced is: *"When skip-counting, except when you skip-count by 1, it is not possible that there are consecutive numbers since the multiples of a number derive from the beginning number that is added all along, so that it is impossible that two multiples are also two consecutive numbers, for instance 2-4-6-8... or 3-6-9-12..."*. In this case also the proof appears to be influenced by the procedural formulation of the statement.

The other five students, although not reaching to a complete proof, make some steps in the first of the two indicated directions.The discussion and the comparisons relative to these two proofs allow (under the guidance and with the mediation of the teacher) to bring out deficiencies, analogies and differences with the other texts that had been produced.

The work of these two classes appears to be satisfactory as a whole, both for the quality of the two proofs that were produced, and for the critical capacities demonstrated by most of the students towards the unsuccessful attempts.In our opinion, it also provides some elements for further studies, especially *the hypothesis that in the case of a statement produced (or assumed) by a student, its proving process may naturally evolve from it as a textual "development" of the statement itself.*

6.The problem of the approach to the proof as an algebraic calculation

The overall positive result of the "arguing" approach to the proof, illustrated at point 5.2, raises the issue of the opportunity of a fast transition to the proof as a calculation carried out on the formula expressing the elements on which the property to be proved is to be verified.

It is apparent that for properties such as those considered in the last two stages of our teaching experiment, the algebraic formalism makes easily accessible to the students proofs that would otherwise present many difficulties, while the proof of the property considered at point 5.2 would not be made any easier by the availability of algebraic formalism. There may be imagined therefore both a development of classroom activities such as that hypothesised in our teaching experiment (where the choice of the statements from stage 2 on lends itself to the best use of algebraic formalisms), and a development based on other types of statements (like some contained in Euclid's Elements) that may be proved also without resorting to algebraic formalism.

On the basis of our experience, it would appear to us that the conquer of algebraic formalism and its use for proving, would involve the students in discussions and considerations regarding conventions, transformation rules, etc. that may distract them from the logical mechanics of the proof, without, on the other hand, producing extended learning results (as far as the ability of autonomously using algebraic formalism for proving). It would appear to us moreover, that the influence on the algebraic formalism on the development of the proof is very strong and may give place to a development of the proof linked to the transformation mechanism and not the analysis of the property to be proven. We would like to mention, in this respect, an episode occurred in a class where, by the end of the teaching experiment described at point 4. only one student manages to autonomously use algebraic formalism for proving. During the teaching experiment, the student must prove that the sum of two consecutive odd numbers is divisible by 4. He gives a proof in words, writing that *"the sum of two consecutive odd numbers is like taking an even number, take away one from it, and sum it to the same even number and add one. So it is the sum of two even numbers and therefore is divisible by four "*. At the end of the teaching experiment he was asked to prove the same property using algebraic formalism; he writes:

" $2n + 1 + 2n + 3 = 2n + 2n+ 4 = 2n + 2n + 2 + 2 = (2n+2)x2$ ".

On the other hand, it is through the very use of the algebraic formalism as a tool for the expression of properties and as a calculation tool when proving arithmetic properties that the students may grasp the powerfulness of the tool that has been introduced and may make the first significant experiences of its "transformation function".

A compromise solution for this problem appears then that of using different classroom activities to approach the two objectives: development of demonstrative reasoning as an arguing and logical experience; introduction to the algebraic formalism and "exemplary" use in carrying out proofs as "algebraic calculations"

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Proving to explain

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Abstract: It has been suggested that in teaching proving the explanatory potential of proofs and proving should be emphasized. My recent research has indicated that students are more likely to prove to explain than they are to prove to verify, but that there are important aspects of proving and explaining that must be considered; these include the degree of formulation of the proving, and the alternative of explaining by analogy.

Introduction

Gila Hanna (1989) has suggested that teachers should be aware of the potential that proofs have of explaining as well as verifying mathematical statements. Others (e.g., de Villiers 1991, 1992) have conducted empirical studies which lend support to this suggestion. In my recent research I have been attempting to develop a clearer description of the reasons proving is used by students in problem solving. Chief among these reasons, or needs, are explaining, exploring, and verifying. A distinction between my work and that of Hanna and de Villiers, is that they referred to proofs which were presented to students as explanations, while I am more concerned with the explanatory power of the proving students do themselves. It should also be noted that I use "proving" to describe any deductive reasoning, even if no proof is produced.

Hanna described *explaining* in this way:

> I prefer to use the term *explain* only when the proof reveals and makes use of the mathematical ideas which motivate it. Following Steiner (1978), I will say that a proof explains when it shows what "characteristic property" entails the theorem it purports to prove. (p. 47)

This characteristic, of revealing the underlying principles on which the proof rests, is undoubtedly a part of what makes proving a useful way of explaining for students.

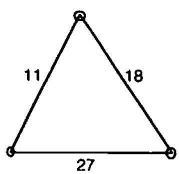
This paper reports some results from my research into students' proving in problems solving which relate specifically to proving to explain. I will distinguish between formulated and unformulated proving to explain, and provide examples of proving of different degrees of formulation. I will also give an example of explaining by analogy which will illustrate the conflict between explaining by proving and explaining by analogy, under the influence of social constraints. I will conclude by describing some implications for the use and acceptance of explanations based on proving.

Formulation is one of the most important characteristics of proving when used to explain. Formulation refers to the knowledge or awareness, on the part of the prover, that s/he is proving. It could also be described as the degree to which the proving is thought-of and thought-out. Formulation is related to two other characteristics of proving: its articulation, and the hidden assumptions made while proving. The extent and clarity of the spoken or written articulation of proving has implications both for the possibility of the proving being interpreted by others, and for the formulation of the proving. Being aware of one's own proving, and being able to articulate that proving, are interrelated. Articulating proving assists in formulating, as articulation makes aspects of proving tangible. At the same time formulated proving is more easily articulated. All proving involves some hidden assumptions. These assumptions can range from wrong assumptions, through implausible and plausible assumptions, to assumptions which are known within a community. The formulation of proving reveals hidden assumptions, making the presence of wrong or implausible assumptions less likely. Articulation and hidden assumptions provide valuable clues to formulation, in addition to being important characteristics of proving in and of themselves.

Explaining in problem solving

The results reported here are taken from a larger research project on proving, involving observations and interviews with high school and university students engaged in problem solving. The general object of this research project is the investigation of the needs which proving addresses for mathematics students. Other aspects of this project have been reported in Reid (1994), Kieren & Reid (1994), and Kieren, Pirie, & Reid (1994). The examples below are taken from problem solving session which involved four university students, Rachel, Eleanor, Ben, and Wayne, working on the Arithmagon problem (from Mason, Burton & Stacey, 1985; see Fig. 1). I will describe briefly the activities of the four participants, pausing to provide more detail and analysis of episodes of explanation.

A secret number has been assigned to each corner of this triangle. On each side is written the sum of the secret numbers at its ends. Find the secret numbers.



Generalize the problem and its solution.

Figure 1: The Arithmagon problem

Partially formulated proving to explain.

The four participants were seated at an "L" shaped table in such a way that Rachel and Eleanor could easily work together, as could Ben and Wayne. Eleanor and Ben were seated closest to the bend in the table. Rachel and Eleanor began by setting up systems of equations and solving them. They arrived at the solution after about four minutes. Ben guessed the solution within 30 seconds of being given the prompt. He and Wayne then worked independently, trying to find patterns in the solved puzzle. Ben concentrated on relations between the numbers, while Wayne attempted to make use of geometric properties of triangles.

Once everyone had a solution they compared notes. Although Ben initially claimed to have no idea how he had found the answer so quickly, he eventually reconstructed a plausible explanation (Long dashes, —, indicate short pauses):

(A1) Ben: You know how I did that? The number here had to be less than 27, and less, it had to be less than 18, the number here, right, — had to be less than 18. And the number here had to be less than 11,— right?

Note that while there is a lot left out of his explanation, Ben is fairly articulate in explaining how one might limit the possible cases to a number small enough to make testing all of them feasible. The basis of the constraints, the justification for the use of "had to be," is a deduction from a hidden, but plausible, assumption that the secret numbers are all natural numbers. This partially formulated proving to explain was fairly successful as such, although it left Eleanor and Rachel with a need to explore its workings in more detail. Ben's later proving was less formulated, and less successful as explanation, as the next example will show.

Unformulated proving to explain

Rachel and Eleanor tried to use Ben's method to solve another triangle, and tried to see if there are other constraints that would help them determine the secret numbers exactly. Ben watched their efforts, and after a few minutes he claimed that any triangle could be solved by his constraints method. In an effort to make him aware of his assumption that only natural numbers could be used, I suggested that he solve a triangle with the values 1, 4, and 12 on the sides (The secret numbers are 7.5, 4.5, and $^{-}3.5$) .

(B1) Ben: On the sides, 1, 4, 12. Well that's 0 or 1. One of them has to be 0 — No, That's impossible — Because, I mean if this one is 0, that one has to be 1, that one has to be 3, this is adds up to 3. If this one is 0, this one has to be 4 and that one has to be 1.

(B2) Wayne: Who said it's got to be 0 though?

(B3) Ben: Well, Yeah — It still shouldn't matter — if you go down on the number line you still have to go up on the number line

Ben's initial comments (B1) are similar to those quoted above, in line A1. He explains why the triangle is impossible, by reasoning deductively from the implicit assumption that the secret numbers are natural numbers. When Wayne questions his hidden assumption, Ben immediately offers further explanation (B2). Note that this explanation has a different character from the ones he has offered before (A1 & B1). It is much less articulated, making it difficult to judge how aware Ben was of his reasoning. His language suggests that his proving is based on an image of the relationship between the values. These features lead me to characterize this explanation as unformulated proving.

Unformulated proving is not very useful as explanation, as is illustrated by Ben's continuing attempts to explain:

(C1) Eleanor: But this doesn't have to be 0

(C2) Ben: But even if it is, like let's say negative 4 and negative 3, right? You still have to get this to be 4 it has to be 7, all right? It's still *minus*. So it will still be like, 3. — You know where I'm coming from?

(C3) Eleanor: Say it again.

(C4) Ben: The difference, the difference between these two is still always going to be 1, right? No matter if you represent it with negative or adding.

Eleanor's request to "say it again" marks the failure of Ben's unformulated proving to explain to her. Ben has based his argument (in C2) on a hidden assumption, which in this case is wrong. He seems to believe that the difference between the two secret numbers is 1. This is true in the case where one of them is zero, which he had just been considering. The two numbers he names,

$^-4$ and $^-3$, have a difference of 1, and these numbers do not work. In fact, if the difference must be 1, there is no way that a difference as large as (12–4) could occur. This provides the basis for Ben's belief that the puzzle can not be solved.

Ben seems determined to explain why the triangle has no solution. There are several needs interacting in this case. In order to convince Eleanor, he needs to be able to explain the situation to her, and to do so he needs to explore it more thoroughly than he had to in order to verify for himself that it could not be solved. His failure to convince Eleanor had the effect of undermining his verification, which caused him to shift from explaining what he had verified to Eleanor, to exploring the now reopened question of whether of not the triangle could be solved.

Ben's unformulated proving did not work as an explanation for Eleanor, in spite of her willingness to listen carefully, and to work through Ben's ideas. The next example shows how Eleanor's receptiveness and Rachel's ability to formulate her proving combined to produce an explanation.

Formulated proving to explain

Eleanor, Ben, and Wayne continued to work together, exploring the situation inductively. They discovered two interesting properties: 1) The sum of the numbers on the sides is twice the sum of the secret numbers; $(a+b+c) = 2(x+y+z)$. 2) The sum of a secret number and the number on the opposite side is the same for all the secret numbers; $(a+x) = (b+y) = (c+z)$. During this time Rachel had been working independently, exploring using algebraic derivations. After twenty minutes Rachel announced that she had found a formula: $x = \dfrac{a+b-c}{2}$. Ben and Wayne immediately began to verify it inductively, but Eleanor asked for an explanation, "How did you get that?"

<u>Transcript</u>		<u>Rachel's writing</u>
(D1) Rachel:	X plus Y equals A	(1) $x+y=a$
(D2) Eleanor:	Yeah	
(D3) Rachel:	Y plus Z equals B and Z plus X equals C.	(2) $y+z=b$
(D4) Eleanor:	Yeah	(3) $z+x=c$
(D5) Rachel:	And then just add A and- Add the first two equations. —	
(D6) Eleanor:	And you come up with?	
(D7) Rachel:	This. — That's right, right? That's what I got?	
(D8) TK:	Work it through cleanly for her.	
(D9) Rachel:	Oh, OK. So, 1 plus 2 is X plus Y equals A, Y plus Z- Did I add?	(1) $x+y=a$
(D10) TK:	Yeah.	+
(D11) Rachel:	Oh it doesn't matter which two you add up.	(2) $y+z=b$
(D12) TK.	It doesn't matter.	
(D13) Eleanor:	We do come up with something.	

Transcript		Rachel's writing
(D14) Rachel:	So X plus 2Y plus Z equals A plus B	$x + 2y + z = a + b$
(D15) Eleanor:	Yeah.	
(D16) Rachel:	But, X plus Z is C, right?	
(D17) Eleanor:	Right.	
(D18) Rachel:	So-	
(D19) Together:	-2Y plus C-	$2y + c = a + b$
(D20) Eleanor:	Yeah. OK	
(D21) Rachel:	-equals A plus , and um,	
(D22) Eleanor:	Yeah, So 2Y equals A plus B minus C	$2y = a + b - c$

Rachel's explanation to Eleanor is quite formulated. She articulated her steps clearly, both in her writing and her speech. She was aware of the structure of her own reasoning, as is indicated by her observation that she is free to choose any pair of equations to add together (D11). Her only hidden assumptions are the basic rules of algebra and arithmetic which she can safely assume are known to and shared by Eleanor.

One might expect that Rachel's formulated proving would be the preferred form of explanation for this group. The next example shows that this was not so, and indicates the importance of both the clarity of the explanation, and the receptiveness of those to whom it is offered, to the acceptance of an explanation.

Explaining by analogy versus explaining by proving

After Ben and Wayne had verified Rachel's formula, and Rachel had explained it to Eleanor, Wayne wondered why it is necessary to divide by 2. Ben, Rachel, and Wayne all offered explanations:

(E1) Ben:	You know why you divided by 2, is because-
(E2) Rachel:	Because there's two sides.
(E3) Ben:	No. No, it's because-
(E4) Wayne:	There's two other points, to be solved for, no?
(E5) Ben:	No. No. No. We found out that Y, X + Y + Z is half of the outside points.
(E6) Wayne:	That's right!

Rachel's explanation (E2) for the division by 2 is quite correct, and based on the proving she had done in deriving her formula. The brief statement she was able to make was not, however, sufficient to communicate anything to Ben and Wayne. Even though her statement was did not explain anything to Ben and Wayne, I would consider it to be an example of using formulated proving to explain. The proving, however, was all done ahead of time, and she merely assumes it in her explanation.

Wayne's explanation (E4) is an example of an explanaticn by means of a weak analogy. The number 2 is involved both in the division by 2, and in the number of vertices to be solved, once the first is known, but that is the only connection between them. It is interesting that, even though

Wayne had been the first to voice a need to understand the division by 2, but at this point he seems more anxious to suggest his own explanation than to hear Ben's.

Ben's explanation (E5) is a strong analogy. The analogy is between two equations with variables, instead of between an equation and a state of affairs (as in the case of Wayne's analogy). This strength is likely to have led to Wayne's acceptance of Ben's explanation over his own (E6). Ben is referring to the relationship $a+b+c = 2(x+y+z)$, which he had discovered inductively with Eleanor and Wayne (see above).

The explanations which were rejected were a weak analogy (Wayne's, E4) and a deductive explanation which could be taken to be a weak analogy (Rachel's, E2). The students preferred the strong analogy, which was based on several points of connection. This is sensible, as a strong analogy could have (and in this case does have) the potential to be developed into a deductive proof.

It is worth noting that even though Rachel's explanation was the most thought out, and based on deduction rather than analogy, which might suggest it was a more certain explanation, it was apparently not even considered by the others. This illustrates a weakness of proving versus analogy for explaining. Proving is a process which must be formulated to be communicated, and must be followed with some care to be understood. In this situation the social dynamic did not afford Rachel the opportunity to make her case clearly. Ben's analogy (E5), on the other hand, could be understood immediately by Wayne and Eleanor, who were familiar with the context to which he was making links. Rachel could also see these links after Eleanor showed her the formula which was being referred to.

Conclusions

The examples above illustrate the range of formulation which proving to explain can cover. While Hanna (1989) points out that some proofs are so formal that they do not explain, proving can also fail as explanation because it is not sufficiently formulated. Further, proving to explain has disadvantages compared to explaining by analogy, in contexts where articulation of the proving is difficult.

The deductive explanations described above involve both unformulated proving and formulated proving. They show that unformulated proving is not very successful in explaining to others, although it might be explanatory to the person proving. Unformulated proving lacks the quality of clarity, which explanation to others requires. Formulated proving is more successful as a way of explaining. Its main weakness is the time and attention it requires of the listener. This can make it useless as explanation in social contexts which do not allow for extended explanations.

The main rival of proving for explaining is the use of reasoning by analogy. Explaining by analogy is more or less successful, depending on the strength of the analogy. A strong analogy might be accepted in preference to a deductive explanation. A weak analogy, however, seems to leave room for a better explanation by formulated proving. Some explanations by analogy make connections which could be established deductively, as was the case when Ben explained the division by 2 in Rachel's formula (E5). It should be noted, however, that I have yet to observe students attempting to transform an analogy in this way. The question of the relationship between analogy and deduction could be a useful focus of further research.

In thinking about explanation it is essential to keep in mind that more than the intent to explain is required for the success of an explanation. Any means of explanation must also involve articulation of sufficient clarity to allow others to understand it. In addition those others must be willing to devote the time and attention necessary to understanding the explanation. It is the combination of these factors which makes an explanation, and which puts certain constraints on the circumstances in which proving can be explaining.

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Beyond the Computational Algorithm:
Students' Understanding of the Arithmetic Average Concept

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This study examined 250 sixth-grade students' understanding of the arithmetic average. It was particularly designed to provide in depth information about students' knowledge of arithmetic average with respect to computational algorithm and conceptual understanding through conducting a fine-grained cognitive analysis of students' written responses. Results of this study showed that 90% of the students knew the "add-them-all-up-and-divide" algorithm for calculating average. However, only about a half of the students showed evidence of having conceptual understanding of the concept. This study suggests not only that the arithmetic average concept is more complex than the simplicity as the computational algorithm suggests, but also implies that the average concept should be taught beyond the computational algorithm.

In the age of information and technology, society has an ever-increasing need for data in prediction and decision making. The National Council of Teachers of Mathematics (NCTM) suggests that "it is important that students develop an understanding of the concepts and processes used in analyzing data." (p. 105, 1989) Arithmetic average is one of the important and basic concepts in data analysis and decision making. Data reported and used in daily life, scientific journals, and public media frequently include average. Statistical analysis and inferences are conducted based almost exclusively on the average and others which are closely related to the average, such as variance. In another words, average is not only an important concept in statistics, but also an everyday-based concept.

Although the average concept seems to be as simple as the computational algorithm suggests, previous research (e.g., Mevarech, 1983; Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988) indicated that both precollege and college students have many misconceptions about the average concept. The misconceptions are not due to students' lack of procedural knowledge of calculating an average, rather due to their lack of conceptual understanding of the concept. For example, Pollatsek et al. (1981) found that college students seemed to have no difficulty computing the average of a set of given numbers, but a large proportion of those students could not solve problems involving weighted average. For example, less than 40% of the college students were able to solve the following problem: "A student attended college A for two semesters and earned a 3.2 GPA (grade-point average). The same student attended college B for three semesters and earned a 3.8 GPA. What is the student's GPA for all his college work?" The most common incorrect answer was 3.5, which apparently resulted from the directly averaging the GPA's for college A (3.2) and for college B (3.8).

The arithmetic average is defined by adding the values to be averaged and dividing the sum by the number of values that were summed. Strauss and Bichler (1988) argued that the simplicity of the computational aspects of the average concept might make it appear to be very straightforward and simple. In fact, most students' understanding of the average concept is the "add-them-all-up-and-divide" algorithm (Shaughnessy, 1992). Previous studies have examined students' misconceptions about the average concept (e.g., Lindquist, 1989; Mevarech, 1983; Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988) and explored the possible instructional approaches (e.g., Mevarech, 1983; Hardiman, Well, & Pollatsek, 1984) to promote understanding of the average concept, but no study directly and explicitly examined students' performance on the average tasks involving computational algorithm and conceptual understanding.

This study was designed to provide in-depth information about students' knowledge of arithmetic average with respect to the computational algorithm and conceptual understanding through conducting a fine-grained cognitive analysis of students' written responses.

METHOD

Subjects

A total of 250 sixth-grade students from the Pittsburgh Metropolitan area participated in the study. The students who participated in the study are judged to be above average in mathematical ability by a group of mathematics teachers and a group of mathematics education researchers in the Pittsburgh Metropolitan area.

Tasks and Administration

Figure 1 shows the multiple-choice average task and the open-ended average task. The multiple-choice average task was administered to the sample with other 17 multiple-choice tasks and students had a total of 15 minutes to complete these 18 tasks. The open-ended average task was administered to the sample with six other open-ended tasks and students had 40 minutes to complete all seven open-ended tasks. In the open-ended average task, students were asked to provide an answer, and importantly, they were also asked to explain how they found their answer. In particular, the open-ended task requires students to find a missing number when the first three numbers and the average of the three numbers and the missing number are presented in a graph. In order to solve the problem, students must have a well-developed understanding of the average concept. Thus, the open-ended task is appropriate to examine students' conceptual understanding of the average concept.

Insert Figure 1 about here

Data Analysis

Each response for the multiple-choice average task was coded as correct or incorrect. In contrast, each student response to the open-ended average task was subjected to a fine-grained cognitive analysis. In particular, each response was coded with respect to four distinct aspects: (1) numerical answer, (2) mathematical error, (3) solution strategy, and (4) representation. The categorization scheme used in this study was adapted from Cai et al. (in press) in their coding of students' responses to a similar open-ended average problem.

RESULTS

Results for the Multiple-choice Average Task

Students appeared to have a little difficulty selecting the correct answer for the multiple-choice average problem. In particular, 88% of the sample chose the correct answer for the task.

Results for the Open-ended Average Task

Numerical answer and mathematical errors. The numerical answer was what the student provided on the answer space on the task, and was judged correct or incorrect. Only about 50% of the students provided the correct answer of 10 for the open-ended average task.

Students who did not give the correct answer of 10 were subject to an analysis of error types. Overall, 126 (50%) students were included in the error analysis. Five different types of errors were identified and those are described below:

Minor error: The student had correct solution process, but they made a minor calculation error, or they gave the total number of cups sold in four weeks (28) as the answer.

Violation of "stopping rule": The student used a trial-and-error strategy, but stopped trying when (a) the quotient was not 7; (b) the remainder was not zero; or (c) the quotient was not 7 and the remainder was not zero.

Incorrect use of computational algorithm: The student tried to directly apply the computational algorithm of the average to solve the problem, but the application was incorrect.

Unjustified symbol manipulation: The student just picked some numbers from the task and worked with them in ways irrelevant to the problem context (e.g., added them together).

Errors cannot be identified: A student's work or explanation was so unclear or incomplete that the error type could not be identified.

Table 1. Students' Mathematical Errors in Solving the Open-ended Average Task

Error Types	% of students (n= 126)
Minor error	11
Violation of "stopping rule"	10
Incorrect use of average concept	34
Unjustified symbol manipulation	24
Errors cannot be identified	21

Table 1 shows the percentage of students who made mathematical errors in each category. Of the five types of errors, the largest percentage, one-third, of the students tried to directly apply the computational algorithm to solve the problem. For example, a student added the numbers of cups sold in week 1 (9), week 2 (3), week 3 (6), and the average (7), then divided the sum by 4, the student then gave the whole number quotient (6) as the answer. Obviously, this student appeared to know the computational procedure of calculating an average (i.e., "add-them-all-up-and-divide"), but he/she appeared to not know what should be added and what should be divided and divided by.

Fourteen students, which is about 11% of the students who had an incorrect answer made minor errors. For example, some students made a minor calculation error. Some others carelessly put the total number of cups sold in four weeks as the answer. Although these fourteen students had incorrect answers, their solution processes suggested that overall, they had a pretty good understanding of the average concept.

Ten percent (10%) of the students violated the "stopping rule" when they used a trial-and error strategy. For example, one student first guessed that Week 4 sold 8 cups, then he added 9, 3, 6 and 8 up, got 26. When he divided 26 by 4, he got quotient 6 and remainder 2. Then he tried number 11, 9 + 3 + 6 + 11 = 29, and 29 ÷ 4 = 7 with remainder 1. The quotient is 7, so he stopped trial and put 11 as the answer. However, he ignored the remainder of 1. Like in this example, the student appeared to know the algorithm for calculating the average, but he did not have clear idea when the guess-and-checking should stop. Error analysis indicated that over one half of the students who did not have the correct answer showed evidence that they knew the computational algorithm of calculating average.

About a quarter of the students simply add up some numerals given in the problem, then put the sum in the answer space. Those student responses show evidence that they did not understand the average concept, even the problem at all.

Solution strategy. Strategies are goal-directed, mental operations that are aimed at solving a problem (Simon, 1989). Appropriate use of strategies is essential to successful problem solving. Three solution strategies were identified, which are described below:

Strategy 1 (Leveling): The student used visualization to solve the problem. Generally, students viewed the average (7) as a leveling basis to "line up" the numbers of cups sold in the week 1, 2, and 3. Since 9 cups were sold in week 1, it has two extra cups. Since 3 cups were sold in week 2, additional 4 cups are needed in order to line up the average. Since 6 cups were sold in week 3, it needs 1 additional cups to line up the average. In order to line up the average number of cups sold over four weeks, 10 cups should be sold in week 4.

Strategy 2 (Using Average Formula): The student used the average formula to solve the problem arithmetically (e.g., $7 \times 4 - (9 + 3 + 6) = 10$ or algebraically (e.g., $(9 + 3 + 6 + x) = 7 \times 4$, then solve for x).

Strategy 3 (Trial-and-error): The student first chose a number for week 4, then checked if the average of the numbers of cups sold for the four weeks was 7. If the average was not 7, then they chose another number for the week 4 and checked again, until the average was 7.

More than a half of the students (130 out of 250 or 52%) had an clear indication of using one of the above strategies. Within those who had clear indication of using one of the above strategies, the strategy 2 (using average formula) was the most frequently used and the leveling strategy was the least frequently used. In particular, 59% (77 out of 130) of them used average formula to solve the problem, 35% (45 out of 130) used trial-and-error strategy, and the remaining 6% (8 out of 130) used leveling strategy.

Representation. Every student who provided an explanation was subject to examination of their representations. This included 244 (98%) students' responses. The representation was examined according to the way students represented their solutions (Janvier, 1987). Four categories were used to classify the representation: verbal, pictorial, arithmetic, and algebraic. If a student mainly used written words to explain how he/she found the answer, then the response was coded as a verbal representation. If a student mainly used a picture or drawing to explain how he/she found the answer, then the response was coded as a pictorial representation. If a student mainly used arithmetic expressions to explain how he/she found the answer, then the response was coded as an arithmetic representation. If a student mainly used algebraic expressions to explain how he/she found the answer, then the response was coded as an algebraic representation.

Two-thirds of the students used the arithmetic representation, but only 2% of the students used the algebraic representation. A quarter of the students used verbal/written words to represent their solutions. Nearly 10% of the students used the pictorial representation to explain how they found their answer.

Since symbolic representations, especially the algebraic symbolic representation are more abstract than verbal and visual representations, it is reasonable to hypothesize that students who used symbolic representations performed at a higher level than those who used verbal and visual representations because of the abstract nature of mathematics. In the remaining of this section, how students' performance is related to their use of representations are examined.

The open-ended average task was administered with other six open-ended tasks, so students' performance[1] on those six open-ended tasks were used as their performance measure to examine the relationships between performance level and the use of representations. As shown in Table 2, the students who used an algebraic representation had the highest mean score, those used arithmetic representation the second highest, those used visual representation the third, and those used verbal representation the last. An analysis of variance (ANOVA), shown in Table 3, suggested that there were statistically significant differences among students who used various representations ($F(3, 240) = 4.19$, $p < .01$). Post hoc analysis showed that the students with the algebraic representation performed significantly better than those who used the arithmetic representation ($t = 2.65$, $p < .05$), than those who used the visual representation ($t = 3.47$, $p < .01$), and than those who used the verbal representation ($t = 3.80$, $p < .01$). The significant differences also existed between the students who used arithmetic symbolic representation and those who used visual representation ($t = 2.48$, $p < .01$), and between the students who used arithmetic symbolic

[1] Each response to the six open-ended tasks was scored according to a holistic scoring scheme, ranging from 0 - 4. The maximum score on the six open-ended tasks is 24. The detailed description of the holistic scoring can be found in Lane (1993) and Silver & Lane (1993).

representation and those who used verbal representation ($t = 2.96$, $p < .01$). There was no significant difference between students who used visual representation and those who used verbal representation.

Table 2. Students' Mean Scores on Solving Six Open-ended Problems

Representation	Algebraic (n=5)	Arithmetic (n=161)	Visual (n=16)	Verbal (n=62)
Means	19.20	17.03	14.19	13.61

Table 3. Summary of the ANOVA Analysis for Students' Performance

Source	DF	Sum of Squares	Mean Squares	F-value	Pr > F
Model	3	157.47	52.49	4.19	.01
Error	240	3005.52	12.52		

Relatedness on Performing Two Average Tasks

Table 4 shows the results of students' correctness of answers on the two average tasks. A chi-square analysis indicated a significant difference between the two tasks with respect to the distribution of students with the correct answer or incorrect answer, χ^2 (1, N = 250) = 14.08, $p < .005$). In particular, although nearly 90% of the students who had the correct answer for the multiple-choice average task, only 50% of them had the correct answer for the open-ended average task. In fact, 40% of the students had the correct answer for the multiple-choice task, but an incorrect answer for the open-ended average task. One-tenth of the sample had both answers wrong. Interestingly, a few students had the correct answer for the open-ended task, but an incorrect answer for the multiple-choice one.

Table 4. Distribution of Students' Correct and Incorrect Answers in the Two Tasks

		Multiple-choice Average Task	
		Correct Answer	Incorrect Answer
Open-ended Average Tasks	Correct Answer	120 (48%)	5 (2%)
	Incorrect Answer	101 (40%)	24 (10%)

DISCUSSION

This study examined a group of 250 sixth-grade students' performance on solving two average tasks. The results of this study suggest that a majority of the students knew the "add-them-all-up-and-divide" algorithm of calculating average. In fact, nearly 90% of the students selected the correct operations when they were asked to choose appropriate

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operations for calculating the average of five scores. However, only about a half of the students showed evidence of having a conceptual understanding of the concept.

Error analysis of students' responses to the open-ended average task suggested that a fairly large proportion of the students attempted to use the average algorithm directly to solve the problem, as used in calculating a simple mean. Obviously, students cannot use the average formula directly in order to solve the open-ended average problem, instead, they should "reversibly" use the average algorithm. Such "reversibility" of using the average formula exhibited a deep understanding of the average concept and a flexible application of the concept in the problem situation.

In addition to the error analysis, students' solution strategies and representations were also examined. In particular, the results of this study showed that students used different solution strategies and used various representations to show their solution processes. The results of this study showed that the use of solution strategies and representations reflected the level of students' mathematical problem-solving performance (Janvier, 1987). Higher performers tended to use more abstract representations than the lower performers. The students who used an algebraic representation performed better than those who used others. The analysis of students' solution strategies, errors, and representations was possible because of using the open-ended type of task. This study suggest the value of using the open-ended task to capture students' misconceptions of average concept and problem-solving processes.

Shaughnessy (1992) indicated that most students' understanding of the average concept was the "add-them-all-up-and-divide" algorithm, because the computational procedure was all they were ever taught. This study not only suggested that most students' understanding of the average concept is only the computational algorithm, but also suggested the need for teaching the average concept beyond the computational algorithm. For example, the average concept may be introduced as that "the average of two or more numbers is related to the process of EVENING OFF of columns of cubes" (Bennett, Maier, & Nelson, 1988), before the formal algorithm is taught. The height of the evened off columns is the AVERAGE of these original columns. The emphasis should be on "averaging" rather than on "average" and averaging is the evening-off process.

Teachers may provide students opportunities to solve various types of average problems, such as the "weighted average problems" and the ones like used in this study so that students will experience the application of the average concept in various situations. It is not enough for students to obtain a correct answer; importantly, they should be asked to explain their thinking and reasoning. Hence, they will experience the conceptual aspect of the concept beyond the computational algorithm. Given the importance of the average concept in statistics and daily life, it is necessary to explore the effective ways of teaching the average concept with conceptual understanding in future mathematics education research.

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Figure 1. Average Tasks

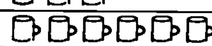
Multiple-choice Average Task

Which operations should you carry out to solve this problem?
On five tests in your math class your scores are 98, 63, 72, 86, and 100. What is your average score?

a. add, then multiply
b. add, then divide
c. divide only
d. multiply, then subtract

Open-ended Average Task[2]

Angela is selling cups for the Mathematics Club. This picture shows the number of cups Angela sold during the first three weeks.

Week 1	
Week 2	
Week 3	
Week 4	?

How many cups must Angela sell in Week 4 so that the average number of cups sold is 7? Show how you found your answer.
Answer: ____________

[2] This task was adapted from QUASAR project. For reasons of confidentiality, the context and numbers embedded in the problem were modified, but the mathematical structure is identical. For more information about the QUASAR project, please write to Dr. Edward Silver, LRDC, University of Pittsburgh, Pittsburgh, PA, 15260, USA.

LEARNING PROBABILITY THROUGH BUILDING COMPUTATIONAL MODELS

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Abstract

While important efforts have been undertaken to advancing understanding of probability using technology, the research herein reported is distinct in its focus on model building by learners. The work draws on theories of Constructionism and Connected Mathematics. The research builds from the conjecture that both the learner's own sense making and the cognitive researchers' investigations of this sense-making are best advanced by having the learner build computational models of probabilistic phenomena. Through building these models, learners come to make sense of core concepts in probability. Through studying this model building process, and what learners do with their models, researchers can better understand the development of probabilistic learning. This report briefly describes two case studies of learners engaged in building computational models of probabilistic phenomena.

Introduction

In the Connected Probability project (Wilensky, 1993; 1994), we explore ways for learners (both secondary and post-secondary) to develop intuitive conceptions of core probabilistic concepts. Computational technology can play an important role in enabling learners to build intuitive conceptions of probability. Through building computational models of everyday and scientific phenomena, learners can build mental models of the underlying probability and statistics. Even learners not usually considered good at mathematics and science can build models that demonstrate a qualitatively greater level of mathematical achievement than is usually found in mathematics classrooms. "Emergent phenomena", in which global patterns emerge from local interactions, are authentic contexts for learners to build with probabilistic parts. By giving probabilistic behavior to distributed computational agents, stable structures can emerge. Thus, instead of learning probability through solving decontextualized combinatoric formulae or being consumers of someone else's black box simulations, learners can participate in constructionist activities - they design and build with probability.

As part of the Connected Probability project, we have extended the Starlogo parallel modeling language (Resnick, 1992; Wilensky, 1993) and tailored it for building probabilistic models. The Starlogo language is an extension of the computer language Logo that allows learners to control thousands of screen "turtles". These turtles or computational agents have local state and can be manipulated as concrete objects. Through assigning thousands of such turtles probabilistic rules, learners pursue both forwards and backwards modeling. Forwards modeling involves exploring the effects of various sets of local rules to see what global pattern emerges, while in backwards modeling learners try to find an adequate set of local rules to produce a particular global effect. In this report, two case studies of probabilistic modeling projects are presented.

Theoretical Framework: Constructionism and Connected Mathematics

This research is organized and structured by a theory of Connected Mathematics (Wilensky, 1993). Connected Mathematics responds to a prevailing view that mathematics must be seen as "received" or "given" and graspable only in terms of formalism *per se*.

Connected Mathematics is situated in the constructionist learning paradigm (Papert, 1991). The Constructionist position advances the claim that a particularly felicitous way to build strong mental models is to produce physical or computational constructs which can be manipulated and debugged by the learner. As described by Wilensky (1993), Connected Mathematics also draws from many sources in the mathematics reform movement (e.g., Confrey, 1993; Dubinsky & Leron, 1993; Feurzeig, 1989; Hoyles & Noss, 1992; Lampert, 1990; Schwartz, 1989; Thurston, 1994).

A Connected Mathematics learning environment focuses on learner-owned investigative activities followed by reflection. Thus, there is a rejection of the mathematical "litany" of definition-theorem-proof and an eschewal of mathematical concepts given by formal definitions. Mathematical concepts are multiply represented (Kaput, 1989; Mason, 1987; von Glaserfeld, 1987) and the focus is on learners designing their own representations. Learners are supported in developing their mathematical intuitions (Wilensky, 1993) and building concrete relationships (Wilensky, 1991) with mathematical objects. Mathematics is seen to be a kind of sense-making (e.g., Schoenfeld, 1991) both individually and as a social negotiation (e.g., Ball, 1990; Lampert, 1990). In contrast to the isolation of mathematics in the traditional curriculum, it calls for many more connections between mathematics and the world at large as well as between different mathematical domains (e.g., Cuoco & Goldenberg, 1992; Wilensky, 1993).

The Role of Technology

The idea that mathematics is not simply received and formal implies a vision for how technology can be used. Not to simply animate received truth (e.g., by running black-box simulations) but instead as a medium for the design of models by learners. Under a traditional formalistic framework, mathematics is "given" and technology is seen as simply animating what is already known. In Connected Mathematics, knowing is situated and technology provides an environment in which understanding can develop. Learners literally construct an environment in which they then construct their understanding.

Because, when learners build computational models, they articulate their conceptual models through their design, researchers can gain access to these conceptual models (see e.g., Collins & Brown, 1985; Pea, 1985). The researcher is given insight into the thinking of the learner at two levels: as model builder and as model consumer.

Building Models vs. Running Simulations

Computer based simulations of complex phenomena are becoming increasingly common (see e.g., Rucker, 1993; Stanley, 1989; Wright, 1992a, 1992b). In a simulation, the learner is presented with and explores a sophisticated model (built by an expert) of a subject domain. The user can adjust various parameters of the model and explore the consequences of these changes. The ability to run simulations (or pre-built models) interactively is a vast improvement over static textbook-based learning with its emphasis on formulae and the manipulation of mathematical tokens. Stanley (e.g., Shore et al, 1992) has demonstrated that curricular materials based on simulations of probabilistic phenomena can be very engaging to secondary students and teachers. But, in simulations, generally, learners do not have access to the workings of the model. Without access to the underlying structures, learners may perceive the model in a way quite at variance with the designer's intentions. Furthermore, learners cannot explore the implications of changing these structures. Consequently, their ability to develop robust mental models of these structures is inhibited. A central conjecture of this research is that for learners to make powerful use of models, they must first build their own models and design their own investigations. It is only by exploring the "space" of possible models of the domain that learners come to appreciate the power of a good model. To support users in building useful models, a number of powerful modeling environments have been designed. (e.g., *STELLA* - Richmond & Peterson, 1990, Roberts, 1978; *Starlogo* - Resnick, 1992; Wilensky, 1993; *Agentsheets* - Repenning, 1993; *KidSim* - Smith, Cypher & Spohrer, 1994).

Extensible models

In the spirit of Eisenberg's use of "extensible applications" (Eisenberg, 1991) extensible models are pre-built models or simulations that are embedded in a general purpose modeling language. This combined approach has many of the advantages of both simulation and model building: there is a rich domain model to be investigated, access is given to the structure of the model, users can modify this structure, and even use it as a basis for building their own models and tools.

The challenge for such an approach is to design the right middle level of primitives so that they are neither (a) too low-level, so that the extensible model becomes identical to its underlying modeling language, nor (b) too high-level, so that the application turns into an exercise of running a small set of pre-conceived experiments.

Probability

The domain of probability (and statistics) has been an ongoing focus of research within the Connected Mathematics program. There are many reasons to recommend probability as a content domain. Among these are:

• There is a considerable literature attesting to the difficulty people have with understanding probability (e.g., Kahneman & Tversky, 1982, Nisbett et al, 1983, Konold, 1991). Standard instruction has been shown to provide little remedy. Educators have responded to this research by advising students not to trust their intuitions when it comes to probability and to rely solely on the manipulation of formalisms. As a result, learners construct brittle formal models of the core probabilistic concepts and fail to link them to everyday knowledge. Connected Mathematics provides an alternative to this formalistic stance. It asserts that powerful probabilistic intuitions can be constructed by learners (Wilensky, 1993; 1994; forthcoming). By taking up such a challenging domain, a strong proof of the value of Connected Mathematics can be demonstrated.

• Computational environments can open doors to new ways of thinking about probability. Computational environments allow users to construct stable products (e.g., normal distributions, see below) using random components. This construction would be very difficult to do without computational environments. From a constructionist perspective, this ability to build meaningful products from random components is a prerequisite for making sense of the core notion of randomness.

• Particularly in the area of probability and statistics, the educational goal should emphasize interpreting (and designing) statistics from science and life rather than mastery of curricular materials. In order to make sense of scientific studies, it is not sufficient to be able to verify the stated model; one needs to see why those models are superior to alternative models. In order to understand a newspaper statistic, one must be able to reason about the underlying model used to create that statistic and evaluate its plausibility. For these purposes, building probabilistic and statistical models is essential.

• Many everyday phenomena exhibit emergent behavior: the growth of a snowflake crystal, the perimeter pattern of a maple leaf, the advent of a summer squall, the dynamics of the Dow Jones or of a fourth grade classroom. These are all systems which can be modeled as composed of many distributed but interacting parts. They all exhibit non-linear or emergent qualities which place them well beyond the scope of current K-12 mathematics curricula. Yet, through computational modeling, especially with parallel languages such as Starlogo, pre-college learners can gain mathematical purchase on these phenomena. Modeling these everyday complex systems can therefore be a motivating and engaging entry point into the world of probability and statistics.

The Case of Normal Distributions

As part of my efforts to create learning environments for probability, I have used a carefully selected set of materials (consisting of newspaper clippings, probability puzzles and paradoxes, core probability concepts and computational tools) to stimulate learners to pursue their own investigations and design their own computational tools for pursuing their inquiry.

One such example is Alan, a student with a strong mathematical background who nevertheless felt that he "just didn't get" normal distributions. Using a version of the parallel modeling language Starlogo which was enhanced for focusing on probability investigations (Wilensky, 1993; 1994), Alan developed a model for explaining his question, "Why is height (in men) normally distributed?" Alan's theory was that perhaps "Adam" had children which were either taller or shorter than him with a certain probability. If this process was repeated with the children, then a distribution of heights would emerge. To explore what kinds of distributions were possible from this model, Alan built a "rabbit jumping" microworld. Taking advantage of the parallel modeling environment, Alan placed sixteen thousand rabbits in the middle of a computer screen. He then gave each rabbit a probabilistic jumping rule. (In Alan's model, the location of the rabbit corresponds to a person's height and a jump corresponds to a set deviation in height). The first such rule he explored was to tell each rabbit to jump left one step or right one step each with probability 1/2. After a number of steps, the classic symmetric binomial distribution became apparent. Alan was pleased with that outcome but then asked himself the question: what rule should I give the rabbits in order to get a non-symmetric distribution? His first attempt was to have the rabbits jump two steps to the right or one step to the left with equal probability. He reasoned that the rabbits would then be jumping more to the right so the distribution should be skewed right. His surprise was evident when the distribution stayed symmetric while moving to the right. It didn't take too long though before he realized that it was the different sized probabilities not the different sized steps that made the distribution asymmetric. This example, while seemingly elementary, captures many facets of the model building approach to learning about complexity:

- The question was owned by the learner
- Theories were instantiable and testable
- Buggy theories could be successively refined
- The modeling environment did not limit the directions of inquiry

The environment provides a suitable set of syntactic primitives so that his model was easily built. It provided a suitable set of conceptual primitives that guided Alan's investigation. In particular the parallelism of the modeling environment puts a focus on the relationship between micro- and macro-aspects of the problem. Typically, distributions are learned and classified by their macro- features (e.g., mean, standard deviation, variance, skew, moments) but the realization that distributions are emergent effects of numerous micro-level interactions is lost. This is a key point since 1) the concept of distribution is central to probability and statistics and 2) this failure to connect levels makes distributions seem like formal received mathematics, mathematics to be memorized and understood solely through formulae. In contrast, Alan constructs distributions and is able to link their macro-properties to the micro- rules he has given them.

GPCEE - The Case of the Gas in a Box

Harry is a science and mathematics teacher in the Boston public schools. He was very interested in the behavior of gas particles in a closed box. He remembered from school that the energies of the particles when graphed formed a stable distribution called a Maxwell-Boltzman distribution. Yet, he didn't have any intuitive sense of why they might form this stable asymmetric distribution. He decided to build a model of gas particles in a box using the Starlogo modeling language.

The model Harry built is initialized to display a box with a specified number of gas "molecules" randomly distributed inside it. The user can then perform "experiments" with the molecules.

The molecules are initialized to be of equal mass and start at the same speed (that is distance traveled in one clock tick) but at random headings. Using simple collision relations, Harry was able to model elastic collisions between gas molecules, (i.e., no energy is "lost" from the system). The model can be run for as many ticks as wanted.

By using several output displays such as color coding particles by their speed or providing dynamic histograms of particle speeds/energies, Harry was able to gain an intuitive understanding of the stability and asymmetry of the Boltzman distribution.

Harry's story is told in greater detail elsewhere (Wilensky, forthcoming). (Originally, Harry had thought that because gas particles collided with each other randomly, they would be just as likely to speed up as to slow down. But now, Harry saw things from the perspective of the whole ensemble of particles. He saw that high velocity particles would "steal lots of energy" from the ensemble. The amount they stole would be proportional to the square of their speed. It then followed that, since the energy had to stay constant, there had to be many more slow particles to balance the fast ones.

This new insight gave Harry a new perspective on his original question. He understood why the Boltzman distribution he had memorized in school had to be asymmetric. But it had answered his question only at the level of the ensemble. What was going on at the level of individual collisions? Why were collisions more likely to lead to slow particles than fast ones? This led Harry to conduct further productive investigations into the connection between the micro- and macro- views of the particle ensemble.

GPCEE as an Extensible Model

Harry's story is not the end of the tale. Harry collaborated with me in making his model into an extensible application. We call the model GPCEE (Gas Particle Collision Exploration Environment). Once GPCEE became a publicly accessible model, we were struck by its capacity to attract, captivate and engage "random" passers-by. People whose idea of fun did not include working out physics equations nonetheless were mesmerized by the motion of the shifting gases, their pattern and colors. And they were motivated to ask questions - why do more of them seem to slow down than speed up? What would happen if they were all released in the center? in the corner?

As a result, many more learners used the GPCEE model and, since it was an extensible model, they extended it. Among the extensions that users built were tools for measuring pressure in the box and viscosity of the gas, pistons to compress the gas, different shapes for the container, different dimensional spaces for the box, diffusion of two gases, different geometries for the molecules (e.g., diatomic molecules with rotational and vibrational freedom), and sound wave propagation in the gas. In order to build the computational tools for these extensions, users had to build conceptual models. They came to ask such questions as: What kind of thing is pressure? How would you build a tool to measure it?

It is clear that GPCEE is both a physics simulation, one in which experiments difficult or impossible to do with real gases can be easily tried out, and an environment for strengthening intuitions about the statistical properties of ensembles of interacting elements. Through creating experiments in GPCEE, learners can get a feel for both the macro- level, the behavior of the ensemble as an aggregate, and its connections to the micro- level – what is happening at the level of the individual gas molecule. In the GPCEE application, learners can visualize ensemble behavior all at once, sometimes obviating summary statistics. Furthermore, they can create their own statistical measures and see what results they give at both the micro- and the macro- level. They may, for example, observe that the average speed of the particles is not constant and search for a statistical measure which is invariant. In so doing, they may construct their own concept of energy. Their energy concept, for which they may develop a formula different than the formula common in physics texts, will not be an unmotivated formula the epistemological status of which is naggingly questioned in the background. Rather, it will be personally meaningful, having been purposefully designed by the learner.

The necessity of creating his own summary statistic led one learner to shift his view of the concept of "average". In the GPCEE context, he now saw "average" as just another method for summarizing the behavior of an ensemble. Different averages are convenient for different purposes. Each has certain advantages and disadvantages, certain features which it summarizes well and others that if doesn't. Which average we choose or construct depends on what about the data is important to us. {how we wish to make sense of the data.}

Having shown the GPCEE environment to quite a few professional physicists, I can attest to the fact that although they knew that particle speeds fell into a Maxwell-Boltzman distribution, most were still surprised to see more blue particles than red -- they had formal knowledge of the distribution, but the knowledge was not well connected to their intuitive conceptions of the model In a typical physics classroom, learners have access either only to the micro level - through , say, exact calculation of the trajectories of two colliding particles, or only to the macro- level, but in terms of pre-defined summary statistics selected by the physics canon. Based on this example, it would seem that it is in the interplay of these two levels of description that powerful explanations and intuitions develop.

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ALGEBRA AS A PROBLEM SOLVING TOOL: ONE UNKNOWN OR SEVERAL UNKNOWNS?

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Abstract

An experiment conducted with students of different academic levels (13 to 17 year olds), concerning different types of problems in algebra, brings evidence to the implicit criteria which lead students towards an engagement in problems in one or several unknowns. The analysis also brings out the differences between the two types of reasoning from the point of view of management of the relations in the problem and what they require of the student. The results of this experiment unite with the epistemological analysis of the construction of algebraic knowledge in the context of problem solving.

1-Introduction

This work is part of an ongoing research project[1] which aims at better understanding the conceptual base which underlies the student's construction of algebraic knowledge. In being interested in the continuities and discontinuities which one can locate in the reasoning used by the students (Bednarz et al. 1992-a, 1992-b, in press), our research program makes a close study of the *nature* of problems generally presented in algebra and leads us to elaborate a *grid of analysis* in terms of relational calculations (Vergnaud, 1982), which enables us to keep track of the relative complexity of the problems and to understand and anticipate certain difficulties observed with the students (Bednarz & Janvier, 1994). This grid forms an essential basis to analyse procedures used by students and problems capable of favoring the emergence and development of algebraic reasoning. The work which we present here aims exactly at analyzing this question at the level of the passage from the reasonings based on algebra of one unknown (or several unknowns) to that based on algebra of several unknowns (or one unknown).

2-From algebra in one unknown to algebra in several unknowns

The use of algebra in several unknowns usually comes into the school curriculum in tenth grade (15 to 16 years old), after two years of introduction to algebra which favors, in the context of problem solving, the resolution in one unknown.This gradation rests on certain presumptions, linked above all to the solution tool, which is the equation, guiding the choice of problems given to students in different academic levels, and in particular the

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choice of the initial problem which forces the student to a resolution in one unknown[2] . The results of experiments conducted with students in different academic levels, concerning different types of problems, questions this a priori gradation based on the equation, and questions the relevance of this hierarchy proposed in learning between resolution in one unknown and resolution in several unknowns (Bednarz & Janvier, 1994).

The study of the passage from one type of reasoning to the other in problem solving has recently been approached by Radford (1994) in light of the changes which are required in the system of mathematical knowledge associated with several unknowns, Sn, in comparison with the system of knowledge associated with one unknown, S1, each of these systems being characterized by their own concepts, their problems to solve and their specific solution methods. The comparison of solution procedures for one or several unknowns, in relation to certain specific problems, allows showing certain profound changes implemented at the level of solution methods in the passage from one way of reasoning to another. Indeed in system S1, to be able to express the many unknown quantities to which the situation makes reference in terms of only one unknown, the students must transform the problem, and this transformation is a source of difficulties (Radford, 1994, p. 79). On the other hand, in the solution methods used in S1, the relationships between quantities do not have an interchangeable role, therefore it is necessary for the students, if such relationships and their sequence are not directly formulated, to anticipate a hierarchy in the relations and to choose the order in which the relations will be used. In contrast, in Sn the management of relations happens otherwise: the insertion of several unknowns avoids the transformation of the problem for the students, but using several equations will raise other difficulties (cf op. cit., p. 77). Difficulties appear, at least in the sample considered, in conferring a real algebraic status to each of the unknowns, the students limit to using these as descriptive elements of the relations of the problem without giving them an operational mobility (cf example 5, op. cit., p. 77).

This work seems to suggest, in relation to certain problems, that the solution procedures in S1 are more difficult for the student than those in Sn from the point of view of the management of the relations. However at the same time this first exploration shows a certain resistance on the part of the students in the passage to Sn. In order to better understand the difficulties linked to each of these modes of reasoning,we propose, in this work, to further push the study of solution procedures used by students in different

[2] It could be wothwhile to keep in mind that the algebraic concepts of unknown and variable are logical and epistemplogical differents (for a deep discussion about the radical differences of both concepts, see Radford, forthcoming 1 and 2).We will deal here with the concept of algebraic unknown only.

academic levels with regard to problems presenting different relational structures (Bednarz, Janvier, 1994). More specifically, our objectives are the following:
to further characterize the elements of the problems (structure of the problem, nature of the relations, linking of these relations, nature of the data ...) capable of further favoring the emergence and the use of algebraic procedures in several unknowns or one unknown.We wish to try to distinguish, from the relational structure of a problem (Bednarz & Janvier, 1994), implicit criteria used by the students which allows them to identify a problem given in one or several unknowns. This work is based on an analysis of problems and of procedures, and has been conducted with a double perspective, historico-epistemological and didactical. This analysis will allow us to shed light on alternating situations to the usual gradation one unknown- several unknowns proposed in teaching .

3-Some epistemological considerations

The history of mathematics shows us that during a long time solution methods remained based on a reasoning of only one unknown: this is the case of the methods of single and double false position (Radford, 1994). This is the case of algebra according to Diophantus, but it is also the case in mediaeval mathematics.Thus the historical record suggests that the second unknown is a late concept in the development of algebraic ideas. To our knowledge, it is found for the first time in the "extrait du Fakhrî" by Aboû Bekr Alkarkhî, written at the beginning of the 11th century in Bagdad (Woepcke, trad. 1853), and later in the 14th century, in the Trattato di Fioretti d'Antonio de Mazzinghi (ed. Arrighi, 1967), where it is designated by the term "quantita", the first unknown having been designated by the "cosa". An important thing to say in relation to the second unknown in the works cited is that it serves to make problem solving simpler. Mazzinghi, for example, uses it for the first time in problem 9 of the Trattato:" *Find two numbers so that when we multiply them it makes 8 and so that the sum of their squares is 27* ", a problem which he already knows how to solve (he gives two other solutions, one of which is geometric, cf. op. cit, p. 28-29). The "quantita", or second unknown (and the "cosa", first unknown) never serve to translate the wording of a problem, as one does in the algebra taught in school today, but to find a quasi-symbolic expression for the numbers sought for .Thus, in problem 9, Mazzinghi represents the numbers sought for by "la cosa plus la racine de la quantita" and the "cosa moins la racine de la quantita". The second unknown thus appears as having a purely heuristic role: its finality is that of procuring an alternative method of resolution, more direct and general (which is likely the case of the first unknown whose emergence procured a shorter and more direct method in comparison to the Babylonian methods of false position (cf. Radford, forthcoming).We find for the second unknown what we found for the first one.

The second unknown thus appeared as a new object hierarchically subjugated to the first unknown. At the beginning it does not have a life of its own.Thus Mazzinghi introduces the "quantita" at the beginning of problem solving, but the calculations were conducted so as to get rid of the "quantita" and to find an equation in terms of the first unknown. It is only much later that the two unknowns acquired the same role: beginning with the first problems, Stifel in his Arithmetica Integra (1554) strove to show that the second unknown could have the same role as the first unknown. In his work in particular, one finds solution procedures which lead to equations including uniquely the second unknown, the first having been eliminated at the beginning of the calculations.

Let us remark finally that the historical analysis suggests that the structure of the problems in which the second unknown emerges are problems where the second unknown is not expressed directly in terms of the first unknown.These are the problems which do not lead to relations of the type "y=g(x)", which is the case of relations of comparison such as " the amount of a person is 3 times the amount of the other ", but which lead to relations of the type "f(x,y)=a", which is the case in particular of relations of sums such as these involved in the problem 9 of Mazzinghi.

4-The experimental study

We saw briefly how reasonings evolved and to what kind of problems this evolution was articulated. It is on this structure of the problems that the experimental study will make the link with the historical study.

In order to better understand the type of problems capable of favoring an eventual engagement of several unknowns, and the differences for the students between the reasonings of one unknown versus several unknowns thus involved, different problems were considered for the purpose of experimentation, based on the grid previously developed by the team (Bednarz & Janvier, 1994). We considered three types of problems.

- problems of **unequal partitioning** (in which a known whole is distributed in many parts), such problems being generally presented at the time of the introduction to algebra[3] , and which bring into play relations of comparison, g(x)=y, between unknown quantities, such as the following problem: "*The three daughters of Mr. Beaulieu together received $181 for their work. Marie received $37 more than Paule, and Danielle received $14 more than Marie. How much did each receive?* " (Problem a).

[3] These problems are generally considered by their authors as problems of one unknown.

We have varied certain elements within this class of problems which are capable of affecting the engagement of the student in these problems and their solution (Bednarz & Janvier, 1994). We will return more specifically to certain of these elements which concern us here at the time of the analysis.

- problems only bringing into play **relations of sums**, $f(x,y)=a$, such as the following problem: "*The three daughters of Mr. Beaulieu received $181 for their work. The difference between the amounts of money that Nadine and Diane received is $37 and the difference between the amounts of money that Francoise and Nadine received is $14. How much did each receive?*" (problem b).

- problems bringing into play **transformations of quantities**, such as the following: "*Together Luc and Michel have $11.90. Luc doubles his amount of money while Michel increases his by $1.10. Now together they have $17.20. How much did each have at the beginning?*" (problem c). This third class allows us, by changing the nature of the relations between quantities before and after transformation , $g(x)=y$ or $f(x,y)=a$, to see the influence of the nature of relations in an another type of problems..

These problems were experimented with students from different academic levels (2 groups per level), for secondary 2 and 3 (13 to 15 year olds) at the moment of introduction to algebra in one unknown in the academic curriculum, for secondary 4 and 5 (16 to 17 year olds) at the moment of and after introduction to algebra in several unknowns in the curriculum.

5-Analysis of results

-With certain problems, <u>the students used solution procedures which are removed from those favored by the school curriculum</u>. Thus in secondary 2 and 3, certain problems (fig. 1 and 2) are going to even more likely be perceived by the students as problems with several unknowns, which will be shown by a writing and a working out in terms of several unknowns (XS), or by recourse to a procedure of the type "false one unknown" (F1), as in the following example (problem d, figure 1: *three children play marbles. They have all together 198 marbles. Pierre has 6 times more marbles than Denis and 3 times more marbles than George. How much marbles did each child has ?*

Pierre: $3x+6x$

Denis: x

George: x $\quad\quad$ $3x+6x+x+x=198$

In this latter case, the student sees two unknowns, but not having the tools for operating, will use the letter x to refer to two different magnitudes which will be treated independently.On the opposite, certain problems are going to evoke in the students from secondary 4 and 5 solution procedures in one unknown, which will be manifested by a direct writing in terms of one unknown (X1) or by recourse to a procedure of the type "false several unknowns"(FX) (the student writes several variables as the school demands, but he thinks, as seen in his solution, in terms of only one). Thus, in the following example (problem a) the student conceives the problem in one unknown:

Paule: p   Marie: m+37   Danielle: d+37+14

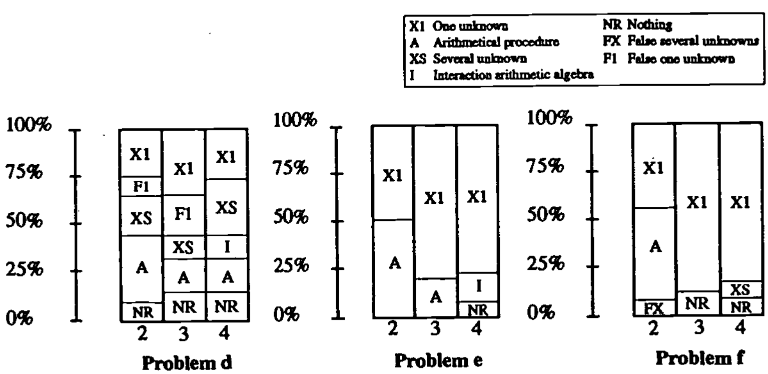
p+m+37+d+37+14=181

(if the m and the d are replaced by p, we have indeed an equation in one unknown).

- The analysis of the reasoning profiles used by the students at different academic levels in each of the problems allows one to bring into evidence the implicit criteria which guides the students in their engagement towards a procedure in one unknown or several unknowns:

The nature of the relations between the unknown quantities and their linking appear to be determinants here. Thus the problems of unequal partitioning (cf fig. 1) bring into play, according to the grid developed by Bednarz & Janvier (1994), a sequence of relations of "well type" (such as problem d) which, more than those in which the quantities are generated directly from the same quantity (such as problem e :" *three chidren play marbles. They have all together 198 marbles. Pierre has 6 times more marbles than Denis and George has 2 times more marbles than Denis. How many marbles did each child has?*" and problem f : "*....George has 2 times more marbles than Denis and Pierre has 3 times more marbles than George...*"), evoke at all levels an engaging of several unknowns.

Figure 1: Patterns of reasonings to problems d, e, f.



In the same way problems bringing into play the relations of the type sum or difference (fig. 2) between unknown magnitudes f(x,y) (such as problems b or c) evoke more of an engagement in several unknowns on the part of the students than are brought into play for the same type of problem with the relations of the type comparison (such as problem a or the following: "*Luc has $3.50 less than Michel. Luc doubles his amount of money while Michel increases his by $1.10. Now together they have $17.20. How much did each have at the beginning?*")

Figure 2: Patterns of reasonings to problems a and b.

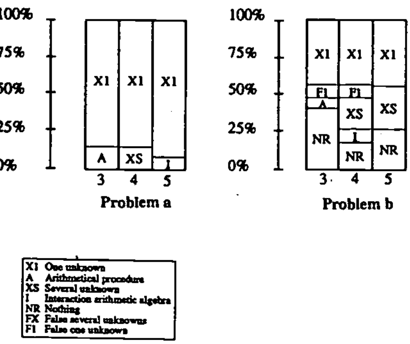


Figure 3: Performance to problems g and h.

	Sec. 2 (13-14 years old)	Sec. 3 (14-15 years old)		Sec. 4 (15-16 years old)	
		Group A	Group B	Group A	Group B
Problem h	64%	24%	84%	32%	76%
Problem g	80% (high level)	52% (weak)	97% (high)	54% (weak)	97% (high)

The referring of magnitudes appears to be another implicit criteria which will be a determinant in the representation which the student will make of the problem, at the moment of introduction to algebra. Thus the following problems do not al all evoke the same type of reaction on the part of the students: "*An inventory has been made of sporting goods. In the second and third warehouses, 288 balls were counted. If there are 4 times more balls in the second warehouse as in the first, and 7 times more balls in the third warehouse than in the second, how many balls are there in each warehouse?*" (problem g, in which the numbers refer to the same entity) and "*Three kinds of sporting goods were counted in a warehouse. When the snowshoes and hockey sticks were counted, the total was 288. If there are 4 times more snowshoes than balls, and 7 times more hockey sticks than snowshoes, how many of each type of sporting goods are there in the warehouse?*" (problem h identical to the preceding, but in which however the numbers refer to different entities). The second problem, further seen by the students as a problem in several unknowns in which the common generator does not immediately appear for the students, will thus be much less successful than the first (figure 3). We can observe ici the same tendancy at all grade levels and for all the groups.

Conclusion

Our results join the historical record in suggesting that the structure of problems in which several unknowns emerge are problems which cannot be directly expressed in terms of one unknown. Our results however give evidence here beyond of other implicit more subtle criteria which guide the student in this engagement (sequence of relations, in the case of problems bringing into play the relations of comparison, "referent "of numbers ...).These results bring back into question the usual hierarchy which one finds in teaching between resolution in one unknown and resolution in several unknowns and provides paths for both a choice and a more appropriate gradation of problems..Our analysis of reasonings show finally that for students the unknowns are always used to translate the problem.

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NEGOTIATING CONJECTURES WITHIN A MODELING APPROACH TO UNDERSTANDING VECTOR QUANTITIES

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Abstract
Within a modeling-based curriculum, one small group of students developed a model of the vector relationships among the multiple forces acting on an object moving at constant velocity on an inclined plane. This paper reports the analysis of how that group interpreted the posed problem, generated and negotiated multiple conjectures, devised strategies for analyzing the data, and confirmed their sense of various conjectures. The results of this classroom case study illuminate the difficulties students encounter in representing and analyzing a vector quantity such as force. These results provide evidence to support the grounding of student's analytic activities in physical phenomena, with particular importance of the boundary conditions as sense-making points, while engaging students in a non-linear, cyclic pattern of interpreting, hypothesizing and confirming conjectures.

Introduction and Theoretical Framework

The usual definition or description of a vector is as a quantity having both magnitude and direction. Physical quantities such as displacement, force, velocity and acceleration are ordinarily described as vector quantities. The concept of force as a multi-dimensional, vector quantity is thus quite distinct from the more familiar scalar quantities such as length and temperature. The measurement of physical quantities such as force and velocity, according to Freudenthal (1993), requires three constitutive components: "a concept of *equivalence*, in order to assign the same measure to equivalent objects, a way to *compound objects* which extends to the addition of their measures, [and] a unit measure" (p. 77). Freudenthal argues that the equivalence relation for both force and velocity is translation. Thus, velocity as a vector quantity can be freely moved along a linear translation, as in the horizontal velocity of projectile motion. With forces, for example, translation along the line of action does not change the measurement of the force: "the point where one pulls the cord does not matter" (Freudenthal, 1993, p. 80). Thus, linear translation defines the equivalence relation between any two forces. The measurement of a force also requires a compounding action that allows for the addition of any two forces. Freudenthal claims that students must corroborate *why* vector addition accounts for multiple forces acting at a point; this claim has been substantiated by other research (Doerr & Confrey, 1994).

In an earlier study with high school physics students on how impulse forces affect the speed of motion, White (1983) found that the students did not use the formalism of vector diagrams to solve problems involving change in the direction of motion. Rather than having a unified concept of vectors, the students used speed (or magnitude) and direction independently. The geometric representation of a vector focused students' attention on the directionality of the quantity, but may have left confusion as to what the length represents. Furthermore, the scalar rules of arithmetic do not transfer to the addition of the magnitudes of vectors. Since understanding force and motion requires simultaneously understanding both direction and magnitude, White concluded that the process of understanding force and motion as vector quantities is complex. This suggests that when compounding vectors, as described by Freudenthal, students may encounter difficulties in simultaneously grasping both the magnitude and direction of the vector quantities. In this study, the students' understandings of both the equivalence relation and the compounding action on force vectors are examined in a modeling curriculum that is centered on student construction of understanding through the interplay between physical experimentation and the use of multi-representational analysis tools.

The problem-solving activities found in a typical mathematics or science classroom suggest a linear relationship between physical phenomena, mental (conceptual) models, and mathematical representations. The mathematical representations are often unduly limited to the symbolic representations of algebra and the subsequent manipulation of those symbols. Mathematical modeling is sometimes described as iterations of such a linear problem-solving approach: understand the particular phenomenon to be modeled; define the context and constraints; identify the key variables; explicitly define the relationships among the variables; translate those relationships to an appropriate computer implementation; analyze and interpret the results; and then refine the model and one's understanding through an iterative process by repeating the above steps (Edwards & Hansom, 1989).

In contrast, Lesh, Surber, and Zawojewski (1983) reject the linearity of the Polya-type problem-solving stages that proceed in a uni-directional process from givens to goals. Based on their research in the Applied Mathematical Problem Solving project, Lesh et al. (1983) argue for a non-linear progression through different phases of the modeling process: interpretation, integration/differentiation, and verification. They note that in their study students spent an overwhelming amount of time in the first phase, refining their understandings about the problem. They go on to argue that these phases do not necessarily occur in any given order and that there is a mapping cycling that occurs within each phase. These mapping cycles are the cognitive processes by which students map their perceptions to their cognitive models, transforming their models and mapping back to the perceived problem situation. Thus, Lesh et al. (1983) posit a "spiraling model evolution [that] is characterized by the occurrence of repeated mapping cycles while simultaneously the qualitative level of understanding increases" (p. 132). Linn, diSessa, Pea, and Songer (1994) extend this notion by arguing for a long-term perspective for the refinement and articulation of students' models as these models evolve into more sophisticated forms.

In this paper, I will argue that the use of everyday materials and first-hand experience with objects strengthens the connections among actions, concepts and representations (including graphical, tabular, geometric, and symbolic). In contrast to those theories which argue for a uni-directional movement from concrete to abstract, this case study seeks to elucidate the theory that student understanding is constructed through the dialectic between grounded activity and systematic inquiry as put forth by Confrey (1993). In her theory, Confrey argues that mathematics evolves from actions with concrete materials and that systematic inquiry stabilizes and extends the use of the mathematics. Such a continuing dialectic provides a theoretical foundation for a non-linear modeling process and implies the examination and re-examination of conjectures and representations. This study closely examines the students' actions with physical materials, their generation of representations and hypotheses, and their further inquiry with those representations. This paper presents an analysis of one small group's interactions as the students make their own concepts explicit, hypothesize relationships, test their ideas, and ultimately develop a model for analyzing the role of friction in the motion of objects on inclined planes.

Description of the Study

This study is part of a larger research project that investigated a modeling approach which integrates three components: building representations and relationships from physical phenomena, exploring and extending conjectures through a simulation environment, and developing and validating solutions through the iterative use of a multi-representational analysis tool. The approach used by the students in this study alternated and cycled through these activities. This larger study was designed to understand how these components are interrelated, to explore the extent to which the components are effective in improving students' content knowledge about force, motion, and vectors, and to learn how this modeling process can be used by students. This paper addresses a sub-unit examining the role of friction on an inclined plane and shows how one group of four students devised strategies for analyzing the data, generated and negotiated conjectures, validated conjectures, generated new conjectures, and used the tools and empirical data.

Curricular Unit and Setting

The instructional approach to the unit is based on the notion of providing an essential question to motivate the inquiry and guide the students. The overall essential question for this unit was: "How will an object behave if it is traveling down a frictionless inclined plane? How can you predict the behavior of such an object for any randomly chosen angle of incline?" This question in turn generated four sub-units, each with its own essential question: the resolution of a vector into its horizontal and vertical components; the effect of multiple vectors (e.g. forces) acting on an object; the relationship between force, mass and acceleration, and the role of friction as it affects the motion of an object. The essential question for the fourth unit is: "When an object of known weight is pulled up an inclined plane at a constant velocity, what is the relationship between the force of friction and the angle of inclination?"

This sub-unit was designed around the gathering of data from a physical experiment, and the mathematical (symbolic, graphical, tabular, and geometric) analysis of the data. The multi-representational analysis component was supported through the Function Probe software (Confrey, 1992). The unit was designed to include extensive student discussion and reflection, collaborative work, small and large group tasks, and individual assignments. Students were consistently encouraged to explore their own ideas and to make sense of physical phenomena in a context of interactions with their small group, the entire class and their teachers. This approach to the design of the unit builds on earlier work on projectile motion done by the mathematics education research group at Cornell University (Noble, Flerlage, & Confrey, 1993).

The setting for this study was an alternative public school located in a small urban community. The administration and teachers are actively engaged in curricular and instructional change in mathematics and science. This study took place in an integrated algebra, trigonometry, and physics class with 17 students in grades 9 through 12, who had elected to take the course. The class was divided into five small groups of 3 to 4 students. The class was team taught by two experienced mathematics and physics teachers. One of the most important aspects of the classroom was the role that the teachers took as guides and facilitators for student inquiry. There were no textbooks used by the students nor did the teachers give lectures.

Data Sources and Analysis

Each class session of this unit was audio- and video-taped, and during the small group work, one selected group was video-taped. Copies of the computer work generated by all the students were collected for analysis. Extensive field notes were taken by the researcher during the class sessions. The video-tapes of class sessions were reviewed and transcribed for more detailed analysis. The small group provided a setting within which to analyze and observe how the students go about interpreting the posed essential questions, generating and negotiating their conjectures, devising their strategies for analyzing the data, confirming the sense of one or more conjectures, and using the tools and their empirical data.

The sub-unit began with two simple physical experiments done in the whole-class setting. The first experiment began by simply pulling a block attached to a spring scale across a horizontal surface at constant velocity. The question posed to the students was: "what is the force of friction in this situation?" In the second experiment, the students measured the force required to pull a block of a given weight along a surface at varying inclines and at a constant velocity. The essential question for this second experiment was: "how does the force of friction vary with the angle of incline?" In both cases, ordinary spring scales, ropes, and blocks of wood were the apparatus.

Results

The first experiment generated considerable discussion regarding the relevant variables: does the velocity at which the block is moving matter? What about the initial acceleration of the block? Does the force vary with the surface area of the block? How does the force change with the weight of the block? What exactly is being measured by the spring scale? The students identified that the block did not move at a constant velocity throughout the experiment, because the block began at rest and hence must accelerate to reach a given velocity. However, the students agreed to focus on the behavior of the block while it was moving at a constant velocity. By pulling the block across the surface at varying velocities, they observed that the force required to move the block was independent of the velocity of the block. The students increased the weight of the block and observed that now a greater force was necessary to move it at a constant velocity. The class then measured the force required to move various weights at a constant velocity. The students quickly came up with the relationship that the ratio of the measured force to the weight of the object was a constant; each group generated an appropriate symbolic expression for this relationship, such as $Ff/Fw = \mu$, where μ is the coefficient of friction, Ff is the force of friction (in this case, the same as the measured force) and Fw is the weight of the object. As the small groups reported their results, several students questioned whether or not the force that was measured on the spring scale was in fact the force of friction. One student persisted in questioning this and asked about all the forces that were acting on the object. This led to an argument from a force diagram which showed both the normal force and the weight of the object in balance and the measured force and the force of friction in balance.

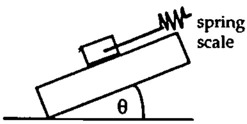

Figure 1. Experimental Setup

In the second experiment, the class measured the force required to pull a block of a given weight up an incline at varying angles with a constant velocity (see Figure 1). The essential question posed for this situation was: "how does the force of friction vary with the angle of incline?" Unlike on the horizontal surface, where the measured force equaled the force of friction, it was not immediately clear what was being measured by the spring scale. Hence, the problem for the students became an understanding of what exactly is the measured force and, from that, can they find a relationship between the force of friction and the angle of incline. Four students, Aria, Alycia, Paul and Sally, were members of one group which investigated this problem. Their first step was to enter the experimental data into a Function Probe table (see Figure 2). The first conjecture was posed by Paul, who hypothesized that the coefficient of friction depends on the weight, the kind of surface material, and the angle of incline. Paul created a third column in the data table for the measured force divided by the weight and then graphed that variable versus angle. This graph, Paul claimed, represented the coefficient of friction (mu) versus the angle. But Aria objected to this line of reasoning, arguing that the coefficient of friction is a property of the materials of the two objects and must remain constant.

Table

θ angle	m measured
0.00	9.00
11.00	11.40
22.00	13.70
33.00	14.10
44.00	15.20
54.00	15.90
65.00	16.10
75.00	16.00
85.00	14.90
90.00	14.10

Figure 2. Table of Experimental Data

P: No, wait. For a horizontal surface

AR: Hm, hm

P: mu is a factor of weight and surface. So on the inclined plane, it's a factor of weight, surface and angle.

...

AR: Well, no, it isn't. It isn't mu.

P: Well, but it is then.

AR: No, but it isn't. Because it's, it's, the mu stays the same. It has to stay the same. It's the same two objects. It's just the angle that's making the numbers different. It's not changing mu.

P: But mu could be a function of angle, the same as it could be a function of

AL: Because when it's at a different angle, it has a different weight.

Paul was using the measured force as the force of friction and arguing that the ratio of the measured force to the weight of the object must give the coefficient of friction as a function of the angle of incline. But there were two emerging challenges to his argument. First, Aria asserted that the coefficient of friction (or mu) must remain constant, since it is a material property. Second, Alycia suggested that the weight changes as the angle of incline changes. This comment was temporarily not responded to, but soon turned out crucial to their understanding of what is being measured by the spring scale. At this point, the teacher suggested that they focus on finding the force of friction. The students turned their attention to understanding more about the measured force, and Paul began to argue that the measured force cannot be equal to the force of friction:

P: OK. What I would say, is no, that's not mu. Because what we measured wasn't the force of friction.

AR: But it could have been

P: What we measured, no, we measured the force of friction and, part of the force of friction and part of the weight of the object. Cause if we like let go of it, let go at that point, it would fall down

AR: I see. Right. And when it's across the horizontal you're not taking any of the weight, you're just taking the friction. Sort of.

P: Right. Yeah. So.

AR: I see your point.

P: The surface is holding up the weight entirely. So what we need to do is one of those little dimensional things and figure out the component that's horizontal. (pause) Right?

AR: OK.

The new conjecture, qualitatively stated by Paul, is that the measured force is part of the force of friction and part of the weight of the object. Paul then asserted that the force of friction must be the horizontal component of the measured force. The other group members soon brought in alternative considerations. Sally argued from their data table back to the physical situation and asserted that because of the weight, the friction would be less when the block was more vertical: "I think that the friction would go down as the [angles in the] table went up. But I'm not totally sure that that's what it did." Aria confirmed this argument by examining the situation at 90 degrees where the measured force is equal to the weight of the object and there is zero friction. Alycia reasserted her earlier conjecture that the friction has to do with the force parallel (i.e. that component of the weight which is parallel to the inclined surface), not the horizontal component of the measured force as suggested by Paul. Up to this point, the students had been using the data table and the physical situation (particularly using

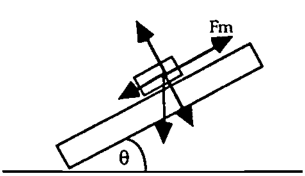

Figure 3. A Force Diagram

both the horizontal and the vertical case) to support their arguments. Their next step was to make their arguments from a force diagram (see Figure 3). Then they came to the tentative assertion that the force of friction changes as the angle of incline changes because the amount of the weight that is pushing down into the ramp changes with the angle of incline:

AR: The problem with this is, the force measured changes. So if the force measured is opposite the force friction, they have to be the same. But, but how can the friction change? And what if it's at 90?

AL: The friction has to change because like the more you're pushing down on an object, the more friction there is. And the more that you're, like the higher you get to 90, then the less it's pushing into the ramp

P: Right! Right! Um, this changes

AL: So of course it has to change.

AR: Friction has to change

P: The force perpendicular changes, which is the force weight acting on, between the two surfaces which means that friction changes

AL: How much is it pushing into the ramp.

AR: OK. Friction changes

Alycia has articulated the crucial qualitative relationship between the force of friction and the amount of the weight pushing into the ramp. Aria was now convinced that the force of friction does not equal the measured force and for the first time Paul stated a relationship between the magnitude of the forces:

AR: I think with this that the problem I have is that the force friction obviously changes as it goes up, the angle, but the force measured can't be the same as the force friction.

AL: Why not?

AR: Because imagine it at 90 degrees. You're just hanging there. The force measured is the weight of the object. But there's no friction. There can't be at 90 degrees. So, how do we get from the force measured to the force of friction?

...

P: The force of friction should be separate from the weight of the object

AR: Yeah, that's right.

P: From the weight, it should be, it should be in terms, the force of friction should be the weight, should be the amount you're pulling minus the amount of the weight of the object that you're pulling

So they have rejected the conjecture that the measured force equals the force of friction, and a new conjecture regarding the relationship of the three forces has been made by Paul. But, Alycia challenged Paul and Aria to make that relationship more explicit:

AL: Wait, do an example, (pause), do it with numbers.

But neither Paul nor Aria knew quite how to proceed:

P: Do you think you could do it?

AR: No

P: I don't know, I don't really know how to start

AR: I don't know where, I don't know where

But the group continued to follow this line of reasoning, with Aria making more explicit the relationship that Paul articulated above:

AR: Yeah, what the scale, what the force measured is, is it's the amount of weight that you're holding and it's the amount of friction. (pause)

AL: Amount of weight that you're holding

AR: Yeah.

AL: Well, amount of weight, because the ramp is pushing up on it, so you're not holding like the complete weight of it.

AR: Right! The point is that you're not holding the complete weight.

The group found the force of friction at an angle of 45 degrees: they calculated the portion of the weight that was parallel to the angle of incline (force parallel: $F_{||} = F_w * \sin 45$) and subtracted that result from the measured force at 44 degrees. Aria articulated the relationship for the force measured as the force of friction plus the force parallel, and they explicitly wrote $F_m = F_f + F_{||}$ and $F_m = F_f + \sin\theta\ F_w$. From this, they returned to the Function Probe environment, where they quickly computed the force of friction as the measured force minus the force weight times the sine of theta. They then graphed the force of friction versus the angle of incline and Aria suggested that the data looked like a cosine curve, which they quickly fit with the equation $f=9\cos\theta$. They identified 9 as the amount of friction at zero degrees. Alycia's suggestion that their graph looked like a line wasn't investigated. Aria was puzzled by the fact that although they had used the sine in this analysis, their final relationship used the cosine function. Now that they had the force of friction, Paul returned to his earlier conjecture about the coefficient of friction as a function of the angle and created a column for the force of friction divided by the weight. At this point, the teacher shifted the activity from the small group interactions to whole class discussion.

Each of the small groups reported its analysis to the whole class. All of the groups had generated the relationship given above for the force of friction, the force parallel and the measured force. One group conjectured that relationship for the force of friction versus the angle might be linear; they had begun to look at first differences but were unable to confirm a pattern. Another group realized that the constant 9 in the equation above was the weight of the object times the coefficient of friction, which they labeled as the "initial friction." The whole class discussion then brought closure to the initial question that was being investigated, as the class agreed that the force of friction on an inclined plane is given by the coefficient of friction times the weight of the object times the cosine of the angle of incline, or $F_f = \mu\ F_w \cos\theta$.

Discussion

This episode began with an inquiry into the role of friction for an object on an inclined plane. The students began by investigating the force required to pull a block at constant velocity across a horizontal surface and then they investigated this same event on the inclined plane. From this seemingly simple, concrete event, the students identified numerous factors that would need to be taken into account: the velocity of the block, the size of the surface area, the weight of the block, and an understanding of what exactly was being measured by the spring scale. When the event took place on the inclined plane, the group was faced with the considerable challenge of making sense of the measured force and its relationship to the force of friction. Their initial conjecture that the measured force equaled the force of friction was rejected as they began to argue that the measured force must include both the force of friction and part of the weight of the object. This led to a new conjecture about the relationship between the magnitude of the force of friction and the part of the weight of the object that exerted a force against the plane.

Several group members confirmed their claim for this relationship by arguing from their physical sense of what happened at the vertical and horizontal conditions. A geometric diagram and a set of calculations for a specific case then led to a symbolic representation of a relationship between the force of friction and the measured force. This representation was thoroughly grounded both in a geometric argument for the direction of the force vectors and in a qualitative understanding of the physical phenomena. This interplay between the grounded activity and the systematic inquiry then led to further investigation with the symbolic expression. The students moved the expression into the table window in the Function Probe environment and created a graph from which they generated an algebraic curve fit for their final relationship. They demonstrated both a qualitative and quantitative understanding that only part of the weight of the block pushed directly into the plane (and was therefore directly related to the magnitude of the force of friction), while part of the weight was acting parallel to the plane directly counteracting the measured force. The students brought together both the magnitude and the direction of the weight of the object in their qualitative argument for understanding the measured force. However, expressing that relationship quantitatively through the compounding of the force vectors generated multiple conjectures, the negotiation of new conjectures and confirming arguments from the boundary conditions and a particular case of the physical experiment. Through a non-linear process, showing the richness of the dialectic between the grounded activity and the systematic inquiry, the students created a compact model for the relationship of the force vectors acting on an object on an inclined plane that included geometric as well as graphical, tabular and symbolic representations.

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PREFERRED PROBLEM SOLVING STYLE AND ITS EFFECT ON PROBLEM SOLVING IN AN ADULT SMALL GROUP MATHEMATICAL PROBLEM SOLVING ENVIRONMENT

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Abstract

In this paper we discuss research aimed at identifying preferred problem solving style and its effect in mathematical investigations, through a study of a group of thirty seven Open University undergraduate mathematics students. They were undertaking small group mathematical problem-solving activities in the academic years 1993 to 1994 as part of their first year Foundation Course. The evidence suggests that gender linked preferences are directly relevant to mathematical problem-solving in small groups for the mature students involved in the research study.

1 Introduction

The rationale for different mathematical problem solving styles has been advanced by Pask (1976) who identified two distinct problem solving strategies that could be used in mathematical problem solving, namely holistic and sequential. Research by Parsons (1990, 1994) and Scott-Hodgetts (1986, 1987) suggested that gender was an important factor in determining the preferred problem solving strategy either holistic or sequential that might be adopted. The cause for concern here is how important gender would be in differentiating the perceived problem solving style adopted; and its assessed effect on the solution of a geometric mathematical investigation involving Pick's theorem. The powerful differential influence of social norms in relation to gender in British society are well established in mathematics education by researchers such as Burton (1992,1994), Hammersley and Wood (1993), and Isaacson (1992). Hence a plausible hypothesis would be a link between women and a preferred sequential problem solving style.

The research methodology used in this investigation of preferred problem-solving strategies was primarily ethnographic, but a wider variety of evidence including quantitative data was collected as well.

Quantitative evidence was also collected from a questionnaire, administered after the ethnographic interviews. In addition it was decided to analyze the students' work for evidence of either holistic or sequential working after both ethnographic interview and questionnaire had been completed and to relate this analysis to the students own perceptions of the problem solving style that they thought they were using. A problem solving investigation involving Pick's theorem was thus assessed for the quality of the mathematical process and mathematical product involved in the students solution. Problem solving protocols suitable for such mathematical investigations are noted by Mason (1984) and by the Mathematics Foundation Course Team of the Open University (1994). The thirty seven students in the main study reported here, were requested to work together in small cooperative groups of three or four on a range of numerical, geometric, abstract and applied mathematical investigations that had been selected on a continuum from closed to open in terms of the type of solutions suggested in a manner similar to that outlined by Parsons (1994). Initial conclusions from this data will be presented here and represent a small part of a much larger research study aimed at investigating the effect of gender variables in adult undergraduate mathematical problem solving investigation environments.

The data which will be presented here will be :-

(i) Reports by the students of their preferred learning style.

(ii) Examination of the learning style used in a problem deduced from marking the students work.

(iii) Actual assessment scores to indicate whether students were disadvantaged or not using that problem solving style.

2 Analysis of the Qualitative Data

The twelve women in the mathematical problem solving investigation groups all expressed a preference for a sequential problem solving style.

The twenty six men in the mathematical problem solving investigation groups however said they used a mixture of problem solving styles, both holistic and sequential. This suggests a gender bias with regard to likely mathematical problem solving strategy. The powerful nature of this preference for different problem solving styles is illustrated by the following extracts from interviews with the students.

The comments of the women problem solvers are particularly revealing, and four typical responses are given. The students I shall call Meg and Lina worked in groups which were informed about the nature of holistic and sequential problem solving styles. Meg worked in an all women problem solving group and Lina in a mixed gender problem solving group.

MEG

"I think I break it down and do it sort of step at a time. It seems easier. You can just sort of start on a lowish sort of level like building up bricks, you know like building bricks. You start with just a little bit and build on from there. We were all simply breaking the problem down into little bits."

LINA

" I started you know to build it up from the problem because you miss out something if you try and specialise and try to do the whole problem. It is better to take a specific example or category and see what is happening, you know. In fact that is why I missed out on that last problem because they were going for a general formula while I was trying to build it up in small pieces."

The students I shall call, Diana and Stella both worked in groups which were not informed about holistic and sequential problem solving styles.

Diana worked in an all women problem solving group and Stella a mixed gender problem solving group.

DIANA

"What we were doing was going in small little steps and I came to the conclusion that you did have to tackle them in that way. But I also while I'm doing it try to get an overall picture of where I'm going. This is quite often where I do have a problem in problem solving, trying to think ahead of myself, and yet I'm still working step by step systematically when I'm trying to get an overall picture. I am however overall using a step by step approach to get this overall picture."

STELLA

"I think I use trial and error. I think I was taking little bits and building up. I think I always do that when I problem solve now. I mean before I wouldn't have had any idea how to approach it so I can appreciate how people feel, because they look at it and think I can't do that. If somebody starts me off I can go, but I need that guidance."
The responses from the men, however, varied and two typical responses that reflect the responses from men in both informed and non-informed groups with respect to holistic and sequential problem solving style are given.
The student I shall call O'Hara came from an all men problem solving group, and the student I shall call Azhar from a mixed gender group that included Lina.

O'HARA

" I prefer to work in small stages towards a solution because it gives us a layout by starting systematically so that a pattern will emerge and you see the pattern very easily; you know you see the trend, I mean where you are heading to. You know it gives you clear stages of development that you can follow. Not everybody in our group worked in the same way."

AZHAR

"The method I used varied but was pretty much holistic. That was how the problems were meant to be done in the first place, because I could see most of what was involved. I tend to run ahead with my ideas presuming a lot of mathematical things, you know, which someone like Lina with a lesser mathematical background would need to have explained to her. As to our group it's been myself who did the directing. Lina she's more lacking in self confidence and so she doesn't join in, she has no ideas or mathematical knowledge, she tends to stick to just doing one thing. I think she is very negative influence in the group mathematically."

3 Analysis of the Quantitative Data

The quantitative data from the questionnaire developed during a pilot study was used to substantiate the inferences deduced from the ethnographic interviews. The questionnaire was administered immediately after the ethnographic interview and the results it yielded with respect to the students perceptions of their preferred problem solving style are given in table 1 below:-

TABLE 1: Preferred problem solving style

	Men	Women
Sequential	15	12
Neither Sequential or Holistic	1	0
Holistic	9	0
Total	25	12

These result appear to confirm Parsons (1994) and Scott-Hodgett's (1986,1987) claim that the men could use both holistic and sequential mathematical problem solving strategies, whereas the women exclusively would use a sequential problem solving style.

4 Objective analysis of the students use of problem solving strategy when solving Pick's theorem

The students actual work was assessed for the type of problem solving style used when solving Pick's theorem. It was graded for both mathematical process and product and a summative score obtained.

The purpose of the analysis of problem solving style used was to see how accurately the students perceived problem solving style matched the reality of the actual problem solving style observed. The results of this analysis is shown in table 2 below :-

TABLE 2: MEN

		OBSERVED PROBLEM SOLVING STYLE			
		HOLISTIC	BOTH	SEQUENTIAL	UNKNOWN
PERCEIVED	Holistic	7	0	1	1
PROBLEM	Neither	1	0	0	0
SOLVING	Sequential	7	4	4	0
STYLE	Total	15	4	5	1

TABLE 2: WOMEN

		OBSERVED PROBLEM SOLVING STYLE			
		HOLISTIC	BOTH	SEQUENTIAL	UNKNOWN
PERCEIVED	Holistic	0	0	0	0
PROBLEM	Neither	0	0	0	0
SOLVING	Sequential	0	3	9	0
STYLE	Total	0	3	9	0

5 Problem solving mean performance by gender grouping

An analysis of the actual problem solving scores in the different gender groups is given in table 3 :-

TABLE 3: MEN

Type of Gender Group	Mean Problem Solving Score	Number
All Men	37.0	15*
Mixed Gender	39.9	9
	Total	24

TABLE 3: WOMEN

Type of Gender Group	Mean Problem Solving Score	Number
All Women	43.9	9
Mixed Gender	43.3	3
	Total	12

* Note one student did not submit mathematical problem solving investigations work for assessment.

6 Discussion

The analysis of the interview data suggests that the adult undergraduate mathematical investigation problem solvers in the research project used specific gender orientated problem solving styles. The students' perceptions of their own problem solving was that the men could use either a sequential or holistic problem solving style whereas the women would prefer to always use a sequential problem solving style. These results were confirmed to a significant degree by an analytic analysis of the students scripts for the actual problem solving style used. Certain interesting points were noted to occur in this objective analysis. Firstly, more of the men were actually using a holistic problem solving style than thought they were.

Secondly, three of the women in one of the all women groups who were alerted to the possibility of using both problem solving styles in a worked example, did in fact use both holistic and sequential problem solving styles, but were more often using a sequential problem solving strategy.

At this stage a claim may be made that adult women are likely to prefer to adopt a sequential problem solving style when undertaking mathematical problem solving investigations. Men are more likely to use either holistic or sequential problem solving approaches and their style may depend on the type of problem selected from a finite set of possible problems available ranging from closed to open.

It is interesting to note that an assessment of the students mathematical problem solving ability when using these problem solving styles gave the higher average score for women working in all women groups (87.8%) the next highest average score to women working in mixed groups (86.6), then the next highest average score for men working in all men groups (79.8%) and finally the average score for men working in mixed groups was (74.6%). These scores suggest that the women were not disadvantaged when using a sequential problem solving style on the mathematical investigation involving Pick's theorem.

It does however indicate the powerful nature of gender variables in mathematics problem solving. Further interview data will be presented at the Nineteenth International Conference of the Psychology of Mathematics Education.

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PROPORTIONAL REASONING BY HONDURAN TOBACCO ROLLERS WITH LITTLE OR NO SCHOOLING

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There is a continuing interest in out-of-school mathematics, only a small amount of which has focused on proportional reasoning. This study, in the Nunes et al. tradition, investigates the proportional reasoning abilities of Honduran tobacco rollers. Ten workers with one to four years of schooling and at least 15 years of experience in tobacco rolling were interviewed. Questions were intended to explore their backgrounds, their ability to solve simple proportion problems, to solve problems involving reversing the way they usually solved proportion problems, and to solve proportion problems in new contexts. Instrumental versus relational understanding was also explored. Results here support the Nunes et al (1993) results.

Research on learning mathematics in informal, out-of-school settings is a somewhat recent phenomena, primarily undertaken by psychologists and anthropologists (e.g., Ginsburg, Posner, & Russell, 1981; Lave, Murtaugh, & de la Rocha, 1984; Nunes, Schliemann, & Carraher, 1993, Saxe, 1988). This research is also of interest to mathematics educators: Questions of the generalizability of mathematical learning to unfamiliar situations, of the role of context in learning mathematics, of the types of mathematics that can be learned informally verses the mathematics that is dependent on mathematical symbolism for learning, all can affect the organization of the curriculum and planning for instruction.

The development of the ability to reason proportionally is an area of particular interest. Nunes et al. (1993) point out that if those who have learned mathematics out of school can solve proportional problems and transfer their learning to new situations then perhaps everyday mathematics is not inherently different from school mathematics. They examined the proportional reasoning abilities of Brazilian construction foremen and of Brazilian fishermen with little formal schooling. They found, contrary to popular thinking that proportional reasoning needed to be learned in school, that both groups could reason proportionally, and that some fishermen were able to reverse their thinking in proportional situations, and were able to generalize to other situations.

In this study, we examined reversibility and generalizability of the proportional reasoning of a new group of people with little formal schooling. In addition, we wanted to examine the manner in which proportional reasoning develops, and the depth of understanding the people had of proportional situations, that is, whether people reasoned instrumentally or relationally (Skemp, 1978).

Method

During a pilot study, Honduran villagers who were small shop owners, construction and masonry workers, and tobacco rollers were interviewed. All of the workers lived in small villages where the first author had previously worked as a Peace Corps volunteer. One of the tobacco rollers appeared to have developed methods for mentally solving rather complex proportions. We decided to focus on tobacco rollers for this study.

Ten Honduran tobacco rollers, nine females and one male, were subjects for this study. They were located with the assistance of three local people known to the first author. All participants had between zero and four years of education. Only one had been to school in 25 years; she attended an brief adult literacy class and claimed not to have learned much. Seven of the ten earned a living exclusively from work in tobacco, and had worked full time at least half of the last year. The remaining three had done mostly hired housework or some coffee harvesting in addition to their tobacco work, but all three had worked with tobacco in the prior six months.

The participants were asked background questions regarding the amount of schooling they had, the types of jobs they had had, and the years they had worked with tobacco and the specific tasks with which they had experience (e.g., buying tobacco, rolling cigars, and selling cigars). They were then asked a series of questions organized into three categories: Beginning and intermediate level questions explored their understanding of proportional situations within the context of tobacco; advanced level questions also examined reversibility and transfer to new situations. Examples of beginning level questions are: How many pounds of tobacco are needed to make a tarea (250) of cigars? If you roll 35 cigars each hour, how many can you role in four hours? If you sell a certain cigar at 5 cigars for 2 Lempiras, how much would you charge for 20 cigars? Examples of intermediate questions are: How many wheels (50 cigars) is 1750 cigars? Which yields a higher profit per cigar, a wheel for 5 Lempiras, or 5 cigars for a Lempira? Examples of advanced questions testing the ability to reverse reasoning are: How much do you pay for an arroba (25 pounds) of tobacco? How much is that per pound? How many pounds is that per Lempira? [weight/price rather than price/weight] Examples of advanced questions that tested transfer are: On the north coast, in order to make pasteurized milk, the factories buy canisters of fresh milk and then use a filtering process to take out the thicker ingredients (cuajada] and some fat. They buy 20 canisters of milk in order to produce 800 liters to sell in the public. How many canisters would be needed to produce only 200 liters? How many liters would 15 canisters produce? Advanced questions were not asked if the interviewee had much difficulty with earlier questions.

All interviews were tape-recorded, transcribed, and translated into English. The transcripts were then examined for data related to each of the research questions.

Results

The Development of Proportional Reasoning as a Function of Experience.

The local residents who helped locate subjects for the study were asked to identify some tobacco rollers who were just learning the work, some who had been working a few years, and some who had been working for many years. However, we found that rolling cigars by hand is being replaced by mechanized methods in larger towns and cities, and no one is learning to roll by hand any longer. The least experienced tobacco roller had worked with tobacco for 15 years. It was therefore not possible to compare workers at different levels of experience. Nonetheless, another related and unexpected question did arise. Since all of the interviewees had at least 15 years of experience with tobacco, it might have been expected that most if not all of them would have developed relatively advanced strategies for solving proportions involving tobacco. Such was not the case. One interviewee could not think about and solve questions dealing with quantities she had worked with repeatedly, whereas others were able to develop methods, in addition to memorization, to successfully solve problems. For example, C, with 15 years of experience, was asked the following:

S. Suppose you have 1750 cigars. 1750. How many tareas (250 cigars) is that?
C. 4 tareas plus 750. 4 tareas, 7 tareas. Yes 7 tareas.
S. Now counting wheels of 50, how many wheels of 50 are there in 1000?
C. In 1000 there are 20. In the 3 tareas there are 15. 15 wheels.
S. So with 1750 cigars, how many wheels would that be?
C. 35 wheels.

Why could some people reason better than others with these questions? Two factors seemed to play a role: other work experiences and education level. Only three or four people interviewed had worked in areas other than tobacco that might have given them other opportunities to learn to reason proportionally. These people were more successful in answering the interview questions. Also, the three who had four years of schooling were also more successful, and made use of a wider variety of approaches and mental strategies than cigar rollers with almost no formal schooling. This result was surprising, considering that it had been so long since any had been to school. However, it may have been the case that those who exhibited better reasoning ability were kept in school longer.

Instrumental versus Relational Understanding.

Proportions with familiar and unfamiliar numbers within the tobacco context. Nine of the ten cigar rollers were able to solve very familiar proportions, although two relied heavily on memorized ratios. Each person interviewed had a set of numerical relationships that had been memorized. They used these relationships in solving slightly unfamiliar proportions. M. is an example of an individual who had some difficulty solving problems that involved unfamiliar quantities. She was asked how many wheels of 50 cigars are in 1750 cigars. She answered that 40 wheels is 2000, and that in 3000 there are 60 wheels, thus using memorized ratios and indicating that she knew it would be between 40 and 60 wheels, but she could not answer for 1750. She was then asked about 1500 cigars, and responded: "1500. In 1000 cigars, there are 20 wheels I said, right? So there are like 30 wheels."

As the problems became less familiar, five of them continued to be capable of applying familiar ratios to solve these problems. When asked how many groups of 20 cigars there are in 250 cigars, G. first stated that there are five groups of 20 in 100 cigars, then said: "Well, 5, 10 groups of 20. 10. 10 groups of 20. So it's 12 groups of 20 with 10 cigars more."

Finally, as the numerical relationships became the most foreign and complex (but were still discussed within the context of tobacco), only four of the people interviewed could correctly reason through to an exact solution. For example, R. was asked how much to charge for 20 cigars if 5 cigars cost 2 Lempiras. He said that cigars were never sold in groups of 5, but proceeded to say that it would be 8, by counting how many five's there were. He thus unitized the ratio 5:2 and used norming (Lamon, 1993) to comfortably solve the proportion. Another frequently used approach when working with unfamiliar numbers was to convert the given ratio to a unit ratio. For example, three people, when solving the proportion that involved 6 cigars for 2 Lempiras, used instead the ratio 3 cigars for 1 Lempira. Typically in the literature, proportions that cannot be solved by taking integral multiples of a ratio are viewed as more challenging than proportions that can (Hart, 1981; Nunes et al., 1993). Four of the ten interviewed were able to completely solve these types of proportions, while a fifth gave estimates using correct reasoning.

Reversibility. Most of the mental calculations that this group of cigar rollers did in their daily tobacco work required them to repeatedly take multiples of a give ratio. A typical example of this was when a person calculated how many tareas could be formed from 2000 cigars by repeatedly adding groups of 250 cigars, or when a worker figured out how much to charge for a tarea (250 cigars) if a wheel of 50 cigars sold for 6 Lempiras, by devising a method to add 6 Lempiras to itself the appropriate number of times. It was much less common for a tobacco roller to need to take a fraction of (both components of) a ratio, starting with a given ratio and finding a given portion of that larger ratio. Thus, for this group of workers, testing for reversibility largely consisted of simply seeing whether they could take a larger ratio and simplify it down to a smaller ratio. An important special case of this was exploring whether they were able to find a unit ratio, given a non-unit ratio. To more completely test for reversibility, workers needed to be tested on their ability to transform a non-unit ratio into a unit ratio using more complicated "multiplication factors" than simply half the given ratio.

Six tobacco rollers were asked to calculate the price of one pound of tobacco, given the price for an entire arroba (25 pounds) of tobacco. Because they were normally accustomed to calculating the value of a larger amount of tobacco, given the price of a smaller weight of tobacco, this problem required them to reverse their usual calculation procedures. When the six who were asked this question given a price of 50 Lempiras per arroba, four could answer 2 Lempiras. These four were then asked the price per pound based on L35.00 per arroba. If they were able to solve this, it was by estimating, based on the fact that L25.00 per arroba meant one Lempira per pound. D, for example, said that it was about 1.25 Lempiras per pound, and when asked how she obtained this, she said: "You know that 25 Lempiras per arroba, it comes out at one Lempira per pound. So adding on the other 25 centavos, it comes out at 1.25 or 1.50 Lempiras, around there it comes out. (So it's not exactly 1.25 Lempiras?) No. Since it's 10 Lempiras more than the 25 Lempiras." Three who answered this question correctly were then asked how many pounds of tobacco they would be given for one Lempira, and were able to solve this successfully. The worker who answered these questions and also questions in other contexts correctly was also able to solve an out-of-context problem involving reversibility.

<u>Solving proportions outside of the context of tobacco</u>. Nine of the ten people interviewed were asked proportional reasoning questions that were not in the context of tobacco or cigars. Of the nine, six were able to discuss and correctly answer at least basic out-of-context problems, all of which involved coffee harvesting and milk production. Of the six, five were able to answer problems that could be solved by taking integral multiples of a unit ratio. For example, D was asked how many gallons of coffee she would need to pick in order to earn 50 Lempiras, if she earned 5 Lempiras for every 2 gallons. She used a unit ratio of L2.50 per gallon, then calculated L3.00 x 20 and adjusted by subtracting L0.50 x 20.

D. . . . It comes out at 2, uh, 20 gallons, what you'd have to cut in order to earn the 50 Lempiras.

S. And what numbers did you make use of to get 20? how did you calculate it?

D. You see, I multiply. L2.50 each gallon. So, at L2.50, 50 Lempiras, take away the 10 Lempiras, you end up with the 20 gallons.

Because of the non-integer unit ratios involved and the strange context, interviewees might have been expected to perform poorly compared with earlier problems where these two factors were not present, but this was not the case. The tobacco workers performed approximately as well outside the domain of tobacco as they did within the context of tobacco. For example, Y was asked how much she should be paid for picking 4 1/2 gallons of coffee beans if she earns 8 Lempiras for every 3 gallons she picks. She used decomposition, unitizing, and norming to answer:

Y. He pays 8 for 3. It goes up by one, 1 1/2.

S. Yes, 1 1/2. From 3 to 4 1/2. Now you pick 4 1/2. How much does he have to pay?

Y. 12.

S. He pays 12. How did you get that just not? What numbers did you work with?

Y. You split up the 8 for 3. Then with 1 1/2 more.

Two others answered this question in a similar fashion. In a different problem, G. could correctly tell how many liters of milk could be produced from 15 milk canisters if 20 canisters yield 800 liters of processed milk, but she was unable to explain her reasoning..

Another new-context problem involved using foreign units of weight and money. Three workers who had answered previous questions successfully were asked about a tobacco worker in Portugal who pays 120 escudos for 20 kilos; how much per kilo. Only one could even begin to solve this problem; the others said they were not familiar with kilograms.

Discussion

This study adds support to the results of the Nunes et al. (1993) studies of the use of proportional reasoning among workers who have had little schooling but have been afforded opportunities by their work for learning to reason proportionally. In particular, it supported their findings that many workers could reason correctly when called upon to solve problems requiring them to reverse their usual way of solving proportion problems, and to transfer their understanding of proportion to unfamiliar situations. We found that the types of strategies used to solve proportion problems, whether or not they involved reversibility, were similar. In the Nunes et al. (1993) study, two strategies were found to account for 94% of the responses of Brazilian construction foremen who were asked to solve proportional problems involving scales. The first, used 34% of the time, they called "hypothesis testing." It involved hypothesizing what scale was used in a given blueprint and then calculating the unknown value in the proportion to verify whether the expected result was correct. In this study of tobacco workers, this approach was also frequently used in order to avoid reversing or inverting familiar procedures. D.'s estimate of the price per pound given 35 Lempiras per arruba, described earlier, is an example of this strategy. The second strategy, used about 60% of the time and called "finding the relation" involved taking a given ratio and expressing it as a unit ratio, then applying this unit ratio of other ratios (unitizing the unit ratio, in the words of Lamon, 1993), by repeatedly adding the unit ratio or by multiplication. Because this approach could be used to solve unfamiliar proportions, it could be considered relational understanding. This approach was also quite common in the study of tobacco workers. Although less than 60% of the tobacco workers showed relational understanding (either through their use of finding a relation or using another method), this may have been due to the difference in education level between the tobacco workers and the foremen.

In the Nunes et al. study (1993) of the proportional reasoning ability of Brazilian fishermen, there was no significant drop in accuracy or change in methodology used when problems of the same type were presented in the context of agriculture rather than in the familiar context of fishing. Almost 80% of the proportions related to agriculture were solved by taking integral multiples of a ratio; a figure almost identical to results when problems of the same type were asked in the context of fishing. The percentages of correct responses in the Nunes et al. study were, again, somewhat higher than corresponding results in the Honduran study. Only 47% of the tobacco workers' responses to proportion problems in unfamiliar contexts were correct. More unfamiliar context problems in this study included proportions that could not be solved by taking the integral multiple of a unit ratio, but even when these problems were excluded, the percentages of correct answers is not significantly altered. On the one hand, these results confirm the findings of the Nunes et al. study that there was not a great deal of difference between tobacco workers' accuracy in solving tobacco problems as compared to their accuracy in solving problems in other contexts. On the other hand, the percent of problems answered correctly (in any context) by the Honduran tobacco workers was consistently lower than corresponding percentages for the Brazilian fishermen. These differences in results between the two studies cannot be explained by difference in work experience. Both the Honduran tobacco workers and the Brazilian fishermen had relatively few opportunities outside of their respective fields to develop proportional reasoning. Again, one possible explanation for the difference in results might be the education level of the people interviewed. This explanation deserves further study.

Lesh, Post, and Behr (1988) described five levels of conceptualization students display as they progress from pre-proportional reasoning to true proportional reasoning. At the lowest level of reasoning, additive strategies are incorrectly used instead of correct multiplicative strategies. We found some instances of additive reasoning: When M. was asked how much to charge for a tarea (250 cigars) if 50 cigars sell for 6 Lempiras, she used the familiar fact that since the current market price was 5 Lempiras for 50 cigars, which yielded 25 Lempiras for 250 cigars, then a price of 6 Lempiras for 50 cigars must yield a price of 26 Lempira for 250 cigars. The second, third and fourth stages of proportional reasoning are marked by increasing awareness of the multiplicative nature of a proportional relationship. Seven of the ten tobacco workers showed varying degrees of understanding that their work with tobacco involved multiplicative situations. Several used unitizing (forming composite units) and norming (interpreting a given situation in terms of a composite unit), processes identified by Lamon (1993) as critical to the ability to reason proportionally. Only the fifth stage identified by Lesh et al., that of writing a mathematical sentence similar to "A is to B as C is to D" was not observed here.

Our results suggest that although the types of mathematical experiences an individual has had do have an influence on what mathematics is learned and what methods a person develops to solve problems in mathematical situations, some individuals clearly develop more complex and powerful methods than others. It appears that the degree of exposure to a mathematical situation is not the only influence on a person's ability to solve unfamiliar problems and generalize to new contexts. Clearly identifying other factors that affect a person's ability to generalize would clearly contribute to our understanding of how mathematics is learned, and how it should be presented in the mathematics curriculum.

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CAN YOUNG CHILDREN LEARN HOW TO REASON PROPORTIONALLY? AN INTERVENTION STUDY

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This study examines the possibility that young children can learn how to make proportional judgements by using the 'half' strategy in terms of part-part relationships.One hundred and eigthy children (6 to 8 years old) were given a pre- and a post-test (Bruner & Kenney, 1966). Children of each age were equally divided into two control groups and a training group. Children in the training group were taught to use the 'half' strategy in part-part terms to solve a proportional task (Spinillo & Bryant, 1991). No significant differences were found among the three groups in the pre-test. However, in the post-test children who received the training gave significantly more correct responses and proportional justifications than those in the control groups. Only these children did significantly better in the post-test than in the pre-test. The conclusion was that children of these ages can be taught how to make proportional judgements. The use of 'half' in such part-part relations may be an important boundary that helps children in making proportional judgements. This is in fact a new way of looking at proportional reasoning in children.

The concept of proportion is considered a late acquisition (e.g., Piaget & Inhelder, 1975; Inhelder & Piaget, 1958). However, there are evidences that children as young as 6 years can: (a) make proportional judgements by using the 'half' boundary (e.g., Spinillo, 1990; Spinillo & Bryant, 1991); and (b) be taught about proportion (e.g, Siegler & Vago, 1978; Muller, 1979; Brink & Streefland, 1979).

Spinillo (1990) and Spinillo & Bryant (1991), for instance, found that 6-year-olds can make proportional judgements. In these experiments children used the half boundary ('more than half', 'less than half' and 'equal to half') to plot out the first-order relations which were established in part-part rather than in part-whole terms. Children's use of this boundary in part-part terms could be the basis for the initial understanding of proportion.

Singer & Resnick (1992) explored the representations children use in reasoning about ratios. The main question in this study was whether children who did not fully mastered proportional reasoning were part-part or part-whole reasoners when solving a proportional task. The conlusion was that children's representations are generally based on the parts. This is an important information about the way children initially deal with proportion, and this is in accordance with Spinillo & Bryant's (1991) findings.

Other studies (e.g., Muller, 1979; Spinillo & Bryant, 1991) have also showed that children make some proportional discriminations on the basis of relative codes such 'greater than', 'smaller than' and 'equal to'. This may suggest that they use perceptual abilities to make such discriminations and that the initial grasp of proportion should be sought in the realms of perceptual understanding. A similar patern of result was found by Lovett & Singer (1991) in a experiment about the perceptual and quantitative conceptions children have about probability. It was established that perceptual strategies are preferred when either perceptual and quantitative strategies could be adopted.

One can ask whether children could be taught to reason proportionally if they were given concentrated experience with part-part comparisons that cross the 'half' boundary or which explicitly involve 'half'. This idea was tested in an intervention study. We used non-numerical tasks in order to explore proportional judgements without emphasizing oomplex computational skills.

METHOD

Subjects

One hundred and eighty children aged 6, 7 and 8 years, attending elementary school.

Experimental Design and Procedure

Group 1 (Control Group - CG1) - 1st Session: Pre-test - Bruner & Kenney's (1966) fullness task. 2nd Session: they were asked to solve a proportional task (Spinillo & Bryant, 1991). They received no explanation or feedback on the correctness of their responses. 3rd Session: Pos-test - Bruner & Kenney's task.

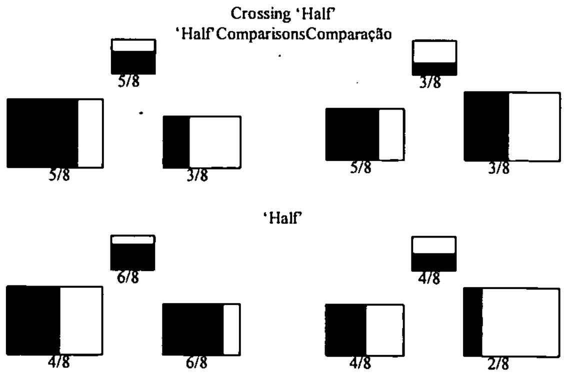
Group 2 (Control Group - CG2) - 1st Session: Pre-test - Bruner & Kenney's task. 2nd Session: Pos-test - Bruner & Kenney's task.

Group 3 (Training Group - TG) - 1st Session: Pre-test - Bruner & Kenney's task. 2nd Session: Training procedure - children were taught to use the 'half' strategy in part-part terms in a proportional task (Spinillo & Bryant) different to that given as a pre- and a post-test. 3rd Session: Pos-test - Bruner & Kenney's task.

The task used in the pre- and in the post-test (Bruner & Kenney, 1966) consisted of comparisons between two of glasses filled with different levels of water. Children were asked which container was fuller than the other or whether they were equally full and to justify their responses.

In the task presented to Group 1 and to Group 3 (Spinillo & Bryant, 1991) children had to judge which of the two large rectangles was represented in a small picture. The rectangles were choices and the picture was the standard (Figure 1).

Figure 1: Examples of the material and trials in the task (Spinillo & Bryant, 1991). Crossing 'Half' Comparisons (5/8 vs 3/8) and 'Half'Comparisons (4/8 vs 6/8 and 4/8 vs 2/8)



In the training group (Group 3) the experimenter explicitly taught children to use the 'half' strategy in part-part terms. They were provided with feedback and explanations about how to use the 'half' boundary to decide about the equivalence or non-equivalence between two choices and a standard. Examples:

(1) Correct choice selected:

Trial crossing the half boundary (Crossing 'Half' Comparisons):
5/8 black (standard) 3/8 black vs 5/8 black (choices)

E - "Yes, that's right. You have chose the rectangle with more black than white as in the picture (standard). It cannot be the other one because the picture shows more black than white and this (3/8 choice) has more white than black. There is more than half black in the picture and less than half white."

Trial explicitly involving half ('Half' Comparisons):
2/8 black (standard) 2/8 black vs 4/8 black (choices)

E - "Yes, that's right. This one (2/8 choice) matches with the picture because in both there is less than half black and more than half white. It could not be the other one (4/8 choice) because it shows half black and half white. Black and white are equal inside the rectangle. In the picture they are different: there is more white than black."

(2) Incorrect choice selected:

Trial explicitly involving half: ('Half' Comparisons)
6/8 black (standard) 4/8 black vs 6/8 black (choices)

E - "No. It is the other one. You know why? Because in the picture there is more black than white and in this one that you choose there is half black and half white. This (6/8 choice) is the correct one because it matches with the picture. In both there is more than half in black and less than half in white."

Trial crossing the half boundary (Crossing 'Half Comparisons):
5/8 black (standard) 3/8 black vs 5/8 black (choices)

E - "No, they do not match. Well, it could not have been this one you choose (3/8 choice) because there is more white than black in it, while in the picture there is more black than white. In the picture there is more than half black and in this one (3/8 choice) there is more than half white. Look at this one here (5/8 choice): it has more than half painted in black and less than half in white. Now, look at the picture (5/8 choice): it matches with this (5/8 choice) because in both we have more than half black and less than half white. It looks different because the everything in the picture is smaller than in the large rectangle. Got it?"

RESULTS

The responses in the pre- and in the post-test were coded for the number of correct responses and for the types of justification given in each trial.

Correct responses

Table 1 shows the general scores in this experiment. It can be noted that the training procedure had a strong effect on children's performance.

Table 1: Percentage and mean number of correct responses

AGE	CG1				CG2				TG			
	PRE		POST		PRE		POST		PRE		POST	
	M	%	M	%	M	%	M	%	M	%	M	%
6	9.7	54	9.6	53	10.8	60	11	61	9.2	51	13.7	76
7	10.8	60	12	67	8.7	48	9.2	51	9.1	51	16.1	89
8	9.7	54	9.5	53	10.4	58	11.6	64	10.7	59	15.2	84

The Age term had no effect on children's performance in the pre-test (p=.2633, Kruskall-Wallis - One-way ANOVA) and in the post-test (p=.1513).

In the pre-test, the number of correct responses did not differ significantly between groups (Mann-Whitney: CG1 vs CG2, p = .7109; CG2 vs TG, p=.4134;and CG1 vs TG, p=.6002), whereas there were marked differences between groups in the post-test. This was due to Group 3 (TG) children being more successful than those in the other two groups (p <.0001). No significant differences were found between the two control groups (p=.8059).

The performance in the pre- and post-test was compared by means of Wilcoxon. This revealed that there were no significant differences between pre- and post-test for the children in Group 1 (p=.7998) and in Group 2 (p=.3454). In contrast, Group 3 children performed significantly better in the post-test than in the pre-test (p <.001). This was particularly so at the ages of 7 and 8 years old

Thus, prior to training the performance among ages and groups was much the same. This suggests that children had a similar initial understanding of proportion in the fullness task. However, after training, Group 3 (TG) children gave significantly more correct responses than children in the control groups. This means that they were able to benefit from instruction and that the training procedure improved children's performance.

Justifications

These were classified into three types. The 6.480 justifications were analysed by two independent judges. The reliability of coding assessment between them was 90.46%.

Justification I - no justification or irrelevant justification. Examples:"I don't know."; "Because I knew it."; "I gessed."

Justification II - non-proportional justifications. Children's judgements were based on the quantity of water in the container rather than on the proportion. Examples:

"This is fuller because it has more water than the other."; "The water here is high and in the other it is low."; "Both has water in the top, but this one is fuller because the glass is bigger."

Justification III - proportional justifications. The children compare the empty space to the space occupied with water (part-part comparisons). In some of these justifications children explicitly mentioned the 'half' boundary. This was more often produced after training. Examples:"This is fuller because it has more water than empty space. In the other there is more empty space than water."; and "The water is different. There is more water here than in here. But they are equally full because they have half air and half water."

Table 2 shows the incidence of types of justification in each age separately. Children after training produced more proportional justifications than those in the control groups.

Table 2: Incidence of each type of justification.

6 YEARS

JUST	CG1		CG 2		TG	
	PRE	POST	PRE	POST	PRE	POST
I	2	1	10	18	2	0
II	305	306	305	300	322	163
III	53	53	45	42	36	197

7 YEARS

JUST	CG 1		CG 2		TG	
	PRE	POST	PRE	POST	PRE	POST
I	4	5	4	5	7	1
II	278	240	329	315	312	128
III	78	115	27	40	41	231

8 YEARS

JUST	CG 1		CG 2		TG	
	PRE	POST	PRE	POST	PRE	POST
I	11	4	3	2	7	6
II	314	310	308	264	271	120
III	35	46	49	94	82	234

Children in the two control groups produced few proportional justifications in the pre- and post-test. However, after training, Group 3 (TG) children offered more proportional justifications than in the pre-test. Kolmogorov-Smirnov analyses revealed that the frequency of the justification types differs significantly between groups in the post-test (TG vs CG1: $p<.01$; e TG vs CC2: $p<.01$). This was because Group 3 (TG) children gave more proportional justifications than the children in the control groups.

DISCUSSION AND CONCLUSIONS

The main conclusion derived from this experiment was that children as young as 6 years can be taught to make proportional judgements. The use of 'half' in part-part terms seems to be an important step in helping young children to reason proportionally.

The training procedure improved performance in both ways: number of correct responses and number of proportional justifications. It is interesting to note that the proportional justifications were established in part-part terms by comparing the empty space to the space occupied with water. This occurred because in the task used for training children was shown how to compare the black part of the rectangles with the white one. Hence, children tranfered what they learned in a situation to another. We may say that the training procedure led children to represent the proportional relations in part-part terms. This result is in agreement with those reported by Singer & Resnick (1992) that 'children's representations are generally based on the parts'(p: 244).

It is important to note that children had arrived at the task with some understanding of proportions (see Spinillo & Bryant's study) and that the training procedure had drawn their attention to establish the first-order relations in part-part terms by using the 'half' boundary in a task (fullness) that in the original study invited to part-whole forms of solution. Prior training, in Bruner & Kenney's fullness task children paid attention to the whole (volume of the container) rather than to their parts (water and empty space). This occurred because the question about the fullness of the glasses invited children to treat the problem in part-whole terms. This probably made the task difficult for them. However, after training, children were able to adopt part-part relationship to solve the task. This helped children to reason proportionally by comparing the water with the empty space, as it can be seen in the proportional justifications children used in this study. As stressed by Spinillo (1990) and more recently by Singer & Resnick (1992), it is probable that many proportional problems may easily be solved by means of part-part relations.

Of course one must be cautious about generalizations: other tasks which involve coordination between separate non-complementary dimensions (e.g., balance scale, projection of shadows, speed problems, best buy problems) are not open to solution by part-

part relations. Thus, this type of training is effective in tasks in which the two parts that form a whole can be diretly compared. This study leads to a new way of looking at proportional reasoning and this might provide some educational implications for the teaching of proportion in elementary school.

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A FIFTH GRADER'S UNDERSTANDING OF FRACTIONS AND PROPORTIONS

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This paper reports an analysis of a child's action in a study that investigated relationships between understandings of fractions and proportional reasoning ability among fifth grade children. Katie's understanding of fractions influenced her ability to deal with proportional tasks. However, she was also capable of solving problems that involved proportional relationships without using fractions. This finding raises questions concerning the way current school mathematics curricula are organized.

Introduction

Children's understandings of multiplicative concepts such as multiplication and division operations, fractions, ratios, and proportional reasoning have long been the focus of mathematics education research. In recent years, more and more studies have investigated children's informal understandings of these concepts. The findings from the extant research indicate that children's rich mathematical experiences have not fully been taken advantage of as we teach these concepts in schools.

In an earlier study, four second grade children's understandings of simple fractions such as one-half was investigated with a specific focus on children's unit concepts. (Watanabe 1991, in press). According to the study, young children's understanding of simple fractions is influenced by their unit related concepts. More specifically, children's schemes for coordinating units appeared to have an important influence on their understandings of fractions. In addition, it was also reported that children's meanings of fractions were closely tied to contexts in which fractional quantities were needed. As a result, some participants held contradictory meanings of fractions in different settings without experiencing cognitive conflict.

In a separate study, two fifth grade children's problem solving strategies were investigated as they engaged in both routine and non-routine proportion problems (Lo and Watanabe, 1993 a & b). It was found that these children, and many other children who participated in the preliminary stage of the research, were able to solve proportion problems using a variety of strategies. However, these children's problem solving strategies were influenced by, among other things, their understanding of fractions.

In order to further investigate relationships between children's understanding of fractions and their proportional reasoning ability, research was conducted involving fifth grade children. In this paper, an analysis of the problem solving activities of one of the participants in this investigation, Katie, will be presented.

The Setting

The research involved 16 fifth grade children, 7 boys and 9 girls, attending a suburban public elementary school in a Mid-Atlantic state in the United States of America. Based on their performances on the first interviews, 6 children, 2 boys and 4 girls, were asked to participate in two additional interviews. Katie was one of the girls who were asked to participate in the follow-up phase.

Tasks

The analysis of four tasks posed during the first interview will be presented. In the Cuisenaire rod task, each participant was given several Cuisenaire rods: 15 white (1 cm), 8 red (2 cm), 5 light green (3 cm), 6 purple (4 cm), and 2 dark green (6 cm). After the children were asked to compare some rods, they were given three pictures, one at a time. The children's task was to find how many light green rods would be needed if they were to cover the whole picture only using light green rods. Five light green rods provided were not sufficient for covering each picture, and the children were instructed not to re-use any rods. The second part of the task involved the following question: (shown a picture similar to the three they had worked on) Suppose this picture needed 27 red rods to cover. Can you find how many light green rods would be needed, without covering the picture?

The half-shaded task was similar to common textbook exercises. The children were shown 16 different partially shaded designs and asked to select those that were half-shaded. However, many of the figures involved unusual partitioning patterns (see Figure 2 for examples). The children were then asked to justify their selections.

The one-half cookie problem involved three shapes that were obtained by partitioning three congruent squares in three different ways, shown in Figure 1. The interviewer demonstrated that two copies of each shape could be arranged to cover the identical square. They were then given one copy of each shape. The question was then posed: If these three were your favorite kind of cookies, which one would you pick if you were very hungry but you were allowed to have only one. If the participant responded by selecting one of the three shapes, s/he was then reminded of the initial demonstration. The interviewer then asked the children if that information helped them decide which one of the three pieces were the largest.

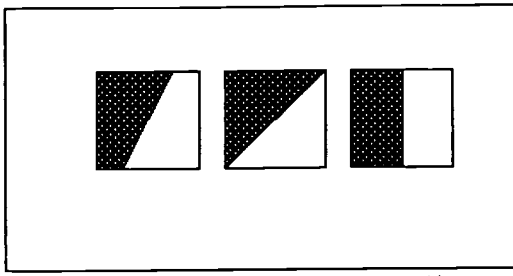

Figure 1 Three shapes used in the one-half cookie problem.

The comparison of shares task was adapted from Lamon (1993). Each question involved one group of boys and one group of girls getting different numbers of pizzas. The participants were to select persons in which group get more pizzas when they were shared equally within the groups. The combinations of numbers used were: (1) 3 boys with 1 pizza vs. 7 girls with 3 pizzas, (2) 5 boys with 2 pizzas vs. 9 girls with 4 pizzas, (3) 5 boys with 3 pizzas vs. 7 girls with 4 pizzas, and (4) 16 boys with 5 pizzas vs. 8 girls with 3 pizzas.

Case Study of Katie

Observations

On the Cuisenaire rod task, Katie was able to use the many-as-many coordination (Watanabe, 1991). Throughout this task, she used the relationships 3 purple rods = 4 light green rods and 3 red rods = 2 light green rods. This type of coordination was the most sophisticated scheme identified among the participants of the earlier study.

When Katie was asked the second question, "Can you find out how many light green rods would be needed if a picture needed 27 red rods to cover?", she immediately responded that it could be figured out. When she was asked to proceed, she began by creating 9 groups of 3 rods. The three rods in each group were not necessarily of the same color. Then the following exchange took place:

Katie: Nine. And one left over. I did 27 into 3 is 9. [Katie started with 28 rods by mistake.]

Int: With no leftover, right? 3 times 9 is 27 [removes the extra]. So, how many light green rods?

Katie: Nine.

Int: Why?
Katie: Because ... No, wait, no not nine. It will be ... about 4.
Int: Why?
Katie: Because if you divide this in half [draw an imaginary horizontal line to separate 9 groups into two], because each 3 is 2 green, that will be four and one (group) leftover. 5 if we had 10.
Int: I'm not sure why you are dividing.
Katie: Because, umm, 3 red blocks are 2 green blocks. So, this (one group) is like 2 green blocks. So, there were nine groups, and they will have to be split in half for 2 [a little perplexed look].
Int: So, [take 3 red rods] three of these are equal to...
Katie: 2 of these (light green).
Int: So, each of these (groups) is worth 2 green ones. How many is that altogether?
Katie: 18?
Int: Is the answer 4 or 18?
Katie: 18.
Int: Why did you change your mind?
Katie: Because, if 27 red blocks cover the whole thing, then there will probably have to be a little less green blocks because they are, red blocks you need more of them. They are small. Green blocks, they are bigger and take up more space. So, you are not going to need as much.

Katie's initial response was given with confidence. Her immediate action to create 9 groups of 3 blocks shows that she had a clear goal in mind. Furthermore, the fact that she did not use 3 blocks of the same color in each group indicates that she was operating with numbers as mathematical objects for her. However, once she had constructed the 9 groups, she appeared to lose sight of her goal; she was unsure of what to do with this relationship. Even though she was able to say, "three red blocks are two green blocks," she did not use the relationship successfully. It is not clear why she decided to divide 9 by 2 once she realized that the answer of 9 did not make sense. However, it is clear that, by the end of above exchange, she had constructed more sophisticated meaning for the task, including the inverse relationship between the sizes of rods being used to cover the picture and the numbers of rods that would be needed.

On the half-shaded task, she initially picked all half-shaded figures except three, see Figure 2. While explaining why the pictures she picked were half-shaded, she often used the justification, "if you fold the picture along the line they (the parts) will match." This metaphor (Lakoff & Johnson, 1980) did not work with the three half-shaded figures she did not pick. However as she began explaining why figure (a) was not half-shaded, she switched her metaphor. Consider the following exchange:

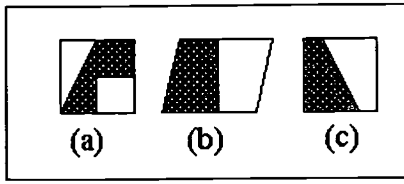

Figure 2 Three figures that were not selected by Katie during the half-shaded task.

Katie:	This (shaded triangle) would have to be here [pointing to the unshaded square], but it has to be a square. And, it's different.
Int:	If the picture is just this [cover the side that are partitioned by a diagonal], is it half colored?
Katie:	Yeah.
Int:	What about if the picture is just this [cover the other side]?
Katie:	Yeah.
Int:	Why?
Katie:	Because it looks like it splits right in the middle. *If you cut it and turn that over, it will fit*. [emphasis added]
Int:	What about the whole picture?
Katie:	Yes.

When Katie switched her metaphor from "folding" to "cutting" (italics above), she was able to determine the figure was half-shaded. It appears that the "folding" metaphor, although it is dynamic, has a limited range of motions. The "cutting" metaphor, on the other hand, gave her more freedom to move her image. As a result, she changed her answers and decided both figures (b) & (c) were also half-shaded.

Katie's response to the half-cookie question was surprising, considering her rather sophisticated responses to the earlier tasks. When she was asked which shape she would pick, she initially chose the trapezoidal shape. However, when she was asked to justify her choice, she decided that the triangular figure was the largest. When she was reminded of the initial demonstration that two copies of each shape would form the congruent squares, she concluded that it had no implication for her decision of which of the three shapes was the largest. Even though she was successful in identifying half-shaded figures correctly, and even though she was able to use sophisticated reasoning which reflected her understanding of proportional relationships, she was not able to reason that all these shapes were one-half of the congruent square, thus, equal in size. It is conjectured that her understanding of fractions were highly contextualized, and she did not construct fraction context for this task. Therefore, she used more naive responses as her primary rationales.

Furthermore, it is conjectured that a part of the reason she did not consider this task to be fraction related is that her understanding of fractions was based on a part-to-part comparison. Even after she shifted her metaphor from the "folding" metaphor to the "cutting" metaphor, her intention was still to compare the two *parts*. If the two parts are equal in size, then one of them represented a half of the whole. However, for Katie the cookie task did not involve comparison of two parts. Since her meaning of one-half was based on part-to-part comparison rather than part-to-whole comparison, this task, which did not involve complementary parts, was not the setting for fraction related reasoning for Katie.

Katie's responses to the comparison tasks also showed interesting contrast. For example, on the first task, she used the relationship 3 persons to a pizza as the basic pattern of sharing. As a result, the group with 7 girls and 3 pizzas will be left with one whole pizza for one person. She was able to reason from this that the girls would get more.

For Problem 2 (5 boys with 2 pizzas & 9 girls with 4 pizzas), she decided that the girls would get more. When she was asked to justify, she again assigned 3 girls to a pizza, leaving one whole pizza. She then said the girls would get one third of a pizza plus one of 9 slices from the last pizza. The following exchange then took place:

Int:	How do you know they (girls) get more?
Katie:	Well, you could, umm... They (boys) will get 2/5 of a pizza, and they (girls) will get ... [changed her mind, and decided the boys get more] Because they (girls) will get 1/27. Because of 1/3 of a pizza and a get 1/9. It's not 1/27. It is 1/12, is less than 2/5.
Int:	How did you know they (boys) get 2/5?
Katie:	Because it's 2 pizzas, and each person got 1/5 from this pizza, and 1/5 from this pizza.
Int:	What about these (girls)?
Katie:	Well, if there were 9 girls, and that can be divided evenly into 3, then they will each have ... If you divide one pizza into 3, then you have 1/3 of a pizza, ...
Int:	What if there was only 1 pizza? How much would each get?
Katie:	(immediately) 1/9.
Int:	What if there were 2?
Katie:	2/9.
Int:	So, if there were 4?
Katie:	4/9.
Int:	Not 1/12?

Katie: Hmm.... They will get 1/12 of a pizza. Because if there are 9 girls and 4 pizzas, each will get 1/3 of a pizza, and 1/9. And that will be, hmm... 2/12.

What is significant in this exchange is that her reasoning became less sensible when she began using fraction terms. She was able to determine the boys' shares to be 2/5 of a pizza. However, when the same reasoning led her to the girls shares of 4/9, she still decided that 2/12 was the correct share. When Katie encountered a previously unfamiliar problem situation, she resorted to an algorithmic (faulty) approach of adding two fractions.

Moreover, Lamon (1994) showed that, as children develop more sophisticated mathematics, their partitioning strategies will utilize larger "units" of sharing. Katie's strategy of 1/3 plus 1/9 is certainly efficient. However, it appears that, to find the total share, this approach created a mathematical problem which was beyond the constructions she had made at this point. It is curious that she rejected the answer of 4/9 as the shares for the girls even though she used the equivalent reasoning for determining the boys' shares. It is conjectured that 1/9 was not a unit for Katie, thus, she was unable to operate with 4/9 constructed by joining four 1/9's.

Discussion

There were several significant moments in Katie's first interview. The importance of the "cutting" metaphor in the half-shaded task showed the importance of partitioning activities in fraction experiences. If children are to develop such reasoning, they must have experiences with partitioning themselves. The insufficiencies of pre-partitioned figures that dominate typical textbooks/workbooks have been pointed out by other researchers, and the current study provide additional evidence.

Katie's response to the half-cookie problem shows how complex an understanding of fractions is. Katie was, by no means, an exceptional case. Of the 16 participants, 7 other students also responded similarly. Moreover, all of these children were able to identify correctly almost all half-shaded figures. Further studies are needed to investigate this inconsistency.

Finally, Katie's ability to reason proportionally, and her inability to do so when she began using fraction language, raises an important question concerning school mathematics curricula. The typical sequence of topics in school mathematics begins with whole numbers. It then moves on to fractions and decimal numbers. Ratios are then introduced, while proportions are usually discussed quickly and with the main focus on the procedure (cross multiplication) of solving proportions. The observations of Katie's problem solving activities reported in this paper suggest that children are capable of reasoning proportionally even before they receive any formal instruction on ratios or proportions. In addition, it appears that some children may be able to reason proportionally even without fraction language. In fact, it may be possible for children to construct fractions as ratios based on their problem solving experiences. Most of school mathematics curricula focus on the part-whole interpretation of fractions. A few experimental programs utilize the operator construct of fractions. However, few, if any, curricula begin with proportional reasoning. The observations reported in this paper suggest that problem solving experience involving proportional relationships may become a basis for constructing fraction knowledge. Further research of this approach may be fruitful.

Acknowledgement

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PARTICIPATORY, INQUIRY PEDAGOGY, COMMUNICATIVE COMPETENCE AND MATHEMATICAL KNOWLEDGE IN A MULTILINGUAL CLASSROOM: A VIGNETTE.

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An inquiring participative pedagogic code and the demands it makes on pupils' communicative competence can, at times, inadvertedly turn in on itself, validating diverse pupil perspectives at the expense of developing their mathematical knowledge. In practice, such effects can be obscured from teachers. In this paper, one incident with one teacher, and her reflections on her teaching, are woven into an analytic narrative vignette (Erickson, 1986) that instantiates and illuminates the above claim.

INTRODUCTION

An inquiring participative pedagogy is often driven by democratising intent, with twin goals of moving away from authoritarian approaches to teaching, learning and knowledge, and improving socially distributed access and success rates. That such pedagogy can turn in on itself, reducing mathematical knowledge development, highlights what in South Africa is well know as the 'democracy-development' tension[1]. This tension permeates key policy documents (ANC, 1994; NEPI 1993) that now frame the process of reconstruction and development in South Africa.

Policy research has had centre stage in educational research activity in South Africa since 1990. What is interesting and pertinent to this paper is that often, the impetus for education policy directives is informed by two poles: abstract research on the one hand, and the common sense of relevant stakeholders on the other. This is not to deride either, but to signal that the research which I will describe here, brings a third and important dimension. Through an in-depth analysis of the actions and reflections of one teacher, and a good teacher at that, we gain insight into the complexities of teaching mathematics in multilingual classrooms in ways that embrace democratic ideals. Insight, that is, into the very real and concrete challenge in education of working the democracy-development tension.

While this paper is clearly situated in and motivated by the South African change process, its problematic is more widely shared. In many and diverse contexts, mathematics teachers face multilingual classes as well as the challenge of developing approaches to learning, teaching and knowledge appropriate to rapidly changing global and social processes.

SOME RESEARCH BACKGROUND.

The vignette is part of a field research study that seeks a critical understanding of the complexities of teaching mathematics in multilingual classrooms through teachers' knowledge of their practice. It is broadly underpinned by a social theory of mind, and draws on the theoretical work of Vygotsky (1986, 1978), Lave (1988) and Lave and Wenger (1991), and Bernstein (1993). It investigates both what teachers in multilingual contexts say about their work and what they do. The design is qualitative and interpretive and involves a strategic opportunity sample (Rose, 1981) of six teachers, two each from three different kinds of multilingual contexts in South Africa. Data collection has been through initial individual semi-structured interviews, video-recording and observation of each teacher for at least two hours, individual unstructured video-reflection interviews and three follow-on group workshops. Analysis has entailed working both across what the six teachers say and do, and then also within each teacher[2].

Much has been written about the complexities of analysing, validating and reporting qualitative field research (Erickson, 1985; Rose, 1982; Hitchcock and Hughes, 1989; Maxwell, 1992, Woods, 1985). Erickson recommends a 'leap to narration' as a way of stimulating analysis with the 'analytic narrative vignette' being the foundation of an effective report of such research.

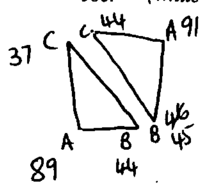
The vignette presented here is focussed on a key aspect of the working knowledge of one teacher in this study - Sue[3] - who wants her pupils to see mathematics as 'something you can talk about' and 'have your own ideas about' and 'not something you just do'. Mathematics is not simply about getting answers it is also about 'asking questions'. Her pedagogical approach to the development of mathematics knowledge is a participative, interactive and inquiring one. Her classroom is both multilingual and communicatively rich.

A VIGNETTE: REFLECTIONS AND AN INCIDENT

The incident[4] - From stretching to labelling angles.

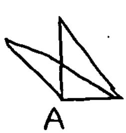
Joe, Std 6, is 'reporting' to the class, his explanation for a worksheet question 'Is it possible to draw a triangle with two obtuse angles?'

Joe: *(While talking, he draws the following two triangles on the board)*



I said all the A's must be like more than .. they must, uh, be the biggest in the triangle, um, so that if, uh, if this A here, say, is like 89, .. and then these are say 37 and *(mumbling to himself, ya, ya)* 44, ya. And then in this one, number two, .. it will be an obtuse angle. I said 91 and this is 44, .. and this here is 46, no *(crosses it out and puts 45 - all 'labels' are outside the triangle).* And I said like if A, if A is going to stretch, .. if A is going to stretch *(pointing to 91)* then these two angles here... if it has to stretch then these two, like these two they are going to contract.

(He draws another 90 degree angle below, and re-explains:)



If this here, if this is A, if A is here now miss and if it has to stretch, like these two we gonna have to (()) them both ... if this is 90, and you if you, if you, if it is gonna (()), turn to be lets say 110 or something, .. *(drawing the obtuse angle)* then this one here *(pointing to top angle)* will be smaller than it was before, it was before, so, so if it was, say, 40 here then it is going to be 30 here, uh .. then A is going to be taking that 90 degrees, uh, that 10 degrees, let's say B had ... uh uh if if if one angle stretches then, uh, the the two angles, the two other angles have to contract.

Sue: OK what do other people think? Any questions? Rose?

Rose: Isn't that triangle the same as the other one if you measure (())

Joe: I was just doing an example, I forgot what angles I was using in my book [] but [] they are supposed to add up somewhere near to 180 degrees [].

After some teacher-mediated interaction between Rose and Joe during which Rose is able to clarify that her question is whether the one triangle is the 'same as the other turned upside down', Sue says:

Sue: I think Joe maybe the first problem is that you haven't shown these angles on the picture and lots of people do this - they write the angles outside the picture. OK Now you know what you mean and I know what you mean and maybe some people know what you mean. But to be clear (she writes the angle sizes inside the triangle), do that. Put it inside [] Now, are these two triangles the same just turned upside down?

As she continues interacting with the class to ensure they understand that while the triangles 'look the same', they are not. So, Joe in not 'wrong'. The bell rings but she continues:

> [] it does not really matter what they really measure - we still get what he is trying to tell us because he has shown us and example of what he has done [] we will come back to this tomorrow.

At the outset, ie while observing the lesson in process, this episode caught my interest. Firstly, I was impressed by Joe's dynamic, relational conception of the angles of a triangle - how angles change in relation to each other. Yet he struggled to explain himself clearly, to find the words and illustrations to express his ideas publicly. Rose's question and Joe's response suggested that they did not understand each other and Sue's mediation focuses on clarifying Rose's question, and then on how to label angle size clearly, on estimated angle values in the diagram, and away from the actual mathematical content of the task - away from 'stretching' angles to labelling them.

In my field notes that day I noted this as an instance of problematic communicative competence - of a difficulty with mathematical English - so as to ensure I discussed it in the reflective interview with Sue.

After watching the video myself prior to the interview (space constraints preclude providing adequate evidence), I was interested that while Joe battled to explain himself to the class, earlier he had managed to convey his reasoning - albeit with lots of particularist language and pointing (in a restricted code in Bernstein's terms) - to both Sue and his partner. Sue is not entirely happy with his explanation, whether 'it covers all possibilities' and suggests he tries to 'start with an obtuse angle like 125 degrees'. In the recap in the lesson the next day, Joe's partner volunteers and summarises his reasoning quite clearly to the class. Sue's question about starting with 125 degrees does not resurface.

From my perspective as researcher, this incident promised to provide insight into (1) differential communicative competence within and across learners and (2) how teacher actions are shaped by problematic communication i.e into a nest of problems pertinent to the research project as a whole.

Sue's opening point in her reflective interview is:
(She, like me, had looked at her video before this interview)

> ... the thing that worries me the most is that I am not sure whether, I am not sure to what extent it helps them learn. **I think that talking to each other is not unproblemmatic.** I think a lot of the kids don't listen. Maybe they are too young. I think. **You can see it with the questions** [] they'll ask a question and say I dont

> understand and then the one who is up will try to explain and it doesnt really help but they are being polite and they are not quite sure and they say 'OK fine'. I am not sure they understand. (VI6, 7-28, my emphasis)

As we observe the video she elaborates:

> With the Std 5's, because their language is much weaker, and they work in partners, I encourage them to first try it and then they talk about it - **but I do question the way they talk to each other ... to what extent they really challenge each other** [] If you are not sure of your own ideas then it is hard to challenge and **if they are not great at explaining, they don't understand each other.** I encourage them to do it, but often they don't.
> (VI6, 234-259, my emphasis)

She particularises her concerns later in the interview when we view the incident presented below:

> He doesnt really answer her question. **They are not communicating - and that happens a lot!** He can't hear her question and she can't hear his explanation. (VI6, 641-644, my emphasis)

and comments derisively on her actions:

> **now I am deflecting more** (VI6, 651-652)

Sue's opening general comment pertains to the incident mirrors my observations and concerns about pupil communication and her 'deflecting', but it is I, the researcher, who brings it into focus in the unstructured reflective interview. The study is concerned with teachers' knowledge. So, it is pertinent that Sue and I are not similarly interested by this incident, though we share concerns that it illuminates. The incident thus could stimulate discussion through differences between teacher and research interests and orientations.

THICKENING THE DESCRIPTION:
Contextualising the incident and reflections

(Supporting data will be in the full version of this paper).

Sue's school is well-resourced. The vast majority of pupils are black and not native English-speakers. Most teachers (including Sue) are white and English-speaking. A culture of professionalism and inquiry permeates school and staffroom. Sue's notions of teaching and learning are thus supported in her school.

The incident occurs in a 37 minute lesson. For 23 minutes, the 16 pupils work in pairs on part of a worksheet designed to elaborate the concept of the angles of a triangle. One task is: **Draw a triangle with 2 obtuse angles. If this is impossible, explain why.** Joe's 'stretching angles' occurs in the 11 minutes devoted to 'explaining to others' what they had done.

While pupils are working on the worksheet, Sue gets round to the whole class. Her interaction, is predominantly in the form of questions that encourage learners to articulate their thinking, present their reasoning and to question each other. In her individual interactions she also pushes Joe and Rose on their responses: She suggests Joe tries starting 'with a triangle that has an obtuse angle', say 125 degrees and asks Rose if her explanation will hold 'in all cases'.

In Lave and Wenger's (1991) terms, there is opportunity for both **talking about and within** the mathematical practices in this class. A **code of enquiry** is evident in Sue's actions and in pupil utterances. In addition she fosters **pupil-pupil interaction**. Pupils interact with each other while on task, and then during report back. Pupils have learnt that they are expected to ask themselves why, to explain and ask why of others and to interact verbally with each other. In contrast to many mathematics classrooms where I-R-F interactions and teacher-initiated questions predominate and, in Campbell's (1986) terms, pupils 'go for an answer', Sue's lessons are better described by pupils 'going for a question'.

Diverse pupil orientations to the task emerge. Some, including Rose, start with an obtuse angle and then try to draw a second obtuse angle, and so forming a quadrilateral, not a triangle. Others have the more deductive explanation: if two angles are more than 90 degrees, there will be more than 180 degrees and so a triangle is not possible. However, there is **differential competence** across pupils in exressing their thinking in written and verbal form. Sue knows this, and says she sometimes avoids asking pupils who can't explain clearly since they confuse others - and that there is obviously a hidden message here for such pupils.

During report back time, Rose also reports her explanation. As indicated above, it is different from Joe's. Again space limitations here preclude evidencing that she presents the product, the whole quadrilateral at the start, not the process of her thinking. Joe asks her a question and as with his report, they struggle to communicate. No other report is given.

In the interview, Sue talks about her interactions with Joe. She feels his explanation was not as general as the others - that is why she pushes him to try starting with 125 degrees. So Sue has a notion of what a more generalised and hence more mathematically powerful response is. In the class, it is implicit in her individual interactions with Joe and Rose. But is does not surface publicly for the whole class during report back.

DISCUSSION

In the complexity of teaching mathematics in a multilingual classroom with a participatory, inquiry code and the demands it makes on communicative competence, we see that:

(1) Pupils sometimes struggle to formally explain their thinking.
(2) Pupils are often unable to communicate and engage each other effectively - their questions are often confusing and restricted to points of clarification.
(3) These communicative difficulties shape Sue's actions to the detriment of pupils' mathematical knowledge - deepening notions of generalisation are not made publicly available.

Sue is aware of communication difficulties - and on reflection sees her 'deflection' to labelling. To what extent can she be aware of their effects?

The discussion of these issues draws across disciplines from the work of Pimm (1992, 1994) on reporting mathematical investigations, from Vygotsky (1978, 1986) and particulalry Wertsch's (1984, 1991) elaboration of the Zone of Proximal Development (ZPD) and briefly from Bernstein pedagocic codes (1993).

An obvious explanation for Joe difficulties explaining his thinking, is that he is not a mother tongue English-speaker. His task is thus one of double attention - to a new mathematical idea and to a language he is still learning. Sue is aware of this but 'doesnt know how to move him on - I dont know how to develop the language'.

However, Pimm's (1992, 1994) work suggests that this the issue is not simply about access to English. Reporting mathematical thinking, even for mother-tongue English-speakers, is not a simple process because of the linguistic demands entailed. 'Skills of reflection and selection' and a 'sense of audience' are important to successful report back. While Joe displays a sense of audience by trying to recount the process of his thinking, his loose selection of angles is confusing. Rose has less sense of audience in that she does not convey the process of her thinking.

Sue is, to some extent, aware of the issues of selection and audience. In the interview she talks about needing to assist pupils structure their explanations, and her 'deflection' is about how to draw better so that others can understand you. Joe's reporting skills are important but not the key issue here.

Pupils in this class struggle to hear and engage each other. Sue's insights from her reflections on her practice are very illuminating. What she understands is how hard it is for them to step out of their own ideas and frames to engage others' mathematical thinking. The question begging is: Can pupils have the vantage point that one expects of the teacher i.e. a vantage point from which to interpret and engage a range of ideas different from your own, and so deepen their mathematical knowledge?

As hinted at earlier, Wertsch's elaboration of the Vygotsky's ZPD explains Sue's insights. Wertsch distinguishes three components to functioning in the ZPD: situation definition; intersubjectivity and semiotic mediation. It is when situation definition is not shared that mediation is required for intersubjectivity to be esablished. Joe and Rose do not share the same situation definition. They start differently. Their orientations to the task, their objects of attention are different and they struggle to see past their own to engage with each other. That is perhaps why we cannot fathom where their questions are coming from and why Sue's insights are spot on.

There is so much potential for deepening pupils' mathematical knowledge in this situation, precisely because of Joe and Rose's diverse approaches. But it requires Sue to mediate these differences publicly, to bring ot attention these different orientations and starting points and their relative mathematical strengths. For here, discussion is not about who is right or wrong - both approaches make sense and answer the question - it is about how they are similar and different and perhaps too which is the better mathematically.

The issue, however, is that Sue would like them to engage each other. She wants them to ask each other more effective questions, perhaps like those she asks Joe and Rose as she interacts with them individually while they are on task. Perhaps the assumption and hope is that deepening mathematical knowledge will happen through this pupil-pupil engagement.

The irony here is that Sue's desire for pupils to engage is simultaneously part of and undermined by participatory pedagogy - and this she does not see! She deflects to teach labelling and does not refocus back onto the mathematical substance of the task. An interesting question to pose is whether this deflection is a choice and then not refocussing a choice? It is arguable that her participatory project is overdetermining and precludes her from focussing on the content, from mediating across differences and from evaluating the substance and content of what the pupils offered. All she can do is validate and clarify what they offer. Mathematical possibilities are lost.

In this we can see that pupils' difficulties engaging each other are perhaps more than metacognitive on the one hand, and their ability to express their mathematical thinking on the other. Bernstein's (1993) analysis of pedagogic codes suggests that when pupils do not perform in ways expected, it is less their inherent ability and more their not 'realising' what it is they are meant to do. If Sue herself does not engage (mediate and evaluate) publicly with the substance of their ideas, her desire that her pupils do so may continue to be thwarted.

CONCLUSION

A culture of meaningful inquiry, pupil-pupil interaction and multiple perspectives on mathematics is encouraged and achieved in this class both while they are working on a mathematical task and when they publicly report on their work. This is no mean accomplishment in mathematics. The practice includes both talking within and about mathematics. However, it is precisely when perspectives are not shared that public/ whole-class pupil-pupil communication takes on form (it looks interactive and engaging) rather than substance (questions are restrictive and confusing). The strong claim is that, at times, the pedagogical code over-determines the mathematics. The unintended consequence of this is to impede possibilities for more demanding intellectual engagement with the mathematical task at hand and hence the development of mathematical knowledge.

The implication for teaching is that while the **withdrawal** of the teacher as continual intermediary and reference point for pupils enables this classroom culture, her **mediation** is essential to improving the substance of communication about mathematics. That is, both are required, and managing the tension, finding the balance the challenge!

This analysis troubles me: in South Africa this is the year of 'delivery' with a new government under pressure to right the wrongs of apartheid. We might well see the balance tip in favour of development. This would be a tragedy, not only because of the gains Sue has made in her classroom, but because democracy and development are inter-related - the one is dependent on the other. It is in Sue's classroom where participation and diversity are encouraged and enabled, that possibilities for exciting mathematics lie. We need to continue to work towards this happening.

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NOTES

1. Democracy implies participation and is linked to equity and redress; development implies growth and improvement.
2. For fuller description of the research, and the initial interviews, see Adler (1995)
3. All names (teacher and pupils) have been changed.
4. Key to transcript symbols:
   (bracketed italics: researcher commentary within an extract from the data
   (()) - inaudible utternace
   [] - utterances edited out
   .. - short pause
   ... - longer pause

PEER INTERACTION AND THE DEVELOPMENT OF MATHEMATICAL KNOWLEDGE

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This paper reports on a qualitative study of a group of junior secondary pupils collaborating on a mathematical task in their mathematics classroom. The pupils' interaction is both beneficial and problematic in relation to their development of mathematical knowledge. This can be understood using Vygotsky's construction of the Zone of Proximal Development as a relationship between scientific and spontaneous concepts and between teacher and learner. It is argued that teaching needs to be conceptualised as integral to learning for peer interaction to be most beneficial.

1. MOTIVATION FOR THE STUDY

The study is located within a context of potential educational change in South Africa. Its purpose is to investigate ways in which small-group work in mathematics can contribute to classroom practices which are "learner centred, non-authoritarian, and which encourage the active participation of students in the learning process" (ANC, 1994: 69).

Initial research influences on the study include mathematics education research which argues that interaction in small groups can increase possibilities for conceptual growth (Wood and Yackel, 1990; Hoyles, 1985), as well as research which has shown that teachers' control over knowledge and discourse in classrooms can be detrimental to learning (Edwards and Mercer, 1987; Barnes, 1969). These studies suggest that peer groups, which provide more equality and allow pupils more control, may be preferable to teacher-directed learning.

On the other hand, teachers working with small groups are continually confronted with tensions between allowing pupils control over the learning situation, and the development of mathematical content knowledge. This is not only a pragmatic tension. Vygotsky (1986) distinguishes the development of systematic knowledge from spontaneous knowledge, and argues that formal instruction is necessary for the former. This paper argues that a conception of teaching as making connections between spontaneous and scientific concepts can help resolve this tension.

2. METHODOLOGY

The group under investigation consists of three Std 7 (14-15 year old) pupils, two girls and a boy. The group was chosen because they were observed to interact well together. They were part of a class of 17 pupils, divided into four groups.

The pupils were given a worksheet which required them to investigate certain properties of area and perimeter on a geoboard. The work lasted for a week (6 fourty-minute lessons). Each group in the class was was audio-taped for the whole week. Transcripts were made of the tapes of the chosen group. The transcripts constituted the major source of data for the study and were supplemented by classroom observation, the pupils' rough notes made while they were working on the tasks, and their final, individual answers to the worksheet questions handed to the teacher at the end of the week for assessment.

An in-depth, interpretive analysis of the transcripts was undertaken. Initially, I attempted to categorise regularities in the pupils' talk and interaction, but this proved unproductive, since I needed to capture the development of the pupils' mathematical knowledge in relation to their interaction during the week. So I focused the analysis on an unexpected method which the group developed to calculate area. I charted the progress of the method for the duration of the week, documenting how it was produced and maintained through the interaction, and the consequences for the development of the pupils' mathematical knowledge.[1]

3. A METHOD TO CALCULATE AREA

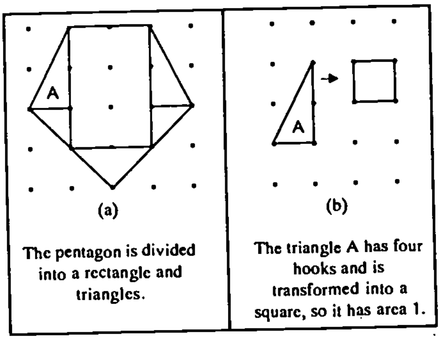
Most of the group's work centred on an innovative, yet problematic method to calculate area. The method, in its most general form can be described as follows (see figure 1):

For a complicated shape, ie: not a rectangle

1. Divide the shape up into squares or rectangles, and triangles.
2. Calculate the areas of the rectangles.
3. Count the hooks in the triangles, including those on the perimeter and inside the triangle, and make a rectangle with the same number of hooks, also on the perimeter and inside the rectangle.

[1] The analysis draws on various parts of the transcript. Space limitations prevent their inclusion here. The complete transcript can be found in Brodie, 1994.

4. Calculate the area of the rectangle formed in this way. It will have the same area as the original triangle.
5. Add up the areas found, to get the area of the original shape.



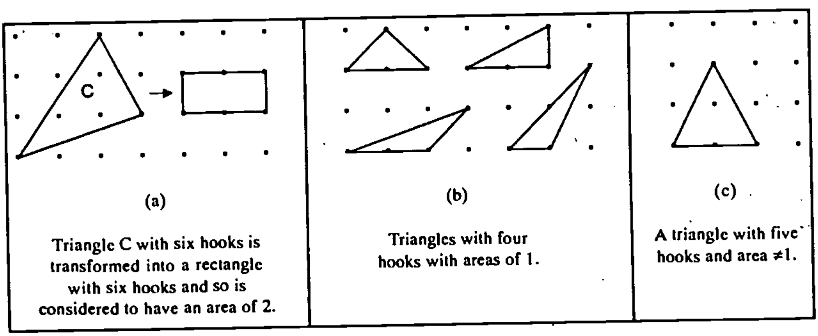
(a)

The pentagon is divided into a rectangle and triangles.

(b)

The triangle A has four hooks and is transformed into a square, so it has area 1.

Figure 1



(a)

Triangle C with six hooks is transformed into a rectangle with six hooks and so is considered to have an area of 2.

(b)

Triangles with four hooks with areas of 1.

(c)

A triangle with five hooks and area ≠1.

Figure 2

Step 4 of the method (as I have articulated it above) is problematic, as it is based on a mistaken notion of conservation of area (figure 2a). For most of the time the group works with a restricted version of this idea, that triangles containing four hooks can be transformed into a square and thus have an area of 1 (figure 1b). This version is found to work in almost all cases (figure 2b). The limitation that the hooks inside the shape are included is important. The triangle in figure 2c does not have an area of 1 because it has 5 hooks, not 4.

In developing the method the pupils engage in mathematical activity including: forming hypotheses; checking the method against particular shapes; modifying and adapting it in the face of new evidence; defending it against challenges; and using it to help them begin more difficult tasks. The pupils also experience emotions as they develop their method - joy and excitement when it is successful; appreciation of their own work; and frustration and disappointment when they experience setbacks (see Brodie, 1994 for a more detailed discussion).

On the other hand, the mathematics that the pupils develop is seriously flawed, and, more seriously, their mistaken notion of conservation of area is not dealt with in any substantial way throughout the week, neither by the group, nor by the teacher. The group's inability to deal with their problems on their own arises from inequality in their interaction and their inability to negotiate intersubjective meanings. This is discussed elsewhere (Brodie, 1994). The teacher's interaction with the group is discussed here.

4. TEACHING AS "FACILITATING LEARNING"

The pupils spoke to the teacher predominantly about triangles with four hooks which can be transformed into squares and which have an area of 1 (figure 2b). She was intrigued as to the apparent validity of their idea in most of these cases, because she knew that it could not be generally true. She tried to engage the pupils in thinking about the general validity of their theory, by providing counter-examples and asking for justification of their ideas. She did not tell them directly that part of their method was wrong, or suggest ways out of the difficulty.

Her interaction can be described as trying to facilitate thinking in the pupils, in the hope that they might develop their ideas further, or change them, rather than giving them direct guidance. This approach is consistent with the view which asserts that direct teaching is more likely to inhibit learning than foster it, and that the teacher should attempt to suppress her mathematisations in order to allow the pupils to develop theirs.

In the case of this group however, this teaching strategy allowed the pupils to continue to work with their flawed method. The worksheet tasks and the pupils' interaction were not sufficient on their own to enable the pupils to deal with their mistake and to move on to systematic notion of area. The transcripts show a number of instances where the teacher could have provided more direct guidance for the pupils in order to foster such mathematical development. A conceptualisation of teaching using Vygotsky's (1978) Zone of Proximal Development (ZPD) will expand on this.

5. THE ZONE OF PROXIMAL DEVELOPMENT

The ZPD expresses a relationship between a learner and a more knowledgeable other. Vygotsky (1978) defines it as:

> "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers." (1978:86)

Vygotsky posited the ZPD as a challenge to the Piagetian perspective that real learning can only occur after the prerequisite development has taken place (1978). Rather, he considers instruction, which is a teaching-learning relationship, to be essential for development. New developmental levels are aimed for and attained with the teacher's assistance.

Central to the notion of the ZPD as it functions in schooling is Vygotsky's distinction between spontaneous and scientific (or systematised) concepts (Vygotsky, 1986). Spontaneous concepts are developed in, and derive their meaning from everyday activity and interaction. Scientific concepts develop through formal instruction and form part of a knowledge system. Their meaning derives from being part of this system as well as from instantiation in everyday concepts. Systematised concepts permit flexibility of thinking and generalisation of meaning and provide control and structure for unsystematised everyday concepts.

The relationship between spontaneous and scientific concepts is to be found in the ZPD. Vygotsky writes:

> "These two conceptual systems, developing "from above" (scientific) and "from below" (spontaneous), reveal their real nature in the interrelations between actual development and the zone of proximal development. Spontaneous concepts that confront a deficit of control find this control in the zone of proximal development, in the co-operation of the child with adults."
>
> (1986:194, quotes in original, my brackets)

The synthesis of spontaneous and scientific concepts forms true concepts (Vygotsky, 1986), which are flexible, and meaningful in relation to individual experience and generalised systems of knowledge.

It is clear that mathematical knowledge forms a system of scientific concepts[2]. However, children develop many spontaneous mathematical concepts in everyday life. Research into everyday concepts (Carraher et al, 1985) has shown that these concepts remain unseen, unheard, and even blocked by school mathematics, and that a synthesis into true concepts rarely occurs successfully in schools.

Small-group discussion is one way of bringing spontaneous concepts into the classroom. However, the evidence from my study shows that this is not sufficient. Spontaneous concepts need to be explicitly and carefully brought into connection with scientific concepts.

6. MAKING CONNECTIONS

The spontaneous area concepts that the pupils bring to the geoboard tasks can be seen in their method. They divide shapes into smaller areas, they perceive a relationship between hooks and area, and they think that a shape can be transformed while conserving area, as long as the number of hooks remains the same.

The mathematical processes that the pupils engage in (section 3) can also be labelled spontaneous, as they have not been explicitly and formally taught. The pupils use the processes of generalising and specialising, forming hypotheses, and drawing conclusions from evidence. However, they use them imperfectly. On a number of occasions they ignore contradictory evidence, and this allows the perpetuation of the faulty method. For example, a serious challenge is posed to the method, when the group cannot find a triangle with 7 hooks and an area of 2½

[2] Exactly what constitutes the scientific concepts of mathematics is contested. A detailed discussion of the debates is beyond the scope of this paper. My assumption, consistent with a Vygotskian perspective, is that mathematical knowledge is culturally and historically constructed and situated.

units[2] as predicted by the method[3]. The pupils do not make use of this contradiction, they are wedded to the connection between the hooks and the area and this overrides the contradictory evidence..

Different teacher intervention strategies may have critically affected the development of the method and the pupils' conceptions of area. The value of contradictory evidence, the relationships between general and more specific theories, the need for justification of their ideas, and the need and role for a method, could all have been usefully discussed with this group. In this way, by bringing the spontaneous reasoning that the pupils did engage in into contact with more systematic mathematical reasoning, the teacher could have led their development. More direct teacher challenges to the pupils' linkage of hooks and area, and their mistaken notion of conservation of area, could have enabled the pupils to make progress with the area concept, by giving them access to scientific concepts and forming a basis for the development of true concepts.

This suggests that the teaching-learning relationship requires special consideration. In the case of this group, a particular approach to teaching, that of facilitating learning with a reluctance to intervene directly, did not enable the pupils to make progress. This suggests that a shift needs to be made from viewing the teacher as only a constraining influence in learner-centred activities. The teacher can and should be a powerful enabling influence, mediating and leading conceptual development in smallgroup work. Teaching can and should be viewed as integral to learning, rather than as a disruption of an otherwise spontaneous process. This requires a reconceptualisation of teaching as both listening and talking to pupils, to bring their spontaneous concepts into the classroom, and to give them access to scientific concepts, so that they can construct true concepts.

[3] The method was extended to include triangles which could not be made into a rectangle (most triangles with an odd number of hooks), by allowing a half triangle at the end of the transformed rectangle.

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INTERACTIVE COLLABORATION AND AFFECTIVE PROCESSES IN THE CONSTRUCTION OF MATHEMATICAL UNDERSTANDING

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This paper reports on how affective processes play a key role in the development of mathematical thinking. The study was part of a larger research project on the effects of a social constructivist teaching environment using a cyclical learning model based on 'experiencing', 'discussing', 'generalising', and 'applying' with undergraduate teacher education students. Students were encouraged to collaboratively construct understandings through small-group and whole-class discussion. All sessions were student-led. Classes were videotaped and students' participation analysed. Comparisons of pre- and post-course scores on attitude and belief questionnaires indicated an increase in positive attitudes and beliefs. Interactive collaboration also precipitated heuristics which supported positive changes in both affective and cognitive states.

Introduction

There is considerable research which establishes various forms of peer learning as an effective method of learning (Gooding and Stacey, 1993). Research has also highlighted how affective states can have a positive or negative effect on learning (McLeod, 1993, 1989). Positive emotions can accompany construction of new ideas while negative emotions associated with a problem blockage may result in students being upset or making wild conjectures (Wagner, Rachlin, & Hensen, 1984, cited in D. B. McLeod, 1993). The learning environment within which the curriculum is established is instrumental in assisting students to handle their emotions and develop positive attitudes. Indeed, it is claimed that a class that shares their solutions and becomes a group of validators will assist students to overcome negative feelings (Cobb, Yackel, & Wood, 1989).

It is impossible to "compartmentalise the cognitive aspects separately from the effective aspects" of learning mathematics (Southwell, 1991, p. 1). Many teacher education students enter university with negative attitudes towards, and low levels of, mathematics. In order to assist these students, a constructivist approach to teaching and learning mathematics has been developed and presented as an alternative to ones commonly experienced. The key constructs of the approach are built around an *experiential learning cycle* adapted from Jones and Pfeiffer (1975) which uses principles of *cooperative learning* and the *problem-centred approach* of the Purdue Mathematics Project (Wood, Cobb & Yackel, 1992). Diagrammatically, the Learning Cycle can be represented as shown in Figure 1.

$$\begin{tikzcd} \text{Applying} \arrow[r] & \text{Experiencing} \arrow[d] \\ \text{Generalising} \arrow[u] & \text{Discussing} \arrow[l] \end{tikzcd}$$

Figure 1. The Learning Cycle.

Stations in the cycle are understood as follows:

1. Experiencing. Learners must be actively involved in their own learning. They must engage in activities which engender their mathematical thinking. These activities may involve physical action with materials but will involve mental action. Learning must involve "doing" in order to be effective.

2. Discussing. Reactions and observations arising from the experiences need to be shared with fellow learners and other members of the community and talked about in order for them to be evaluated and, perhaps, validated against the taken-as-shared knowledge of the learner's community. Explanation, justification, and negotiation of meaning through communication will help the learner establish this knowledge.

3. Generalising. Learners need to develop for themselves, through individual construction and interaction with their communities, hypotheses which indicate the current state of their understanding. These hypotheses, or generalisations, will then be tested for viability through their application to other problematic situations or further communicative discourse. It is these generalisations which form the basis for the learner's next experience.

4. Applying. Planning how to use the new or revised learning and actually applying it to contextual situations will not only validate it as viable knowledge (or suggest rejection of it as non-viable) but will also provide the learner with another experience which could be used to commence yet another cycle (Perry & Conroy, 1994, pp. 5-6).

The role of classroom interaction in developing student's mathematical problem solving has been demonstrated by Schoenfeld (1987). A key feature of the authors' approach to interactive collaboration is the construction of a set of social norms within the class. The following norms were developed:

1. Activities will consist of problems for the students. That is, it is assumed that the students may not be able to obtain solutions or even know where to start immediately.

2. When working in small groups, students are expected to develop solutions to the activities cooperatively and to reach consensus on these solutions. The teacher is expected to circulate among the groups, observing their interactions and encouraging their problem-solving attempts.

3. Students are expected, as a small group, to explain and defend their solutions or attempts at solutions to the whole class. Other students are expected to indicate their agreement or disagreement and to encourage alternative solutions.

4. The whole class is expected to see itself as a community of validators and is expected to work towards a solution or solutions which can be taken-as-shared. It is not the teacher's role to validate solutions.

Each student was expected to record in a journal his/her reactions to the course, attempts at solutions to the activities, and any other feelings or concerns they may have had. The students were encouraged to make and record summaries of discussions and their generalisations. Mason et al (1982) point out that by recording the process we follow in solving a problem, we afford ourselves the opportunity to look back over the process, refine our thinking, and store our ideas for later use.

Theoretical background on affect in mathematics

According to Silver (1985) one's affective state influences decision making and contributes towards determining one's actions. Affective states may be attributed to a variety of causes (Mandler 1989). Responses to a block during the problem-solving process in mathematical problems causes emotional states which can be immediate and short-lived or intense and globally encompassing. Goldin (1988) asserts that we give students too little experience with the intensely positive affective states of pleasure, confidence, and satisfaction in mathematics; and for many students the first step in achieving needed cognitive and affective mathematical reconstruction is the interruption of incessant negative feelings created as responses to mathematical encounters.

Method

Students elected to participate in the course in order to achieve a level of competency required for registration as a primary teacher in New South Wales. Most students were adults and had minimal background in high school mathematics. In paired groups the students worked through the phases of the Learning Cycle by way of a series of mathematical problems. All classes were videorecorded using two cameras. Videotapes were analysed for students' reactions to their classes, and any movement in affective variables and/or achievement. Data was also collected from students' assignments, reflective interviews, journals and surveys. It was anticipated that the students' learning would involve positive changes in affective as well as cognitive states. Two procedures were used to investigate affective changes. First, comparisons were made between students' responses to attitude and belief questionnaires before and after the course. Second, video recordings of students' responsiveness in class and written records in their journals provided case study data on how attitudes changed. A final examination tested cognitive changes.

Part A: Attitude and Belief Changes

Three questionnaires were used:

1. An *Attitude to Mathematics* questionnaire was administered. It consisted of 24 statements which were in no set order but could be divided up into three subsections. Part A consisted of nine items that were particularly likely to have been influenced by participation in the course such as, "I get satisfaction from solving mathematics problems" (item 6); Part B consisted of nine items that were generalisations that may have been influenced by participation in the course such as, "I find mathematics fascinating and fun" (item 3); and Part C consisted of six items that referred to past experiences such as, "I did not look forward to mathematics lessons at school" (item 5). Students

responded by choosing one of five categories - strongly disagree, disagree, neither agree nor disagree, agree, and strongly agree - and items were scored from 1 to 5 respectively with some items being scored in reverse order. Scores were added so that total scores could range from 1 to 120 (that is 24 x 5), and a neutral score was 72 (that is 24 x 3).

2. A *Beliefs about Mathematics* questionnaire with six items such as, "mathematics is computation" had a similar response format.

3. Similarly, *Beliefs about Mathematics Learning* had six items such as, "children are rational decision-makers capable of determining for themselves what is right and wrong".

4. A similar questionnaire *Beliefs about Mathematics Teaching* was used with six items such as, "the role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge."

There were entry and exit results for 37 students, from three different classes taken by two different teachers and in two different modes (that is, two classes attended weekly for three hours for 14 weeks while two classes attended the same number of hours but in three blocks of two weeks spread over one semester). Class sizes varied from 8 to 16 (too small to make inter-class comparisons).

A two-tailed *t*-test was used to compare the results before and after the course for each of the four surveys.

Results

Table 1 gives details of the before and after scores for each of the questionnaires. There was a statistically very significant increase in attitudes to mathematics which were closely linked with the course (Part A and Part B of *Attitude to Mathematics* questionnaire) and a statistically significant increase in perceptions of attitudes to mathematics of past experiences, in beliefs about mathematics, beliefs about mathematics learning, but not about beliefs about mathematics teaching.

Discussion

In light of the candidly honest revelations of students' anxieties and fear of mathematics, these results suggest that the course had a positive effect on students' attitudes and perceptions to mathematics. The fact that there was no change in beliefs about mathematics teaching may be a reflection of students' already changed attitudes from other mathematics curriculum subjects which also incorporate constructivist approaches to teaching at the University.

A further case study analysis considered *how* attitudes may have been changing.

Part B: Case Studies

The authors have been able to identify positive change points in students' affective states indicating progress towards more effective encounters with the content of the course. Presented below are examples of change points evidenced in excerpts from journals and video recordings.

Table 1
Attitudes and Beliefs

Questionnaire		Before		After		*t*-score	Probability
	n	mean	st. dev.	mean	st. dev.		
Attitudes to mathematics (total)	30	74	20	81	16	2.65	.013**
Attitudes to mathematics (Part A)	35	26	6	31	6	4.29	.000**
Attitudes to mathematics (Part B)	32	28	7	31	6	2.08	.045*
Attitudes to mathematics (Part C)	37	17	7	19	6	2.33	.025**
Beliefs about mathematics	35	19	2.8	21	2.3	2.53	.016**
Beliefs about mathematics learning	36	24	2.2	25	2.4	2.37	.023**
Beliefs about mathematics teaching	34	20.6	2.3	21.3	1.7	1.33	.193

Note. * significant difference in means of pre- and post-treatment results at 0.05 level;
** significant difference in means of pre- and post-treatment results at 0.01 level

Student 1 and Student 2: The following students were mature-aged female students who had not completed higher level mathematics at school, who showed a lack of school mathematical knowledge, who indicated a strong dislike for mathematics, and who showed a degree of fear about doing mathematics.

Background. Student 1 (S1) and Student 2 (S2) worked together in a pair. S1's score on the questionnaire *Attitude to Mathematics* went from 41 to 62. S2's score on the questionnaire *Attitude to Mathematics* went from 42 to 46 with increases in all areas of attitudes and beliefs except the immediate classroom effects.

Early Anxiety and Fluctuating Feelings. Like most, these two students entered the course and openly admitted their reservations and anxieties about mathematics. It is clear that their feelings fluctuated; it was not until the fifth or sixth week after considerable struggle that they became more positive about mathematics. The following extracts have been taken from their journals.

Week 1

S1: Today we had our first maths lesson (the subject I have been dreading) . . . quite frankly the thought of maths scares the hell out of me. It is my biggest fear in teaching. I wonder if the children I will teach in the future will suffer because of my insecurities.

S2: With much trepidation and reservation I approached the [maths] class even harbouring a certain resentment for having to participate in a subject which I despise and abhor. . . . My feelings towards maths have been deeply ingrained since my childhood and I cannot attribute them to either a teacher and teaching methods or an horrific experience that has left an indelible mark on my psyche, I have just little mathematical process and even less interest in cultivating dexterity.

Week 2

S1: Well after such a successful first week, I attended this week with enthusiasm. This was short lived. Although it wasn't a complete disaster, I do feel it is not going to be as easy as I first thought. . . . However, I am persevering.

S2: The way by which our team attempted the solution was futile and frustrating and thoroughly annoying. We did not achieve the answer nor did we get anywhere near close. Yes, yes, we did attempt it and used what to the best of our ability was the most practical way of achieving the solution. . . . When the easiest and most common method was explained, I still did not understand and I could not possibly use this method myself.

Week 3

S2: I am fighting a sense of panic and being forced to deal with this in front of a group of [other students]. . . . I have nothing positive to say about today's lesson. I'm too distraught and disgruntled.

At all times, students in the class were aware that they needed to make the mathematics their own and to reproduce it themselves if they were to be successful. On many occasions class empathy for the struggle resulted in the class overcoming a constraint in order to make progress. Interactive collaboration and peer support offered comfort and reassurance as now described.

Week 4

S1: My partner also had a tough time with this. I was really concerned for her. . . . We should not have to succumb to tears because no one else is making sense. . . We did not do [the homework] because we both felt defeated.

Through such empathy, the learners bonded together and became willing to share their feelings and understandings leading to further affective and cognitive critical-changes (Geoghegan et al, 1994).

Week 9

S2: The dreaded "T" word. That snuck up on me today as I sat unsuspectingly contemplating [the success of our last assignment]. Not quite as daunting as a previous experience, we managed to come to terms with the concept of Trigonometry . . . When it is explained to me by one of my peers in a really basic fashion I am able to understand and it actually seems to make sense.

All students have an immense store of mathematical knowledge but they need to recognise that they have it and be encouraged to recall it, and to be confident to use it (Goldin, 1988). It would seem that the class collaboration assisted students by providing them with empathy, and time to think verbally and analytically. S2 had a high positive self-esteem in language, she was able to listen to others, say ideas in her own words, make links with verbal ideas, construct ideas, and summarise what was being discussed. These were pathways for her to move away from negative global structures and move towards a useful heuristic for understanding the problem. At least for this occasion, she was able to receive some sense of pleasure in finding a "solution." Goldin (1988) asserts that success can evoke continued application of the successful method; it blossoms into pleasure as the method continues to work. Interestingly, S2 seriously considered withdrawing from the course in week 3 but remained, and gradually displayed considerable development in confidence and understanding. It was through peer encouragement and interactive collaboration that this student's doubts were allayed and a feeling of empowerment derived. Goldin (1988) states that *encouragement* is a vital step on the pathway to positive feelings towards mathematics.

Week 7

S2: What a positive experience. Today is the first day where I felt I was part of the class and could almost 'keep up'. The fact that we passed the assignment also has left me on a high and I never actually felt that I would achieve the sensation of satisfaction about maths. . . . perhaps my attitude is changing. . . . I walked out of the class happy with the lesson and my own performance.

Week 10

S2: I thought I was making some connections and even got excited at the fact that I was doing the work unaided at one stage. . . well we are moving much too quickly but I am starting to feel a little more au courant with tan, cos and sin.

Discussion

The students, by and large, were responsive to the approach and developed positive dispositions which kept them on track towards meeting the challenge of the course content. S1 and S2, like most students, attributed their lack of mathematical ability and lack of self-esteem in mathematics to "a *lack of endowment*" (Week 1, journals). By the end of the course, students had recognised their own use of mathematics, they had seen some relevance after all in mathematics, and had conceded some successes which they were willing to own. Overall it can be said that *interactive collaborative work* was a very important support for students and a main reason for their attitude changes and success.

Conclusion

Interactive collaboration within the learning environment allowed the students to feel comfortable with the approach and what it was attempting to do. The cooperative, problem-centred approach facilitated the mathematics learning of many of the students and developed in them a confidence in their own abilities to get started on mathematical problems and persevere with them. We have seen how over a period of time increasing success and positive feelings associated with a supportive classroom have assisted learning. Different types of affects, from positive to negative—short term feelings, general attitudes, and long term beliefs—were all involved in the learning. The paper has further illustrated the links between the heuristics of learning mathematics and the development of positive or negative affective variables.

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GROUPWORK WITH MULTIMEDIA IN MATHEMATICS: THE ILLUMINATION OF PUPIL MISCONCEPTIONS FROM A VYGOTSKIAN PERSPECTIVE

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ABSTRACT

This paper outlines the way in which the National Curriculum Council sponsored multimedia package "World of Number" (Shell Centre et al, 1993) was used as the focus for a group activity with a Year 9 (age 14/15 years) mathematics class. The classroom research was carried out in a South Yorkshire comprehensive school during the Spring Term of 1994. The class was engaged in work which involved graphs of relationships and close attention was paid to relationships between distance, speed and time in a variety of contexts. The research methodolgy involved the recording of the classroom interaction on videotape and a micro-analysis of the resulting discourse. Using Vygotsky's notion of the function of egocentric speech, this analysis illuminated the misconceptions of one pupil in particular. The results of this analysis are the focus of this paper.

BACKGROUND

The group involved in this project was a Year 9 top set of approximately thirty pupils. The project was designed to fit in with the planned scheme of work for the Spring Term of 1994 when the group was due to do a unit of work on graphical interpretation involving graphs of motion. The overall plan was based upon the theme of graphical interpretation and the aim was to integrate activities both on and off the system with aspects from the planned scheme of work over a two week period. The topic was introduced as a whole class activity, more fully detailed in Hudson (1994a), and in the following lesson one of episodes from the unit *Running, Jumping and Flying* was introduced to the whole class with the aim of setting the context and giving the pupils a sense of what to expect in terms of the future activities on the system. The unit is made up of video clips of various examples of motion, several of which are sporting events from the Seoul Olympics as detailed in Figure 1. Each sequence has two or three graph options associated with it. For example, in the sequence shown in Figure 2, the chosen axes in the bottom left hand window are height and time. Other choices might be distance against time and speed against time. This would give three graphs to choose from in the bottom right hand window. The combined choice is illustrated in the top right hand window.

Following the whole class introduction some groups began working on the activities at the system. A group size of three had been agreed with the class teacher, with the aim of creating the conditions for effective interaction. Each group was allocated an initial period of thirty minutes for intensive work at the system. The practical limitations were eased considerably by the use of two systems. In addition to the original laser disc package the school also had the use of the CD ROM version. This provision enabled four groups to carry out the multimedia-based activities in a one hour lesson and for each group to have a turn over the period of a single week.

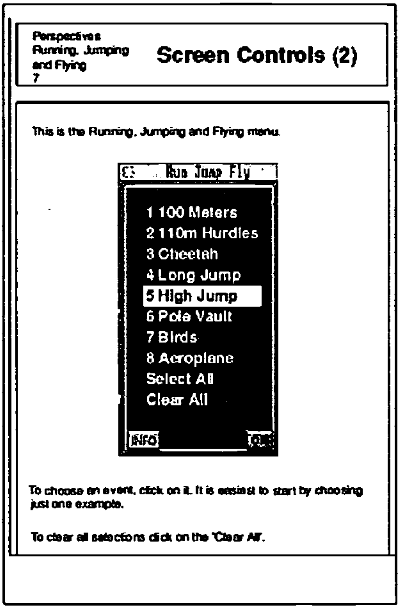


Figure 1

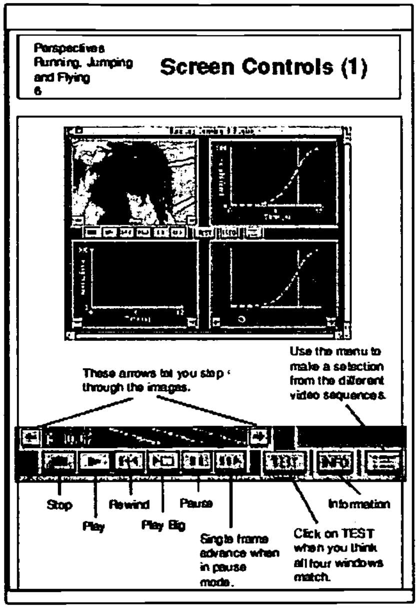


Figure 2

MULTIMEDIA-BASED ACTIVITIES

The main aims of the multimedia-based activity were to promote discussion and require time for reflection which was consistent with one of the preliminary findings of the evaluation conducted by the National Council for Educational Technology, as reported by Hughes (1994), of the need for "reflective moments". The activity was structured in such a way as to encourage the following process: select and view a video sequence, think about the distance-time graph, sketch the graph, compare graphs, choose which fits your ideas, explain to each other why a particular graph does or does not fit, test out choice on the system and finally repeat the process with a different choice of axes. In summary this was a cycle of observation, reflection, recording, discussion and feedback.

RESEARCH METHODOLOGY

The approach to the analysis of the classroom discourse was particularly influenced by the work of Mercer (1991), Edwards and Mercer (1987) and also that of Teasley and Rochelle (1993). The focus of the study reported upon by Mercer is the content and context of educational discourse from a theoretical perspective strongly influenced by the work of Vygotsky (1962). He describes the analytic methods adopted as being similar to those of ethnography, "in that we were similarly concerned with the minutiae of what was said and done; and we were interested in participants' accounts and interpretations of what they said and did". The method employed by Mercer involved the complete transcription of all the discourse recorded on videotape. In addition, any information on the physical context and non-verbal communication, which was necessary to make sense of what was said or done, was added alongside the relevant section of the transcript. This information is described as *context notes* which might include reference to the physical 'props' of the classroom, such as equipment, drawings, texts and computer-screen representations invoked by speakers to support the discourse. However this work was conducted in classroom situations in which the focus was mainly upon the interaction between teacher and pupil.

In reflecting upon this approach, the need for an interpretive framework through which to analyse the data arising from peer interaction soon became evident. The approach adopted by Teasley and Rochelle was found to be particularly resonant and was consequently adapted to form the chosen framework. Teasley and Rochelle report on a study which is intended to illustrate the use of the computer as a cognitive tool for learning that occurs socially. The study is concerned with the question of how students construct shared meanings in relation to modelling activities, in the context of a Newtonian microworld. This microworld is a computer package which is described as "a graphical and dynamic simulation of a physicists' mental model of velocity and acceleration". They outline a theoretical perspective in the tradition of Vygotsky, in that it is based upon a view of learning as a fundamentally social activity i.e. that understanding is built through social interaction and activity and that concepts and models are social constructions resulting from "face-to-face participation" in activities.

A framework for the analysis of collaboration is outlined, which the authors argue involves not only a micro-analysis of the content of students' talk, but also how the pragmatic structure of the conversations can result in the construction of shared knowledge. In order to understand how social interaction affects the course of learning, Teasley and Rochelle argue that it requires an understanding of how students use coordinated language and action to establish shared knowledge, to recognise any divergences from shared knowledge as they arise, and to rectify any misunderstandings that impede joint work.

The notion of "a shared conception of a problem" is a central one and this is used as the basis of what is described as a *Joint Problem Space*. It is proposed that social interactions in the context of problem solving activity occur in relation to a Joint Problem Space (JPS). This is defined as a shared knowledge structure that supports problem solving activity by integrating goals, descriptions of the current problem state, awareness of available problem solving actions and associations that relate goals, features of the current problem state and available actions.

A number of "structured discourse forms" are described which conversants use in everyday speech to achieve mutual intelligibility. These utilise language, bodily action and combinations of words and actions. It is proposed that students use the structure of conversation to continually build, monitor and repair a JPS. They also describe some *categories of discourse events* that they have used in their analysis such as turn taking, narrations and coordinations of language and action.

A fuller account of the application of this framework is detailed in Hudson (1994b). The focus of the main section of this paper is on how this analysis illuminated the misconceptions of one pupil in particular.

ANALYSIS OF VIDEO TAPE TRANSCRIPTS - NEIL: AN INDIVIDUAL CASE STUDY

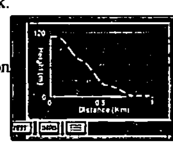
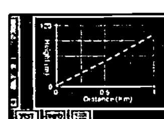
This section focuses on Neil's utterances who was a member of a group with Philip and Jonathan. In this first example of classroom discourse Jonathan and Neil are responding to what they have seen on the video, which involves a cheetah chasing its prey, and also to the questions posed by the teacher researcher (TR). It begins with Jonathan and Neil's responses to the question posed, about what the graph of distance against time might look like:

> *N: Distance is ... It goes up - the distance doesn't it? Well like along. Time.*
> *TR: What's happening to the distance?*
> *P: It's getting greater.*
> *N: It's going up. Higher.*

Neil appears to be confused, when he asserts that "It goes up - the distance", seeking acceptance or confirmation with the question "doesn't it?". He continues with the utterances "Well like along" and "Time". From these it would appear that Neil is very confused in his thinking. This elicits an attempt to clarify matters, with the question "What's happening to the distance?". Philip replies correctly that "It's getting greater". However Neil's response suggests a confusion between the graph itself, which is indeed *going up* the page, and the actual distance which is *increasing*. A feature of Neil's thinking is this lack of distinction between the motion itself and its abstract graphical representation.

In a later episode of an aeroplane landing, this aspect of Neil's thinking is again evident. The axes are initially set on height against time.

Group 3: Philip B, Neil and Jon'n
Episode 4: Aeroplane (0.58.18)

1	P:	It doesn't start off ... Watch this. Height against time.	Philip runs the video.
2	N:	Speed against time that.	The axes are set on height against time.
3	P:	Yes but no. We've got to choose which height against time is the right one.	Trying to clarify the task.
4	J:	Let's have a look.	Referring to graph option 1. 
5	N:	Yah! Oh! How come it does all the wavy lines? It goes straight down. It doesn't go up and down does it?	 Making a diagonal downward wavy motion. Making a diagonal downward smooth motion.
6	I J:	Well change it! Have a look ...	
7	I P:	No but the nose goes up, doesn't it?	
8	I N:	No! That's not it!	Referring to graph option 2. 
9	I J:	That's not it!	
10	N:	It's taking off that, isn't it?	

Philip's response Neil's initial statement appears to be contradictory. He replies "Yes but no". By this he may have been indicating that, "yes", the graph showing is the correct choice to fit the speed against time axes but that, "no", it is not addressing the current problem which is "to choose which height against time is the right one." In doing so, Philip is attempting to establish a shared understanding of the problem or a Joint Problem Space. Neil's response to the video sequence would seem to be based upon an expectation of a smooth line, which probably reflects the more simple models from his past experience. However Philip observes that the nose of the aeroplane "goes up" on landing. The final comment in this section from Neil, displays evident confusion between what he interprets from the graph and what he observes by watching the video sequence, which is clearly of the plane landing. The fact that the graph is rising from left to right suggests to Neil that this is the flight path of the aeroplane taking off.

It would seem that Neil's misconception is related to the fact that he is describing the picture that he sees on the page i.e. "It (*the line*) is going up (*the page*). Higher (*up the page*)". The inability to distinguish, between the abstract representation of the motion pictorially and the motion itself, would explain why Neil interpreted this graph as showing the aeroplane taking off.

Neil's difficulties appear to stem from his use of speech and in particular from the lack of distinction he makes between the situation that he is describing, and its abstract representation in the form of the graph. For example, this can be highlighted in the following utterances of Neil, taken from the interaction above:

N: Yah!
Oh! How come it does all the wavy lines?
It goes straight down.
It doesn't go up and down does it?

When Neil refers to "it" doing "all the wavy lines", he would appear to be referring to the graph, though he does not make this clear. However, in the subsequent utterances, he seems to refer to the aeroplane when he talks about "it" going "straight down" in contrast to it going "up and down". Later in the same episode the group is considers distance against time.

Group 3: Philip B, Neil and Jon'n
Episode 4: Aeroplane (continued 1)

11	J:	Do you want to change that one?	Referring to the choice of axes.
12	P:	Yeh, I've done that. It's distance against time now.	
13	N:	Distance is going down? No! How could it be going down - distance? Oh, it's just landed. But its time's going up!	
14	P:	What?	
15	J:	The distance? It can't ... can't ...	
16	N:	... go down. It just goes up.	
17	P:	I know it can't.	
18	N:	So, why does it look like that then?	Looking at graph option 2.

Neil's stream of utterances at line 13 form a narration of his current thinking, which once again appears to be very confused. He seeks to interpret the graph in terms of the possible motion of the aeroplane. His first utterance relates to a perception of the distance *going down* rather than *decreasing*. Once again, Neil fails to make a clear distinction between the situation and its abstract representation in the form of the graph. In the first utterance from this interaction, he uses "it" to refer to at least two aspects:

Distance is going down?
No! How could it be going down - distance?
Oh, it's just landed.
But its time's going up!

Firstly he uses "it" to refer to the distance, then in the following utterance refers to the aeroplane and finally talks about "its" time going up. He seems to dismiss this as a possibility but then refers to the fact that the plane has "just landed". By prefixing his sentence with "Oh", he seems to imply that the distance *going down* might be linked with the plane landing. This might suggest a confusion between the height and the distance (*going down*). However the notion of going down in this case would appear to have been transferred from (going down) the page to (going down) in mid-air. The evident inability to distinguish between the abstract graphical representation and the motion itself would be consistent with his previous thinking. He concludes with the utterance "But its time's going up!" which seems to emphasise his state of confusion.

Neil's use of language throughout is resonant with the function of speech as outlined by Vygotsky (1962). According to Vygotsky's theory, which was based upon a critique of that of Piaget, speech can be considered to have two particular forms which he describes as egocentric and communicative respectively. The notion of communicative speech is based upon Piaget's idea of socialised speech. However Vygotsky proposes that both egocentric and communicative speech are social, but that it is their functions which differ. The function of communicative speech, as implied in its description, is for the purpose of communication with others. On the other hand, the function of egocentric speech is as an instrument of thought itself. He develops his view of the function of egocentric speech, by arguing that all silent thinking is "nothing but egocentric speech".

In particular, many of Neil's utterances are resonant with Vygotsky's description of egocentric speech. From observations based on his own experiments, Vygotsky notes that children resort to egocentric speech when faced with difficult situations. From these observations, he concludes that egocentric speech and silent reflection can be functionally equivalent. He argues further that egocentric speech is the genetic link in the transition between vocal and inner speech, and that it is this transitional role that lends it such great theoretical interest. He proceeds to highlight how the conception of speech development "differs profoundly" in accordance with the interpretation given to the role of egocentric speech. The resulting picture of the development of a child's speech and thought is thus from the social, to the egocentric and finally to inner speech. Thus the direction of the development of thinking is not from the individual to the social (as argued by Piaget), but from the social to the individual.

In supporting his argument, Vygotsky describes "an accident" which occurred during the course of one of his experiments, which he suggests provides a good illustration of one way in which egocentric speech may alter the course of an activity. He recounts a young child who was drawing a "streetcar" when the point of his pencil broke. Nevertheless, he tried to complete the circle representing the wheel by pressing down on the pencil very hard. However nothing showed and the child muttered to himself, "It's broken." He then put aside the pencil, selected a paint brush instead and proceeded to draw a broken streetcar after an accident, continuing to talk to himself from time to time about the change in his picture. Vygotsky uses this incident of the child's accidentally provoked egocentric utterance as an example to show how it "so manifestly affected his activity that it is impossible to mistake it for a mere by-product, an accompaniment not interfering with melody". Vygotsky develops his argument by describing how, from his observations, egocentric speech at first marked the end result or a turning point in an activity, then was gradually shifted towards the middle and finally to the beginning of the activity, taking on a directing, planning function and raising the child's acts to the level of purposeful behaviour. He compares this process to the well-known developmental sequence in the naming of drawings. A small child draws first, then decides what it is she has drawn. At a slightly older age, she names her drawing when it is partially completed. Finally she decides beforehand what she will draw.

Neil's egocentric utterances are provoked in response to the examples of motion and also to the possible graphical representations of these which he sees on screen. From the episode involving the cheetah, it can be seen that Neil describes the distance as going up, when it is in fact increasing. As indicated earlier, the graph of distance against time could be described, quite reasonably, as going up the page. However, Neil does not distinguish between his descriptions the motion itself and those of its abstract graphical representation. In the later episode of the aeroplane landing, he is now faced with a situation which involves vertical motion, for which the use of the term going down would be appropriate and for which, in more general situations, it would be quite appropriate to describe an aeroplane taking off as going up. In a similar way to Vygotsky's example of the streetcar, Neil's interpretation seems to be affected by his previous egocentric utterances, when on viewing the graph which shows a diagonal line, rising from left to right, he responds by saying "It's taking off that, isn't it?".

Neil's confusion is exacerbated by the fact that the graph which he sees on screen is not a simplified idealised version but a realistic representation of the downward motion of the nose of the aircraft, which is not uniformly smooth. With apparent reference to the graph, he asks, "How come it does all the wavy lines?" and adds, with seeming reference to the aeroplane, that "It goes straight down. It doesn't go up and down does it?" Subsequently he describes the distance as "going down" and his thinking would appear to have been affected by his previous egocentric utterances with regard to the aeroplane. The notion of the "distance going down" now seems to be transferred from the abstract graphical representation to the situation itself, and Neil exclaims, in what appears to be a series of entirely egocentric utterances:

No! How could it be going down - distance?

Oh, it's just landed.
But its time's going up!

CONCLUSION

Forman and Cazden (1985) observe that when they sought to explore Vygotskian perspectives for education, they immediately confronted questions about the role of the student peer group. In this study the process began by exploring the role of the peer group and led to the perspective representing the starting point of Forman and Cazden's enquiry. Forman and Cazden point towards Vygotsky's notion of internalisation, by which the means of social interaction, especially speech, are taken over by the child and internalised and how development proceeds when interpsychological regulation is transformed into intrapsychological regulation. The case study of Neil offers particular resonance with Vygotsky's notion of the function of egocentric speech and also illuminates how misconceptions can develop as a part of this process. These findings point towards the need for monitoring the processes of peer interaction in such contexts involving the use of multimedia and/or computer systems. The experience from this study suggests that the role of the teacher in this process is a crucial factor, made more effective by an awareness of the potential for the development of such misconceptions as those highlighted in the case of Neil.

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Tell me who your classmates are, and I'll tell you what you learn:

conflict principles underlying the structuring of the math class

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Tracking and grouping are still widely viewed, among math educators, as the best way to deal with students of different levels and achievements. In this paper we report on a research study into the differences between the mathematical achievements of the students participating in a project of teaching mathematics in heterogeneous classes and others who study in heterogeneous homerooms classes while being allocated to ability groups in mathematics. Fundamental differences between the two groups of students have been found.

Background

In Israel, tracking and grouping are widely viewed as the best way to deal with students of different levels and achievements. Recently, however, discouraging results from studies of tracking, on one hand, and new evidence of the promising potential of cooperative learning in mixed ability groups on the other, have prompted attempts to cope with diversity of students within the mathematical classroom. In this paper we will report on a recent project of teaching mathematics in heterogeneous classes based on those attempts. The implementation of the project is accompanied by a research study into the differences between the mathematical achievements of the students participating in the project and others who study in heterogeneous homerooms classes while being allocated to ability groups in mathematics.

The project guidelines

The teaching was conducted according to four central strategies: 1. Whole-class discussions 2. Small heterogeneous group learning 3. Small homogeneous group learning 4. Larger homogeneous group learning. Each of these strategies was employed in response to different needs for interaction among the learners. Whole-class discussions gave the teachers a chance to create an appropriate learning atmosphere and develop norms such as: listening to classmates, legitimization of errors, freedom to express ideas as well as to indirectly deal with issues like the mathematical conversation, conceptions about what is mathematics, and the ability to cope with conditions of vagueness (Gooya, 1994; Davis, 1989).

The justification for small-group interaction within the mathematics classroom is widely described in many recent papers. However, Cobb (1994) identifies 4 types of small-group interactions. (1) univocal interaction (2) multivocal interaction (3) direct collaboration (4) indirect collaboration. Cobb maintains that the type of interaction is heavily dependent on the diversity of the mathematical abilities of the participants. Moreover, the established type of interaction is quite stable and determines the types of learning opportunities arising within the group. As such, we may say that different group combinations provide the learners with different learning environments.

Cobb points out that productive small group interactions involve both indirect collaboration and multivocal interaction, which at first glance imply homogeneous grouping. "Homogeneous grouping, however, clashes with a variety of other agendas that many teachers rightly consider important including those that pertain to issues of equity and diversity" (Cobb, 1994, p. 207). Since within the same small-group the type of interaction very rarely changes we have chosen, in our project, the strategy of alternating two basic types of small groups: "homogeneous" groups and "heterogeneous" groups, so that each child is simultaneously a member of two groups (sometimes even more). A student who is the mathematical authority in one setting may play the role of the "follower" in another. Experiencing the position of the "helper" on a regular basis in one context influences the student's ability to become an "active" listener" in another and in turn influences his/her role as the "helper". No doubt, opportunities for indirect and multivocal interaction are created.

The fourth strategy of class organization, large homogeneous groups, provided opportunities to bring topics discussed in the homogeneous small-groups to semi-class discussions while exploring unforeseen avenues emerge from these discussions and planning future activities for the groups. The teachers also used this type of organization whenever they thought that their intervention was crucial to the quality of the groups' work.

Ability grouping versus Mixed ability classes: a glimpse at past research

The degree of influence of school grouping methods upon the individual student's scholastic achievements is a central issue in the educational system. Ability grouping, one of the most common of these, is justified by the need to adapt content, pace and teaching methods to students functioning on different levels and as a means of improving the scholastic achievements of all the students (Dar, 1985; Slavin, 1988; Sorensen & Hallinan, 1986, 1990). On the other hand, not a few theoretical approaches disagree about whether placing students into ability groups is the correct method for dealing with the heterogeneity of classes. Most of these approaches stress learning as an individual process nourished by interpersonal interaction. (Bandura, 1982; Corver & Schiene, 1982; Voigt, 1994), and argue that the learning group makes a critical contribution to the student's progress. For example, Kerckhoff (1986) compared the achievements of students with the same placement data in schools of various levels, as well as students in different ability groups within the same school. He found that the type of school and the level of the ability group have a statistically significant effect on the students' achievements in mathematics and reading. The general controversy surrounding this issue is even greater with respect to the subject of mathematics. The justification offered for the need to organize students into ability groups is that mathematics is "graded", "linear", "structured", "serial", "cumulative", etc. - making it difficult to work with groups of students on different levels of ability. And indeed, the central concepts used by the supporters of ability grouping are "ability to learn mathematics" and "the hierarchical nature of the subject" (Ruthven, 1987). They view students' ability as the central explanatory factor in differentiating their achievements in mathematics (Lorenz, 1982).

It is therefore questionable whether ability grouping advances us toward the goal for which it was designed, or whether it actually defeats this purpose. In empirical research on ability grouping there are two utterly different traditions: 1. Studies comparing the achievements of students in ability groups to those of students in heterogeneous classes. 2. Studies comparing the achievements of students in higher-level ability groups to those of students in lower-level groups in the same school.

Studies of the first sort have generally concluded that there is no difference between the two methods from the standpoint of the average achievements of the students in general (Slavin, 1990). Nevertheless, when the differential effect of ability grouping was examined, the results were different: the results reveal an interaction between tracking and the students' achievement levels: On the average, students in the higher ability groups gained more than equally able students in heterogeneous classes, while students in the intermediate and lower ability groups gained less than students of the same ability in heterogeneous classes.

However, most of these studies have a common methodological problem, the selection problem: the possibility that the groups being compared differ not only in the treatment they received (ability grouping versus heterogeneous classes) but also on some other relevant characteristics (Abadzi, 1984; Kilgore, 1983).

Since it is so difficult to perform random experiments in the educational system, while post-hoc comparisons between schools with and without ability grouping pose methodological problems, the second type of research became more prevalent. This type of research focuses on the differential effects on learning at the various levels of ability groups. The question studies of this sort ask is whether the gap between the better and the weaker students after being placed in ability groups for some time is different from the gap that would be expected on the basis of the previous differences between them. In other words, does the placement of students in ability groups differing in their initial level lead, in and of itself, decrease or increase the differences between students' achievements (Slavin, 1990). The most prevalent finding of these studies is that ability grouping does have an effect on achievement, and that this effect is in the direction of increasing the gap between the students in the various ability groups (Alexander, Cook & McDill, 1978; Gamoran & Barns, 1978, 1986; Gamoran & Marr, 1989; Oakes, 1982; Sorenson & Hallinan, 1986).

In these studies the main methodological problem is to separate the effect of belonging to groups at different levels from the effect of the initial differences between the students placed in these groups on final achievements. Moreover, in this case the selection -- that is, the initial difference between the students placed in the various groups -- is inherent in the very notion of ability grouping and cannot be considered a "mishap". Therefore the differences in achievement between groups at different levels reflect both the effect of the initial differences between the groups and the possible effect of the grouping itself. This is an interesting case in which it is in principle impossible to investigate a treatment effect using an experimental design.

The regression discontinuity design

The above described methodological problem can be overcome in the cases where the students were divided into group levels by setting agreed-upon cutoff points. Having this information allows us to perceive the students who stand immediately on the two sides of a cutoff point as identical from the view point of the selection criterion. Following their mathematical development for a period of time can put light on the influence of ability grouping. This research design is known as the quasi-experimental regression discontinuity design (Cook & Campbell, 1979; Kahan, Linchevski & Igra, 1992; Linchevski, Kahan & Dantziger, 1994). The use of this design enables us to estimate the effect of the initial differences among the students: This effect is estimated by the regression line of the posttest on the pretest within each group level, while the effect of the group level is estimated by the discontinuity between the regression lines within consecutive group levels. A design of this sort was used successfully by Abadzi (1984, 1985) for investigating the effect of track levels. The research described in the present paper uses this discontinuity regression design to investigate the effect of ability grouping in mathematics in Israeli junior high schools and to compare it to the effect of studying in the mixed ability classes participating in our project. The research consists of four longitudinal studies. The first one evaluated the effect of grouping after a year and after three years (henceforth study 1). The second one evaluated the effect of learning in heterogeneous classes, on students of different ability levels, after a year and after two years (henceforth study 2). The third one has been examining in depth the mathematical thinking and performance of students who were initially at the cutoff points and were therefore subsequently more or less arbitrarily assigned/ or hypothetically assigned to two distinct ability groups. The fourth one compared the mathematical achievements of 2 groups of students studying at the same school and who were arbitrarily assigned to heterogeneous classes or ability grouping. In the following sections we will report on the first two large-scale studies only.

Research design

Sample

Study 1: A sample of 9 junior high schools, with a total of 1677 students, was taken from among the schools that satisfied the necessary conditions: 1) heterogeneous homeroom classes and ability grouping in mathematics, 2) students were allocated according to a clear agreed-upon criterion.

Study 2: All the 12 junior high schools which participated in the heterogeneous project were included, with a total of 1730 students.

Tests: Achievements in mathematics were measured by tests constructed according to the topics covered in school detailed in the mathematics curriculum (Ministry of Education and Culture, 1968) and were validated by experts.

Variables

Study 1: Four variables were defined for each student: Ability group level (level 1 "high" to level 4 "low") and placement score (henceforth pretest) served as independent variables, while the achievement test scores in mathematics after one year and after three years of ability grouping (henceforth posttests) served as the dependent variables.

Study 2: In study 2, however, since the students were actually studying in heterogeneous classes, the implementation of the research design required a procedure of creating **hypothetical ability groups**. Therefore, at the beginning of the seventh grade, without bringing this to the knowledge of any of the parties, the students were allocated into "hypothetical ability groups". The placement procedure was done exactly the same as the school had used in the past, and consequently the students' distribution was shelved. The hypothetical group level and the hypothetical placement scores (henceforth pretest) served as the independent variables, while the achievement test scores in mathematics after one year and after two years of studying in heterogeneous classes served as the dependent variables.

Calculation of effects

The grouping/hypothetical grouping effect and the pretest effect were calculated separately for each of the schools. The overall grouping effect, , was defined as equivalent to , 1<j<m-1, where m indicates the number of ability groups in the school and Hj is the effect of ability group. Similarly, the overall effect of the initial differences between the students, was defined as Pj, 1<j<m where Pj is the overall effect of the pretest for ability group j. The two effects can easily be calculated using a multiple regression equation to predict posttest scores on the basis of the pretest score and ability group level. Thus the overall effect of grouping in each school is $(m-1)b_{LL}$, where b_{LL} is the regression coefficient of the ability level variable. The overall effect of the pretest score is (Xmax - Xmin)bp, where bp is the regression coefficient of the pretest score variable and (Xmax - Xmin) is the range of the variable within the specific school.

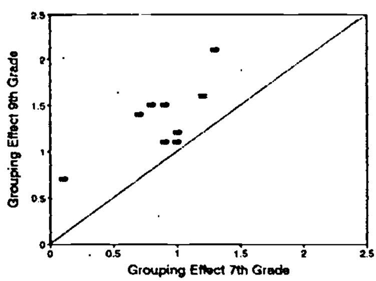
Results

Study 1: The overall effects of ability grouping on achievements in mathematics at the end of the seventh and the ninth grades, for the students who remained in the same group for the entire period, are presented in Table 1 and in Figure 1, separately for each school.

Table 1: The grouping effect and the ability effect in achievments in mathematics at th end of the 7th grade and at the end of the 9th grade.

School	7th grade		9th grade	
	Ability Effect	Grouping Effect	Ability Effect	Grouping Effect
1	1.3	1.4	2.2	0.7
3	1.5	1.1	2.2	0.9
5	1.2	0.7	2.1	0.1
6	1.4	1.5	1.5	0.8
9	0.6	1.6	1.2	1.2
10	2.2	1.1	2.2	1
12	0.7	2.1	1.7	1.3
14	1.2	1.2	1.5	1
15	0.8	1.5	1.1	0.9
Median	1.2	1.4	1.7	0.9

Figure 1: A comparison between the grouping effects in the 7th grade and the 9th grade.



As can be seen from the Table, the overall effect of group level in each school was already positive at the end of the seventh grade, after one year of ability grouping. The effect size varied considerably among the schools. At the end of the seventh grade, the effect ranged from 0.1 to 1.3 SD, with a median of 0.9 SD. At the end of the ninth grade it ranged from 0.7 to 2.1 SD, with a median of 1.4 SD. In each of the schools the grouping effect was greater at the end of the ninth grade than at the end of the seventh grade (see Fig. 1). Figure 2 presents examples of three cases of regression discontinuity of the posttest scores on the pretest scores at the end of the seventh grade: School #5, where there was a negligible grouping effect; School #6, which had the median absolute value of the group level effect, as well as the median ratio between this value and the size of the pretest effect; and School #4, where there was a negligible pretest effect and an especially large grouping effect.

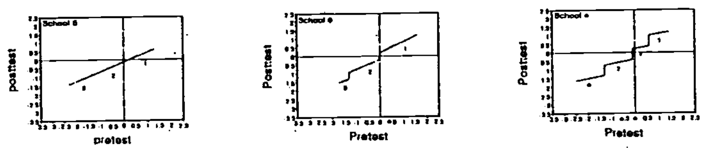


Figure 2: Representative cases of discontinuity in post-test / pre-test grades

the increase in the grouping effect over the three years of junior high school was not only absolute, but also relative to the effect of the initial differences. At the end of the seventh grade the grouping effect was less than the effect of the initial differences in 8 of the schools and equal to it in the remaining school, while at the end of the ninth grade the grouping effect was greater than the effect of the initial differences in most of the schools.

Study 2: The overall effects of the hypothetical grouping on achievements in mathematics at the end of the seventh grade, for students studying in heterogeneous classes is presented in Table 2.

Table 2: The results of the regression at the end of the 7th grade inside the schools

School	No. of Students	No. of Groups	Hypothetical Grouping Effect	Standard Deviation	Ability Effect
1.1	93	4	0.36	0.27	2.75
1.2	64	3	0.42	0.28	3.17
2	247	3	0.26	0.16	2.30
3	196	3	0.60 *	0.20	2.75
4	91	4	0.12	0.23	2.71
5.1	94	4	0.87	0.20	3.45
5.2	64	3	-0.10	0.17	2.78
6	187	4	-0.09	0.17	3.15
7	240	3	0.34	0.18	3.11
8	155	3	0.40	0.21	3.57
9	66	3	-0.26	0.34	2.20
10	132	3	-0.10	0.18	2.90

As can be seen from Table 2 in eight out of the 12 schools the effect was positive and in four of them negative. It means that in some of the schools the gap between the students was greater than the one expected on the basis of the initial differences and in some smaller. The effects of the hypothetical ability grouping and the effects of the initial differences, after two years of studying in heterogeneous classes, for four schools, is shown in Table 3.

Table 3: The results of the regression at the end of the 8th grade

School	No. of Students	No. of Groups	Hypothetical Grouping Effect	Standard Deviation	Ability Effect
3	180	3	0.52 *	0.21	2.7
5.1	93	4	0.36	0.21	3.0
5.2	57	3	0.24	0.18	2.8
9	66	3	-0.10	0.31	2.7

* Significant

The effects of the hypothetical grouping at the end of the seventh and the ninth grades were small in themselves and also small in comparison to the effects of the initial differences. Since they were not statistical significant and not consistent in direction we might even interpret them as no effect.

Discussion

The purpose of the study was to examine the effect of ability grouping in mathematics and of studying in heterogeneous classes on the variance in students' scholastic achievements. The findings show that placement in ability groups has a clear effect on the students' achievements. The direction of the effect was consistent across all 9 schools in the sample: The differences between the scholastic achievements of the students at the different group levels at the end of the first and the third years were greater than would have been predicted by the data at the time of placement. In other words, if two students with the same pretest scores, close to the cutoff point, were randomly placed in groups at different levels, then the scholastic achievements of the student in the higher group would be greater than those of the student placed in the next lower group. The consistency of the effect across the 9 junior high schools studied (which constitute independent replications) indicates, in our opinion, that the phenomenon is a general one. The findings with regard to the heterogeneous classes do not show a clear effect. In some of the schools the gap increased and in some decreased. Those findings can be interpreted in several ways. Although all the investigated schools belong to the same project this, probably, does not in itself lead to identical mathematical environment. The need to challenge each child, for example, for as long as possible requires intra-class differentiation. Investigation into the fundamental differences between the two types of schools is necessary.

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WHAT IS THE MOTIVE OF MATHEMATICS EDUCATION?

An attempt at an analysis from a Vygotskyan perspective

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Abstract

The presentation sets out from an analysis of the motive of mathematics education made from activity theoretical assumptions. The origins of 'maths-difficulties' are discussed from this analysis and related partly to the concept 'proceptual divide', coined by Gray and Tall as predicting success or failure in mathematics at an early primary level, and partly to results of Swedish phenomenographic investigations, illustrating that 'proceptual divides', eliminating scores of pupils continuously, also appear after the first one at the primary level. The result of these studies lead to two hypotheses 1) Neither the kind of flexible proceptual thinking predicting success, nor the rigid procedural behaviour predicting failure is inborn. Both dispositions might be the result of teaching. 2)To avoid 'proceptual divides' the motive of mathematics education must be changed from activities giving 'surface knowledge' to activities with the motive of producing social competence.

Introduction

Some years ago I put the question: 'What use is maths?' to five 8-year old Swedish pupils who were at the end of the first grade. All of the children thought that the main reason for learning 'maths' was so they 'could do adding and ... take away ... . or so that they could 'write ... some numbers'. Asked when they would need this knowledge, they just sighed, this was the same when I asked if their parents ever needed 'maths'. One child finally came up with the thought that parents *have* to know 'maths' because 'otherwise they *don't know it*!' (Neuman, 1989) This child illustrated a deep familiarity with the motive for traditional 'maths education' given at school: to give people the 'surface polish' necessary for posing as men/women of education.

Freudenthal (1983) proposed a didactical phenomenology with a different motive. 'Since our mathematical concepts, structures, and ideas have been invented as tools to organise the phenomena of the physical, social and mental world' (p ix), he says, we ought to help pupils step into the continuously ongoing 'learning process of mankind'. Mellin-Olsen (1987) underlines that the pupil must recognise the tasks the teacher gives him 'as a component of an Activity' (p 105) and Vygotsky (1979) claims that children should be helped to invent an arithmetic which is relevant to them.

The main aim of this presentation is to articulate the *motive* of the first kind of pedagogy, here called 'surface pedagogy', and to show how this, as a whole – not as methods used for teaching single topics – is what causes the large amount of failures in mathematics. A pedagogy akin to the theories of Freudenthal, Mellin-Olsen and Vygotsky mentioned above, but related to phenomenography, will be used to contrast this 'surface pedagogy'. Both these kinds of pedagogy will be analysed from the so-called 'activity theories' inspired by Vygotsky.

Phenomenography, activity theories and the non-dualistic ontology

According to phenomenographic 'non-dualistic' ontological assumptions, which like Freudenthal's didactics are related to phenomenology, reality is neither seen as a representation of the world, nor as a subjective construction of a world not accessible to us through our senses. Consciousness is regarded as consisting of individual-world, or object-subject relationships, and our 'conceptions' – the units we study in phenomenography – are also thought of in this way. An object, e.g. a 'maths problem', is never, according these assumptions, seen as a problem per se, but rather as a problem experienced by someone, e.g. by a pupil. Similarly a subject, e.g. a pupil, is seen as a subject whose awareness is always directed towards something, e.g. towards a 'maths problem' and towards experiences that this problem makes him aware of (Marton and Neuman, 1990). These latter experiences, or conceptions, of the problem depend on what the pupil experiences as the activity within which it has arisen.

Non-dualistic assumptions do not characterise phenomenology alone. To view individual consciousness as being a participant in social consciousness – or in Vygotsky's words, in 'our co-knowledge' (Vygotsky, cited in Leont'ev 1981) is another way of expressing non-dualistic ontological assumptions. Vygotsky's colleague and follower, Leont'ev (1981), elaborated Vygotsky's thoughts into so-called 'activity theories', related to the concepts: *activity*, *action* and *operation*. An activity is related to a *motive*, while actions are directed towards *goals* subordinate to this motive. The consciously performed actions are transformed little by little, into unconscious *operations encapsulated in more complex actions*, and so on. These theories form the ground from which the activities and motives of mathematics education will be scrutinised.

The proceptual divide

The part of the theory which concerns transformations of actions into operations, mirror recent theories on procedures transformed into objects, e.g. Sfard's (1991) theory of 'reification' and the theories about 'procepts' put forward by Gray and Tall (1994). Procepts is the denotation that these researchers use as an amalgamation between procedure and concept, which is the result of an encapsulation process. Through detailed videotaped interviews with pupils between 7 and 12 years of age, they have been able to illustrate how elementary pupils, regarded by their teachers to be 'above average', display 'proceptual thinking', while others, regarded to be below average, display 'procedural thinking' (p 9). Children who display proceptual thinking acquire 'elementary procepts' through a lot of flexible strategies used to derive new 'facts' from a few initial ones. Conversely, children who display procedural thinking use rigid and tiring counting procedures and hardly ever derive new facts from the few facts they eventually learn. Gray and Tall refer to an early 'proceptual divide' (p 9), which differentiates between prospective success and failure in mathematics.

In a pilot study of a phenomenographic investigation of how school beginners conceive of addition, subtraction and of number relations (Neuman, 1987), 31 pupils, aged between 8 and 13 and regarded by their teachers to perform below average in mathematics, were interviewed. The result of this simple study support the theories, which later were put forward by Gray and Tall. In this study the pupils almost always displayed a procedural behaviour pattern. This was the same when 14-17 year old pupils and adults with 'maths difficulties' were interviewed.

Conversely, procedural behaviour was hardly ever identified among the untaught school beginners interviewed in the main study. They found a lot of creative ways of solving word problems within the range of 1–10, similar to the problems given to the older pupils. Sometimes these inventions resulted in answers which were far from correct and even very peculiar. Yet, if viewed from the child's perspective, they all held a certain logic (Neuman, 1987, 1989, 1994). Many inventions, however, ended up in correct, and sometimes brilliantly worked out answers (Neuman, 1992). These inventions could be studied when the children formed them concretely, with the help of their fingers, structured and grouped in specific ways (and not merely used for keeping track). Some children explained, after immediate answers, that they 'thought with their hands', afterwards they concretely illustrated how this had been done. Others described the same strategies in words, without allusions to their fingers. These untaught children seemed to conceive of the problems they met in the interview as mirroring problems they might meet in everyday life, and which therefore ought to be solved in the simplest possible way.

In a 2 year teaching experiment (Neuman, 1987, 1993, 1994), in which two of the classes that had been interviewed were followed, the children took part in activities with the motive of preserving and developing this conception of why one has to learn 'maths'. The experiment set out from a game related to a story of 'The Land from Long Ago' where no 'maths' existed, but where problems demanding some kind of 'maths' continuously turned up. In this game the children experienced mathematics as a subject that we need, and that we have produced – and still produce – through common problem solving. Through the way in which the pupils solved problems using united efforts, they began, little by little to model numbers within the range of 1–10 as divided up into two parts. This had already been done by some pupils in the interviews at the start of school. For all of the pupils this concrete modelling was transformed after a while into unconscious operations, which were encapsulated within more complex, conscious actions, when the pupils later had to deal with addition and subtraction above the 10-limit.

The 11 pupils in these two classes, who had not been able to give any correct answers to the subtraction problems given in the interview at the start of school were interviewed again at the end of grade 2. By then they all used flexible 'proceptual' thinking in all problems, even in addition and subtraction within the range of 1–100, thus passing the 10-limit. In a control group of 13 pupils with similar results at the start of school, only one child could solve all the problems in this way, and 6 pupils continuously displayed procedural behaviour, except for in a few problems within the range 1–10, which had probably been learned by heart.

The results of these studies give birth to three hypotheses worth further investigation. First: 'procedural behaviour' is not inborn but rather a result of traditional 'surface pedagogy'. Secondly: 'proceptual thinking' is not inborn either but is, contrary to 'procedural behaviour', rooted in informal learning. Thirdly: 'proceptual thinking' is possible to *teach*, within a pedagogy where the motive is to arrange situations that are experienced as relevant by the children themselves. This is a motive characteristic not only of phenomenographic pedagogy, but also – and first and foremost – of the 'realistic teaching tradition' (see e.g. Streefland, 1989). Pramling (1994), however, discusses three tenets which she sees as specific to what she calls a 'developmental' phenomenographic pedagogy: 1) to set out from knowledge of children's thinking, 2) to have a specific methodological knowledge, 3) to have confidence in children's ability to learn from each other. The third tenet is well known from socio-constructivist research (Cobb, Wood and Yackel, 1991; Murray, Olivier, and Human, 1991 et al.), while the first two are more specific to phenomenography. An important aim of phenomenographic interview studies is to produce a knowledge of children's 'conceptions'. Using this kind of knowledge is what the first tenet implies. The second tenet – to have *'methodological knowledge'* – means teaching in the 'thought provoking' ways in which phenomenographic deep-interviews are carried out. An example of teaching applying the latter two tenets was given in my 1994 PME-presentation. Examples of deep-interviewing, will be given in this years' presentation.

Proceptual divides eliminate scores of pupils also at later levels.

The motive for presenting these examples, however, is not primarily to illustrate the nature of phenomenographic interviews: The main motive is to elucidate how a 'surface pedagogy' seems to give birth to proceptual divides also after the point where, at an early primary level, pupils who 'think' are differentiated from pupils who 'count'. To judge from Crowley, Thomas and Tall (1994) algebraic teaching further exacerbates the differences between proceptual and procedural thinkers. Here, however, the denotation 'procedural thinkers' is not used for students using *'counting procedures'* but for those displaying a 'process-oriented' behaviour of another kind, i.e. for those who rigidly *'must'* put the operation on the left and the answer on the right of the equals sign. Thus, even pupils who express 'proceptual thinking' – in the sense that they have left the use of *counting procedures* – might later be 'lost'. This does not only happen when algebra is introduced, but at many other junctures too, e.g. when the pupils meet fractions or multiplication/division of decimals. It seems as if scores of pupils are eliminated at every such 'proceptual divide'. The young, 'creative' mathematicians' we met at the start of school; at the age of 7, evidently seem to have little possibility of ever becoming familiar with the subject of mathematics, and still less of becoming good mathematicians.

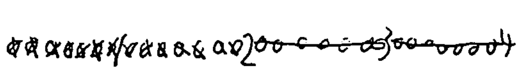
One 'proceptual divide' seems to occur already at the first introduction of division, the examples illustrating the nature and roots of difficulties that eliminate pupils at such divides are taken from a study, in which pupils in grade 2, 3, 4 and 6 were interviewed about their conceptions of quotitive and partitive division (Neuman, 1991). An astonishing observation in this study was that 'proceptual' 3rd graders, interested in forming derivation strategies to extend their store of 'known products' in multiplication, could sometimes – after having been introduced to division – give up attempts to make sense of the problems they solved, in their strife to invent such strategies. Conversely, the 6th graders could often give up the easy way of using a 'known product' in division, for the more laborious method of putting the numbers together according to an algorithm. Another confusing phenomenon was that this kind of behaviour was related to partitive division only.

My interpretation of this behaviour was that it was rooted in the way division is usually introduced in Sweden: through 'naked number sentences' and the rule: 'Find out how many times the divisor goes into the dividend'. 'Naked number sentences' related to this instruction do not demand any understanding of the different roles played by the divisor in quotitive and partitive division. When the pupils became acquainted with the word problems in the interviews the rule to 'measure the dividend by the divisor' did not fit in with the role played by the divisor in partitive division. It is, then, senseless to ask: 'Find out how many times 7 children go into 28 marbles' in order to find out how many marbles each of the seven children will get if they have 28 marbles to share. As many as six of the ten pupils in grade 6 – who immediately solved the quotitive division using a known product (or in one case using a derivation strategy) and who illustrated in other ways too, that they knew 'the multiplication table' – tried an algorithm for the partitive problem. (That algorithms could be used, even if they weren't understood, seemed to be self evident to all the pupils.) The 'proceptual' 3rd graders, however, did not yet reflect upon the senselessness of 'measuring a number of marbles with a number of children'. They were too occupied with facilitating the repeated addition they used to divide. In the next section one 3rd grader and one 6th grader will exemplify the nature of the difficulties that many pupils seemed to experience when they met partitive division introduced as a part of the 'surface pedagogy' representative of traditional teaching.

Pitfalls at the introduction of division – two examples

A phenomenographic interview is thought to be a learning situation for the interviewer as well as for the interviewees, a situation in which the interviewer learns about the conceptions of the interviewees, and where the intention is that the interviewees – through the questions posed by the interviewer – will become conscious of the 'pre-conceived' ideas from which they have set out.

Ever since the first interviews that were made, I had seen drawings and heard utterances made by 2nd and 3rd graders who had not yet been taught division, these made me aware of the fact that before pupils are introduced to division they find it natural to 'measure the dividend by the divisor' also in partitive division. The 2nd graders could, for example, solve the task with the 28 marbles and the 7 children, saying: '7 marbles, one marble each; 14, two each...' and so on, finally arriving at '28 marbles, four marbles each'. Or they could make drawings mirroring this thought (see below). They formed ratio-tables through which the dividend was 'measured', not by the number within each part, but by *the number dealt out in each round*, the number the divisor represented to these pupils in partitive division.



Erik says, after having marked seven marbles: 'I'd better write a one, so I remember that they've got one each.' After having marked seven more he writes 2, and so on ...

Karin shares out 28 marbles between 7 boys

Kalle, however, is a 3rd grader who some weeks ago *was* introduced to division. When presented with the task in which 28 marbles are divided among 7 children he immediately answers 'Four marbles each'. When asked how he knew the answer, he replies:

K: First I do ... seven and seven, that's fourteen ... then fourteen and fourteen's twenty-eight.

I wonder if it is just the task: 'Find out how many sevens there are in 28?' that Kalle solves through his addition, which he has made easier by 'doubling'. Or has he thought of a dealing out situation of the kind the 2nd graders illustrated: 'seven marbles in each round' until the marbles run out. A long dialogue ensues:

I: Why did you take seven?
K: 'cause ... there were seven boys that were sharing
I: Now, when you've taken this first seven, what do you do with them then?

K: Mmm
I: What happens to them? ……… when you've taken them? ……
K: When I've taken seven marbles??
I: Mm mm ………
K: Then ……… well, then there're … well, what can I say?

Kalle goes quiet. He does not seem to be aware of what lies behind his decision to take seven four times. I try to confront him with an experience of a fair share situation, trying to find out if he experiences the semantics of the problem at all, and Kalle tries to step away from the rule he has been taught for division. When replacing it with an everyday experience of sharing, however, he first thinks of a situation where seven marbles are dealt out to each child, a dealing out experience related to quotitive division. The dialogue goes on like this:

K: Then another's got to take,then ……………………

Kalle becomes silent a second time. I interpret his answer to mean that one boy has got seven marbles and that the next one now gets a similar amount. To keep the dialogue going I say:

I: And then another boy gets to take seven marbles?
K: Yes ……… just a minute ……………… Yees, yeeees ………

Kalle might already have continued in his line of thought, dealing out the last 'two sevens' to two more boys, when he says 'just a minute …'. Then he might have become aware that three boys do not yet have their share. He is quiet for a long while. Then he says:

K: But it might be wrong …………

Kalle becomes silent again. Finally I say:

I: Might it be wrong?… You wrote four marbles each. And I wonder, how did you know that?
K: Mm m. 'Cause ……… mm mm …… It can't be that …………
I: Can't it?
K: Well ……… Yeees! 'Cause I've taken seven four times!

Kalle again seems to have found an acceptable point of departure for his line of thought. But what does it look like? Is it: "Seven marbles four times, one marble to each boy in each round"? Or is it: 'This is the kind of problem you can solve through working out what makes twenty-eight in the 'seven-times table'? I want to know, and ask:

I: 'And what did you do when … Each time you took seven … what did you do with them? …
K: I put them together

Kalle is back in his repeated addition again, I try to explain in a better way, what I want to know:

I: But now, if you had thought in this way in a real fair share situation …… that you'd do that …… Here they are …… the twenty-eight marbles lets say …… in a little bowl. Then you take seven, and those seven boys are sitting here (I point around the table). What would you do with the seven marbles you took?

Again Kalle is forced to turn back to reality, and again the 'dealing out' experience related to quotitive division is thematised: "Seven in each part – seven to each boy".

K: I give 'em to a boy ……
I: To a boy, all seven? ………
K: Nooo ………………

Kalle probably remembers that he has already rejected this idea, and a new theme is actualised:

K: No, I'd take four marbles ………

The new theme is closely related to the conception 'What might 1/7 of 28 be?', this was also expressed by some 2nd graders. But Kalle probably doesn't estimate what 1/7 of 28 is and check this estimation, as the 2nd graders who solved the problem had done with great effort. He certainly sets out from the answer four, that he got when he first solved the problem. Maybe he then tried to check if all the 28 marbles would be used up if he took four at a time. Anyway, he stops speaking again. It is difficult to add four seven times, and maybe Kalle thinks this laborious addition is not worth the trouble. Nevertheless, he hesitates again at the answer 'four' that he has written on his paper, asking 'is that right?'. And again I assert that it is correct, before I resume the dialogue:

I: But I just want to know how you actually think this out ... and then see that it's four each? That's twenty-eight marbles .... First you took seven ...
K: Mm m
I: So the seven boys are sitting here .... Just put them in the middle of the table ... ? Or what are you doing with them? ......
K: I'm putting 'em in a pile and I'll take another seven ......
I: Yes, but these poor boys, they were supposed to share the marbles, they can't be very pleased that the marbles are here in four piles? (I point to the middle of the table four times)
K: Noo .... I'll take all the marbles .... in ... ... yes, yeah ... I'll take all the marbles (inaudible) ...... then share them out ...... 'n' then there'll be ...... four each ......
I: How do you know that then? What do you do to share them out? ......
K: I'll put 'em ...... mm ...... one to every first like that ...... then I'll do it as long as I can ...... that'll be four to each of them ....

In the end, the experience of 'giving out one at a time' is thematised. But is it related at all to the idea of seven in each round, which would make it sensible to solve the problem through the repeated addition of seven? I find it difficult to formulate more questions, without leading Kalle to the idea 'seven in each round'. Finally I say:

I: Mm mm ...... I don't really know now ...... really what you mean ....Because all twenty-eight were there, weren't they? (Again I point to the middle of the table four times.) Did you take them again? Did you put them back in the bowl?
K: No, I take one from the pile and give it to one boy ... (What Kalle now says is impossible to hear on the tape, but according to my protocol Kalle first pretends to take one of the four piles indicated by me, sharing out one marble at a time while he points to each of the boys in turn, as he imagines them sitting round the table) ...'n then I take ... (now indicating the next pile).

Eventually Kalle seems to relate his repeated addition of 7, compressed through a 'doubling', to an experience of 'giving out seven each round, one to each boy'. If this is so, the interview situation has become a learning situation for him, making him aware that adding seven four times can be a meaningful way of solving a partitive division task in which 28 marbles are to be shared between 7 boys.

The older the pupils were, however, the harder it was to make them conscious of an experience of sharing as related to partitive division. For example, Marita, a 6th-grader, directly solves a quotitive problem in which 42 buns are packed into bags of 6 buns each, answering 'Seven, since seven times six is 42'. In the partitive problem, however, – in her case 4 children with 28 marbles to share – she tries a long division. She puts the dividend and the divisor in the wrong places, arriving at the answer 0.14 marbles each. Understanding that this cannot be the right answer she supposes she has chosen the wrong algorithm. So, she tries a multiplication algorithm, arriving at 112 marbles each, which she also experiences as unreasonable. Finally, for some reason she multiplies 28 by 6. Then she does not know how to handle the carried digits, and gives up, saying that she cannot solve the problem. When asked if she cannot find any other way of solving it – instead of bothering with 'these' (the column algorithms) – she says 'Well, seven ...', but immediately changes her mind, adding 'No, that's not right either'. Asked by the interviewer why it is not right, she says again: 'No, it can't be like that, can it'. As the interviewer is eager to find out what lies behind the correct answer 'Seven', she again insists: 'But you actually said seven ... you must have thought it out in some way ... how were you thinking?' – 'I just thought 7 x 4 is 28 ...', Marita answers, still rejecting this answer. My interpretation of her behaviour is that an answer acquired with the help of the rule 'Find out how many times 4 children go into 28 marbles' does not make sense to her. She probably finds it safer to use an algorithm, which she knows ought to lead to the right answer when correctly performed, even if one does not understand why.

Discussion

To avoid 'proceptual divides' and to give all pupils the idea of mathematics as a subject which is meaningful, thrilling and full of beauty, the motive of the *whole activity* called 'mathematics education' must be changed and not only a few methods used to teach a few single topics. From the first day at school, maths activity must have the motive of giving pupils an experience of themselves as participants in the never ending learning process of mankind, in which we together invent mathematical concepts, and ideas as tools for organising phenomena in our world.

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A PRELIMINARY REPORT OF A FIRST-GRADE TEACHING EXPERIMENT: MATHEMATIZING, MODELING AND MATHEMATICAL LEARNING IN THE CLASSROOM MICROCULTURE

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Abstract

The reported study is part of a year long first-grade teaching experiment conducted with six-year-old children which focused on the development and implementation of instructional activities. These activities were developed in collaboration with the classroom teacher and were intended to support the students' development of additive and subtractive thinking strategies. The discussion first elaborates constructs used in the analysis procedures that emerged from previous research efforts to coordinate psychological and sociological perspectives. We then provide an overview of the current research efforts which includes outlining the data analysis procedures and summarizing two of the five instructional sequences. Preliminary findings of student learning indicate that the students made considerable progress during the first half of the school year. In particular, the students appeared to shift from predominantly using counting strategies to routinely using non-counting strategies (e.g., going-through-ten, referencing doubles). Final comments point to subsequent analysis which will account for the students' mathematical progress.

Introduction

This paper provides an overview of a first-grade teaching experiment that was conducted during the 1993-1994 school year. Theoretically, the overall aim of this research effort was to investigate the possibility of accounting for mathematical learning in the social situation of the classroom by coordinating sociological and psychological perspectives (e.g., Balacheff, 1990; Cobb & Bausersfeld, in press; De Corte, Greer, & Verschaffel, in press). As part of the investigation, particular attention was given to the development and use of instructional activities based on the theory of Realistic Mathematics Education (RME) (cf. Gravemeijer, 1994; Streefland 1991; Treffers, 1987). In this discussion we summarize the project by giving a brief description of the data corpus and two of five instructional sequences that were developed in collaboration with the classroom teacher. We then outline the analysis procedures and present some of the preliminary findings. Before we begin this discussion, it is necessary to situate this work in the previous as well as our ongoing research efforts.

Previous research efforts have focused on the more general classroom social norms for both whole class discussions and small group work. Such norms for whole class discussions included "*explaining and justifying solutions*, *attempting to make sense of explanations given by others*, *indicating agreement or disagreement*, and *questioning alternatives in situations where a conflict in interpretations or solutions has become apparent*" (Cobb & Yackel, 1993, p. 11). These norms are not specific to the mathematics classroom but apply to any subject matter area. That is, one would hope that the teacher might expect students to explain their reasoning when they engage in discussions of historical events or works of literature. These social norms have as their psychological correlates,

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the teacher's and individual student's beliefs about their own and others' roles in the classroom (Cobb, Yackel & Wood, 1993). For instance, a student's response to the teacher's request for an explanation (the teacher's expectation) might implicitly indicate the student's obligation to explain his solution process rather than merely state a result. Prior analyses indicate that, during the course of whole class discussions, the individual students reorganize their beliefs about their own roles as they participate in the interactive constitution of the norms that both constrain and are shaped by their beliefs (Cobb et al., 1993). In general, the analysis of social norms delineates the evolving participation structure realized in the classroom (cf. Erickson, 1986; Lampert, 1990).

A second phase of the prior research focused on the normative aspects of classroom discussions that are specific to the students' mathematical activity. These norm have previously been called sociomathematical norms (Yackel & Cobb, 1993). Sociomathematical norms have as their psychological correlates individual students' beliefs and values about engaging in mathematical activity. These norms include what *counts as a different mathematical solution*, *what counts as a sophisticated mathematical solution, what counts as an efficient mathematical solution, what counts as an acceptable mathematical explanation* and *what counts as an acceptable mathematical justification.* It is claimed that students develop specific mathematical beliefs and values as they participate in the interactive constitution of these sociomathematical norms. For example, after the student has explained his or her thinking, the teacher might ask other students if they solved the task in a *different* way--the teacher's explicit expectation for the students' contributions. It is in such situations that what counts as a different mathematical solution is renegotiated (Yackel & Cobb, 1993). In particular, the teacher and students clarify what is a difference that makes a mathematical difference as they respond to contributions that are proposed as being different.

Our current research efforts address a third aspect of the classroom microculture: the taken-as-shared mathematical practices established by the classroom community. The corresponding psychological correlate is taken to be individual students' mathematical conceptions and actions (Cobb & Yackel, 1993). Specifically, students develop their individual mathematical ways of knowing as they contribute to the establishment of communal mathematical practices such as ways of posing tasks, interpreting tasks, solving tasks, and symbolizing. Conversely, these practices both enable and constrain students' mathematical ways of knowing. These mathematical practices are said to be taken-as-shared rather than shared because there are typically significant qualitative differences in individual students' mathematical interpretations. These interpretations fit for the purposes at hand in that differences do not necessarily become apparent as students attempt to coordinate their activities.

The Teaching Experiment

Data Corpus

Data consists of videotaped individual interviews of all 18 six-year-old children in the class in September, December, and January; video-recordings of an additional interview conducted with ten of the children in November; video-recordings of 47 mathematics classes made using two cameras; the students' worksheets; and daily field notes that summarize the events of the classroom.

Interview Sessions

To document the students' mathematical activity over time, each of the eighteen first-grade students were interviewed in September, before the first instructional sequence began; in December at the completion of the second instructional sequence; and in January. Two researchers were present during these interviews, one researcher conducted the interview with the student while the other researcher completed an observation log to document the student's activity. The tasks used included horizontal number sentences, hidden collections tasks (cf. Steffe et al., 1983), thinking or derived fact tasks, and context problems. Although the interviews were structured so that these tasks could be presented systematically, the interviewer varied the task selection and follow-up questions to explore students' current conceptual understandings.

Instructional Sequences

The two instructional sequences that were developed between September and December are called *Patterns and Partitioning* and the *Arithmetic Rack.* The development of the first of these sequences was precipitated by the preliminary analysis of the pre-interviews conducted in September. Six of the 18 students had considerable difficulty in using their fingers as perceptual substitutes for collections when they attempted to solve additive tasks involving quantities of five or less. As a consequence, in collaboration with the classroom teacher, we developed an initial instructional sequence that highlighted flexible finger patterns, spatial patterns, and conceptual partitioning and recomposing collections of up to ten items. The Arithmetic Rack instructional sequence was developed once this preliminary sequence had been completed. Before outlining the Arithmetic Rack instructional sequence we clarify the domain-specific instructional theory of Realistic Mathematics Education (RME) that underpinned the development of instructional activities.

In the elaboration of RME proposed by Gravemeijer (1994; in press-a; in press-b) an instructional sequence involves (a) informal problem solving situations that are experientially real to students and in which they can engage in informal mathematical activity, and (b) the development of student-generated *models of* their informal mathematical activity. These models, which might consist of drawings, pictures, non-standard notation, computer graphics, or conventional notation, subsequently take on a life of their own and become (c) *models for* mathematical reasoning. This sequence of idealized development supports students' transition to (d) more formal mathematical activity in which their use of conventional symbols signifies the conceptual manipulation of abstract and yet personally-real mathematical objects. It should be stressed that this idealization involves a series of conjectures about how the activities might support student mathematical development when they are realized in the classroom. These activities are not, however, "pre-programmed" (Gravemeijer, in press-a; in press-b) and do not constitute what is often referred to as ready-made curricula. Further, the process by which these activities are *realized* in interaction in the classroom depends on individual students' contributions. For analytical purposes, a major goal is to understand how the activities are interactively constituted between the teacher and the students as they engage in mathematical activity.

Arithmetic Rack Instructional Sequence. The intent of this sequence was to support students' development of additive and subtractive thinking strategies. The rack consists of two rods on each of which are five red beads and five white beads (Treffers, 1990; Gravemeijer, in press-a; see Figure 1).

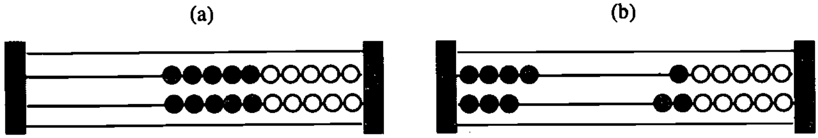

Figure 1. (a) The starting configuration in which all the beads are placed to the right, and (b) the arithmetic rack showing a collection of seven beads.

To use the arithmetic rack, the student moves beads from the right to the left either by counting individual beads or by moving several beads at a time. For instance, a student might show seven (beads) by moving three beads on the top and three beads on the bottom, and then move an additional bead.

The arithmetic rack was designed to fit with the students' previously-documented non-counting strategies. Initially, we hoped that students might create perceptually-based numerical composites of some type when they acted with the rack. Suppose, for instance that a student has made a collection of nine beads by moving five beads on the top row and four beads on the bottom row, and wants to add more beads to make 16 (see Figure 2a). To complete the task, the student might first move five beads on the top row and then move two more beads from the bottom row (see Figure 2a). Alternatively, the student might first move one more bead on the bottom row and then move three on the top row and three on the bottom row (See Figure 2b).

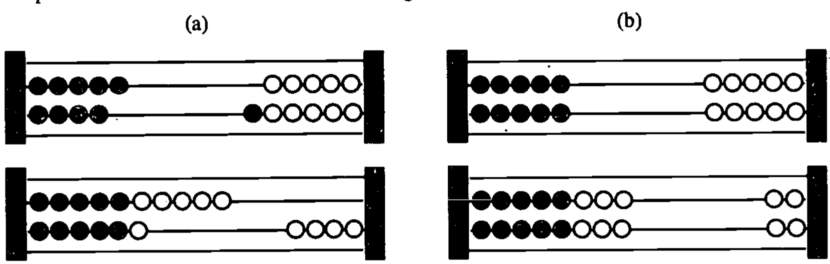

Figure 2. Strategies for showing 16 using the arithmetic rack: (a) going-through-ten, and (b) referencing doubles.

From the observer's viewpoint, the use of a going-through-ten strategy is implicit in the student's activity in the first example, and the use of a doubles strategy is implicit in the second example. The instructional challenge is then to guide the emergence of such aspects of children's activity as explicit topics of conversation in the classroom.

The Arithmetic Rack Instructional Sequence was developed over a 9 week period and involved for a total of 25 lessons. A sequence of activities was developed that included *flashing and showing various numbers*, *bingo*, *addition and subtraction problem situations*, *imagery addition and* *subtraction, imagery bingo*, and *anticipation addition and subtraction*.

The initial activities were situated within the scenario of a double decker bus (van den Brink, 1989), that had developed during the Partitioning and Patterning Instructional Sequence. The top and bottom rows on the arithmetic rack signify passengers on the upper and lower levels of the bus, respectively. Thus, various combinations of beads represent different ways in which people might sit on the bus. For example, tasks that involve *showing various numbers* might entail the teacher telling the students that three people were on the upper level and four were on the lower level and asking the students how many people were on the bus. A follow up activity might involve the students showing different ways seven people could be sitting on the bus.

The *addition and subtraction problem situations* differ from these initial tasks in that the teacher first might tell the students that a certain number of people are on the bus, and then tell them that some people either got on or off the bus. The students' task is to determine how many people are now on the bus. For example, the teacher might tell the students that ten people are on the bus and then six more people got on the bus. A student who has shown the initial ten as five on the top and five on the bottom might move the beads in one of several ways. He or she could move five more beads on the top and one bead on the bottom, and then enumerate the resulting configuration as 16 without counting. It is important to stress that both here and elsewhere in the sequence, students are obliged to explain both how they acted and why they chose to do so. In this regard, the students' activity is located within a classroom microculture that considers mathematical activity to be eminently discussable.

In subsequent activities, the students were first encouraged to imagine and later to anticipate the number of beads (people) that must be added or subtracted to make a certain number. For instance, the students might be asked to show a certain number of the people on the bus. They might then be asked to imagine how they would move beads to show that, say, 11 people got off the bus before actually doing so.

In the final set of instructional activities, explicit reference was made to neither the double decker bus nor the arithmetic rack. Instead, tasks were posed in a variety of contexts such as *money tasks* using various denominations of coins, *purchasing various items*, *the single decker bus*, *money in a piggy-bank*, and *cookies in a cookie jar*. For example, to pose tasks using the scenario of the cookies in the cookie jar, the teacher might draw a cookie jar and indicate the number of cookies in it by writing a numeral on the jar. She might then explain that a certain number of cookies have been placed in or taken from the cookie jar. The students are to determine how many cookies are now in the cookie jar. While these tasks did not explicitly reference the arithmetic rack, some of the students chose to use the arithmetic rack whereas others produced purely conceptual thinking strategy solutions. By this point in the sequence, students were encouraged to record their solution processes so that other children could understand their thinking. These records, which combined conventional and non-conventional elements, then became an explicit topic of conversation in whole class discussions.

With regard to RME, it was hoped that the arithmetic rack would initially serve as a *model of* the double decker bus scenario as students solved various tasks. Later it was conjectured that the arithmetic rack would function as a *model for* their arithmetical reasoning as they solved a variety of problems until, eventually, the students might use various thinking strategies without either imagining or using the arithmetic rack.

Data Analysis Procedures

The data analysis procedures progressed in three phases. The first phase of analysis focused on the individual interview sessions for all 18 children and sought to determine the nature and the quality of their solution methods. As a part of this process, changes in the students' solution methods between interview sessions were documented. The analysis drew on psychological constructs developed by Steffe et al. (1983) and Steffe and Cobb (1988), as well as those developed by Neuman (1987). Constructs developed by Steffe and his colleagues were used to identify the various counting strategies the students used. Neuman's work proved to be relevant in that she has reported detailed accounts of non-counting solutions that involve patterning and grouping. In the second phase of analysis, in-depth case studies were developed of four of the students. The case studies provide snap shots of the four students' conceptual progress over time and complement the summary of all 18 students' mathematical activity.

The third phase of analysis, which is currently in progress, involves analyzing the videotaped mathematics lessons from September through December together with and three lessons conducted in January. The normative aspects of the classroom participation structure will be documented first. This will include identifying the general social norms and sociomathematical norms to account for the classroom microculture. This will then be taken as the local social situation within which both the classroom mathematical practices and the four target students' individual mathematical meanings evolved.

Findings

Analysis of the interview sessions indicates that all 18 students made considerable progress in the ways that they interpreted and attempted to solve a range of tasks. One of the most profound finding relates to the shift that many of the students made from using counting strategies to using non-counting strategies (e.g., going-through-ten, referencing doubles). Further, the flexibility with which the children used these strategies to solve a wide variety of tasks indicates that they carried the significance of acting on arithmetical objects. During the September interview session, all of the students employed various counting strategies that ranged from counting-all using their fingers to counting-on or counting-down, subvocally. At least six of the students experienced considerable difficulty when they attempted to solve small number sentences (e.g., 5 - 3 = ___) posed in a story context. Further, only two students spontaneously employed non-counting strategies to solve large number sentences (e.g., related 6 + 8 = ___ to 6 + 7 = 13). Four other students employed non-counting strategies for small number sentences when prompted to do so by the interviewer (e.g., related 4 + 3 = ___ to 4 + 4 = 8). By way of contrast, during the December interview session, 11 students used non-counting strategies routinely to solve all or almost all the tasks and a further 5 students used non-counting strategies some of the time to solve a range of tasks.

These developments can be illustrated by considering the solution methods employed by one of the students, Lori, in the September and December interview sessions. In the September interview session, Lori frequently counted-on or counted-down to solve tasks by using spatial patterns or using sophisticated finger patterns to know when to stop counting. For example, Lori gave an answer of 13 for the number sentence 7 + 6 = ___. When prompted, she explained, "Um, I have a 7 and ...I...I remembered the pattern how that would be one dot here and one dot here and one dot here and one dot here and that would make 4." She went on to explain "So I count 7...6...8, 9, 10, 11 [as she pointed to make a spatial pattern for four and said the number words 8, 9, 10, 11, synchronously] 12, 13 [pointed to two more locations as she said the number words 12 and 13]." By way of contrast, during the December interview session, Lori routinely used non-counting strategies such as referencing doubles and going-through-ten to solve a variety of tasks. For instance, she immediately gave an answer of 17 to the horizontal number sentence 8 + 9 = ___. She first explained that she "broke-up" the eight into seven and one (i.e., 7 + 1 = 8) and added the 1 and the 9 to make 10. She then said that she had 7 left and that made 17 (i.e., 10 + 7 = 17). Lori appeared to have made a significance shift from counting and referencing spatial patterns in the September interview session to routinely employing non-counting strategies in the December interview session. Her progress is representative of the progress made by most of the other students in the classroom.

Conclusion

In this paper, we have attempted to provide an overview of a first-grade classroom teaching experiment that was conducted during the 1993-1994 school year. These preliminary remarks have been offered, in part, as a way of documenting these research efforts. We have provided a general summary of the students' progress over the first several months of the school year in which two instructional sequences, the Patterns and Partitioning and the Arithmetic Rack were developed in collaboration with the classroom teacher. The findings reported here do not, however, document the *process* by which the children's arithmetical reasoning evolved. The next phase of the analysis will involve accounting for the students' conceptual progress as they participated in and contributed to the evolution of the classroom mathematical practices established by the classroom community. In completing this analysis, we hope to articulate possible ways of supporting young children's arithmetical learning as it occurs in the social setting of the classroom.

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CLASSROOM SOCIOMATHEMATICAL NORMS AND INTELLECTUAL AUTONOMY

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The analysis presented in this paper focuses on normative aspects of mathematical discussions that are specific to students' mathematical activity. This extension of our previous work on general classroom social norms that sustain inquiry-based discussion and argumentation to sociomathematical norms places special emphasis on the mathematical aspects of the mathematics classroom. In the process, we clarify how students come to develop a mathematical disposition and account for students' development of intellectual autonomy in mathematics. In addition, the teacher's role as a representative of the mathematical community is clarified.

The purpose of this paper is to set forth a way of interpreting classroom life that aims to account for how students develop specifically mathematical beliefs and values and consequently how they become intellectual autonomous in mathematics. To that end, the focus is on classroom norms that we call sociomathematical norms (Yackel & Cobb, 1993). These norms are distinct from general classroom social norms in that they are specific to the mathematical aspects of students' activity.

Sociomathematical Norms

In the course of our work, we have collaborated with a group of second- and third-grade teachers to help them radically revise the way they teach mathematics. Instruction in project classrooms typically consists of teacher-led discussions of problems posed in a whole class setting, collaborative small group work between pairs of children, and follow-up whole class discussions where children explain and justify the interpretations and solutions they develop during small group work. The approach we have taken reflects the view that mathematical learning is both a process of active individual construction (von Glasersfeld, 1984) and a process of acculturation into the mathematical practices of wider society (Bauersfeld, in press).

Our prior research has included analyzing the process by which teachers initiate and guide the development of social norms that sustain classroom microcultures characterized by explanation, justification, and argumentation (Cobb, Yackel, & Wood, 1989; Yackel, Cobb, & Wood, 1991). Norms of this type are, however, general classroom social norms that apply to any subject matter area and are not unique to mathematics. For example, students should ideally challenge others' thinking and justify their own interpretations in science or literature classes as well as in mathematics. In this paper we extend our previous work on general classroom norms by focusing on normative aspects of mathematics discussions specific to student's mathematical activity. To clarify this distinction, we will speak of *sociomathematical* norms rather than social norms.

Sociomathematical norms include normative understandings of what counts as a *different solution*, a *sophisticated solution*, and an *efficient solution* and what counts as an *acceptable explanation and justification*. Issues concerning what counts as different, sophisticated, and efficient

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solutions involve a taken-as-shared sense of *when* it is appropriate to contribute to a discussion. In contrast, the sociomathematical norm of what counts as an acceptable explanation and justification deals with the actual *process* by which students contribute. In this paper we restrict the discussion of sociomathematical norms to the latter.

Since teachers with whom we collaborated were attempting to establish inquiry mathematics traditions in their classrooms, acceptable explanations and justifications had to involve described actions on mathematical objects rather than procedural instructions (Cobb, Wood, Yackel, & McNeal, 1992). For example, describing manipulation of numerals would not be acceptable. However, it was not sufficient for a student to merely describe personally-real mathematical actions. Crucially, to be acceptable, other students had to be able to interpret the explanation in terms of actions on mathematical objects that were experientially-real to them. Thus, the currently taken-as-shared basis for mathematical communication served as the backdrop against which students explained and justified their thinking. Conversely, it was by means of mathematical argumentation that this constraining background reality itself evolved. We will therefore argue that the process of argumentation and the taken-as-shared basis for communication were reflexively related.

The theoretical constructs were developed by analyzing data from a second-grade classroom in which we conducted a year-long teaching experiment. Data from the teaching experiment include video-recordings of all mathematics lessons for the entire school year and of individual interviews conducted with each student in the class at the beginning, middle, and end of the school year. Copies of students' written work, and field notes are additional data sources. Sociomathematical norms are established in all classrooms regardless of instructional tradition. In this paper we limit our discussion to classrooms that follow an inquiry tradition because our purpose is to indicate the potential of the constructs for clarifying how students might develop mathematical beliefs and values that are consistent with the current reform movement and how they become intellectual autonomous in mathematics.

Theoretical Perspectives

Our theoretical perspective is derived from constructivism (von Glasersfeld, 1984), symbolic interactionism (Blumer, 1969), and ethnomethodology (Leiter, 1980; Mehan & Wood, 1975). We began the project intending to focus on learning primarily from a cognitive perspective, with constructivism as a guiding framework. However, as we attempted to make sense of our experiences in the classroom, it was apparent that we needed to broaden our interpretative stance by developing a sociological perspective on mathematical activity. For this purpose, we drew on constructs derived from symbolic interactionism and ethnomethodology. We were then able to account for and explicate the development of general classroom social norms. These same constructs proved critical to our development of the notion of sociomathematical norms.

Bauersfeld (1988) and Voigt (1992) have elaborated the relevance of interactionist perspectives for mathematics education research. A basic assumption of interactionism is that cultural and social processes are integral to mathematical activity. This view, which is increasingly accepted by the mathematics education community (Cobb, 1990; Eisenhart, 1988), is stated succinctly by Bauersfeld (in press).

> [T]he understanding of learning and teaching mathematics ... support[s] a model of enculturation rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using mathematics. The many skills, which an observer can identify and will take as the main performance of the culture, form the procedural surface only. These are the bricks of the building, but the design for the house of mathematizing is processed on another level. As it is with cultures, the core of what is learned through participation is *when* to do what and *how* to do it. ... [T]he core part of the school mathematics enculturation comes to effect on the meta-level and is 'learned' indirectly. (p. 24)

In this view, the development of individuals' reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meanings.

Voigt (1992) argues that, of the various theoretical approaches to social interaction, the symbolic interactionist approach is particularly useful when studying children's learning in inquiry mathematics classrooms because it emphasizes the individual's sense making processes as well as the social processes. Thus, rather than attempting to deduce an individual's learning from social and cultural processes or vice versa, it treats "subjective ideas as becoming compatible with culture and with intersubjective knowledge like mathematics" (Voigt, 1992, p. 11). Individuals are therefore seen to develop their personal understandings as they participate in negotiating classroom norms, including those that are specific to mathematics.

The construct of reflexivity from ethnomethodology (Leiter, 1980; Mehan and Wood, 1975) is especially useful for clarifying how sociomathematical norms and goals and beliefs about mathematical activity and learning evolve together as a dynamic system. Methodologically, both general social norms and sociomathematical norms are inferred by identifying regularities in patterns of social interaction. With regard to sociomathematical norms, what becomes mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. At the same time these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity. It is in this sense that we say sociomathematical norms and goals and beliefs about mathematical activity and learning are reflexively related.

The Interactive Constitution of What Counts as an Acceptable Explanation and Justification

To elaborate the notion of sociomathematical norms we consider how the teacher and students in an inquiry mathematics classroom interactively constitute normative understandings of what counts as an acceptable explanation and justification and thus extend and clarify their taken-as-shared basis for communication. Viewed as a communicative act, explaining has as its purpose clarifying aspects of one's (mathematical) thinking that might not be apparent to others.

A Mathematical Basis for Explanations

A preliminary step in children's developing understanding of what constitutes an acceptable mathematical explanation is that they understand that the basis for their actions should be mathematical rather than status-based. Developing this preliminary understanding is not a trivial matter, especially since children are often socialized in school to rely on social cues for evaluation and on authority-based rationales. For example, in many classrooms it is appropriate for a child to infer that his answer is incorrect if the teacher questions it. In the classrooms that we have studied, one of the expectations is that children explain their solution methods to each other in small group work and in whole class discussions. However, most of the children were experiencing inquiry-based instruction for the first time and has little basis for knowing what types of rationales might be acceptable. In their prior experience of doing mathematics in school their teachers were typically the only members of the classroom community who gave explanations. They were therefore accustomed to relying on authority and status to develop rationales. For example, early in the school year one child attempted to resolve a dispute about an answer during small group work by initiating a discussion about who had the best pencil, then about which of them was the smartest. This attempt to use status rather than a mathematical rationale to resolve the disagreement is consistent with the way many children interpret traditional mathematics instruction, as arbitrary procedures prescribed by their classroom authorities, the textbook and the teacher (Cobb, Wood, Yackel, & McNeal, 1992; Voigt, 1992). In project classrooms teachers capitalize on situations that arise naturally in the classroom to make children's reasons an explicit topic of discussion. In general, such interventions are successful in establishing the expectation that rationales should be mathematical.

Explanations As Descriptions of Actions on Experientially-Real Mathematical Objects

A more complex issue than establishing that mathematical reasons should form the basis for explanations, is *which types* of mathematical reasons might be acceptable. Here reflexivity is a key notion that guides our attempt to make sense of the classroom. We argue that what constitutes an acceptable mathematical reason is interactively constituted by the students and the teacher in the course of classroom activity. In the classroom studied the children contributed to establishing an inquiry mathematics tradition by generating their own personally meaningful ways of solving problems instead of following procedural instructions. Further, their explanations increasingly involved describing actions on, what to them were, mathematical objects. In addition, children took seriously their obligation to try to make sense of the explanations of others. As a consequence, explanations that could be interpreted as relying on procedural instructions or used language that did not carry the significance of actions on taken-as-shared mathematical objects that were experientially real for the students were frequently challenged. These challenges in turn gave rise to situations for the teacher and students to negotiate what was acceptable as a mathematical explanation. The following illustrative episode, which occurred two months after the beginning of the school year, clarifies how the sociomathematical norm of what is acceptable as a mathematical explanation, is interactively constituted.

Example: The episode begins as Travonda is explaining her solution to the following problem.

Roberto had 12 pennies. After his grandmother gave him some more, he had 25 pennies. How many pennies did Roberto's grandmother give him?

At Travonda's direction, the teacher writes

$$\begin{array}{r} 12 \\ \underline{+\,13} \end{array}$$

on the overhead projector. Thus far, her explanation involves specifying the details of how to write the problem using conventional vertical format. She continues.

Travonda: I said, one plus one is two, and 3 plus 2 is 5.

Teacher: All right, she said ...

Rick: I know what she was talking about.

Teacher: Three plus 2 is 5, and one plus one is two.

Travonda's explanation can be interpreted as procedural only in nature. She has not make explicit reference to the value of the quantities the numerals signify nor clarified that the results should be interpreted as twenty-five. Furthermore, in repeating her solution, the teacher modifies it to make it conform even more closely to the standard algorithm by proceeding from right to left. Several children simultaneously challenge the explanation.

Jameel: (Jumping from his seat and pointing to the screen.) Mr. K. That's 20. That's 20.

Rick: (Simultaneously) Un-uh. That's twenty-five.

Several students: That's twenty-five. That's twenty-five. He's talking about that.

Jameel: Ten. Ten. That's taking a 10 right here ... (walking up to the overhead screen and pointing to the numbers as he talks). This 10 and 10 (pointing to the ones in the tens column). That's 20 (pointing to the 2 in the tens column).

Teacher: Right.

Jameel: And this is 5 more and it's twenty-five.

Teacher: That's right. It's twenty-five.

Both Rick's challenge that the answer should be expressed as twenty-five, rather than as two single digits, and Jameel's challenge that the 1's signify tens and the 2 signifies 20 contribute to establishing the sociomathematical norm that explanations must describe actions on mathematical objects. Further, by acknowledging the challenges and accepting Jameel's clarification the teacher legitimized the ongoing negotiation of what is acceptable as an explanation in this classroom.

As a communicative act, explanation assumes a taken-as-shared stance (Rommetveit, 1985). Consequently, what constitutes an acceptable explanation is constrained by what the speaker and the listeners take-as-shared. But, as the above example shows, what is taken-as-shared, is itself, established during class discussions. Further, our analyses of discussions across the school year document that what is taken-as-shared mathematically evolves as the year progresses. Here, Jameel's clarification assumes that the conceptual acts of decomposing twelve into ten and two and of decomposing thirteen into ten and three are shared by other students. Individual interviews conducted with all of the children in the class shortly before this episode occurred indicate that for a number of students this was not the case. Thus, while Jameel's explanation made it possible for him to orient his own understanding to Travonda's reported activity, it may have been inadequate for others.

Explanations as Objects of Reflection

When students begin to consider the adequacy of an explanation for others rather than simply for themselves, the explanation itself becomes the explicit object of discourse. During classroom discussions, it is typically the teacher's responsibility to makes implicit judgments about the extent to which students take something as shared and to facilitate communication by explicating the need for further explanation. As students' understanding of an acceptable explanation evolves, they too may assume this role. To do so, they must go beyond making sense of an explanation for themselves to making judgments about how other children might make sense of it. This involves a shift from participating in explanation to making the explanation itself an object of reflection. This shift in students' thinking is analogous to the shift between process and object that Sfard (1991) describes for mathematical conceptions. In the same way that being able to see a mathematical entity as an object as well as a process indicates a deeper understanding of the mathematical entity, taking an explanation as an object of reflection indicates a deeper understanding of what constitutes explanation. Analysis of the classroom data document that as the school year progressed a number of students made this shift and focused on the explanation itself as an object. Children began to challenge each other's explanations on the basis of clarity and potential to be understood by others in the class.

Intellectual Autonomy

The development of intellectual and social autonomy are major goals in the current educational reform movement, more generally, and in the reform movement in mathematics education, in particular. In this regard, the reform is in agreement with Piaget (1948/1973) that the main purpose of education is autonomy. Prior analysis shows that one of the benefits of establishing the social norms implicit in the inquiry approach to mathematics instruction is that they foster children's development of *social autonomy* (Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, & Perlwitz, 1991). However, it is the analysis of *sociomathematical norms* implicit in the inquiry mathematics tradition that clarifies the process by which teachers foster the development of *intellectual autonomy*.

Students who are intellectually autonomous in mathematics are aware of and draw on their own intellectual capabilities when making mathematical decisions and judgments (Kamii, 1985). The link between the growth of intellectual autonomy and the development of an inquiry mathematics tradition becomes apparent when we note that, in such a classroom, the teacher guides the development of a community of validators and thus encourages the devolution of responsibility. However, students can only take over some of the traditional teacher's responsibilities to the extent that they have constructed personal ways of judging that enable them to know-in-action both when it is appropriate to make a mathematical contribution and what constitutes an acceptable mathematical contribution. But, as we have attempted to illustrate throughout this paper, these are the types of judgments that the teacher and students negotiate when establishing sociomathematical norms that characterize an inquiry mathematics tradition. In the process, students construct specifically mathematical beliefs and values that inform their judgments. For example, Jameel's challenge to the teacher when he said, "one and one is two" rather than "ten and ten is twenty," illustrates that children are capable of making judgments about what is appropriate mathematically. Further, Jameel's challenge indicates that he had developed the belief that mathematical explanations should describe actions on experientially-real mathematical objects. Examples such as this show that it is precisely because children can make personal judgments of this kind on the basis of their mathematical beliefs and values, that they can participate as autonomous members of an inquiry mathematics community.

Significance

The notion of sociomathematical norms that we have advanced in this paper is important because it sets forth a conceptual framework for talking about, describing, and analyzing the *mathematical* aspects of teachers' and students' activity in the mathematics classroom. Sociomathematical norms are intrinsic aspects of the classroom's mathematical microculture. We have demonstrated that these norms are not predetermined criteria introduced into the classroom from the outside. Instead, they are continually regenerated and modified by the students and the teacher through their ongoing interactions. As we have shown, in the process of negotiating sociomathematical norms, students in classrooms that follow an inquiry mathematics tradition actively constructed personal beliefs and values that enabled them to be increasingly autonomous in mathematics.

The notion of sociomathematical norms is also important for clarifying the teacher's role as a representative of the mathematical community. The question of the teacher's role in classrooms that attempt to develop a practice consistent with the current reform emphasis on problem solving and inquiry is one of current debate. Many teachers assume that they are expected to assume a passive role (P. Human, personal communication, August 1994). However, we question this position. As we have stated previously,

> The conclusion that teachers should not attempt to influence students' constructive efforts seems indefensible, given our contention that mathematics can be viewed as a social practice or a community project. From our perspective, the suggestion that students can be left to their own devices to construct the mathematical ways of knowing compatible with those of wider society is a contradiction in terms. (Cobb, Yackel, & Wood, 1992, pp. 27-28)

The analysis of sociomathematical norms indicates that the teacher plays a central role in establishing the mathematical quality of the classroom environment and in establishing norms for mathematical aspects of students' activity. In this way, the critical role of the teacher as a representative of the mathematical community is underscored.

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A FRAMEWORK FOR ASSESSING TEACHER DEVELOPMENT[1]

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This paper describes a framework for analyzing children's mathematical behavior in three contexts: (1) a non-routine problem task working with a partner in a classroom setting; (2) an open-ended task-based interview about how they built those solutions ; and (3) a series of problems in a second task-based interview involving fractions. The framework was the basis for an assessment of the impact of a long-term teacher development project in an urban district according to (1) the processes by which the students build solutions: their use of heuristics, models built, representations constructed; (2) language used to communicate solutions; (3) students' ability to be metacognitive: generating descriptions, explanations, and predictions for other mathematical problems, and reflecting on their own problem-solving capabilities; and (4) the richness and depth of their solutions.

For the evaluation of an in-service teacher education intervention, it was necessary to go beyond the usual test scores, and to create a framework that gave greater emphasis to those goals judged to be most important. Two guiding principals were: 1) the focus of meaningful evaluation of teacher development should be the student, since the ultimate goal is to improve student learning; and 2) student performance must be examined according to a number of dimensions including cognitive, metacognitive, and affective domains. A complete story would certainly include a thorough discussion of the teacher development intervention itself, the design and implementation of the assessment, and its results. Since that discussion far exceeds the constraints of this forum, this paper will be limited to describing the framework of the assessment, with a brief discussion of the context, and an analysis of a specific example of children's work as an illustration of one of the dimensions that were chosen .

If one accepts the premise that the goal of instruction is not simply to get students to master rules and procedures, but rather to encourage deeper and higher order understandings, documenting mathematical "success" in students must be done in a way that goes beyond merely examining standardized test scores (Maher, 1991). While standardized test scores have provided a relatively cheap, fast and easy to use format for testing, they measure, to a large extent, recall and the applications of facts, rules, and procedures. Romberg, Wilson, and Khaketia (1991) conducted studies comparing the types of evaluation advocated in the 1989 National Council of Teachers of

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Mathematics report *Curriculum and Evaluation Standards for School Mathematics* (1989) with existing standardized tests. They concluded that traditional tests "are not appropriate instruments for assessing the content, process, and levels of thinking called for in the Standards" (p.3). According to Lesh, Lamon, Behr, and Lester (1992), assessment information should be taken from a variety of contexts, including clinical interviews and classroom observations that focus on the mathematical behaviors exhibited by the children during these activities.

<u>What are some of these key mathematical behaviors?</u>

Characteristics of deeper and higher-order student understanding, identified by Lesh and Lamon (1992) as particularly important to document, include students' ability to: (1) go beyond the problems given, generating descriptions, explanations, and predictions for other mathematical problems, (2) investigate the structural properties of specific models or complete systems; (3) use a variety of alternative representation systems that go beyond the execution of procedural rules; (4) construct, refine, plan, monitor, and assess their own thinking; (5) investigate similarities and differences among various problem situations; and (6) form accurate and productive beliefs about their own mathematical ability.

Lester (1982) asserts that, since the ability to use a variety of heuristics is useful in solving mathematical problems, it is important to notice not only when students use heuristics, but also the appropriateness of their use. Various heuristics used by students engaging in authentic problem-solving situations that have been documented (Landis and Maher, 1989; Goldin, 1982) include strategies such as thinking of a simpler problem, working backwards, looking for patterns, and drawing diagrams.

Behaviors which reflect the mathematical representations children have built can also be an important indicator of their depth of knowledge. Representations, according to Davis and Maher (1990), "[i]n general... mean primarily mental representations..." (p.65), although, paper, pencil, and physical materials such as manipulatives can be used by the student to demonstrate their thinking. In order to assess the mathematical understanding of children, it is useful to analyze the mental representations that they form while solving problems.

<u>In what contexts should these behaviors be studied?</u>

"...[M]athematics is increasingly coming to be seen as a social and collaborative act" (Schoenfeld, 1992, p.344). Schoenfeld further points out that mathematical collaboration and communication have a very important role in students developing a sense of "membership in a community of mathematical practice" (p. 344). For students to develop powerful representations of mathematical situations, adequate class time must be provided, and the use of small groups is advocated (Noddings, 1990; Cobb, Wood, & Yackel, 1990; Maher & Alston, 1990;). Vygotsky (1978) suggested that children who work together in small groups, can begin to internalize the talk that occurs within their groups, and challenge themselves and their group mates to provide reasons for their thinking and work. They can begin to monitor their own mental work when they are forced to justify, explain, or defend a solution or process. In addition, there is research to show that as children perform a greater number of group problem-solving explorations, they develop greater persistence, and become less dependent upon the teacher for affirmation and support (Fennema & Leder, 1990). Group work can also provide opportunities for novice problem solvers to see additional, and in some cases, more sophisticated problem-solving strategies used by their peers (Lesh & Zawojewski, 1988).

Clinical interviews with individual students to determine the level of sophistication of their knowledge can provide a valuable source of assessment information. Davis (1984) makes specific reference to the *task-based interview* as a format in which the interviewer makes use of a specific protocol with enough flexibility built into it to address and follow up on the student's responses. By questioning a student in this manner, the interviewer can go beyond gaining information about modes of thought, but also determine the degree of conviction and the method of validation of the student's response (Ginsburg, Kosssan, Schwartz and Swanson, 1983).

How do we capture and document these behaviors?

Analysis of videotapes has proven to be a valuable tool, in a number of studies, for assessing how students think about mathematical problems, and for recognizing student's representations (Alston, Davis, Maher, and Martino, 1994; Kumagai, 1993; Davis, Maher, and Martino, 1992).

DESIGN OF THE ASSESSMENT

The assessment involved a comparison of students taught for three consecutive years by teachers who had successfully completed the in-service project (Strand A) with students taught by teachers who had not (Strand B). All activities were designed to determine the *stability* of the child's understanding, his or her *overall attitude* about mathematics, and *ability* to:

a. *demonstrate understanding* of mathematical ideas using various representations;
b. *make connections* among ideas, problems and representations;

c. *recognize similarities and differences* between different problems and representations;

d. *connect formal concepts and procedures* to other circumstances, especially those involving real-life or everyday situations; and,

e. *reflect metacognitively* in a useful way about his or her own mathematical activity.

The problem-solving behaviors of both groups of children were carefully observed and videotaped during three activities. The first activity involved small groups of two or three children working together within a whole classroom setting to solve an open-ended mathematical problem. The problem-solving task was non-routine, designed to allow students opportunity to apply various strategies, develop their own notations and representations, and create a convincing argument to defend their solutions. The same mathematics educator, unaware of the Strands which each classroom represented, conducted sessions in the classroom of each of the twelve children.

Activities two and three of the assessment were task-based interviews in which two external interviewers, unaware of the Strands to which each of the students had been assigned, interviewed each of the twelve students. The first task-based interview was open-ended and based upon the child's reflections on the classroom activity. This interview provided an opportunity for the child to talk about the classroom problem-solving activity. The child was encouraged to discuss alternative strategies; justify his own or another student's thinking and/or solutions; and make connections and extensions to other problems. The second task-based interview was designed to present the child with problems whose content is typically associated with the sixth grade curriculum. The particular focus was on a number of ideas relating to fractions. This interview was designed to examine the child's ability to connect formal concepts and procedures to other circumstances, especially those involving real-life or everyday situations.

Twelve sixth grade students were chosen for the study, six belonging to Strand A and six matching students from Strand B. Careful attention was paid to matching each pair of students (one from Strand A and the other from Strand B) according to their ethnicity; the type (socioeconomic status, community ethnicity, etc.) of school neighborhood where they lived; gender; and third grade California Achievement Test scores in both reading and mathematics.

Each of the activities was video-taped with two cameras, one focusing on the child (or pair of children in the classroom task) and the other on the work that the child was doing. The data for analysis for each child included the videotapes of that child in each of the three activities and his or her written work completed during each session.

An instrument was developed and tested for use in the analysis, and analysts are now in the process of analyzing the data for each of the twelve children. Two analysts, independently, complete an analysis of the entire set of activities for each matched pair of children, document their conclusions and develop a rated profile for each child, after which the two analysts compare their analyses, discuss differences and either come to agreement or else document areas of disagreement.

THE FRAMEWORK

In order to assess the children's mathematical behavior from cognitive, metacognitive, and affective perspectives, the following list of dimensions were defined as a basis for analyzing the data:

1. Ability of student to go beyond the execution of procedural rules;
2. Richness and depth of solutions;
3. Content-specific aspects of each problem and/or activity;
4. Student's ability to be metacognitive in a useful way;
5. The nature of the representations that are constructed;
6. The student's ability to investigate the accuracy and goodness of fit of the descriptions that are generated;
7. The student's ability to go beyond the problems given;
8. The effectiveness of the student's use of language and communication;
9. The student's ability to work cooperatively with others;
10. The student's expectation that mathematics is a thoughtful endeavor and that solutions to particular mathematical problems should make sense.

The instrument developed for the assessment was built around these dimensions to guide the review and analysis of the videotaped and written data. It additionally was intended to provide a basis for the establishment of inter-rater reliability.

Several sub-categories of questions in the instrument are associated with each dimension. The analysts were instructed to respond to each question for each child, documenting the response with actual episode(s) from the videotape transcript or excerpts from the written data. One sub-category of questions in the instrument refers to the dimension "Ability of the Student to Go Beyond The Execution of Procedural Rules".

To give concrete meaning to the analysis of this dimension, we reproduce the instructions for the section from the instrument given to the analysts.

1. The child explains fractions mainly in terms of: (check as appropriate)

____ *Symbols without relation to meaning.*

____ *What the symbols mean.*

____ *The meaning of fractions with little or no use of symbols. (For instance, the child can draw pictures, or use concrete materials, but is unable to use symbols in a meaningful way.)*

This section is intended to distinguish between a child mindlessly following rote rules and procedures and one who is in search of meaning. The meaning can be made still clearer by looking at two examples of student written work. The problem task given to the students was:

Jeannie is puzzled by the problem

$$6 \div \frac{1}{2}$$

Can you write something to help Jeannie understand?

The language "Can you write something ..." was carefully chosen to give students no indication of what kind of explanation was being asked for, so that the student's response will give information on what that student thinks is involved in "understanding". Shoshana interpreted this in terms of written algorithms:

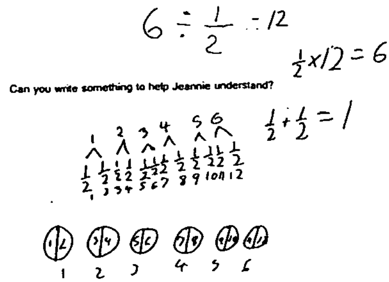
Can you write something to help Jeannie understand?

First you can make the whole number 6 into $\frac{6}{1}$. Next you have to find the recipical of $\frac{1}{2}$ which is $\frac{2}{1}$. Then you multiply numerator by numerator. And denomiter by denomiter.

$$\frac{6}{1} \times \frac{2}{1} = \frac{12}{1} = 12$$

Data from the transcript of the session confirmed that knowing the "rule" or algorithm was what Shoshana meant by "understanding". When asked why the algorithm had the form that it did, she could only answer that that was what the teacher had said; she found the notion of a mathematical justification unfamiliar, foreign, and meaningless.

By contrast, Alec wrote



showing that, for Alec, the mathematics was a meaningful response to a meaningful question. (Indeed, Alec gave both a "real-world" meaning, in terms of "how many half pizzas can you get from 6 whole pizzas", but also gave a mathematician's meaning as "the inverse of multiplication".)

This report has not described the teacher education intervention, nor the final results of the evaluation (which are not yet available). It has dealt with the third part of this work: the construction of a method for assessing the outcome. Preliminary results, however, indicate two things:

- The evaluation gave substantial differences in the performance of different students:
- Based upon the preliminary analysis of the data derived from the instrument, Strand A students (students taught by Project teachers) outperformed their Strand B counterparts (students taught by non-Project teachers) by a wide margin.

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PREPARING TEACHER-CLINICIANS IN MATHEMATICS

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Abstract

We contend that the preparation of specialist teachers in assessment has not received due attention within the teaching profession. This paper outlines one attempt to remedy this situation, currently underway at La Trobe University. The research reported here is very much 'action research' and we begin by describing the requirements for good student assessment using the Initial Clinical Assessment Procedure for Mathematics (ICAPM). Details of the initial phase of development of a professional development program for teacher-clinicians and the program's current state of development are presented, together with implications for the future gained from experience with developing such programs.

Introduction

In recent years there has been considerable attention focussed upon issues of assessment, accountability, and standards (see for example Noss & Dowling, 1990; Ellerton & Clements, 1994). A critical issue in student assessment is the extent to which the use of an assessment tool will directly assist in the advancement of students' mathematical education. We would argue that this has two implications; first the availability of powerful assessment tools, and second, teachers with expertise in the use of such tools.

We would also argue that clinical tools are *the* most powerful available, as Hunting (1993) noted:

> Clinical approaches to assessment beat other approaches hands down in this regard. The reason is that the data source (the student) and the data analyser and interpreter (the teacher-clinician) can engage directly in interactive communications. The teacher-clinician 'reads the play' as the 'play' proceeds (p. 8).

However the mere existence of such tools is insufficient. Hence the need for highly skilled teachers who can make effective use of these tools. In what follows we will discuss the features of a professional development program which aims at creating skilled teacher-clinicians. The approach outlined emphasises the primacy of teachers becoming expert at understanding, at a non-trivial level, what *general* mathematics knowledge structures and competencies might be expected from students of a particular age and background, but not treating this information as normative in respect of any particular child; making interpretations and judgments concerning an individual child's level of competence with respect to particular domains of mathematical knowledge; creating or adapting suitable mathematical learning environments to advance the student's progress in specific directions (Steffe, 1990).

A model program for training teacher-clinicians

The model we use is consistent with paradigms for teacher development which emphasise 'action-reflection' approaches (Jaworski, 1992; Labinowicz, 1985; Schon, 1987;). It is also sensitive to research on teacher change (Fullan & Stiegelbauer, 1991). We believe teachers are best trained initially by using 'hands-on' methods under the supervision of other highly skilled personnel and with the assistance of and support of other 'would-be' specialists. The 'hands-on' stage should not entail working in a school setting to begin with, but would certainly involve practical work with students. Consequently a small group of teachers would undertake clinical experiences associated with appropriate theoretical reflection at a university site. As Hunting et al. (1991) have argued:

> Teachers whose use of numerical concepts and procedures has become routine over the passage of time need to understand the constructive processes of children as they attempt solutions to mathematical problems. They need to observe first-hand the behaviour of children under particular conditions and in particular contexts (p. 172).

The model requires a clinical setting — a facility that provides opportunity for experienced teachers (the would-be clinicians) to observe, unobtrusively, a highly skilled supervisor working with a student. Video records of sessions would be a great advantage because critical utterances and actions would be able to be replayed and discussed following an interview. Following the 'hands-on' stage would be further practical experience, this time an 'on-the-job' stage, where teacher-clinicians would be set the task of building further on their experiences, in a school, more intensively with students, and less intensively with expert supervision. The teacher-clinicians would still meet regularly together to discuss cases, and the supervisor would make visits to the school sites to assist the teacher-clinician 'on-the-job'. In fact, we would see this aspect of the program creating a 'community of practice' (Lave & Wenger, 1991) among its participants. A further outcome of the 'hands-on' stage and the subsequent 'on-the-job' experiences is that the teacher-clinicians develop an improved, abstracted mathematics curriculum (Steffe, 1990). The teacher-clinicians progressively builds up a more complex and functionally detailed general model of students' mathematics knowledge.

A summary of the approach that guides such a program can be encapsulated in the words *observe*, *interpret*, and *act*. We make a distinction between observation and interpretation, since individuals may observe the same event or behaviour and 'see' different things. Interpretation of a sequence of events is an outcome of an assimilation of sensory experiences into an individual's experiential framework. The sense an individual makes of that sensory data is a construction of that individual (von Glasersfeld, 1984). The object of the interpretation is the construction of a model of the mental processes of the student that give rise to the observed behaviour. Our task is to assist teacher-clinicians to attend to, that is notice, those behaviours that reveal clues about the student's mathematical constructions. Noticing is a combination of observation and interpretation. Without an adequate theoretical foundation teacher-clinicians will not attend to those aspects of the sensory data impinging on them that indicate critical qualities of a student's understanding of mathematics.

The need for the teacher to act appropriately seems obvious. We consider that there are three modes of action. One mode is consequent upon interpretation. Having evaluated the situation carefully the teacher-clinician moves to design a teaching strategy that takes advantage of the student's present knowledge state. The second mode of action is a result of observation and *unresolved* effort to interpret. In this case steps are taken to engage the student in further dialogue, possibly involving a different problem, to clarify what the student understands. The third mode results from experience. Experiences of working with children may cause the teacher-clinician to *re-interpret* behaviour. The teacher-clinician's theoretical framework is fluid and developing. Often children's behaviour will not fit existing expectations for that child, or any child, with which she has previously worked. This is usually due to the teacher-clinician's theoretical framework not being adequate enough to account for the possible behaviours that children might exhibit. The re-interpretation is made possible through mental restructuring on the part of the teacher-clinician. As such, mental restructuring is an action. It is a third mode of action and perhaps the most crucial action of all, since teacher growth and effectiveness depends upon it.

It is of great advantage if the process of development of skills of observation, interpretation, and action can be shared between colleagues. This is why we consider it necessary to encourage the notion of a community of practice, in which apprentices and more expert professionals together engage in the craft of clinical assessment and assistance.

In summary then, teacher-clinicians would work with students using interviews designed to allow consideration of interpretations and recommendations for action. The goal of this 'hands-on' work is to bring teacher-clinicians face-to-face with real examples of particular difficulties students have in learning mathematics. Each teacher has to construct a model of the student's mathematical understanding for herself; further, we would expect cases of particular difficulties to link with broader classes of conceptual difficulties.

A pilot program for training teacher-clinicians

In 1991 a pilot program to train teacher-clinicians was established at La Trobe University and undertaken by students as part of their Bachelor of Education course.

The program was designed to fuse the theory and practice of clinical approaches to assessment and assistance in mathematics. Initially the teacher-clinicians spent a high proportion of class time considering the tools and methods of clinical interviewing. This was done by discussing pertinent literature, including protocols of interviews, viewing video records of clinical interviews, and observing experienced interviewers at work. The teacher-clinicians were introduced to the Initial

Clinical Assessment Procedure for Mathematics – Level B (ICAPM-B) which had been developed specifically for use in clinical interviews (Doig, Hunting, & Gibson, 1993; Gibson, Doig, & Hunting, 1993; Hunting & Doig, 1992; Hunting, Doig, & Gibson, 1993a; 1993b). Opportunities were then provided for the teacher-clinicians to interview each other, and as a group, discuss aspects of their performance as they replayed the interview video record.

Practical work with children was an integral component of the program. After negotiations with a nearby school, a number of parents agreed to bring on campus their Year 6 children for an hour after school each week for seven weeks. Thus the teacher-clinicians were able to work in pairs to assess and teach an individual child. In addition to the children arranged by the course lecturers, the teacher-clinicians were each required to work in their own time with a child whom they were responsible for identifying. Work with the second child was conducted off-campus at a time mutually convenient to both parties. A minimum of five meetings were to be held with each child. Teacher-clinicians were required to prepare a case report for each child which detailed background information about the child, pertinent questions and tasks given, together with responses elicited and interpretations of behaviour. A set of recommendations outlining possible further action was also to be provided.

The sessions were structured into a briefing time, an assessment-teaching time, and a debriefing period. The debriefing period was very important as it was here that the teacher-clinicians and lecturers discussed aspects of recent episodes with children. Sometimes the stimulus was a replayed video segment, or a problem that presented itself to one of the teacher-clinicians. At other times the focus was on a teaching technique pertinent to one of the children, or on resources and materials that might be used profitably. Since the 1991 program, an action research methodology (act-reflect-modify-act cycle) has developed the initial program to the current version, last conducted in 1994.

A developed program for training teacher-clinicians

Three years of modifying the program due to suggestions from participants as well as reflections on experience by the authors, has led to many changes to the original concept. A major feature of the 1994 program in Clinical Mathematics Method was that the program was divided into three two-day workshops, spread over a semester. A surviving aspect from the pilot program was that the teacher-clinicians were taken from a strictly supervised interview situation through to operating independently.

The emphasis of the first two-day workshop was on theory and background. This included discussion of the reasons why children fail at mathematics, including major hurdles that students overcome in the course of their mathematical education, and identification of discontinuities between informal and formal mathematics knowledge and their causes. This was followed by an overview of the teaching-learning process, including transmission versus constructivist approaches to learning and teaching, the central role of communication, social interaction, and language, the role of reflection in the development of abstract thinking, and the subjectivity of linguistic meaning, with reference to the work of Cobb, Yackel, & Wood (1992), Steffe (1991), and von Glasersfeld (1990, 1991). Discussion of characteristics of the skilled teacher-clinician centred on an explication of the observe-interpret-act rationale explained above. The major focus of the second day was on the theory and practice of clinical interviews, beginning with a brief history of the technique as a research tool, followed by discussion of its features. Prior to discussion of specific techniques of conducting and recording clinical interviews, the teacher-clinicians observed an interview take place in real time through a one-way mirror. One of the program presenters conducted the interview with a volunteer student. A video record of that interview was available for reference in the subsequent discussion. Observation of the 'live' interview facilitated the discussion on interview techniques because it provided a reference point for many of the techniques discussed. After this the history and use of the Initial Clinical Assessment Procedure for Mathematics – Level B (ICAPM-B) was reviewed. The workshop concluded with a case study of a student, in which the teacher-clinicians watched a video record of an interview, discussed the case, considered a written report that had been prepared for parents of the student, and discussed appropriate strategies for follow-up teaching based on the assessment records and report.

The second two-day workshop, several weeks later, focused on work with individual students, and had as preparation an exercise with the aim of the teacher-clinician critically examining their personal interview skills. The exercise was for each teacher-clinician to video record an interview conducted by herself with a student, and using the principles and techniques of the clinical interview method as a reference, analyse and critique their personal style. A further interview with a different student, using the same set of ICAPM-B tasks was then conducted, for gauging how their interview technique improved, needed modifications, or even proved resistant to change. Another feature of this workshop was the provision of students with whom the teacher-clinicians could practice their interviewing skills. Materials needed for administering the ICAPM-B tasks were made available by the program presenters, and each interview was video recorded for later reference.

On the first day the focus was on assessment and on the second day teaching. The teacher-clinicians worked in pairs for assessment. One took responsibility for interviewing using the ICAPM-B task set; the other acted as an observer and record-keeper. After this the teachers swapped rôles, and a different student was interviewed. Some teachers preferred to replay their own interviews to assist in planning instruction for the following day, when each teacher-clinician had responsibility for implementing two teaching sessions with one of the students they had assessed the previous day. Time was allocated to reflection and review of the assessment data at the end of day one. A plenary session was held after the teaching session had concluded. In this session teacher-clinicians shared their recent experiences and specific issues and problems were contributed and discussed.

The focus of the third two-day workshop was on the construction of a written, diagnostic report based on an initial assessment, as well as consolidation of clinical interviewing skills. The practical value of this exercise was to deal with the eventuality that the clinician in a school situation might not always be able to follow up each student they interviewed with an intensive program of instruction. Hence the importance of preparing a clear and useful communication that could be understood and acted upon by another teacher. To this end students were again available and the teacher-clinicians worked alone conducting an initial assessment with a student. Teacher-clinicians prepared a written report based on their interview observations and records. The reports were than exchanged, so that each teacher-clinician received a report prepared by a peer. Based on the information provided in the report, the teacher then prepared and implemented an instructional session with the appropriate student.

What we have learned so far

Cases referred to mathematics clinics are often accompanied by little specific information. Expertise in assessing the mathematics knowledge of individuals demands the intelligent use of qualitative procedures and skilled interpretation. As Steffe (1984) says:

> The only way that I know to understand the mathematical reality of children is to interpret the language and actions that can be observed as they engage in activity that could be called mathematical and, on the basis of those interpretations, make decisions about what new knowledge the children might construct, how they might construct it, and what aspects of 'old' knowledge need to be refined and consolidated (p. 7).

As we have already said, provision of diagnostic tools for mathematics teachers to use, and the training of teachers with advanced clinical skills go hand in hand, because it is the way in which a tool is applied that will determine its effectiveness. Clinical approaches to assessment have an important advantage over other methods because the data source and the data interpreter can engage directly in interactive communications.

Teachers who have undertaken the program as it has evolved over the past several years have worked with student clients under considerable pressure. They have been placed in situations in which it is often not possible nor appropriate to engage well-rehearsed strategies or adopt tried and true methods. As a consequence it has been common to observe them fall back on ingrained methods such as telling students, or providing direct information rather than questioning, posing easier tasks, or helping them realise the consequences of their mathematical beliefs. In addition, they lack confidence in their ability to interpret situations. They need as many opportunities as possible to develop deeper understandings of the theory and research, and also curriculum wisdom.

Our dilemma is that development of research-grounded understandings for interpreting students' responses to questions and tasks is a life-time professional activity.

We are conscious of the need to provide more one-to-one contact between supervising staff and teacher-clinician. This is difficult, because when the teacher-clinicians are working with students, they need to feel free to attempt strategies and take risks without having someone always looking over their shoulder. More time may need to be spent outside the teaching sessions to review and to plan in consultation with the program leaders. The program needs to be extended so that additional support can be provided to teacher-clinicians in their classrooms and schools as they seek to adapt their skills in their particular contexts.

Teacher-clinicians are seriously hampered by lack of a good mathematical background – what we mean is not that they are unable to do mathematics, but that they do not have a good grasp of the fundamental concepts of elementary arithmetic. It would be advantageous to provide an overview of the key mathematical ideas, as well as the significance of those ideas, that students need to learn in primary schooling and in the early years of secondary education. It would be even more advantageous for teachers undertaking the program to have recently reviewed fundamental principles of the learning of elementary mathematics, where if substantive research is not available, then at least there is some consideration of available curriculum wisdom. An important reason for reviewing fundamental principles of mathematics pedagogy is that over time different pedagogical approaches are stressed and new insights into learning are made. Some review of critical problems faced by students learning elementary mathematics would be beneficial. Thus a common set of understandings about how students learn mathematics needs to be established in order to begin the work of training successful clinicians.

In summary, our experience has raised three major issues. First, that teachers require greater knowledge of mathematics *per se*, and mathematics pedagogy beyond that of their initial training. Second, with structured guidance and 'hands-on' experiences, teachers can gain in confidence and technique to become more effective observers, interpreters, and actors in the drama of advancing children's mathematical development. Third, the act-reflect-modify cycle, whilst effective in developing a professional development program, has no end!

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THE TENSION BETWEEN CURRICULUM GOALS AND YOUNG CHILDREN'S CONSTRUCTION OF NUMBER: ONE TEACHER'S EXPERIENCE IN THE CALCULATORS IN PRIMARY MATHEMATICS PROJECT

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The contradiction posed by traditional curriculum statements and a constructivist view of learning is problematic for classroom teachers. As part of a long-term study into the effects of the introduction of calculators on the learning and teaching of primary mathematics, this paper uses data from four interviews, over a three year period, to report on how the classroom availability of calculators resulted in one infant teacher being confronted by children's sometimes unexpected and surprising knowledge, the tensions created between her previously determined curriculum goals and the children's construction of number, and her reconceptualisation of her own classroom practice.

Introduction

Constructivist theories of learning have gained widespread acceptance amongst mathematics educators. However, whilst it is relatively easy to accept a *trivial* constructivist position that knowledge is actively constructed by the learner, rather than passively received, von Glasersfeld (1990a, p. 27) argues that adoption of *radical* constructivism requires formidable changes in thinking and attitudes. In particular, the task of education "can no longer be seen as a task of conveying ready-made pieces of knowledge to students, nor, in mathematics education, of opening their eyes to an absolute mathematical reality" (von Glasersfeld, 1990b, p. 33). The contradiction posed by traditional curriculum statements, which often appear to be based on the assumption that the goal of mathematics teaching is to transmit prescribed knowledge, and a constructivist view of learning is problematic for classroom teachers (Cobb, 1988; Hart, 1992; Boufi, 1994).

This paper reports on how one infant teacher was confronted by children's unexpected and sometimes surprising explorations of number in the *Calculators in Primary Mathematics* project and the resulting tensions between her previously determined curriculum goals and the children's construction of number.

The *Calculators in Primary Mathematics* project was based on the premise that calculators, as well as acting as computational tools, have the potential to radically transform mathematics learning and teaching by providing a mathematically rich environment for children to explore.

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In 1990, all children at kindergarten and grade 1 levels in six schools were given their own calculator to use freely in class. The project followed these children through to grade 4 level in 1993, with new children joining the project each year as they started school. In total, about 80 teachers and 1000 children participated in the project. Teachers were not provided with classroom activities or a program to follow, instead they were regarded as part of the research team investigating the ways in which calculators could be used in their mathematics classes. Feedback and support was provided through regular classroom visits by members of the project team and through teachers sharing their activities and reflecting on their practice at regular half-day meetings and in the project newsletter.

Findings relating to positive long-term learning outcomes for project children – based on a large scale program of testing and interviews – and changes in teachers' expectations – based on an extensive written questionnaire – have been reported elsewhere (Groves & Cheeseman, 1992, 1993b; Groves, 1993a, 1994a, 1994b; Groves & Stacey, 1994; Stacey, 1994).

Teacher change was a focus for research in two of the project schools. The project was based on models of teacher professional growth which attribute changes in knowledge and beliefs to teachers' reflections on changes in their own classroom practice (Guskey, 1986; Clarke & Peter, 1993). An underlying hypothesis of the project was that the introduction of the calculator would greatly enhance children's development of number concepts and thus confront teachers with the need to re-examine their beliefs and practice. Interviews with seven teachers over a three year period showed that all reported substantial changes in their teaching practice, with all seven believing that their teaching had become more "open-ended" (Groves, 1993b).

This paper attempts to explore the role of the calculator in effecting change by investigating the tensions created for one of these seven teachers, referred to here as Barbara.

Background and methodology

At the time the project commenced, children in Victoria (Australia) commenced school at kindergarten (called preparatory or prep grade) aged between 4 $^1/_2$ and 5 $^1/_2$ years. While there was no centralised curriculum, most schools based their mathematics curriculum statements on state guidelines. Unlike many schools, which developed their curriculum statements in local clusters, the school at which Barbara teaches had just completed a thorough review of its own mathematics curriculum, making substantial changes. For example, it had been decided that the symbols for the four operations would no longer be introduced in kindergarten. Barbara had played a key role in the development of the new mathematics curriculum.

The school, which is located in a middle-class suburb of a large city, has an excellent academic reputation in the local community, where it is sometimes referred to as "the private [i.e. fee-paying] school you send your children to when you don't send them to a private school". This reputation is based on the high quality of its teachers – including Barbara, who is recognised by the Principal as being a particular drawcard for parents of kindergarten children – and the emphasis placed on curriculum development. The author had known Barbara for about 8 years prior to the commencement of the project and had worked with her previously. One of the reasons for inviting the school to participate in the project was the fact that both Barbara and the other infant teacher, referred to here as Sharon, were held in such high regard and were so enthusiastic about participating.

The project commenced in April 1990 – the beginning of term 2 of the school year. All teachers in the project were interviewed at the beginning of their involvement in the project (which in 1990 only took place in June, but in later years early in term 1) and at the end of each year of their involvement. Each of the 30 – 40 minute semi-structured interviews included questions related to a range of issues such as: background information; initial reactions to the project; teachers' expectations of the project; effects of calculators on the children; effects of calculators on mathematics teaching; the support program; and effects of the project on the school. Interviews were tape-recorded and transcribed. For the purpose of data analysis, a set of categories, under headings indicated by the original research questions, were developed using an iterative process. Responses were analysed in terms of these categories. All responses which related to a particular category were recorded, whether or not they were given in response to the question designed to address that issue.

There are many features in common between Barbara's responses and those of the six other teachers whose interviews have been analysed in detail. However, it is in Barbara's interviews that the tensions between existing curriculum goals and teachers' growing awareness of children's conceptual development are best articulated – half of the 66 responses relevant to this analysis referred to curriculum issues. There is no attempt made here to cover all aspects of the four interviews – rather the discussion focuses on the main themes which have emerged from the data analysis. While this paper is based on the interview data, it should be noted that the conclusions drawn are also supported by over 50 classroom observations and video tape of a lesson (see Groves & Cheeseman, 1993a, for excerpts).

Barbara, who had been teaching for approximately 20 years when the project commenced, had recently participated in a professional development program in mathematics and was always eager to change. In line with most practising teachers, she described her teaching in terms of practice rather than theory, listing the use of materials and allowing children to "discover things for themselves and then discuss it afterwards" as the main features of her teaching (Interview 1). Although the project's theoretical framework was consistent with a constructivist view of learning, the project did not present teachers with any particular theoretical framework, but instead attempted to encourage teachers to reflect on their (necessarily altered) practice.

Barbara had never used calculators in the classroom, had no pre-conceived ideas about what might happen, and said that what most interested her about joining the program was "free calculators ... and also being able to talk to other people about it, and the chance to try something that I hadn't tried before" (Interview 1). She was teaching a combined kindergarten/grade 1 class in 1990, and kindergarten only classes in 1991 and 1992.

The role of the calculator

The project team had hypothesised that curriculum goals would be challenged by the presence of the calculator, as children would encounter large numbers, negative numbers and decimals at an earlier age. However, an unexpected outcome was the extent to which Barbara (and most other project teachers) made frequent reference to the role of the calculator in providing opportunities for sharing and discussion about number – for example:

> It certainly encouraged me to talk to the children much more in maths, and discuss how did they do this, why did they do that, and get them to justify what they're doing, which I guess, previously, I hadn't done in maths – much more discussion and sharing. (Interview 1)

Part of this sharing and discussion was a result of a need created by the altered nature of the classroom environment, where children engaged more in exploratory activities, which then required explanations:

> I think the more open-ended things, where they can go and choose a different way to do something – more exploratory activities – I think the calculator has been fascinating in that sort of use. Especially getting children to discuss what we've done – I think it's been a great tool to have there and say "What did you press there?" or "Why did you do that?" or "What did you find out?" or "How did you do that? Let's share your ideas". (Interview 4)

The fact that the children were engaged in, often lengthy, independent explorations also enabled teachers to spend more time sitting with individual children, observing and interacting with them. Barbara (and many other teachers) spoke of the calculator "revealing" children's knowledge:

> You find out what they know – you may not have found out previously – with the calculator – I mean unless you actually sat and questioned them. (Interview 2)

Children were not only exceeding teacher expectations, but were now being allowed to bring into the classroom their mathematical knowledge from outside:

> I really didn't know what I had in mind that the children would do, but I think what has actually happened is that the children have gone ahead much further than I thought they would have.... There won't be an artificial ceiling put on what they're doing ... the brighter children ... they're showing me what they can do rather than me teaching them. I might have not known what they could have done before, because when they're discussing what they've done and how they've worked it out, they're bringing into the classroom what they know from outside of school. (Interview 1)

Nevertheless, Barbara was concerned at the start about those children she described as "weaker" and whether or not, in the long term, they were going to get a lot more from using calculators than they would have otherwise. She also found at the end of the first year that "there are about two or three very timid preps ... who are a little bit anxious from time to time – they do not necessarily feel very relaxed about it" (Interview 2).

Children's construction of concepts related to number

From the outset, the children's interaction with a calculator rich environment challenged Barbara's preconceived curriculum goals – for example:

> I've never taught formally addition before – I haven't actually taught it formally this year, but the children are writing it down formally – with preps. I've done it in grade one, and we've done it orally with preps before, but I haven't encouraged them to write it down, and this time it's just happening. I've said to them "Write down what you've done" and they're just writing down equations about it and I'm encouraging them to draw it and

write it at the same time as using their calculator, so that I can see that they understand it. So that's why that aspect has changed .... So we're coming to the symbol first, and then we're working out what that means. Mostly because of the availability of the calculator, they just go to it straight away, and they bring in their knowledge from at home, and then they discuss it with each other, so everybody's using the sign. (Interview 1)

Children's (often quite sophisticated) construction of number concepts did not necessarily fit well with attempts to "transmit" knowledge (albeit on this occasion it was a parental attempt):

> Tim was interesting because it was 20 take away 2, and he got into all sorts of strife because he was taught at home the business about re-grouping and he didn't quite know what to do because the two was smaller than the zero, so he ended up taking zero from two. Now he's a really smart clever kid and then he got the wrong answer. He said "I don't know how to do it. I know the answer is 18, but I don't know how to do it".... He knew it was wrong and he wrote down all these wrong answers then he said "I'm really mixed up because I know the answer is 18 but I can't get it, I can't work it out." This was very interesting. We did say to him "What is it?", but nobody is really quizzing him to know what was in his head .... You see the fact that it was vertically set out – he couldn't work out why he couldn't do it that way. Yet he knew the answer. (Interview 2)

Some kindergarten children's number concepts were quite remarkable:

> I guess in reading numbers a lot of them are really concerned with "How many zeros are there in a million?" and they are having a go .... This is just the top lot. The other ones are hearing it, and that is what I think is important about the sharing time, they're hearing somebody say how many zeros are important, however many it is. I don't think it has sunk in to many of them, which is fair enough. I think a lot of them are really talking about how you can read those larger numbers. (Interview 4)

By the end of the third year, Barbara had not only accepted the fact that there was a real spread in her class, but was also thinking less in terms of curriculum content being either "covered" or "not yet covered" and much more in terms of mathematical understanding being an "on-going, growing, process by which one responds to the problem of re-organising one's knowledge structures" (Kieren & Pirie, 1994).

Tensions

About 10 weeks into the project, Barbara was already seriously challenged by the conflict between the exploratory nature of working with the calculator and her perceived need to "cover the curriculum":

> I guess we've done a lot more exploratory work too, which I hadn't really done in the past, and I'm finding it just a bit difficult to balance the two. Whether to keep going on the exploratory work, or to come back and do things that are very separate from all of that – like with the preps, making two groups of six and six groups of five and all that sort of thing, which I've always done in the past and there are always some children who find that very difficult – I'm not too sure now whether to continue with the more exploratory work, or to continue with that and play safe by going over the other things about the grouping activities that I'd normally do. I'm a bit torn between the two .... I guess I'm sort of walking the line between two ways of teaching maths. I did start to change after I'd done the [professional development] course, and now, again, using the calculators, I'm starting to change a little bit more .... But I'd just like to have someone else say it's okay to do that – maybe eventually you'll find you don't need to do the more formal things. They're all things that I've considered, but I'm not prepared yet to throw out the more

formal things and go one way. I would like a little bit more talking about that aspect of things in the support group. (Interview 1)

At the end of the first year, although she was happy with what the children had achieved, Barbara was still concerned about the "gaps" in the curriculum:

I also worry that we cover all those other structured things that we always go through – have we covered all the basic things? Actually Sharon and I are the ones who have gone back to a lot of things we might have slipped on. Maybe it's that time of the year. You are thinking it's getting towards the end of the year – are there any gaps? (Interview 2)

From the outset, curriculum goals were also being challenged by the notion that the children were covering topics from higher grade levels – for example:

I am not sticking to what has been the guideline for us in the past. As we started off I tried to stick to what was going on and then go out from there. Then I went further out and now I am almost coming back a bit because there seems to be some controversy between other grade levels. But we haven't been sticking to exactly what's been put down for us to do, we've been going further than that. (Interview 4)

The project team had anticipated that the presence of calculators would have implications for children's representations and recording. However, children's use of calculators also resulted in them making constructions which were in conflict with those of the teacher – for example:

I guess that's something that's been a bit different this year really, we haven't talked so much about "this equals this" and what that means ... other years we spend a lot of time on that....We were so much tuned in previously to make sure that they really understood what equality was: that 7 + 3 = 10 – that 7 and 3 is the same amount as 10. If they are pressing that equals button it's not the same. It doesn't mean the same thing any more, I guess. I don't really know what the children know themselves. We say "is" when we are pressing it, although the children will still use "equals". I guess it is something we have not really addressed a lot. We were horrified to see them writing down 7 + 3 = = = [when using the constant function to count by 3's] and all that sort of thing. When we first saw it we thought "What's going on? Can we have that?" We were worried about it, but we didn't stop them doing it. (Interview 2)

At the end of the third year, she had still found no resolution to this problem:

I think one of the things that has come up as an area of difficulty has been the use of the equals sign. I find I haven't spent as much time as I would have previously – before calculators – talking about what equality means. I don't know whether it is important or not, but I think it is certainly important somewhere along the line. I usually do it a lot in prep and I don't do it much at all now. I somehow can't come to terms with how I talk about that in the equation form and how I talk about it when they are using the constant function. I've tended to back out of it, which I don't necessarily think is a good thing. (Interview 4)

Reconceptualising classroom practice

Based on the work of Clay (1979) and others, many Victorian teachers have made significant changes over the past 15 years in the way they view the learning and teaching of language. Barbara (in common with many other project teachers) expressed the desire to make – and subsequently claimed to have made – her mathematics teaching more like her language teaching:

I've had to really encourage them to share what they've done. I think I've always done that in language, but I haven't really done that very much at all in maths before. What I see in the use of calculators and how it's changed my maths teaching is that I think I'm teaching maths now more in a way that I've been teaching language for a while. I've always taught maths much more formally – still with use of materials, and still with problem solving, but not as exploratory as what I'm doing now with calculators. (Interview 1)

By the end of the second year, a significant change appears to have occurred in Barbara's beliefs about mathematics – she no longer thinks in terms of teaching children the "one right way":

I think that happened last year and it's continued this year. I think that's something that really started with the use of the calculator – the idea of "What did you find out?" and "How did you do it?" and that there isn't one right way. I'm not teaching them that "This is the way you do it" but "How did you work it out?" .... Maths is very much more in line with the way I teach language I think now .... We always do a lot of sharing in language. Now we do in maths, which previously was "This is the way we do it" – everybody will do it and it's either right or it's wrong, and that's it. But now I think, I much prefer to work that way too [in maths]. It feels much more natural and much more the way that I like to teach. (Interview 3)

For Barbara, the emphasis has changed from teacher direction to a more child centred approach, which takes into much greater account children's constructions of the number concepts:

I don't think I've been as concerned as to what the children know – apart from what I want them to know – before. I think it has opened things up a little bit more and I'm more interested in them telling me what they know ... having the calculator in the classroom has helped that – just finding out from children what they know, rather than me teaching.... There will still be times when I do direct things, but the emphasis is much more on them finding out and exploring and then sharing. (Interview 4)

Overall, although she elsewhere refers to other possible causes for change, Barbara believes that the presence of the calculator has helped her achieve the type of change in her mathematics teaching which she had been seeking:

I've done a lot of changing in language and I haven't changed much in maths for ages really ... I think [the calculator] gave me something I was happier with and it encouraged me to share, to build on what the children were finding out – which is what I've done in language – and it made the two areas work a little bit more in a similar style, which I hadn't been able to do before. I think that helped. (Interview 4)

Conclusion

A constructivist view of learning applies to teachers as well as children. In order to move from a transmission view of communication, teachers need to undergo a conceptual revolution of their own, based on reflection on and abstraction from their own experiences of mathematics learning and teaching (Cobb, 1988; Steffe, 1990).

Steffe speaks of the impressive generative power of children working in environments conducive to constructive activity and states that teachers "have an exciting choice between being participants in specifying the generative power of students or taking what their students can learn as being already specified by an *a priori* curriculum" (1990, p. 395).

The *Calculators in Primary Mathematics* project promoted a classroom environment conducive to such constructive activity by children, which in turn could provoke teachers to reflect on their practice within a social environment provided by the support structures of the project.

This paper has shown how the presence of the calculator has assisted one teacher in gaining insights into children's knowledge, challenged her previously held views on the nature of mathematics learning and teaching, and resulted in a reconceptualisation of her classroom practice. Although Barbara is unique in the emphasis she placed on curriculum issues, many other project teachers also welcomed the opportunity for reflection on their own experiences, constructing their own interpretations of the project and correspondingly changing their beliefs and practice.

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TEACHER STRATEGIES AND BELIEFS IN A COMPUTER-BASED INOVATORY CLASSROOM SITUATION: A CASE STUDY.

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Abstract: *We report a case study investigating a teacher's classroom activities within a school-based research project encouraging an inovatory pedagogy with the use of computer technology. The teacher had had six years in which to form her personal pedagogy-in-practice within this setting. Based on transcripts and video recordings of three hours' classroom activity, a semi-structured inteview and the pupils' written presentations of their work, we analyse her intervention strategies and beliefs regarding her pedagogical role, mathematics learning and the role of computer technology as constructed in the classroom. The results show a clear tendency to embed interventions into pupils activity with reference to procedural, social and content-related issues and a number of comments aiming to encourage reflection smaller but comparable to directive ones. Episode analysis, however illustrates how the school context restricts and shapes beliefs and intervention srategy.*

Theoretical framework

From the days when mathematics education research focused mainly on students and their relationship to mathematical concepts as a two-way interaction, there has been important development in the recognition of the teacher's role in the classroom. Initial interest was on the teacher's actions and their bearing on student performance. This was followed by a shift towards aspects underlying these actions, i.e. teacher attitudes, beliefs, intentions and social constructions of their own and the students' roles as integral elements of classroom learning. In subsequent research there was a further discriminating tendency in two respects: that of disaggregating attitudes and beliefs (to mathematics, to mathematics learning, to the role of the student) and that of distinguishing beliefs studied outside classroom situations to those embedded in classroom practice (Ernest, 1989), recently termed "situated beliefs" (Hoyles, 1992).

Here we describe on-going research into the a) beliefs-in-practice and b) strategies to intervene in pupils' learning processes, developed by nine teachers in a classroom environment in Greece, which was designed to facilitate and encourage inovation in pedagogical practice. In particular, we take the case of one of the teachers to discuss the issues emerging from the analysis. The study involved a detailed observation of the teachers' activity during the seventh year of a longitudinal project in a Greek primary school (Kynigos, 1992) where the main aim of the researchers-as-educators was for the teachers to use Logo as a tool with which to setup and develop an alternative pedagogical paradigm with respect to that of the wider educational setting (Kontogiannopoulou - Polydorides and Kynigos, 1993). Our theoretical orientation concerning learning mathematics, teacher education and curriculum innovation is tightly related to the Vygotskian perspective of social construction of knowledge (Vygotsky, 1978) rather than the Piagetian view of learning as a developing re-organisation of knowledge structures in a continual quest for equilibrium in the environment, which was not explicit about the role of the social aspect of that environment. We therefore see knowledge as constructed through social interaction which, in this case, includes using the computer rather than through interaction with the environment and the computer medium alone. Consequently, we perceive didactical inovation as a process whereby teachers construct meanings within the society of the classroom (Confrey, 1986) and reflect on and reorganise their practice (Olson, 1989) rather than meet objective targets expressed or mediated in a way which is disembedded from the classroom practice of the repspective teacher (Brophy, 1986). Despite our appreciation of the French researchers' focus on the reproducibility of didactical situations rich in opportunity for the pupil to construct meanings (Arsac et al., 1991) we try to allow for the complexity of the classroom situation and keep an open mind regarding the observation of the unexpected.

In the present school project, the computer was used to design classroom environments encouraging inovation (Hoyles and Sutherland, 1989, Noss, 1985), i.e. cooperative small group projects, emphasis on pupils' autonomy from the teacher in decision making, cooperation, active thinking and construction of meaning. During teacher education seminars, there were discussions regarding principles and issues related to learning with Logo as a result of classroom practice, but the researchers did not participate in the classroom and the seminars were not intensive and oriented towards "delivering" a prespecified teaching algorithm. The computer was thus used to facilitate the social construction of knowledge (Cobb et al., 1992, Hoyles, Healy and Pozzi, 1992) amongst pupils and with respect to the teachers. Previous research has shown how used in this way, the computer may provide a window to pupils' thinking processes for teachers and researchers alike (Weir, 1986, Noss and Hoyles, 1992, Hoyles and Noss, 1992b). Regarding the present research it was thus used as a window to the teachers' intervention strategies and their beliefs in practice (Hoyles, 1992).

Background to the Study

Prevailing educational practice in Greece, as well as the social and educational context of the particular study, inevitably influence and shape the teachers' beliefs and conceptualisations about education in general and their own role in particular. The Greek educational paradigm is characterised by an emphasis on content, abstract knowledge, teacher-centred approaches and a lack of systematic pragmatic orientation (Kontogiannopoulou - Polydorides and Kynigos, 1993, McLean, 1990). The domain of each subject matter is strictly defined and presented in the (until recently) unique curriculum textbook. The teacher's role is seen as that of transmitter of the information presented in the book, and the students' role as the receivers of this information, who must memorise it and be able to re-produce it on demand. Learning is thus seen as an individual, rather than a cooperative or group experience. Not surprisingly, perceptions of computer use and relates policies have been technocentric and with little relation to educational priorities and development ("Astrolavos" report, 1992, Kontogiannopoulou - Polydorides and Kynigos, 1993, Plomp and Pelgrum, 1992).

The school project in question, which began in September 1986 and has been based on classroom activity from the outset, involves teacher education and a "Logo curriculum" development. From year 3 to 6 inclusive (i.e., children aged 8 to 12), all 24 teachers of the school are taking part, each one responsible for the participation of all the children in his/her class (500 children in total). The main component around which the program is organised has been called an "investigation". Throughout the four year period, the children engage in informal collaborative investigational work for one teaching period a week and compose a written presentation (including the problems they encountered, ways in which they solved them and how they worked together in their group) on each of their "investigations", which typically lasts for 5 to 6 weeks. Finally, one teaching period is given to the oral presentation of the projects by each group of children, followed by discussion.

Regarding the teachers' role, the explicit focus of the "investigation" hour, as mediated during the teacher education seminars, was to use the technology to set up an unconventional classroom practise; the teachers were left to develop strategies for a pedagogy encouraging collaborative investigations. The explcit pedagogical objectives of the "investigation" hour were: a) cooperation, b) active thinking and c) initiative (the third objective was later reformulated to "autonomy from the teacher"). During the first three years of the project the focus on content was restricted to Logo, followed by systematic suggestions (but nothing more) from the researcher-educator to gradually make more explicit references to school mathematics content which happened to be used during the investigation and to the pupils' written expression. Details of the project's outline, educational objectives, working structure, classroom setup and "taught" content can be found in Kynigos, 1992. Studies involving children's use of programming ideas can be found in Kynigos et al., 1993, Kynigos, in press and involving their learning process in Kynigos 1992, 1993, . Studies of related issues in school settings can be found in Hoyles et al., 1992, Hoyles and Sutherland, 1989.

Methodology

In a setting encouraging an inovatory (with respect to the wider educational paradigm) pedagogy with the use of computer technology and where a long time was given to the teachers to form their personal pedagogies-in-practice, we set out to investigate a) their beliefs as constructed during this specific classroom practice, regarding mathematical learning, their pedagogical role and the role of computer technology and b) their intervention strategies regarding the aspects of the learning situations they refered to, the extent to which they were embedded in the pupils' investigations and the kind of activity they intended to encourage.

Nine of the school's teachers were selected as subjects, chosen so that the classes they teach span all the age groups. They were each observed and videotaped over the three "investigation" teaching periods. The video recordings were used a) to transcribe all their verbalisations in the classroom as well as those of the pupils they addressed respectively, b) to be able to capture and reproduce important episodes including a view of the computer screen and c) to get a feel of the classroom atmosphere in general. The researcher carried the camera and could thus follow the teacher or the action, focus in on computer screens or on people's faces and at the same time, keep a distance so as not to distract. This was made possible with the help of a remote microphone attached to the teacher, so that verbalisations were clearly heard whatever the distance between teacher and researcher. So, the video was used as a combination of a "holistic note taker" and a "silent observer" (Harel, 1991). After the end of the observation periods, semi-strucured inteviews of all nine teachers were carried out regarding their views on and evaluation of the ways in which children learn during the "investigation" hour, how they perceive ther own role and pedagogical strategy and how they compare this kind of pedagogy and learning to the one which goes on during the normal curriculum activities. Verbatim transcriptions of audiorecordings were made. Background data was also collected, i.e. all the pupils' written presentations of their investigations and researcher notes on specific aspects of each particular hour which may have influenced the atmosphere (e.g. a broken down computer).

The data is being analysed in two ways. Firstly, all teacher comments are interpreted by the researchers regarding their intent and characterised accordingly. A first characterisation is related to the aspect of the learning situation the comments refer to and to whether it is embedded in the pupils' activity. The most frequent comments were then characterised with respect to the kind of activity or response they entended to encourage as interpreted by the researchers. The characterisations in this latter part of the analysis were influenced by Hoyles and Sutherland, 1989.

Secondly, episodes important in illuminating further the above characterisations or other aspects regarding the teachers' interventions and in contextualising teacher strategies were identified and used as vignettes (Kynigos, 1993). A vignette involved one or a series of related episodes and is in some cases (as in this report) presented in a descriptive - summative (narrative) way rather than in raw form for efficiency purposes. The teacher interviews and the pupils' presentations of their work served as background data. The interviews in particular were seen as revealing beliefs expressed outside the classroom setting.

Results.

Characterisation of teacher interventions.

The comments were characterised as follows:

Those embedded in pupil activity, with respect to which aspect of the learning situation they refered to, i.e. A the interactions between a group, B preocess - related, C mathematical content associated with the normal curriculum maths, D techie fact, E Logomaths (programming) and being in control of the computer.

Those not embedded in pupil activity, i.e. addressing more than one group of pupils, providing disembedded information, frontal teaching and phatic communication.

Additionally, the teacher's interaction with a group of pupils was made identifiable in the transcripts and each such interaction was characterised with respect to whether the pupils had requested it or it was the teacher's initiative.

Out of the five kinds of embedded comments, we were interested to analyse the four most frequent ones with respect to: whether they were directive (D), in which case we further labelled them as disciplinary (DD), motivational (DM), nudge (DN) or factual (DF), whether they were reflectional, in which case we labelled them according to the intent to reflect on a previous action (Rpre) or a subsequent one (Rpost) and whether they intended to motivate the pupils.

Table A shows the percentage of each characterisation with respect to the total number of comments made during the three hours of observation of this particular teacher.

	TOTAL	**total D**					**total R**			**M**
			(DD)	**(DM)**	**(DN)**	**(DF)**		**(Rpre)**	**(Rpost)**	
A	9.09%	3.57%	0.00%	3.57%	0.00%	0.00%	4.22%	2.60%	1.62%	1.30%
B	35.71%	23.05%	4.55%	15.58%	2.27%	0.65%	9.09%	7.14%	1.95%	3.57%
C	7.47%	1.30%	0.00%	0.32%	0.65%	0.32%	6.17%	4.55%	1.62%	0.00%
D	8.44%									
E	29.87%	11.69%	0.00%	3.57%	2.60%	5.52%	15.58%	10.39%	5.19%	2.60%
F	3.57%									
G	0.00%									
H	0.00%									
I	5.84%									
		39.61%					**35.06%**			**7.47%**
R	31.91%									
U	68.09%									

Table A

A very large proportion of this teacher's comments are embedded in pupil activity (82.14% not including D) and refer to the interaction within groups (9.09%), to procedural issues (35.71%) and to school and Logo maths content (37.34%). Furthermore, there seems to be more or less a balance between directional comments aiming at specific pupil activity and those aiming to encourage pupils to reflect on their past or future actions (39.61% versus 35.06%). In fact, if the procedural comments are excluded (where we have 23.05% directional vs 9.09% reflectional comments), the picture is even more in favour of the reflectional comments (3.57% vs 4.22% for A, 1.30% vs 6.17% for C, 11.69% vs 15.58% for E). It is also particularly interesting that regarding school mathematics content, her comments seem to mainly encourage pupils to reflect. We notice a relatively small proportion of comments intending to motivate the pupils, which would favour the argument that they were engaged in their work. Finally, this teacher would seem to be rather active and tend to be in control in interacting with the pupils since she more or less intervened twice as many tmes on her own initiative than as a result of the pupils request.

Embedded teaching: an episode

This is an episode involving three interventions to a group of 8 year-old pupils, made by the above teacher over a period of two sessions. Her intention seems to be to take advantage of their plan to make a planet next to their rocket, in order to encourage them to investigate how to consruct a circle with the Logo turtle.

The group have finished their "rocket" project, with another hour and a half to go before the end of the investigation. The teacher encourages them to enrich their project in the remaining time, and they suggest making a planet. The teacher agrees and asks them how they will make a circle. The pupils at first say they don't know how, and thus decide they will make a square planet! The teacher urges them to think about how they could make a circle and they decide to try moving the turtle a bit and turning it a bit, many times. They type in moves and turns alternately, but with no pattern to the input quantities. The teacher does not intervene for the rest of that hour. The following week during the first 15 minutes or so, the pupils continue in the same way. At some point however, they have a sequence of equal inputs to the turn commands. The teacher intervenes at this point, suggesting to the pupils to look for a pattern. About 15 minutes later she comes back and asks how they're doing, only to discover that they still have not come to the desired conclusion, i.e. constant turns and moves. She accepts their efforts and asks whether they can predict what shape will result from them, at which point they say that it won't look very much like a circle, as it will have straight bits. She then suggests that they had better rethink about their turns, and points out on the screen the result of their one relatively successful sequence, asking them to rethink their strategy and to compare the results they have had so far. She also asks them not to erase their previous commands, so that they can later reflect upon them, and finally she feels she has to spell it out for them by saying: "see if in this sequence where the turns were the same it was more like a circle". Eventually the students come to the desired conclusion and announce it to her, but they do not change their figure accordingly, nor do they mention the planet in their written essay!

The teacher is obviously attempting here to encourage problem solving activity and some autonomy on the part of the students, even though their final reaction is disapointing. At the beginning she urges them to investigate an interesting problem which has stemmed from their own, self-initiated goal, even though their initial reaction is to avoid it entirely. She allows them plenty of time for trial and error, and intervenes at the crucial moment, when they have approximated a solution (although it turns out that it was by chance) nudging them towards the right direction. Finally, and when they are running out of time, she becomes more heavy-handed and directional, explicitly suggesting that they should reflect and predict, pointing out to them which specific sequence will give them the clue for their solution, and showing them the resulting computer feedback.

She thus seems to be implementing a strategy to influence the learning environment, both to discourage an unreflective use of Logo (observed and highlihted in related research, Leron 1985, Noss and Hoyles, 1992) and to help the children to focus on the interesting and powerful ideas that they use in their projects. However, there does seem to be a negotiating problem here. Nudges and "light" encouragement seemed to bring not much result in pupils perseverence in investigating a mathematical pattern or even in forming a theorem - in - action, to use Vergnaud's term (Vegnaud, 1982). She thus gradually resorts to more directive methods in order to produce the desired result, and still the students do not seem to have perceived this as a useful learning experience. It is interesting to note that such directive interventions as were used towards the end of this episode differ widely from directive interventions as initial teaching strategies, with no previous effort towards influencing students into active problem solving of the own. This is a difference which the quantitative analysis used above is not sensitive to, although certain insights can be reached by comparing the percentage of directional and reflective interventions each teacher has used. The episode also highlights the teacher's use of the computer as an educational tool. She points the students' commands out on the screen, relates them to the computer feedback and urges them to keep their approximatory efforts on screen, so that they can later compare them to the commands that were closer to the desired result, and reach their own conclusions.

During the subsequent interview, the teacher said that her experience with Logo had influenced her overall teaching, that she wished the principles of co-operation, initiative and problem-solving could be carried on to other curriculum subjects also, although she thought that was very difficult, and that the children have benefited greatly from their experience with Logo, which they also transfer to other subjects (e.g. they had learned how to work effectively in small groups etc.). Her own perception of her interventions was that they mainly concerned group dynamics: when asked to describe what she does during the classroom sessions, she said that she walks around the classroom, intervening only when asked to, or when she observes a group of students who are either arguing or caught into a cycle of doing the same thing again and again and getting nowhere, and that when she does intervene, her first concern is to get the children to co-operate. Thus, her beliefs and perception of her actions expressed ouside the classroom situation presents interesting differences to the intervention strategy indicated by her comments in the classroom. This finding corresponds with the research mentioned earlier (see also Sosniak et al., 1991) distinguishing beliefs and beliefs-in-practice and warrants further analysis.

Conclusions.

The case study highlights the complexities of trying to observe and interpret teacher interventions in a classroom situation, since analysis based on a detailed characterisation of each intervention benefits from the in-depth illumination of contextual aspects of specific intervention situations which may be important in forming an overal profile of the teacher's pedagogy. The evidence seems to point to some distance between the teacher's espoused beliefs as expressed outside the classroom setting and those underlying her actual pedagogy, as for instance, the difference between her classroom activity and the importance she attributed to the teacher's role in shaping the interaction within pupil groups. This teacher seems to have constructed an inovatory pedagogy characterised by interventions embedded in pupils' activity and referring to interaction within groups, procedural issues, Logomaths and school maths. Within this context, however, refering to mathematics content and negotiating the validity of mathematical investigation seems infrequent and just one amongst several priorities.

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LEARNING TO TEACH: FOUR SALIENT CONSTRUCTS FOR TRAINEE MATHEMATICS TEACHERS

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ABSTRACT

This factor analytic study maps out four salient constructs for trainee mathematics teachers and proposes an alternative theoretical model for the interpretation of attitudinal research. The sample was drawn from eight cohorts undertaking PGCE courses in the university sector. Views of different aspects of teaching and learning to teach were elicited on two occasions using Likert-item questionnaires (the second questionnaire being a refinement of the first based on results obtained). Empirical results from both questionnaire administrations uphold three theoretical a priori *ideas of theory-practice relationships, apprenticeship learning and pedagogical content knowledge but also elaborate these from the trainees' perspective. In particular, pedagogical content knowledge appears to comprise at least two sub-components. It is suggested that the emergent constructs are worthy of further elaboration through qualitative work and from a social representations perspective.*

INTRODUCTION

This study is part of a larger investigation of how trainee mathematics teachers develop knowledge and expertise within school-based training. To understand trainees' learning it is important to understand conceptions of teaching and learning to teach and how views about these concepts intersect with training experiences..A wide literature asserts that attitudes, beliefs and expectations structure teachers' decisions and classroom behaviours (Cooney 1985, Clark and Peterson 1986, Bromme and Brophy 1986, Brookhart and Freeman 1992, Thompson 1992). These writers suggest that entry beliefs affect cognitive change in teacher education and that orientations then determine the professional knowledge acquired and used by teachers. Such accounts have in common the underlying view that dispositions to act reflect the psychological states of individuals and primarily reflect theoretical models from the professional literature.

However, as a prerequisite for understanding trainee learning, little appears to have been done to map out representations held by trainees as a group against those held by teacher educators. How do trainee teachers conceive of important aspects of learning to teach and how do these constructs relate to our *a priori* ideas? This study therefore sets out to chart some ideas from the professional literature against the representations of teaching and learning to teach held by trainees, with attitudes as indicators of those representations. It aims to clarify useful constructs for describing trainee teachers' views to important aspects of their training and, simultaneously, to develop a reliable instrument for charting these. A large scale survey was considered necessary to generate the statistical information needed to meet these aims.

Although various constructs could have been chosen, this study focuses on three ideas which permeate the literature and which arose in a previous study (Meredith 1992), namely; pedagogical content knowledge (pck), theory-practice relationships (tp) and apprenticeship learning (a).

The relationship between theory and practice appeared to have high face validity for trainee teachers involved in the pervious study and as a construct.was characterised by different aspects of the traditional-radical continuum proposed by Wilkin (1990 p.7). Theory embedded in and emerging from practice is contrasted with theory as a product, independent of and informing practice.

The previous study also indicated that trainees' perceptions of knowledge for teaching mathematics could be usefully explored (see Meredith 1993). This construct, referenced to mathematics, was characterised by different aspects of Shulman's pedagogical content knowledge which is '*that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding*' (Shulman 1987 p.8). Teaching as a process leading to the transformation of mathematical knowledge and an understanding of learners was contrasted with a static view of teaching as the transmission of pre-existing subject knowledge.

Finally apprenticeship learning emerged as a third idea which seemed especially important for those trainees who were undertaking a majority of their training in school. The notion of peripheral participation (Lave and Wenger 1991) and learning through association with more experienced others was contrasted with a more individualistic stance which involves being left alone to learn to teach through trial and error and by reference to past personal experience.

METHODOLOGY

A Likert-item questionnaire, to explore trainees' attitudes towards these ideas was designed in a series of stages. The initial sources of statements for the present study included:

- previous trainees' verbatim comments;
- trainees' comments edited for parsimony and clarity;
- Likert-item statements which previously 'worked well' (for instance, eliciting a range of responses or appearing to resonate strongly with trainees);
- Likert-item statements modified to reflect trainees' responses;
- ideas in the literature.

Together these sources generated around 80 items. Criteria of clarity, simplicity and specificity were used to select a smaller sub-set and these items were then checked for ambiguity and predicted balance of response. After any necessary rewording, item pools of 10 statements for each construct, worded equally in positive and negative directions, were formed. The final questionnaire therefore comprised 30 statements with each item pool evenly distributed throughout.

Respondents were asked to indicate the extent to which they agreed or disagreed with each statement choosing from strongly agree (SA), agree (A), unsure (U), disagree (D) and strongly disagree (SD). Two versions of the questionnaire (A & B) were produced to control for possible sequencing effects. Version B presented the same statements but in reverse order to Version A. Following its first administration, the questionnaire was modified in line with results and re-administered to the same cohorts seven months later.

The subjects of the survey were trainee mathematics teachers on one-year Post-graduate Certificate in Education (PGCE) courses in eight English University Departments of Education (UDEs). This was a large opportunity sample from the university sector representing 27·8% and 22·1% of that trainee population (214 and 170 questionnaires returned for the first and second administrations respectively). The samples comprised every student in each of eight cohorts who was present on the day of administrations and consented to participate and included 148 trainees who completed the questionnaire on both occasions.

ANALYSIS

The data was analysed using the Factor and Frequencies sub-programmes of the Statistical Package for the Social Sciences (SPSS). The main results were obtained from factor analysis with the descriptive statistics used as a basis for refining the Likert items.

Factor analysis is a widely accepted technique for identifying meaningful factors which summarise the patterns of association within the data in a parsimonious way amenable to interpretation. Using this approach items which measure the same construct load on the same factor and are distinguishable from those which measure other distinct factors; the underlying assumption being that factors are uncorrelated. Factor analysis also indicates the minimum number of factors which underlie the variables and the strength of the relationship between each variable and factor. Analysis was iterative and exploratory throughout, in line with the aim of refining and clarifying concepts.

RESULTS

First Administration

The completed questionnaires were coded for 34 variables, the Likert items being scored on a 5 point scale ranging from −2 (SD) to 2 (SA) with 0 indicating uncertainty. It was unnecessary to reverse this scoring for negatively directed statements because item inter-correlations indicate the real relative direction of all statements. A full range of responses was elicited on all but 3 items, the missing categories on these being either SA or SD. The correlation and anti-image correlation matrices confirmed sufficient strength in the relationships between the items and established the appropriateness of a factor model.

Two tests, the percentage of total variance attributable to each factor and the scree plot of eigenvalues (total variances) against each factor indicated that four or five factors were necessary to represent the data. Full solutions for four and five factors were therefore obtained using principal components extraction and orthogonal rotation. Solutions from oblique rotation showed no appreciable correlations between factors and agreed well with those from the orthogonal rotation, suggesting that both the four and five models were generally robust.

Comparison of the 4-factor and 5-factor solutions (restricted to items with loadings > 0·4) revealed identical factor structures except for Factor III (on the 4-factor model) which subdivided into two new factors on the 5-factor model (re-labelled III$_5$ and V$_5$). The relationship of the two models can be represented as follows:

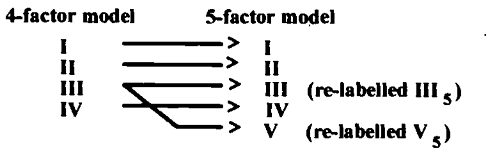


FIGURE 1-The relationship of the 4-factor and 5-factor models

Subsequent analyses confirmed the greater stability and consistency of the 4-factor model. However Factors III$_5$ and V$_5$ were retained because they reveal distinct but weak aspects of Factor III. Table 1 shows the relationship of the questionnaire items to the final solution for the 4-factor model after deleting items with loadings < 0·4 (to ensure that only relatively pure items were included). Because it was the weakest factor and comprised a disparate collection of items, Factor IV was subsequently abandoned. The three remaining factors are interpreted below together with Factors III$_5$ and V$_5$.

TABLE 1–Rotated Factor Matrix for 4-Factors [non-load and low (<·4) load items removed]

Q	‡	Likert-item Statements	Factor I	Factor II	Factor III	Factor IV
28	tp	Ultimately, theoretical aspects of training contribute as much to good teaching as classroom experience	–·66			
5	a	Teaching is a craft skill which is best learnt on the job	·63			
7	tp	The best training is practical and drawn from the 'chalk face'	·61			
19	tp	Educational theory does not hold up in the classroom.	·52			
25	tp	Practical experience of teaching is an adequate training in itself.	·46			
11	a	I prefer to be left alone in the classroom to experiment with my teaching.	·44			
22	tp	A theoretical framework helps teachers to understand and explain learners' development in mathematics	–·42			·34
14	a	Faced with a difficult classroom situation I would try to deal with it in a similar way to the usual class teacher.		·63		
29	a	For me, learning to teach mathematics involves imitating a model teacher of mathematics.		·60		
8	a	You cannot learn to teach by copying an established teacher.		–·58		
13	tp	I should be able to justify my teaching decisions by reference to theory.		·47		
15	pck	My task as a mathematics teacher is to transform mathematical ideas to make them understandable.	·37	·45		
27	pck	My understanding of mathematical ideas has changed in the process of teaching.			·66	
6	pck	My experience of teaching has made me re-examine some of my own mathematical knowledge.			·63	
3	pck	Mathematical knowledge is more important in teaching than knowing how to provide appropriate mathematical activities for pupils.			–·59	
18	pck	Apart from subject matter there is no other knowledge base for teaching mathematics.			–·53	
9	pck	The mathematical mistakes of most learners tend to be arbitrary and illogical.			–·51	·37
23	a	I learn best by making and reflecting upon my own mistakes.				·66
24	pck	It is very hard to anticipate which topics learners will find difficult				·57
21	pck	The ability to analyse mathematical reasoning processes is a necessary part of mathematics teaching.				·48
10	tp	The most valuable educational theory develops from within practice.	·33			·40
		% of variance explained	12·9	10·2	9·0	7·6

Notes:

1. ‡ The original classification of each statement:
   tp theory–practice relationships
   pck pedagogical content knowledge
   a apprenticeship learning
2. Item loadings on the factors are given to 2 d.p.
3. The direction of the loadings for factor III in the oblique solution were reversed in the orthogonal solution because they emerged from the orthogonal rotation 180° out of phase. However, the orientation in the oblique solution is consistent with the meaning of factor III as described later. The signs on the loadings for Factor III have therefore been reversed for all orthogonal solutions and tests reported.

Factor I - Teaching-Craft (TC)
5/7 items comprising Factor I are classified as theory-practice suggesting that this factor represents the relationship of theory to practice entailed in particular views of the nature of teaching. The positive pole sees teaching as a practical craft most appropriately learned through induction as an apprentice on the job (Q5 & Q7). The existence of an underpinning body of professional knowledge seems to be denied (Q28 & Q19); learning to teach is highly idiosyncratic and individualistic and not about acquiring shared knowledge (Q11). Taken together then, these items seem to assert the absence of any implicit or explicit theoretical model for teaching. The positive end of this factor therefore appears to be concerned with teaching solely as a craft skill requiring neither propositional knowledge nor intellectual activity but rather, practical 'know how'.

The contrasting (negative) pole sees theory as upheld at the classroom level (Q19) and values its importance for informing practice (Q28). The existence of a shared body of knowledge which cannot be acquired in isolation or by apprenticeship alone is implicitly asserted (negative responses to Q5, Q7 Q11 & Q25). The negative pole therefore appears to be represent teaching as an intellectual activity drawing upon a collective body of professional knowledge.

Factor II - The Apprenticeship Ideal (AI)
Factor II agrees partly with the apprenticeship construct, having 3/5 items classified as such (Q14, Q29, Q8) These items see learning to teach as a modelling exercise and represent a stereotypical view of apprenticeship entailing observation and imitation of a 'master'. The other items (Q13 & Q15) seem to relate to the trainees' understanding of the elements to be learned and aspirations for the outcomes of training. This may indicate the transitional position of the trainee; imitating a role model is a means to an end and not an end in itself.

Implicit in all items loading on Factor II is the image of an ideal practitioner, possessing both practical expertise and theoretical knowledge. The positive pole of this factor therefore seems to reflect a highly optimistic view of the process and product of apprenticeship based on an idealistic image of the expert teacher. This is termed the Apprenticeship Ideal. It is consistent with the proposition that, by virtue of their position, apprentice learners 'can develop a view of what the whole enterprise is about, and what there is to be learned' (Lave and Wenger 1991 p.93) but probably represents a less realistic position.

Factor III - Pedagogical Content Knowledge (PCK)
Factor III is comprised entirely of items classified as pedagogical content knowledge and is therefore unequivocally concerned with the hypothesised construct. It also appears to be a bi-polar factor. The positive pole concurs with Shulman's idea that a trainee's own understanding of subject matter is transformed through teaching (Q27 & Q6) and learning to represent the subject appropriately for learners (Q3). Learning to match mathematics to the needs of intelligible learners is seen as more important than subject knowledge (Q3 & Q9) which is insufficient in itself (Q18). Hence knowledge of mathematics is located within the knowledge needed for teaching.

The contrasting (negative) pole can be interpreted as a view of the trainees' own subject knowledge as unchanged and unexamined but of paramount importance. Thus the only knowledge really needed for teaching is that of the subject. A view of mathematics as a rigorous, necessarily given body of knowledge may also be associated with this pole, explaining the notion that learners' mistakes are random and irrational. The negative pole of this factor therefore seems to represent being closed to new ways of thinking about mathematics, both at the trainees' own level and for the purposes of teaching. This amounts to the view that there is little to learn in teaching mathematics.

Although based on a limited number of defining items in what appears to be a less stable model, interpretations of Factors III_5 and V_5 illuminate possible sub-components of the pedagogical content knowledge construct. Table 2 shows the relationship of questionnaire items to the final solution for factors III_5 and V_5 from the 5-factor model.

TABLE 2–Extract from the Rotated Factor Matrix for 5–Factors Showing Factor III_5 and V_5 [non-load and low (<·3) load items removed]

Q	‡	Likert-item Statements	Factor I	Factor II	Factor III	Factor IV	Factor V
6	pck	My experience of teaching has made me re-examine some of my own mathematical knowledge			0·82		
27	pck	My understanding of mathematical ideas has changed in the process of teaching.			0·78		
3	pck	Mathematical knowledge is more important in teaching than knowing how to provide appropriate mathematical activities for pupils.			−0·44		0·33
18	pck	Apart from subject matter there is no other knowledge base for teaching mathematics.					0·65
9	pck	The mathematical mistakes of most learners tend to be arbitrary and illogical					0·63
26	a	On the job training offers little room for developing innovative methods	0·35				0·41
30	pck	Pupils find Algebra difficult mainly because it has not been explained clearly enough to them					0·34

‡ The original classification of each statement

Factor III_5 - Self-referenced Learning (SRL)

This factor appears to represent aspects of pedagogical content knowledge which concern the trainees' own knowledge and understanding of mathematics. The association of Q3 with Q6 and Q27 suggests that changes in the trainees' own understanding may be connected with thinking about teaching tasks. Hence, the emphasis is on the trainees' own learning of and about mathematics through teaching the subject. This therefore is a representation of the trainee as a learner of mathematics.

Factor V_5 - Learner-referenced Teaching (LRT)

Factor V_5 complements Factor III_5, emphasising pedagogical content knowledge as it relates to teaching based either on subject content or on meeting learners' needs. Thus the positive pole stresses teaching driven by subject knowledge whilst the negative pole, in contrast, links pedagogical knowledge to understanding learners and knowing how to respond to their difficulties. This dimension therefore describes the trainee as a teacher, either of mathematics or of learners.

Sequencing and Acquiescence Effects

The correlation matrix from a full and unrestricted factor analysis with the questionnaire version listed as the first variable revealed no apparent sequencing effects and examination of the positive/negative balance of the mean scores for all items found negligible sequencing effects.

Second Administration

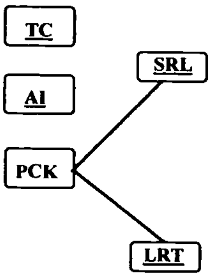
A second 31-item version of the questionnaire was formed by preserving the most strongly loading items on Factors I, II, III_5 and V_5. 11/15 of these 'core' items were retained unchanged since the initial descriptive statistics indicated that parametric assumptions were met. The remaining 4 items were amended slightly to shift the balance of response. New items, attempting to capture facets of the factors as characterised, were then added so that each of the 4 factors had 7 associated items.

Tests on the second questionnaire showed improvements on all criteria and response sets were more evenly distributed as a result of rewording particular statements. Overall, the results from the second administration supported the interpretations proposed with all but four items loading as predicted. The interpretation of TC and AI were strongly upheld by the emergent factor structure and SRL was the strongest factor in the solution supporting the hypothesis that this is a sub-dimension of PCK. Interestingly LRT split into 2 sub components, possibly indicating 2 further dimensions of LRT and thence PCK.

CONCLUSION

Robust and stable factor solutions over 2 administrations demonstrate the validity of four emergent constructs and provide the following strong and sensible model on which the final scale instrument is based:

FIGURE 2-Emergent Constructs



KEY:

TC Teaching Craft
AI Apprenticeship Ideal
PCK Pedagogical Content Knowledge
SRL Self-referenced Learning
LRT Learner-referenced Teaching

—— incorporated into the final questionnaire

The factor structures of the emergent constructs match many aspects of the hypothesised constructs but also capture subtle aspects of trainee thinking which go beyond the *a priori* categories. In particular, from the trainees perspective it appears that:

- ideas about the relationship of theory to practice are related to particular views of teaching; as a craft skill devoid of theory at one extreme and as an intellectual activity, grounded in theory, at the other.
- views of apprenticeship relate to a representation which is stereotypical, optimistic and based upon the image of an ideal, expert teacher.
- trainees' views about PCK represent their openness to learning within teaching which may relate to two sub-domains; the trainees' own learning of mathematics and learning to teach mathematics.

Finally by reinstating 'attitudinal objects' on which attitudinal measurement was originally based this study attempts to work within a theoretical framework which goes beyond the individual. Social representations theory sees attitudes as *'consequences of participation in social life'*

(Duveen and Lloyd 1990 p.3). From this perspective, cognitive representations (reflections of the social world which are shared amongst those who share the same environment) are distinguished from response dispositions which may reflect an individual's personal experience and distinguish him/her from others in the group (Jaspars and Fraser 1984). It is the first of these, now neglected in teacher education literature, which constitutes social reality for the group and influences individual behaviour. In revealing subtle differences in the representations shared by trainees and teacher educators this study points to the importance of social representations for understanding trainee learning.

Current in-depth qualitative work seeks to develop these theoretical ideas whilst elaborating the empirical findings from this study.

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TEACHERS' MATHEMATICAL EXPERIENCES AS LINKS TO CHILDREN'S NEEDS

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As part of our on-going support for teachers implementing a problem-centered approach to mathematics teaching and learning, we run in-service programmes for teachers. Some teacher groups have very weak mathematical backgrounds and very rigid and instrumental perceptions of the nature of mathematics and mathematics teaching. The technique of posing problems that are challenging to the teachers themselves and then encouraging reflection on their experiences has proved to provide a driving force and a network of connections that enable us to address in a very limited time a number of major issues, ranging from perceptions about the nature of mathematics to the practicalities of classroom organisation.

Introduction

Research on different in-service (INSET) programmes and attempts to identify the reasons why some programmes are more effective than others, have led to various descriptions of the immediate outcomes of successful programmes, e.g. Joyce and Showers (1980); Kinder and Harland (1994). Working back from these desired outcomes it is clear that successful INSET programmes should address at least two main issues: firstly, teachers' perceptions (beliefs and attitudes) and secondly, the skills that teachers need for day-to-day classroom activities. Researchers generally agree that both of these are essential before lasting effects can be observed at classroom level: "To master a new approach we need to explore and understand its rationale, develop the ability to carry out the new strategies, and master fresh content" (Joyce & Showers, 1980:380). "They have to understand, at the level of principle, what they are trying to achieve, why they are trying to achieve it" (Rudduck, 1991:92), but "... changed awareness is no guarantee of changed practice." (Kinder & Harland, 1994:36).

There are different ways in which an INSET programme may attempt to address these two issues, depending on which perceptions and which skills are addressed.

We believe that, for mathematics teachers from kindergarten to twelfth grade, the *perceptions* that radically influence their classroom practice concern

- the *nature* of mathematics
- the way mathematics is both *learnt* and *applied* in life
- children's mathematical thinking
- the aims of school mathematics
- how children best learn mathematics, given particular aims.

The necessary *skills* are clearly those that enable the teacher to create and sustain on a daily basis the learning environment which will support the type of learning in children the teacher has come to accept as desirable.

Our own perspectives of the above matters are based on a socio-constructive view of knowledge, and on our continuing research on young children's thinking and on environments which seem to support their thinking. We try to implement these ideas in the classroom through a problem-centered approach to mathematics learning and teaching (Murray, Olivier & Human, 1993).

INSET programmes with similar views on mathematics education may use a particular technique as part of their programmes: Such programmes expose teachers to *doing* mathematics at their own level as a vehicle to encourage teachers to reflect on the nature of mathematics and mathematics learning (e.g. Simon & Schifter, 1991; Hadar & Hadass, 1990; Corwin, 1993). However, much like the ELM programme (Simon & Schifter, 1991), we take it one step further: We actively use teachers' mathematical experiences as the core around which we construct the rest of the programme.

For the purposes of this paper, we limit our discussion to two-day INSET workshops for lower elementary teachers with a very low perception of their own mathematical abilities, who possess only limited skills and little explicit understanding of basic whole number arithmetic.

Organisational information

The number of participants for such a workshop has varied from 34 to 47, and consisted mainly of K–3 teachers, with a sprinkling of upper elementary teachers who function as subject heads for mathematics. Most of the teachers had only had school mathematics up to 9th grade, and some had left school after tenth grade. They all had at least a three-year teachers' diploma. During their school and college years they had, at least for mathematics, been exposed to quite rigidly traditional views of mathematics as a series of set formulae which had to be memorised and then applied to the appropriate word problems. Although it appeared during the workshops that the teachers possessed strong intuitive powers for solving problems they did not experience as school-type problems, these thinking skills had never been sanctioned and initially most teachers were embarrassed to talk about them.

It seems sufficient to attempt only the following in the workshop:

1. Addressing teachers' perceptions about the nature of mathematics and how mathematics is learnt and practiced (used).
2. Addressing teachers' perceptions about their own mathematical abilities and how they (can) do mathematics.
3. Describing and justifying a problem-centered approach to mathematics learning and teaching.
4. Sharing information on some basic guidelines for establishing a problem-centered learning environment in the classroom.

According to our basic technique, the activities that address teachers' personal views (points 1 and 2) also supply us with direct links to children's thinking and children's needs (which we present by means of many examples of children's work and videos of children solving problems), which lead directly to points 3 and 4.

We now briefly elaborate on the main ideas to be covered under points 3 and 4.

A problem-centered approach

In brief, the problem-centered approach implies that the teacher regularly poses problems to her students that the students do not experience as routine problems, and that they have to construct solution methods for the problems with the tools that they have available (theorems in action, number knowledge at different levels of development). Students are expected to share ideas, to discuss, justify and explain among themselves. Although students may (and should) experience classroom events as informal and child-centered, the teacher plans the classroom activities and tasks in accordance with a simple but important set of guidelines that we have been able to formulate through longitudinal research in problem-centered classrooms:

- Certain simple but powerful activities that help students to develop a flexible number knowledge, which directly influences the solution strategies they construct
- Different word problem types that suggest different computational methods—if some problem types are omitted, certain methods will not be constructed. Teachers therefore design their word problems from a list of basic problem types so that the different meanings of the four basic operations and fractions are all covered
- Students mainly learn through voluntary interaction with each other, and not through listening to the teacher, but the teacher has to know that social-type information still has to be supplied to her students (e.g. recording skills, and knowledge involving measurement). The ability to distinguish between the logic of a solution method and the way in which it is recorded is essential for a teacher.

The problems posed to teachers in the workshops

When we use the teachers' own mathematical experiences and their reflections on these experiences as a laboratory to provide clues to (or empathy with) children's needs, the basic assumption is that adults' and young children's responses to novel mathematical situations are sufficiently similar to use in such a way. Simon and Schifter state this categorically: "Teachers' learning can be viewed in much the same way as mathematics students' learning." (1993:312). Although we accept this as probably generally true, we have found that teachers (and other adults) only respond in ways that can be used as departure points for children's thinking when the cannot solve the problems automatically (or mechanically).

The choice of problem for a particular audience is therefore crucial. It is important that the problem situation makes sense, even though it may be ambiguous (this is discussed later). We have never used a puzzle-type problem or investigations, since we do not know whether a situation that has no clear connection with any syllabus content will have the same powerful effect on the teachers. We know of INSET programmes where investigations have been used very effectively, but the informal lore among the teachers themselves has it that when teachers only experience enjoyment with problems that they do not relate to a syllabus, they view these experiences as add-ons, "something you do every Friday."

Furthermore, it should be possible to solve at least the initially-posed problem(s) by direct modeling, i.e. by drawing or a sketch, because direct modeling enables a person to easily resolve an incorrect choice of a drilled method at a deep level. A logical refutation often only serves to strengthen *existing* beliefs about the nature of mathematics.

It should be kept in mind that the problems used create powerful situations only because they suit these particular audiences — other audiences may need different problems.

Some problems and how they are used

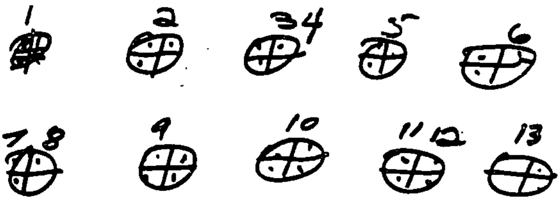
The problem is always presented to the group as a whole, teachers are encouraged to consult with each other, to leave their seats and move around if needed. The presenter moves around, trying to maintain a very low profile, but identifying a variety of different conceptualizations of the problem. Different teachers are then requested to explain on the overhead projector how they had conceptualized and solved the problem. This is followed by a general discussion, eliciting from the teachers the links that need to be established to future topics, or illuminating and emphasising points that will be referred to again. It must be emphasised that these discussions are very thorough and that the teachers really share not only their mathematical thinking but also especially their feelings and fears; i.e. all the factors which could have inhibited or supported their thinking.

❑ The apple tarts

Mrs Daku bakes small apple tarts. For each apple tart she uses $\frac{3}{4}$ of an apple. She has twenty apples. How many apple tarts can she bake?

The most common solution methods generated by the teachers are:

1. Incorrect choice of a drilled method.
   Eva: $20 \times \frac{3}{4} = 15$
2. Direct modeling of the situation.
   Twenty apples are drawn and each is divided into $\frac{3}{4}$ and a $\frac{1}{4}$; the $\frac{3}{4}$ pieces are counted (giving twenty tarts), then the remaining quarters are grouped. Sometimes the remaining quarters are grouped into threes, giving another six tarts with two quarters left; sometimes the remaining quarters from each group of three apples are immediately dealt with.
   Lillian solves the problem in this way, having first reduced the problem to ten apples and afterwards doubling the answer.



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3. Numerical approaches which closely model the problem structure.

   Cassius: "Twenty apples give me at least twenty tarts. With the twenty quarters I make five apples. Five apples give me at least five tarts. With the five quarters left I make one apple. There are two quarters left. I have twenty plus five plus one tarts, and half an apple left."

   Sibongile: "Three apples give four tarts. How many groups of three in twenty? Six groups of three is eighteen. So eighteen apples is equal to 6 × 4 tarts. That is twenty-four. There are two apples left. That's another two tarts and half an apple left."

   Beauty: "I thought about the kitchen. First I cut all the apples into quarters and then I find out how many groups of three I can make. So I do 20 × 4 = 80; 80 ÷ 3 = 26 remainder 2." Beauty knew she could make twenty-six tarts, but she needed prolonged discussion and an inspection of one of the direct modelers' drawings to decide what the remainder of two signified.

These discussions generate a great deal of excitement among teachers, especially when they are informed that from the formal point of view the problem involves division with a fraction, which is only introduced in the local seventh grade school syllabus.

The following important perspectives arise naturally out of the whole episode:

1. Attempts to classify the problem type and choose an operation made the problem more difficult for some teachers, and an incorrect choice of operation prevented some teachers from solving the problem. Teachers who simply responded to the structure of the problem itself and who tried to make sense of the situation, using the tools they had available and felt confident with, were invariably successful.

   This very important perspective serves as a link to the next session during which examples of young children's responses to problems are studied, and comparisons made between the child's view and the adult's view of problems which seem quite routine to adults. Mistakes that children make when they feel forced to "choose an operation and apply a procedure" are discussed extensively.

2. The tools that were used to solve the problem are identified: a knowledge of fractions, of whole numbers and of some recording skill. This is elaborated on in a later session, where the development of young children's number concept is studied, and a practical demonstration, with some teachers acting as children, and videotaped classroom scenes give some ideas of suitable number concept development activities. The role of the teacher regarding the "social knowledge" component of mathe-

matics, and the development of communication skills, both verbal and written, are also discussed.

3. The teachers are asked to reflect on how they went about solving the problem: *when* did they talk to one another, about *what* did they talk, what was the main *effect* of these discussions on their thinking processes and *why*, etc. These issues touch on the classroom culture, the didactical contract between teacher and students, the nature of knowledge and how knowledge is constructed (individually and socially), some characteristics of a good learning environment for mathematics, research-based information on young students' own perceptions of what constitutes a good learning environment, etc.

❑ Mr Sishuba

Three friends help Mr Sishuba to do a job of work. Two of the friends work for the whole day, but the third friend only works for half the day. Mr Sishuba gives them R60 all together. How should they share the money?

The most common solution methods are:

1. An attempt at proportional sharing.
   Lungie: "There are three friends, so divide sixty by three. That gives twenty. But the one worked for half the day, so take R10 away from him and give it to the other two. So two get R25 each and one gets R10."
2. A different attempt at proportional sharing.
   Nonzuzo: "Give the friends R20 each, but then take away R10 from the one. Divide the R10 into three equal portions and share them out. Two friends get R23,66 and the one friend gets R13,66."
3. A proportional sharing out, directly related to the time worked.
   Maud:

R60 = 

= 1 day	1 day	½ day
R8	R8	R4
R8	R8	R4
R8	R8	R4
R24	R24	R12

4. A proportional sharing out in which the units are first equalised.
   Tom: "The three friends worked for two-and-a-half days altogether. I make that five half days. Five goes into sixty twelve times. So it's R12 for half a day. Two friends get R12 + R12 each, and the other one gets only R12."

The discussions here revolve mainly around the fact that the problem can be interpreted in different ways according to the practical situation. Teachers who use the first solution method are often adamant that the remaining two friends had to work harder to finish the job after the other one had left, and should be paid accordingly. Another argument is that the third friend had abandoned them and should be punished as a result.

We use this problem as an introduction to the role that the context plays, i.e. therefore not only the basic mathematical *structure* of the word problem but the (supposedly irrelevant) details of the setting itself. When this idea has been discussed briefly, the following problems are posed in succession as a further elaboration:

Mr Bengu has fifteen pails of water with which to irrigate his $2\frac{1}{2}$ beds of vegetables. How should he share the pails among the $2\frac{1}{2}$ beds?

Mother has a big jug of cooldrink which holds ten glasses. She thinks that she may not have enough cooldrink for all the children, so when she pours out the cooldrink, she only fills the glasses $\frac{3}{4}$ full. How many of the $\frac{3}{4}$-glasses can she pour from the jug?

After a very brief discussion, teachers quickly see that Mr Bengu's problem is the same as Mr Sishuba's, yet it is very easy to solve and much less ambiguous than Mr Sishuba's. They then notice that the cooldrink- and apple tart problems have the same structure, but insist that the cooldrink problem is much more difficult than the apple tart problem.

This activity leads to the study of the different word problem types. Teachers are now asked to reflect on how students may respond to some word problems, and are then asked to make up word problems for particular problem types, keeping in mind the variables that affect young children's understanding of a particular word problem. A number of very important ideas emerge during this discussion. For example, different cultures and different backgrounds should not be ignored in the mathematics classrooms, but should actually be subjected to discussion and comparison. Also, the very limited ideas that children have of the world around them are discussed.

❑ Fractions

Whereas the previous problems were selected to provide mathematical experiences for the teachers themselves, the following activity aims at demonstrating the use of children's *different* methods to create learning opportunities for the class. Teachers are given a set of simple sharing problems that lead to fractional parts, and are requested to solve them in ways that they think young children might use.

For example, one of the problems is: *Share ten sausages equally among six friends.*

Through different direct modeling strategies, teachers generate the following solutions:

one and a half and a sixth
one and four sixths
one and two thirds

The resulting discussion then revolves around using this situation to start students thinking about equivalent fractions. The main message here is that the children themselves generate mathematical ideas that are sufficiently rich to initiate and support discussions about new topics, provided the teacher chooses a suitable problem.

Evaluation

Free-format evaluations invited from teachers at the end of a workshop were unanimously positive and enthusiastic. Most of the teachers mentioned that they now "knew where to begin" in their own classrooms, but that they desired a follow-up workshop in approximately six months' time. About 10% of the teachers involved suggested that the workshop be spread over three days, not to deal with more issues but to give them more opportunity for discussion and reflection. About half the teachers responded in person as well, stating that the workshop was the most meaningful and radical training experience that they had ever had.

We mentioned in the beginning that workshops with any chance of lasting influence probably need to address the two main issues of beliefs and skills. Since the teachers' free comments mentioned both these issues extensively, there is at least the possibility that the workshops were to some extent successful. It is, unfortunately, the case that no workshop can really be evaluated until its effects on classroom practice can be observed. Changed classroom practice is yet again heavily dependent not only on the quality of the workshops, but also on factors like peer, principal and supervisor attitudes and support.

Conclusion

It has been proved possible to identify some problems, which when posed to teachers during a workshop, will supply them with mathematical experiences that can serve as links to both the basic principles of a problem-centered approach, as well as to the practicalities of classroom organisation and the flow of classroom activities. Where the basic principles of a problem-centered approach involve a particular perspective on the nature of knowledge and on how knowledge is acquired, the teacher's own experience when solving a problem can encourage reflection on what mathematics is, how mathematics-related learning takes place, and the factors that encourage or hinder such learning. These reflections can then help the teacher to understand her students' needs.

It therefore seems that personal feelings of incompetence and anxiety which have been caused by rigidly formalist mathematics teaching may be turned to good account if handled correctly, and need not be a liability at all.

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TEACHER PROFESSIONAL GROWTH PROCESSES
AND SOME OF THEIR INFLUENCING FACTORS

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The study of teacher professional development has frequently been limited in its subjects, its tools, and the duration of the research. This paper describes some of the results of a study to examine teacher change through a multiple perspectives approach. The guiding aim of the study was the comprehensive portrayal of the change process in teaching practice, knowledge and beliefs and valued teaching outcomes as experienced by four junior secondary mathematics teachers from three different schools who participated in the Australian ARTISM professional development program. The four case studies identified a high degree of individuality of each change process that is influenced by variables related to the teacher himself/herself, his/her school and the structure, content and organisational conditions of the professional development enterprise itself.

Theoretical Background

The professional development process by which teachers change their classroom practices and their knowledge and beliefs about their subject as well as their role as teachers is fundamentally a learning process.

The identification of teacher change with a learning process has been explicitly modelled as "teacher professional growth" (Clarke & Peter, 1993). The knowledge about the nature of this learning process itself, the factors involved and their relation to and influence on each other is still incomplete. Previous research in the area of professional development has identified three types of factors that can influence the individual change process significantly:

1) Factors that are determined by the age-related life period of the teacher, his/her biography and present cognitive-developmental stage (Oja, 1989) as well as teacher characteristics and responses to innovation (Joyce & McKibbin, 1982; Doyle & Ponder, 1977);
2) Factors related to school culture and collegiality and the degree of support from the school administration for professional development (McLaughlin 1991; Fullan, 1990 and 1988; Sparks & Loucks-Horsley, 1990);
3) Factors related to the organisational and structural conditions of professional development enterprises such as time, financial ressources, competencies of the staff developer(s) and the content of the inservice program (Fullan, 1990; Little, 1984).

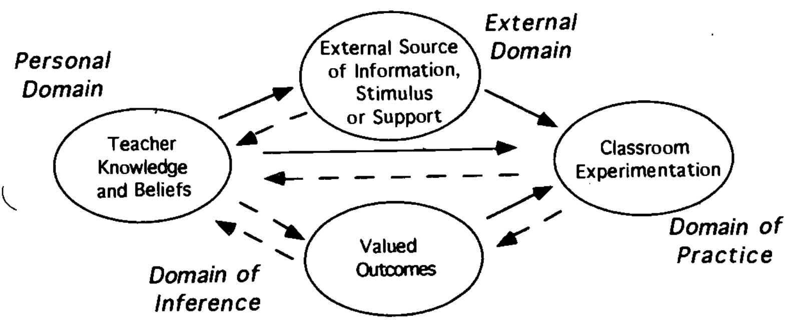
Most studies that investigate the professional development of teachers who were involved in an inservice program rely mainly for their data on the teachers' retrospective impressions and perceptions of the process and degree of their personal change (Loucks & Melle, 1982). The teacher's perspective on the outcomes of a professional development program is only one of several perspectives, although certainly an important one. For a comprehensive portrayal of the individual change process, this study required access to the perspectives of all members of the community participant in this process of teacher change. The approach of this study is a multiple perspective on teacher change that focuses on the teacher and his/her perception of change, but also includes the perspectives of the students, the mathematics coordinator, the school principal, the researcher, the developer, and the educational consultant as well in order to provide a more detailed and more comprehensive picture of the quality of the teacher change process.

The context of this study was the Australian ARTISM (*Active and Reflective Teaching In Secondary Mathematics*) professional development program for secondary mathematics teachers. The program was intended to make the participating teachers aware of current developments in the learning and teaching of mathematics while acknowledging the factors inhibiting implementation and providing appropriate support for individual teacher change processes. This support included school visits by the presenters (who were also the program developers) between the seven ARTISM sessions to discuss individual teacher experiences with new classroom strategies. The ARTISM program was predicated on the belief of the program developers that change will arise from the classroom experiences of teachers who have undertaken to field-test new techniques. Therefore the application of the key content of the sessions in the teachers' own classrooms was an essential element of the course (Clarke, Carlin, & Peter, 1992).

Conceptual Framework

The study seeks to gather information about the teacher change process on three levels: (1) change in their classroom practice, (2) change in their knowledge and beliefs about mathematics and the teaching and learning of mathematics, and (3) change in valued outcomes in their classrooms. These factors have been identified as analytic domains in a recent model of professional growth:

Figure 1
The Clarke-Peter model of professional growth (Clarke & Peter, 1993, p. 170)



(solid line = *enactive* mediating process; broken line = *reflective* mediating process)

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The Clarke-Peter model invokes two distinct categories of construct: *analytic domains* and *mediating processes*. The four analytic domains which characterize the model are:

The *Personal Domain* - Teacher Knowledge and Beliefs
Teaching practice is in large part the enactment of individual teacher's knowledge and beliefs regarding their subject, effective instruction, student learning, and the socio-political environment of the school setting. The *Personal Domain* is concerned with the knowledge and beliefs underlying practice.

The *Domain of Practice* - Classroom Experimentation
The enactment of teacher knowledge and beliefs takes the form of classroom practice. Where the classroom situation is perceived as a problematic or challenging one, teacher classroom practice becomes classroom experimentation.

The *Domain of Inference* - Valued Outcomes
Those professional outcomes to which the teacher attaches value constitute the mediating domain by which classroom experimentation is translated into changed teacher knowledge and beliefs. These valued outcomes may include student learning, teacher satisfaction, teacher planning effectiveness and efficiency, reduced teacher classroom stress, and increased student and teacher classroom enjoyment.

The *External Domain* - Sources of Information, Stimulus or Support
Teacher classroom experimentation and teacher reflection may both be stimulated by external sources. These external sources might be an inservice program, professional reading, faculty meetings, or informal conversations with colleagues.

The mediating processes translate growth in one domain into another. These mediating processes can be classified as being either *enaction* or *reflection*. The term "enaction" has been chosen to distinguish the translation of a belief or a pedagogical model "into action" from simply "acting". Acting occurs in the *Domain of Practice*, and each action represents the enactment of something a teacher knows, believes or has experienced (Clarke & Peter, 1993, p. 169-170).

The perspectives which must contribute to the comprehensive portrayal of the change process can be identified with the specific roles played by individuals within the change community. Data collected from each of these individuals offer a distinct perspective on each of the analytic domains listed above. The members of this community, whose common focus is the realisation of teacher professional growth, are listed in Table 1. Individuals can be characterized by their roles within the change community, the perspective that they represent in terms of the change process, and the significance of that perspective.

Table 1

Multiple perspectives on teacher change and their significance (Peter & Clarke, 1993, p. 4)

Roles within the change community	**Perspectives on teacher change**	**Significance**
Teacher	Active participant in the change process	Person whose learning is the object of study
Students	Members of the group through whom the change process is enacted	Persons whose activities embody a major goal of the change process
Subject Coordinator	Curricular gatekeeper	Person who initiates and sustains change opportunities within the subject domain
School Principal	Administrative gatekeeper	Person who authorizes access to change opportunities and affirms professional outcomes
Consultant	Interpreter and facilitator	Person who translates the program's substance into inservice practice
Program developer	Defines the aims of change	Original source of the external stimulus embodied in the program
Researcher	Describes and analyses teacher change processes	Person whose concern is the monitoring of the change process

Methodology

If the value of these multiple perspectives is to be realized, then research methods have to be identified to access the insights, observations, and experiences of all individuals listed in Table 1. This study employs a variety of research techniques to do this.

The data collection commenced at the start of the development of the ARTISM program. Observational data were gathered from those members of the change community involved in the implementation of the program, and during the related school visits which occurred between the inservice sessions.

Interview, questionnaire, and observational data focusing on four case study teachers from the three participating schools and students from one of their mathematics classrooms were collected during the implementation of the ARTISM program and for a period of twelve months after the official program was finished.

A variety of different research instruments have been developed to take into account the preferences that individuals might show in responding to these tasks. The body of instruments developed and applied in the study include:

* structured interviews
 (with teachers, principals, maths coordinators and program developers and the consultant);
* questionnaires with open-ended as well as multiple choice items
 (completed by students and teachers);
* observations of inservice sessions, school visits and classrooms;
* teacher journal entries.

The four case studies yielded a comprehensive insight in the individual changes processes in terms of their different qualities and structures in addition to a variety of factors that influence those changes processes. This paper focuses only on some selected aspects of the individual change processes that were particularly highlighted through the multiple perspectives approach employed in this study.

Case Studies of Individual Change Processes

The Case of John - Classroom Experimentation as a Stimulus for Change

John's[1] mathematics teaching before his contact with the ARTISM program was characterized by a teacher-centred classroom approach. The lessons were designed similarly and consisted of teacher explanations on the board followed by individual student work out of the textbook.

For John the classroom experimentations required by the ARTISM program were a crucial stimulus for change. While initially he was sceptical of changing his valued classroom practices, later after the second ARTISM session he decided to explore some of the suggested strategies and was very satisfied with the results. From that lesson onwards John continued with the exploration of new strategies and ideas. His attempt to carefully adapt, rather than to uncritically adopt the promoted alternative strategies lead to the enhancement of his preferred classroom results. For example classroom control was certainly an important issue for John. In implementing ARTISM ideas in his teaching he was always conscious of maintaining classroom control. One of the main outcomes for John was the experience that the changes in his classroom practice did not interfere with the control of his class but enabled him and his students to extract greater enjoyment from their maths lessons.

John's changed maths classroom can be characterized as student-centred with the students working in pairs or groups for most of the time on tasks that require them to apply mathematical skills and knowledge to their daily life.

John's individual change process was based on changes in his classroom practice through the exploration and implementation of ideas and strategies from the ARTISM program. He evaluated the results of these explorations according to his valued outcomes in terms of his maths teaching and found that the student learning did not suffer but even improved, that classroom control could be maintained and that the students and he himself enjoyed the maths lessons much more than before.

[1] All teacher names have been changed to insure their anonymity.

As a result of this he changed his knowledge and beliefs about mathematics teaching and learning which lead him to rewrite the year 10 maths curriculum together with a colleague who had had similar experiences. Furthermore he tried to adapt the assessment procedures according to his changed maths teaching and came up with a new assessment policy that is now applied in the whole school. Within his school John became a leading figure in the reform of the school maths curriculum.

The Case of Anne - Perceived Structural and Organisational Teaching Conditions Presenting a Barrier to Change

Anne's professional development process due to her participation in the ARTISM course was radically different compared with John's. Her maths classroom before ARTISM was based on teacher explanations of the content and student work from a textbook. Like John originally Anne did not perceive a personal need for professional development and for changing her maths teaching. She took part in the program because the maths coordinator of her school expected her to participate. From Anne's point of view structural and organisational conditions at her school were limiting factors in terms of her maths teaching. She argued that the large classes with thirty and more students, the attitude of the students (*"the boys don't want to learn maths"*), their intellectual abilities and the lack of equipment and time stop her from changing her maths teaching. Although Anne trialed some of the activities that were introduced during the ARTISM sessions, these explorations did not lead to changes in her knowledge and beliefs about mathematics teaching and learning.

The Case of Bill - Change as an Attempt to Adapt his Classroom Practice to Existing Knowledge and Beliefs

Bill originally had been trained as a primary teacher and had worked in a primary school for the first three years after his graduation. He is highly involved in many sporting activities and a trained international tennis umpire. He decided to work in a secondary school because this allowed him to become involved in a higher level of school sport.

Working as a teacher in a secondary school Bill was struggling with a conflict between his knowledge and beliefs about maths teaching and learning and his classroom practice. Through his primary training he was already aware of the current approaches towards a reform in mathematics teaching in Australia, but he had problems implementing his beliefs accordingly into his maths teaching. Bill was very interested to participate in the ARTISM program. He found that his existing knowledge and beliefs about the teaching and learning of mathematics were confirmed and "refreshed" and he engaged in the exploration of the introduced classroom activities and strategies. The results of these explorations matched his predetermined valued outcomes and encouraged him to further explorations. For Bill the content of the ARTISM program had a stronger effect on a cognitive than on a practical level. Bill always checked whether new ideas and strategies that were suggested during the ARTISM course matched with his existing beliefs and valued outcomes before he implemented them into his teaching. Other than John, Bill never tried to integrate and link the new strategies and acticities to an overall concept that underlies his maths teaching. Therefore his different approaches to change parts of his maths teaching were isolated from each other. Furthermore Bill did not develop continuity in terms of the implementation of new strategies and activities.

The Case of Steve - Change as Adaptation to a Reformed Mathematics Curriculum

Steve was working at the same school as John. In his school Steve had the reputation of being a brilliant mathematician, but especially prior to the ARTISM program Jane, the maths coordinator of his school, was worried about Steve's lack of pedagogical content knowledge and pedagogical knowledge. Jane had got the impression that Steve could not always relate to the needs of his year 7 students. Steve himself perceived a need for professional development and has articulated explicit expectations in terms of his participation in the ARTISM course that all address pedagogical content issues. Before Steve was ready to modify his classroom practice and to implement elements of the ARTISM course he evaluated and expanded his knowledge and beliefs and also his valued outcomes as a result of the information on teaching and learning mathematics provided by the ARTISM program. A major factor that contributed to Steve's change process was the fact, that the mathematics curriculum at his school had been changed after the end of the ARTISM program. All year 7 units had been rewritten by the maths coordinator, so Steve's classroom behaviour to a large degree was guided by the new units and assessment practices. While ARTISM certainly impacted on Steve's knowledge and beliefs, the change of his actual classroom practices was based on adaptations to the reformed school mathematics curriculum.

Conclusion

All four teachers had the same external input for professional development through the ARTISM program. Nevertheless their individual change processes vary depending on the level of support and collegiality they experienced in their schools, their biography, their individual "ages and stages of adult development" (Oja, 1989) in addition to their personal characteristics and responses to innovation. Among the variety of factors influencing the individual teacher change processes the level of collegiality provided by their peers and the degree support of their principals seem to be critical factors in terms of the individual development processes.

The case study data suggest an obvious link between the individual change processes and external conditions determined by the school culture. Both the principal and mathematics coordinator at the school where John and Steve work understood staff professional development as important aspect of their responsibilities. They formally acknowledged the achievements related to the ARTISM program of the participating teachers by providing time release and positive feedback. Furthermore they have seen the involvement of their school in the ARTISM course as a crucial part of the internal school development. Both encouraged and supported the reform of the school mathematics curriculum and the school assessment policy initiated through John and his colleague Max by making these issues the topic of staff meetings and staff professional development days. Ultimately Steve's change process benefitted to a large degree from the changes in the maths curriculum and individual units. Cooperation among teachers of the same subjects and year levels in the planning and teaching of their lessons is much appreciated and encouraged in this school. John and an his colleague Max have been working together for many years and their cooperation is acknowledged in the time-table which enables them to teach the same year level. The maths coordinator actively tried to engage in cooperation with Steve (who was teaching in the same year level with her) trying to benefit from his extraordinary content knowledge while offering him support with questions related to pedagogical (content) knowledge.

The principals and maths coordinators and the school climate at the other two schools were much less supportive in terms of the individual change processes of Anne and Bill. Both teachers lacked opportunities to work together with colleagues and did not receive much support or acknowledgement from the principal, because both principals did not feel responsible for the individual change processes of their staff. While the maths coordinator at Bill's school tried to establish a culture of support and shared goals among the maths staff to facilitate a better environment for change, the maths coordinator at Anne's school understood teacher professional development as an individual task for each teacher. Therefore he perceived that his responsibility as a faculty coordinator was mainly to dissiminate information about available inservice activities and to determine the staff who should attend a particular program or activity.

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TEACHING MATHEMATICS AS AN EDUCATIOANAL TASK - TEACHERS' VIEWS ABOUT SOME ASPECTS OF THEIR PROFESSIONAL LIVES

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ABSTRACT

Following Conney and Shealy (1994) it is indicated that a broader theoretical framework is required in order to deal with the essential aspects of teaching mathematics. Such a framework should relate to the goals and the problems of the profession as well as to psychological elements in the cognitive and the affective domain. The paper presents and analyzes some teachers' views about such essential aspects.

The number of studies about mathematics teachers is increasing. At PME 18, only, there were 2 working groups and 12 research reports which focused on mathematics teachers. This is, of course, in addition to research reported in books (for instance: Grouws 1992, Houston 1990) and mathematical education journals. Each study is carried out within a certain theoretical framework in which the nature of mathematics or the nature of learning has a central role. Inevitably, such a theoretical framework is a result of the researcher's preferences and values. I do not object this tendency. On the contrary, I strongly support it. I mention it only because this fact is not emphasized explicitly by most of the researchers. They aim to desirable outcomes in teacher education as if "desirable" is absolute and does not depend on their own particular belief systems. For instance, if a study focuses on a teacher's change in the direction of the socioconstructivist approach to teaching (Boufi, 1994) then it is implicitly assumed that this approach is better than the traditional approach. If a study focuses on teachers beliefs about the use of computers in mathematics education (Bottino and Furinghetti, 1994) it is assumed that "teachers who are interested in constructing knowledge find in computers answers to their needs" (p.118). Finally, if a study focuses on the role of problem solving in a teacher's everyday practice (Fernandes, 1994) it is assumed that problem solving is the essence of learning mathematics. Thus, an educational research emerges, very often, out of the researcher's educational *credo* . This implies that if you do not accept the constructivist paradigm, or if you have a formalistic approach to mathematics, or if you do not believe that meaningful learning is important to our technocratic society which is based mainly on technical training - then the above studies might become pointless for you.

This is true, of course, about this study as well. Its title is borrowed from Freudental (1973) and it is assumed that teachers, perhaps not exactly as other professionals, should have, in addition to the fact that teaching is their way to make their living, some educational philosophy, a belief that they have some educational mission or a social destiny. All these will have certain impact on their teaching practices, as well as their beliefs about the nature of mathematics and about the nature of learning. Since the teacher trainer community is interested in changes in the mathematics teaching practices, it is important to map the above mentioned conceptions about mathematics teaching as an "educational task." On the other hand, we all know that changing somebody's views and behavior is not a simple task. It is hard, as we all know from therapy, even if somebody is interested in changing themselves. Therefore, if they are not interested - change becomes almost an impossible mission. In an ETS report (Focus, 1992,) the title of which is "Teachers: The Key to Success," Robert Davis (p.19) speaks about teacher development. "People don't understand what it is like," Davis says. "I tell people that this is a lot more like psychoanalysis than it is telling somebody a new recipe...Our people have worked with some teachers for seven years now." Bearing this analogy in mind I was looking for research and treatment frameworks borrowed from therapy. It seems to me that the most suitable framework for research and treatment is the group therapy. The Webster's Ninth New Collegiate Dictionary characterizes group therapy as a therapy "in which several patients discuss and share their personal problems." If you take a way the words that have, for some people, a negative connotation ("therapy," "patient") and replace them by words which are relevant to our context you get: **A framework in which several teachers discuss and share their personal problems related to their profession.** My belief was that such a framework would help me to reveal the participants' views about problems in mathematics education and would be a starting point for a future change in case a desirable interaction will take place. I called this framework: A workshop for discussing problems in mathematics education. My **research aims** were to expose teachers' views about the meaning of their professional lives and about their problems.

Methodology

Two groups of teachers were formed. The first one was a group of 14 volunteers (10 females). In order to form it, a letter of invitation to a workshop for discussing problems in mathematics education was sent to about 200 high school teachers in Jerusalem. The second one was a group of 22 teachers (18 females) who participated in an in-service teacher training program. A compulsory part of this program was the workshop for discussing problems in mathematics education. The teaching experience in the two groups was between 5 to 30 years with a mean of 15 in the first group and a mean of 19 in the second group. This is in contrast to samples investigated in most of the studies on teachers where, usually, perspective teachers are involved. Each of the above groups met 7 times. Each meeting lasted 90 minutes. The research tool was the group discussion. In order to make the teachers talk about their problems I prepared a few questions which were supposed to serve as a trigger. I asked the teachers to answer the questions in writing. My purpose in doing it was to let everybody express their views and not only those who participated in the discussion. The written answer also helped me to "navigate" the discussion. I tried to encourage someone, whose point of view was relevant, to elaborate on his or her written answer. Once the teachers started talking I made myself passive and remained so, as long as the discussion went on and was relevant to the main theme. The discussions were video-taped and transcribed. The written answers were analyzed as well as the transcriptions of the video-tapes.

Results

In this section I will present analysis of some written answers as well as analysis of two excerpts from the video-tapes.

A question: *In your opinion, what are the most bothering problems in mathematics education?*

This question was posed only to the first group, N = 14. The written answers were classified to the following categories (each respondent stated more than one problem. The number in parenthesis is the number of answers in the category):

1. Students' lack of motivation, anxiety and repulsion (13).
2. Lack of prerequisites from previous stages. Poor arithmetical and algebraic skills (10).
3. Teaching heterogeneous classes and the size of the classes (9).
4. Lack of time to cover the curriculum (6).
5. Students' lack of ability to cope with topics which require thought, abstraction and imagination (5).
6. Bad text books (4).
7. Geometry and word problems (4)
8. The emphasis on mathematical techniques instead of mathematical thinking (3).

I have not mentioned categories which had only one answer. There were 4 like that. However, I would like to mention 2 answers which seemed interesting to me. One of the teachers (female) said: *Mathematics should teach people how to think. I feel that this is a too heavy responsibility on my weak shoulders.* Another one (male) said: *I do not know how to implement all the wonderful ideas that are suggested in the pedagogical literature, how to avoid stagnation and how to create motivation in my students.* I consider these two answers as exceptional answers. They both express self dissatisfaction which very often motivates people to improve - an everlasting project. In the first statement, in my opinion, an "existential modesty" is expressed. In fact this is one of the principles of the liberal education. When saying this I am using terminology borrowed from the ethical domain and I no longer restrict myself to the cognitive domain. As I claimed earlier, research in mathematics education is determined very often by the researcher's preferences. A distinction between the cognitive domain and the ethical domain seems to me quite artificial. If you read carefully the Curriculum and Evaluation Standards (1989) you realize immediately that it is impossible to distinguish between them. By the way, the above two teachers have extremely good reputation as mathematics educators. I would like to elaborate now on the above 8 categories of answers. I ordered them by their size. However, there is also a clear thematic direction in this order. They go from the everyday, immediately apparent problems to problems the characterization of which depends on analysis and reflection. If you consider the community of mathematical education researchers, as reflected in the research literature, you will probably found out that this community is bothered mainly by cognitive problems. It does not ignore motivation but it is implicitly assumed that the problem of motivation will be solved together with the solution of some cognitive problems as intellectual curiosity, relevance to everyday life of the students or intellectual challenge. One of the most bothering problems for mathematical education researchers is the quality of learning. The fact that learning mathematics became procedural, rather than conceptual. Teaching and learning mathematics have become an activity in which students try to acquire mathematical procedures which will help them to solve routine problems in some tests, the purpose of which is to select students for a higher stage of learning. Therefore, the community of teacher educators, which is extremely dissatisfied with this tendency , emphasizes so strongly other elements in mathematics learning such as problem solving and student construction of knowledge (for instance, Boufi, 1994; Cooney&Shealy, 1994; Fernandes&Vale, 1994). In the above group of teachers (N = 14) only 3 teachers related to the problem which is so central for the mathematics researcher community (*instead of mathematics we teach mathematical techniques,* wrote one of the teachers). In this case, I would like to avoid judgmental statement and only to point at the gap between the mathematics teacher community and the mathematical education researcher community. Cooney&Shealy (1994) advocated that teachers should be reflective. The above gap should call the mathematical education researchers for reflection as well.

In the discussion that followed the written answers to the above question I presented the comment about implementing the ideas of the pedagogical literature as a starting point. Here is one excerpt: I: *What are the wonderful pedagogical recommendations that you mentioned in your written answer?* Teacher 1: *To motivate the student to become an active learner, to participate with the teacher in knowledge construction, to solve problems and to test hypotheses.* Teacher 2: *What are you talking about?* I: *He told you; to motivate the student to participate in knowledge construction and so on.* Teacher 2: *We'll never cover the curriculum.* Teacher 3: *Our problem is that because of the present situation we give up.* Teacher 4: *Yes. We do give up.*

There are two types of reactions to the first teacher statement. In the first one you notice denial of the pedagogical literature. Teacher 2 knew about it. However, he thought it was irrelevant. A typical reaction of teachers who think that the mathematical education researchers have nothing to offer to the field because they know nothing about the field. The "wonderful ideas" cannot be implemented because we, the teachers, have to cover the curriculum before we do anything else. This is our real mission and therefore we do not have time to deal with all the "wonderful recommendations of the pedagogical literature." This reaction is mixed with anger. The teacher is angry with the mathematical education researchers who pretend to know how to deal with the field problems but, as a matter of fact, have no idea what is really going on in the field. The reaction of teachers 3 and 4 is different. They also believe that it is hard to implement the recommendations of the "pedagogical literature." Nevertheless, they are not hostile to mathematics educators. They admit that the fact that nothing is done to improve the situation is due to their weakness. They gave up. Thus, the two reactions differ from each other in the sense of taking responsibility. The first one develops anger against the agents of change and improvement - the mathematical education community. The second one takes responsibility and admits the teachers' weaknesses as a cause to the fact that things do not improve. (In the above analysis we rely on some parts of the video-tape which are not represented in the excerpts.)

A (two part) question: a) *Why do we teach mathematics?* b) *What is really achieved by mathematics teaching?*

This question was presented to the above 2 groups (N = 14, 22) The classification of the answers gave the following categories (the numbers in parenthesis indicate the numbers of answers of this type in the two groups, respectively. Each teacher gave more than one answer to each question). a) *Why do we teach mathematics?*

1. In order to develop thinking (mathematical, analytical , logical). (10, 16)
2. Mathematics is a tool in other disciplines. (7, 15)
3. General education. (5, 1)
4. Because it is interesting, beautiful and enjoyable. (3, 1)
5. It teaches accuracy and order. (3, 5)
6. It is a part of the matriculation exams. (1, 1)
7. It is a tool to solve everyday problems. (1, 0)
8. It develops certain virtues (as intellectual integrity, initiative, creativity, ability to cope with challenges in life). (0, 5)
9. It is essential for survival in the in the society of our time. (0, 5)
10. I like to teach mathematics. (0, 1)

b) *What is really achieved by mathematics teaching?*

1. Development of thinking (mathematical, analytical, logical). (7, 8)
2. Acquisition of mathematical techniques. (4, 5)
3. The matriculation exam. (5, 4)
4. General education. (4, 2)
5. Decreasing mathematical anxiety. (5, 0)
6. Enjoyment. (1, 2)

About 2/3 of the answers in the 2 groups mentioned development of thinking as one of the goals of teaching mathematics. But does not the mathematical education community expect every mathematics teacher to say it? It is hard to tell why the other 1/3 did not mention it. It is possible that some teachers are not aware of the educational potential of mathematics? On the other hand, it is possible that they are aware of it, but also being aware that it is not achieved they avoid mentioning it. This possibility is well reflected when comparing the real achievements to the declared goals. The ratios are 10:7 and 16:8 in the two groups, respectively. This implies that at least some teachers are convinced that this goal is not achieved. Again, if we consider the views of the mathematical education researcher community about this matter, at the current situation in typical schools, we will realize that its majority does not believe that development of thinking has been achieved. (I cannot establish this claim here but even reform documents as the above mentioned Curriculum and Evaluation Standards for School Mathematics (1989) and also Professional Standards for Teaching Mathematics ((1990) indicate that the current situation is quite poor and therefore a visionary reform is urgently needed.) If this is the case and the mathematics teachers are not blind then two different reactions are possible. The first one is *frustration* and the second one is *denial*. I claim that both occur quite often in the mathematics teacher community. Usually, people do not like to admit that they are frustrated. Especially, if the frustration is associated their profession. On the other hand, there is a tendency to state sublime principles as goals of somebody's profession. Thus, you can see unrealistic claims about the goals of teaching mathematics as in categories 8 and 9 above and unrealistic achievements as in category 5. At the same time you can see an attempt to deny prosaic goals as in category 7 or prosaic achievements as in categories 2 and 3. Only about 1/3 in the first group and about 1/5 in the second group mentioned mathematical technique acquisition and matriculation exam as achievements of teaching mathematics. However, everybody in the educational system (students and teachers) know that almost all the efforts in the classrooms and in homework assignments at the high school level are directed to the matriculation exam, for which mathematical techniques are the most crucial element. Since this is not such a sublime goal many teachers avoid mentioning it as an achievement. The reason I have presented the results of the two groups separately is that, as a moderator, I felt the two groups were different. Is that difference reflected also in the written answers? It is not so simple to tell, also because the samples are quite small, as in many studies on teachers. Therefore, the following claim should be considered with some caution. The two groups are quite similar in stating the goals of mathematics which are within the consensus (categories 1 and 2). There is a difference in categories which are less central (3, 8 and 9). In the second group there is a tendency toward a slightly exaggerated rhetoric (intellectual integrity, survival) whereas in the first group the tendency is toward a less "loaded" vocabulary (general education). On the other hand, in stating the achievements there is a tendency in the second group to be more positive (categories 1, 4 and 5). If this is the case, has the fact that the first group was a volunteer group (namely, people who really cared about their profession) has something to do with it? One answer, *I like to teach mathematics*, deserves a special attention. It relates to the personal aspect of the professional life, an aspect which is usually abandoned when educators are concerned. Would not a declaration *I like to be a mathematics teacher* more desirable for the educational system than some of the above rhetorical statements?

Here is another excerpt from the discussion that followed the written answers.

I: *Let's ignore for a moment the restrictions imposed on us by the educational system and talk about things that we would like to do in case there are no restrictions.* Teacher 2: *Alright. Each year I devote one lesson to self-learning. I tell them to open the book on a certain page; there is a theorem on this page. I give them instructions how to prove it. They get working sheets. With one class I failed even with this. So, I told them: let's play games the same way as the Scouts. I brought them some puzzles. For instance, given a set of numbers - what is the relation between the numbers? I did it with weak students, usually, they refuse to learn. They were turned on by this. Please, they said, bring us this stuff each lesson.* I: *do you do this only once a year?* Teacher 2: *If I do it once a year I am happy.* Teacher 3 *(ironically): Activities like this do not make the students virtuosos of exam questions. They do not bring them closer to the highest mark they can get on the matriculation exam. On the matriculation exams, questions which require real mathematical thinking never occur.* Teacher 4: *And if they occur everybody will scream that there is no time to teach mathematical thinking.* I: *The moment Ron (teacher 1 in the above first excerpt) started to speak he aroused strong objection, but after a few minutes I realized that, as a matter of fact, you agree with him.* Teacher 4 (the one who claimed above that teaching people how to think was too heavy responsibility for her): *I think it is pretentious to think that we can teach 11-th graders how to think. This is something that should have been taught to them immediately after they were born. The goal which I want to achieve is that students will know that there are information sources, that they have access to these sources and that they have the tools to use them. This is what I know.* I: *If I understood you correctly, teaching how to think is not a task that you are ready to undertake.* Teacher 4: *I do not want to be blamed in case I fail.* Teacher 5: *The trouble is that the majority of the students have prejudice against mathematics. This is a result of the fact that at the elementary level the teachers do not know how to teach. If mathematics were taught in a more enjoyable way then the students would have come to the junior high level with better attitude to mathematics.*

It turns out that teacher 2, who acted in the beginning of the discussion as if he knew nothing about "wonderful pedagogical ideas", actually knows something about them. His denial in the beginning was a kind of protest. When he eventually admits that he uses some alternative pedagogy, again one can notice anger in his reaction (*if I do it once a year I am happy*). Here the anger is against the educational system by which his teaching style is determined. He knows how to motivate even weak students but because of the educational constraints he can do it only once a year. My comment as a moderator indicated lack of awareness in some of the teachers. In the beginning of the discussion some of them did not distinguish between what they would like to do in case there are no external constraints and between what they can do under these constraints. This might cause unconscious frustration or unconscious anger. The workshop can serve as a channel to express anger and frustration, something which is recommended by some psychologists. Notice the comments of teacher 4. Her approach represent an anti-rhetorical attitude. If you do not undertake pretentious missions you do not fail and thus , you avoid the gap between goals an achievements. Notice also the tendency of teacher 5 to blame the elementary level of the illnesses of the secondary level. The analysis of the video-taped discussions showed increasing *awareness* of the problems, awareness of different views and also more *harmonious conceptions* of the goals of mathematics teaching.

Conclusion

The aim of this study was to better understand the professional life of mathematics teachers. It was indicated that a conceptual framework to deal with this theme should include general psychological notions. Cooney and Shealy (1994), for instance, used the notions *reflective* and *adaptive* . In addition I suggested to use *frustration, anger, denial, awareness, credo* and more. The findings and the analysis point at some inner conflicts that mathematics teachers have and also at a gap between the ways the mathematics teacher community and the mathematical education researcher community view mathematics teaching and learning.

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