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AUTHOR	Meira, Luciano, Ed.; Carraher, David, Ed.
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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following research papers: "The Impact of Meaning on Students' Ability to Negate Statements" (T. Barnard); "A Study of the Secondary Teaching System about the Concept of Limit" (L. Espinosa & C. Azcarate); "Difficulties Teaching Mathematical Analysis To Non-Specialists" (M.M.F. Pinto & E. Gray); "Reasons to Be Formal: Contextualising Formal Notation in a Spreadsheet Environment" (J. Ainley); "Articulation Problems between Different Systems of Symbolic Representations in Linear Algebra" (M. Alves-Dias & M. Artigue); "Analgebraic Interpretations of Algebraic Expressions: Functions or Predicates?" (H. Bloedy-Vinner); "Students' Responses Utilizing the Procedural and Structural Aspects of Algebra" (C. Coady); "Word Problems: Operational Invariants in the Putting into Equation Process" (A. Cortes); "A Case Study of Algebraic Scaffolding: From Balance Scale to Algebraic Notation" (J.T. da Rocha Falcao); "The Absolute Value in Secondary School: A Case Study of 'Institutionalization' Process" (J.J. Perrin-Glorian); "The Gap between Arithmetical and Algebraic Reasonings in Problem-Solving Among Pre-Service Teachers" (S. Schmidt & N. Bednarz); "The Influence of Problem Representation on Algebraic Equation Writing and Solution Strategies" (K. Stacey & M. MacGregor); "The Development of Elementary Algebra Understanding" (E. Warren); "Algebraic Thinking in the Upper Elementary School: The Role of Collaboration in Making Meaning of 'Generalization'" (V. Zack); "Developing Clinical Assessment Tools for Assessing 'at Risk' Learners in Mathematics" (R.P. Hunting & B.A. Doig); "Analysis of Errors and Strategies Used by 9-Year-Old Portuguese Students in Measurement and Geometry Items" (G. Ramalho & T. Correia); "Teachers' Conceptual Framework of Mathematics Assessment" (L. Rico, F. Fernandez, F. Gil, E. Castro, A. del Olmo, F. Moreno, & I. Segovia); "Qualitative Features of Tasks in Mathematical Problem Solving Assessment" (M. Santos Trigo & E. Sanchez); "The Influence of Teachers on Children's Image of Mathematics" (L. Brown); "Listening to Students' Ideas: Teachers Interviewing in Mathematics" (M. Civil); "Analyzing Four Preservice Teachers' Knowledge and Thoughts Through Their Biographical Histories" (D. Fernandes); "Professors' Perceptions of Students' Mathematical Thinking: Do They Get What They Prefer or What They Expect?" (Y.B. Mohammad-Yusof & D. Tall); "What Are the Key Factors for Mathematics Teachers to Change?" (E. Pehkonen); "Teachers' Awareness of the Process of Change" (V.M.P. Santos & L. Nasser); "A College Instructor's Attempt to Implement Mathematical Problem Solving Instruction"

(M. Santos Trigo); "Designing Computer Learning Environments Based on the Theory of Realistic Mathematics Education" (J. Bowers); "Passive and Active Graphing: A Study of Two Learning Sequences" (D. Pratt); "Between Equations and Solutions: An Odyssey in 3D" (M. Yerushalmy & M. Bohr); "Understanding and Operating with Integers: Difficulties and Obstacles" (R.E. Borba); "Games for Integers: Conceptual or Semantic Fields?" (A.C.C. Souza, A.L. Mometti, H.A. Scavazza, & R.R. Baldino); "Cognitive Science and Mathematics Education: A Non-Objectivist View" (L.D. Edwards & R.E. Nunez); "Overcoming Limits of Software Tools: A Student's Solution for a Problem Involving Transformation of Functions" (M.C. Borba); "Graphs That Go Backwards" (T. Noble & R. Nemirovsky); "The Graphical, the Algebra, and their Relation: The Notion of Slope" (S. Rasslan & S. Vinner); and "Visualizing Quadratic Functions: A Study of Thirteen-year-old Girls Learning Mathematics with Graphic Calculators (T. Smart). (MKR)

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Editors:

Luciano Meira
David Carraher

Universidade Federal de Pernambuco
Mestrado em Psicologia Cognitiva, CFCH, 8° Andar
Av. Acadêmico Hélio Ramos, s/n, Cid. Universitária
50.670-901 Recife PE
BRAZIL

E-mail: LMeira@Cognit.Ufpe.Br / DCar@Cognit.Ufpe.Br
Fax: 55-81-271 1843

With the collaboration of Marcelo Borba, Jorge da Rocha Falcão, Angel Gutiérrez, Carolyn Kieran, Lilian Nasser, Teresa Rojano, Analúcia Schliemann, & Judith Sowder.

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Christina Machado

The cover painting reflects the integration of the PME Community with the warmth of the tropical sun and beaches of our home town and host site of PME 19: RECIFE.

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RESEARCH REPORTS

The Impact of 'Meaning' on Students' Ability to Negate Statements

Tony Barnard
King's College London

This paper reports on a study to investigate students' capabilities for handling logical structures in mathematics, in particular in negating statements involving quantifiers. Undergraduates, both at early and later stages of a university course, were asked to negate a variety of statements set in everyday and mathematical contexts. It was found that, even after two years at university, one in three students could not negate apparently simple statements. Comparison of the performances of the two groups showed that the ways in which they differed reflected characteristics of the parallel transitions in the nature of the mathematics encountered and in the intellectual development of the students.

Introduction

Mathematical discourse at university is permeated with structures of the form "Suppose A is not true. This is the same as saying that B is true". Consideration of equivalent ways of expressing the falsity of a given statement, such as "for all $x > 0$, $a < x$" or "p divides ab implies p divides a or p divides b", occurs abundantly in both exposition and construction of mathematical proofs. Thus the ability to negate statements correctly is fundamental to meaningful mathematical communication at this level. Students who have difficulty with such structures may willingly accept, learn and reproduce instances of these in a mathematical argument, but they will be missing the point of such an argument in that it will have contributed little to their overall understanding of what is going on in the mathematics.

In an attempt to gain insight into the difficulties students have with 'negations', lists of statements of the following kinds were drawn up.

1. x satisfies P, for all x in X.
2. x satisfies P, for some x in X.
3. x and y satisfy P.
4. x satisfies P and Q, for all x in X.
5. A implies B.
6. There exists x in X such that $S(x,y)$ is true for all y in Y.
7. Given x in X, there exists y in Y such that $S(x,z)$ is true for all z in Z (the 'limit' definition structure).

These statements were set both in everyday contexts and mathematical contexts, and students were tested on their ability to negate them. The students were drawn from two groups: students in the first term of their first year, and a mixed group of second and third year students who had completed at least one year of formal mathematics.

The most notable finding was perhaps the sheer number of wrong answers, even with what many lecturers would regard as "just common sense". Thus for statements 2, 3 and 4, generally less than half of the first year students tested gave the correct answer. For statement 6, the number of correct answers was less than 1 in 4. The performance of the second and third year students was markedly better: generally 2 in 3 correct for each of statements 2, 3 and 4, and just under half correct for statement 6. However, the prevalence of such errors among students engaging with the more advanced mathematics of an undergraduate course was still far from ideal.

Subsequent interviews with students and consideration of the most common incorrect answers suggest that among the underlying causes of difficulty in performing negations are the following:

- logical structure,
- lexical representation (language, symbols),
- contextual influences,
- level of abstraction,
- degree of complexity.

It will be argued that ability to cope with these difficulties is related to progress in the transition from a descriptive view of mathematics, grounded in a practical domain in which objects and meanings of words are the dominant constructs, to one of definition and deduction, grounded in a theoretical domain in which symbols and words themselves are predominant. This aspect of mathematical ability is discussed in (Tall, 1994).

The test

Six lecturers in the mathematics department of a UK university were asked to run the test with their classes in the first term of the academic year. The total numbers of students involved were 78 from the first year and a further 78 from the second/third years. Before distributing the papers, the lecturers gave an explanation/reminder of the meaning of the word 'negation', following a prepared briefing sheet of notes and examples. Each student was then given a paper containing the following three sets of questions (figures 1, 2, 3).

For each of the following statements, circle the letter beside the statement below it which is its negation.

1.1 All people living in Cheltenham watch 'Neighbours'.
- A. No people living in Cheltenham watch 'Neighbours'.
- B. Some people living in Cheltenham watch 'Neighbours'.
- C. All people living in Cheltenham don't watch 'Neighbours'.
- D. Some people living in Cheltenham don't watch 'Neighbours'.

1.2 Some students stay awake at lunchtime.
- A. All students stay awake at lunchtime.
- B. Some students fall asleep at lunchtime.
- C. No students fall asleep at lunchtime.
- D. All students fall asleep at lunchtime.

1.3 Linford Christie and Sally Gunnell can run fast.
- A. Linford Christie and Sally Gunnell cannot run fast.
- B. Neither Linford Christie nor Sally Gunnell can run fast.
- C. Either Linford Christie or Sally Gunnell or both can run fast.
- D. Either Linford Christie or Sally Gunnell or both cannot run fast.

1.4 Long John Silver always has a briefcase and an umbrella.
- A. Long John Silver is sometimes either without a briefcase or without an umbrella or without both.
- B. Long John Silver is always either without a briefcase or without an umbrella or without both.
- C. Long John Silver is sometimes without a briefcase and without an umbrella.
- D. Long John Silver is always without a briefcase and without an umbrella.

1.5 What goes up must come down.
- A. What goes down must come up.
- B. What goes up must stay up.
- C. If something doesn't go up, it needn't come down.
- D. If something goes up, it needn't come down.

1.6 There is a station on the London Underground whose name contains no letters of the word 'MACKEREL'.
- A. There is a station on the London Underground whose name contains some letters of the word 'MACKEREL'.
- B. There is a station on the London Underground whose name contains all the letters of the word 'MACKEREL'.
- C. There is no station on the London Underground whose name contains all the letters of the word 'MACKEREL'.
- D. For any station on the London Underground, there is a letter of the word 'MACKEREL' which is not in the name of the station.
- E. For any station on the London Underground, there is a letter of the word 'MACKEREL' which is also in the name of the station.

1.7 For any lecture room, there is a time of day such that all students able to attend lectures at that time can fit into the room.
- A. There is a lecture room such that, for any time of day, there are students able to attend lectures at that time who cannot fit into the room.
- B. There is a lecture room such that, for any time of day, all students able to attend lectures at that time can fit into the room.
- C. For any lecture room and any time of day, there are students able to attend lectures at that time who cannot fit into the room.
- D. For any lecture room, there is a time of day for which there are students able to attend lectures at that time who cannot fit into the room.

Figure 1 : Negating statements in everyday contexts

For each of the following statements, circle the letter beside the statement below it which is its negation.

2.1 For all integers a, $a^2 \geq 0$.

A. There does not exist an integer a satisfying $a^2 \geq 0$.
B. $a^2 < 0$ for all integers a.
C. There exists an integer a such that $a^2 < 0$.
D. There exists an integer a such that $a^2 \geq 0$.

2.2 There exists a real number x such that $\log(x) = -1$.

A. There exists a real number x such that $\log(x) \neq -1$.
B. There does not exist a real number x such that $\log(x) \neq -1$.
C. $\mathrm{Log}(x) = -1$ for all real numbers x.
D. $\mathrm{Log}(x) \neq -1$ for all real numbers x.

2.3 $\mathrm{Sin}(x) > 0{\cdot}1$ and $\cos(y) < 0{\cdot}9$.

A. $\mathrm{Sin}(x) \leq 0{\cdot}1$ and $\cos(y) < 0{\cdot}9$.
B. $\mathrm{Sin}(x) \leq 0{\cdot}1$ and $\cos(y) \geq 0{\cdot}9$.
C. $\mathrm{Sin}(x) \leq 0{\cdot}1$ or $\cos(y) \geq 0{\cdot}9$.
D. $\mathrm{Sin}(x) > 0{\cdot}1$ or $\cos(y) < 0{\cdot}9$.

2.4 For all $x \in X$, $x^2 \geq 1$ and $x^3 \leq 8$.

A. Given $x \in X$, either $x^2 < 1$ or $x^3 > 8$.
B. There exists $x \in X$ such that either $x^2 < 1$ or $x^3 > 8$.
C. There exists $x \in X$ such that $x^2 < 1$ and $x^3 > 8$.
D. For all $x \in X$, $x^2 < 1$ and $x^3 > 8$.

2.5 If $u > 7$, then $v = 3$.

A. If $u \leq 7$, then $v \neq 3$.
B. If $u > 7$, then $v \neq 3$.
C. '$u > 7$' does not imply '$v = 3$'.
D. '$u \leq 7$' does not imply '$v = 3$'.

2.6 There exists a positive integer m such that $m + n \geq 5$ for all positive integers n.

A. Given any positive integer m, there exists a positive integer n such that $m + n < 5$.
B. Given any positive integer m, there exists a positive integer n such that $m + n \geq 5$.
C. There exist positive integers m and n such that $m + n < 5$.
D. There does not exist a positive integer m such that $m + n < 5$ for all positive integers n.
E. There exists a positive integer m such that $m + n < 5$ for all positive integers n.

2.7 Given a prime number p, there exists an integer x such that $pa < x$ for all positive integers a.

A. There exists a prime number p such that, for any integer x, there is a positive integer a satisfying $pa < x$.
B. There exists a prime number p such that, for any integer x, there is a positive integer a satisfying $pa \geq x$.
C. Given a prime number p and an integer x, there exists a positive integer a such that $pa \geq x$.
D. Given a prime number p, there exists an integer x such that $pa \geq x$ for some positive integer a.

Figure 2 : Negating statements in mathematical contexts

For each of the following statements, write its negation in the space below it.

3.1 All people living in Neasden have black hair.

3.2 Some TV programmes are good.

3.3 Kylie Minogue and the Loch Ness Monster can sing.

3.4 Donald Duck always wears glasses and a hat.

3.5 Where there's a will, there's a way.

3.6 There is a tree in England whose number of leaves is not equal to the number of words in any book.

3.7 For any textbook, there is a price above which the number of students who can afford the book is less than the number of copies in the bookshop.

Figure 3 : Formulating the negation of statements

Responses of the students

In each of the boxes in the tables below, the upper italic figure relates to the first year students and the lower figure relates to the second and third year students.

	1	2	3	4	5	6	7
Section 1	*58* 81	*46* 76	*44* 65	*37* 60	*60* 68	*24* 49	*32* 50
Section 2	*53* 73	*50* 65	*53* 67	*42* 65	*32* 50	*18* 42	*40* 44
Section 3	*60* 82	*62* 79	*35* 69	*31* 54	*33* 55	*24* 32	*12* 29

Table 1 : Percentage of students giving correct response to each section

	1.1	1.2	1.3	1.4	1.5.	1.6	1.7	2.1	2.2	2.3	2.4	2.5	2.6	2.7
A	*7* 5	*10* 5	*15* 6	***29*** **47**	*8* 6	*25* 11	***25*** **39**	*18* 9	*15* 13	*2* 3	*10* 4	*15* 8	***14*** **33**	*4* 7
B	*11* 4	*27* 10	*26* 17	*7* 8	*11* 8	*16* 7	*9* 3	*12* 10	*17* 11	*26* 17	***33*** **51**	*29* 18	*7* 4	***31*** **34**
C	*15* 5	*4* 3	*2* 3	*16* 11	*12* 9	*10* 9	*12* 18	***41*** **57**	*7* 2	***41*** **52**	*12* 11	***25*** **39**	*8* 8	*15* 8
D	***45*** **63**	***36*** **59**	***34*** **51**	*26* 10	***47*** **53**	*6* 6	*27* 6	*7* 2	***39*** **51**	*9* 4	*20* 10	*9* 12	*26* 13	*21* 14
E						***19*** **38**							*17* 14	

Table 2 : Number of students choosing each option (*N*=78)
(correct responses in **bold**)

The most common underlying error was that of negating a single part of the statement which had, for the student, a dominating presence. For the statements set in everyday contexts, this point of focus was often the section of the main verb. For example, in 1.2, by far the most common error was to solely convert "stay awake" to "fall asleep". However, in 2.2, where the section corresponding to "stay awake" was the less tangible "$\log(x) = -1$", the errors were more evenly distributed between solely converting "$\log(x) = -1$" to "$\log(x) \neq -1$" and solely converting "There exists" to "There does not exist". Similarly, in 1.6, 53% of the first year students and 23% of the second year students solely converted "contains no letters" to either "contains some letters" or "contains all letters", whereas in 2.6 the logically corresponding errors, C and E, were exceeded in popularity by the error of choosing D, the statement which converted "There exists" to "There does not exist" as well as "$m + n \geq 5$" to "$m + n < 5$". This behaviour was also widespread in section 3 where the students had to construct their own statement. For the negation of 3.6, 21% of the first year students changed only "not equal" to "equal", and even a very high proportion of the 'correct' answers consisted merely of the replacement of "a" by "no" after "There is".

Where students operated on one component of the statement with no relation to the others, and this was not a negation of the main verb, it was usually a transposition of two quantifiers. On being asked why they were focussing on just one part of the statement, typical student responses were "I was going for something a bit different", "I just want to make it not true ... minimum statement to make it false". This is the kind of behaviour that might be expected from students operating in the unifocal (Case, 1985), or unistructural (Biggs and Collis, 1982), mode of a developmental stage.

A possible explanation could be related to the opposing needs for coming to a conclusion, and for conclusions to be consistent. Students operating at a higher level of sophistication, for whom consistency was a factor of relative concern, were less likely to jump to hasty conclusions.As might be expected, the students' difficulties were greater with those statements which were more complex logically, such as statements 6 and 7 which were longer and had more than one quantifier. With a short term working memory of limited capacity, successful operation with these statements may require a chunking strategy and/or use of symbolic notation to mentally compress the components. As one student put it, "I think there was too much in that one"! However there were also complexities not related to logical structure. For example, "stay awake at lunchtime" in 1.2 was more complex linguistically than "are good" in 3.2. This variation was likely to be less significant to students more proficient in abstract reasoning, and could partially explain the different relative performances of the two groups at 1.2 and 3.2. For the first year students the percentage of correct answers for 3.2 was 35% greater than that for 1.2, while for the second and third year students the corresponding figure was only 4%.

Contextual influences

It will be noticed that the increases in success rates of the second and third year students over the first year students for the first four statements of each section were greater for sections 1 and 3, where the statements were set in everyday contexts, than they were for section 2, where they were set in mathematical contexts. (The increase for 3.2, which was slightly less than that for 2.2, may be related to the remarks of the previous paragraph.)

A possible explanation for these phenomena may lie in the role played by truth value. Students with less facility in abstract reasoning are generally less able to throw off the 'real world' true/false dimension when contemplating a given statement. For example, they are more comfortable writing down a statement they know to be true than one which they know to be false. A student comment on 1.5 and 3.5 was "I found them hard because they were phrases that you knew". For such students, more grounded in the practical than in the theoretical domain, the truth or falsity of a statement was a matter of relative importance and probably had a greater influence on their perfomances at negating statements than it did for students with a greater facility in abstract reasoning. Furthermore, this differential effect was likely to be greater with statements set in concrete everyday contexts than with more abstract statements where, for students whose abstract thought was more fragile, the true/false dimension had less immediacy.

Relative difficulties with statements set in everyday contexts and those set in mathematical contexts with concise symbols were also reflected in the following contrasting student remarks. While discussing her difficulty with 2.7, one student said, "It is harder with numbers than with the worded sentences because you've got the mathematical language as well, that you have to be thinking of. At the same time you have to think what $pa < x$ actually is, rather than in the common sense case." On the other hand a second student, whose best performance was on sheet 2, said, "(There was) less to keep in mind".

There is one final statistic which, though not surprising, does have its merits. Five lecturers were also given the 21 statements. While the percentage of students who gave correct answers in all 21 cases was 1%, the percentage of lecturers who achieved this was 100% !

Conclusion

Although the statements were chosen to have the same logical structure from section to section, there was no significant correlation of logical structure in the students' responses. The error patterns that did emerge arose rather from factors such as (a) complexity, (b) single, or unrelated multiple, operations, and (c) links with meaning via dominant phrases and truth value. For students at an early stage of development in detached theoretical thinking, the various components of a statement were likely to have attached weightings of importance, or presence, derived from a network of associations and meanings in their base of experience. They were less able to shake off logically

irrelevant associations than students who had progressed further in the transition to the stage where it is the weightless words themselves which are the dominant feature.

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A Study on the Secondary Teaching System about the Concept of Limit

Lorena Espinoza and Carmen Azcárate

Mathematics Education Department
Universidad Autónoma de Barcelona, Spain

The mathematical knowledge of 'limit' and the transformations it undergoes in order to be taught are studied in this paper; also, the mathematical activities performed with limits and the mathematical models used around this concept are described. The results obtained in this phase make an analytical instrument to allow us to approach in better conditions the teachers' knowledges and teaching methodologies, so that some didactic phenomena which are present in secondary teaching and learning of limits can be identified, explained and even predicted.

1 Introduction

The present report accounts for the results of the first part of a research aiming to study the mathematical secondary teaching system for the concept of limit.

Most researches undertaken in mathematics teaching related to this concept have focused on the study of the students' conceptions and on the epistemological problems linked to its learning process. Especially remarkable in this area are the works by B. Cornu, 1985 and A. Sierpinska, 1987, [5],[6],[7],[8],[14],[19],[20],[21],[22],[23]. Our own research has been developed under a systemic standpoint and out of the three main components of the complete teaching system, i.e., knowledges, teacher and pupil [13], it mainly focuses on "the mathematical knowledge of 'limit' itself" and on "the teacher".

This work follows a research line which takes mathematics education as "the science of the specific conditions for conveying those mathematical knowledges which are useful for human institutions to operate" (Brousseau, 1993). In this view, it is concerned with the study of mathematical knowledge involved and with the transformations it undergoes in order to be taught, as well as with the mechanisms and operations employed for the mentioned conveying [2], [4]. Here is where the variable Teacher assumes a remarkable importance since it is her/him who will, in the end, transmit to the pupil the decisions taken towards a teaching goal by the institution [4].

2 The Conceptual Framework

We take as a starting point the fact that in order to understand and interpret how the system of mathematical education works, how disfunctions are generated and developed within it, and also how to detect some didactic phenomena, it is first necessary to study the mathematical knowledge actually taught by the teaching systems [6]. To this purpose, the specific

mathematical activity carried out by the didactic systems has to be analysed. This implies selecting an epistemological model for the mentioned activity [15]. In our hereby account, we use the anthropological model of the mathematical activity proposed by Chevallard [11], or more generally, of the institutionalised mathematical practices, the one which considers them as a human activity of studying the domain of mathematical problems, putting forth an interrelation between creation and evolution of the problems' domains, the building of mathematics study techniques and the recursive development of the associated theories. Furthermore, we use some concepts of the theory of didactic transposition [9], of the relationship with knowledge [10] and of the didactic momentums [13], as analytical tools, as developed by the same author.

The kernel theory which supports this standpoint considers that mathematics actually taught at school is different from the one built up or used by the specialists; this late enduring a series of transformations and adaptations in order to be taught [9], [16]. The distance between an object of mathematical knowledge and its "correspondent" teaching object is often very large and, even some times surprising. Without the analysis of the mathematical knowledge actually taught and not having in hand good and explicit epistemological models which would allow this analysis, it comes to be very difficult to visualise any phenomena, and any didactic phenomena, in particular [1], [4], [15].

3 Design of the Study Program for Research

During the phase of the work we are presenting here, the strategy aimed at analysing the mathematical activity developed about limits in secondary education textbooks, as related to the concept of limit, targeting at:

1. Different contexts in which the concept occurs
2. Description and classification of the studied and proposed techniques
3. Correspondence between the theoretical tools appearing in textbooks and the actually performed activities
4. Highlighting those activities not actually developed which could nonetheless be developed using the proposed theoretical tools
5. Elicitation of those mathematical models implicit within the activity performed with limits
6. Identification of some didactic phenomena actually present in this development.

Finally, using the results derived from this analysis, formulating some conclusions and explanations about the spotted didactic phenomena, which will serve as hypothesis for the research on course.

4 Study Methodology

Three official Spanish textbooks are selected in order to be analysed by means of the study program proposed by Dr. J. Gascón [16], based on some theories developed under the same didactic paradigms, consequently matching the model of the mathematics activity sustained in this study.

5 Main Study Results

From the analysis of the selected textbooks we have developed the following sections:

1. Kinds of Problems: Description of the techniques employed and delimitation of the Fields of Problems

   Three grand classes of types of problems were essentially found within the global activity:

   (a) Algebraic handling of limits

   This is the first and most important, in view of the amount of exercises covered within this field of problems and of the time dedicated to them. The technique consists of: T1: producing a result by means of consecutive algebraic handling of a given expression and by applying different theorems about limits. The algebraic handling bears more importance or interest than achieving a result, not only for grading purposes, but basically because should the expression be not "adequately" handled the result would be almost impossible to draw.

   (b) Graphical representation of Functions. These functions are in general continuous except in a finite set of discontinuities (in most cases, a maximum of 3), and they have an algebraic analytical expression. The technique consists of: T2: Algebraic handling of a functional expression (almost always algebraic) in order to locate the points of discontinuity. Then, calculating the limit on those points. Last, drawing a graphical representation of the function.

   (c) Study of "slightly different" Functions. These functions are often absolute values, integer parts and functions which are defined in slices. The technique consists of: T3: Reducing those expressions to handling able algebraic expressions and then calculating the limits.

2. What id not done but could be done with limits? Some unrest arises to see how the theoretical developments explicitly shown by textbooks would be good enough to perform various activities which are not actually performed, although, by some mysterious reason, they are highly valued as far as learning evaluation is concerned, both in secondary education system and in cognitive education researches:

   (a) Related to Graphics: There is a lack of elementary reading technique to read a graph of which the analytic expression of the function is not known.

   (b) Related to Discrete condition and Successions: Being limits a powerful tool for the purpose, no relationship is built up between continuous and discrete. No work is done with the succession of function images, which would allow to link succession and function, through limits, even though the mathematical model which eases this relationship is elicited during the theoretical discourse.

   (c) Related to Numerals, Real numbers: No work is done with the conception of real numbers as limits of successions, even though the body of real numbers is characterised as ordered and complete (density property). No work is done with the idea that 3.999... equals 4 and with the idea that three points following a numeral actually represents a limit.

3. Some Didactic Phenomena.

During the didactic analysis some quite surprising, and even contradictory, facts and situations arose. Those situations seem to

indicate the presence of some phenomena, since they appear under the same aspect in all three textbooks:

(a) While dealing with limits, various theoretical models about this concept are explicitly presented which will never be required or used later on, while developing the actual mathematical activity proposed to the students. Those models appear to be purely ornamental, only meant to emphasise the fact that the matter been studied and worked about limits is complex, abstract, and consequently, important.

(b) Along with the theoretical development various activities could be performed, which would enhance the usefulness and meaning of this mathematical tool, and it would also help to justify introducing those rigorisms in secondary school; nevertheless, they are nor performed.

(c) The actually performed mathematical activity is very clear and simple. It lacks any complexity: the exercises do not contain any ϵ, δ, notation nor any deep abstractions; problems deal basically with calculation of limits solvable by means of techniques which are clearly explained during the theoretical discourse.

(d) The techniques being taught and being used for the mathematical activity bear such a severe rigidity that it almost blocks any work linking one with the other, or modifying them to derive one from the other. They are so much specific that the study activity about fields of problems comes to be atomised and restricted to just a few fields, leading to the loss of an integrated sight [15], [13].

(e) There is only one mathematical environment where the concept of limit is developed: the study of functions. No relation with numerals is shown, as opposite with what was done up to the seventies [18]. The concept is not presented as a suitable tool to read a function graph of which the analytical expression is unknown. Most of the functions are of algebraic nature, except some transcendent and trigonometrical ones.

(f) Formalising the concept of limit in mathematics emerges as a need in order to provide real numbers with continuity [17]. It moulds a tool associated with continuum, with convergence. By contrast, textbooks systematically portray it as linked with the concept of discontinuity.

4. Some possible explanations for those phenomena.

The premise implied states there is an implicit model of function in the secondary teaching system which forces the limits to be considered as sheer algebraic computing, and consequently, free of any difficulty coming from its analysis, that is: Function ¡¡synonym¿¿ Algebraic Expression Any intricacy in its study derives from the algebraic handling itself. This is the reason why a function is different from a graph and also different from a real situation. Also, there is the other implicit model which deals with limits as synonyms of function limits. In this case, limits will be limits of algebraic expressions and they are approached as solving an algebraic problem.

As a result, the following facts are drawn:

(a) The relationship between function and graph is of consequence nature, not by definition. That is: given a function, we can elicit its graph, but a given graph does not represent nor has any associated functional feature.

(b) There are no functions without algebraic analytical expressions, consequently, since numbers are not functions, no such things as 3.999... = 4 are considered.

(c) The students consider the following expressions as analytically different:

$$\frac{x^2-1}{x-1}, \quad x \neq 1 \quad \text{and} \quad \mathrm{x}+1, \quad \mathrm{x} \neq 1$$

(d) On building a graph, the students engage themselves in calculations and building of tables; they do not use discontinuity points nor asymptotes, in order to create the graph. These are studied as something unrelated to the drawing technique.

(e) Almost absolute lack of any activity related to the continuity of functions (just one exercise appears among the recapitulation problems).

5. Some epistemologic and didactic obstacles

(a) In order to formulate the concept of limit the concept of real number is required, but in order to define the real number the concept of limit is required as well [17]. Limit (convergence) refers to items which do not yet exist since they have not been defined.

(b) From an epistemologic standpoint the idea of limit cannot be conceived as unlinked from the idea of real number; both ideas were formalised almost simultaneously. Nevertheless, in secondary teaching system numerals can only be scarcely approached (limits of numeric successions) since it would mean to approach real numbers, which are still a mysterious matter in secondary teaching [3], [8], [14], [19], [20], [23].

(c) From a mathematical standpoint, limit of a succession is a simpler thing that limit of a function, since it is a discrete item. From an education standpoint, limit of a function is simpler since it is easier to be elicited.

6 Some Final Conclusions and Remarks

1. The techniques being taught bear a severe rigidity in solving problems about limits. The fields of problems being studied are atomised and almost completely mutually unlinked, making it difficult to engage into a deep study of those problems [15].
2. There is a uniformity in both theoretical and practical activities as far as the concept of limit is concerned.
3. There is a conflict between: what the secondary teaching system states that should be taught and learned in mathematics, as can be seen in the theoretical discourse of textbooks, and what is actually done, as can be detected through the activities submitted by those same textbooks to the students.

As a possible explanation to these phenomena, the suggestion is put forth that there is a hidden mathematical model of limit not elicited because it is considered as very limited and scarcely analytical, although it is the actual concept being used and also the one which characterises the kind of activities developed in the classrooms: that is: limit as synonym of function limit, and even more, function as synonym of algebraic expression. This way, what is

really studied is the limit taken as an operator assigning a number to each function. Nothing aside from algebraic expressions is being considered as study object.

Finally, one of the important contributions to research in mathematics teaching being brought up by this section of the study, is the elicitation of how knowledge to be taught constitutes a new construction being produced under different paradigms and interests. Models of mathematical knowledge do not match with models of taught knowledge, and consequently it embraces a new epistemology, an epistemology characteristic of mathematics education. Therefore, although there exists an interdependence between epistemologic and didactic problems, the weight of mathematicians in designing a teaching participation must be conditioned and supervised by teachers.

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Difficulties teaching Mathematical Analysis to Non-Specialists

Marcia Maria Fusaro Pinto
Departamento de Matematica
Universidade Federal de Minas Gerais
Belo Horizonte: Brasil

Eddie Gray
Mathematics Education Research Centre
University of Warwick
Coventry: UK

This paper reports the effects of teaching mathematical analysis to students who are to be teachers of elementary school children, yet who take analysis as the final summit of their mathematical studies at university. The students concerned divided into three groups. A tiny minority understood the formalities of the subject and the need for logical proof, the majority attempted to learn definitions by rote but in the main failed to understand the underlying concepts, and the remainder used inappropriate concept images from earlier mathematics. This paper questions the rationale of teaching formal analysis at degree level for those who are not specialist mathematicians.

Introduction

This paper considers the almost insignificant effect that a course in analysis had in changing the quality of mathematical thinking of a group of students who, training to be elementary and secondary school teachers, follow the course as a high point of their university degree programme. Evidence from written assessment and individual interview shows that only a tiny minority of the students are moving in a direction that would eventually enable them to utilise the formal aspects of mathematics. The majority did not recognise the need for formality. It was a surprise to find some students, even at this level, attempting to generalise from the particular; despite their extensive work with real numbers, their concept image had not expand to take in the notion of the concept definition. Knowing the concept definition by heart did not guarantee that they understood the concept (Vinner, 1992). Their experience prior to meeting the formal definitions not only affected the way in which they formed mental representations of the concepts (Tall, 1992), but frequently became manifest through their efforts to resolve problems with an inappropriately "evoked concept image" (Tall & Vinner, 1981).

A high proportion of pre-university mathematics teaching tends to emphasise calculation and manipulation of symbols to get "answers". In such an atmosphere the acquisition of the concepts has an intuitive basis which is founded upon experience (Tall, 1992). Such a paradigm contrasts starkly with that utilised to develop advanced levels of mathematical thinking; formal definitions give rise to concepts whose properties are reconstructed through logical deductions.

The study of analysis may be seen as an attempt to introduce the student to the formality that is the hallmark of the working mathematician; the general thought patterns of the students are encouraged to change from a mode which relies extensively on the formation of concepts through the encapsulation of process as concept (Gray & Tall, 1994), to a mode which is structured within the realms of concept definition. However, the transition from one form of thinking to the other is a difficult one. Though mathematicians use definitions and formal language in a meaningful way to compress

mathematical arguments, the learners method of thinking about mathematical concepts can depend on more than the form of words used in a definition.

Vinner (1992) has outlined students possible responses to cognitive tasks associated with the implied use of definition: the desirable one in which the student is not supposed to formulate a solution before consulting the concept definition, and a more usual model where the respondent is unaware of the need to consult the formal definition but places emphasis on a concept image. In the instances considered in this paper we show there was little very little evidence of the former but a considerable emphasis on the latter. But perhaps it is impossible to avoid the mathematical tensions that arise between the mathematcs tutors' desire to introduce students to the rigour of mathematical proof and the student perceptions that may be dominated by other considerations:

> "When I got the piece of work back my main concern was with what I had got. Unfortunately being so preoccupied with other things I am doing....I am fully aware of the fact that the things I did last year and even last term are going to be out of my head unless I think about them again. What I said to you earlier about relating everything, well it just goes against that philosophy... basically I have a problem of relating..." (Third year undergraduate student)

The context

At the end of a first course in Analysis, 20 students, all following a four year course leading to a teaching degree with mathematics as their main subject, were given written tasks that required a demonstration of their understanding of the use of definitions introduced during the course. Though there were three items within the package of assessed work we will consider student responses to the first, a problem which focused on the their understanding of real functions and the definitions associated with differentiability and continuity. As a result of the analysis of the students efforts seven students were invited to take part in more detailed individual interviews.

This first item invited students to:

Explain why the function $f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ is discontinuous for *all* $x \neq 0$.

The students written responses showed that the majority of them tried to avoid as much as possible the use of formal language; they worked mainly with an image and/or tried to use a dynamic or procedural version of the definition. In their responses it was possible to identify the coexistence of these characteristics with older images that had remained unchanged by the new theory. In other instances it was possible to identify "incorrect" images constructed on a misunderstanding of the theory.

Image using

Though all of the students had been taught the concept definition, only one student used it to solve the problem. By far the greater majority of students provided evidence of attempts to reconstruct a proof through a concept image, without reference to the concept definition; in some cases with verbal reconstruction (Figure 1):

When $x \in \mathbb{Q}$, x^2 also $\in \mathbb{Q}$, so that function would be continuous. However, it can be proved that between every two rationals lies at least one irrational number. Since $f(x)=0$ for $x \notin \mathbb{Q}$, inbetween each point evaluated at x^2, $x \in \mathbb{Q}$, the graph drops to zero, since there are irrational values inbetween, hence the function is discontinuous for all $x \neq 0$.

Figure 1: Verbal reconstruction of the proof for continuity

One attempted to be more explicit about the image (Figure 2):

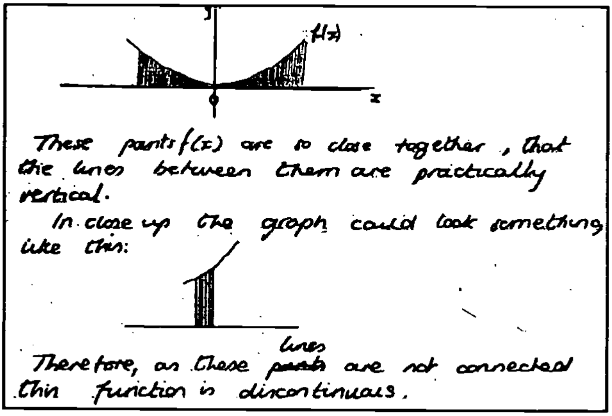


Figure 2: An evoked concept image of continuity

Though this student was interviewed later, no further insight into his image was forthcoming. However, some was gained from the interview with another student. Asked to draw pictures of functions that could not be differentiated this student drew the graphs shown in Figures 3 and 4.

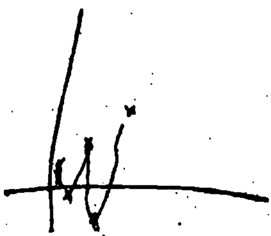


Figure 3. Student image of a non-differentiable and discontinuous function

As she drew Figure 3 the student commented:

> "I still think if you could differentiate at a point [pointing to a cross]... if you joined those together [joining up the crosses] like that you could still find the gradient at a certain point....you can have the gradient between two of those points, that would be the gradient if it was a straight line."

28

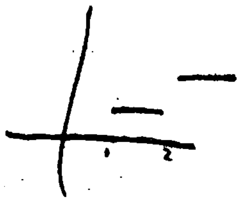


Figure 4. Student image of a non-differentiable and discontinuous function

As she drew the second graph the student was asked if it was possible to join the points:

> "No... because I've seen one similar to that [Figure 3] on a graphical calculator, and I've seen that one as well [Figure 4]."

The student was asked if she could provide an example of the formula for the function she had seen drawn in the calculator, but she replied:

> "I can't remember. It wasn't exactly that [Figure 3] it was similar. It had lots of little bits there [in the calculator] and then got wider."

Three isues would appear to arise from the students efforts to compensate for their inability to providere the appropriate concept definition:

- The image associated with the "linked points" of a graph prevents any formal association between the concept definition and an appropriately formed concept image.
- Such an image may be reinforced by misconceptions that arise from an automatic use of graphic calculators and computer programs; initially students may not associate the relationship between the graph, the defined function and the associated procedure (see also Hunter, Monaghan & Roper, 1992).
- Students learn images and intuitive ideas by rote; some seem not to worry about basic foundations upon which to relate knowledge meaningfully.

Reconstructing the definition

The images some of the students constructed differed significantly from the ideals that mathematicians would wish to be constructed from the definition. The statement "Between two rationals there exists an irrational; between two irrationals there exists an irrational" was translated by some students to mean:

> "since there is always an interval around each rational p/q where x is an irrational..."

> "there is always an interval around some rational x where x is irrational and.,."

Such representations provide an example of student's imprecision in the use of mathematical language and their difficulty in dealing with quantifiers which may arise from interpreting theorems in such a way.

Helped by a "redefined" model:

The $\mathbb{R}$ must $\therefore$ be ordered...rational.... irrational....
rational ... irrational.... etc.

one student simplified his arguments to prove that the function given in the first question is continuous at x=0. This student could conclude:

let α be the $\mathbb{R}$ either side of $x = 0$ (ie $(\pm \alpha)$)
from earlier explanation 0 will lie between two irrational numbers (ie $\pm \alpha$ · irrational numbers)

$$f(0+\alpha) = f(\alpha) = f(\text{irrational}) = 0$$
$$f(0-\alpha) = f(-\alpha) = f(\text{irrational}) = 0$$
$$\therefore f(0+\alpha) \to 0 \quad \text{and} \quad f(0-\alpha) \to 0$$

$\therefore$ at $x = 0$ $f(x)$ is left and right continuous

During the interview this was placed into a context by the student:

> "We had worked out in class that between any two rationals you always find an irrational, between any two irrationals you'll always find a rational, so from that I deduced that if you took two rationals you'll always be able to find an irrational in between, so I put down on my assignment that it was alternating between rationals and irrationals, which is wrong I think...Why do I think it is wrong? To be absolutely honest with you I haven't really looked at it properly to work it out which I know I should, but all I remember is thinking that I was right when I did the question."

Such a student would require some considerable time to synchronise his model with the proposed theoretical model. These students will not have this time.

The individual interviews confirmed the evidence received from assessment. Each interview started with a series of common questions to establish students understanding of the formalities and central concepts arising from the analysis course. The students selected for interview (N=7) were drawn on one hand from those whose written work had shown evidence of the interplay between personal description and a concept image and those who, on the other, displayed the inappropriate use of a concept image.

Space precludes presentation of the "formal" questions but the following synthesis will allude to them and highlight the most important issues that arose from the interviews.

- None of the students gave the formal definition of continuity and neither could they state how to calculate correctly "the derivative of f at a point in the domain D where $f: D \to IR$.".

- Since the student's examples of differentiable and non differentiable functions were the same as those given for continuous and discontinuous functions, it is hypothesised that their concept images of these notions were the same (Vinner & Tall, 1981). Their confusion over these two ideas could be seen even when they attempted to provide a formal definition:

  " A function is continuous if it can be differentiated at every point within a range"

"A continuous [function], you can differentiate that.....if you have two points on it, it is continuous between the two points then you can differentiate that"

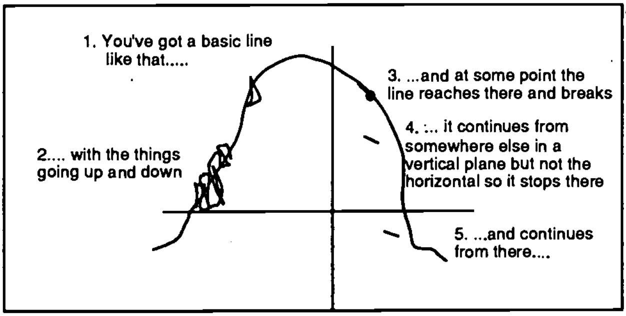
- Whether or not a function could be defined at a point determined whether or not it was continuous for some students

"...where you pick two points and a point between can be defined as well. You've got a curve which continues because whichever point you pick there's always another point on the line, there's no gaps in the curve."

"Continuity is every single point has another value"

Some others had a confused image that they could not synthesise in words:

"I don't know the definition but I know that it's where all the points if you drew them in a graph all points.....well they are not up and down all over the place".



Unable to write the definition of continuity one student indicated that the images of continuous function she possessed were from graphical work (Fig. 5):

"I can vaguely describe what a continuous function is on a graph".

Figure 5: A students evoked concept image of a continuous function

This student's attempt to describe such a function with her graph were almost indecipherable. However, she did indicate that

"I am just remembering a few things but it is not coherent at all".

Discussion

It seems that a great problem in dealing with mathematics lies in the fact that the theory was constructed upon aims that students do not achieve. Partially, this is because the composite theory is not made explicit but hidden behind the formal language and apparently clear hierarchies which mathematicians use to present the subject matter. Students have difficulty linking the language and the sequential steps of the hierarchies to form an overall theory encompassed within an understanding of the reasons for its formation. Many, destined to acquire definitions by rote learning, attempt to support these through intuitive ideas and the reproduction of procedural aspects of the theory. Even though they may be given intuitive experiences to support the formal aspects, being unable to understand the relationship, they evoke previously established concept

images which are not good enough to build upon. They acquire definitions with no supporting content; they evoke images from within the school mathematics curriculum.

Additional evidence for such an hypothesis could be found in the students' efforts to classify numbers as rational or irrational. One student who had no understanding of the difference and who made little effort to obtain any, stated that:

> "I always look these up when I need to know what they are. I've got a list of all the different symbols and what things mean and I usually refer to that when I need to know, but it hasn't stuck yet."

Another, though aware of the definition, preferred to use a concept image when analysing $0.\dot{9}$:

> "If you rounded that up it would be a rational number."

His explanation of this comment indicated that he did not understand what one means:

> "I don't know, it's just like .999... is too close to 1 but I don't know whether that makes any difference to a rational or an irrational number being so tiny. I'm just guessing."

A third had difficulty classifying zero as rational or irrational but even though he attempted to work with the formal definition he failed because the latter was misunderstood.

> "...zero isn't it ? I don't know...Maybe it's an irrational. I'm not really sure whether you can have division by zero....Zero divided by zero, normally you can't have zero on the bottom of a division line because it's undefined, so therefore it can't be defined as p over q so it must be irrational."

This evidence of students rote learning, both of the definition and the concept image, must be placed alongside additional evidence which illustrates that students knowledge of mathematical concepts may take on a variety of identities (Duffin & Simpson, 1993). We suggest that though such variety may be strongly associated with students conception of real numbers, the real numbers may still not be natural in the sense used by Duffin & Simpson even for students at this level.

Conclusion

This paper presents some evidence that arises from the mismatch that can occur when students who are not candidates for advanced mathematics are faced with the rigours of advanced mathematical thinking. The vignettes serve to support the evidence provided by Vinner (1992) but we would wish to look more closely at the longer term prognosis for the mathematical development of the students considered. Although only one student provided evidence of a reasonable understanding of the place of concept definition in analysis, all of the students described within this paper achieved at least pass grades in their assessed work–largely through a kindly interpretation of the marks.

From the evidence of the assessment and the individual interviews the students may be seen to fall into one of three groups:

- A very small group (N=2) which seemed to be moving towards a formal understanding of the subject matter using the formal definitions meaningfully or recognising the need for formal language and logical proof.

- A second, much larger group, (N=10) who, though they evoked the use of a concept image to support personal description, did not effectively use formal definitions. The majority of these students revealed that they had initial difficulties interpreting problems in the context of the theory. Such difficulties could be manifest through the limited considerations they gave to crucial aspects of the problems, for example, considering rational cases but not irrational ones, or arguments augmented with superfluous–in the sense that they provided more than the necessary–repetitive considerations.

- A third group of students (N=8) used inappropriate concepts images formed from earlier mathematical conceptions which remained largely unchanged as a result of the course in analysis. Such students attempted to establish a formal result by generalising from specific cases or they displayed an inability to link procedural and conceptual images of function and graphical representation.

The laudable desire to lead these students towards the formality of mathematics was thwarted for two reasons. Not only do they not appear to be ready to start the course– and thus the assumptions underlying the move to formality were not met–but, more importantly, they will have no opportunity to consolidate their knowledge to the point where concept definition and concept image have appropriate associations. When faced with formal aspects of a theory which they do not construct for themselves, students can ignore not only its convenience but also the arbitrary and respective reasons for each theoretical construction and each definition; important links can be missed and such deficiency will give way to a collection of fragments which bear little relationship to each other.

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REASONS TO BE FORMAL:
contextualising formal notation in a spreadsheet environment

Janet Ainley

Mathematics Education Research Centre
Institute of Education, University of Warwick

This paper addresses the early stages of children's introduction to the use of variables in formal algebraic notation. We conjecture that some of the difficulties encountered by children in this area may be accentuated by their lack of appreciation of the purpose or power of formal notation. A teaching approach is described which aims to situate the use of formal notation in meaningful contexts. Case study evidence from children working with this approach, using graphical feedback in problem solutions, is used to suggest links to other areas of cognitive research, and to refine questions for future study.

Background

In a recent survey of the learning and teaching of school algebra, Kieran (1992) cites a number of research findings which indicate the relative success of computer-based environments in developing children's understanding of variable in the early stages of learning algebra. Kieran attributes this success largely to the procedural nature of the programming involved. The use of variables in Logo is mentioned particularly as being accessible because it lends itself to procedural interpretations. Kieran also comments on the fact that although there has been a great deal of research into children's learning of algebra, there has been little research into the teaching of algebra or the content and presentation of what is taught. This paper reports on research which involves an innovative approach to the introduction of the use of variables to primary school children which may suggest an additional explanation for the relative success of children working in computer-based environments. We conjecture that the lack of any sense of *purpose* for the use of formal algebraic notation in traditional approaches to beginning school algebra may contribute to children's difficulties in accepting formal notation. Activities based around working with a computer often involve pupils in *using* variables, for example within Logo or BASIC programming, in order to achieve particular effects, so that the algebraic notation is a means, rather than an end in itself.

Approaches to contextualising algebraic notation

The idea of contextualising formal notation is not, in itself, a new one. Word problems offer a way of both giving meaning to algebraic expressions, and linking work in algebra to children's experiences of arithmetic problems. However there is considerable evidence that representing word problems as formal equations presents major difficulties for pupils. (Kieran (1992)). Generally such word problems have a single solution, which may be found through a number of different approaches. Describing the problem situation in an algebraic form may be high on the teacher's agenda, but not on that of the pupils.

'Investigations' offer another approach to introducing formal algebraic notation in meaningful contexts. Typically in such an activity the child might be required to explore a number pattern arising from a practical situation, and then asked for the hundredth number in the pattern, or a method for finding any term in the sequence. The aim is to encourage the child to generalise the pattern in the form of an algebraic expression. This approach has been characterised by Hewitt (1992) as 'train spotting', since the learner's attention is generally focused on pattern spotting rather than on the situation from which the investigation arose. From the child's point of view, it is difficult to see any purpose in formalising the pattern in algebraic terms: a verbal description of the pattern, or a generic method for calculating values, may seem just as efficient for giving the solutions required.

An alternative approach to formalising

One focus of our research in the Primary Laptop Project has been children's use of spreadsheets as a mathematical tool. Early studies indicate that the children's ability to interpret and understand graphs has been enhanced through working in a spreadsheet environment (Ainley (1994)). In order to exploit this potential, we have developed a teaching approach (illustrated crudely in Figure 1) which we have called *active graphing* (Ainley and Pratt (1994a)). Children are encouraged to enter data they collect in experimental activities directly into a spreadsheet, and graph this data regularly during the course of the experiment, thus enabling the graph to be used as an analytical tool. This means that the physical experiment, the tabulated data and the graph are brought into close proximity. Research evidence from data-logging projects (e.g. Mokross and Tinker (1987)) supports our conjecture that this proximity is important in supporting children's understanding of the conventions of graphing, and their ability to interpret complex graphical representations by relating them to the activities from which they arise (Pratt (1994)).

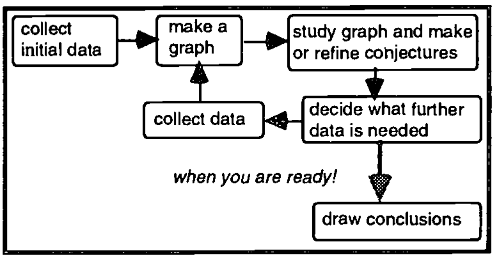
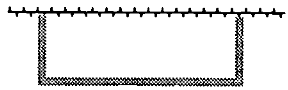


Figure 1: The active graphing process

Since the spreadsheet is an environment in which an algebra-like notation is used, we were interested to explore whether an active graphing approach could be used to introduce children to the power of generalising through formal algebraic notation. In order to do this, we selected activities which lent themselves to this approach, having a practical element so that children could begin by collecting data, but in which the underlying mathematical structure was accessible to the children. Two other key features of the activities were that the outcome was not obvious, so that there was some point in using the active graphing approach, and that the practical activity was rather tedious, so that children would be encouraged to look for short cuts.

With these criteria in mind we selected a number of optimising activities. We have reported elsewhere (Ainley and Pratt (1994b)) on the use of one of these as a whole class activity which gave us some insights into situations which prompted the need for formal notation. Here we focus on one pair of children working on a second activity, known as *The Sheep Pen*, shown in Figure 2.

A farmer has 30m of flexible fencing. She wants to make a rectangular pen for her sheep against a stone wall.



What length and width should she make it to enclose the largest area?

What if she had a different length?

Figure 2: The Sheep Pen Activity

Methodology

In this stage of the Primary Laptop Project our research is essentially exploratory, rather than addressing clearly focused research questions. We are interested in exploring the range of mathematical activities that are possible for children who have continuous and immediate access to computers, and identifying areas for more focused research in the future.

The case study material used in this paper was collected in a research setting removed from the classroom. Eight pairs of children (chosen by the researchers) worked on the activity with one of the researchers acting as 'teacher', introducing the activity, responding to the children's questions and occasionally intervening. The sessions were recorded on video tape, with the second researcher also taking field notes.

Jordan and Stellios were both aged eleven and in their final year at primary school. They were described by their class teacher as being of average ability, but neither of them were particularly highly motivated in mathematics. They had not been introduced to formal algebraic notation; but they were familiar with using a spreadsheet and had experience of an active graphing approach in the context of experimental work.

Working through the active graphing process

Like most of the pairs we observed, Jordan and Stellios began by working practically on the activity, using an art straw cut to 30 cm long to model the fencing. They bent the straw, measured the length and width of the pen, and set up columns on the spreadsheet to record their results. They knew that they could use and replicate a formula to calculate the area of the pen, and since the focus of the activity was not on understanding the calculation of area, we helped them where necessary to get this formula working correctly.

When they had collected several pieces of data, the researcher intervened to encourage them to look at a graph, shown in Figure 3. Jordan was able to discuss the meaning of the graph but at this stage,

his attention was on particular points, rather than on the relationship between length and area. However, it is clear from the boys' responses to further questions that they were aware of the overall shape and pattern of the graph.

STEL: *If I put eight and a half, where would that be? How would we write that?*
RES: *Where do you think 8.5 would appear as a cross?*
Stellios points between 8 and 9 for the width, and at about 100 for area. They put in 8.5 as the width, and Jordan bends the straw and measures the length as 14. The area appears on the spreadsheet.
STEL: *Highest! That's the best so far!*
Jordan makes a chart again to check the position. The length is actually measured incorrectly, so this point looks higher than 8 or 9.
JOR: *There it is* (pointing to the graph)
STEL: (pointing lower) *I thought it would come around there.*

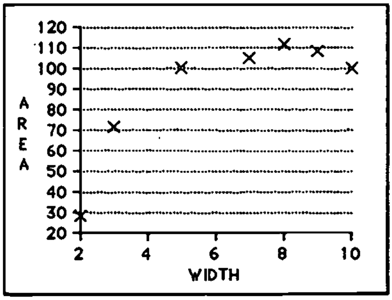


Figure 3: Graph of measured data

The boys were confident to make predictions based on the graph, but they had not yet seen the shape of the graph clearly enough to realise that some of their measured data was inaccurate. For some other pairs, *irregularities in the graph*. provided feedback which stimulated them to question their results, and either re-measure, or change to *calculating* the length of the sheep pen for a given width. For Jordan and Stellios, looking at *extreme values* was the stimulus to use *calculate* data rather than measuring. This was a pattern which we came to recognise in other pairs. It is quite awkward to bend the straw accurately for such a small width, and also the small numbers involved make the calculation relatively simple.

STEL: *Try a width of point 5.*
JOR: *What's the length?*
STEL: *Oh er 19, 29*
RES: *How did you work that one out Stellios, because you didn't measure that one did you?*
STEL: *If the ruler's 30, half and half is one and the rest is 20, no 29.*

Once Stellios had described his method, the boys wanted to use it to check the other values they had already entered. Thus the method used initially for finding a single value developed into a *generic method* which they could use repeatedly. At this point the researcher intervened to suggest that the boys might 'teach the computer' their method for calculating further data. This was a metaphor which was familiar to the children from their work with Logo.

RES:. .. *What you are trying to do is to tell the computer how to work the length out, given some width.* (pointing to cell B11 in the width column) *So if you knew what that width was, you're trying to work that length out* (pointing to cell A11, in the length column.)

...
JOR: *You have to add these together* (pointing vaguely at the length and width column). ... *double it* (pointing to the width).
STEL: *How do you double it? ..*
JOR: *and then you work out the length.*

STEL: zero point five add zero point five or something
JOR: .. yeah but they don't know .. (pointing at width cell)
JOR: I know B eleven, (typing) *B 11, B11, ... right B 11, add, ... B11 add, oh no, B11 times 2.*
STEL: oh yeah times 2
JOR: so then that doubles it , and
STEL: add A 11
JOR: B11 times 2 add ..
STEL: add A11 equals C11
JOR: No we need to ...if there's 30 in the ruler right, it's all doubled though, we need to tell it how to work out what's left.

The boys' initial attempts to formalise their method show a number of significant features. Jordan has a clear picture of the calculation he wants to express, but has to overcome two hurdles in order to formalise it. The first is to express 'double it', which he quickly resolves as 'times 2'. The second is more difficult. Having doubled the width, he then needs to find a way to express 'what is left' from the original 30 cm. In working on this, the boys quite confidently use 'B11' as a placeholder for a width which they don't yet know. This step in formalising does not seem to present an obstacle for them, but as they try to resolve the problem of how to find 'what is left' Jordan reverts to a generic example. His use of the cell reference as a placeholder is not yet secure.

The boys continued to work on their problem for several more minutes, occasionally touching the keyboard, but mainly trying out ideas verbally. At one point, they deleted the formula they had typed, and the researcher took the opportunity to ask them to recap what they have done.

JOR: So far we've got, from here we've got B11, anything that's in B11
STEL: times it by 2
JOR: times it by 2 so it doubles it
RES: ...OK
JOR: We need to tell it like, we want to tell that there's 30 over there, if we times, say it was 5, times by 2 it becomes ten, and what, and tell it to know how much is left on the ruler.
RES: Right . How do your calculate what's left? What do you do when you do it in your head?
JOR: Well if it was, if it was ...
STEL: What's left ...is it that little r thing? Is it remainder?
JOR: if it was, if it was 7, you double the 7 to 14 it would go in there but there's 16 left ...
RES: What have you done to work that 16 out?
JOR: I know that 14 add 16 is 30

Here we see that although Jordan still reverts to a generic example when he cannot resolve the problem of finding what is left, his grasp on B11 as a placeholder has changed. He spontaneously talks about *anything that is in B11*, indicating he has some understanding of the cell reference as a variable. It is interesting that each time he goes to a generic example he chooses different values to work with.

Stellios' interjection about *remainder* at first seems confused, and indeed we watched the tape several times before we noticed what he was saying. He seems to be making a link between the phrase 'what's left over' and memories of division problems, where he has learnt to record the

remainder' with a 'little r', e.g. 25 ÷ 3 = 8 r 1. He seems to be using a direct-translation approach (Chaiklin (1989)), but the translation is not from a given word problem, but from the boys' own verbal formulation of their calculation method. This direct-translation approach continued to prove an obstacle in their attempt to devise a formula.

About ten minutes later, they decided they needed to include 30 in their formula. They typed **=30 B11*2 -** . They seemed to have a sense here that they must start with the length of the straw, but they were trying to translate 'take it away from', and they could not see which operation to use. They quickly deleted this formula and typed **=B11*2-30**

STEL: .. You can't take 30 from ...um
JOR: times it by 2 take it from 30
STEL: times it by 2 and take it from 30
They try putting in 13 for the width and get length -4 and area -52.
JOR: its probably 52
STEL: the minus, shouldn't have put the minus in
JOR: I don't know
JOR: B11 times it by 2 take it from 30 ... but this looks like take away 30, and we don't ... It should have been 4, so its nearly right.

At this point, the boys had been working on the problem of teaching the computer their method for around thirty minutes. It is tempting to interpret their position at this point as failure to move from their generic method to a formal algebraic expression. However, from the language that Jordan uses it would seem that he has accepted the cell reference as a variable which he can operate on. We felt that his difficulty lay in attempting to make a direct translation from their verbal formulation, which cannot be reconciled with the arithmetic structure required by the spreadsheet. Their verbal formulation for the method of calculation followed closely the physical process which they had gone through, choosing the width and bending the straw this amount at both ends ('B11 times 2'), then measuring the length of straw left between the two folds ('take it away from 30'). We decided to intervene, offering them a slightly different physical model with the aim of redirecting their attention from the verbal formulation and back to the physical situation. The results were dramatic.

RES: Let's think of it in a different way ... Here's our length of fencing, which is 30 (holding up straw). *Let's imagine cutting off our two widths. So we're starting with the 30 and instead of folding, lets cut them off. ...*
JOR: If we start with 30, take away B11 times 2
At this point Jordan typed in the correct formula (**=30 - B11 * 2**), filled down the column, and they began to enter more values for the width.
JOR: we virtually did that, but it was the other way round.

The boys then worked excitedly, entering values to try to find the maximum area, and using decimals to home in on where they thought it would be. In a second session the following day, they

worked on the more general problem, using different starting lengths, and enjoying producing graphs showing smooth curves.

Discussion

In analysing the work of Jordan and Stellios, and of other pairs working on the Sheep Pen task, we see a number of factors which seem to contribute to their success in formalising. Their familiarity with the spreadsheet environment enables them to accept a cell reference as a placeholder in increasingly sophisticated ways. Initially, they used it as little more than an alternative **name** for the value of the width. Later, Jordan at least used it as a placeholder for a **potential number** soon to be realised, (*JOR: ... yeah but they don't know ..* (pointing at width cell)). Finally, he seemed to be using the cell reference as a placeholder for a **range of numbers**, that is, as a variable (*JOR: So far we've got, from here we've got B11, anything that's in B11*). It is worth noting that these children were familiar with entering and immediately replicating given formulas. As a result, they tended to see these as two parts to the same process. Thus they have an image of a physical location not only for the cell into which they will enter a particular number, but also for the column of cells into which they may enter a whole range of numbers.

Tall (1992) refers to a formal algebraic expression of a relationship as a **template**, a potential arithmetic relationship waiting to be realised. Some children may only be prepared to accept the use of a symbol as placeholder within the template if that potential can be immediately realised, i.e. it can be immediately turned into a number. Later, children, may accept a greater distance between the use of symbolisation and its realisation as a number. Such children are further on the way towards reification, when they must accept that the symbolic expression is itself something that can be manipulated and used (as if the distance between potential and realisation had become infinite).

When working with a spreadsheet, it is difficult to identify those children who have reached this final level of sophistication in their thinking, since those with more limited views of the nature of the cell reference may also be able to successfully create a formula to model their rule. We conjecture that the extent to which children are able to express a **verbal generalisation** of the rule they are trying to formalise may give some indication of whether or not they have taken this final step in their thinking. In the Sheep Pen problem, such a verbal generalisation might be signalled by describing their rule in terms of the width of the pen, rather than by using generic examples. Much of Jordan and Stellios' discussion of the problem focuses on creating a formula: they repeatedly use the cell reference, and so it is often unclear how far they have moved towards such a generalisation.

In analysing the tapes of pairs working on this activity, we were impressed by the perseverance the children showed in working towards a formalisation of their rule. Jordan and Stellios spent about thirty minutes on this stage of the activity without noticeably losing motivation or moving off task for more than a short period, even when their attempts were apparently unsuccessful. Although

they often talked in terms of operating on numbers or cell references, their hand movements indicated that their thinking was clearly grounded in images of folding and measuring the straws.

Even when situated in investigations or word problems, formalising is often a separate process from the main activity which has been externally imposed by the teacher. In contrast, within active graphing activities, formalising has a clear purpose: to generate more data. This larger quantity of data enables you to work on the problem, and the accuracy of this data can be seen from the feedback given by the graph. We conjecture that such activities give children a sense of the purpose and the power of formalising. They realise that unlike their teacher, the spreadsheet simply will not be able to interpret non-formal rules, such as 'take away from'. It is our belief that this experience of *using* formalising contributes to children's success in understanding variable. In common with other computer based environments, children's thinking is supported by feedback given by the computer on their attempts to give a formalisation. Further, there is an external referent, the physical situation in the case of the active graphing problems, or the functioning of the program in the case of programming. This broader context allows for alternative formulations to be developed, and so offers an escape route from the trap of direct translation from a single formulation.

Our analysis also raises a number of questions which we hope to address in further research.

- How do children perceptive of the nature of the cell reference in their formalisations?
- How interactions with the spreadsheet support them to move towards generalisation?
- What are the factors which influence children's ability to transfer from spreadsheet notation to traditional algebra?
- What kind of activities might support this transfer?

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ARTICULATION PROBLEMS BETWEEN DIFFERENT SYSTEMS OF SYMBOLIC REPRESENTATIONS IN LINEAR ALGEBRA

Marlene Alves Dias, Michèle Artigue, Equipe DIDIREM, University Paris 7

Abstract: *This article deals with the issue of flexibility between the cartesian and parametrical viewpoints in linear algebra. Firstly, we present the notions of setting, register of representation and viewpoint which constitute the theoretical basis of this article. Then we come to our project of research and the methodology we have set up to analyse flexibility. Finally, through the analysis of a written test, we show the difficulties first year students encounter before the flexibility issue. We also show that for problems that can be solved by only manipulating techniques, the lack of flexibility both technical and conceptual leads the students to mistakes which show important losses of meaning.*

I. INTRODUCTION

The disappearance of linear algebra rudiments from secondary school programmes in France (since 1989), has resulted, at the university, in an awareness of learning difficulties in this field. Since 1987 studies on the analysis of these difficulties as well as experimentation of didactic engineering were developed (Robert & Robinet, 1989), (Dorier, 1990), (Rogalski,1991). Some of the identified difficulties can be formulated in terms of **flexibility**, a notion which is now recognised in mathematics didactics as a key element of conceptualisation.

It seems necessary, here, to distinguish two types of flexibility, according to whether or not flexibility operates within cognitively hierarchical structures.

• the first type corresponds to a **hierarchical flexibility**. It is the case, for example, in E. Dubinsky's research (Dubinsky, 1991) which is built around the "process-objet" duality of mathematical concepts and where "encapsulation" and "disencapsulation" processes allow mathematical work to sail between the two levels. A.Sfard's research (Sfard, 1991) similarly emphasises on the double dimension "operational" and "structural" of mathematical concepts and the necessary flexibility between these two dimensions, even though, the first dimension is the necessary preliminary to the second one.

We can also find this kind of flexibility in D.Tall's research who underlines the two reading levels which can be associated to the same mathematical symbol via the notion of "procept". One can finally find it at stake in the three levels distinguished by Hillel and Sierpinska (Hillel & Sierpinska, 1994) in the reference to Piaget and Garcia's work in a recent research on linear algebra.

• The second type corresponds to a **non-hierarchical flexibility**. Such a flexibility is particularly considered through analysis in terms of "setting" as introduced by R.Douady (Douady, 1986, 1992) or in terms of "register" as introduced by R.Duval (Duval, 1993) as well as in terms of "changing viewpoint " used by a few authors, (Rogalski, 1991) for instance.

Our research in linear algebra situates within this global problematics of cognitive flexibility. More particularly, our interest will be focused on the second aspect of flexibility: the one of non-hierarchical flexibility.

II. SETTING, REGISTER AND VIEWPOINT NOTIONS

1. Setting notion

This notion was introduced by R.Douady in her thesis and based on an epistemological analysis emphasising on:

- the duality of mathematical concepts, first implicit then explicit **tools** of mathematical activity before they take the status of **object** and are studied as such;

-the role played by **changes in settings** in the mathematical production activity.

This epistemological analysis leads her to transpose these features into the didactic field through the notions of **tool/object dialectic** and **setting games** (Douady, 1986,1992)

Therefore, a **setting** is defined as being "made of objects of some mathematical branch, of relationships between these objects, of their eventually various formulations and of mental images associated with these objects and relationships [...] Two settings can comprise the same objects and differ in mental images and/or in terms of developed problematics". The **change in settings** " is a means to obtain different formulations of a problem that, though not necessarily equivalent, allow a new access to the difficulties encountered and the elaboration of means and techniques which did not appear necessary in the first formulation. Anyhow, **translating one setting into another often leads to unknown results, to new techniques, to the creation of new mathematical objects, in fact to the improval of the initial setting and the other intermediary settings used".**

Setting games, as organised by teachers, are didactic transpositions of these processes. They are seen in the developed theory as privileged means to raise "cognitive desequilibrium" and also to allow the overcoming of these and the reach of higher equilibrium.

Therefore, the setting notion emphasises the idea that the same concept is meant to function in various environments and that its functioning in each one of these environments offers specific features. The existing differences are just means and tools of mathematical creation.

As far as linear algebra is concerned, introducing the first concepts (generated space, linear dependence and independence, equality and intersection of subspaces) is often made by only using the R^n subspaces. Moreover, teaching favours the two and three dimensions which allow an emphasis on the game between the algebraic and geometrical settings and give way to cognitive flexibility which later become more metaphorical in higher dimensions or in more general spaces. In our study, we consider two settings: the algebraic one and the geometrical one.

2. Register notion

The setting notion is about the whole functioning of a mathematical concept while the register notion, which comes from the linguistic area, is more particularly about the symbolic representations according to which it can be represented and studied. R.Duval underlines the role played by this semiotic dimension in the conceptualisation process. In other words, the distinction between object and semiotic representation, which depends on the possibility to associate to the same concept many

different representations and to carry out conversions between these representation systems, is considered as a strategic knot in the conceptualisation process.

For Duval, the **semiosis**, that is to say the semiotic representation's apprehension or production, and the **noesis**, that is to say the conceptual apprehension of an object are inseparable. He defines a **representation register** as "being a semiotic system which makes possible the three basic cognitive activities that are linked to semiosis.

1 - **Forming** a **representation** identifiable as a register representation. This implies selection of features and data in the represented content, selection which is done according to the units and forming rules of the register in which the representation is produced.

2 - **Treating** a representation, that is to say transforming this representation in the same register where it was formed.

3 - **Conversing** a representation, that is to say transforming it into a representation of another register while keeping the whole content of the first representation or only a part of it".

R.Duval underlines that, as far as teaching is concerned, activities concerning the formation and treatment of representations are present but conversion activities are often neglected, as if conversion tasks between two registers were automatically mastered by someone who knows each register, separately.

In our study, the following registers of semiotic representations will be more particularly considered: intrinsic symbolic representation, coordinates representation, equation representation, matricial representation.

3 - Viewpoint notion

Mathematicians' work requires other kinds of flexibility, particularly, what we call "changing viewpoints" and is not so easy to define in a general way. Therefore, linear algebra seems to require flexibility between what we call the "cartesian" viewpoint and the "parametrical" viewpoint.

Such flexibility acts both in the geometrical and the algebraic settings and, even if it relies on flexibility between semiotic representations, it does not seem reducible to a mere semiotic flexibility as it involves more global aspects. For instance, the vector subspace notion may appear under a parametrical viewpoint with a stress on generating elements which characterise the subspace elements or under a cartesian viewpoint with a stress on algebraic equations which characterise the subspace.

Of course, in that case, the cartesian/parametrical flexibility puts at stake flexibility between representations, particularly between:

- intrinsic parametrical representations, such as: $A = \mathrm{lin}\{a,b\} = \{v/\ v = \alpha a + \beta b\}$;
- explicit parametrical representations under the form of a table, such as:

$A = \mathrm{lin}\{(1,0,0), (0,1,0)\} = \{(x,y,z) \in R^3 / x = \alpha , y = \beta\}$;

- intrinsic cartesian representations, such as: $A = \{ v/ T(v) = 0 \}$, T being a linear operator;
- explicit cartesian representations (by homogenous and linear equation systems), such as:

$A = \{(x,y,z) \in R^3 / z = 0\}$.

But we hypothesise that flexibility between the parametrical and cartesian viewpoints, which involves for instance the idea of duality, goes beyond a mere control of these semiotic conversions.

4. A study on linear algebra focused on cognitive flexibility between registers

K.Pavlopoulou's thesis (Pavlopoulou, 1994) is directly situated in the prospect developed by R.Duval. Its deals with the learning of elementary vector notions: linear combinations, linear dependency and independence in R^2 and R^3. Three semiotic representation registers are considered:

- the graphic register (G): in which a vector is represented by an arrow in R^2 and R^3;
- the symbolic writing register (S): in which a vector is represented by the linear combination of any two or three vectors in R^2 or R^3;
- the table register (T): in which a vector is represented by a column matrix with two or three lines.

An analysis of beginners' linear algebra textbooks shows that, in general, different registers coexist but conversion problems between registers are not explicitly set up in terms of learning. Moreover, there are kinds of conversions highly privileged.
K. Pavlopoulou organises a didactic sequence with students in difficulty (those who have failed their traditional programme). Her purpose is to emphasise the co-ordination between registers by following a classical experiment scheme: experimental group, control group, pre-test, post-test. She confirms the difficulty of a spontaneous building of conversion knowledge and proves the positive effect of the experimental didactic sequence, positive effect which goes beyond pure conversion tasks.

III. OUR RESEARCH PROJECT ON FLEXIBILITY IN LINEAR ALGEBRA

In our research, we try to study, more particularly, articulation problems between different systems of symbolic representations in linear algebra in the frame of the global study of flexibility between two viewpoints: the cartesian and parametrical viewpoints.
This project is based on a first piece of research (Dias, 1993) concerning the evaluation of a didactic engineering product (Dorier, Robert, Robinet, Rogalski, 1994) on linear algebra for first year university students. Our evaluation was focused on the central notion of the experimental teaching: the rank notion. Our attention was, then, draw to the difficulties students had found with the articulation of cartesian and parametrical viewpoints required to solve problems of determining vector system ranks and vector space representations. These difficulties were at the root of our present problematics.

1. The global project

In order to tackle these problems, we cross different approaches:
1 - an a priori mathematical analysis of both technical and conceptual knowledge linked to this flexibility, for the different notions and tasks involved in a first course of linear algebra;
2 - an analysis of the way these flexibility problems are taken into account through a study of curricula and textbooks, including a comparative study of the French and the Brazilian situations;

3 - a study of the cognitive functioning of students aiming at the identification of key-stages and difficulties in this area;

As far as methodology is concerned, our research is based on an analysis table of flexibility, issued from the first part of the research and on diagnostic tasks which aim at evaluating the students capacities concerning flexibility.

Our research is meant to emerge to a didactic engineering project aiming at a more efficient management of these issues since the first year at university.

2. The analysis table of flexibility

The purpose of this table is taking into account, as we said before, the flexibility between cartesian and parametrical viewpoints. It is obvious that this flexibility is based on flexibility between the different registers of representations associated to these viewpoints. It is also based on quite a number of conceptual and technical knowledge.

The analysis table is meant to be a tool useful for analysing the knowledge linked to the flexibility which is necessarily or potentially at stake in elementary linear algebra:

- according to the involved linear algebra notions;
- according to the tasks that are usually encountered at this level;
- according to the variables of these tasks, particularly the representation registers at use.

• At the **level of notions**, we distinguish the following notions:

- vector space; - vector subspace and operations between subspaces (including linear combination, generated subspace, identity, intersection, sum, direct sum of subspaces, supplementary subspaces); - basis and dimension (including linear dependence and independence, rank); - linear application (including kernel and image, isomorphism, linear operator's matricial representation); - linear equations system.

• At the **level of tasks**, for instance, as far as the notion of vector subspace and of operations between subspaces are considered, we distinguish the following tasks:

- Check with the definition whether a vector space's subset is a subspace or not; - Describe the solution's subspace of a linear and homogeneous system; - Determine whether an object defined in a certain way belongs to a subspace defined in another way or not. - Demonstrate that a vector is or is not a linear combination of some given vectors; - Check whether a vector belongs to the subspace generated by other vectors or not; - Characterise the subspaces generated by given vectors; - Find a generating part of a set of given vectors or a given subspace; - Move from a kind of representation to another; - Demonstrate that a subspace is included in other one or that they are equal; - Determine the intersection of two subspaces; - Determine the sum of two subspaces; - Demonstrate that two subspaces are in direct sum; - Demonstrate that two subspaces are supplementary.

• And at the **level of the variables of the task**, for instance, for the task: "Demonstrate that two subspaces are equal", we distinguish the following variables:

- type of space: R^2 and R^3, R^4, R^n isomorphic spaces, others;
- types of given representations: parametrical or cartesian with sub variables in order to take into account the different kinds of possible representations en each category (cf.II.3);
- types of required representations: idem
- dimensions of space and subspaces involved :
- flexibility: compulsory or potential
- flexibility knowledge: with distinction between technical and conceptual knowledge (see below)

3. Analysis of University first year exam.

- **The context**: We have used this analysis table of flexibility in order to analyse a written exam taken in 1992/93 by 113 students after their first semester course, at the university of Lille. The linear part of this exam consisted of four questions:

In R^4 are given the following vectors:
a = (0,-1,1,0); b = (2,1,1,0); c = (0,0,3,1);
d = (2,0,-1,-1); e = (1,0,1,1); f = (1,0,0,1)
1) What is the rank of the vector system {a,b,c,f}?
2) Give a parametrical representation and a linear equations system for lin{a,b,c,d}.
3) Determine:) lin{a,b,c} ∩ lin{a,e};) [lin{a,e}+lin{a,c,e}] ∩ lin{a,b,c};
4) The system {2y+2t = α; -x+y = β; x+y+3z-t = γ; z-t = δ, have a solution for all $(\alpha,\beta,\gamma,\delta)$? Justify your answer without any calculations.

In this report, we have chosen to analyse questions 2 and 4 because they are more significant in showing the difficulties of flexibility both at technical and conceptual level.
For question 2, the variables of the task are the following:

- type of space: R^4;
- type of given representations: 4 vectors represented by their coordinates in the canonical basis and an intrinsic symbolic notation of the generated subspace
- type of required representations: a parametrical representation and a cartesian representation;
- space and subspace dimensions: 4 and 3
- compulsory/potential flexibility: If the expression "find a parametrical representation" is understood as: "find a minimal parametrical representation", flexibility is strongly necessary. But students can produce the trivial parametrical representation {xa+yb+zc+td=0/ x,y,z,t $\in$ R} and solve the associated linear system (xa+yb+zc+td=v, in order to find the condition of $\alpha - \beta - \gamma + 3\delta = 0$ for $v = (\alpha,\beta,\gamma,\delta)$) which gives directly the cartesian representation. If so, flexibility remains necessary but it is reduced.
- flexibility knowledge: here it appears tightly linked to knowledge related to the resolution of linear systems, more precisely to the relations made between resolution conditions/cartesian representation, rank of the linear system/rank of the vectors system, number of necessary parameters/number of necessary equations with the fundamental theorem linking these two numbers.

Moreover this necessary flexibility can function at different levels. It can function at a technical level, encapsulated in some way in algorithmic processes or at a more conceptual level.

The same type of a priori analysis can be applied to question 4 but, in this question, as the answers have to be justified without any calculation, a conceptual level of flexibility is required.

- **The data analysis**

Question 2: For this question, 34 different procedures were identified. Here, we shall focus on the procedures P and Q which are in some way the typical erroneous procedures and correspond to 38% of the answers. With some local variations, the procedure P is the following:

- to write the matrix whose lines are given by the coordinates of a,b,c,d; - to write the associated linear system: {-y+z = 0; 2x+y+z = 0; 3z+t = 0; 2x-z-t = 0}, seen as a cartesian representation of the subspace; - to apply the familiar Gauss method to this system. This leads to a parametrical representation depending on one variable, as there are infinite many solutions, for example: x = z, y = z, t = 3z.

These cartesian and parametrical representations are internally coherent but they are incoherent with the results of question 1 (rank(a,b,c,f) = 4) and the obvious independence of each pair of vectors.
The erroneous procedure Q is similar (with columns instead of lines) and it leads to similar results.
The procedures described in the a priori analysis represent only 38% of the answers, that is to say 43 students, half of them just giving the trivial parametrical representation.
It is worthwhile noticing that among the 57 students who first looked for the rank of {a,b,c,d} and correctly found 3 by using the familiar Gauss technique, very few were able to correctly exploit this result in order to give a minimal parametrical representation. Some of them, for instance, give the relationship: d = a+b-c as a cartesian representation of the subspace, or this one: d = ax+by-cz as a parametrical. Most of them jump to P or Q procedures.
We have found only 6 students who tried to check their final results, that is to say, the number of parameters to be used and the number of equations to be found, as expected by the didactic contract. Among these students, only one had got the correct representations but he failed to identify which one was the parametrical representation and which one, the cartesian representation. So, he provided the following wrong justification: "We are in R^4, where lin{a,b,c,d} is represented by three independent linear equations therefore dim(lin{a,b,c,d}) = 1. Only one parameter is sufficient". The five other students which had used P and Q procedures also found: dim(lin{a,b,c,d}) = 1. They suggested the relation: dim(lin{a,b,c,d}) = n-r as a means to justify such a result, n being the dimension of R^4 and r the rank of the vectors system.

Question 4: Only 20 students gave correct answers to question 4 and once more we were surprised by the variety of procedures used by students. Among them, 55 recognised the given system as associated to the equation: xa+yb+zc+td = 0, but many did not know how to use this result to give a right and justified answer. This is understandable, taking into account the results obtained for question 2 and the fact that a conceptual flexibility was compulsory here. The attachment to Gauss technique was so important that 13 students used it explicitly and 7 students used it implicitly without respecting the instructions.

IV. CONCLUSION.

These results confirm our conviction that flexibility between cartesian and parametrical representations has a fundamental role to play in the learning of elementary linear algebra and that this flexibility cannot be reduced to abilities of a mere semiotic type. It has both conceptual and

technical components which intertwine in the solving process, the conceptual dimension playing an essential role in anticipation, control and interpretation processes.

These results also show that this flexibility is not of an easy access and that students tend to reduce it to its most algorithmic aspects and, as a consequence, to be trapped by all kinds of possible formal skid. This confirm our hypothesis that flexibility competencies cannot be left to the student personal effort. They have to be explicitly taken into account in the teaching process and managed in the long run. The ambition of our research project is to provide tools in order to better understand how this flexibility is or can be at stake in a first course of linear algebra and to manage it more efficiently.

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ANALGEBRAIC INTERPRETATIONS OF ALGEBRAIC EXPRESSIONS - FUNCTIONS OR PREDICATES?

Hava Bloedy-Vinner

Hebrew University, Jerusalem, Israel

Abstract: Algebraic language is analyzed and compared to natural language. The term <u>analgebraic</u> is defined. A conceptual framework is suggested for students' interpretations of algebraic expressions. New explanations for various phenomena, including the "students and professors" reversal error, are given, illustrated by students' written response and interviews.

1. Introduction

This paper presents a part of a study which suggests a conceptual framework for dealing with phenomena related to students' difficulties with the symbolic language of algebra. Kaput (1987) discusses the influence of natural language rules on translation errors in algebra. In my study I try to systematically analyze the structure of algebraic language, compare it to the structure of natural language, and learn about the influence of the latter on the understanding of algebraic language.

The term <u>analgebraic</u> (Bloedy-Vinner 1994) will be used to refer to modes of thinking related to improper use of algebraic language. The word <u>algebraic</u> here will refer to correct use of algebraic language. The definition of analgebraic depends on the mathematical context. In this paper I will discuss analgebraic mode of thinking in the context of interpreting algebraic expressions.

2. Comparing algebraic and natural language

As we shall see, analgebraic interpretations of algebraic expressions are related to erroneous analogy between algebraic and natural language. To explain this analogy let us start with a comparative analysis of both languages. The structures to be analyzed and compared are natural language sentences e.g. "She likes Bill's friends", and equalities and inequalities as $x-2=8y$ or $2+5^2>3$. The constituents of these structures in both languages are:

1. Primitive nouns like "she", "Bill" in natural language, and numbers or letters like 2, 5, x, y in algebraic language.

2. Complex nouns like "Bill's friends", $x-2$, or $2+5^2$, which contain other nouns as constituent parts. When some of the constituent nouns are replaced by empty places, e.g. "__'s friends", $\_\_-2$, or $\_\_+\_\_^2$, we get functions. <u>Functions create complex nouns</u> when

substituting nouns for the empty places. In algebra we use letters instead of empty places and get algebraic expressions, which are functions and create numbers (complex nouns) when substituting numbers for the letters.

3. When we replace nouns in a proposition by empty places we get predicates, like "__called", "__likes__", __=__, and __<__. (Relation is another term for two-place predicate.) <u>Predicates state something about nouns</u> which are substituted in them.

The structural similarity we have seen points to some correct analogy between natural and algebraic language. To describe the erroneous analogy made by students, we need to look at some differences between both languages. We shall look at two aspects:

1. The aspect of richness: Natural language is rich in noun types and in predicates, while algebraic language has one noun type only, numbers, and two (two-place) predicates, equality and inequality. When we use natural language to make algebraic statements, we have many additional predicates like "__ is positive", "__ is even" etc. To express these verbal predicates with algebraic symbols, we have to do with algebra's two predicates. The richness of algebraic language, on the other hand, is obtained from its ability to compose functions, and to create nouns which are much more complex than those of natural language.

2. The aspect of precision: Natural language can be ambiguous and has vague meanings, while algebraic language is unambiguous and precise.

It turns out that students often make an erroneous analogy and use algebraic language as if it had the properties of natural language with regard to both aspects of richness and precision. In the following sections we shall see this erroneous analogy and how it results in analgebraic interpretations of algebraic language.

3. Analgebraic interpretations of algebraic expressions

In this section I will discuss one form of analgebraic interpretation of algebraic expressions. As we shall see, this interpretation can explain phenomena related to students' difficulties with pure algebraic tasks as well as with word problem translation tasks. This form of analgebraic interpretation consists of two misconceptions which are related to each other:

1. When we consider an algebraic expression as a function, the

origin and the image should be conceived as two separate entities. Instead, students sometimes identify these entities and see them vaguely as one entity, being changed by the function, like a growing child, or an object which is painted but remains the same object. For example, in the expression $|x|$ the origin and the image are vaguely conceived as one changing entity (rather than a pair of separate entities), so that x becomes positive by the function, and still remains x. The vagueness here is one example of erroneous analogy between the languages.

2. An algebraic expression should be interpreted as a function which creates a new number. Instead, students may interpret it as a (one-place) predicate, stating something about x. For example, $|x|$ is interpreted as a predicate stating that "x is positive". This misconception is related to the previous one: the origin and the image are conceived as one entity, and an obvious property of the image, in this case positiveness, is attributed to x. Thus the expression is interpreted as the predicate "x is positive", rather than a function creating a new number which is positive.

With this conception, the student borrows algebraic expressions and uses them as predicates rather than functions. The result is an "enrichment" of algebraic language by additional predicates. This is another erroneous analogy between algebraic and natural language, by which algebraic language is made similar to natural language, namely, rich in predicates.

4. Method

The purpose of this study was to examine to what extent students were algebraic or analgebraic in their interpretations of algebraic expressions in the sense described above. For this purpose I compiled the questionnaire which is presented in figure 1.

The questionnaire was administered to Israeli students at a university preparatory course. These students had taken 3-5 unit matriculation exams in mathematics (a unit is one weekly hour during 3 years of high school). They answered the questions after having restudied the related material in the course, at the end of which they repeated the matriculation exam. By a rough estimate, more than half of high school graduates are on their mathematics level or below. Results will be given for two groups: Group SCI preparing to study science at the university, repeating 4 or 5 unit

exams, and group SOC preparing to study social studies, repeating the 3 unit exam. Some students in the SOC group were interviewed after answering the questionnaire.

Figure 1: The questions administered in the study.

1. Which of the following forms mean "x is negative"? (you may circle more than one answer):
   a. $-|x|$ b. $-x$ c. $x<0$ d. $-x^2$ e. $-x<0$
2. The temperature a tonight was negative, but was still 5 degrees higher than the temperature b of last night. Which of the following equations expresses the claim made above?
   a. $a=b-5$ b. $-a=-b+5$ c. $-a=b+5$ d. $a=b+5$
   e. $-a=-b-5$ f. $-a=b-5$ g. None of the above.
3. The teacher asked the students to compose a table for the function $y=0x+2$. Danny said that x was always zero and y was 2. What is your opinion? (correct, incorrect, explain).
4. What are x and y's solutions in the equation: $0x+5y=10$?
5. What do you think of the following statements:
   a. In $x+10$, 10 was added to x, therefore it is now larger.
   b. In $10x$, x was multiplied by 10, therefore it is now larger.
6. Write an equation to represent the following statement: There are 6 times as many students as professors at this university. Use S for the number of students and P for the number of professors (Rosnick & Clement 1980).

5. Results

For all questions, answers were classified into 3 categories: algebraic, analgebraic, and other errors (which may be analgebraic too, but in a different sense). These categories will be described specifically for each question. Distributions of answers are reported in table 1.

Questions 1 and 2: These questions deal with expressions with a minus sign. For both questions the analgebraic category includes answers demonstrating the two misconceptions described previously: first, the origin and the image of expressions with a minus sign ($-|x|$, $-x$, $-x^2$, $-a$) are vaguely conceived as one entity, as if x changes and becomes negative. Second, the expression is interpreted as the predicate "x is negative" rather than a function creating a new number. (These misconceptions should not be confused with that of seeing $-x$ as negative regardless of the value of x.)

In **question 1** the student is given expressions (functions) and inequalities (predicates), and is required to circle those which have the meaning of the verbal predicate "x is negative". The algebraic category includes $x<0$. All other answers are analgebraic,

because they interpret expressions as a predicate. Most of the students chose inequalities and expressions together. This implies that they understood that the question was about x and not about the result being negative, and that they did not understand that expressions were functions and not predicates. Note especially the answer $-x<0$ (chosen by 33% in SOC): this response vaguely combines $x<0$ and $-x$; because of the identification between the origin x and the image $-x$, the student is not aware of the contradiction between "x is negative" and $-x<0$.

Table 1: The distribution of the answers.

question	group	algebraic	analgebraic	other errors	no answer
1	SCI n=50 SOC n=33	34% 9%	66% 91%	0% 0%	0% 0%
2	SCI n=49 SOC n=33	43% 21%	45% 61%	10% 15%	2% 3%
3	SCI n=50 SOC n=33	68% 18%	32% 73%	0% 9%	0% 0%
4	SCI n=48 SOC n=34	65% 50%	21% 32%	9% 18%	6% 0%
5a	SCI n=49 SOC n=40	10% 17.5%	90% 77.5%	0% 0%	0% 5%
5b	SCI n=49 SOC n=40	12% 12.5%	88% 77.5%	0% 0%	0% 10%
6	SCI n=49 SOC n=33	84% 52%	14% 45%	2% 3%	0% 0%

To see that students really meant to state something about x (and not about the result of the expression) let us look at some interviews (**S** is student, **I** is interviewer):

Interview 1: (Written response. a. $-|x|$ b. $-x$ c. $x<0$ d. $-x^2$.) **S:** In (b) we can see that it is minus x, therefore it is negative, x is negative, this is what we're asked about in problem 1. **I:** Who? **S:** The x. **I:** Yes. **S:** In (c) it's also negative because it says that x is less than 0.
Interview 2: (Written response: a. $-|x|$ b. $-x$ c. $x<0$ e. $-x<0$.) **S:** (Pointing at $-x<0$)...this is also negative! **I:** What? **S:** x. **I:** Why? **S:** Because it's less than 0, and negative too.

Question 2 deals with the meaning of $-a$ in the context of translating a word problem. The algebraic category includes the response "(g) none of the above, it should be $a=b+5$ and $a<0$", and

"(d) a=b+5" (or (d)+(e) of students who noticed that these were equivalent). The category of other errors includes the response "(a) a=b−5" (or (a)+(b)) which is a reversal error (discussed in section 6.) All responses with one equation with −a are in the analgebraic category. These students see the origin a and the image −a as one entity, the temperature, and use the expression −a to translate the predicate "a is negative". to illustrate this claim let us look at one of many similar interviews:

Interview 3: (Written response: c. −a=b+5.)
S: The temperature tonight is −a. **I:** −a. But is says here that the temperature was a. **S:** But it was negative! **I:** That is why you wrote −a?. **S:** Yes. **I:** a itself represents a negative or a positive number? **S:** Negative. **I:** That's why you wrote minus here? **S:** Yes.

Questions 3 examines the interpretation of 0x as a predicate stating that "x is 0", which follows from seeing the origin and the image as one entity which changes and becomes 0. Answers like "wrong" or "wrong, x can be any number" were included in the algebraic category. Answers which justify Danny's statement were included in the analgebraic category. About one third of these answers gave explicit explanations like: "correct, when we multiply a number by 0 it becomes 0".

For **question 4** the answer x=0, y=2 is a familiar phenomena. It was classified as analgebraic. The misconception revealed in question 3 can be one of its explanations. (Other errors include the answer: y=2, x has no solution.)

Question 5 examines the meaning of 10x and x+10. Answers to both parts were classified in the same way: the algebraic category includes answers like "wrong" or "wrong, x is not larger, the result of the expression is larger." The analgebraic category includes answers justifying the given statements (for all x or for positive x). These statements imply that the origin and the image of the expression are one entity: x is conceived as changing and becoming larger. More than half of these answers expressed this conception explicitly: "true, multiplying x by 10 makes it larger", or "it's correct only if x is positive. If it is negative, it becomes more negative", or "When we add 10 it does not change x because there is no relation between x and the 10; but when we multiply x by 10 we add x's to it, therefore it becomes larger." The last quote shows that the student really thinks that x becomes larger, and that it is not just a matter of vague formulations.

6. Back to "students and professors" reversal error

The misconception just described may lead to an interpretation of algebraic expressions which can explain the reversal error in the "students and professors" problem of Rosnick and Clement (1980) (see question 6). All explanations proposed in literature before 1993 for this error, were based on interpreting letters as objects, sets of objects, word abbreviations, or labels rather than numbers, and on the influence of problem word order (Rosnick & Clement 1980, Clement et Al. 1981, Davis 1984, Mestre 1986, Kaput 1987). MacGregor and Stacey (1993) claim that students represent on paper cognitive models of compared unequal quantities, which do not depend on problem word order. Crowley, Thomas, & Tall (1994) claim that the order of symbols in the equation depends on process vs. concept orientation of the student.

In my study I found evidence for the above explanations of the reversal error, but also for a new explanation, related to the misconceptions described in this paper: Letters are perceived as numbers; in 6S the origin S and the image 6S are conceived as one entity, the number of students, which is changing and becoming 6 times larger, so that S is now 6 times larger. This leads to the interpretation of 6S as the predicate "S is 6 times larger". (Note that this is a vague one-place predicate, not paying attention to the question larger than what.) The answer 6S=P is interpreted as a table with the (unequal) numbers of students and professors on both sides. The answer 6S+P adds up the number of students 6S and the number of professors P. Both answers include 6S as stating the predicate "the number of students is 6 times larger". Thus students who cannot use algebra's predicates to translate the (two-place) predicate "S is six times as large as P", use 6S as a (one-place) predicate, erroneously enriching their algebraic language, making it like natural language.

Unlike other explanations in literature, this explanation works well for reversals of other arithmetic operations as well.

In **question 6** correct equations were classified as algebraic, while all reversals, e.g. 6S=P, 6S>P, Q=6S+P, were analgebraic. Other errors include non reversal like S=P or 6P>S.

Let us look at some interviews for illustration of the above claim:

Interview 4: (Written response: 6S>P.)

S: Here I understood that <u>the number of students is S</u> and that S

is 6 times as large as the professors, so the number of professors is P and 6S is larger than P. **I**: That is, 6S represents that... **S**: That this is the number of students.
Interview 5: (Written response: 6S students, P professors, Q the whole population, Q=6S+P.)
S: I took Q as the whole university population, then it equals the number of students plus the number of professors. **I**: OK, and what is S in this problem? **S**: The number of students!

7. Conclusion

In this paper I dealt with an analgebraic mode of thinking in the context of interpreting algebraic expressions. It consists of two misconceptions: the identification of origin and image as one entity, and the interpretation of expressions as predicates rather than functions. We saw how these interpretations explain certain phenomena, including reversal errors in translation.

The results of the research show that there is a high rate of analgebraic thinking. We should remember that the study was performed in a population of high school graduates and that the questions were administered to them after finishing the algebra and functions chapters at a university preparatory course. This shows that normal instruction does not uproot these misconceptions, and that special treatment is needed. It seems that the conceptual framework given here can explain the misconceptions and their origins and set the ground for treatment suggestions.

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STUDENTS' RESPONSES UTILISING THE PROCEDURAL AND STRUCTURAL ASPECTS OF ALGEBRA

Carmel Coady University of Western Sydney, Nepean

Cognitive research aimed at determining the components of algebraic thinking in students has become the focus of attention for many mathematics educators. One finding that has seen general agreement among such researchers is that mathematical concepts may be acquired in two ways: i) 'procedural' (Kieran 1992) or 'operational' (Sfard 1992) or 'process' (Dreyfuss 1990 and Dubinsky 1991) and ii) 'structural' (Kieran 1992 and Sfard 1991) or 'object' (Dubinsky 1991), with Sfard stating that a 'deep ontological gap' exists between the two.
The purpose of this paper is to examine this 'gap' by analysing the responses of four students to two mathematical questions selected primarily because their solutions may be obtained by utilising either a 'process' or an 'object' view of algebra. The four students mentioned above were part of a much broader study of algebraic concepts conducted using first-year university students who had recently completed six years of secondary schooling. The results obtained from these supposedly 'experienced' students of algebra appear to indicate that, although they have had repeated exposure to both aspects of algebra, this 'gap' still exists with the possibility of a 'bridge' between the two, in many cases, being extremely remote.

Introduction

It is the firm belief of several educationalists involved in mathematical research that students acquire algebraic concepts from both a 'procedural' (or process) perspective and a 'structural' (or object) perspective (see for example Kieran 1992, Sfard 1991 and Dubinsky 1990). For the purposes of this discussion, Kieran's definitions of these terms have been adopted. She defines the term 'procedural' to infer "... arithmetic operations carried out on numbers to yield numbers" (p. 392), while the term 'structural' incorporates a set of operations performed on algebraic expressions rather than numbers. Features of both conceptions are listed by Sfard (1991): "... the structural conception is static, instantaneous and integrative, the operational is dynamic, sequential and detailed" (p. 4).

The characteristics of the acquisition of algebraic concepts in association with the notion that 'transition' from a 'process' conception to an 'object' conception is not achieved quickly or easily (postulated by Sfard, cited in Kieran 1992) provided the impetus for the present study. The research questions formulated were:

a) To what extent are these two views of algebra still prevalent in older students who have completed their fundamental algebraic instruction?

b) Given that both aspects exist, is there empirical evidence that mirrors Sfard's descriptions?

c) To what extent do students appreciate the connection between the two conceptions, or are they seen as representing two totally separate 'categories' of algebra?

Methodology and Sample

In order to address the research concerns mentioned above, one hundred and twenty eight first year university students were given several questions requiring written responses intended to elicit their understanding of algebraic concepts from the 'procedural' and/or 'structural' standpoint. Ten students were asked to attend a follow-up interview of approximately one hour's duration, during which each was required to repeat the questions while verbalising their reasons. Each of the ten interviews was audio-taped and later transcribed verbatim. It is not the intention of this paper to report on the overall results of the entire sample but rather to present four mini case studies that clearly distinguish between the two conceptualisations being explored.

Frank, Rod, Barbara and Paul, the four students discussed in this paper, were enrolled in mathematically-based science degree programs. All were 18 years of age and had just completed the 2-unit mathematics course during their final two years of secondary school. This particular mathematics course consists of substantial calculus and algebra components with students being exposed to the associated material for six, forty-minute periods (or equivalent) per week. Because one of the primary goals of secondary school education is to promote and instil in students the structural aspects of algebra, it seemed reasonable to include in the sample those students who had completed their elementary algebra training and who had also been exposed to algebraic concepts in a variety of contexts over several years.

For the purposes of this paper, two questions only have been selected and the students' responses to these, both written and verbal, are now analysed and discussed in some detail. It must be noted that both the questions chosen could have been solved quite readily employing techniques involving either or both aspects of algebra under consideration.

Results and Discussion

Question 1: Given $V = \pi r^2 h$.

a) Find V, given $\pi = \frac{22}{7}$, $r = 2$; $h = 7$

b) Find V, given π = same as in a), r = same as in a), h = double the value in a).

c) Find V, given π = same as in a), r = half the value in a), h = same as in a).

d) Find h, given V = same as in a), π = same as in a), r = half the value in a).

The four responses given to this question together with some suggested reasons are considered collectively, as all were identical.

Each of the four students chose to substitute numerical values for the variables V, r and h (where appropriate). Since all treated each of the four parts of the question separately, tedious and repetitive calculations became a feature of the responses given. Prompting by the interviewer to re-examine the question, in the hope that the relationships existing between each part of the question would be identified, was ignored. The interviewer even suggested that an alternative method of solution may be applicable. However, this was also rebuffed with the students stating emphatically that in order to answer the question, numbers corresponding to the conditions stated in each part had to be used. Hence all students, although actively involved in the processing of an algebraic statement containing numbers, were completely unaware that their workload would be considerably reduced had the relationships between the variables used and also within the question itself been identified and utilised.

Question 2: Determine the effect on the

a) volume of a sphere ($V = \frac{4}{3}\pi r^3$)

b) surface area of a sphere ($A = 4\pi r^2$)

if the radius is doubled.

This question evoked qualitatively dissimilar responses from all four of the students being discussed here. An analysis of their answers together with the verbal reasoning behind them revealed distinctive cognitive patterns underlying each response.

Frank

Frank's answer to this question was quite banal - "it will increase". When urged to try and quantify this response in some way, Frank's denial was insistent:

> *No we cannot work out by how much unless it [r] is defined.*

He made absolutely no attempt to select possible values for r as he was convinced that numerical values had to be stated in the question before the magnitude of the effect could be determined. Frank's superficial answer seemed to indicate that he could not fully engage in the processing needs of the question and, therefore, chose 'the easy way out'. However it could be argued that Frank had in fact fulfilled the requirements of the question as he interpreted it, since he did state the 'effect' on the volume/surface area. Furthermore, the fact that additional probing proved futile, appears to indicate that Frank's logical skills did not extend much beyond the obvious.

Rod

Rod's answer to both parts of this question was that the volume (or surface area) "doubled". When asked the reason for this, Rod replied (with regard to part b)):

> *Well ... if the radius of the sphere had doubled, you'd get a larger sphere so therefore the surface area would be two times as great.*

Rod's working showed that he had merely inserted a '2' into the formula. Therefore, although he had no knowledge (in terms of a numerical value) of the surface area of a sphere with radius r, he knew that if the radius was doubled, then the sphere became larger by 'twice' the amount. In other words, he multiplied the original (albeit unknown) volume/surface area by two. Clearly Rod had seized upon one aspect of the question only and had deduced (incorrectly) a conclusion based on this single piece of information.

Barbara

Barbara's method of solving both parts of this question involved the substitution of values for r. Her choice of this method appeared to be based on intuition as the following extract (also typifying her response to part b)) indicates:

> *I would substitute values here, first doubling the radius. OK we'll use $r = 2$ and $r = 4$.* [With the aid of a calculator she obtained $V = 33.5$ and $V = 268.$ .respectively]. *So, if you double the radius of a sphere the volume of the sphere is increased by approximately 8 times.*

Of particular interest is the wording of Barbara's conclusion: " ... the volume [or surface area] is increased by approximately 8 times [4 times for the surface area]". The use of the word 'approximately' obviously results from the computations made with the aid of the calculator. When asked whether she could state the 'exact' effect on the volume/surface area of a sphere if the radius is doubled, she replied:

> *No as whatever values were chosen for r, decimals would be involved and only an approximate answer could be given.*

She appeared uncertain as to whether the approximate value would change given a different set of values for r and hence was reluctant to pursue this line of thought. The most logical explanation for this was that she had reached a conclusion that she felt satisfied the requirements of the question and therefore saw no reason to explore other possible solutions.

Paul

Paul was able to spontaneously generalise the phrase 'the radius is doubled' into symbols ($2r$) and then to substitute this expression into both formulae. Furthermore, he was capable of correctly interpreting his answer, exhibiting complete confidence in his belief that the use of the abstract 'object' $2r$ would result in the correct answer.

> *Substitute 2r and square that* [$(2r)^2$] *equals* $4\pi.4r^2$ *so that just is timesing the surface area by 4.*

Paul's ability to use and perform operations with the generalised expression $2r$, without the need for a concrete referent such as the substitution of numbers, suggests that he is quite comfortable with the structural properties of algebra. His immediate recognition of the quantitative effect that doubling the radius has on the volume/surface area would appear to indicate that his logical skills are relatively advanced. In order to examine the extent of these skills, Paul was asked to re-do Question 1. Surprisingly, his response still centred on the use of numerical substitution, even though during this latter phase of the interview, Paul was again prompted to look for relationships between the different parts of the question. However, Paul could still not identify any relationship between the parts, nor did he use a variable expression to obtain a solution. Arguably the cuing effect of the explicitly stated values for π, r and h may have dominated any impulse to generalise although this appears contrary to his 'object' orientation demonstrated previously in Question 2.

[It should be noted here that Frank, Rod and Barbara were not asked to re-do Question 1 after having attempted Question 2. Given their solutions to the latter together with their accompanying reasons, it was felt that any further investigation of Question 1 would serve little purpose.]

Thus it appeared from the responses given by Frank, Rod and Barbara to both questions, that the manipulation of numbers in some form or another was mandatory if they were to achieve a conclusion. However, Paul's responses to both questions indicated a clear dichotomy in terms of the solution methods used. As stated earlier, this may be attributed to the nature of the questions asked, although this inference does lose some of its credibility as further prompting failed to elicit any association between his two solution procedures.

Conclusion

This paper has examined both the procedural and structural aspects of solving algebraic problems from a student perspective. The discussion above clearly illustrates that both means of acquiring algebraic concepts exist at this educational level, with the 'procedural' aspect of algebraic learning, requiring that *numbers* must be manipulated, predominating.

Frank, Rod and Barbara provided responses to both questions that manifested the features of a procedural approach as described by Sfard since their answers tended to be dynamic, in that they

could seemingly 'change' given a different set of circumstances, and sequential, as the solutions were characterised by step-by-step rather detailed procedures. Frank knew that if the radius increased then rationally the volume/surface area would increase. However, he was unable to quantify his conclusion as he was still locked into the procedural stage where he required the substitution of 'given' values for r. Rod's answer could be classified as slightly more sophisticated in that he knew that the resulting volume/surface area would increase by "double the amount", after having multiplied the 'unknown' volume/surface area by two. Barbara, on the other hand, while still needing to work with numbers, was successfully able to integrate all pieces of given information. This successful integration and hence completion of the problem would also help to explain her hesitation in corroborating her initial conclusion with any additional numerical support. Finally, with regard to these three students, it seems reasonable to conclude that, although they each felt the need to 'process' in order to solve the algebraic problems, they clearly displayed differing degrees of 'procedural' competence. This is evidenced by Frank's and Rod's use of a single piece of information only in Question 2, while Barbara was able to hold all relevant pieces of information while formulating an answer.

Paul's responses to Question 2 tended to be static (he did not feel the need to justify his conclusion any further), and instantaneous (as he was able to spontaneously generalise "double the radius" to $2r$) resulting in a complete integration of the question. Once again Sfard's description of the structural approach to algebraic problem-solving has been verified. However, his inability to use this method when answering Question 1 is somewhat disturbing, perhaps hinting at the possibility that, although a 'transition' from one conception to the other may occur, each may then continue to develop separately with the initial link between them being forgotten or even lost completely. This adds a further dimension to Kieran's (1992) statement: "The transition from a "process" conception to an "object" conception is accomplished neither quickly nor without great difficulty" (p. 392). In fact, as demonstrated by this small study, some students may never accomplish this 'transition', implying that Sfard's 'deep ontological gap' between the two aspects may unfortunately never be totally bridged. This is further exemplified by the responses of the fourth student, Paul, who was successfully able to manipulate numbers as well as algebraic expressions. Hence the 'gulf' between the procedural and structural aspects of algebraic problem solving appeared to have been 'bridged' to some extent. However in Paul's case, this could be viewed as a 'one-way' crossing only as he

was unable to immediately generalise from the particular in Question 1 and thus reduce the otherwise necessary but laborious calculations. This hints at the possibility that these two perspectives, 'process-object', may develop independently of each other with a student often becoming proficient in using either one or the other interpretation but lacking (or forgetting) the two-way connection between them.

In conclusion, a final point that should be stressed is that the results of this study have at least two far-reaching implications for the teaching profession. First, it appears that one of the major goals of secondary school teaching, that of instilling in students the structural properties of algebra, is not being achieved for all students. Thus the potential for these students to acquire the necessary thought processes required for advanced mathematical thinking must be limited, at least to some extent. Secondly, while a thorough development of both the procedural and structural conceptions is desirable, the affinity between the two should not be de-emphasised once structural competence has been achieved. Both play important roles in mathematical activity (Kieran 1992) and hence their recognition and an awareness of the influence of their interconnection should be continually reinforced so that a deeper understanding of the principles underlying mathematics is secured.

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WORD PROBLEMS: OPERATIONAL INVARIANTS IN THE PUTTING INTO EQUATION PROCESS

CORTES Anibal
LabPSYDEE, 46 rue St-Jacques, Paris, France

Errors made by 9th and 10th grade students in putting word problems into equation were analysed and classified. The classification of all errors recorded during the experiment has resulted in only three categories: **a** - Errors in the construction of mathematical correspondences. **b** - Errors that concern the concept of equivalence and that of the unnknown. **c** - Errors in the construction of a "calculable" mathematical object.

The analysis of relationships between errors and the mathematical properties violated allows the identification of inherent operational invariants for the putting into equation process.

Research in cognitive psychology concerning word problems has often focused on the analysis of reasoning by analogy, for example Bassok and Holyoak (1989); Clement J. (1988).

In international publications devoted to mathematical education many authors analyze the resolution of word problems. The passage from natural language to an algebraic expression was analyzed by several authors in terms of syntactic and semantic translation, MacGregor and Stacey (1993) have reviewed this research and focus their work on reversal error.

Several authors analyse the mathematical problem posing processes, Silver E.A. (1993) has reviewed this research.

Other authors construct methods for the resolution of problems, see Filloy E. and Rubio G (1993). Rojano T. and Sutherland R. (1993) recommend writing intermediate mathematical expressions and then construct, by substitution, the equation of the problem. The intermediate expressions become thus explicit, but the cognitive process underlying the writing of these expressions remains unexplored.

The algebraic solving process is not analysed in this article, because: the solving of equations or systems of equations is made by means of algebraic transformations that end to other equations that are not, in general, related with the text of the given problem; there is therefore a **detour behaviour** in the algebraic solving process. This conclusion allows us to analyze the putting into equation process independently of the associated algebraic solving process. The operational invariants that guide thought in the solving of equations can be found in Cortés (1993).

In this article the putting into equation process (notably the implicit cognitive work) is modeled in terms of operational invariants.

The theoretical framework and the experimental work.

Our theoretical framework is based on the "Conceptual Field Theory" (Gerard Vergnaud, 1990). Cognitive behavior is modeled in terms of "schèmes". The concept of schème was introduced by Piaget and later was further elaborated by Vergnaud, in order to find a model for the acquisition of complex knowledge, in particular scientific knowledge. According to Vergnaud (1990): *"a schème is the invariant organisation of behaviour (action) for a certain class of situations... A schème is made of four different kind of ingredients: operational invariants, inference possibilities, rules of action, goals. The representational part is essential".* The operational invariants are mainly: implicit concepts (concepts-in-action) and implicit theorems (theorems-in-action).

The analysis and the classification of word problems that appear in secondary school text books, as well as the analysis and the classification of elementary cognitive mathematical tasks necessary for putting word problems into equation, provide indices on the implicit nature of implied cognitive processes. The analysis of relationships between errors and the mathematical properties violated allows for the identification of inherent operational invariants for the putting into equation process.

The experimental work: for the past several years we have focused our investigations using 7th through 10th grade students. In this article, however, we will only discuss the results from the **9th and 10th** grade classes: 25 word problems were given to students (5 different tests comprised of 5 problems each) the resolution of which implies the construction of first degree equations.

The resolution of these problems implies: a) The possibility to construct a single-unknown equation directly. b) The construction of a system of two equations (two unknowns). c) The construction of a system of several equations that can be reduced to a system of two equations. d) The construction of second degree equations that can be reduced to a first degree one by simplification of terms.

For some problems it is necessary to write and to transform formulaes. Problems concerning inequations and the the study of numerical functions will not be approached in this paper.

Errors in the putting into equation process

All the errors observed in the putting into equation process can be classified into the following categories:

a - Errors in the construction of mathematical correspondences.

b - Errors that concern the concept of equivalence and that of the unknown.

c - Errors in the construction of a "calculable" mathematical object.

a - Errors in the construction of mathematical correspondences.

After reading the problem text students are faced with the construction or with the identification of useful mathematical functions: a number (given or unknown) corresponds to **only one** number (given or unknown). Each of these correspondences is a particular case of a numerical function in which the algebraic expression is not known. The search for these correspondences is guided by the necessity to construct one or several equations that will allow to solve the problem.

For example: *A person has 120F more than a second person. When they have both spent 360F, the first person has twice as much money than the second one. How much money did each person have before making their purchases?*

26% of students succeed: $y + 120 = x$; $x - 360 = 2(y - 360)$ (x represents the money of the first person and y represents that of the second person). The problem makes reference to additive processes that unfold in time, in which there are initial states (x and y) which, after the expense of money (transformation), will correspond to the final states (x-360 and y-360). The conceptualization of these processes is necessary to construct a correspondence between these states. For example: the initial state y for the second person will correspond to a state x, greater (120F more) than the first; the functional relationship remains to be constructed. We observe that a correspondence (sometimes evident) built correctly in natural language, can drive to an erroneous numerical function. For example, many students propose $x + 120 = y$ instead of $x = y + 120$

30% of students make errors in the construction of these correspondences and write false equivalences, for example: $x + 120 = 360 * 2$

or a system of equations $120 + x - 360 = 2y$; $x - 360 = y$

The next error: $x + 120 + y - 360 = 2x + y$ clearly shows the meaning of the **"summary of the problem text"** of the written equation: some students do not make a rupture with natural language.

Sentences are sometimes perceived in an ambiguous manner, for example: *One pays a sum of 1750F with 24 bills of 50F or 100F. How many bills are there of each kind?* Some students write the following equations: $24x = 1750$ and $24y = 1750$. This particular analysis of the problem leads to absurd numerical solutions: the students do not check the plausibility of the results obtained. Written equations do not translate relationships of the problem and thus the meaning of the symbols x and y shift from the meaning of a number to that of a unit: "bills of 50F and 100F" respectively.

Conclusion: These errors always lead to a false equivalence: it is the written trace that one analyzes. Therefore, in the analysis of implicit processes of thought it is necessary to go beyond the written equations. The identification of relevant correspondences implies the respect of the fundamental constraint of mathematical functions: only one image. Consequently, there is an operational invariant: **the concept-in-action of mathematical function** expressed in terms of correspondences between sets (modern definition of function). The students have never seen this definition; a concept-in-action designates implicit operational knowledge.

b - Errors that concern the concept of equivalence and that of the unknown.

Once pertinent correspondences (contained explicitly or implicitly in the problem text) are identified, students are faced with mental construction (implicit) and with writing the equations. The introduction of the "equal" sign establishes a rupture with natural language. Each equation has, in general, the meaning of an equivalence between magnitudes, and the terms of this equivalence must therefore respect a constraint of homogeneity: to have the same units and the same meaning.

An equivalence can be constructed: a) By the equality of two functions. b) By the substitution of given numbers into a function. c) By the substitution of given numbers and functions into another function. The mathematical functions giving origin to a first degree equation are, in general, also of the first degree and of one or several variables; for example: $y= 3x$, $y= 5x-20$, $3x + 4y= z$..........

b - 1 - The functional relationship between variables is not constructed.

For example in the problem: *One pays a sum of 1750F with 24 bills of 50F or 100F. How many bills are there of each kind?* From the first sentence one can construct a correspondence: 24 bills corresponds to a sum of 1750F; a numerical function can not be immediately constructed. It is necessary first to conceptualize that there is an unknown number x of 50F bills and an unknown number y of 100F bills; and that the number of bills will total 24, mathematically expressed as: $x + y= 24$. Moreover, the total sum (1750) must contain two sums of money: S1 comprised of 50F bills and S2 of 100F bills (the equation is: $S1 + S2= 1750$). It is also necessary to construct that S1 corresponds to x number of bills following the numerical function $S1= 50\ x$, similarly $S2= 100\ y$. Equations that model the problem are constructed from these numerical functions. This cognitive work is generally implicit. Several students write the following equations:

$50x + 100y= 1750$; $24\ (x + y)= 1750$ (instead of $x + y= 24$).

The second equation "summarizes" a correspondence between sets (24 bills corresponds to 1750F): the function $z= 24= x + y$ (total number of bills) is not constructed. Also, the equation $24(x+y)=1750$ does not respect either the homogeneity of the units and the significance of all its terms, or the homogeneity of the significance of symbols inside a system of equations: x can not be a "object or a unit" in an equation and a number of objects in the other.

Another exemple: *A rectangular piece of land has a perimeter of 110m. By decreasing its length 1m and increasing its width 1m, its area is increased by 4m². What were its initial dimensions?*

30% of students succeed: $2x + 2y= 110m$, $(x-1)\ (y + 1)= xy + 4m^2$. An analysis in terms of initial and final states can also be made for this problem. There is a correspondence between linear measures and perimeter and there is another correspondence between linear measures and area. The corresponding numerical functions are formulaes that the student is supposed to know. Some students write: $(x-1) + (y + 1)= 110 + 4.2$. In this example 4.2 represents $4m^2$: for some students a

length and an area have the same unit. There is therefore a failure in their concept of area and in the conceptualization of the relationships involved in the problem. In this equation areas and perimeters are processed indiscriminately: these errors are conceptual. This is similar for the following errors: $(x-1) + (y + 1)= 110 + 4$ or $x-1 + y + 1= 4m^2$ or $(x-1) + (y + 1)= xy$. These equations are not equivalences because they do not respect the constraint of homogeneity.

Another exemple: *A gardener wants to plant a surface with tulips, in which there would be 3/10 yellow tulips, 2/10 red tulips and 30 black tulips. How many yellow tulips did the gardener buy?* Some students write "x number of yellow tulips; y number of red tulips" and then the equation: $(3/ 10)x + (2/ 10)y + 30= n$. In this equation the significance of the unknowns shifts to that of objects or a unit "tulips" and the numerical fuctions $x= (3/10)n$ and $y=(2/10)n$ are not constructed. This type of error is very frequent.

b - 2 - Some errors are due to a failure to check the functional relationship between variables.
For the sentence *"the length is 20 m greater than the width"* many students propose $L + 20= l$ (instead of $L= l + 20$), this equation has the meaning of the "summary of the problem text". To check the validity of the written equation implies checking the functional relationship between variables by means of numerical examples: the student (or the expert) will give to variable l a numerical value (for example10) and then calculate the value of L and verify that "L is 20m greater than l". These errors are due to a failure to check: the numerical function underlying the equation is not constructed.

b - 3 - The homogeneity constraint of units is not respected.
For example: *A rectangular field has a perimeter of 5.28Km. Calculate its dimensions knowing that the length is 220m greater than the width?*
Some students, starting from the following equivalences (often implicit):
$y= x + 220$ and $5.28= 2 (x + y)$
write: $\mathbf{5.28}= 2 (2y + \mathbf{220})$.
The homogeneity of units is not respected; in most cases these errors are due to a failure to check.

Conclusion: The equation concept, taking the meaning of **equivalence** between magnitudes, is an operational invariant in the putting into equation process. **The numerical function concept** necessary for the construction of equivalences is also an operational invariant.

In the construction of numerical functions and equivalences, thought is guided by the principle: **the respect of the homogeneity** of terms that constitute the equation. This principle is also an operational invariant.

c - Errors in the construction of a "calculable" mathematical object.

The construction of correspondences, numerical functions and equivalences is motivated and guided by the necessity to construct an equation or a system of equations (in examples analyzed here). Furthermore the writing of a mathematical object will be the outcome of the putting into equation process because it allows the calculation of the numerical value of an unknown or several unknowns: it is, in this sense, a "calculable" mathematical object. The choice to construct a particular mathematical object implies conceptualization of the mathematical properties (of this object) concerning the possibility to provide the type of numerical result that one seeks to calculate. **The construction of a "calculable" mathematical object** is an operational invariant.

c - 1 An erroneous substitution in the construction of a single-unknown equation. The construction of a single-unknown equation often implies the substitution of an unknown by a function. For example in the problem: *A rectangular field has a perimeter of 5.28Km. Calculate its dimensions knowing that the length is 220m greater than the width?* The substitution of the function is made, sometimes, in an erroneous manner: one constructs a function y=f(x) and then f(x) takes the place of x instead of y; it is a conceptual error, for example:

$y= x+220$, $5,28=2(x+y)$ which leads to $5.28= 2\ (x+220+2y)$

one ends thus with a two-unknown equation: a mathematical object non relevant for the solution. A failure to check can lead to the following error:

$y= x + 220$; $5.28= 2\ (x + y)$ leads to $5.28= 2\ (2y + 220)$

c - 2 - Errors that concern the concept of system of equations

c-2-a) The writing of two identical equations. For example in the problem: *One pays a sum of 1750F with 24 bills of 50F or 100F. How many bills are there of each kind?* Some students write a system with two identical equations: $50x + 100y= 1750$; $50x + 100y= 1750$.

In their concept of system of equations students lack mathematical knowledge that would allow them to decide if the written system makes it possible (or not) to calculate the unknowns. The former is a conceptual error.

c -2 -b) Impossibility to solve an "unusual" system of equations .

Students from 9th and 10th grade know the single-unknown equation and the system of two equations as tools for solving word problems. A large number of students are able to solve systems of equations, but only if they are written according to the "usual" script: $ax + by= c$; $a'x + b'y= c'$; they then apply quasi algorithmic procedures to solve them. These students stop after the construction of an "unusual" system of equations, for example:

$y= x + 2x + 4x + 8x$; $y= x + (x + 22) + (x + 44) + (x + 66)$

or $y= (3/ 10)x$, $(2/ 10)x + y + 30= x$ or $L= l + 220$; $5280= 2l + 2L$

Some of these systems can be solved by the substitution of an equivalence into the other. The use of

the linear combination method can imply the rewriting of an equation by means of an algebraic transformation well known by students (for example L= 1 + 220 becomes L-1= 220). This type of dead-end appears in classes where the teaching is focussed on the resolution of systems of the type: $ax + by= c$; $a'x + b'y= c'$.

c -2-c) Dead-end in front of a system of equations the resolution of which mplies algebraic transformations.

In the algebraic treatment of word problems, the detour behaviour can begin with algebraic transformations that lead to a mathematical object that one is able to calculate. However, students often construct equations with several unknowns that they do not transform (in order to put them in the form that they can process). For example: $(x + 5)/ (y + 5)= 9/ 11$; $(x-5)/ (y-5)= 2/ 3$
or $x + y= 50$; $y + z= 29$; $z + x= 35$

Conclusion: These errors show: First, a limited conceptualization of the mathematical properties of written equations (concerning the possibility to provide the type of numerical result that one seeks to calculate). Second, the absence of the checking process.

Cognitive model of the putting into equations process.

Our cognitive model is the functioning of the schème which governs the putting into equation process. Some aspects of this model can be found in Cortes A. (1994).

The resolution of word problems has a purpose: the calculation of the numerical value of unknown magnitudes by means of the construction of a relevant mathematical object: a single-unknown equation (if one wants to calculate the value of only one unknown); a system of two equations (if one wants to calculate the numerical value of two unknowns); a function; an inequation... The choice to construct a particular mathematical object implies the conceptualization of the mathematical properties of this object. **The construction of a "calculable" mathematical object** is a principle that guides thought in the resolution of word problems. This principle provides a means to select relevant mathematical relationships among the whole range relationships.

After reading the problem text, students are faced with constructing or identifying useful mathematical correspondences: a number (given or unknown) corresponds to only one number (given or unknown). Each of these correspondences is a particular case of a numerical function in which the algebraic expression is not known. Consequently, there is an operational invariant: **the concept-in-action of numerical function.**

Then, students are faced with the implicit or explicit construction of equations that have the meaning of equivalences between magnitudes; the introduction of the "equal" sign

establishes a rupture with natural language. **The concept of equivalence** is also an operational invariant. However, the terms of numerical functions and written equations must have the same units and the same significance and the coherence of the resolution process implies that symbols have the same significance all through the solution process. Consequently we can define a fourth operational invariant, a principle: **the respect of the homogeneity of equation terms and symbol significance**. This principle guides the transformation of correspondences (natural language) into numerical functions and allows then the construction of equations as well as the checking of the validity of these equations: it establishes therefore an essential link between the conceptualization of reality and mathematical modelization.

Conclusion:

Classifying errors according to the mathematical property violated, allows us to classify errors that have different scripts into the same caterory. The type of teaching influences the occurence frequency of certain errors, as well as their script.

The cognitive model that we propose takes into account the most important conceptual aspects of the putting into equation process. The infuence of the teaching process is not analyzed in this article, and neither is the checking of the numerical results and other processes.

The construction of a cognitive model of the putting into equation process is intresting from a theoretical point of view and also from a practical one (e.g. teacher's training).

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A CASE STUDY OF ALGEBRAIC SCAFFOLDING: FROM BALANCE SCALE TO ALGEBRAIC NOTATION

Jorge Tarcísio DA ROCHA FALCÃO[1]
Universidade Federal de Pernambuco (BRASIL)
Graduate Program in Cognitive Psychology

A experience of introducing algebra to a group of 11 low-class and poorly-schooled children from Recife (Northeast of Brazil) by a group of researchers coordinated by the author is reported in terms of case study. This experience consisted in the proposal of a didactic sequence covering a semester, and including four sets of activities: introduction of the two-pan balance scale in order to make explicit some basic principles, passage to simbolic representation and introduction of a new contract (represent first, solve later), "scale-cleaning" using symbolic representation, and simbolic depuration with rewriting. Clinical data suggest important acquisitions in terms of a new representational tool, for which the two-pan balance scale has served as preparatory metaphor.

The experience reported here was conducted by the author during the period from August to early December 1994, in the context of a larger project of assistance to poor children in Recife, Brazil. This project has been supported by grants from European non-governmental organizations, and consists in offering a professional training coupled with school-like activities in language and mathematics. Professional activities offered include the crafting of marionettes and giant puppets, bread production in bakeries, artisenal fabrication of candies and formation of waiters (for boys only). The essential aim of the project is to offer these children an alternative to the streets, by offering them an opportunity of learning a professional tool and having some school support. The activities in mathematics mentioned above were conducted by a group of researchers and students of the Graduate Program in Cognitive Psychology - UFPE, and consisted of three main topics, the first two having been exaustively *negociated* with the group: 1. new Brazilian currency (R$, *real*) and decimal number system; 2. algorithms of subtraction and division and 3. introduction to algebra. Activities concerning topics 1 and 2 were conducted by two associated researchers, assisted by graduate students; topic 3 was under the coordination of the author of this report. The group of researchers was offered complete autonomy in proposing mathematical activities during the semester: there was no mathematics teacher to "negociate" with, no program needs to cover, no curricula prescriptions nor specific time-table to take into account. We developed the complete experience (three topics) in mathematics in 16 meetings that took place once a week, on tuesdays afternoons, in the rooms of the CECOSNE Fondation at Recife. The group of 11 children (6 boys and 5 girls) who participated in the experience reported here was heterogeneous both in age (12 to 17 years) and school level (6th grade to high school); this last aspect, by the way, must be considered cautiously, since high-school students showed poorer level in elementary mathematics in a previous evaluation than elementary 6th and 7th grade ones. Only two among all of the children ventured, upon questioning, to offer a meaning to the word "algebra": the first one, a clever 15 year-old boy, 8th level at elementary school[2], stated that "*Algebra... é o bicho*!" (local popular slang corresponding roughly to: *Algebra... it's the boogie man*!); the other one, a 16 year-old girl, 1st high-school level, wondered if algebra wasn't "*... uma coisa que tem a ver com asa-delta*" (*something concerning hang-gliders* [called in Brazilian Portuguese asas-delta (*delta-wings*) because of their delta-shape]). We discussed with the group the possibility of starting a set of mathematical activities concerning algebra, without offering any previous definition of it (in spite of their insistence in having such a definition). They agreed in starting studying algebra, provided that it wasn't too boring. We proposed, in the next meeting, the first of three main sets of activities, all of them described below.

[1] This study was sponsored by grants from FACEPE (Fundação de Amparo à Ciência e Tecnologia) and CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico).
[2] In Brazilian public school system, algebra is frequentlly introduced by the end of the 7th grade.

The experience with algebra covered 10 weekly meetings of aproximately two hours and a half each, and were all registered by a research assistant.

1. Facing a certain problem and introducing some activities with the two-pan balance scale

1.1. At the first meeting, the following problem was proposed to the group:

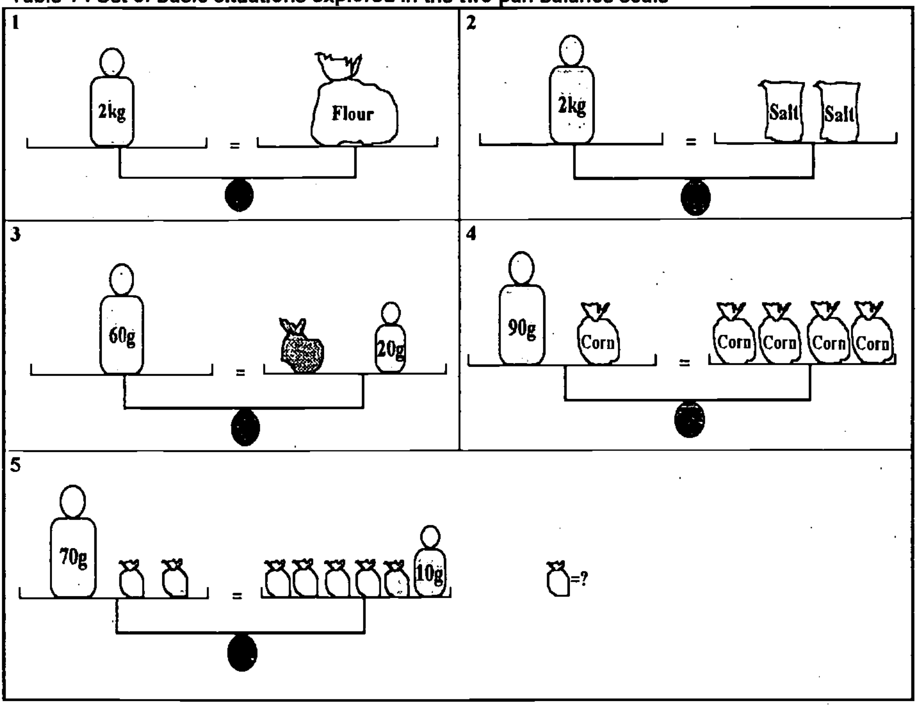
João had 5 bags of marbles and 2 more marbles, and his friend Pedro had 3 bags of marbles of the same type of João's and 6 more marbles. The two boys had, in all, the same number of marbles. How many marbles were there in each bag?

This problem sparked a lively debate between two blocks of opinion in the group: the first one, under the leadership of S., a 15 year-old girl, 1st grade/high school, stated that the whole problem was a trick, since it wasn't possible to have two people (João and Pedro) owning each one a ***different number*** of bags and extra-marbles and, concomitantly, having the ***same number*** of marbles; the second block of opinion proposed that it was possible to find out the number of marbles, provided that we were very patient and lucky and tried a lot of possibilities (a small "sub-group" inside this block of opinion stated that, in fact, we could not find out a precise number, since there wasn't a precise operation to do in order to calculate the number of marbles). In spite of this second bloc of opinion, none of their members tryed to "patiently" find out the number, and the first meeting was over without any answer at all.

1.2. At the second meeting, we proposed to postpone the debate about João and Pedro's problem, and to start thinking about a series of situations concerning the use of the very familiar two-pan balance scale. Five basic situations in the two-pan balance scale were then explored and discussed with the group during this and the next two meetings. It is important to mention that these situations, represented pictorially in the table 1, were presented to the children with an actual scale. Among the set of five situations, situations 3 and 4 were especially discussed, since for many of the children they displayed an inproper, messy set-up, caused by two violations of the two-pan balance scale canonic lay-out: 1. known weights in both pans (situation 3); 2. unknown weights in both pans (situation 4). The group was then motivated to discuss a strategy of "cleaning" the scale, in order to be able to find out the unknown weight. A basic *theorem-in-action* (Vergnaud, 1985) concerning the fonctioning of the two-pan balance scale (•), with its logical consequence (*), was previously explicited in the form of a principle : • [Principle 1] *we have to have equal weights in each of the two pans of the scale in order to have these pans in equilibrium*; * [Consequence] *if the pans of a two-pan balance scale are in equilibrium, then there are equal weights at each pan*. By the end of the fourth meeting, a small sub-group proposed that the right way to proceed, in situation 3, was to take away the 20g weight in the right pan of the scale; the violation of the basic principle of the two-pan balance scale was resolved when a complement of the proposition above was produced in the following terms: we take away 20g from the right pan of the scale, *and* [Principle 2] *we do the same in the left pan, in order to keep the pans in equilibrium*. The (proposital) lack of a 40g weight forced the group to propose an important complement to the most recent principle: in the absence of concrete weights to put in the pans of the scale, we can *make believe* the substitution was done. Despite this important and consensual achievement, the transfer to situation 4 was not direct and immediate, since many of the subjects stated that this situation was very different of situation 3: "*In situation 3, we know the weights, so we can take them away or just to imagine we've done so; in situation 4, we don't know the weight of the corn packages, and we can't do anything upon unknown things*!" (S., 15 year-old girl, 1st grade/high school, the same girl who stated the impossibility of solving João & Pedro's problem). The debate sparked by this restriction was very interesting and intense. In fact, S. didn't have a good answer to the important question asked by a little 6th grade 12 year-old boy: "*WHY can't you take way one corn package from each pan if you know the scale will keep the balance*?" The group was then convinced that the principle of taking weights away (factually or making believe) could be extended to situations where the weight of the

package was unknown, provided that principles 1 and 2 were respected. Situation 5, a combination of difficulties of situations 2 and 3, provoked an unexpected discussion on "procedural order" taken seriously into account by the group: when we have to "clean" known and unknown things in the scale, by which one we begin: knowns or unknowns? They decided, as a social contract (not strictly respected, in fact), to begin always by *known* things. Once these two important principles (1 and 2) explicited and refined by the group, we started the second set of activities, described below.

Table 1 : Set of basic situations explored in the two-pan balance scale

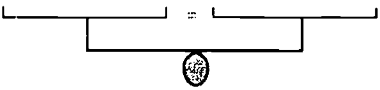


2. Describing scale dispositions and installing a new contract: *represent first, try to solve later*

2.1. At the fifth meeting, the subjects were introduced to a new activity, consisting of representing, in a diagram prepared by the author (see reproduction in figure 1 below), a new set of situations in the two-pan balance scale. This activity was presented to the children as a *scale-dictation*, in analogy with the familiar situation of class-room dictations; they were asked to represent, in the paper, four situations proposed in a real scale, using known and unknown weights. These situations corresponded to the following algebraic structures: $x + a = b$; $2x + y + a = b + y + x$; $2x + y + z + a = 2x + y + b$ and $x + y + a + b = y + c$

At the very begining of this activity, a new *contract* (Brousseau, 1988; Schubauer-Leoni & Perret-Clermont, 1985; Schubauer-Leoni, 1986; Perret-Clermont, 1992) was for the first time introduced: avoid trying to find out the value corresponding to the unknown weigth, trying instead to initially *represent* the situation, with the aid of the scale-diagram. Although everybody gently and immediately seemed to agree, we soon realized how difficult it is, in fact, to postpone the resolution

Figure 1: diagram proposed as auxiliary paper and pencil tool for the representation of scale-situations



of a problem or problem-like situation, *"wasting time with drawing and writing"*, as one of the subjects said. It is very comprehensible: they were under another implicit, older and stronger contract which states that *the longer someone spends in finding the solution of a school-problem, the more "burro" (stupid) he/she is.* As a consequence, the new contract (concerning the priority of representation over resolution) had to be re-taken plenty of times. We also discussed the meaning of the equal sign (=) in the auxiliary diagram above mentioned. They accepted without discussion that this equal sign, in this particular context, did not represent an identity between the content of each scales's pans, but rather the equilibrium of the pans caused by the equality of weights in each pan of the scale. The third and equally important point of discussion concerned the representation of unknown weights in the diagram. Since the group was especially worried about time-spending in the task-solving procedure, it was easy to negociate the introduction of a simplified representation for the known and unknown entities in the scale: we proposed small geometric figures (circles, triangles and squares) to represent the packages (unknown weights), and numbers to represent known weights. This very question of codification generated a very important debate, since one of the subjects (A., 6th grade) decided to use squares as simbolic representation for unknowns, and represented by the same symbol (the square) different packages (corn and flour) put on the scales's pans. This proposal was criticized by L. (1st grade, high school), who called the attention of the group to the non-differenciation of different entities in A.'s representation. L. proposed, then, an alternative representation where different packages were coded by different symbols (squares and a triangle: see figure 2 on the next page). A. argued that *"the teacher had allowed the use of any symbol to represent unknown things"* [what is true] , and he had the right of choose the squares, but he and the rest of the group was easily convinced to adopt L.'s representation. We explaned, then, our third principle: [Principle 3] *different things must be represented by different symbols in the scale diagram.* Later, this principle was refined after a debate caused by some troubles in the representation of a complex scale lay-out: sugar, salt, salt, corn, known weight (first pan), corn, sugar, known weight (second pan). One of the subjects decided, coherently with principle 4, to utilize three different symbols in the left pan, but violated the correspondence *food package ↔ symbol* in the right pan (triangle for sugar in one pan, square for sugar in the other). The debate led the group to refine principle 3 with an addendum in the following terms: [Principle 4]: *once a symbol is choosen for representing something unknown, this symbol cannot be used to represent another unknown entity, and the relation previously established between symbol and thing represented cannot be changed in the context of a particular scale-diagram.* The group was, then, able to represent many situations proposed in the two-pan balance scale; we passed, then, to another set of activities, consisting of representing not more scale-dispositions, but ***problems***, in the same scale-diagram.

2.2. The first problem proposed[3] is reproduced below:

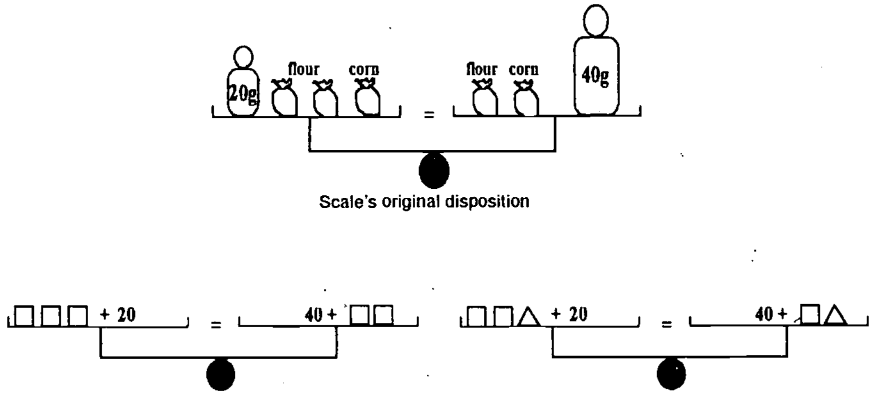
Amanda and Tiane like to collect samples of stationery. Amanda's collection is composed by 70 especially-decorated individual sheets of paper; Tiane has 10 individual sheets and two similar blocks of sheets given by her father. We know that the two girls have the same number of individual sheets of stationnery. How many sheets of stationnery are there in each of Tiane's blocks?

The transposition from scale representation to problem representation required the group to work upon two aspects: 1. To discuss once more the *contract* giving temporal priority to representation over problem solving procedure; 2. To forget the two-pan balance scale *itself*, and start considering the derivated diagram, since the group was facing situations concerning other equalities (e.g.,

[3] This problem is part of a set of problems proposed originally by Lins Lessa, 1994, and adopted in this study.

number of stationnery sheets) than weight equality. After having represented these problems, we passed to another group of activities, consisting of solving the situations represented (values of unknowns in scale dispositions and solutions in problems) through *"scale cleaning-up"*.

Figure 2: Scales's disposition and respective A. and L. propositions of representation.



Scale's original disposition

Representation proposed by A.

Representation proposed by L.

3. The "scale cleaning-up" activity and symbolic depuration

This set of activities covered the three last meetings, and consisted of a symbolic transposition from motoric, effective activity of taking away packages from the pans of an actual scale to an activity of eliminating icons (representing unknowns) graphically, taking into acount principles 1 and 2 (Figure 3 reproduce the activity of scale cleaning proposed by R., 5th grade). All representations previously proposed were then given back to their proposers in order to be "cleaned-up". A four-point procedural sub-contract was stablished for the cleaning-up procedure: 1. Make explicit which icon-unknown would have its value searched; 2. Keep in mind principles 1 and 2; 3. Rewrite the new scale set-up after each round of scale cleaning-up; 4. Reach a final line of rewriting with the format **icon = value.** The piece of protocol on the next page (Figure 3) illustrates well these points. After this activity, many problems were then proposed in order to be represented and then cleaned-up. It's important to mention that, at this phase of new problems, the auxiliary diagram proposed was first reduced to a simplified form, and then reduced to the equal sign, as shown in figure 4. We tried, at this phase, avoid mentioning scales explicitly, talking instead about principles (especially principles 1 and 2) aplicable to representational situations. We also discussed ways of refining representational propositions, and the group was able to propose two main refinements: 1. Substitution of a series of icons of one type by a numeric coefficient folowed by the icon (e.g., 3□ instead of □ □ □); 2. Substitution of the connector "and" by the operator + (plus), in the transposition from natural language to representational language. We proposed to add a final procedural item of contract in combination with the two itens above: representing the familiar icons (triangles, circles and squares) by some specific letters, those at the end of the alphabet: **x**, **y** and **z**. This substitution was very well accepted by some of them, since they realized that their representations had rejoined those in mathematics books. The whole work was then completed by an invitation to bring to class their algebra books, in order to work over some algebraic expressions and

problems considered very difficult by themselves. Because of time limits, only one meeting (the last one) was dedicated to this activity, during wich S. was also invited to reconsider the problem of João and Pedro, in order to verify if it had a solution: after having represented the problem, she easily cleaned it up and solved it, with a shy smile of satisfaction.

Figure 3 : reproduction of R.'s protocol

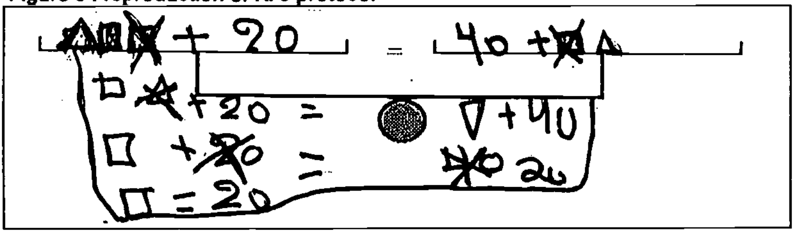
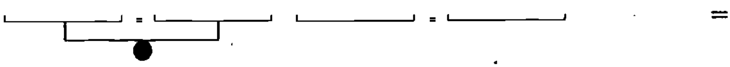


Figure 4 : simplification of auxiliary representational diagrams offered during the didactic sequence



Auxiliar diagram 1 Scale-disposition dictations and cleaning-up	**Auxiliar diagram 2** Problem dictations and cleaning-up	**Auxiliar diagram 3** New problems, cleaning-up and rewriting

4. Discussion

This work represents an incursion of the author in the terrain of mid-term didactic projects directed to school-like groups, without the methodological comfort provided by experimental and quasi-experimental designs.

A certain set of ideas presided the didactic sequence reported here in its main traits. First of all, the idea of an epistemological gap between arithmetic and algebra (Vergnaud et al., 1988). This gap (which dialectically shares epistemological relevance with the idea of continuity (Da Rocha Falcão, 1992; 1993)), can assume many aspects, one of the most important concerning explicit and implicit *contracts* undergoing arithmetic and algebraic procedures. In fact, the arithmetic procedure implies an immediate search for solution, represented by the calculation of intermediate values in order to reach a final answer. Algebraic procedure, differently, postpone the very activity of solution's search and begins by a formal transposition from empirical domain or natural language to an specific representational system. Because of this, much energy was directed in the didactic sequence presented here to the negociation and installation of a new contract: *represent first, try to solve later*.

Symbolic representation is a key psychological aspect in the development of algebra and many other *conceptual fields* (Vergnaud, 1990) in mathematics because of two points: first, it is not a result or superstructure of operational structures, as proposed in the context of Piagetian theory (Piaget, 1970; 1975) but rather a constituent of concepts, with operational invariants and situational links that gives socially shared meaning to knowledge (Vygotsky, 1985)); second, it opens to a particular individual a wide range of symbolic cultural tools that, as *cultural amplifiers* (Bruner,

1972), enables one to access new instances of conceptual construction. So, representations provide ***metaphors*** that can be useful as pedagogical tools in the context of an effort of *didactical engineering* (Artigue, 1988); these metaphors help in amplifying pre-existing schemes (Vergnaud, op.cit.), since they provide semantic links between structured knowledge and new pieces of information. In this process of enrichment of meaning, a quite important psychological sub-process is represented by the explicitation of *theorems-in-action* (Vergnaud,1994), upon which are established many practical competences exercised in daily life. The proposal of the two-pan balance scale represents an effort of offering a metaphor of algebraic equivalence between equations, based in the conservation of a pre-stablished fonctional equality between each side of an equation. The construction of meaning for the equivalence of equations (essential aspect for the comprehension of algebraic algorithms) is initially connected to the familiar idea of *equilibrium*, in the context of a culturally familiar artifact, the balance-scale. This idea of equilibrium is frequently poorly explicited, although people can make a competent use of a two-pan balance scale in order to sell or buy fish in Brazilian popular markets; nevertheless, equilibrium as theorem-in-action is based upon two *explicitable* principles (see section 2. As a metaphor, the balance-scale offers a context of cultural fonctionning where complex mathematical concepts (algebraic equivalence and algorithmic manipulation) can be initially rooted in competences and theorems-in-action (Schliemann and cols, 1992), enriching pre-existing schemes. The balance-scale also offers a support for symbolic representation, which semantically and syntactically sets the fondations for the introduction of algebraic formalisms. This is one of the reasons why we have passed, very quickly, from the actual, concrete balance-scale to a scale-diagram and to an even more abstract diagram (figure 4). This passage is also important because of a central point concerning the use of metaphors in general: if it is valuable to introduce metaphors in the effort of scheme enrichment, it is equally important to leave them behind *as soon as possible*, in order to avoid an undesirable over emphasis on the scaffolding, so to speak, at the expense of hiding the architectural structure one is interested in analyzing. I other words, the concept of algebraic equivalence can not be reduced to the idea of balance on a two-pan balance-scale. I would finally say, quoting once more G.Vergnaud, that "(...) symbolic systems can be "conceptual amplifiers" (...), provided we never forget that they can be misleading, that their use raises specific difficulties, and that they are not the real thing in mathematics" (Vergnaud, 1987, p.232).

The reflection above leads to the last point to be discussed here: ***what did the children learn after this semester-long work?*** Did they understand algebraic equivalence? Did they build up the concept of algebraic variable? Was the passage from principle 2 to algebraic script-algorithm of equation processing sucessfull? Is the competence shown in algebraic problem solving at the last meeting indicative of effective scheme improvement? These are complex and important questions. First of all, scheme improvement cannot be assimilated to the simplistic, false dichotomy of being or not being able to do something; a scheme, as an invariant organization of behavior for a certain class of situations, made of operational invariants, goals, expectations, anticipations, rules of action and inferences, cannot be reduced to a frozen competence disconnected from its socio-cultural ecology, its situated meaning (Meira, 1993). It is time for cognitive psychology to leave behind "general problem solvers", universal algorithms and "central" heuristics: cognition is not an intransitive, decontextualized entity (Lave, 1988). So, there is not an easy and unique answer for the question that opens this paragraph: a careful, multi-task and long-term evaluation must be done in order to assembly elements of answer. Nevertheless, clinical data immediately available seems to allow the following two points in terms of possible achivements due to the didactic sequence reported here:

1. A new contract (*represent first...*) was established; it does not mean that other contracts were simply substituted, but we seem to have succeed in negociating their social allowance for a new one.
2. A new representational tool (the diagram), their two operational principles and procedural sub-principles are now available for a certain class of problems.

These two points touch the very core of a new, incipient and workable scheme, upon which the pedagogigal effort of teaching the basics of algebra goes on.

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THE ABSOLUTE VALUE IN SECONDARY SCHOOL. A CASE STUDY OF "INSTITUTIONALISATION" PROCESS.

Marie-Jeanne Perrin-Glorian
Equipe DIDIREM, Université Paris 7

Abstract

The present research intends to study the process of "institutionalisation", i.e. all that the teacher uses to give to the mathematical knowledge of the students a status according to what is expected by the institution at this grade of school, and to identify relevant variables on the side of students and on the side of teachers, in relation with the knowledge at stakes. Here the knowledge is the absolute value. We analyse the change in the French curriculum, the choices of two teachers, and we compare the same lesson done by the same teacher, in the same week, for 2 different classes : a "good class" and a "weak class", we look for differences in the students'work, differences in the discourse of the teacher. An effect of the differences in the knowledgre of students and in their work is that the same lesson of the teacher can be a clarification for some of them and an abstract discourse getting very few links with their own activity for others.

1. Problematics and methodology

1.1.The problem

In our previous researches (1990, 1991, 1993), we identified from observations of classes and interviews with teachers and students, some phenomena especially perceptible in "weak" classes, namely :

- something like an opposition between a logic of learning and a logic of success : the desire of getting a short-range success for the students may impede learning and long-range-success ; it looks like the teacher gives to the students the ways to solve exercices instead of obtaining a real learning from them.
- the difficulty to find a balance between the construction of the sense of the mathematical concepts and the acquisition of basic mechanisms as algorithms
- the inclination of teachers to reduce mathematical teaching to teaching of algorithms.

Those phenomena are related to contrainsts (time, students themselves who ask for algorithms, the need for the teacher to get some successful results for the students and so on) and lead to the "no-learning" of some students. The contrainsts affect especially institutionalisation (Brousseau 1987), namely all that the teacher uses to give to the students' mathematical knowledge a status according to what is expected by the institution at this grade of school.

This process of institutionalisation is on the teacher's responsability. It takes various forms and appears on several occasions in the class : during the lecture, conclusions of problem solvings, remarks, recalls, but also for instance through the choice made by the teacher of the exercices given, especially for evaluation.

A very important point is the articulation between this institutionalisation and the sense actually involved by the students during activity of problem solving. Even if students use with sense in problem solving some tool that we can identify as a mathematical concept, the choices for the teacher are quite tightened : without institutionalisation, most of the students remember only the context of an activity and cannot use the same concept to solve another problem, but after the lesson, when definitions and formalisms are given, we may often observe a loss of sense for some students.

For example, on the one hand, after an activity to learn fractions from sharing rectangles, some students think that they learned to share rectangles, so it is not surprising if they did not use those fractions to deal with lengths for instance, but, on the other hand, after the lesson, when fractions are written with numbers, we can observe errors like "one sixth is the double of one third" and so on.

The present research intends to study this process of "institutionalisation" and to identify relevant variables on the side of students and on the side of teachers, in relation with the knowledge at stakes.

On the side of the knowledge, we study its place in the curriculum, in handbooks (which knowledge is aimed to, what types of exercices are offered, what relationships with other knowledge in the same grade, in previous grades, in future grades) and the evolution of this place. We study also the choices of the teachers : organization of their teaching, types of exercices, evaluation...

On the side of students, we pay attention to the links they make between problem solving and the teacher's lesson : we try to identify the knowledge used by students in problem solving, by themselves or with the mediation of the teacher, how this knowledge is modified by interactions between students or under the influence of the teacher. We also pay attention to home work, the way students prepare the tests, how they learn for these tests and by these tests and we search a possible relationship between specific successful problem solving for instance and global success during the school year.

1.2. Methods

In order to do this study, we have chosen to make some cases studies by observing several classes on the same mathematical topic. We are collaborating with a sociologist to study the interaction of cognitive and social factors.

We have chosen the 10th grade (15-16 years old) because it is in France the first year of "lycée" (10th to 12th grades) and the last year before orientation into scientific, literary or economic sections. We have taken into account three variables : the mathematical topic, the teacher and the class.

We selected two mathematical topics that are new at this level : the absolute value (including the absolute value function) and homothety. The first one is a bit marginal in the curriculum at this level. The second one has a more important place : it is an opportunity to use the vectoriel calculus which has been introduced two years before, about translation but only with addition : the multiplication of a vector by a number is new in this class.

For the teacher, one variable which we selected is his experience : for the mathematical topics selected, we planed to observe teachers who are used to teach in lower grades and others who are used to teach in higher grades with the hypothesis that the first ones will be more attentive to the consolidation of previous attainments and the second ones to the preparation of future knowledge.

For the class, we intended to observe the same teacher in two classes of the same grade but not with the same knowledge : one considered as a "good" class and one as a "weak" class.

We have got observations in the classes, students'tests, interviews with teachers and with students...

The research is still in progress and here we present one topic : the absolute value (analysis of the new curriculum, choices of 2 teachers) and the observation of one lesson of the same teacher in two classes of different levels.

2. The teaching of the absolute value

2.1. A new presentation of the absolute value in the French curriculum.

2.1.1. The classical teaching of this topic before 1990.

In the precedent curriculum, the term "absolute value" was introduced (up to 1986) in the 6th grade in the same time as relative numbers : it was defined as the number without the sign, namely a relative number has a sign and an absolute value, a number and its opposite have the same absolute value. In the 6th and 7th grades, it was used to express the rules of operations on relative numbers, but it was (up to 1988) actually studied and used on and after the 8th grade (13-14 years old), with regard to relative numbers and points marking on a graduate line, but also functions (that were introduced on the 8th grade too), solutions of equations.

It was introduced and treated from a numeric and an algebraic point of view. The definition of $|x|$ was given in one of the following ways: "the positive number among x and -x", or "$|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$." It was used to make some exercices a little more difficult, for instance resolution of equations and inequations. We found in 8th grade exercices like

"Find all the real numbers x such that $|x| < 3$; $1 < |x| < 2$; $|x| > x$; $|x| < x$; $|x - 1| < 2$;

$|2x+9| < 3{,}5$; $|4x-5| = 3$". Sometimes, it was required to place these numbers on a graduate line.

There were sometimes more difficult exercices even in 8th grade, like :
"Calculate the rational numbers x such that $|\frac{2x-3}{3x-2}| = 2$" or "$|\frac{2x-3}{3x-2}| \leq 2$" ;
"Find all the x such that $|x^2-4| + |x-2| = 0$", "$||x-1|-1| = 1$", "$||x-1|-1| \leq 1$"
"Interpreting absolute values as distances between points on a graduate line, find x such that $|x-2| + |x-3| = 5$", $|x-2| + |x-3| > 2$".

The notion of function was introduced in this grade and we found also exercices like :
"Consider the function f defined by f: $\mathbb{R} \rightarrow \mathbb{R}$ f : $x \rightarrow |x-2|$

1) Calculate f(2) ; f(-5,5) ; $f(\frac{3}{5})$; f(0) ; f(-1)

2) Let x be a real number superior or equal to 2. Compare real numbers (x-2) and $|x-2|$
Prove that the real number 3,5 is the image by f of one and only one x superior or equal to 2 ; find this x.

Same question for $\frac{3}{4}$; 5,5 ; 1

3) Let x be a real number inferior to 2. Compare then real numbers (x-2) and $|x-2|$
Prove that the real number 3,5 has one and only one antecedent inferior to 2 ; determine this antecedent.

Same question for $\frac{3}{4}$; 5,5 ; 1.

4) Is the function f a bijection from $\mathbb{R}$ onto $\mathbb{R}^+$?"

In the 10th grade, the properties of the absolute value were restated at the same time as approximations, but in the exercices on approximations, expressions with inequalities were more used than the absolute value. Nevertheless, there was a large use of the absolute value in the exercices about functions : " Study and draw the graph of the functions $f(x) = |-2x^2+5x+3|$;
$f(x) = x^2-|x|$; $f(x) = \sqrt{|4-9x|}$; $f(x) = \sqrt{|x|-1}$; $f(x) = \sqrt{|x|+|2x-4|}$; $f(x) = \sqrt{2x+|x-1|}$ " ...

However, the official instructions and commentaries on the curriculum of 10th grade precised since 1982 about the absolute value : "the essential point is to be able to interpret $|b-a|$ as the distance between the points a and b, relations such that $|x-2|<1$ or $|x-2|<1/100$ with intervals the centre of which is 2, to be able to do some simple majorations using the triangular inequality... The sudy of some piecewise affine functions is a reasonable objective. Other examples accumulating absolute values, as the study of the fuction $|x-|x-1||$ or the solution of the equations $||x|-3| + |2+|x|| = 1$ or $|\frac{2x+5}{x-1}| = \sqrt{(2x+5)^2}$ are dependant on gratuitous technics and can do nothing but repulse students."

2.1.2. *The new curriculum*

The absolute value caused a lot of errors for students many years after its introduction (see for example Duroux, 1983 or Chiarugi, Fracassina, Furinghetti, 1990). In particular, students hardly accept that $|x|$ may be -x, and when they have to study $|x-2|$, they distinguish the cases $x>0$ and $x<0$. These errors seemed to be related to the early definition of the absolute value as the number without sign. So this notion has been considered as difficult and of no real use for this grade, and now, students (the first were those who began secondary school in 1986), meet the absolute value for the first time in the 10th grade (since 1990), as a distance on the real line and as a particular function (the theme of functions is important for this grade and it is also new : before, there are linear and affine functions but the notion of function is no longer introduced in the first years of secundary school). Exercices like those above have disappeared from 8th grade, we now find some of them in the 10th grade. Some others cannot be offered even in this grade because there is no longer definition of function, bijection and so on and the only functions composed with an absolute value that are now to be studied in this grade are of the type $|x-a| + b$.

In the new handbooks, the absolute value is defined as the distance between x and 0. The distance between numbers is said to be the usual distance between the points getting these numbers as abscisses on a graduate line : "the largest minus the lowest"). Some handbooks define $|a-b|$ as d(a,b) before defining $|a|$ as the particular case where b = 0. They don't speak of invariance of distance by

translations to justify that the two definitions given for $|x|$ when $x=a-b$ are the same. According to the official instructions, the links between absolute value and definitions of intervals are reinforced, equations and inequations like $|x-a| = b$ or $|x-a| \leq b$ are first solved in a geometric way.

Does this new curriculum allow to avoid the difficulties described above ? It is difficult to know. We have some informations by the evaluation made by APMEP[1] (1991). But as the competences expected from students at the end of 10th grade are quite low, there are few questions about the absolute value. The relationship between absolute values and distances seems better. Nevertheless, students succeed the translation from distances into absolute value for the definition of intervals the center of which is positive but the other cases and the translation in terms of inequalities are not yet mastered : among 1800 students,

- 86% can tranlate $d(x;7) \leq 3$ in $|x-7| \leq 3$ but 48% only can tranlate the same in $4 \leq x \leq 10$;
- 42% can tranlate $|x+5| \leq 1$ in $d(x;-5) \leq 1$ and 29% in $-6 \leq x \leq -4$;
- only 29% can translate $-2 < x < 2$ in $d(x;0) < 2$ and 28% in $|x| < 2$.

This evaluation took place the first year after curriculum had changed and perhaps teachers had not yet completely changed they way of teaching this notion which was not new in this grade before.

Chiarugi and al. found only 39% students 14 years old but 80% 17 years old able to draw on the real line the interval such that $|x| < 3$.

It is difficult to compare the other questions, except the better success in numerical questions than in algebraic ones : 41% gave a right answer for the 4 questions (yes, no, I don't know):

if $a = -5$; $b = 25$; $c = -13$, then $|a-b| = 20$; $|c| - |b| = -12$; $a + |b-c| = 33$; $|a-b| - |a+b| = 10$,

but only 20% gave a right answer for the 4 questions : whatever reals numbers are a and b, we can say that $|a-b| \leq a$; $|a+b| \geq a+b$; $|a+b| \geq |a| + |b|$; $|a+b| \leq |a| + |b|$.

For the function aspect of the absolute value, 60% can draw the right graph for g defined by $g(x) = |x|$ and 53% can complete the variation table

x	-5	7
g(x)		

2.2. What type of problems is the absolute value a suitable tool for ?

The absolute value is a tool to have shorter formulations, to define functions with only one algebraic expression, instead of several ones, as an example it is a convenient way to express square roots : $\sqrt{(f(x))^2} = |f(x)|$ or positive magnitudes as distances, areas... but one may always avoid it by considering several cases. Moveover, to solve equations or inequations, you have to remove the absolute value and distinguish cases. But, since there is no longer formalisation of the notion of limit in secondary school, the expressions the students have to consider can often be written easily without the absolute value; so it is not very useful for problem solving. So, the status of this concept makes it difficult to introduce as an implicit tool to solve a problem. Morever, it is nearly impossible to find a problem which is not given in a mathematical form and for which the absolute value is a suitable tool for solving it (see for instance the problem analysed in 3.1.). If we want that the students use this concept, we are nearly obliged to ask them to do it by exercices which need translations between two langages, for instance intervals defined by their ends or by their centre and radius like these :

[1] Association of teachers of mathematics

valeur absolue	distance	droite des réels	encadrement	intervalle
$\lvert x-2\rvert \leq 3$	$d(x; 2) \leq 3$	-1 0 1 2 5	$-1 \leq x \leq 5$	$x \in [-1 ; 5]$
$\lvert x\rvert \leq 7$				
$\lvert x-1\rvert \leq 3/2$				
	$d(x; 5) \leq 9/2$			
			$1 \leq x \leq 4$	
				$x \in [-7 ; -2]$
$\lvert x+3/2\rvert \leq 2$				
			$1,4 \leq \sqrt{2} \leq 1,5$	
				$\sqrt{3} \in [1,73 ; 1,74]$
$\lvert \pi - 22/7\rvert \leq 10^{-2}$				

In other exercices as study of functions, the function to study is given with an absolute value and the student has to distinguish different cases to study it. It is difficult to find a problem in which the student has to use absolute value to define a function because it is always possible to do it whitout the absolute value, except if the student has to enter this function in a computer or a calculator, for example to draw the graph. The absolute value provides also means to make counterexamples : it gives the only function studied at this level that is continuous and not differentiable everywhere. For this, it is used, as square roots, to produce functions more difficult to study and to give the opportunity of studying the sign of algebraic expressions, as we saw in the analyse of handbooks.

2.3. The choices of two different teachers

We observed the lessons prepared on this topic by two teachers : teacher A has a long experience of teaching in the first years of secondary school : he had teached for nearly 20 years in a "collège" (6th to 9th grade) and 2 years in a "lycée" before the observation ; teacher B has always been in a "lycée" (for about 18 years) and he teaches in "Terminale C" (the beginning of calculus, the last year of secundary school for students who wants to begin scientific section) since many years. The 2 teachers said that they follow the official instructions and define absolute value from distance.

Teacher A started from distance in various situations, namely "in space, (then in a plane, on a line, on a graduate line), you have a point A, the unit of length being cm, find the points M (resp. M', M") such that AM = 3 (resp. AM' ≤ 4,5, AM" > 2)" and calculation of some gaps between temperatures.

Then he defined d(x,y) as the distance MN where M and N are respectively the points the abscisse of which are x, y and precised $d(x,y) = x - y$ if $x > y$ and $d(x,y) = y - x$ if $y > x$. He gave "technical" exercices (some like the one above and resolution of equations and inequations, both from an algebraic and geometric point of view).

Then the students had to solve 2 problems in small groups: "M. Dupont" (see below) and another one about distances and geometric transformations (as an example one question was : "On the graduate line (d) with the reference (O, I), A has an abscisse 3 and M an abscisse x, P is the symetric of O around M, write the distance AP as a function of x"). In these problems, students met the absolute value as a function. Students also played on a computer with a game of targets in which they had to guess an interval. Lastly, properties of the absolute value were studied.

Teacher B started from the distance only on a graduate line. First, he studied only the numeric and algebraic aspects of the absolute value, with its properties, he gave the same definition as teacher A and insisted too on the geometric way of resolution for equations and inequations. The absolute value as a function was studied two monthes later in the chapter about functions. At this moment, the problem "M. Dupont" was given to solve at home.

For the evaluation, their choices were not the same to test the numeric and algebraic aspects of absolute value : teacher B asked to complete a table of 4 columns (absolute value, distances, intervals and drawings, inequalities: some exercice like in § 2.2). Teacher A asked questions about distances

and graduate line in a previous test but without absolute value. Both of them gave equations and inequations to solve but teacher A asked questions with numerical difficulties, unlike teacher B, who used entires numbers or 2,5 ; 1,5 ; 0,25 ...

Moreover, the 2 teachers gave a problem that needed more research from the students : teacher A gave a problem which looked like "M. Dupont" and teacher B one which looked like the one above about geometric transformations.

We saw that technical exercices were easier in class B than in class A and so the results were better. In the "good class" of teacher A (see below), 12 students on 32 (37,5%) succeed (except perhaps errors on numbers) for $|x - \sqrt{3} + 1| = 3*10^{-2}$; $|3*10^{5} - x| < 5*10^{-5}$, 8 make an error on signs, 12 make errors on powers, 7 make errors linked to absolute value. We see that, when the geometric representation is not easy to draw and to read, many students fail.

In class B (a "good class" too), for the table, on 33 students, 18 (54,5 %) do at most one error (9 have no error in translations, 9 fail only in the translation of $d(x,1) \geq 3$ by inequalities), among the others, 9 do an error with sign. But, as there was 21 answers to give, on the whole (i.e. 693), 96% are corrects. For equations $|x-2| = 3$; $|x+5| = 2$; $|x+0,5| = 3,5$, 26 students on 34 (76,5%) have 3 right answers. For inequations $|x-3| \leq 1,5$; $|x+1| \leq 3$, 27 students on 34 (79,4 %) have 2 right answers, 23 (67,6%) succeed the 5 questions.

3. Observation of the same lesson, by the same teacher, in two classes : a "good class" and a "weak class".

3.1. The problem and the actual aim of the lesson.

The following problem was given by the same teacher, Teacher A, in the same week, to the students of his two classes of 10th grade, one of good level and one of weak level.

Mr Dupont works in a society, the head office of which is in Paris and which has branches in Rouen, Yvetot and Le Havre, all located, in this order, along the same road from Paris to Le Havre. The distances are : Paris-Rouen : 110 kms, Le Havre-Yvetot : 50 kms, Le Havre- Rouen : 85 kms.
On Monday and Saturday, Mr Dupont has to go in the head office,
on Tuesday and Thursday, he must go in Rouen,
on Wednesday in Le Havre
and on Friday in Yvetot.
He comes back home every night.
Where should Mr Dupont live if he wants to drive the less as possible ?

This teacher has a somewhat innovative practise in the 2 classes : during one session, students are offered problems involving the new concepts as implicit tools ; they are organized in small groups to solve them. During the next session, the teacher directs a synthesis, and, at the same time, gives the lesson : definitions, explanations ...

The problem "Mr Dupont" was given just after the definition of the absolute value from distances. Students were organized in small groups of 3 or 4, half a class at a time. They got about one hour and forty five minutes for solving this problem in which a first question required "how many kilometers has Mr Dupont to drive if he lives in Paris ? if he lives in Barentin, 17km West from Rouen ?" and there were two other questions with a different organization for the week of Mr Dupont (with in particular a case in which the minimum was reached on a whole interval).

This problem involves distances on a line and seems a good problem to use absolute value. It was the reason why teacher A chose this problem. But an a priori analysis shows that students are sure to find the good answer even if they try only the given towns. It is not because the problem is bad, but a necessity of this type of problems : if you considere a sum of absolute values of polynomials of first degree, you have a piecewise affine function and there is necessarily an end of an interval where the minimum or maximum is reached. The best you can do is that this minimum is reached on the whole of one interval. One needs to express the function only to prove but not to find.

So, if you want that the students define a function, you have to require it. Anyway, when defining the distance covered by Mr Dupont during one week, students have no reason to use absolute value : to prove, it is easier to have the function defined on intervals on which it is an affine function. It is only in the case where you need only one algebraic expression, for example to use a computer that the absolute value will be an economical way.

We now analyse the realization of this situation in two classes.

From the synthesis realized in another lesson by the teacher, we can see that he was going towards two objectives : first, to use the sense of variation of affine functions in a proof and restate in this occasion the difference between affine and linear functions, second to introduce a table to present the different algebraic expressions for the distance.
If the origine of the graduate line is in Paris and x the abscisse of Mr Dupont's home :

	Paris		Rouen		Yvetot		Le Havre
Monday		2x		2x		2x	
Tuesday		2(110-x)		2(x-110)		2(x-110)	
Wednesday		2(195-x)		2(195-x)		2(195-x)	
Thursday		2(110-x)		2(x-110)		2(x-110)	
Friday		2(145-x)		2(145-x)		2(145-x)	
Saturday		2x		2x		2x	
week	1120	1120-4x	680	4x+240	820	8x-340	1220

3.2. Development of the students' research in different groups.

Three groups of four students were observed during their research, each in a different half-class, two in the "weak class" and one in the "good class".

For the students the main difficulty was to put the problem in a mathematical form. They drew a line and place the towns, but did not think of using a variable for the abscisse of the house. In fact, they thought that Rouen was the good place (and it was) and, after some tests, tried to find a symetry. But the symetry was good if you were not too far from Rouen (in fact as far as Yvetot : 35 kms) and was wrong after.

It was the teacher who asked a mathematical proof and said that the house will be represented by a point M with an abscisse x. This order of evolution was somewhat similar in the three groups but the development in time was quite different : in the good half-class, the intervention of the teacher came after about half an hour when in the weakest of the three half class, it came about one hour later.

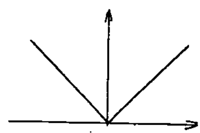
All groups thought of representing the road by a graduate line because the precedent lessons about distance insisted on this representation, but the weakest groups spent plenty of time to discuss how to do that (choice of origin, unit...). So the three groups spent very inequal time to try to write the distances with "x", and the work was to complete at home. But, on the next lesson, theweakest students had not done it, so the synthesis made by the teacher came after an actual research for the best students and nearly nothing in the algebraic domain for the weakest ones.

3.3. The synthesis of the teacher in the two classes

When observing the same teacher in two classes, we started from the hypothesis that the teacher fits to his students and we were expecting some differences.
We detected two main differences between the 2 classes in the synthesis :

- time is not managed in the same way : there are more digressions in the class of lower level. Moreover, in this class, those digressions are caused by an error or an insuffisantly accurate answer of a student ; they give an opportunity to restate previous lessons. In the other class, they give an opportunity to anticipate further lessons. Paradoxically, more difficult questions may be asked in the class of lower level.

For example, a student said that a curve like this was a parabol because it had a symetry, the teacher expected that this proposition would be refused because the equation was of degree 1. It seems that the students were not able to say that, in the other class, one student gave this argument so a little time was spent for this question but we don't know how many should have been able to do it.



- there is more heuristic discourse in the class of lower level, but it is more algorithmized.
We said that this teacher has an innovative practise. Moreover, he gives much place to heuristics in his discourse. We studied the non mathematical discourse in the two classes during the synthesis of the session "Mr Dupont" with the method presented in PME in Assisi by C. Chiocca, E. Josse and A. Robert. We were expecting the rate of explanations higher in the good class. In fact it was a little higher in the weak one but it was cut in smaller units and more repetitive, we said more algorithmized.

Conclusion

About a marginal content as absolute value, we saw that the choices made by teachers, who said to follow the official instructions, were quite different, including evaluation. We are now going on this research by observing 5 teachers, including teachers A and B in 5 classes, on absolute value, equations, inequalities. We are going to construct a common test for these five classes, and see the evolution in the different classes of some students getting the same results at the beginning of the year in numeric and algebraic domains, with reference to the test that, now, in France all students pass at the beginning of 10th grade.

Another question is the differences in the way the same teacher conducts the class, according to the level of students. Here, we have a case study. Will the differences observed be the same for another mathematical concept ? for another teacher ?

Nevertheless, WE think that we may retain as a general fact the important differences in the actual work of students during activities aiming at giving sense to mathematical concepts : for some of them, the work during the class prepares the understanding of the lesson , for the others, it is far from this... Then, what help to give to students during their research for making it more suitable, but without reducing this research to nothing ?

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SYLVINE SCHMIDT, UNIVERSITÉ DE SHERBROOKE
NADINE BEDNARZ, UNIVERSITÉ DU QUÉBEC À MONTRÉAL, CIRADE

THE GAP BETWEEN ARITHMETICAL AN ALGEBRAIC TYPES OF REASONING IN PROBLEM-SOLVING AMONG PRE-SERVICE TEACHERS

ABSTRACT: *The management of teaching situations means that teachers are confronted with a number of choices as to the approaches they are to favour in an introductory algebra context whereby connections to arithmetic are put to good use. The choices a teacher makes are strongly influenced by the relationship which he or she maintains from the outset with these two fundamental areas of knowledge. Three groups of future teachers (164 students) were questioned with a view to analyzing to what extent these students were able to shift back and forth between these two areas of knowledge within the particular context of problem solving. Interviews on either an individual basis or in a dyad format were conducted with a number of subjects, and have served to bring out the gap which emerges as the different types of reasoning are deployed.*

RÉSUMÉ: *L'enseignant, dans la gestion des situations didactiques, est aux prises avec des choix à poser en regard des situations et approches à privilégier dans le contexte d'une introduction à l'algèbre et d'une saine articulation avec l'arithmétique. Ces choix sont fortement marqués par le rapport que lui-même entretient a priori avec ces deux domaines de connaissances fondamentaux. Trois groupes de futurs enseignants (164 étudiants) ont été interrogés en vue d'analyser le passage qu'ils sont à même ou non de faire entre ces deux domaines dans le contexte particulier de la résolution de problèmes. Des entrevues individuelles et dyadiques conduites auprès de quelques sujets pointent l'écart qui existe entre les raisonnements mis en place de part et d'autre.*

A dissociation between arithmetic and algebra has been observed to varying degrees among students who have received instruction in algebra (Lee and Wheeler, 1989); such observations have served to show that these students do not in any way appear to see the relevance of algebra for arithmetical situations that could call for this other type of reasoning, no more than the same students appear to see the importance of making use of arithmetic for occasionally determining the un-truth of an algebraic statement. This necessary, functional dialectic involving two areas of knowledge that are essential to any meaningful grasp of algebra (Chevallard, 1989-90) thus appears to go completely unnoticed by students.

The dissociation, among students, which arise between these two universes obliges us to examine not only the nature of the teaching situations involving arithmetic and algebra which students undergo but also the relationship which the teacher him-or herself maintains with these two basic areas of knowledge in which he or she is called on to act and interact. The choices which a teacher makes concerning the structuring of classroom learning situations depend among other

things on the relationship which he or she has with the knowledge to be taught (all forms of knowledge reinterpreted according to the postulates and cognitive experiences of the person). It is through such choices that the instructor thus refers to the play and interplay of issues underlying the didactic contract present in the classroom setting to organize the operations whereby a student takes hold of knowledge. Thus the relationship to knowledge manifested by the instructor inescapably produces an impact on the relationship which the student maintains toward the knowledge being taught, in this instance arithmetic and algebra. But what exactly does the relationship to the field of algebra involve? And what, then, those relationship to the field of arithmetic entail? And are teachers capable of establishing a dialectic between these two fields?

In this study, we have chosen to take a closer look at the future teachers with whom we have been in contact. On the other hand, algebra as a whole encompasses too large a field to serve as a subject of research for us. Hence, we have narrowed down our choice to problem-solving for two reasons: 1) the importance which has been attached to this area of activity in, precisely, the development of algebra in elementary and secondary teaching programs; 2) the experience which future instructors have acquired in these two fields.

OBJECTIVE OF THIS RESEARCH

The fundamental objective of this research is concerned with examining the modes of problem-solving which student instructors make use of in arithmetical or algebraic contexts to identify the resistances and eventual difficulties which arise in the shift from one type of approach to the other. Do student instructors perceive the relevance of moving on to algebra? Does arithmetic appear as a useful tool to fall back on on occasion? The following questions have guided our research: are pre-service teachers capable of easily moving back and forth between each type of reasoning (from arithmetic to algebra and vice versa) and can they do so spontaneously according to the different problems which justify using one mode or other or when called on to do so? Do they provide evidence of resistances in this movement from one type of handling to the other?

METHODOLOGY

With the foregoing in mind, during an initial phase we presented a written test made up of eight problems (four "arithmetic" problems

and four "algebra" problems) to students from three different teacher education programs (future elementary school teachers (n=66), secondary school teachers (n=33) and remedial teachers (n=65) who will be called on to work with students having learning problems in both elementary and secondary school settings). The schema which Bednarz and Janvier (forthcoming publication) elaborated to analyse problems was used to develop the test. This first step enabled us to sort out those students who essentially relied on arithmetic to solve the eight problems contained in the test from those who mainly used algebra, as opposed, finally, to those who mixed their use of arithmetic and algebra. Eight students from each of the three groups were then selected to participate in an individual interview. In addition, in a later phase, a number of subjets were requested to participate in dyadic interviews in which one student with "arithmetical tendencies" and another student with "algebraic tendencies" were involved together in problem-solving. The latter type of interview is useful in that it offers, on the one hand, a new angle from which to view the distinctions between arithmetical and algebraic types of reasoning evidenced by the misunderstandings characterizing each partner, and serves, on the other hand, to shed light on the difficult articulation of these two fields among pre-service teachers. We shall return to a number of the observations which we have derived from these interviews once an overview of the results of the written test has been presented.

RESULTS

The results of the written test provide evidence of a dichotomy between arithmetic and algebra among pre-service teachers. Thus, the great majority of future secondary school teachers (SEC) confine themselves to algebra, even when dealing with "arithmetical" problems (see fig. 1).

On the other hand, few students from the group of future remedial teachers (REM) make the shift to algebra whenever the situation requires it (see fig. 2), altough their duties will eventually involve them with students with learning difficulties, in connection with algebra in particular. Finally, the group of futur elementary school instructors (ELEM) appear to be the best prepared for playing off both of these fields.

When the interviews are used to examine the gap between the "arithmetical" students from their "algebraic" counterparts, the

KEY: arithmetical procedures | algebraic procedures

other (no answer or ambiguous answer)

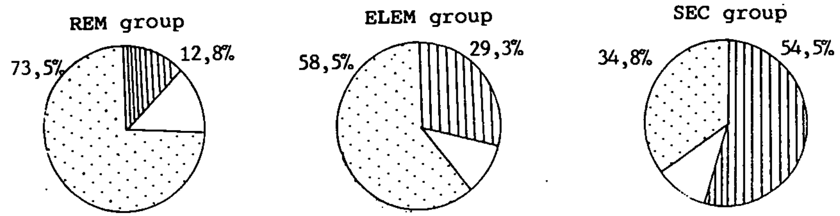

Figure 1

Average percentage of arithmetical and algebraic procedures used by each group for solving "arithmetical" problems

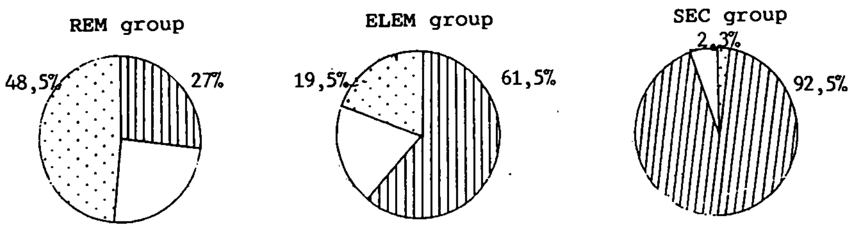

Figure 2

Average percentage of arithmetical and algebraic procedures used by each group for solving "algebraic" problems

resistances which arise in the shift from one field to the otherbecome easier to make out. These resistances can be grouped according to: 1) the nature of procedures used to solve problems in arithmetic and algebra, and 2) the kind of control which is brought to bear on both situations.

1. Reasoning based on states versus reasoning based on relationships

Our research has enabled us to better understand the gap which exists between algebraic problem-solving procedures and a number of arithmetical procedures used by pre-service teachers, principally involved in problems having no affixed states (see the problem of "Luc and Michel" in the appendix). To solve this problem, Mirielle, for example, an "arithmetical" subject, will basically focus on the differences which occur between the amounts ascribed to Luc and Michel before and after transformation of these sums, an approach wich Éric, the "algebraic" subject, will have great difficulty

understanding, as is shown in the following excerpt taken from a dyadic interview:

Excerpt of a dyadic interview:
Éric (EC), "algebraic" subject; Mirielle (MV), "arithmetical" subject

L		M	(notes taken down by Mirielle on her sheet of paper)
?	3.50	?	
4.20	.40		
	3.10		
	1.10		
	4.20		

MV: Ok. "Luc has $3.50 less than Michel does" (writes down L, M and 3.50, as above). Now to start with, I suppose that...

EC: Michel has at least $3.50.

MV: Well, let's say..., yeah, you could say that. Ok, "Luc doubles his money"..., Well, when you get down to it, I go about it more using the difference between the two. I know that he, here, there's 3.50 separating them. Uh, "Luc doubles his money whereas Michel increases his money by $1.10." So I know that here there was an increase of $1.10. But I don't know that here. Here, I don't know the amounts that they had (writes down the two "?" (question marks)).

EC: Ok.

MV: What I do know is that there was a difference and that afterwards, I've got Luc who's now got 40 cents less than Michel (writes down ".40"). So I know that the difference between these two (draws an arrow between $3.50 and $.40), is $3.10.

EC: $3.10, you say...

MV: A difference of $3.10, and I know already that... $1.10, here there was an increase of $1.10. So normally that should give the amount...

EC: ...that Michel had.

MV: Here, that Luc had.

Whereas Mirielle reasons in terms of the gap between Luc and Michel's respective amounts, Éric on the other hand tries to fix the states involved, particularly the amount belonging to Michel, as may be seen when he comments, "Michel has at least $3.50". In contrast with the reflections of her algebraic counterpart, Mirielle provides clear demonstration of the quite explicit distinction which she is able to make between what Éric's interpretation and her own mode of reasoning: she works off of differences whereas he thinks in terms of states. The "structural" type of arithmetical reasoning adopted here and which has been previously identified among certain students in the study prepared by Bednarz and Janvier (forthcoming publication), appears to be completely beyond the grasp of the algebraic subject, who is unable to comprehend the underlying logic of this procedure:

"I could see (her do it), but I don't understand why she did that and I don't understand why it worked. I mean, where did she come up with that? Like, the way she takes $.40 from $3.50, I don't get it" (Schmidt, 1994, p.377). We had the opportunity in another dyad to observe how misunderstanding of what is involved in an arithmetical type of reasoning based essentially on relationships can lead an "algebraic" subject to reinterpret a solution derived from this kind of reasoning in terms of states:

Excerpt of a dyadic interview:
Jacinthe (JL) "algebraic" subject; Nadine (NL) "arithmetical" subject

JL: She (meaning Nadine) assumes that Luc is equal to 0. In other words, it's as though Michel had 0 plus three-fifty. All right. Then she adds a dollar-ten. Luc has got 0.

NL: But it's not that... When you get down to it, I'm not saying that Luc is equal to 0. What I am saying is that Michel starts out having three-fifty more (than Luc) and then adds a dollar-ten, which gives him four-sixty. If Luc had gone the same way as Michel, he should have had three-fifty less, really, but instead he's come up only 40 cents short (of Michel)...

JL: Uh huh, that gives him four-twenty to start with...

NL: To start with... Does that make sense to you?

JL: Yeah, because, look here. You said that Michel wound up with four-sixty, all right? You know that he's got 40 cents more than Luc. So... You know he winds up with four-sixty but that he's got 40 cents more than Luc, so then you have to say that Luc's the one who winds up with four-twenty.

NL: But to start with...

JL: Sure, I can tell you it works, but... It's like you found the amount Luc winds up with. All right. It's as though you were saying that Michel winds up with four-sixty and Luc winds up with four-twenty.

NL: I'm not saying thaht Michel winds up with four-sisty. He's got 40 cents less, I mean more than Luc...

Play with relationships does not figure in the mode of reasoning demonstrated by the "algebraic" subject, and when she was required to explicate the procedure of the "arithmetical" subject, she tended to transform the stated relationships into states: "Luc is equal to 0", "Michel had 0 plus three-fifty", "Michel winds up with four-sixty and Luc winds up with four-twenty". The type of reasoning at work here is in keeping with algebraic-type problem-solving which, by positing "x", fixes states and organizes relationships around this symbolic substitute but also operates at some remove a "structural" type of reasoning which is capable of developing independently of states.

Observations involving various students have shown us that "arithmetical" subjects do not shift easily to algebra, for they do not perceive the relevance of this field for problem-solving. Conversely, th "algebraic" subjects we interviewed do not make the shift back to arithmetic easily, and are unable to comprehend the meaning and consequences of the operations performed solely on relationships or transformations by "arithmetical" subjects.

2. The kind of control which is brought to bear in arithmetic and in algebra

A comment by an "algebraic" subject who was required to state his views concerning an arithmetical and an algebraic solution to the "arithmetical" problem of "The pool" (see apendix) provides a clear illustration of what distinguishes the kinds of control which are made use of arithmetic and algebra:

> "Now, with this one (the arithmetical solution), 400 litres divided by 40 minutes gives you a flow of 10 litres per minute: seeing as how the first faucet produces 24 litres per minute and the second 14 litres per minute, it would seem... to be a comprehension problem. To her way of thinking, it's as though she sees... I imagine a pool that's being filled and is going to overflow. You've got to drain it. Now, with this one (the algebraic solution), (24 X 40) minus (x X 40) = 400, the answer is less obvious (to come by), because you could take away this problem and deal with any other problem you want.
>
> Whereas, the other one (arithmetical solution) seems to me stick more to the problem. This one (algebraic solution) is more abstract. Once you've got that down (pointing to the equation), you could forget the problem for all intents and purposes... You know, it's easier to see you're just doing math... you're just replacing it by algebra... Whereas here (arithmetical solution), you're referring to the question the whole time."

In arithmetic, contextual meanings serve to guide the approach adopted in problem-solving and to reassure subjects as to the correctness of the types of reasoning used. In algebra, meaning relating to context appears to play less of a role, if any; the subject has no choice but to develop new criteria of validity. But do subjects actually do that when they move on to algebra? What elements contribute to the certitude among algebraic subjects that the right types of reasoning have been used? Our observations have led us to distinguish between two types of "algebraic" problem-solvers among pre-service teachers: 1) those who embark upon an instability as to the meaning they give to variables and a lack of relective involvement in the way they handle problem-solving, and 2) those who bring constant control to bear on the development of algebraic

calculations and who themselves readjust according to certain errors they have made. Whereas the behaviour of the first group of subjects presents a lack of control (the reasons for this remain to be determined), these results lead us to examine the type of control exercized by the second group of subjects and offer a number of new research paths in this field.

CONCLUSION

Our research shows that dissociation of algebra and arithmetic exists among many future teachers. It is possible to explain this phenomenon in part on account of the gap which exists between the types of reasoning deployed in each of these areas (particularly so between the "structural" type of arithmetical reasoning and algebraic reasoning). These results give us pause for reflection on the capacity of these future teachers to comprehend the strategies students make use of in introductory algebra, to reckon with these strategies, and to install a productive articulation between arithmetic and algebra among students throughout their secondary school studies.

APPENDIX

Problem "Luc and Michel": Luc has $3.50 less than Michel. Luc doubles his sum of money whereas Michel increases his by $1.10. Now Luc has $.40 less than Michel. How much money did each have to begin with?

Problem "The pool": To fill a pool with a capacity of 400 litres, two faucets are opened simultaneously: one to fill it and the other to empty it. With the two faucets working, it takes 40 minutes to fill the pool. How many litres per minute can be emptied by the second faucet if the first faucet pours out 24 litres per minute?

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THE INFLUENCE OF PROBLEM REPRESENTATION ON ALGEBRAIC EQUATION WRITING AND SOLUTION STRATEGIES

Kaye Stacey and Mollie MacGregor
University of Melbourne

This paper investigates the possibility that the mental model which a student constructs of a problem situation affects the equations written and the solution strategies used. A series of problems was presented to 166 Year 9 and 10 students in such a way that different mental models of the same problem situation were constructed. Success rates and strategies for solving were affected by the mental model, and a psychological set could be induced which tended to affect the perception of subsequent problems. As they worked, many students extended their mental models to encompass further features of the mathematical structure. For this reason, students' use of algebra was not hindered by initial construction of a mental model incompatible with algebra.

As teachers and researchers know only too well, the formulation of algebraic equations to represent a problem situation is very difficult for many students. However, if students are to derive any real power from the algebra they learn in school, they must be able to take a problem situation and formulate useful expressions and equations from it. Only after this step has been completed correctly can the routine algebraic procedures which they are taught be used for solving the problem.

Research relevant to students' difficulties in formulating equations comes from several sources. The extensive research on one-step arithmetic word problems (for summaries, see Fuson, 1992 and Greer, 1992) has established that the semantic structure of problem situations influences both task difficulty and children's strategies, even amongst sets of problems which only involve one mathematical operation (e.g., subtraction). Although there has been a good deal of research and theory-building in the field of one-step arithmetic problem solving, there is no general agreement about the nature of the knowledge and processes involved in modelling a situation mathematically (Fuson, 1992). As Greer (1992) has pointed out, "psychological complexity" (p. 276) frequently underlies what on the surface appears to be a simple relationship.

An increasingly sophisticated series of studies has linked aspects of the verbal presentation of word problems (mainly arithmetic problems for young children) to task difficulty. For example, factors such as the numbers involved, problem length and readability, and the degree to which the semantic relations between the quantities in the problem are made explicit and easy to process (De Corte, Verschaffel & De Win, 1984; Lewis & Mayer, 1987) are known to affect success rates.

Cognitive psychologists have investigated comprehension processes for algebra word problems, concluding that students sometimes use schema-driven approaches which direct them to identify information to fill "slots" in the schema, sometimes make direct translation of words to symbols, and sometimes "read" information from various forms of mental representation of the problem situation (Hinsley, Hayes & Simon, 1977; Paige & Simon, 1966). MacGregor and Stacey (1993) showed how "reading" information from intuitive mental representations of comparisons of two quantities explains one common error in formulating algebraic equations. An extension of this

earlier work to the study of the effect of mental models on algebraic equation writing and the selection of solution strategies is described in this paper.

Different verbal descriptions of the same problem situations were used to encourage students to form different mental representations of the same problems. One set of mental models is compatible with algebraic solutions of the problems; the other is not. The study set out to explore:

(a) whether different verbal presentations of the same situation could lead students to construct different mental models;

(b) to what extent students moved between the models;

(c) how the mental models affected students' success in obtaining answers, the strategies they used and the equations which they formulated.

OUTLINE OF THE INVESTIGATION

Procedure

The items shown in Figure 1 were included in two forms of a pencil-and-paper test. The test was given in two schools to 166 students in Years 9 and 10 who had been learning algebra for three or four years. Test papers were randomly distributed in each of the seven classes participating, half the students receiving Test A and half receiving Test B. At the beginning of each test paper there were instructions to write an equation for each item and to solve it.

Construction of the items

The four items each describe problem situations where the size of a part is to be found, given information about the total amounts and various comparisons between the parts. Items 1, 2 and 3 have the same underlying mathematical structure; Item 4 is a variation. Items 1A and 1B (see Figure 1) are both valid, simple, complete natural language descriptions of the same problem situation. They pose the same question and are designed not to differ on any readability characteristics. Item 1A describes the problem as a sum of parts. It was expected that students would construct from this description a mental model reflecting the sum of parts structure and as a consequence begin to solve the problem by noting that two equal quantities plus 5 give 47. The first step towards a solution based on this model is to subtract 5 from 47. The sum of parts model is totally compatible with an algebraic solution, such as $x + (x + 5) = 47$, so $2x = 47 - 5$, etc. It also involves easy arithmetic. In contrast, Item 1B describes the same situation as a division into parts. It was expected that students would construct a different, but equally correct, mental model and consequently tend to solve the problem by a strategy of "share equally, then adjust". They would first allocate Mark and Jan equal amounts (usually $23.50 each) and then try to adjust the amounts by giving some of Jan's money to Mark. The first step in a solution based on this mental model is therefore to divide by 2. This method of solution is not compatible with a solution by algebraic equations. The arithmetic necessary is harder than for the sum of parts model. It was predicted that students working from a sum of parts model (Test A) would find the problem easier than those working from a division into parts model (Test B),. Evidence as to which mental model students constructed would be obtained from the first symbols they wrote down or the first calculation they did: subtraction for the sum of

parts model, division for the division into parts model.

TEST A

1A. Jan has $\$x$. Mark has \$5 more than Jan has. Altogether they have \$47. How much has each person got?

2A. A group of scouts did a 3-day walk on a long weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. The total journey was 80 km. How far did they walk on Saturday?

3A. Jeff has to wash 3 cars. The second car takes 7 minutes longer than the first one, and the third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?

4A The three sides of a triangle are different lengths. The second side is 3 cm longer than the first side, and the third side is twice as long as the first side. The side lengths add up to 63 cm altogether. How long is the first side?

TEST B

1B. \$47 is shared between two people, Mark and Jan. Jan gets $\$x$. Mark gets \$5 more than Jan gets. How much does each person get?

2B. A group of scouts walked a distance of 80 km on a 3-day weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. How far did they walk on Saturday?

3B. Jeff has to wash 3 cars. The second car takes 7 minutes longer than the first one, and the third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?

4B. The three sides of a triangle are different lengths. The side lengths add up to 63 cm altogether. The second side is 3 cm longer than the first side, and the third side is twice as long as the first side. How long is the first side?

Figure 1. The two versions of the test

Items 2A and 2B, and 4A and 4B, were also designed to prompt construction of different mental models (sum of parts versus division into parts) for the same situation, although with three parts rather than two. In these items, the sum of parts model is again compatible with an algebraic solution and also leads to a solution by arithmetic reasoning with first step subtraction. However, no

solutions readily follow from the division into parts model: no algebraic solution is available and the arithmetic reasoning required to find out how to adjust equal amounts to meet the conditions is very difficult. (In fact it was achieved by only one student, Martin whose work is shown in Figure 2.) In both Items 2 and 4, the prompting of the division into parts model was lessened a little, deliberately - the key word "shared" was not used, and in Item 4 the first statement ("The three sides of a triangle are different lengths") might suggest that division into three equal lengths is not a good way to start. Of course all items can (eventually!) be solved by a guess and check strategy, with either or neither mental model, and it was expected that some students would use this when other methods were not immediately obvious to them.

Items 3A and 3B are identical. It is possible that a "mental set" created by the previous two problems on Test B would affect students' perception of its structure and consequently their choice of first operation. On the other hand, if students did not perceive Items 2 and 3 as having the same structure and hence did not transfer the previous solution method, Item 3 should be equally easy in both forms of the test.

RESULTS AND DISCUSSION

Table 1 shows the percentages of correct answers for each item for each test, regardless of the method used. Table 2 shows the number of students carrying out subtraction or division as their first written operation on each item, and whether or not their final answer was correct. Some students began an item in one way (e.g., dividing 80 by 3 in Item 2B) and then changed strategy (e.g., to a sum of parts subtraction or a guess and check method) to obtain their final answer. However, as the purpose of Table 2 is to help identify mental models prompted by problem descriptions, only the first operation written down was used to classify responses. We acknowledge that, in some instances, students had probably already replaced their first mental model by another more promising one before they began to write.

Table 1
Success rates on four items ($N = 166$)

Version	n	Item 1	Item 2	Item 3	Item 4
A	83	73%	73%	73%	61%
B	83	67%	64%	63%	63%

Evidence for different mental models

Evidence that the different forms of presentation of items did indeed lead students to construct different mental models is obtained by comparing overall success rates and the first operations written. As shown in Table 1, there were more correct answers in Test A than in Test B. Combining data in Table 1 for all items, the chi-square test shows an association significant at the 5% level

between the test version and success on these items [$\chi^2(1, N = 664) = 4.17$, $p = 0.04$]. Since the two groups were well matched, it is reasonable to conclude that the difference in difficulty was due to the different presentations.

Table 2

Numbers of correct and incorrect solutions related to first operation written down and version of test

	Test A						Test B					
	Subtract		Divide		Other[a]		Subtract		Divide		Other[a]	
	✓	×	✓	×	✓	×	✓	×	✓	×	✓	×
Item 1	34	3	18	15	9	4	22	0	30	27	4	0
Item 2	42	8	3	3	16	11	37	9	5	14	11	7
Item 3	44	6	3	3	14	13	38	5	2	11	12	15
Item 4	34	5	4	3	13	24	31	6	4	11	17	14

[a]Category "Other" includes answer only, guess-and-check method, method not clear, and no answer.

Note. For Item 4, initial division by 3 but not by 4 is classified as "Divide".

Table 2 links the problem presentations to the mental models constructed. It shows that 104 responses to Test B indicated division as the first operation whereas only 52 responses to Test A did so. Similarly, subtraction was indicated more often for Test A (176) than for Test B (148). The association between the test version and the first operation chosen is highly significant [$\chi^2(1, N = 366) = 18.6$, $p = 0.0001$]. This finding supports the hypothesis that problem presentation is a factor influencing which mental model is constructed.

Moving between the different mental representations

The results above support the hypothesis that the different test forms promoted the construction of different mental models. However, as Table 2 shows, over one third of Test A students appear to have used the division into parts model in Item 1. On the other hand, many Test B students worked with a sum of parts model, particularly in Items 2B and 3B. We propose that the inability of the division model to provide satisfactory answers to the more complex items encouraged students to extend their division representation to include sum of parts aspects. As students continued to confront the mathematical structure underlying the set of problems, their mental model of the structure was further elaborated and other aspects became dominant. Mental models are fluid rather than static, and take on extra features as the solution process continues.

The approximately 10% difference (8 students) in success rates on Items 1, 2 and 3 shown in Table 1 can be seen as being due to the greater persistence of the division into parts model by the Test B students (see "Divide ×" columns in Table 2). Analysis of the incorrect answers leads us to propose that students for whom the division into parts model was strong were more likely to be

satisfied with incorrect answers such as \$23.50 and \$18.50 for Item 1, and (80+3)-7-11 for Item 2.

Since the wording of Item 3 is identical in both tests, the discrepancy between success rates on this item (see Table 1) is likely to be caused by the psychological set favouring the division into parts model induced by the two previous items. The written working on the test papers shows that most students had used the same procedure for Item 3 that they had used for Item 2, often written in exactly the same format. Examples of students' work are shown in Figure 2. These and other Test B students perceived Items 1, 2 and 3 as having identical structures, although the first two were presented as a division into parts and the third was presented as a sum of parts. This is further support for the fluidity of mental models.

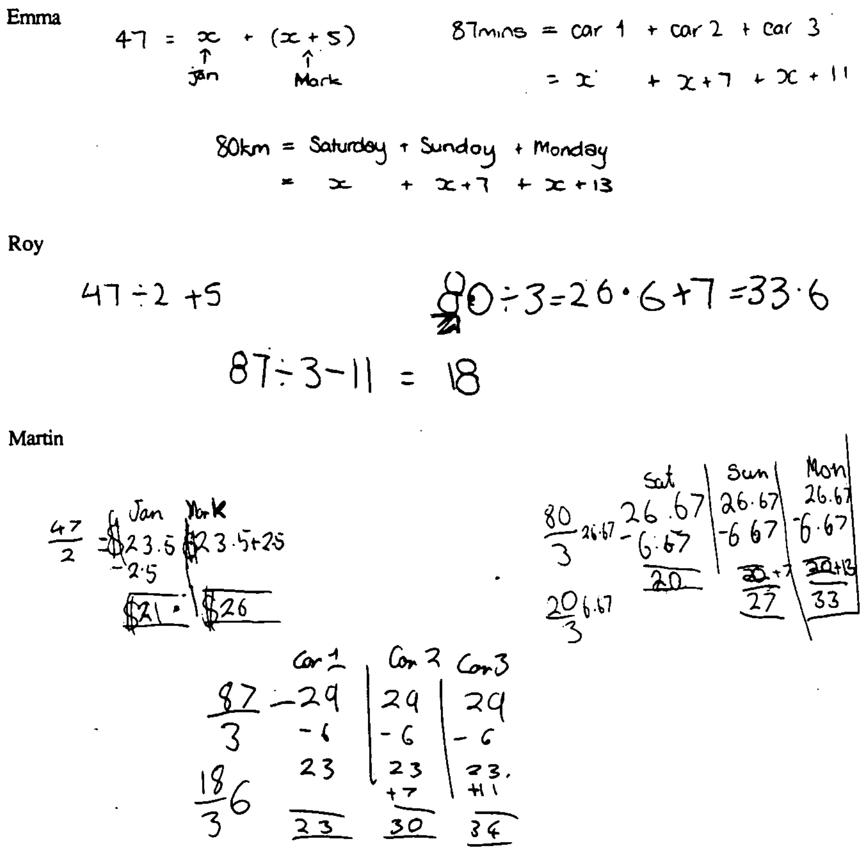

Figure 2. Three students' approaches to Items 1B, 2B and 3B

On Item 4, despite the "sum of parts" format of the previous item 3B, and the statement that the three sides of the triangle were different lengths, many Test B students showed continuing evidence (see Table 2) of the division into parts model. It is therefore puzzling that Item 4A was not easier than Item 4B. Some students with a sum of parts model for Items 2A and 3A did not adapt it for Item 4A. Another puzzling feature is why so many students chose division for Item 1A. Possibly the fact that the problem is about two people and a sum of money prompted them to think immediately of the action of sharing.

Use of algebra

Despite the explicit instruction to write an equation for each item, there were only 49 equations (or sets of simultaneous equations) written for Test A and only 41 for Test B out of a possible total of 664 equations (4×166). The small number of equations written (14% of the possible total) make it unwise to draw firm conclusions whether students doing Test A (which prompted the model compatible with algebraic solutions) found it easier to construct equations than students doing Test B. However, there appears to be no difference. There was no evidence in the equations or attempted equations written by Test B students of any tendency to try to express algebraically a "share equally then adjust" strategy based on the division into parts model. It seems that the students who were prepared to write equations were able to access mental models incorporating the sum of parts, as is required for an algebraic solution.

Knowledge of algebra, or willingness to use it, varied considerably between classes. In one of the two schools, most students avoided algebraic methods. In the other school, an equation was written (but not necessarily used) in approximately 25% of responses. However even in this school, Test A students were no more inclined than Test B students to try to use algebra. In these relatively simple situations, there were frequent difficulties with algebra which seemed unrelated to the mental model chosen. Some students omitted one or more terms when writing an equation, for example, writing $A + 7 + A + 13 = 80$ in Item 2A or 2B. Students frequently used algebraic letters only to record information, (e.g., $A + B + C = 80$) and then used unrelated processes (arithmetic reasoning or guess and check) to solve the problems. It seems that some of these Year 9 and 10 students see algebra as a language for expressing mathematical relationships but only a few realise that it is also useful for problem solving.

CONCLUSIONS

The different presentations of the problems tended to cause students to construct different mental models, evidenced by statistically significant differences in solution strategies as well as in success rates. We have also shown that a mental set or schema induced by one problem can affect the perception of subsequent problems. There is evidence that as students worked with the same mathematical structure through the series of problems, most of them readily extended their mental representations to encompass other features of the structure. Our data support the view that comprehension and problem solving are intertwined processes - comprehension is not a first isolated step.

As predicted, the group of students who were more likely to have constructed the mental model unhelpful for algebra wrote slightly fewer equations, but the data were not conclusive and further investigation is required. There were no instances where students tried to write equations based on the inappropriate model. Regardless of the test version, students who wrote equations had access to the sum of parts model; indeed the routine of the algebraic solution may itself prompt it.

The investigation of mental models related to common mathematical structures is a useful direction for future research. Within even simple problem situations such as the ones used here, there is a complex web of relationships between quantities which different students will perceive with different emphases and interpretations. It is important for teachers to appreciate the variety of mental models their students may construct and to appreciate that routine procedures (such as solving linear problems algebraically) are compatible with only some of these models. Students need to know that there are alternative models of a situation, and that their initial perceptions of underlying structure may not be the most useful.

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THE DEVELOPMENT OF ELEMENTARY ALGEBRAIC UNDERSTANDING

Elizabeth Warren

Australian Catholic University, McAuley Campus

The role of reasoning skills per se in the learning of mathematics has received little attention. Yet the importance of such processes in mathematical learning has been often acknowledged. In the algebraic domain, a recent approach for introducing elementary algebra involves generalising from both visual patterns and tables of data. The difficulties that children experience with these generalising processes have been well documented. But there remains a need to explore not only the contribution these generalising abilities make to understanding the variable concept but also the specific reasoning processes that are associated with this particular mathematical learning. This paper begins to explore these issues. A number of tests were administered to 355 students. Logical reasoning, analogical reasoning, patterning and spatial visualisation all contribute to the algebraic domain. The results also indicate that both generalising abilities contribute to predicting understanding of a variable concept, with the ability to generalise from tables being more accessible to most students.

INTRODUCTION

A recent approach for introducing the variable concept has focused on the developmental patterns that represents the transition from arithmetic to algebra (Mason, Pimm, Graham, & Gower, 1985; Pegg & Redden, 1990). This approach entails introducing algebra by looking at patterns, creating tables, describing the pattern, and "short handing" these descriptions into algebra. A number of research projects have reported the difficulties students experience with this approach. For example: in their attempts to generalise most children could not express a generalisation, disregarded all patterns when trying to generalise, and tended to use a procedural approach in reaching a solution (Ursini, 1991); students experience many difficulties when expressing relationships clearly in either natural language or algebraic notation (MacGregor & Stacey, 1993); and arithmetic incompetence and fixation with a recursive approach seriously obstruct progress (Orton & Orton, 1994). The focus of many researchers in the algebraic domain has seemed to be on how students' specific knowledge, especially that of the novice, influences the nature of the processes they use.

Many researchers contend that mathematical competence requires both comprehensive knowledge structures and general reasoning processes (e.g. Champagne, 1992; English, 1992). The importance of fostering general reasoning processes in all areas of mathematical curriculum has been widely documented. Such processes allow one to learn more mathematics, and to solve

mathematical problems throughout life (Fennema & Peterson, 1985). Yet there has been a paucity of research focusing on the role of these reasoning processes, especially in the learning of algebra. It seems that different modes of algebraic representation involve developing an array of powerful reasoning processes. The exact nature of these processes and the identification of those associated with particular mathematical learning needs to be explored (Champagne, 1992).

This paper reports on part of a larger study, which explores childrens' understanding of early algebraic concepts. This paper begins to investigate how developmental patterns relate to understanding the variable concept, and the reasoning processes students apply in algebraic learning, with a particular focus on those used in interpreting, and translating symbolic and visual representations. In mathematics, Lipman (1985) claims that spatial thinking, analogical and logical reasoning, classifying and hypothesising, and an ability to perceive patterns and generalise from them, all influence mathematical learning. A preference for a visual or symbolic approach to solution is also claimed to play a role (Presmeg, 1986,1992). Thus this phase of the study was exploratory in nature and attempted to identify any relationships existing between students' general reasoning processes (namely spatial visualisation, spatial orientation, logical analogical, patterning, and a preference of a visual or symbolic approach to solution), and their understanding of pre-algebraic and early algebraic ideas (namely, generalising from visual patterns and tables of data, and understanding the variable concept).

THE STUDY

Methodology

Since the aim of this study was to explore relationships between general reasoning processes and understanding pre-algebraic and early algebraic ideas, a correlational research design was utilised (Isaac & Michael, 1985). Seven written tests were developed. These consisted of six reasoning tests, including one test for ascertaining a preference for visual or symbolic approach to solution, and one algebra test. Each test measured a different process and understanding.

Logical reasoning, analogical reasoning, and pattern generalisation were each measured by a separate test, each comprising ten items. Since spatial reasoning, according to Tarte (1990), consists of two distinct component, spatial visualisation (the ability to mentally manipulate, twist or invert a visual stimuli) and spatial orientation (the ability to change ones perceptual perspective when viewing an object), two tests were used to measure these aspects of the spatial reasoning process. All items for these five tests were adapted from a wide range of

commercially available materials (e.g. Kit of Factor-referenced cognitive tests).

The test for measuring preference for visual or symbolic approach to solution was developed from a number of well established sources (e.g. Krutetskii, 1976; Moses, 1982; Suwarsono, 1982). The problems demanded a minimum application of mathematical knowledge but relied heavily on students' general reasoning processes. All items could be solved by both visual and non-visual means, and each solution was scored accordingly with a score of 0 awarded for a non-visual solution and 2 for a visual solution.

To test children's algebraic understanding, a number of different item types were developed. These items tested children's ability to: complete patterns and tables and generalise from this data to an algebraic expression, and understand the variable concept in a variety of contexts. Questions were drawn and adapted from a range of sources (e.g.Kuchemann, 1981; Quinlin, 1992). An earlier study, (English & Warren , in press) reported that there was no significant correlation between the ability to generalise from a pattern and understand the variable concept. This was an unexpected result in need of further probing. It was felt that perhaps the original algebra test was either too short or too narrow. As a consequence, the algebra test, for this study, was expanded and modified, with a greater emphasis placed on the measurement of the patterning component.

Nature or the sample

Since the study was concerned with children's development of beginning algebraic concepts, children were chosen from Grade 8 and Grade 9 (mean age 13 years and 4 months) as these are the two grades when algebra is formally introduced in the Queensland curriculum. The sample comprised of 355 children drawn from two coeducational schools in the Brisbane metropolitan area. Children attending both of these schools are representative of diverse socio-economic and cultural backgrounds. All seven tests were administered to each student.

RESULTS

Reliability of the tests

The reliability of each test was determined by calculating the Cronbach alphas. Table 1 presents a summary of these results.

Table 1

Reliability analysis scale for the tests

Component	No of items	Cronbach alpha
1.Logical	10	.61
2.Analogical	10	.55
3.Patterning	10	.84
4.Spatial visualisation	10	.64
5.Spatial orientation	20	.60
6. Visual approach	10	.52
7.Algebra (patterning)	16	.86
8.Algebra (tables)	16	.86
8.Variable concept	28	.89

Considering the number items in each test these reliability coefficient were regarded as more than adequate.

Intercorrelations among the variables

As each test measured different reasoning processes and understandings, the aggregated results from each test were used to ascertain relationships between these variables. A Pearson correlation analysis was used to identify any intercorrelations.

The algebra test consisted of three distinct components: generalising from visual patterns, generalising from a table of numbers, and understanding the variable concept. Given the emphasis on students' ability to generalise from patterns and tables of data in their early algebraic learning, it was considered important to investigate the extent to which these skills relate to each other and to understanding the variable concept. Correlations were carried out to identify any interactions between these components. Table 3 summarises the results.

Table 2

Intercorrelations Between the components of the algebra test

Component	1	2	3
1. Algebra (patterning)	--	.67*	.54*
2. Algebra (tables)		--	.63*
3. Variable concept			--

* $p < .001$

As shown in Table 2 there were significant correlations between all components of the algebra test. The significant correlation between the ability to generalise from patterns and the ability to generalise from tables was not unexpected, given that these two processes comprise a number of similarities. Of particular interest to this study was the correlation between the variable concept and the ability to generalise from tables compared with the correlation between the variable concept and the ability to generalise from patterns. A stepwise multiple regression analysis was carried out to ascertain the role each generalising skill plays in predicting understanding of the variable concept. Table 3 summarises the results of this analysis.

Table 3

Summary of Stepwise Regression Analysis for Variables Predicting Understanding the Variable Concept (N=355)

Variable	B	SE B	BETA
Step 1			
Algebra (Tables)	.48	.05	.49
Step 2			
Algebra (Patterning)	.19	.05	.21

Note. $R^2 = .39$ for step 1. delta $R^2 = .03$ for step 2
(p at each step is below .05)

The score for the ability to generalise from tables was selected first, accounting for 39% of the variance. The independent contribution of the ability to generalise from patterns

accounted for 3% of the variance. Thus both the ability to generalise from tables and the ability to generalise from patterns are both related to an understanding of the variable concept, with the ability to generalise from tables being a stronger predictor of success.

Various reasoning processes correlated with the three components of the algebra test. Table 4 summarises the results of this analysis.

Table 4

Intercorrelation among the Reasoning Processes and the Components of the Algebra Test

	1	2	3	4	5	6	7	8	9
1.Logical	--	.35	.31	.35	-	-	.38	.38	.36
2.Analogical		--	-*	-	-	-	-	.30	-
3.Patterning			--	.38	-	-	.30	-	-
4.Spatial Visualisation				--	-	-	.31	.33	.32
5.Spatial Orientation					--	-	-	-	-
6.Visual approach						--	-	-	-
7.Algebra (patterning)							--	.67	.54
8.Algebra (tables)								--	.63
9.Variable concept									--

* Correlations below .3 have been omitted

The spatial orientation reasoning process and a preference for a visual approach to solution failed to correlate significantly with any components of the algebra test. Both logical reasoning and spatial visualisation correlated significantly with all three components of the algebra test. Analogical reasoning was significantly correlated with an ability to generalise from tables, and patterning with an ability to generalise from patterns.

DISCUSSION

This research raises a number of issues regarding the teaching of algebra. Firstly logical reasoning, analogical reasoning, patterning and spatial visualisation seem to have some bearing on success in the algebraic domain. Yet little opportunity exists in our current curriculum for development and fostering of these reasoning processes.

Secondly, even though the ability to generalise from patterns contributes to an understanding of the variable concept, the ability to generalise from tables is a stronger predictor for success. In fact, there seems to be considerable overlap between these abilities

with the ability to generalise from tables being the more accessible of the two (Both tests comprised a total of 24 marks with the mean score for the patterning component being 11.77 and the mean score for the table component being 13.49). Thus generalising from tables is perhaps a more feasible means of introducing the variable concept. Once understood, this skill could be subsequently drawn upon when generalising from patterns.

Thirdly, generalising from tables and generalising from patterns have specific reasoning processes that need to be developed and fostered. Both draw on the logical and spatial visualisation reasoning processes. But specifically, the development of Analogical reasoning seems to be related to interpreting and generalising tables and the development of the patterning reasoning process seems to be related to generalising from visual patterns.

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ALGEBRAIC THINKING IN THE UPPER ELEMENTARY SCHOOL: THE ROLE OF COLLABORATION IN MAKING MEANING OF 'GENERALISATION'

Vicki Zack, St. George's School and McGill University
Montreal, Canada

Twenty-five children in a Grade 5 elementary school classroom worked alternately alone and together to solve one pivotal non-routine problem and other assigned related problems. Guided by the teacher who was observing their reasoning, nudging and learning with them, a number of the students came to see the structure of the problem, identified key ideas, and were able to express them algebraically.

The creators of the National Council of Teachers of Mathematics (NCTM) Standards envision classrooms in which students take charge of their learning, debate alternate solutions and develop connections and meanings as they speak together. Vygotsky's famous concept, the *zone of proximal development*, suggests that "skills ... and understandings are achieved in interaction with others before the children can do them on their own" (Newman et al, 1989, p. 15). For the past six years I have been a homeroom classroom teacher and researcher in a Grade 5 elementary school classroom (10-11 year olds). The students in our school have been tackling non-routine problems throughout their elementary school years. In this paper, I would like to describe how in the 1993-1994 year, a number of children working together progressed further than had children in my class in any previous year, in regard to algebraic thinking.

I will suggest some possible reasons for the difference. In the 1993-1994 year, when I assigned a problem I had presented in prior years, a new strategy was put forth and a few of the children were seen to use each other's ideas as springboards to a greater extent than had been the case previously. I endeavored to build upon the children's emerging ideas by providing other problem-solving challenges, aiming thereby to promote reflection and connections. Also, due to my participation in a graduate course on algebra taught by Lesley Lee (January to April, 1994, Concordia University), I had exposure to ideas about algebra I had not reflected upon extensively before, such as algebra as generalised arithmetic (Mason et al., 1985) and different definitions of what algebraic thinking might be (Davis, 1985), and to ideas I had not encountered previously, such as Bednarz's distinction between educators she calls *algebra types*, i. e. those who always think in terms of equations, and those who are *arithmetic types*, i. e. those who endeavor to connect with children's (hopefully) rich experience with arithmetic and to use it as a foundation for learning algebra (Nadine Bednarz, personal communication, February 2, 1994). The focus of the paper will be on the evolution of the children's understanding as they worked on figuring out the number of diagonals in a decagon. Specifically, I will discuss the algebraic expressions which some managed to generate for a number of the problems. Aspects which are integral but which can only be touched upon briefly in this paper are those dealing with my own growth in awareness, and with the mediating roles played by peers for each other, such as rephrasing, interpreting, highlighting, resisting closure, and serving as a receptive audience.

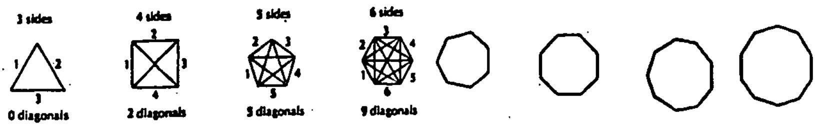
Classroom set-up, and assigned problems

St. George's is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds; the population is pre-dominantly English-speaking. The total class size in the 1993-1994 year was 25; however I always work with half-groups (12 or 13 children in each group) of heterogeneous ability. Problem solving is at the core of the mathematics curriculum in my classroom; non-routine problems are drawn from various sources. Mathematics class periods are 45 minutes each day (and are at times extended to 90 minutes). Problem solving is the focus of the entire lesson three times a week. In class the children often work in heterogeneous Groups of Four selected by the teacher. The children work first with a partner (2-some), and then when the pairs are 'ready', two pairs discuss the solution to the problem together as a 4-some. When the Groups of Four teams have all completed their deliberations, the entire group (12-some) meets to discuss the problem.

In addition to the in-class problem-solving sessions, each week the children also work on one challenging problem at home (Problem of the Week), and are expected to write in their Math Log about all that they did as they worked the problem. The children present their Problem of the Week solutions to a partner, to their 4-some, and to the group of twelve. The focus in this paper is on a series of connected, and increasingly demanding, Problem of the Week problems. The wording and sequence of the assignments were as follows:

- *Tunnels:* **'Nine prairie dogs need to connect all their burrows to one another in order to be sure that they can evade their enemy, the ferret. How many *tunnels* do they need to build?'** (Moretti et al, 1987, T-81, revised) (February 7, 1994)

- *Decagon Diagonals:* **How many diagonal lines can be drawn inside a figure with 10 sides?** (April 25, 1994)



- *25-Sided- , 52-Sided Polygons*: **How many diagonals would there be in a 25-sided polgon? in a 52-sided polygon?** (May 16, 1994)

- *Tunnels revisited:* **Can you write a number sentence or general rule for the *Tunnels* problem?** (May 25, 1994)

The children are videotaped on a rotating basis as they work in their groups of two and four. All the presentations done at the chalkboard are also videotaped. Much of the class session is conducted by the children. Data sources include: focused observations, videotape records, student artifacts (copybooks), teacher-composed questions eliciting opinions (written responses), and class discussions regarding research topics. In the 1993-1994 year, after each problem-solving discussion was concluded, I sounded out the students' reactions via a response sheet I used (the Helpful Explanation sheet) in which I asked the children whether they found a peer explanation of the solution helpful (Zack, 1994).

From drawing & counting, to detecting patterns, to a generalized algebraic expression: Solving the *Decagon Diagonals* Problem

The pivotal focus in this paper will be upon the solution to the *Decagon Diagonals* problem. I will indicate briefly how the students progressed and made connections to the other problems. I have assigned the *Decagon Diagonals* problem for the past 3 years. Every year but this one there have generally been three strategies (S) used:

> S#1 (used by most of the children): drew all the diagonals and counted them; the children often used different colours to help them distinguish diagonals belonging to each vertex. (Sometimes, the students make an **organized list** charting all of the combinations, and count them.)
>
> S#2 (used by some): drew some diagonals and then saw a pattern in the number of diagonals within one polygon (for example for the 10-sided one you would have: 7, 7, 6, 5, 4, 3, 2, 1, 0 = 35). The children continued the pattern without continuing the drawing of the diagonals, and added the numbers together.
>
> S#3 (used by only 2 to 4 children each year): saw a **pattern of differences** between the number of diagonals contained in each polygon
>
> triangle   square   pentagon   hexagon   septagon
>
> 0   2   5   9   14 ...
>
>   2   3   4   5
>
> and then continued the pattern without drawing any diagonals.

This year, one student in each group of 12 or 13, (Jerome in Group I, and Cathy in Group II) arrived at the answer by counting the number of diagonals emanating from a vertex, multiplying that number by the number of sides, and dividing by two. I will call this **number sentence** strategy, Strategy #4 (S#4).

Let me begin by sketching what happened in Cathy's group (Group II). Cathy wrote the above-mentioned procedure in her Math Log. During the discussion in a 3-some (one child was away), Cathy presented her solution to Abe and to Linda, saying that it was "weird" but it worked. Abe stated: "Oh, I see how you did it"; he did follow her thinking but later when he tackled the subsequently assigned *25-Sided-, 52-Sided Polygon* problem, used the same strategy he had used for the *Decagon Diagonals* problem (S#3). (Abe only appreciated the power of Cathy's approach later, as a result of further discussion. It must be stressed that Cathy and Jerome did not themselves see the potential of their strategy until later.) Linda (who used a variant of S#4, and whose answer was incorrect but whose strategy was closely linked to Cathy's way) tried to follow Cathy's thinking. Although it was fascinating to see that Linda sensed the two key ideas (i. e. divide by two, subtract 3, discussed in a section below) when she credited Cathy with helping her understand, it was nevertheless also evident that Linda was confused. David (who had used S#1 and had had the correct answer) heard Cathy present her strategy when the 12 children gathered in the large group; he at that point only commented that it was interesting: "Oh, that's neat." However, David was later seen to use Cathy's strategy for the *25-Sided, 52-Sided Polygon* problem, and was then seen to delve further, to extend and to engage in inquiry (*Tunnels-- Algebraic*, May 25, 1994). Susan (who had used S#1 correctly) was yet another who did not at first feel that Cathy's strategy was helpful (cf. her comments on the Helpful Explanations sheet), but was then seen to use it for her work with the subsequent *25-Sided-, 52-Sided Polygon* problem.

In the second group, the strategy Jerome used at first in his Math Log was S#3; however, the pattern which Cathy had seen and documented in her Math Log was one which Jerome "saw" while the May 4 discussion with all 12 children in his group (Group I) was occurring at the chalkboard. The videotape shows him studying the assignment page in his Log, and then saying:

> Oh-- I just saw another pattern. Well, every time. . . what happens is . . . if you see like the five has two from each one, well from two--, from two times five it's ten, but this is actually five so it's half of ten- and then over here it was seventy but it's only 35 . . . so it's always half

And we hear Gina saying "half" simultaneously with Jerome saying "always half", and then hear Jeff saying: "Ya, ya, if you multiply it by 10 and then divide by 2". (Jeff's "it" is the 7 diagonals emanating from one vertex in a decagon.) Despite the seeming abstruseness of Jerome's oral explanation, Jeff was able to follow it, and to **interpret** it for others. When analyzing this portion of videotape one sees that Jeff not only interpreted, he also focused on salient points, for example **highlighting** the fact that one need only attend to the diagonals emanating from <u>one</u> point and proceed from there. Thus we can state that the number sentence expressed in Strategy #4 at this point was as follows:

> S#4: Number of diagonals emanating from a vertex, times the number of sides, divided by 2.

When I asked the group whether anyone would like to try to express it algebraically (at which point Valerie groaned, and Tamara stated, "I should have taken notes"), Jeff put it into a generalized expression, writing upon the chalkboard as follows:

$A \times S \div 2 = $ # of diagonals

(with A standing for the number of diagonals from one vertex, and S the number of sides in the polygon)

There were three children in Jerome's 12-some who I now feel were in sync with Jerome's solution: there was Jeff, who said in his Helpful Explanation sheet that Jerome had "helped him by explaining better the pattern I saw but could not put my finger on", and who wrote the generalized expression; Micky (who used S#1 but whose answer to *Decagon Diagonals* was incorrect) who as I will show below adapted Jerome's idea/Jeff's expression for his subsequent algebraic solution of the *25-, 52-Sided Polygon* problem; and Gina (who used S#1 and whose answer to *Decagon Diagonals* was correct) who understood the aspect of "double-counting" and explained it clearly to her partner, Indira, as well as the rest of the group. During this session I strove to **highlight** the importance of the notion of "double-counting" to the group, but it is not clear to me how many of the children, Gina included, understood the reasons for my emphasis.

It is of import to note that I stressed to both groups that what had transpired was unique in that no other student had ever previously come up with the Cathy/Jerome-Jeff strategy. (See my correction which follows in Note 1.) Nor had I! I stressed to Group I (the Jerome-Jeff group) that no one in any of my Grade 5 groups had in the past ever come up with an algebraic expression such as theirs. My comments may have influenced some of the children to look more closely at Cathy's or Jerome-Jeff's strategy than they might otherwise have done.

Key ideas expressed algebraically: "Seeing" the structure of the problem

I needed to oblige the children to see that what they wanted was an expression which could be used no matter what the number of sides, and so I assigned the second problem, the *25-Sided Polygon, 52-Sided Polygon* problem: **How many diagonals would there be in a 25-sided polgon? in a 52-sided polygon?** As the class was dispersing, Micky quietly asked me: Do we have to draw all the diagonals? and I turned the question back to him: <u>Do</u> you have to draw all the diagonals?

When the children submitted their Math Logs, I noted that none used S#1, nine used S#2, one (Abe) used S#3, and ten used S#4. Of those who used S#4, some used arithmetic expressions (Group II: Cathy, David, Susan, Linda, Carrie; Group I: Anne), and others expressed the equation algebraically and then solved it (Group I: Micky, Jeff, Jerome; Group II: Bruce). Micky wrote and explained the following

notation in his Log-- (Z - 3) x Z ÷ 2, by saying: "Each time I do a problem like this I don't want [to have to] draw an "X" sided figure. I know that a [vertex] connects with all of the other [vertices] exept for 3, itself, the [vertex] to the left and right. . . You subtract 3 from the amount of total sides . . . then . . . here's the rule: (Z = no. of sides) (Z - 3) x Z ÷ 2 = no. of diagonal lines in figure."

It was as they solved or discussed the solution to this problem that some of the children and I began to see the structure of the 'diagonals' problem, both the *Decagon Diagonals* and the *25-, 52-Sided Polygon Diagonals* one. I will note the key ideas, the order in which the students attended to them, and how some of the children understood the key ideas and could represent them algebraically:

Key ideas

(1) **divided by 2** :

Many children spoke of "overlap", or "double-counting", or "so it's always half" but they were not necessarily able to express it in a number sentence as ÷ 2

(2) **minus 3** :

At first the children attended to the number of diagonals emanating from a vertex (see Jeff's "A"). Subsequently while solving the *25-Sided, 52-Sided Polygon* problem, they became aware that the number of diagonals was derived from the number of sides in the polygon 'minus 3', and were able to say why; for example, we saw above Micky's explanation of his reasoning in his Log, and Cathy wrote in her Log: "You always take away 3 becase two make the two lines to the side and one is the vertex."

During the discussion at the table (3-some), and then again during the 12-some discussion, Jeff said to Micky: "Oh, I understand, the minus 3 is to get my A" hence making the connection between Micky's notation and his own. It is important to note that my attempt to point out to Jeff that in order to solve a problem, one must use one letter (eg. Z, or other) and state the second variable in terms of the first variable met with a confused reaction on Jeff's part. He did not see why he could not use two unknowns, A and Z. This was perhaps because he already knew the solution to the problem and hence did not need to actually solve the equation. So although Jeff could see the connection between aspects of his and Micky's respective notations ('Your Z minus 3 is my A'), he did not see the need to express the relationship of one variable in terms of the other, as in Z representing the number of sides, and (Z - 3) representing the number of diagonals.

Some derived the expression in collaboration with a partner, i.e. it was a symmetrical relationship in that each was close to the solution, and needed only a nudge to get to the final expression (David and Bruce). In another case, the relationship was asymmetrical in that two children explained the expression, and the listener agreed with it (Micky and Jeff to Hosni). In two cases where I thought the students would arrive at the algebraic notation, they could not do so, and could not understand it when someone else presented it (Gina, Tamara).

Making connections, and writing meaningful algebraic expressions: The *Tunnels* problem revisited

I then wanted to see whether the children could make connections between the afore-mentioned problems and the *Tunnels* problem, which was the first they had solved in the series (Feb. 7). The strategy which the children had used for *Tunnels* was S#1, i. e. draw all the diagonals and count. Two children explicitly made a connection between the diagonals problems, and the *Tunnels* problem (Jeff and Bruce). Upon the conclusion of the *Decagon Diagonals* discussion, Jeff said: "Look, Vicki, I just turned to the *Tunnels* problem. It's just the same, but you have the sides" (May 4, 1994). It was at the end of May that I asked the children to look back at the Tunnels problem, and then asked: **Can you write a number sentence or general rule for the *Tunnels* problem?** (*Tunnels--Algebraic*, May 25, 1994).

Cathy, Jerome, and Jeff, the three architects of Strategy #4, seemed to stick closely to the original expression for *Decagon Diagonals* when writing their statement for *Tunnels--Algebraic* in their logs. For example, Jerome's statement in his Math Log is as follows: (S - 3) x S + 2 + S. The three arrived at their statements independently of each other. When I at first looked quickly at Jerome's statement, I was sure it was incorrect. It did not make as much sense to me as did David's Math Log entry for example (see below), and I had difficulty figuring out why it worked. Jerome listened to his partner Michel when Michel suggested that Jerome could use the 'other' rule "but instead of taking off 3, you take off 1." Jerome later saw that Jeff and Micky's collaborative effort had also rendered the corresponding expression: (S - 1) x S + 2.

Micky saw that there was a connection between the *Decagon Diagonals* and the *Tunnels* problems; in the Tunnels problem the connection does go to the left and the right. Micky kept pushing to see: How do you put the 3 back? Although Jeff's expression in his Math Log at the outset was the same as Jerome's (above), and Jeff seemed for a while very content to stick with it when conferring with Micky, it was Micky's insistence on determining 'how to put the 3 back' (it is Jeff who points out that it is not 3 but rather 2 which needs to be 'put back') and Micky's dogged resistance to closure that led them both to work together and arrive at the creation of "(S - 1) x S + 2 = tunnels". Jeff's cryptic note in his Log next to this equation reads: "Best way *". Jeff is thrilled with this creation; his pleasure is captured on the videotape as he says quietly but exultantly: "PERFECT!"

David's algebraic expression for the Tunnels--Algebraic problem was Z - 1 x Z + 2: "[It's] 1 instead of 3 because you can go to the sides." Although the notation varied somewhat, there were 6 other algebraic notations which represented the same idea as David's (Abe, Bruce, Susan, Sheree as well as David in Group II; Micky and Jeff in Group I). I regret that at no time did I display all the diverse notations in order for the children to see and discuss the range of possibilities. The children were seen to push for supporting arguments, and for meaning; they suggested that some ways of expressing the idea made more sense than did others. For example, a

number sentence such as 22 x 25 + 2, worked out as 22 x 12 1/2, met with a response that it did not make sense because it would mean that the sides of the polygon had been cut in half.

The achievements of all the children are worthy of mention. There were nine students who were able to use algebraic notation and of those, five seemed aware that the use of a general rule was powerful. Four students were able to make connections to previous problems and to use a number sentence (equation) to express the solution for *Tunnels* but used an arithmetic (eg. $8 \times 9 + 2 = 36$) and not an algebraic expression. The other twelve were able to understand aspects of the problems even though they could not yet express the solutions algebraically. Most importantly, the children contributed to each other's learning; no one child was able to achieve alone the goal regarding algebraic expression. I the teacher appreciated in retrospect the extent to which the novel strategy (S#4) had helped me to see how a generalised equation for these problems would be derived; the children's discussions regarding what I have here designated key ideas helped me see what the components of the equation meant.

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Note 1:
As I finished mentioning to the children that their use of the number sentence was unique, it suddenly struck me that another child, Mario, might have made just such an attempt two years previously. And when I went back to consult the Math Logs from that year (1991-1992), there it was. It is important to stress this instance because as I reflected upon it, it became clear that Mario's ideas never reached their potential because some essential elements were missing. For one, I the teacher could not follow Mario's reasoning as presented in the Log (although there was obviously a vital germ of an idea there that stayed with me over the two year interim), and as a result could not nudge him further in his Log, nor help support his ideas when he presented to the class. Prior to 1993, the students presented their Problem of the Week solution only to the group of 12, and did not discuss with a partner and then in a 4-some; hence Mario did not have a peer to hear him out, who might have attempted to follow his reasoning. When I consulted the videotape of that session, I discovered that Mario gave his presentation to the group of 12 just before recess, and so the children in the group may have been attending more to the fact that they would soon be at recess (recreation) than to Mario's ideas.

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DEVELOPING CLINICAL ASSESSMENT TOOLS FOR ASSESSING "AT RISK" LEARNERS IN MATHEMATICS

Robert P. Hunting

La Trobe University

Brian A. Doig

The Australian Council for Educational Research

Research-based tools for assessing students' mathematical strengths and weaknesses are important to good pedagogy. We report a project for developing clinical tasks for assessing "at risk" students. The history and evolution of clinical assessment methods is traced from their origins in Piagetian research, and related developments in psychological assessment noted. A process for validating clinical tasks entailing content relevance and representativeness, theoretical validity, process analysis, and useability is outlined. We briefly discuss two examples from our recent work to highlight the need for explicit links to research.

The purpose of this paper is to provide some background to a project aimed at developing research-based tools for clinicians to use in the initial stage of assessing mathematical strengths and weaknesses of students, and provide examples of some tasks. Effective strategies are needed for helping individual students who are not realising their mathematical potential in the regular classroom setting. The training of teachers with advanced clinical skills, and the provision of assessment tools for them to use, go hand in hand, because it is the way in which a tool is applied that will determine its effectiveness.

The clinical interview as a research tool, and as a diagnostic and teaching tool

It is generally agreed that the origins of the clinical method as a formal educational research tool coincided with Piaget's early investigations into children's thinking (Ginsburg & Opper, 1969). Neither of the two most widely used research methods of that time - naturalistic observation or standardised testing - were considered suitable for studying children's cognitive functioning.

In a clinical interview a dialogue or conversation is held between an adult interviewer and a subject. The dialogue is centred around a problem or task which has been chosen to give the subject every opportunity to display behaviour from which mental mechanisms used in thinking about that task or solving that problem can be inferred. An idealised description of the method is provided by Opper (1977).

Several variations of the same task may be presented to probe the strength and limits of the construct thought to underlay the subject's response, and to provide additional insights into that subject's mental functioning. Because of the dependent relationship between the subject's responses and the interviewer's questions, no two subjects will ever receive exactly the same interview. It follows that interviews can vary greatly across subjects in any one experiment. Basic to research involving clinical interviews are analyses of verbal protocols and non-verbal communications (Davis, 1991; Ginsburg, Kossan, Schwartz, & Swanson, 1983; Resnick & Ford, 1981).

The clinical method has been the mainstay of cross sectional status studies conducted by genetic epistemologists and cognitive psychologists in the Geneva tradition (for example, Inhelder & Piaget, 1958; Lovell, 1971; Noelting, 1980; Thomas, 1975). Repeated use of clinical interviews has provided powerful case study data bearing on questions of why children fail to learn mathematics (Allardice & Ginsburg, 1983, Erlwanger, 1975). In addition, clinical interviews have been used in longitudinal constructivist teaching experiments (Cobb & Steffe, 1983; Hunting & Korbosky, 1990; Hunting, Davis, & Pearn, in press; Steffe & Cobb, 1988; Wright, 1989).

Good teachers from the beginning of time have used similar strategies to the clinical method, precisely because the teaching process involves efforts on the part of teachers to understand the mathematical realities of their students. As Cobb and Steffe (1983) have said: "The actions of all teachers are guided, at least implicitly, by their understanding of their students' mathematical realities as well as by their own mathematical knowledge" (p. 85). Recent didactic literature oriented around the clinical interview as a teaching strategy is exemplified by Labinowicz (1985), who argued that the dominant form of paper and pencil testing in the United States did little to assist the teacher make decisions about what to do next with their students. He proposed a clinical form of assessment that allowed teachers to follow the children's thinking as they worked through tasks presented in the context of materials.

Diagnostic assessment often uses clinical approaches similar to those used for research purposes. The diagnostic interview is the point of entry at which information is gained that is needed to assess a problem, a relationship is initiated that will facilitate communication by the interviewee, and where the client's further relationship in a program of visits is facilitated (Pope, 1983). In the field of mathematics education Ginsburg et al. (1983) identify two phases in the diagnostic process. In phase one available data are assembled from parents and teacher, including information about the curriculum the student has been experiencing. This data is used, along with a standard set of general items, to help broadly identify the student's problems. The second phase involves specifying more precisely the nature and possible source of the student's difficulties. Diagnostic assessment is akin to tailored testing (Lord, 1980), whereby items are selected contingently using available estimates of examinee status. However, in contrast to tailored testing, where the emphasis for diagnosis is derived from formal scoring and psychometric analyses, diagnostic assessment has been categorised as *impressionistic* (Cronbach, 1984). As Ginsburg et al. (1983) have reminded us,

> "There is little agreement on a taxonomy of general mathematical disabilities. Any particular child is likely to have a mixture of conceptual and procedural difficulties contributing to math learning problems, as well as more general learning problems and emotional difficulties" (p. 46).

The clinical movement in assessment.

Renewed interest in clinical approaches to assessment of learning in mathematics have coincided with recent emphases on *action-reflection* models of teaching (see for example, Schon, 1987) and orientations to psychological testing that admit more qualitative approaches such as dynamic assessment (Feuerstein, 1979; Gupta & Coxhead, 1988; Lidz, 1987, 1991) and individualised

assessment (Fischer, 1985; Frederiksen et al., 1990). A further breakthrough in mathematics assessment occurred in the 80s when several authors re-discovered Piaget's clinical interview techniques (Donaldson, 1978; Ginsburg, Kossan, Schwartz, & Swanson, 1983; Hughes, 1986; Labinowicz, 1985). Labinowicz' textbook, in which he reported in detail the responses of young children to clinical interview tasks, was a significant advance.

Other assessment tools.

Clinical methods and tools are one of a range of assessment alternatives world-wide that are being trialled and evaluated in efforts to improve student learning of mathematics (Anastasi, 1990; A. C. E. R., 1994; de Lange, 1987; Leder, 1992; N. C. T. M., 1993; Niss, 1994a; 1994b; Romberg, 1992; Izard & Stephens, 1992). Other alternatives include student portfolios and journals, investigations, open-ended questions, observations, performance tasks, and student self-assessment (Grouws & Meier, 1992). Recent work on curriculum and assessment in Australia (AEC, 1990; 1994a; 1994b) has been driven primarily by a desire to monitor standards, provide accountability measures, and to improve reporting (Board of Studies, 1994). As Cronbach (1963) pointed out, these purposes are not the same as that of improving learning and teaching, and as such have somewhat different qualities. The advantage that clinical assessment methods have over instruments designed to serve administrative regulation is that the data source (the student) and the data analyser and interpreter (the teacher-interviewer) can engage directly in interactive communications. The teacher-interviewer "reads the play" as the play proceeds. Moreover, the primary concern of the assessor is to better understand the knowledge state of the learner.

The development process

Traditional approaches to the question of test validity have embraced three major categories of validity evidence: content-related, criterion-related, and construct-related (APA, 1966). The testing field has moved to recognise that validity is a unitary concept (Anastasi, 1990; Cronbach, 1984). Because content- and criterion-related evidence contribute to score meaning, they have come to be recognised as aspects of construct validity (Messick, 1989). Yet, as Messick (1989) says, "in applied uses of tests, *general* evidence supportive of construct validity usually needs to be buttressed by specific evidence of the relevance of the test to the applied purpose and the utility of the test in the applied setting" (p. 20). In the context of developing clinical tools for assessing the mathematics knowledge of a student in an interview setting, there were several areas which we felt needed careful attention during the process of task construction and development. These are content relevance and representativeness, construct or theoretical validity, process analysis, and useability (Hunting & Doig, 1992).

Content relevance and representativeness. In order to highlight skills or understandings which may have been overlooked in the construction of provisional task sets, analyses were conducted to

determine links between each task and corresponding content cells of three major curriculum statements. These were

- A National Statement on Mathematics for Australian Schools (AEC, 1990);
- the Curriculum and Evaluation Standards of the National Council of Teachers of Mathematics (NCTM, 1989); and
- the mathematics portion of the United Kingdom National Curriculum (DES, 1988).

However, no attempt was made to include tasks which link to every content cell of any of the curriculum statements, or indeed to content cells common to all three statements. The tasks were designed to reveal the processes of students' mathematical learning, so the emphasis was to capture major psychological subdivisions, as we currently understand them. Since a task may tap multiple aspects of a student's knowledge, it was our goal to choose the least number of tasks that would maximise information about students' breadth and depth of knowledge. The goal of parsimony is especially important due to time constraints surrounding assessment interviews.

A second procedure, complementary to the first, involves submitting the provisional task set to a panel of mathematics education experts familiar with relevant curriculum content and related research. The panel was asked to consider, as well as construct representation, relevance of task context, format of protocol, appropriateness of vocabulary, and adequacy of logical branches for tasks that have alternative pathways dependent on student response. They were also invited to comment on interrelations between tasks and clusters of tasks.

Theoretical validity. It was our aim to provide supporting rationales for each type of task. Rationales have their bases in the research literature of mathematics education, as pioneered by Labinowicz (1985). A major contribution of this exercise was to delineate known psychological boundaries and emphasise the significance of cognitive functioning with respect to the domain of mathematics relevant to the student. Through this analysis recent research findings can be manifested in tasks not otherwise forthcoming from an analysis of traditionally conservative curriculum documents. Regular and routine revision of tasks are needed as new research findings throw light on students' mathematical behavior.

Process analysis. This phase entailed documenting typical student responses to each task type, with an interpretive commentary that attempted to link student behaviours to available theoretical constructs. This analysis represents a potentially rich source of data for clarifying existing theories and results about students' mathematics learning. The documentation of typical student responses is conducted through training programs designed to prepare skilled clinicians (Hunting & Doig, 1994).

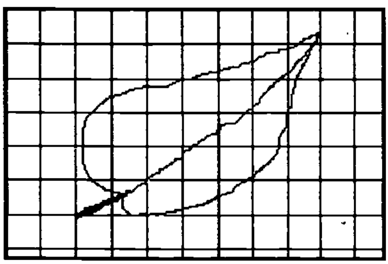
Useability. Teachers who have undertaken specific training in the implementation of clinical assessment procedures in mathematics were asked to comment on the utility of the procedure. They were invited to comment on the format and sequencing of the tasks for ease of use, as well as accompanying checklists or data sheets used to record student responses.

The need for a research base for assessment tasks

We argue that assessment tasks require rationales based on research and literature into the mathematical learning processes of students, and that these rationales need to be made explicit to teacher-clinicians so that they understand the responses of students against some theoretical base. We have explained elsewhere (Hunting & Doig, 1992; 1994) the critical link between the assessment tool and the skill level and knowledge base of the person who uses that tool. Historically, test developers have neglected to acknowledge results from educational research into the learning and teaching of school subject matter. Recognition of the need to make explicit the research base that supports assessment items or tasks is a serious weakness in current efforts to design national and state assessment instruments (Ellerton & Clements, 1994).

We present here two examples from our Level B assessment task set (Hunting, Doig, & Gibson, 1993). These examples illustrate the importance of incorporating research results into the design and development of assessment tasks. The first example is taken from the Measurement section, and deals with area. The second example is from the Geometry and Spatial Sense section. Our instrument development model assigns a high priority to provision of a theoretical justification for assessment tasks used. Other examples are discussed in Hunting, Doig, & Gibson (1994).

Example 1: B6.4 Measuring an irregular region. For this task the clinician begins by showing the student the graphic shown below, and says: "This diagram shows a leaf placed on 1 cm grid paper. What is the approximate area of the leaf? How did you work out your answer?"



Webb and Briars (1990) recommended that tasks for assessing students' measurement skills should involve the student actually measuring something. In their opinion items of the type where there is a pencil drawn alongside a ruler and the student is asked "How long is the pencil?" are inadequate because they assess only whether a student can read scales rather than measure using a particular unit. A criticism of usual practice by Dickson, Brown, and Gibson (1984) is addressed with this particular task. They argued that measurement presented in schools is more precise than it occurs in real life. A related task is B7.5 in the Geometry and Spatial Sense section. Here the student is told. "A dog is tied to a five metre rope which is attached to a stake in the ground. Make a sketch on this grid paper to estimate the area and shape of the ground on which the dog can walk. Tell me how you work out your answer as you do it".

Example 2: Task B7.6 Making a net from a three dimensional model. A three dimensional model of a cube is placed in front of the student. The interviewer shows how the cube can be unfolded without fully showing the net. The interviewer says, "Draw how this cube would look if it was unfolded and laid flat". After the student has responded, the interviewer asks, "Why did you draw it that way?"

Many examples of drawing and recognising nets of cubes can be found in teaching and test material. Piaget and Inhelder (1967) found that children shown a series of correct and incorrect nets of solids, were able to choose the correct one by guessing rather than demonstrating genuine understanding. Piaget and Inhelder found that asking a child to draw the net after being shown the model revealed intentions much more intelligently motivated. Piaget and Inhelder (1967) found that children who had been given experiences in folding and unfolding paper shapes were two or three years advanced on children who had not had those experiences. Children who do not perform well on this task would likely benefit from activities of transforming two dimensional shapes to three dimensional shapes and vice versa.

Some final comments

Clinical approaches to assessment have advantages for the classroom teacher wanting a deeper understanding of their students' knowledge of mathematics. Two critical aspects of clinical assessment methods are, first, the quality and appropriateness of the task with which the student will engage, and second, the skill of the interviewer in eliciting responses, and interpreting those responses. The clinical interview is a tool from research that can be applied powerfully in practice because the methodology is closely attuned to a fundamental activity of teaching and learning -- interactive communications. It's power derives partly from the incisive nature of the task, and partly from the potential for the interviewer to use the task to uncover conceptual strengths and weaknesses of the student. Subtle differences in task presentation and structure can elicit responses from students that reveal different aspects of conceptual understanding. For example, in the measurement task above, th·: directions are comparatively specific. More specific would be to provide square shaped tiles with instructions to use the tiles to cover the region. Less specific would be the questions, "How could this grid be used to find the area of this leaf? or "Why would we want to put this leaf on a grid like this?" Several challenges face this approach to assessment. Mathematics education is a young science where research into many aspects of the learning of school mathematics curriculum is negligible or non-existent. Our analyses of tasks and their research bases raises many questions needing sustained disciplined inquiry. Teachers need specific training in clinical assessment techniques and principles, and school organisational practices need to change to allow clinical assessments and follow-up strategies to be conducted effectively.

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ANALYSIS OF ERRORS AND STRATEGIES USED BY 9-YEAR-OLD PORTUGUESE STUDENTS IN MEASUREMENT AND GEOMETRY ITEMS

Glória Ramalho, Instituto Superior de Psicologia Aplicada
Teresa Correia, Instituto Superior de Psicologia Aplicada

Abstract

An experiment is described involving 9-year-old portuguese students responding to the geometry and measurement items included in the IAEP survey. One of the major aims of this study was to detect the errors that were made in those items and the misconceptions underlying them. The children were interviewed on an individual basis. A brief discussion of the difficulties found is presented.

The major motivation for this line of research was to elucidate the difficulties met by 9-year-old portuguese students while solving the mathematics test included in the I.A.E.P. survey, in view of this country's low performance in 1991 (Ramalho, 1994). This presentation will address the errors made and the strategies used by 21 pupils while solving the items regarding the topics of Measurement and Geometry that were included in that survey. We are aware of the time lag existing between the test administration in 1991 and the present study whose results pertain to 1994: students were not the same and it is not possible to argue that there were not any changes in these two years. Moreover, in the current study we were constrained to a smaller number of students due to the methodology that we considered as appropriate to use.

Nevertheless, in spite of the limitations that we just acknowledged, we found it interesting i) to examine the difficulties met by current pupils; ii) to compare their general achievement with their 1991's colleagues; and finally, iii) to investigate their strategies and possible misconceptions.

THEORETICAL FRAMEWORK

In terms of theory of psychological development we subscribe Bickhard's (Bickhard, 1980, Campbell & Bickhard, 1986) nonstructural proposal of levels of knowing . This perspective conceives of "developmental stages as a hierarchy of levels of knowing that is generated by iterating the basic knowledge relationship. (...) The hierarchy of knowing levels has an invariant sequence in any domain. The knowing levels can thus be used as the basis for a new, nonstructural definition of stages" (Campbell & Bickhard, 1986, p. 51). More specifically, the reflective iteration of the knowing relationship consists of a process of making knowledge that is only implicit in the organization of the interactive competencies of one level, explicit at the next higher level of knowing, which in turn will have further implicit properties knowable at the next lowing level,. and so on.

This general view of development is compatible with the perspective of "conceptual fields" introduced by Vergnaud (1990). Like Bickhard, this author rejects the model of developmental stages pointing to the evolvement of general logical structures. His contention integrates a partial order organization of chidren's competences and conceptions, rendering the analysis of cognitive development dependent on both the specific epistemology of the particular contents and the analysis of the subject's experience. The research priorities which Vergnaud maintains for the field of Mathematics education imply the recognition of a varied class of possible problems, the careful examination of its structure and of the cognitive operations necessary to deal with them.

Methodological considerations

We would like to recall here that the present investigation follows the application of a standardized method in the context of the IAEP survey. As mentioned above, the main goal of the international study was to characterize the educational systems and the cultural environments favorable to success in the Mathematics domain. The test was built so that it could rank students' performance within each country and relate that performance with contextual factors potentially favorable to success. The use of the survey method was adequate to the objectives so defined, but has a limited value in the scrutiny and understanding of the cognitive processes involved (Ginsburg, 1983) and, consequently, the conceptions underlying the solution of the items that were included.

More specifically, as we aimed at a better understanding of the cognitive processes, we sought out to use protocol methods which could elicit students immediate accounts of their mental steps while atempting to solve the questions.We used the revised clinical

interview (Ginsburg, 1983) consisting of flexible questioning of individual children, with concrete object support. Even if the resulting protocols may not be able to model all of the thought processes at stake in the problem solving activities, they are at least recognized as important contributions to the study of such mental paths (Hart 1985, Ginsburg 1983). Moreover, as the discovery type of the revised clinical interview allows for the researcher to perform a naturalistic observation of nonantecipated results and a flexible exploration of their meaning, we thought of this type of protocol method as the most adequate to the objectives and constraints of the present investigation.

METHODOLOGY

Sujects and Instrument

This study included 21 9-year-old students attending two elementary schools in Lisbon.

The children were presented the 15 items covering Measurement (9) and Geometry (6) included in the Second International Assessment of Educational Progress. The pupils were individually subject to a revised clinical interview as referred to above. In the end of each interview they were also asked to answer the questionnaire enclosed in that survey and addressing their family and school contexts.

This presentation covers the results found for eight of these items. The criterion for this selection was the low performance obtained. Therefore, we will summarize our findings with respect to the five Measurement items and the three Geometry items in which the difficulties were greater in the current study.

Procedure

The items were presented in the same order to each of the children by one of the present researchers. As they answered, they were asked about the way they had gotten to that solution, through the question "How did you get to that result?". The interview would proceed in a way that was dependent on the answers given. The time of each interview varied between 10 and 30 minutes and the interviews were audio-recorded.

Data Analysis

The tapes were transcribed and the resulting protocols were descriptively analyzed in several different ways, in accordance with the research interests already mentioned above. In order to do that:

1) we first classified the results under four categories: i) correct answer; ii) correct answer after the student was urged to read the question once again; iii) correct answer after some interaction with the interviewer; iv) correct answer not reached;

2) we made a comparison between 1991 results and the present ones, considering as correct answers only the categories i) above;

3) we attempted to identify the types of errors made and the misconceptions underlying those errors.

RESULTS

Current student performance

Current sudents' achievement in each of the items is presented in Table 1. We can see that children performed the worst in items 5, 7, 8, 14 and 15, which were selected for a more careful analysis that will be presented below.

TABLE 1 Frequency Distribution (%) of Students Answers to Measurement Items

Items	1 Finish a pattern involving squares		2 Select the figure with larger area among different figures dividided in unitary blocks		3 Determine how to balance two sets of marbles		5 Determine the length of one side of a square given its perimeter		7 Solve a problem involving hours and minutes		8 Relate an object's volume with the quantity of that objecto in a box		11 Read a temperature below zero on a thermometer		14 Measuring a segment when zero-point of ruler is not at the end of segment		15 Find the distance around a given rectangle	
Schools	I	II	I	II	I	II	I	II	I	II	I	II	I	II	I	II	I	II
Answers A	72.72	100	100	100	100	90	18.2	30	45.5	50	54.5	70	72.7	80	45.45	60	63.6	50
B	9.1	-	-	-	-	10	-	10	-	-	9.1	10	27.3	20	-	-	-	-
C	18.18	-	-	-	-	-	81.8	60	-	-	36.4	20	-	-	18.18	10	36.4	50
D	-	-	-	-	-	-	-	-	54.5	50	-	-	-	-	36.36	30	-	-

A - Correct Answer
B - Correct Answer after urged to read the question once again
C - Correct Answer after some interaction with the interviewer
D - Did not reach Correct Answer

I - School in the center of Lisbon I - N = 11
II -School at the periphery of Lisbon II - N = 10
N = 21

Table 2. illustrates the outcomes of the administration of the Geometry items. Achievement was poorer in the items 4, 6 and 12, which will be given more attention in the discussion that will follow.

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TABLE 2.Frequency Distribution (%) of Students Answers to Geometry Items

Itens	4 Count the faces of a solid figure		6 Identify a circle from its basic properties		9 Complete a pattern involving triangles		10 Identify figures with line symmetry		12 Visualize the rectangular faces of a geometrical solid		13 Identify a rectangle in a figure	
Escolas	I	II	I	II	I	II	I	II	I	II	I	II
Respostas												
A	81.1	80	27.27	20	100	90	81.8	90	45.45	30	100	90
B	-	10	-	30	-	-	9.1	10	45.45	60	-	10
C	9.1	10	63.63	50	-	10	9.1	-	9.1	10	-	-
D	9.1	-	-	-	-	-	-	-	-	-	-	-

A - Correct Answer
B - Correct Answer after urged to read the question once again
C - Correct Answer after some interaction with the interviewer
D - Did not reach Correct Answer

I - School in the center of Lisbon I - N = 11
II -School at the periphery of Lisbon II - N =10
N = 21

Comparison between current and previous results

The contrast between current and previous results is shown in Table 3, and shows a somewhat remarkable change. In nine of the 15 items (6 Measurment items and 3 Geometry items) there was a notorious improvement and the performance in two other items was kept mostly the same. The results were only worse in four of the items.

TABLE 3. Items' Levels of difficulty - in the 1991 survey and in the present study

Itens	Measurement	Geometry
1	85.7 (85) →	-
2	100 (90) ↗	-
3	95.2 (82) ↗	-
4	-	80.95 (55) ↗
5	23.8 (39) ↘	-
6	-	23.8 (39) ↘
7	47.6 (59) ↘	-
8	61.9 (56) ↗	-
9	-	95.23 (15) ↗
10	-	85.71 (77) ↗
11	76.19 (59) ↗	-
12	-	38.09 (50) ↘
13	-	95.23 (94) ↗
14	52.38 (21) ↗	-
15	57.14 (35) ↗	-

Errors and Strategies

Starting from the most difficult items to the easiest ones, **item 5 (Measurement)** asked for the length of one side of a square, given the distance around it. Five students answered well immediately. Some of the others (2) got to respond correctly just by reading the question.once again. But most of them (15) reached a right answer after some degree of interaction with the interviewer. This interaction showed that there were students who could not understand what was meant by "the total length of the square's sides", and some would identify it with length of one side. Some others would seem to be confusing a square with a cube. Once this problem got straightened up children had their answers right. Quite often they used the strategy of multiplication to get to the length of the side: "if the total is 20, than 5 times 4 is 20, and so the answer is 5."

Item 6 (Geometry) required the identification of a circle, among a set of geometrical figures, from a basic property: that all of its points were equally distanced to its center. The geometrical shapes seemed not to be very well known among the children, particularly the circle. But there seemed to exist also a problem with the drawing: the shapes were indicated by points, and their center was also identified by another point (P); the figures were not clear and were perceived as fuzzy and belonging to one only drawing, rather than representing different entities. Only 5 youngsters identified the correct answer right away. Three other responded well after being urged to read the item once again. Most of them had trouble in understanding what was asked and in devising a strategy to answer it. The most common strategy utilized was to draw a few lines connecting the points which defined the figure and its center. They drew in this way for all the shapes, but usually chose to draw specific lines, like diagonals in the case of parallelograms, which allowed for more than one possible answer.

The number of rectangular faces in a triangular prism, where the hidden faces were indicated, was demanded in **Item 12 (Geometry).** As in all the other items that were included, the figure was first and the question came last. Eight pupils were right in their first response. Many others (11) were asked to look at the item once again, and gave the right answer just by reading it a second time. Apparently, the triangular faces were most catching in the drawing, and when they came to read the problem they understood it as regarding the triangular faces, rather than the rectagular ones.

In **item 7 (Measurement)** half the pupils anwered correctly on a first answer basis, and the other half did not get to respond well. This item inquired the students about the time of a train departure once it was known that it had been a few minutes late. What seems to have been the problem here was their understanding of the word "late". For some of them, leaving "late" meant leaving "before", or else "going back, and therefore, subtracting".

Measuring a segment when the zero-point of the ruler is not at the end of the segment, which was asked in **Item 14 - Measurement,** was not a trivial problem for 10 students, three of whom did not get to solve it well, even after some interaction with the interviewer. The errors that we found regarded either focusing on the numbers rather than the standard unit of measurement (they would just look at the ending point of the segment), or starting at a point other than zero (they started their counting from one on, at the beginning of the segment).

Item 15 (Measurement) asked the children to find the distance around a given rectangle given the length of its two dimensions, which were indicated on the drawing that was presented. Thirteen students had no trouble in answering correctly. Some of the other pupils found some difficulty in understanding the meaning of "distance around", getting it confused with the notion of area. The figure displayed, once again preceding the text, helped that confusion: apparently the indication of the side's length led them to the computation of the area.

In **item 8 (Measurement)** students were asked to select the dimension of an object that would allow for less objects fitting in a given box. Most students did not show any difficulty in solving the problem. Two children got it right just by reading the item once again, and six students used direct proportionality (less objects - smaller objects) which led them to the wrong answer. Once they reflected on the results of each step of their reasoning they easily corrected their answers.

Counting the faces of a solid figure which did not indicate the hidden faces was the content of **Item 4 (Geometry).** Although this item showed to be relatively easy for these students, it is noteworthy that four of them still had troubles in solving it. Apparently the difficulty regarded the fact that the hidden faces were not drawn: they would count the faces in the drawing and just add one more.

DISCUSSION

In the first place we would like to emphasize the precipitation shown by most children in giving quick answers to the questions that were posed. This haste had a strong impact on their (poor) identification of what was asked in each of the items. We tend to attribute it to an apparently pervasive notion among these students that quick answers are an indication of smartness and good achievement. In our view, this notion brings serious limitations to their thought capacities and to their development of reflective power.

There were misunderstandings about some of the words utilized in the items (e.g. "distance around", "late"), as well as problems in the figures' display: the fact that these latter always preceded the text, that they sometimes were fuzzy or else gave too much

stress to some of their aspects, together with the problem mentioned above, appears to have constrained the youngsters' performance. In the same manner, the multiple choice format, unusual to portuguese students this age, led them in some sense to direct their reasoning to conform to one of the alternatives, before they centered it on the problem itself.

In the Measurement items we also detected some other difficulties at least on their first approach to the problems: i) with respect to the determination of a segment's length we verified Hiebert's findings, namely that students were "focusing on the numbers rather than the standard unit of measurement, starting at a point other than zero and leaving gaps" (Hiebert, 1984, quoted in Boulton-Lewis et al., 1994); ii) regarding the connection between volume and quantity we identified strategies that made use of direct relationships: "less" objects was understood to imply "smaller" objects.

To conclude, we think that this methodological approach, already explored by Hart (1985), which combines an extensive survey with a more intensive approach, in this case, the revised clinical interview, has allowed for interesting outcomes concerning both the specification of the overall performance of the large group, and the elucidation of the difficulties that were met.

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TEACHER'S CONCEPTUAL FRAMEWORK ON MATHEMATICS ASSESSMENT

L. Rico (*); F. Fernández (*); F. Gil (**); E. Castro (*); E. Castro (*); A. del Olmo (**); F. Moreno (**); I. Segovia (*).

(*) Granada University. (**) Almeria University. SPAIN

SUMMARY: *Whenever we want to make use of a test or a questionnaire to collect information for taking any kind of decision, we have to decide about which one could be the most appropiate for our task. For this work, we have designed and made use of a specific tool to study the way in which mathematics teachers conceptualize and value the students' assessment. The result has been a system of categories which shapes mathematics teacher's ideas and concepts about assessment. This work ends with a discusion about the results and their interpretation.*

INTRODUCTION

Over the last thirty years, many changes have been taking place in the organization and structure of the Educational System in most developed countries, and mainly in Compulsory Education; one example is the almost constant reform of Curricula, being Mathematics teaching always in the heart of it (Howson & Kahane, 1986).

Curricular changes in Mathematics have been various, each time being focused on different areas of the Curriculum (Begle, 1968; Fey, 1980, 1992; Howson, Keitel & Kilpatrick, 1981; Steiner, 1980). However, those changes in assessment methods have been very limited and controlled, being more frequent in theory than in practice; the debate about assessment is very recent and, for the moment, its changes are not very well known (Romberg, 1989). The ideas of changes and reforms in Mathematics Education are not having almost any effect on assessment due to two main reasons: the social impact of assessment, with its effects on pupil's promotion, and the continuous practice of examinations and external tests (Kilpatrick, 1979; Niss, 1993; Romberg, 1992; Webb, 1992). Regarding Compulsory Education, assessment is the area of Mathematics that has experienced less changes, what is supported by teachers. Mathematics teachers are, in fact, a key factor in the determination of the ways and uses of assessment made on our pupils' knowledge of Mathematics (Popkewitz, 1994; Skovmose, 1994; Webb & Coxford, 1993).

Current research in Education has focused on teacher's cognition (Houston, 1990), on teacher's implicit theories as a field of research. However, within the bibliography consulted, we have found very few works about assessment.

WORK PURPOSE

The following work is an **approach to Mathematics teacher's implicit theories about assessment**. These theories are understood as schematic representations of the teaching activity. They are teachers' statements about the empiric world after having examined the information obtained from reality. This study has been based on **descriptive methodology** and we have made use of **surveys** carried out with the traditional tools in this methodology: **the questionnaire**. It is a study of transverse nature made with the help of a small-scale survey, as it will be explained later on.

The goals of this survey have been set out following two phases:

First phase: it was devoted to focus the general aim on a specific central objective; this objective has consisted in **establishing the Spanish Mathematics teachers' current opinions about assessment, i.e. their common concepts, ideas, relationships and valuations.**

Second phase: **it was defined to identify and relate the stages and secondary aspects derived from the central objective**. To achieve our purpose, we have gone through the following stages:

a) the posing of certain key questions in order to reveal the main ideas and functions about assessment in Mathematics.

b) the delimination and application of the categories used to classify the different answers given to those questions.

c) the study of the categories validity and the description of the established system of categories.

METHODOLOGY

Sample. We have followed a **purposive sampling** to choose the appropiate sample for our study: we have selected one by one the different cases that have finally been included in the sample, finding on this way one able to meet our specific necessities.

This sample has finally been made up of 59 teachers, where 24 were receiving initial training at that moment, and the rest were tenure teachers of Mathematics who belonged to Secondary Education and University -10 of them belonged to the area of Didactics of Mathematics. Our purposive sample has consisted of some teachers who were willing to participate in the study in debate. It has covered different teaching levels in which Mathematics assessment has been extremely important.

Tool. At the beginning of the 91-92 Academic Year, we finished the composition of the Survey of Conceptual Framework about Assessment (SCFA). It was a tool to determine the field of ideas and functions normally used by Mathematics teachers with regard to assessment. This questionnaire, which appears in Appendix I, fits the following schema:

Information about the Institution that carries out the study.

Description of the purpose, request for help and thanks to the subject who has participated in the survey.

Eleven consecutive questions. The model of each one consisted in posing a question (first line), continuing with a general sentence that helped to find its answer (second line) and, following, six blank spaces for writing one or several answers to the posed question. The content of the questionnaire saw three different versions and it was finally accepted by the research team, being later on applied as pilot surveys to small groups of teachers who had no relation with them.

Questionnaire structure. The questions that were finally included have the following structure:

Questions 1-5: questions related to assessment in general where:
questions 1 & 2 refer to objectives and aims of assessment;
questions 3, 4 & 5 refer to practical and technical aspects.

Questions 6-10: specific questions about assessment in Mathematics where:
questions 6 & 7 refer to objectives and difficulties of assessment in Mathematics;
questions 8, 9 & 10 propose the consideration of other elements in Mathematics curricula.

Question 11: this question intends to collect information about other aspects not considered in the previous questions.

Application procedure. Although there was no time limit to answer the questions, in all cases it took between 30 minutes and an hour. During the test, where some members of the research team were present witnessing the seriousness of this process, the question paper was applied to individuals in some cases, and to groups in others.

ORGANIZATION AND CLASSIFICATION OF ANSWERS

The second stage, which consisted in establishing the meaning of each posed question, gave the result of a number of different answers. Each answer was expressed by one statement, which could appear once or more times. We have compiled all the given statements as answers to the questions and have organized them according to an alphabetical list.
The general information appears in Table 1.

Question	Frequency	Average of answers per participant
1	235	4
2	129	2
3	131	2
4	191	3
5	92	1.5
6	215	4
7	112	2
8	180	3
9	198	3
10	128	2
11	62	1
Total	1673	28

We have collected **1673** answers in total, which correspond to **543** different statements. It is interesting to note the high number of coincidences among all the answers. The average of statements per participant has been **28** and the average of different answers has been **9.2**.

Our study is focused on determining if these answers fit a system of ideas and concepts able to be classified and systematized. Are there any prior and clear criteria to classify the answers in order to know the underlying idea and the position of the person who gave the answers?

Our next step consisted on determining some criteria to classify the given answers. **The established classification must reveal, in each case, the different concepts involved in the content of the question, and the relationships among them.**

We intended to establish a conceptual structure which determine **the field of meanings of each question** in order to classify them according to different interpretations. The procedure carried out by the research team was as follows:

* the determination of a system of ideas and concepts to elaborate the questions and classify their corresponding answers;

* the establishment of some theoretical criteria to classify the answers, in two stages with corrections;

* the comparison with the classification criteria followed by an experienced teacher.

* the elaboration of a final classification and application to the statements of the criteria derived from this classification. In this classification we did not take into account the results obtained in question 11 for not being significant.

The third stage was devoted to study the reliability of the classification. For that reason, the list and system of categories were submitted to the control of 10 external judges who made their own classification. According to it, our first classification was revised and another one was scheduled following the criteria that were previously established; the percentage of coincidences in the two stages of classification with the different judges -external and internal- was as follows. Table 2: Coincidence in statements classification:

	Average of coincidences	
	First classification	**Final classification**
Question 1.	50.8%	63.2%
Question 2.	57.6%	66.6%
Question 3.	80.0%	95.6%
Question 4.	47.0%	58.9%
Question 5.	56.2%	76.2%
Question 6.	45.7%	58.0%
Question 7.	60.4%	72.1%
Question 8.	57.2%	67.6%
Question 9.	59.3%	73.1%
Question 10.	63.6%	71.3%

The increase in the percentage of coincidences with external examiners regarding the new classification is noticeable. The final average percentage was 70.2%.

OVERALL ANALYSIS OF FREQUENCY

At the end of the third stage, we carried out a description of answers given to each question according to the established system of categories. The final categories obtained in this stage, which appear in Appendix II, were 41. These categories appear with the original question together with the frequency of answers given to each question.

We followed two criteria to analyze the classification before mentioned.

Firstly, and as far as the total number of answers is concerned, we considered their frequency and percentage according to each category and studied the importance of each category in the total number of given answers.

Secondly, and with regard to the relation between the number of answers and the number of participants, we considered the percentage of answers that teachers gave related to a particular category.

According to the first criterion, we have to point out that all the questions were not analyzed with the same precision. In questions 1 and 6, their answers were classified according to 12 and 8 categories respectively. The answers to question 10 were analyzed through 4 different categories; questions 2, 7, 8 and 9 had 3 categories each, whereas questions 3, 4 and 5 had only 2. This can be reasonable if we bear in mind the information obtained through all the answers and the average of answers per subject. Questions 1 and 6 had the highest frequency of answers with an average of 4 per participant. In general, a higher number of answers makes a coincidence with a higher number of categories established for their analysis.

As far as the second criterion is concerned, it should be interesting to mention that all the categories had not got the same percentage of answers with regard to the total number of participants. Therefore, there were 8 categories that obtained a percentage higher than 100%, i.e. on average, each teacher had, at least, written one statement within these categories.
These are the following:

Assessment is used to control.

Examiners should belong to the classroom -internal examiners.

Traditional tools should be used for assessment.

In Mathematics, it is a priority to assess knowledge.

In Mathematics, it is a priority to assess ability.

Difficulties in assessment are due to the student.

Content is the criterion to assess Mathematics textbooks.

Teachers are assessed on their professionalization.

These 8 categories establish a basic profile of the main ideas about assessment in Mathematics, and correspond to 7 of the posed questions; only two of these categories are answers to a

single question (number 6). This profile is consistent; it is not made up of contradictory categories, and offer a conservative and traditional idea about assessment in Mathematics.

Thirteen categories obtained a percentage of 50%-100% in answers. This reflect some frequent but not prior opinions about assessment; the remaining 20 categories had a percentage inferior to 50%.

CONCLUSION

With this work, we have collected data to:

* **describe the nature of current conditions** with regard to Mathematics teacher's knowledge about assessment.

* **identify norms and patterns to be compared to current conditions** in order to explain Mathematics teacher's ideas about assessment.

* **determine the relationships among specific cases** regarding the obtained structured system of ideas, concepts and opinions.

We do believe that we have established a system of categories for the SCFA questionnaire, being also significant the procedure carried out to determine and validate this system. The variety shown by the 1673 statements that were compiled, gave the result of 41 different categories that established the system of concepts and ideas employed to answer the posed questions. Those categories reflect the different interpretations and meanings given by our participants in order to express their knowledge. Moreover, they reveal how complex and rich the system of ideas about assessment is.

We have carried out an analysis of every question and of the total, bearing in mind the frequency and percentage of given answers for each category and for the categories related to a single question. The fact that the categories show different frequencies in their use, and that a certain conceptual framework of assessment is seen as predominant does not mean that these conceptions also belong to a regular group of subjects. However, within the different categories, we can find some patterns of interpretation that can improve the explanation obtained up to now. Therefore, it should be interesting to continue this work with an analysis of the different category groupings obtained through the answers given by our subjects of study in order to explain the dimension underlying in the established categories and reduce the information obtained.

APPENDIX I

The following questionnaire is focused on determining and precising some of the most important questions related to Mathematics Assessment. Please, read it carefully and complete the information that we ask you for. Thank you very much.

1.- What should be object of assessment?

Assessment in education should be mainly addressed to:

2.- Why to assess students?

The aims of students' assessment in Compulsory Education are:

3.- Who should assess students?

Compulsory Education Students should be assessed by:

4.- What tools should be used to assess students?

The most frequent tools to assess students are:

5.- How should the results of assessment be expressed?

Students should receive the results of assessment through:

6.- What should be assessed in Mathematics?

Mathematics pupils' assessment should be addressed to:

7.- What are the main difficulties in Mathematics assessment?

The most difficult aspects in Mathematics assessment are:

Assessment not only affects students but other elements in the educational system.

8.- What criteria do you consider important to assess Mathematics textbooks?

9.- What aspects should be assessed in teachers of Mathematics?

The most important performances to assess Mathematics teachers are:

10.- What results should be assessed in the Centres with regard to Mathematics Education?

11.- What other aspects, not considered before, can be assessed in a Mathematics lesson?

APPENDIX II

Question	Category	frequency
1	It is a priority to assess student's knowledge	27
1	It is a priority to assess student's work	39
1	It is a priority to assess the attitude towards the subject	25
1	It is a priority to assess the student's ability	26
1	It is a priority to assess the student's behaviour	19
1	It is a priority to assess curriculum	9
1	It is a priority to assess teachers	23
1	It is a priority to assess students	13
1	It is a priority to assess content	17
1	It is a priority to assess objectives	16
1	It is a priority to assess means and materials	8
1	It is a priority to assess the educational institutions.	14
2	Assessment is carried out to obtain information	35
2	Assessment is carried out to take decisions	30
2	Assessment is carried out to control	66
3	Examiners should be internal	97
3	Examiners should be external	23
4	Traditional tools should be used to assess	144
4	General tools should be used to assess	32
5	Assessment should consider the way of communication	25
5	Assessment should consider the kind of information	49
6	In Mathematics, it is a priority to assess knowledge	62
6	In Mathematics, it is a priority to assess work	13
6	In Mathematics, it is a priority to assess attitude	25
6	In Mathematics, it is a priority to assess ability	84
6	In Mathematics, it is a priority to assess behaviour	8

6	In Mathematics, it is a priority to assess content	37
6	In Mathematics, it is a priority to assess objectives	6
6	In Matehmatics, it is a priority to assess means and materials	8
7	Difficulties in assessment are due examiners	9
7	Difficulties in assessment are due to students	67
7	Difficulties in assessment are due to procedures	36
8	Criterion to assess Mathematics textbooks is presentation	56
8	Criterion to assess Mathematics textbooks is content	97
9	Teachers are assessed on their personal values	41
9	Teachers are assessed on their scientific and didactic training	53
9	Teachers are assessed on their professionalization	97
10	Centres are assessed on their organization	54
10	Centres are assessed on their projects	29
10	Centres are assessed on their teachers	7
10	Centres are assessed on their students	28

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QUALITATIVE FEATURES OF TASKS IN MATHEMATICAL PROBLEM SOLVING ASSESSMENT

Manuel Santos Trigo* and Ernesto Sánchez
CINVESTAV-IPN

This paper discusses the importance of considering qualitative tasks to evaluate the students' work in mathematical problem solving. Two examples which show some features of two type of tasks are provided. In addition, the initial work shown by some ten grade students is analyzed as a means to show the potential of the use of these tasks in assessing the students work and also as a class material.

Introduction

Recent directions on what type of mathematics students should learn have challenged the idea of learning only collections of procedures or abilities to solve routine problems. Instead, there has been interest to focus on understanding the meaning of mathematical ideas and to search explicitly for different application of those ideas. As the NCTM (1989) indicated students need to develop abilities to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems (p.5). Research on how students solve mathematical problems has suggested that it is important to pay attention to the basic domain knowledge o resources, the use of cognitive and metacognitive strategies, and the conceptualization of mathematics that students bring into the problem solving arena (Schoenfeld, 1992). In this context, the question "What type of mathematical tasks could promote or foster the students understanding of mathematics?" becomes essential to analyze how this students understanding could be achieved. This paper presents an experience that analyses the process of designing qualitative tasks to assess students understanding of mathematics. It also focuses on discussing mathematical ideas that students showed when were asked to solve the tasks. Although some of the tasks were initially thought as a way to evaluate students' ideas of mathematics, there is indication that they can also be used as a means to engage the students in some kind of classroom discussion.

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*Visiting Scholar at The University of California, Berkeley.

The Importance of Qualitative Assessment

Kitcher (1988) presented a view about mathematics called "naturalism" in which he relied on the analysis of mathematical practice to explain the development of mathematics. He proposed that a mathematics practice has five components:

i. A language employed by the mathematician whose practice it is

ii. A set of statements accepted by those mathematician

iii. A set of questions that they regard as important and as currently unsolved

iv. A set of reasoning that they use to justify the statements they accept

v. A set of mathematical views embodying their ideas about how mathematics should be done, the ordering of mathematical disciplines, and so forth (p. 299)

In this context, it is important to relate the students mathematical learning to the practice of doing or developing mathematics. Schoenfeld (1994) stated that "learning to think mathematically means (a) developing a mathematical point of view --valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the goal of understanding structure--mathematical sense-making (p. 10). Indeed, what Kitcher identifies as key issues in the practice of mathematics becomes essential to promote the students learning of this discipline. Here, it is important to design mathematical tasks in which students have the opportunity to use different representation of mathematical situations, to identify meaningful information, and to use mathematical ideas to make progress or solve the problem. It is also important that students use different means to estimate or evaluate the plausibility of their solutions. That is, tasks that can be used as a vehicle to promote the use of language, mathematical content, and diverse strategies to find and discusses diverse solutions or ways to solve the problems.

A Sample of Tasks

Some tasks could involve finding relationships of various events that appear regularly around us. In some cases, the event provides the context to set a specific task. Although many of the tasks are embedded in contexts such as dealing with post office information, painting a school wall, or designing a field track, there are also situations in which the tasks are presented in terms of straight mathematical content, i.e., geometry, algebra, or arithmetic patters. The underlying principle is that all the tasks give the students the opportunity to use basic mathematical ideas to work on different phases of the process of solution. In fact, several tasks were designed by choosing situation that are familiar to the

students and offer the opportunity to use their mathematical knowledge already studied.

Two types of tasks have been developed, 45 minutes tasks in which the students have to explore various mathematical ideas to work on a plan that leads to the solution of the task; and 15 minutes tasks in which there is a direct application of some content. An example and some discussion about the features of each task is provided below.

A. Statement of the Task (45 minutes) (bowl task)

A hemispherical bowl contains some water. The height of the water is 8 cm and the radius of the bowl is 20 cm.

i. To what angle could the bowl be tilted before the water spills?

ii. If the height of the water is h and the radius of the bowl is R, write an algebraic equation to find the angle a .

iii. How would you show that your equation in part ii is correct?

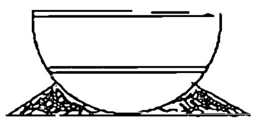


Figure 1

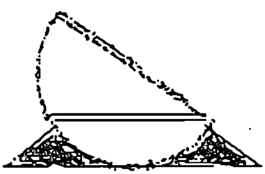


Figure 2

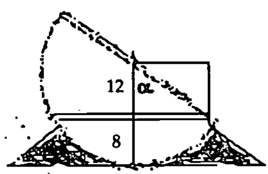


Figure 3

Qualitative features: An important part of this task is to represent the key elements of the task into a diagram that could provide some sense on what the problem is about. Figure 1 is a sketch of the recipient that shows the amount of water. Note that the recipient has a stand that maintains the bowl in a fixed position. Here, students are provided with Styrofoam spheres to help visualize the container. The second figure represents an abstraction of what happens when the bowl is tilted. For example, the water can be thought of being frozen

and the bowl moving. Here, it is possible to observe that the situation could be approached on the plane, instead of a three dimensional representation. Indeed, making the connection from three to two dimensions is a powerful step that allows to locate the data of the problem easily (figure 3). Another important feature of this task is the selection of the angle. For example, in figure 3, the angle a goes from the vertical radius to the line representing the top of the bowl, 90 - a could also be identified as the angle to tilt the bowl.

Dealing with values h and R gives the students the opportunity to analyze various particular cases and observe the behavior of the angle.

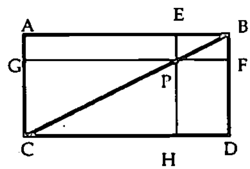
From the figure 3, it is observed that: Sin a = 16/20 which leads to

a = arc sin(16/20), and for the general case,

a = arc sin [(R - h)/R = 1 - h/R]. with $0 \leq h \leq R$

B. Short task (15 minutes) (area task)

In the figure below (rectangle), a point P is an arbitrary point on the diagonal CB. From P two perpendiculars are drawn to AB and AC. These perpendiculars intersect AB and CD in E and H and respectively and AC and BD in G and F respectively.



i. What can you say about the area of AEPG and DFPH? Explain

ii. Explain what happens to both areas when P is moved along the diagonal BC?

Qualitative Features: This is an example of a task in which the context is geometric oriented. It involves ways to examine special cases in which the information given in the problem could provide a sense of what occurs to the areas. For example, if p is located in the middle of BC, it is clear that the are is the same. Now, if segment CB is taken as a reference, it is observed that ΔABC & ΔDBC are congruent (SAS); in the same way ΔCPG is congruent to ΔCPH,

and ΔPBE is also congruent to PBF. By taking into account this information, it is concluded that the area of AEPG must be the same as the area of DFPH.

The above examples show some mathematical ideas that students may use during the process of dealing with the task. An important feature of these type of task is that students have to analyze the given information in order to decide what plan could lead to the solution. That is, there is no direct way to solve the tasks. The approaches shown by the students are used to evaluate their work from a holistic perspective. An example of this evaluation is given further.

It is important to mention that the development or design of a task follows various stages.

i. The task is discussed among colleagues to identify some of the its qualitative features. Here, some changes for the presentation or context of the task may emerge from the discussion and the initial statement or wording of the task is often modified or adjusted.

ii. The next part is to pilot the task with a small group of students to see what students can or can't do with the task. This initial field test gives some indication on whether or not the potential identified initially is real. Then a second revision is done by taking into account the pilot results and a new version of the task is presented.

iii. The next stage is to try this version with different students and then to analyze the results and give specific recommendation for its use.

Preliminary analysis of the students' work

The bowl and the area tasks were given to 12 grade 10th students, they worked individually for 45 minutes in the large task (bowl) and 15 minutes the short one (area). The written work shown by the students was analyzed by considering the type of resources and strategies that the students used to solve or make progress while working on the tasks.

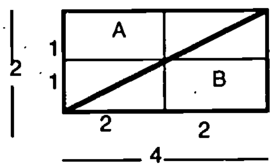
Six students recognized that the area of the two rectangles was the same. One of the students wrote:

> They are equal. The rectangle is split into 2 congruent triangles by the diagonal CB. And CP split the rectangle GCHP into 2 congruent triangles. And PB splits rectangle EBFP in also 2 congruent triangles. So by this point the two sides have equal areas. So if CB divides the large rectangle into two small triangles that have areas. Then AEPG and DFPH must have equal areas.

In explaining what happens to the areas when P is moved along the diagonal BC, this student responded:

The areas will continue to be equal no matter where P is on BC.

In general, the students who recognized that the area was the same provided a similar type of argument to support their claim. It was clear that they have all the resources to justify their arguments, however, they did not provide a reason on why for example the triangles were congruent. One student used a special case to support his answer. He relied on a 4 x 2 rectangle and drew the following diagram:



Area of A = 2 and
Area of B = 2

Two students thought that the area of the rectangles was different but did not show any work to explain their responses. These students wrote comments such as "I was never really good at this area of geometry" or "I do not remember how to do this". Two students did not show any work.

In the bowl problem, the students experienced difficulty in visualizing the relationship that could help them to relate the information given in the problem to the angle. However, it was interesting to observe that all the students use some kind of representation. Five students tried to use a three dimension representation to identify the angle, but they struggled to identify the angle position and failed to make significant progress to the solution of the problem on their own. Only three students were able to represent the angle in a diagram of two dimension. And four students asked for significant help throughout the entire process of solution.

It is important to mention that the students who achieved the solution of the problem (three students) spent significant part of their time trying to understand the conditions given in the statement of the problem. For example, some of the questions that they discussed that seemed to be helpful to understand the components of the problem included:

How important is the shape of the bowl? What about if the container were a parallelepiped? Where should the angle be located? Which data do we have to calculate the angle? Indeed, by discussing these questions, the students were

able to focus and determine a plan and strategy of solution. In contrast, the students who immediately selected a way to approach the problem, experienced serious difficulty not only in identifying which angle to consider but also what data to use. It seems that trying to approach the problem without having a complete understanding of possible relations of the data impeded that the students explored or identified other ways or variants to solve the problem. Some of the students commented (after having worked on the problem) that the bowl problem resulted to be difficult because it was a kind of different type. That is, it was necessary to choose a manageable representation in which the behavior of the bowl could be easily manipulated and they expressed that, in general, the regular class examples do not include this kind of exercise. Although, they thought that it was important to deal with these problems to find interesting applications of mathematics.

Discussion of Results and Recommendations

In general, the students showed significant progress while working on both tasks. However, the use of their mathematical resources often appeared loosely attached to their arguments. This was evident in the use of congruence in the short task. In addition, it seems that students experienced difficulty in presenting the written form of their response. As Schoenfeld (1992) suggested, problem solving activities should encourage students to value both discussions of various approaches and the communication of what is important to support their solutions.

Although few students were able to solve the bowl problem on their onw (30%), it was found that by working on this task, they showed awareness of using various means to approach the solution. For example, thinking of other type of container seemed to help them make sense of the main components of the task (behavior of the water, level, and the angle). It appeared that representing the problem in two dimension was the main obstacle to identify a manageable relationship. Here, it is suggested that students spend more time dealing with tasks that involve the use of representation from three to two dimensions. Although the initial purpose was to explore the type of strategies used by the students while working on some problems, it was interesting to observe that the information of the students work also could be categorized in terms of how they conceptualize mathematics and problem solving and in terms of mathematical disposition to work on this type of problems. For example, it was evident that

some students tried to work on the problems via number grabbing as initial method while other spent more time making sense of the conditions of the problem. It was also clear, that some students showed a better disposition to work on these tasks and those who showed some kind of flexibility in using more than only one approach, eventually were able to solve the problems.

Conclusions

The use mathematical tasks in which students have the opportunity to apply diverse mathematical ideas has been recognized as a necessary step not only in mathematical instruction but also in the students evaluation. Students may initially be reluctant to approach tasks in which they are asked to do more than using procedures while making sense of the information, designing a plan, or to solve the task. This paper shows that they eventually become interested in exploring various ways while approaching the tasks. It seems important that teachers should value and use these tasks on regular basis in their instruction. So, students could accept that dealing with these type of tasks is part of their experiences in the learning of mathematics.

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THE INFLUENCE OF TEACHERS ON CHILDREN'S IMAGE OF MATHEMATICS

LAURINDA BROWN
UNIVERSITY OF BRISTOL, SCHOOL OF EDUCATION, UK

A set of instruments which allowed the exploration of links between the image of mathematics of the teacher and their pupils is introduced. Although, given the constraints of the research design, I had expected no connections to be found the data showed clear evidence for the teacher's influence on their pupils' image of mathematics. In particular, interviews with the pupils revealed little evidence for the negative perceptions of mathematics prevalent in many children and adults. The teachers had been selected because they were effective in the opinion of advisory teachers and the question is then raised 'what is it that these teachers are doing?' In line with a current concern with grounding theoretical and philosophical concerns in mathematics education within the reality of practice my current research attempts to identify the strategies used by effective practising teachers to achieve positive images of mathematics in their pupils.

In 1988 I worked at devising a set of instruments which might allow me to explore whether a particular teacher did, in fact, influence the image of mathematics of their pupils in the same way. I found research studies on:

• children's attitude to mathematics and their perceptions of it
• teacher's views of mathematics and of mathematics teaching
• identifying characteristics of good practice
• working at the complex space which encompasses the children, their teacher and the mathematics

and I read reports (Buerk, 1982 and Vertes, 1981) of teachers with a strong philosophy who apparently influenced their pupils in some way. This reading all fed into the research design. (See Lerman, 1993 for a more up to date survey of the area.)

Research design

Definitions of image and influence

The personal theory (Kelly, 1955, Claxton, 1984) which an individual holds about mathematics at the present time which will include feelings, expectations, experiences and confidences was called the individual's image of mathematics.

An influence of the teacher on children's image of mathematics will therefore be defined as how the children's personal theories of mathematics have undergone a common change or adaption through working with the teacher.

The total of influences of the teacher on a particular child's image of mathematics would be expected to be greater than that of any common changes, but the identification of common changes would help the teacher to identify those of their personal beliefs which are most apparent to their pupils.

In looking for common changes in the set of pupils taught by a particular teacher, I will not, therefore, be considering the difference in adaption of personal theories which might be apparent, say, in the subset of boys and the subset of girls within the pupils.

Choice of teachers

I imposed a number of constraints of which the most important for me were:

• each teacher would have a strong personal philosophy and be considered to be effective either by

advisory teachers or their head of department

• a range of philosophies would be represented such as SMP 11-16 (an individualised workcard scheme), skill in exposition, 'not using a text-book'

• each teacher would have taught a group of pupils for at least a full year before the process started.

In the end I worked with 4 teachers, the fourth one being chosen because their own philosophy of teaching mathematics was undergoing change.

Choice of pupils

With each teacher I worked with one class and asked each teacher to choose 6 pupils, two of whom did respond to whatever they did with them, two of whom did not respond and the other two to make up imbalances such as gender representation. They were not to inform me of the reasons for their choices. Given the definition of influence above i would be looking for similarity of responses between the teacher and all six of their pupils even though one-third of them did not respond! Given this constraint, at this stage of the process, I was sure that I would find no evidence of such influence!

Pre-visit

I felt that it was important to experience the teacher working with the group of pupils and this visit started the fieldwork. My record of this visit was in the form of notes, written at the time, of events that happened in the lesson using reportive statements only eg 'Teacher says 'Now make 22'' (A M Brown, 1987). At this early stage of the process I wanted to avoid other categories quoted by Brown such as interpretative and prescriptive. I had to work quite hard prior to this first visit to get into the habit of doing this. Each teacher was asked not to prepare a special lesson for me to observe, but just to do what they planned to do.

Interview with 6 pupils

Each interview was semi-structured (Walsh, 1985) with a basic script for me to follow from which I could ask contingent questions as in a clinical interview (Ginsburg, 1981). I tape-recorded each interview and later transcribed the tapes. The interviews began with us engaging in some mathematics which was chosen by the pupil from five alternatives. This was partly as a vehicle for us to get to know each other before the 'questions' and partly as a vehicle to allow a choice of mathematical activity to be made which then led to a discussion of why that choice was made and the possibility of learning something about their view of mathematics directly. The five activities were chosen to offer a spread of categories of mathematical experience: practical work/geometry, numerical/algebraic, an investigation, applied/bookwork, a problem.

The work of Hoyles (1985) in: *asking pupils to recall particular episodes* and Thomas (1987) in the formulation of the questions in the affective domain were influential in the design of the script for the interviews with the pupils.

> Opening statement:
>
> *On the table in front of you there are, in fact, five different activities. Although we will not have time, probably, to finish the activity in any sense, could you choose one for us to get involved in?*
>
> After approximately 15 minutes:

Your answer to this question might be the same as your first answer. If instead of asking you to choose an activity for us to get involved in, I'd asked you to choose the one you thought was most mathematical, would you have chosen differently?
Stories:
For this part of the interview I am going to make some statements and, for each one, see what is brought to mind by what I say. Try to remember the event so clearly that you can tell me a story about what happened.
a) Tell me about an activity you have done recently in a maths lesson, and although you probably did not think so at the time, it is brought to mind now when I say, there you are, sitting in a maths lesson and what you are doing does feel like mathematics.
b) Tell me about an activity you have done recently in a maths lesson, and although you probably did not think so at the time, it is brought to mind now when I say, there you are, sitting in a maths lesson and what you are doing does not feel like mathematics.
c) Imagine a time when you felt good in a maths lesson.
d) Imagine a time when you felt bad in a maths lesson.
e) What I am interested in is your image of mathematics. So far you have indicated in your responses to the various statements and activities that maths is ... Is there anything else you'd like to add that has not been covered so far to the question: What is mathematics to you?

Some of the questions might seem long-winded but experience showed that they precipitated direct responses from the pupils without any need for them to clarify what I meant. The precise wording developed over time. The last question proved useful in that I could get feed-back from them about what I thought I had heard.

Interview with the teacher

I kept the interview with the teacher as close as possible to that with the children and also taped them. They engaged with the mathematical activity and then I asked them to describe to me their criteria for choosing the children for interview. The story questions were the same followed by:

• can you say in one word how you feel about mathematics?
• can you say in one word how you feel about teaching mathematics?
• and finally a chance to mention anything you would like to that you do not think we have covered naturally so far.

The technique of asking for stories through which to probe the underlying tenets (Davis and Mason at a seminar *Changing ways*, ATM Easter Course, 1988) of the teachers worked well and I felt comfortable in the interviews which lasted in some cases over an hour.
As the interviews progressed I reflected on the techniques I was using to encourage the teachers to talk more. The most effective technique I called 'summing up'. Clearly, on the later tapes, there are comments from me, in response to a particular story or statement, which are little more than a simple reiteration eg *So you think that* ...(repetition) ... These comments seem to provoke either agreement or disagreement on the part of the teacher followed by clarification and further examples which I now call provoked articulation.

Post-visit

The final visit was to observe the teacher and, as with the pre-visit, to simply note reportive observations (A M Brown, 1987) unless anything from the previous experience in the interviews was brought to mind in which case I would change to interpretative mode (A M Brown, 1987) continuing in reportive mode when sufficient notes had been taken to ensure that the link could be remembered. In practice this was the least satisfactory part of the whole process since those pupils who had been through the interviews wanted to engage with me and I became an active part of each of the lessons and 'observation' was difficult. Having worked hard to observe in the pre-visit in reportive mode I also found it difficult to move into a more interpretative style.

Evidence

A large amount of evidence was collected. There appeared to be evidence for strands linking linking the images of mathematics of the teachers with those of their pupils as follows (diagrams showing the links for Teachers A and C are included as a separate figure on the following page):

- Teacher A through challenging the pupils leaves them with an image of mathematics as initially hard, but easy when sorted out
- Teacher B through using the structure of the SMP 11-16 individualised learning booklets leaves the pupils with an image of mathematics as a set of titles from the booklets
- Teacher C sees mathematics as a framework of ideas which all link with each other and leaves with the pupils an image of mathematics based on using and applying it
- Teacher D and the pupils have a common image of mathematics as enjoyable.

These links came as a surprise to me. I had expected that my conditions on the pupils, namely some who did and others who did not respond to their teachers, would ensure that influence, defined as being an image common to all the six pupils and their teacher , could not be present.

One other surprise was that the vast majority of the 24 children interviewed seemed on the whole to be genuinely engaging with their mathematics. Where were all these pupils who hate mathematics and cannot see the relevance of it? Where are the children lacking in confidence, scared of making a mistake? Certainly not all the children I interviewed would have said that maths was wonderful; some thought it hard at times, others boring at times, but my overwhelming impression was of children working in classrooms where there was a positive experience. They were children who were learning something in mathematics lessons and had a feeling of progress.

Teaching strategies and purposes

So, these four teachers are seemingly capable, on this evidence, of working with pupils so that they end up seeing mathematics in a particular way. This seems to have positive effects on the pupils' performance in mathematics. The most pressing question raised for me by this piece of work was : *what is it that these teachers do?* They have a strong philosophy, yes, but if, say, I am starting to teach

Pupil A2

Mathematics is about organising thing so that they are easier to think about. When things are already easy they are not worth thinking about and not mathematics. It's more demanding than just number.

Pupil A3

When you look at all those numbers, you don't read it, it looks difficult. I think maths is more about problems as well as figures. Sorting out things. Finding easier ways to do things.

Pupil A1

I don't understand it sometimes. I get a bit lost, but in the end I can do it.

Teacher A: influence through challenge

Control, insight and challenge. in one word mathematics is compelling. When kids report 'It's all so easy and you can make it as complicated as you like and it's just as easy' about an extended piece of algebra, that's at the heart of what I'm trying to do with my teaching.

Pupil A4

I think maths is a hard subject.

Pupil A6

I think I felt good it was building those classrooms. Feeling good when I'd finished it, achieving, actually working it out. We'd not really known what to do - all our own work really.

Pupil A5

That making a scale-model was really hard, interesting but hard. I got to understand it a bit better, it took about a week to understand.

Pupil C2

I prefer having to solve it myself. It gives you that satisfaction of not having to take it from a book. I enjoy mathematics. I find it more of a challenge than a chore. The problem-solving exercises would help me because I could imagine how I felt and go logically through the steps.

Pupil C3

Mathematics is problem-solving. In Connect-4 I'd start by experimenting on a smaller grid to see if there's any pattern and be able to predict; maybe changing the number of counters which you have to make a row.

Pupil C1

You've got to actually solve things for yourself which aren't in a book. That's not really what I thought maths was going to be in the earlier years because that was just numeral sort of maths. You can relate it more to things outside, it's not just like a picture on the board, you can imagine it.

Teacher C: influence through philosophy

I think the whole idea of a problem is that you model it and make it solvable.

Mathematics is a framework and mathematics teaching is fun. Fun when you see the children building their own frameworks which are not necessarily you frameworks.

Pupil C4

I think maths is just applying stuff that you have learned in the lesson in reality.

Pupil C6

There was a real problem there. I understood what was happening and there were so many different types of maths used to find the final answer. Maths was numbers to me. I felt that in maths everyone knew the answer but as time's gone on I've discovered that even Teacher C doesn't know all the answers - so maths has changed - you can experiment.

Pupil C5

Maths is using what I already know like trigonometry and measurement.

and know that I want to create an environment in my classroom where children learn from their mistakes rather than feel they have failed when they make one, are there specific techniques which I can try?

In trying to find a way of thinking about what these teachers actually do to achieve the atmosphere in their classrooms which they value I started using a confusing variety of words such as 'strategy', 'tactic', 'skill' and 'technique'. Sometimes the teacher used a specific and repeatable behaviour over again with marked effect and at other times the strategy seemed more nebulous and global such as asking questions. How could I find a way of describing what I was looking for? There seems to be a general richness (lack of consistency?) in the use of words to describe strategies, but in the literature concerned with learning strategies I found a model which proved most useful in describing to me what I was trying to articulate in terms of teaching strategies (for a full discussion see Nisbet and Schucksmith, 1986):

	Characteristics	Examples
Central strategy (style, approach to learning)	Related to attitude and motivational factors	'Planfulness'
Macro-strategies (executive processes closely linked to cognitive knowledge)	Highly generalisable Improve with age Improve with experience Can be improved by training, but difficult?	Monitoring Checking Revising Self-testing
Micro-strategies (executive processes)	Less generalisable Easier to instruct Form continuum with higher-order skills More task-specific	Asking questions Planning

The examples here are concerned with learning strategies, but this hierarchy of strategies seems worth working with as a tool for allowing me to notice teaching strategies. I am not going to put too much energy into their classification beyond these broad divisions since 'a strategy is essentially a method for approaching a task, or more generally attaining a goal. Each strategy would call upon a variety of processes in the course of its operation' (Kirby, 1984 quoted in Nisbet and Schucksmith, 1986). These are distinct from non-executive processes which might be termed skills but any distinction when related to the complex arenas of learning and teaching is as Nisbet and Schucksmith point out easier to maintain in theory than in practice. For my purposes micro-strategies will subsume skills and tactics.

The work I am currently involved in is concerned with interviewing perceived effective mathematics teachers after their first lessons in the school year with classes new to them to find out what strategies they use to achieve the classroom ethos for mathematics which they want. I am not so interested in the management and organisational strategies as those linked to how they teach mathematics. After the initial interviews the work is followed up using a similar set of instruments to those discussed above and working with a class which the teacher has taught for some time.

In the above learning hierarchy applied to teaching strategies I am linking the *central strategy* to the teacher's images of mathematics and mathematics teaching and, as such, gives an overall sense of direction to their work. Such philosophical and attitudinal perspectives built up over time are certainly not easily transferable but do inform the decision-making necessary to apply lower order strategies. In finding a way of talking about what I am observing in such a way that it might be usable by trainee teachers I have started to work on macro-strategies in terms of their purpose as articulated by me or the teachers whilst observing or listening to practice. For a particular purpose the teacher often has a range of strategies which could be applied at differing times and in differing circumstances and, where a particular purpose is shared by a number of teachers, they will have a range of strategies between them. The micro-strategies for a particular purpose might be easily transferable as behaviours but the trainee teacher would still need to work at the level of purpose to begin to integrate the behaviour into a tool to achieve that purpose and will only recognise the micro-strategy as being useful if it conforms to their naive but developing central strategy. To give an indication of what I mean here follows a purpose with a few related strategies:

Purpose: Knowing what they know

At the start of a topic or theme how can you find out what the individual students in your class know and where they find problems so that you can make links?
Strategies:

• invite the students to make posters or write in response to 'Tell me what you know about ...'

• open-ended starter

  eg You're going to be working on area and you invite the students to draw shapes with area 8. This can be constrained by using square dotty paper and inviting the corner of the shapes to be on the points of the grid.

• a pupil offered an explanation of how they had begun to tackle a problem. The other pupils were invited to close their eyes and put up their hand if they had started in the same way. In fact, in the lesson observed, only two pupils did so. An alternative start was requested and the pupils again closed their eyes and put up their hands if this was their way of starting. The process continued with more information being collected and these different starts were then used for further exploration:

  - *What was the aim of the people who drew the radius? (Brown, L, 1992)*

This work is still in its early stages but the framework is providing a useful tool for my observations. In the presentation I will give an update on the work so far with a fuller list of purposes and strategies and would be interested to meet teacher researchers from other countries with a view to exploring similarities and differences.

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LISTENING TO STUDENTS' IDEAS: TEACHERS INTERVIEWING IN MATHEMATICS[1]

Marta Civil
University of Arizona

As teachers work on developing learning environments that build on students' ideas in mathematics, questions such as "how to uncover these ideas?" and "what to do with these ideas?" arise. This report focuses on the experience of a group of teachers as they interview students in mathematics. The teachers' write-ups serve as a spring board for a discussion of beliefs about mathematics and its teaching, and of aspects related to their pedagogical content knowledge. This discussion is placed within the larger framework of change in mathematics education, with a particular emphasis on implications for teacher education.

"It's scary to go into the classroom with the idea of letting the children go in different directions and me following them." This is what Donna, a preservice elementary teacher, said upon reflecting on her experience with an approach to mathematics instruction that moved away from the teacher as imparter of knowledge to the teacher as facilitator of mathematical inquiry. Donna's statement captures what I think many teachers are currently going through as they work on changing their teaching practice. The classrooms described in documents addressing recommendations for change in mathematics teaching and learning (NCTM, 1989; NRC, 1989) are very different from the classrooms that many of us experienced as students and as teachers. If classrooms are to become mathematical learning communities where students and teachers participate in the joint construction of mathematics (Cobb, Wood, & Yackel, 1990; Wilcox, Schram, Lappan, & Lanier, 1991), students' ideas should come to the foreground of class discussion. Teachers may then be faced with a variety of mathematical ideas floating in the room and their role is to probe and guide without leading or imposing their views as to what constitutes the ultimate answer. In order to build on their students' ideas, teachers need to listen to their students in ways that may be quite different from the kind of listening that usually takes places in a mathematics class. This "new" listening is an active listening in which the teacher shows a genuine interest in the students' thinking in mathematics by asking them to elaborate, to explain further, and by involving different students in the conversation.

This report is part of a larger research project that has as a goal to document and analyze the efforts of a group of elementary teachers as they try to bring change to their mathematics classroom. As they move towards an approach that focuses and builds on students' ideas of mathematics, becoming active listeners of these ideas is of key importance. Hence, one aspect of our research focuses on teachers listening to students' ideas about mathematics and on what they do with what they listen. We are looking at this from different perspectives: nature of the

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classroom discourse (through observation and video-tapes); teachers' use of students' writing in mathematics; teachers' use of task-based interviews as a means to learn about their students' thinking. This paper uses the teachers' interview reports as a window into aspects of their beliefs about mathematics and its teaching, their pedagogical content knowledge and their understanding of mathematics.

Background

This research report is part of a teacher enhancement project that was developed to address the needs of teachers who want to become active participants in the reform movement in mathematics education. Thirty one teachers (teaching children ages 8 through 14) constitute our first group of participants. This project has a strong leadership component. Hence, one of the criteria in the selection of these teachers was evidence of their participation and exposure to a variety of reform-oriented workshops and experiences in mathematics. These are neither novice teachers nor unfamiliar with the rhetoric of reform in mathematics education. Many of them have been using alternative teaching strategies and reflecting on the teaching implications of this call for change for quite some time now. This is not to say that there was not a great variability across participants as to their level of awareness of the call for reform and their personal interpretation of this call. Different participants had different needs--often shaped by the circumstances of their school and school district. Furthermore, a constant throughout the institute was the tension between participant as learner of mathematics and participant as teacher of mathematics.

A broad goal of this project is to enhance teachers' understanding of mathematics by engaging them in a variety of experiences as learners of mathematics. During the summer institute, these elementary teachers work in small groups on problems in geometry, numbers and number theory, probability and statistics. Technology, manipulative materials, and a variety of exemplary curriculum resources are integrated throughout the institute. In addition to exploring mathematics, the teachers discuss topics from research on learning mathematics and work on the implications of the reform for their own classrooms and schools. During the school year, the teachers work on implementing aspects addressed in the institute. Support is provided through project staff visits to their classrooms, monthly meetings with some of the teachers, and four all-day reunions during the year.

The instructional approach followed throughout the summer institute is based on a social constructivist view of learning (Cobb, Wood, & Yackel, 1990; Simon, 1994). In our interpretation of this view, we focus on the complementarity of individual and social perspectives on learning (Bartolini-Bussi, 1994; Cobb, 1994). Throughout the institute, the instructors often choose challenging tasks aimed at promoting social interaction in a mathematical context. The teachers share their ideas on the tasks posed, examine different ways to approach them, and

extend upon them. Our aim is to create a mathematics learning community in which participants are responsible for the negotiation of meanings and for deciding on the validity of different methods (Bishop, 1985; Wilcox, Schram, Lappan, & Lanier, 1991). Hence, a key activity during the institute is listening to each other's ideas and pursuing these to advance in the exploration of mathematics. We also explore different ways of finding how and what students think about mathematics. The participants watch videotapes of task-based interviews of children and read and discuss resource materials related to the issues of interviewing students and listening to students' ideas of mathematics. During the school year the teachers have to interview two of their students (individually or together) on a topic and with tasks of their choice. The teachers' write-ups about their interviews provide us with another way to gain an understanding of the many issues involved in changing the teaching and learning of mathematics.

A Discussion of the Teachers' Interview Reports

Letting Go of the Teacher's Hat

With a few exceptions, interviewing students on a mathematics task was new to these teachers. Three factors appeared to play a role in the teachers' writing of the interview reports: the task chosen for the interview (e.g., the choice of a problem-solving task or of a concept-investigation task led to considerably different reports); the teachers' beliefs about mathematics and its teaching; their level of experience with interviewing and reporting findings.

Added to the novelty and difficulty of the interviewing process, we need to keep in mind that this is just a small part of a larger program in which these teachers are being pushed to think about how mathematics is usually taught and how it may be taught. Hence, their interview reports convey a tension between the images they have held for years and the alternate images proposed for the teaching and learning of mathematics. Several teachers wrote about how hard it had been for them to refrain from telling or showing their students how to do something during the interview (cf. Markovits & Even, 1994, for similar observations). Letting go of the "telling mode" can be particularly difficult since the model of the teacher as dispenser of knowledge is very well ingrained throughout our experience in school. Knowing how to ask probing questions without directing students' thinking is usually not an easy task, especially because this kind of questioning is very different from the recitation type questioning that takes places in many classrooms. An interview that had as a goal to investigate the concept of decimal soon turned into a teaching-by-telling interview in which the students were led through the teacher's agenda by a series of questions that remind me of Plato's dialogue, Meno.

Another characteristic common to several reports is the focus on outcomes, especially to point out that the students did reach the "correct answer." This may be an indication of their beliefs about what is valued in a mathematics activity. But it also may be partly due to the difficulty in reporting in writing a thought process, leading to statements such as "they both

thought for a deliberate amount of time and gave the correct answer," with minimal to no description of what this thinking was.

The need to evaluate the students' performance is very much present. But this evaluation is coated with very positive remarks, emphasizing the fact that they "did get it." In reading some of these reports, several of the characteristics of the progressive educator (Ernest, 1991) surface, especially in relation to the minimizing of conflict and the avoidance of errors. Yet, it is through cognitive conflict and by looking into students' "errors" that we are most likely to learn about their thinking. Interviewing in mathematics should enable teachers to prod students' ideas and walk into murky areas. Shying away from these areas may lead to lost opportunities for learning for both teacher and students.

I next focus on what I consider to be two exemplary interviews. These teachers' write-ups are thought provoking and could serve as documents for reflection in teacher discussion groups along the lines of current work on using cases in mathematics teacher education (Barnett & Sather, 1992).

Pedagogical Content Knowledge

Lisa is a seventh and eighth grade (12 - 14 year olds) mathematics teacher with a very solid background in this content area. This became evident during the discussions on mathematics problems and concepts throughout the institute. She appeared knowledgeable, comfortable, and ready to take on the many mathematical challenges that were presented to the participants. During the institute she gained further appreciation for the use of manipulatives as a means to help students gain understanding. Thus, for her interview she had two students work (in individual interviews) on some fraction tasks using square tiles and the number line (this being a model that she routinely discusses with her students). Lisa had the students represent fractions such as $\frac{1}{2}$ and $\frac{2}{3}$ with the square tiles. She was interested in them "seeing" these fractions. The students had no difficulty making up models with the tiles to represent the fractions given. Then, after a very short time on this (each interview lasted about ten minutes), Lisa asked them to represent $\frac{1}{2}$ on the number line. Both students drew a number line from 1 to 10 and marked 5 as $\frac{1}{2}$. In her write-up, Lisa does not share much of her thinking about this response. She writes:

> I tried to ask them again to show me where $\frac{1}{2}$ was and not half of ten. I asked them to draw a number line starting at zero and going to one and then to mark $\frac{1}{2}$. They both appeared apprehensive at first, as if the space between 0 and 1 were sacred ground. They knew that half was in the middle and finally marked the $\frac{1}{2}$ and labeled it.

Several questions seem appropriate here: What model for fraction were these students using? Was their answer to be expected given the sequencing of tasks? What kinds of probes and tasks could be used to further explore their thinking? Using this episode to investigate these and similar questions could prove very fruitful in teacher discussion groups as a means to explore their own understanding of the mathematical content and their pedagogical content knowledge. By asking the students to draw a number line from 0 to 1, the students are able to give the "desired answer." But, what would have happened had she given them a number line from 0 through 6, with marks at every whole number? Would the students have marked .5 or 3 as their choice for $\frac{1}{2}$? As Larson (1980) points out, a number line from 0 to 1 may be interpreted by students as a part-whole model and thus may shed little light on students' understanding of the number line as a model for fractions.

Lisa's report reflects the difficulty to bridge between one's own understanding of the content and the students'. In no more than ten minutes per student, she had them work on three very distinct tasks, involving different subconstructs of rational number and the idea of unit. Just based on her report, it is hard for me to assess how much awareness Lisa has of the many subtleties (from a learning point of view) involved in the tasks she presented. She does write that, prior to this experience, she had assumed that her students knew more about certain concepts such as the number line. Students do find the number line model for fractions to be more difficult than the area (region; part-whole) model (Larson, 1980). Yet, teachers such as Lisa may not have had a chance to revisit the large knowledge base on rational numbers since they were in college working on their teaching degree. And even then, it is not clear how much they did (since time is always a factor) and how relevant they may have perceived that information to be at the time. Now that they have had their own experiences teaching this topic, a discussion of relevant research related to vignettes such as those written by Lisa is likely to be meaningful and help advance their pedagogical content knowledge.

Learning about their students' concepts

Penny teaches third grade (8 year olds) in a middle to upper class school district. She is very aware of what the implications of mathematics education reform are and is clearly a leader in implementing change. She feels isolated in her school because of some of her peer teachers' apparent lack of interest in or concern about discussing mathematics teaching and learning. Judging by her very insightful journal, her attitude throughout the institute, and her teaching behavior, she seems to be in the forefront of reform. Her write-up presents three students' responses to one single task which was part of a larger interview that she administered individually to each of 25 students. The task can be stated very simply:

The teacher shows a red trapezoid (from the pattern blocks collection) and tells the student "this trapezoid represents $\frac{1}{3}$ of the whole, show me the whole."

Penny had interviewed students in mathematics before and had in fact used this task in previous years. It is Penny's ability at probing, listening, and waiting that makes her report a very rich one. In reading it, one learns about these children's images of fractions and how prior experience with the pattern blocks appears to influence their work. One of the students, in looking for the whole, appears fixed on the idea that it has to be a hexagon. Immediately discarding the yellow hexagon as being too small, the student seems at a loss because she is looking for one single piece. When Penny asks her a general question about fractions, the student refers to a pie as the context for her answer. In the context of pattern blocks, this student may have been limited by an overreliance on the yellow hexagon as the whole. In a different context, one of her choice, she may have succeeded in solving a task similar to the one posed (see Mack, 1993, for a discussion on the influence of context when working with rational numbers). These are just suppositions, but the point is that the case presented opens up the door to further investigation. But how does this student finally respond to the task posed? She takes six red trapezoids and arranges them in the shape of a hexagon, with a hole in the middle. This hole creates some discomfort for her, as she says that the shape has to be "all filled in." She then takes a yellow hexagon to fill in the hole and presents that as her answer. Penny probes to make sure that the child is indeed done and concludes the interview telling her that they will look at this problem with everybody in the class later on. What is this student's concept of the whole in the context of pattern blocks? Is only a hexagon acceptable? And more generally, would she accept a pie (or another representation) with a hole as a model for the unit? This image of a continuous region (with no holes) is also shared by another student in Penny's report. This student takes three red trapezoids right away. He appears confident that this is what he needs for the whole. The whole interview is then spent on his trying to decide how to arrange these blocks to make the whole. To Penny's question as to whether there is only one way to arrange these blocks, the student answers that they can go any way as long as they touch. Penny probes this idea of touching and the student insists on it and says that "it's a rule." He finally arranges the three red trapezoids as a larger trapezoid for his answer. This vignette could serve as a motivator for a discussion on discrete versus continuous models for fractions (part-group/part-whole). Penny does address several teaching implications based on her reflection on the overall interview. She discusses what these students knew and what they did not, what they did and what they did not do. Her knowledge of her school program and of her students allows her to situate her findings within the larger context of where these children come from (in terms of mathematical experiences in school) and where they are heading.

Implications

One clear aim of the project in which these teachers are involved is the development of teaching and learning environments where students' thinking about mathematics is encouraged, shared, and explored. To do this, teachers not only need to nurture a safe atmosphere conducive to intellectual risk taking, but also, they need to know how to uncover students' thinking and what to do with it. The experience in interviewing described in this paper is one way for teachers to become more comfortable with how to probe students' thinking in a more controlled environment than that of the classroom. Two questions arise from this experience: What do teachers learn from listening to their students in an interview setting? What do we (as mathematics teacher educators) learn from reading the teachers' reports of their interviews? The first question is hard to address without some follow-up conversation with the teachers on this experience. Some teachers were more explicit than others in their reports and shared some of their views on what they had learned. Several of them commented on the difficulty of not asking directed questions. A few expressed that this interview experience had been eye-opening in terms of what they thought their students understood and what the interview revealed (e.g., Lisa).

The second question has been the focus of this paper and leads to several implications for mathematics teacher education. The choice of a task for an interview is a difficult one. Some teachers chose tasks that did not seem to be challenging enough (or for which they did not know how to probe). A knowledge of their students (what they know, how they learn) and of what makes a task mathematically rich seem necessary ingredients when designing an interview. The most informative interviews are those that set out to investigate a child's understanding of a concept (rather than a problem-solving task). Among those, several teachers based their interview on aspects of rational numbers (e.g., the two cases presented earlier). Yet, this topic was not addressed in the summer institute (other than incidentally). The teaching of rational numbers occupies a prominent place in mathematics in elementary school. To dismiss it in institutes for teachers as something "elementary" that they most likely already know is certainly a mistake as Lisa's and Penny's reports show. The questions raised by their write-ups can lead to a very fruitful discussion in which this topic can be investigated from both a teacher-as-learner and a teacher-as-teacher position.

Learning how to listen to students talking mathematics in an effort to uncover their thinking is a step towards the development of learning environments where mathematics is socially constructed. But teachers need to know what to do with the ideas they hear. An interview setting may give them a chance to reflect on the ideas they uncover. This reflection could be enhanced through a discussion group in which teachers share their findings and discuss teaching and learning implications. What should Penny do next with her third graders based on what she

found out about their concept of $\frac{1}{3}$ in the context of pattern blocks? How many more teachers in our group are unaware of the difficulty for many middle-school students to visualize fractions on the number line?

Uncovering their students' ideas is not enough, as several teachers in this group have realized. They regularly use students' writing in mathematics as one other way to "listen" to their ideas. In the classroom, they are working on changing the nature of discourse by involving their students in the inquiry process. But what to do with their students' writing and with the ideas they advance in the discussion is becoming an issue for some of these teachers. They would like to go beyond the "thank-you for sharing," often followed by little to no further discussion of the student's idea. Engaging teachers in small group discussions of actual students' ideas about mathematics (such as those uncovered through interviews) seems like a necessary next step.

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ANALYSING FOUR PRESERVICE TEACHERS' KNOWLEDGE AND THOUGHTS THROUGH THEIR BIOGRAPHICAL HISTORIES*

Domingos Fernandes
University of Aveiro, Portugal

Abstract

For about 40 hours I met four prospective secondary mathematics teachers aiming at (a) reconstructing, and reflecting upon, aspects of their own academic and personal lives; (b) talking about Mathematics teaching and learning ; and (c) talking about their views of what it means to be a Mathematics teacher. Research data enable one to relate the participants' biographical histories to the development of their current professional identities; it was also possible to get an understanding of the participants' personal traits, thoughts, knowledge and learning strategies. This paper discusses and elaborates upon those results and, consequently, yields reflections on: (a) the preservice education of mathematics secondary teachers; (b) the relevance of the biographical approach in the study of preservice teachers' knowledge and thoughts; and (c) the impact of the biographical approach on the development of preservice teachers' professional identities.

INTRODUCTION

I have been involved in the education of mathematics teachers for about 15 years. During all these years I have learned how difficult it has been for me to have a significant impact on my students -- future mathematics teachers. As a matter of fact, their personal views about mathematics and about its teaching and learning, which are known as being related to their teaching actions, are not easy to change. Besides, I have also realised how their optimism, and even enthusiasm, which they usually show while they are engaged in the preservice program, starts to fade away as they engage in the school routine. We all know that some particular school contexts do not facilitate the professional development of beginning teachers; many of them aren't even supportive of their ideas or pedagogical proposals. These facts might partially explain why it is so difficult for them to bring innovation and change into practice.

I think that I could go on and on building up a long list of difficulties which are inherent to our role as mathematics and teacher educators and which clearly call for our permanent reflection and informed action. However, I do believe that preservice teachers can make a difference in the future development of mathematics education. We need to provide them with more adequate learning environments and, simultaneously, we need to pay closer attention to preservice teachers as persons who are engaged in a life-long process of human and professional development. This means, for example, that we should reflect upon the answers to questions such as: Are we listening to our preservice teachers? Do we really know what they learn and how they learn? Do we take their own formative experiences, their knowledge, values, and beliefs into account ? Are we providing preservice teachers with an education which truly takes into account the contexts in which they are supposed to teach? Do we care about the meanings

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they attach to their preservice program? Are we dealing appropriately with the theory-practice dicotomy?

These are some of the questions that we must deal with if we are to make changes and improvements in the education of mathematics teachers, that is, if we are to provide them with a meaningful and powerful pedagogical atmosphere which can challenge their teaching beliefs and, at the same time, contribute to the development of their professional identities. That is why the underlying idea of the research reported in this paper has to do with the need to get a better understanding of preservice teachers' knowledge, thoughts, and professional identities by means of describing and interpreting the points of view which emerge from their biographical histories.

The following are the fundamental questions which guided the research described in this paper: (a) What meanings do the preservice mathematics teachers who participate in this study attribute to the former experiences that they lived as students both in their College and pre-College education? (b) What are the main characteristics of these preservice teachers' professional identities? That is, what knowledge and thoughts do they reveal about being a mathematics teacher, about mathematics and mathematics teaching and learning and, particularly, about problem solving? (c) What relationships can be found between these preservice teachers' biographical histories and their professional identities?

RESEARCH FRAMEWORK

Beyond the personal reflections and concerns expressed above this research is based upon the following grounds : (a) Recent research work on teachers' knowledge and thoughts that has been conducted in the context of Portuguese education (e.g., Delgado, 1994; Fernandes & Vale, 1994; Ponte e Canavarro, 1994; Vale, 1993); and (b) Work done in the area of narrative and biographical research, not necessarily developed by mathematics educators (e.g., Butt, Raymond, McCue e Yamagishi, 1992; Carter, 1994; Elbaz, 1990; Knowles, 1992; Kelchtermans, 1993).

In reviewing that research I have reinforced the idea that reflecting and coming up with new approaches about the initial education of mathematics teachers must be linked to the development of empirical research which enables one to get to know who preservice teachers are, what they think, what they know, what they learn, and how they learn.

Most of the recent Portuguese research in the mathematics teacher education area is based upon the assumption that teachers' thoughts and knowledge play a determinant role in the development of their teaching decisions and actions and, ultimately, in their teaching practices. It was based on this same assumption that researchers such as Thompson (1992) and Cooney (1985) developed their own research.

What are the main lessons that can be drawn from the results of that research? Firstly, it is clear that teachers and their formative experiences emerge as key players in any changing or innovative process in mathematics education. Secondly, in order for teachers' professional development to be successful both the quality of the programs and our good intents are

necessary; however, they are not sufficient. Thirdly, studying teachers' thoughts and knowledge provides relevant information to be taken into account by teacher educators in the process of improving teacher education programs. Finally, it must be said that implicitly or explicitly all studies acknowledge that we need to pay more attention to the person that every single teacher is.

It is important to highlight some of the shortcomings of that body of research as well:

1. As it should be expected their conceptual frameworks are strongly anchored in research work developed by mathematics educators. There are very few references to research work done in other areas; particularly, in teacher socialization, teacher thinking, and teacher education literatures.

2. In most of the studies the participants were experienced teachers; very few investigated preservice or beginning teachers' beliefs, practices, or knowledge (Abrantes, 1986; Fernandes, 1992; Fernandes e Vale, 1994; Vale, 1993).

3. There are very few references to the teachers' preservice programs and to their relationships with the teachers' beliefs and actions (practices). The role those initial programs can play in investigation and reflection upon teacher education processes has been essentially ignored.

4. Teachers' academic and formative experiences while they were pre-College or College students were not studied as well as their relationships with their current views about mathematics education, about teaching and about learning.

This analysis calls for the statement of other research questions, for the adoption of other methodological and analytical approaches and for the inclusion of other theoretical perspectives. I think that as mathematics educators we have much to learn from research conducted in teacher education by non-mathematics educators. This may help us to fill in some gaps and to overcome some shortcomings like the ones referred to above.

The biographical research perspective is one of those methodological and theoretical approaches that can enrich our work as teacher educators and researchers in mathematics education.

Barthes (1973) stated that narratives are an integral part of people's lives. That is, groups and members of a particular class or professional group express their knowledge, feelings, beliefs, thoughts, values, and experiences through narratives. Ultimately, one can say that people's narratives are a genuine expression of their culture. Thus, one is doing biographies when one uses people's narratives to write or to reconstruct their lives.

For the purposes of this investigation narratives, in writing or oral form, are more or less organized means through which human beings express their thoughts giving meaning to past, present or future events (experiences). Consequently, in their very nature, narratives are personal and subjective. According to Cortazzi (1993) one may state that there are at least three main reasons which justify the adoption of the biographical research approach in teacher education: a) it promotes and facilitates teachers' reflection; b) it enables one to thoroughly

investigate the nature of teachers' knowledge and thoughts ; and c) it gives "voice" to teachers' feelings, lived experiences, and the like.

Asking teachers to tell their experiences or their histories is a means to encourage them to reflect. Under this perspective, teachers and preservice teachers are commonly asked to narrate episodes, experiences, or teaching/learning events (e.g., Bird, Anderson, Sullivan e Swidler, 1993; Carter, 1994; Johnston, 1994; Keiny, 1994; Pultorak, 1993; Rust, 1994; Stoddart, Connel, Stofflettt e Peck, 1993).

Research on teachers' thoughts described in Clark & Peterson' s extensive review of the literature has shown how complex the relationships between teachers' thoughts and their teaching actions really are (Clark & Peterson, 1986). In order to study these relationships researchers have been making extensive use of teachers' narratives (e.g., Butt et al, 1992; Carter, 1994; Elbaz, 1990; Knowles, 1992; Kelchtermans, 1993; Stoddart et al, 1993).

In sum, investigating teachers and preservice teachers' knowledge and thoughts through their biographical histories allows for its contextualization from the inside; that is, through narratives one can learn thoughts and knowledge which are deeply grounded in classroom events experienced by the teachers to which only they can ascribe real meaning (e.g., Cortazzi, 1993; Kelchtermans, 1993).

METHOD

The method adopted in this study follows recommendations by Kelchtermans (1994) and Knowles (1992) who have also used a biographical research approach to understand teachers' thoughts, knowledge, and development. Some of the concepts that they have used as heuristic tools were extensively used in this study. *Critical phase*, *critical person*, *critical event*, *formative experiences* and *teachers' professional identity* are some of those concepts.

Participants

Four participants volunteered to participate in this study: Inês, 23, Regina, 21, Catarina, 23, and João, 24. All four were enrolled in a 5-year program leading to a License in Mathematics Teaching and were seen by their mathematics methods course instructor as good informants. Inês was in her sixth year at the university, João and Catarina in their fifth year, and Regina in her fourth year. Their mathematical ability was considered average (Inês, João), above average (Catarina) and excellent (Regina).

Data Collection and Data Analysis

As suggested by Kelchtermans (1994) data for this research were mainly collected through a "cycle" of three biographical individual semi-structured interviews focusing on: (a) the participants' most significant educational experiences in their family environments and in pre-College schools; (b) the participants' experiences in their university program; and (c) the clarification and reflection upon what had been said in the previous interviews. Participants' views about mathematics and about its teaching and learning as well as their views about what it

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means to be a mathematics teacher were inquired throughout the three interviews. All these interviews as well as the ones mentioned below were audio-taped and totally transcribed.

Two group interviews were also performed. The first one took place in the very beginning of data collection and was aimed at getting to know each other, presenting the research objectives, legitimating the work to be done, and motivating the participants to the importance of their commitment to this research work. The second one took place before the third round of individual interviews and focused on issues such as the participants' preservice education and their experience in this particular investigation. A fourth round of individual clarification interviews was needed and helped both myself and the participants to reflect upon the written reports and to clarify and elaborate on some of their parts.

Data analysis was performed in three main steps: (a) after the first two rounds of individual interviews and the first group interview; (b) after the second group interview and the third individual interviews; and (c) after the final clarification interviews. Thus the analysis was inductive and recurrent in nature. All participants had the chance to analyse the written form of their own narratives for further ellaboration and reflection. As a consequence, all the cases were subjected to some sort of modification.

Each participant's narrative was used to write a case reflecting a vertical analysis. Together all four cases provided the grounds for a horizontal analysis. Each case was organised in two main sections: (a) Genesis of the Ideas; and (b) Features of a Professional Identity. The first one has three sections: Family, Schools, and University. The second one has four sections: Mathematics, Teaching and Learning Mathematics, Becoming and Being a Mathematics Teacher, and Problem Solving.

MAIN FINDINGS AND DISCUSSION

Due to space limitations individual cases are not discussed in this paper. A discussion of the general findings which arose from the horizontal analysis follows in the next sections.

On the Genesis of the Participants' Ideas

All the participants except one (Catarina) grew up in very small, rural villages and belong to families of modest cultural, social, educational and economic backgrounds. For example, Regina's and Inês' fathers had to leave the country to raise the necessary for their families. Those two participants grew up with their mothers in Portugal while their fathers were working in Germany and France, respectively. Maybe because their parents had to struggle very hard to earn their livings all the participants were encouraged to study since they were very young. This is interesting to point out because all the participants' parents but Catarina's parents and Inês' mother did not even get a middle school diploma. They all hold a Grade 4 one.

Education was primarily seen by all families as a means to get social, economical, and cultural promotion. A means through which their children could get a decent life. It was under

this perspective that education was strongly valued in the families of all participants. They were all pushed into getting a College education.

While in Grades 1-6 the participants enjoyed quite different experiences, they all stressed their close and tender relationship with their primary school teachers (Grades 1-4) and remember their moments of fun and recreation with their peers in the school's playground. Their memories of Mathematics learning back in Grades 1-6 are limited to performing computations. They all enjoyed it and they all experienced success in this discipline. When inquired about their teachers' teaching strategies and styles they either said that they didn't remember any or stressed the "fun activities" that they provided in special ocasions of the year (e.g., Christmas, Easter).

Apparently, and according to all the participants' narratives, it is during Grades 7-12 that their relationship with Mathematics and with its teaching and learning starts to evolve in a more visible and significant way. They all pointed out more or less traumatic experiences in the Mathematics classrooms which are still very present in their memories and which ended up being recognised as critical incidents.

For example, Regina gives a strong meaning to the effort that she put into overcoming difficulties inherent to getting classes through TV. She states that what "saved" her was "the method" she used then. It is interesting that in our days she still refers to that experience when she is in trouble.

Catarina's narratives provide us with another example. For two years in a row she didn't like Mathematics because of her Grade-6 teacher. However, because of her Grade-8 teacher she regained confidence in herself and she started to enjoy mathematics again. These and several other examples illustrate that all the participants remember very well experiences lived in their mathematics classrooms. However, they seemed to have some difficulties in analysing the events from different perspectives. Usually, they were inclined to focus on the teacher-student relationship. That is, most of the times they did not blame their teachers' teaching methods; they blame them frequently on the grounds of their attitudes towards the students.

Another aspect which seems to be strongly rooted in the participants' ideas has to do with the number of students who fail in Mathematics. It is seen as something unavoidable and almost natural. Mathematics is difficult, they say, and one fails because either one is working on weak grounds or one doesn't study enough. A constant in their opinions was that they tended to blame the students for their own failure; not the teacher, nor any other external aspect. They only blamed the teacher when his or her rapport to the students was seen as a bad one.

Based upon the participants' experiences in their university program, one can list the following ideas: (a) they see their program for becoming a mathematics teacher as difficult or very difficult; (b) they tend to see education courses as the ones that "everybody is able to do"; (c) they seem to value more the mathematics courses, including methods courses, than courses of a more general nature; (d) they expect to spend 7 to 8 years at the university to complete their 5-year program (the exception is Regina who is one of the best students in the university); (e) they

all feel that they are not being adequately prepared to be a mathematics teacher; (f) they claim for more education courses of a practical nature instead of theoretical ones.

On the Participants' Professional Identities

As it should be expected, the participants' professional identities are still quite incipient. They seemed to have difficulties in reflecting upon the issues that were raised during our interviewing time. For example, it was quite surprising for me to realise that they couldn't mention other role of mathematics teaching beyond providing students with utilitarian tools to function properly in society. The formative, cultural, and scientific roles of mathematics teaching were seldom mentioned. This is consistent with their Platonic and utilitarian views of mathematics.

Mathematics, for these students is a difficult discipline. However, it can be learned by everybody if one works very hard and if one perseveres. This view seems to give little importance to the role that pedagogy and method can play in mathematics teaching and learning. Paradoxically, and on the other hand, they referrred to the need to provide students with more active methods to enhance their motivation and to facilitate their learning. In sum, I expected the participants to possess a more elaborated and sophisticated pedagogical discourse. Apparently, their views are strongly based upon quasi-naive reactions to educational situations and experiences that they have been living.

The preservice teachers who participate in this study seem to consider mathematics teachers as different from teachers in other disciplines. They say that mathematics teachers are seen differently by students and society in general. They apparently link this view of mathematics teachers with the nature of mathematics itself and state that to become a teacher of mathematics is a quite difficult endeavour.

Teaching mathematics, according to the participants, must be diversified to meet students differences in attitudes, capacities and abilities. However, it was clearly a difficult for them to elaborate this idea. That is, to make a difference these preservice teachers miss pedagogical tools which can guide them in the planification of more appropriate mathematical tasks.

On the Biographical Research Approach

In my opinion, based upon this research experience, the use of the biographical approach in the context of preservice teacher education has several advantages. First of all, it highlights the relevance of preservice teachers' past experiences both in family and school contexts; this is important because we need to be aware of the limited impact that our courses or actions may have on the professional development of the future teachers. Secondly, narratives call for preservice teachers' reflection and interpretation of past and present events. This is probably one of its strongest features because, as this study suggests, preservice teachers seem to have difficulties in developing a coherent and elaborate discourse about issues on mathematics teaching and learning. Thirdly, this research approach helps to get a better understanding of preservice teachers' thoughts and knowledge because, in telling and reflecting upon their stories, they are necessarily led to express their thinking about mathematics and education.

Many other things could be said about this research experience. Although it is the first time that I engaged in this approach, I think that I can tell that I learned the importance of knowing more about "my" preservice teachers. As a matter of fact, I learned that bringing up their biographical histories into our discussions might be an effective means to challenge their professional thinking and knowledge. And this is an indispensable component for the development of preservice teachers' professional identities.

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Professors' perceptions of students' mathematical thinking: Do they get what they prefer or what they expect?

Yudariah Binte Mohammad Yusof
Department of Mathematics
Universiti Teknologi Malaysia
Locked Bag No 791, 80990 Johor Bahru
Malaysia

David Tall
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL
UK

In a previous study (Mohd Yusof & Tall, 1994), it was shown that university students in a problem-solving course developed positive attitudes towards mathematics as a process of thinking rather than as a procedural body of knowledge. In this study their teachers are asked to specify the attitudes they ***expect*** *from their students and the attitudes they* ***prefer****. The difference is used to define the professors' "desired direction of change". It is found that almost all attitudinal changes in the problem-solving course are in the* ***desired*** *direction. Six months after returning to standard mathematics lecturing, almost all changes are in the* ***opposite*** *direction – consistent with the hypothesis that professors get what they expect, not what they prefer.*

Mohd Yusof & Tall (1994) studied the attitudinal changes in 44 students following a course in mathematical problem-solving based on the approach of Mason *et al* (1982). (There were 24 male and 20 female students – a mixture of third, fourth and fifth year undergraduates aged 18 to 21 studying Industrial Science (majoring in Mathematics) and Computer Education at Universiti Teknologi Malaysia.) The original study used a 17 item attitudinal questionnaire and showed that students' attitudes to mathematics and problem solving changed in what was considered a positive manner. In particular students' attitudes changed from mathematics as a body of procedures to be memorised to mathematics as a process of thinking.

Here we collect data from the students' teachers to establish their "desired direction of attitudinal change" and further data from the students in a delayed post-test, after six months of standard mathematics lectures. This allows a comparison to be made between the staff's desired change and the actual changes occurring in the students during problem-solving and during a return to regular mathematics teaching. The data from the questionnaires is supplemented by interviews with selected students and staff.

The "desired direction of attitudinal change" perceived by mathematics staff

Members of the Mathematics Department were invited to fill in the attitudinal questionnaire of Mohd Yusof & Tall (1994) twice. On first reading, they were requested to tick the response they *expect* from a typical student. On the second they were requested to put a circle where they **prefer** it to be. Twenty-two members of the department took part, responding to the following questionnaire on a five point scale:

Y, y, –, n, N (definitely yes, yes, no opinion, no, definitely no).

Section A : Attitudes to Mathematics	Section B : Attitudes to Problem-Solving
1. Mathematics is a collection of facts and procedures to be remembered.	1. I feel confident in my ability to solve mathematics problems.
2. Mathematics is about solving problems.	2. Solving mathematics problems is a great pleasure for me.
3. Mathematics is about inventing new ideas.	3. I only solve mathematics problems to get through the course.
4. Mathematics at university is very abstract.	4. I feel anxious when I am asked to solve mathematics problems.
5. I usually understand a new idea in mathematics quickly.	5. I often fear unexpected mathematics problems.
6. The mathematical topics we study at university make sense to me.	6. I feel the most important thing in mathematics is to get correct answers.
7. I have to work very hard to understand mathematics.	7. I am willing to try a different approach when my attempt fails.
8. I learn my mathematics through memory.	8. I give up fairly easily when the problem is difficult.
9. I am able to relate mathematical ideas learned.	

Table 1 : Attitudinal questions to mathematics and problem-solving

Table 2 shows the responses of 22 lecturers in the Mathematics Department and the direction of the desired change from the expected to the preferred attititude. The columns marked "Yes(Y)" have the "total yes" responses (Y+y), with the subset "definitely yes" (Y) in brackets. Similarly for "No(N)".

				Expect			Prefer		
	Attitude	desired change		Yes (Y)	–	No (N)	Yes (Y)	–	No (N)
Mathematics	facts and procedures	$\downarrow^{+++}_{+}$	<1%	20 (8)	0	2 (0)	13 (4)	0	9 (2)
	solving problems	$\uparrow^{+++}_{+++}$	n.s.	19 (9)	0	3 (0)	22 (9)	0	0 (0)
	inventing new ideas	$\uparrow^{+}_{-}$	n.s.	8 (2)	0	14 (1)	11 (3)	0	11 (1)
	abstract	$\downarrow^{+++}_{--}$	<1%	20 (6)	0	2 (0)	7 (0)	0	15 (4)
	understand quickly	$\uparrow^{+}_{---}$	<1%	3 (0)	0	19 (6)	15 (1)	0	7 (1)
	make sense	$\uparrow^{++}_{-}$	<1%	8 (0)	0	14 (2)	19 (3)	0	3 (0)
	work very hard	$\downarrow^{+++}_{++}$	n.s.	21 (13)	0	1 (0)	18 (4)	0	4 (0)
	memorisation	$\downarrow^{++}_{---}$	<1%	15 (5)	0	7 (1)	2 (1)	0	20 (6)
	ability to relate ideas	$\uparrow^{+++}_{--}$	<1%	5 (0)	0	17 (5)	22 (5)	0	0 (0)
Problem Solving	confidence	$\uparrow^{+++}_{-}$	<1%	10 (1)	0	12 (0)	22 (3)	0	0 (0)
	pleasure	$\uparrow^{+++}_{+}$	n.s.	15 (0)	0	7 (2)	21 (4)	0	1 (0)
	only to get through	$\downarrow^{+++}_{-}$	<1%	21 (9)	0	1 (0)	7 (2)	0	15 (3)
	anxiety	$\downarrow^{++}_{---}$	<1%	16 (5)	0	6 (0)	2 (0)	0	20 (5)
	fear unexpected	$\downarrow^{++}_{--}$	<1%	15 (7)	0	7 (0)	3 (0)	0	19 (5)
	correct answers	$\downarrow^{++}_{-}$	<1%	19 (3)	0	3 (0)	6 (2)	0	16 (2)
	try new approach	$\uparrow^{+++}_{+}$	<1%	12 (1)	0	10 (0)	22 (4)	0	0 (0)
	give up	$\downarrow^{++}_{--}$	<5%	16 (2)	0	6 (0)	2 (0)	0	20 (2)

Table 2 : Lecturers' perceptions of students preferred and expected attitudes

The arrow and the plus and minus signs in the second column indicate the direction of movement. The number of plus or minus signs indicates the average weighted strength of response, taking each Y response as 2, y as 1, n as –1 and N as –2. If the average response is 1 or more, the response is considered "strong" and denoted +++ or ---. Between 0.5 and 1 it is denoted "++" or "--", and less than 0.5 it is considered "weak" denoted "+" or "-". For instance, "facts and procedures" is desired to change down from an expected strong agreement (+++) by the typical student to a preferred weak agreement (+) by the lecturers. In line 4, "being abstract" diminishes from an expected strong agreement (+++) to a preferred disagreement (--). The significance of the change is computed using a χ^2 test (with Yates correction) on the number of yes responses (Y+y) and is given as significant (<5%), highly significant (<1%) or not significant (n.s.).

In only four of the cases is the change too small to be statistically significant: the lecturers expect the typical student to believe strongly that ***mathematics is about solving problems*** and prefer it marginally stronger, that ***mathematics is not about inventing new ideas***, but weakly prefer that it should be, that the student has a strong expectation to ***have to work hard to understand***, whilst lecturers have a lower expectation, and that there is ***a weak expectation of pleasure***, but lecturers prefer it to be strong.

One change in direction is statistically significant – that the typical student is expected to ***give up when a problem gets difficult***, but the lecturers prefer the opposite.

Two differences remain in the same direction but the changes are highly significant – an expected strong student belief that ***mathematics is a collection of facts and procedures to be remembered***, which the lecturers desire less, and ***a weak expectation that they are willing to try a different approach*** when their attempt fails, which is preferred stronger.

The remaining ten are both statistically highly significant and have opposite expectances and preferences. The lecturers *expect* the typical student to think mathematics is very ***abstract***, will ***not understand quickly***, will consider that mathematics does ***not make sense***, will ***learn through memory***, will ***not relate mathematical ideas***, will ***not have confidence***, will only solve problems ***to get through*** the course, will show ***anxiety***, will ***fear*** the unexpected, and regard ***correct answers*** as the most important thing. In every case the lecturers *prefer* the student to think the opposite.

Individual interviews with lecturers

Interviews revealed substantial differences in meaning of ideas expressed in the questionnaire from the ideas of "mathematical thinking" in the problem-solving course. For instance, Kilpatrick & Stanic (1989) suggest three different perceptions of problem solving—as means to a focused end, as skill and as art. It soon became apparent that the lecturers see it more as a means to achieve a specific end or a skill to be learned rather than the art of thinking mathematically. "Inventing new ideas" was perceived as original research rather than just ideas new to the individual, as in the following quotation:

To me mathematics is a tool for solving problems. One way of motivating the students is by showing them applications in the real world. In this way they get the knowledge and the skills for solving problems. ... I do not think the students are capable of creating new ideas on their own.

Lecturers are not certain of the problem-solving techniques used in the course:

... I am not sure of these [processes]. I have not thought about them and I don't know how to go about [teaching] them. I think I need to learn more about them before I can implement them. We developed certain abilities to look at problems but we are not sure how those abilities came to be with you.

Instead they show students how to do examples in the hope that they will develop their own techniques:

The experience of making conjectures, generalising and the like, I think the students can get themselves on their own, from doing their project work. We do not have the time to teach them everything.

We tell them how to do it – for example, what are the criteria that should be fulfilled in the formula before they can use it. Normally I explain only part of it then I think the students can complete it themselves. ... I think that is sufficient for the students.

Under the circumstances, I expect students to acquire the mathematical skills and to get high marks in the exam. ... I would want them to become good problem solvers but I am not sure they would be. I myself did not try to get them into becoming one consciously.

Some lecturers genuinely want to change the system but are not sure how to do so:

I would like students not only to see mathematics as a subject that they need to learn and pass in an exam but also as a discipline which enables them to think for themselves. My main aim is not in trying to finish the syllabus but rather in making the students learn the mathematics in a more meaningful way. ... I am not really sure how but I am trying to do it.

To me mathematics is a mental activity but I should say that at this level I presented it more as a formal system. Because we are confined by the syllabus and also depending on the students' background. ... I would like it to change. How do I do that? I don't know.

There are a lot of problems that we face. Firstly the students themselves do not have the motivation in their mathematics learning. Secondly they do not have the confidence in their ability to do mathematics. So we have to deal with these first before we can make them see mathematics as a thinking subject.

I very rarely allow students to think [mathematically]. The problems that we gave them do not require them to use their thinking capability. ... It is due to the shortness of time.

We give them little room to do their own thinking. But we cannot change it because the system does not allow us to do so. So we end up teaching them what they need to know.

The system has been proven a failure. It has not been successful in producing good mathematicians, or engineers that can use mathematics effectively. They only know how to use procedures or computer packages without really understanding why they use them. ...It's all down to the system. We are not training students to discover patterns, or how to prove a statement is true, for example. What we teach them is mainly how to use the procedures.

The change in student attitudes in problem solving and mathematics lectures

To discover how the attitudes of the students changed, the same attitudinal questionnaire was given before and after the Problem-Solving course, then six months later after a semester of standard mathematics lectures. The responses were as follows:

		Before P S			After P S			After Math		
		Yes (Y)	No (N)	–	Yes (Y)	No (N)	–	Yes (Y)	No (N)	–
Mathematics	facts and procedures	34 (18)	8 (2)	2	11 (3)	32 (8)	1	30 (9)	14 (1)	0
	solving problems	27 (10)	16 (4)	1	42 (21)	0 (0)	2	32 (22)	12 (0)	0
	inventing new ideas	21 (4)	21 (6)	2	37 (15)	5 (0)	2	24 (4)	18 (1)	2
	very abstract	25 (13)	17 (0)	2	15 (8)	26 (3)	2	22 (11)	21 (0)	1
	understand quickly	9 (0)	30 (5)	5	20 (3)	21 (2)	3	13 (2)	29 (1)	2
	make sense	22 (4)	22 (5)	0	35 (5)	7 (0)	2	29 (4)	14 (0)	1
	work very hard	37 (15)	5 (1)	2	28 (8)	13 (0)	3	32 (8)	12 (1)	0
	learn by memory	30 (1)	12 (2)	2	11 (0)	31 (7)	2	20 (2)	22 (1)	2
	able to relate ideas	24 (8)	14 (2)	2	35 (11)	8 (0)	1	31 (5)	10 (0)	3
Problem Solving	confidence	26 (7)	17 (2)	1	36 (12)	6 (0)	2	34 (7)	10 (0)	0
	pleasure	43 (25)	1 (1)	0	42 (21)	0 (0)	2	42 (21)	1 (0)	1
	get through	16 (4)	27 (8)	1	4 (0)	37 (17)	3	14 (1)	29 (5)	0
	anxiety	17 (1)	24 (4)	3	6 (0)	36 (9)	2	9 (0)	32 (4)	2
	fear unexpected	30 (10)	12 (3)	2	10 (3)	31 (9)	3	16 (3)	28 (2)	0
	correct answers	21 (4)	21 (3)	2	5 (1)	36 (11)	3	17 (0)	23 (7)	2
	try new approach	42 (17)	0 (0)	2	43 (20)	0 (0)	1	43 (16)	1 (0)	0
	give up	19 (3)	24 (9)	1	5 (0)	37 (20)	2	8 (0)	34 (12)	2

Table 3 : The changing attitudes of students before and after problem-solving and "aftcr math"

Calculating the significance in the change of the total "yes" responses and using a weighted average response as in table 3, we find the following changes:

		desired change		After P S		After math		Total change	
Mathematics	facts and procedures	$\downarrow^{+++}_{+}$	<1%	$\downarrow^{++}_{--}$	<1%	$\uparrow^{++}_{--}$	<1%	$\downarrow^{++}_{++}$	n.s.
	solving problems	$\uparrow^{+++}_{+++}$	n.s.	$\uparrow^{+++}_{+}$	<1%	$\downarrow^{+++}_{++}$	<1%	$\uparrow^{++}_{+}$	n.s.
	inventing new ideas	$\uparrow^{+}_{-}$	n.s.	$\uparrow^{+++}_{-}$	<1%	$\downarrow^{+++}_{+}$	<1%	$\uparrow^{+}_{-}$	n.s.
	very abstract	$\downarrow^{+++}_{--}$	<1%	$\downarrow^{+}_{-}$	n.s.*	$\uparrow^{+}_{-}$	n.s.	$\downarrow^{+}_{+}$	n.s.
	understand quickly	$\uparrow^{+}_{---}$	<1%	$\uparrow^{o}_{--}$	<1%	$\downarrow^{o}_{-}$	n.s.	$\uparrow^{-}_{--}$	n.s.
	make sense	$\uparrow^{++}_{-}$	<1%	$\uparrow^{++}_{-}$	<1%	$\downarrow^{++}_{+}$	n.s.	$\uparrow^{+}_{-}$	n.s.*
	work very hard	$\downarrow^{+++}_{++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.*	$\uparrow^{++}_{++}$	n.s.	$\downarrow^{+++}_{++}$	n.s.
	learn by memory	$\downarrow^{++}_{---}$	<1%	$\downarrow^{+}_{--}$	<1%	$\uparrow^{-}_{--}$	<5%	$\downarrow^{+}_{-}$	n.s.*
	able to relate ideas	$\uparrow^{+++}_{--}$	<1%	$\uparrow^{++}_{+}$	<5%	$\downarrow^{++}_{++}$	n.s.	$\uparrow^{++}_{+}$	n.s.
Problem Solving	confidence	$\uparrow^{+++}_{-}$	<1%	$\uparrow^{++}_{+}$	<5%	$\downarrow^{++}_{++}$	n.s.	$\uparrow^{++}_{+}$	<5%
	pleasure	$\uparrow^{+++}_{+}$	n.s.	$\boxed{\downarrow^{+++}_{+++}}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.
	get through	$\downarrow^{+++}_{-}$	<1%	$\downarrow^{-}_{---}$	<1%	$\uparrow^{--}_{---}$	<1%	$\downarrow^{-}_{-}$	n.s.
	anxiety	$\downarrow^{++}_{---}$	<1%	$\downarrow^{-}_{--}$	<5%	$\boxed{\downarrow^{--}_{--}}$	n.s.	$\downarrow^{-}_{--}$	n.s.
	fear unexpected	$\downarrow^{++}_{--}$	<1%	$\downarrow^{++}_{--}$	<1%	$\uparrow^{-}_{--}$	n.s.	$\downarrow^{++}_{-}$	<1%
	correct answers	$\downarrow^{++}_{-}$	<1%	$\downarrow^{+}_{--}$	<1%	$\uparrow^{-}_{--}$	<1%	$\downarrow^{+}_{-}$	n.s.
	try new approach	$\uparrow^{+++}_{+}$	<1%	$\uparrow^{+++}_{+++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.	$\downarrow^{+++}_{+++}$	n.s.
	give up	$\downarrow^{++}_{--}$	<5%	$\downarrow^{-}_{---}$	<1%	$\downarrow^{--}_{---}$	n.s.	$\downarrow^{-}_{--}$	<5%

Table 4 : Desired changes compared with changes after problem-solving and after mathematics lectures

Note that *the attitudinal changes during the problem-solving course are all in the same direction as the desired change, with the exception of one*: "pleasure" was rated highly each time with positive attitudes changing only from 43 down to 42 (out of 44).

On the contrary, *all but one of the changes during the mathematics lectures are in the opposite direction*. Even the exception—"anxiety"—has an increase in those feeling anxious from 6 to 9, but the weighted average is biased marginally in the oppositive direction by the drop in "definitely not anxious" from 9 to 5.

During the problem-solving course, only four changes are not statistically significant: ***pleasure*, *williness to work hard, willingness to try a new approach*** remain highly rated, whilst ***mathematics is abstract*** has a small improvement from positive to negative.

Three items change significantly: ***ability to relate ideas*** and ***confidence*** both increase, whilst ***anxiety diminishes***. All other items have highly significant changes in the desired direction. Some beliefs are reversed so that after problem-solving students now believe that ***mathematics is more than facts and procedures,*** it involves ***inventing new ideas***, it ***makes sense***, it is ***not learnt just through memory***, there is ***less fear of the unexpected***, it is ***not just getting correct answers***. Others are greatly increased: ***mathematics is more about solving problems***, it can be ***understood more quickly***, and students are ***less likely to give up*** when encountering a difficulty.

However, six months later, after returning to the mathematics course many opinions have reverted back in the old direction. Of these there is a significant ***reduction in belief that mathematics is not just memorisation***, and highly significant ***reversal in belief that mathematics is just facts and procedures***; it is ***less about solving problems, less about inventing new ideas, less about doing the work for reasons other than to get through the course*** and ***less about things other than correct answers***.

Comparing the situation from before the problem-solving course with the status after six months back at regular mathematics lectures, many of the indicators revert back towards their old position. But three problem-solving attributes remain: ***confidence*** and ***unwillingness to give up*** remain significantly improved and ***fear of the unexpected*** is highly significantly reversed. Smaller changes are evident in the belief that mathematics ***make sense*** and that ***it is not necessary just to learn by memory***. (These are improved by a factor that would be significant at the 10% level, marked "n.s.*" in table 5.)

In addition to these changes, there are other items that are given at least "++" or "--" in the final ratings: ***mathematics is facts and procedures***, ***is about solving problems***, students ***work hard***, are ***able to relate ideas***, take great ***pleasure*** in their work, have ***low anxiety***, are ***willing to try a new approach***. All these are attributes carry over from earlier mathematics learning. The emphasis is on procedural aspects, working hard to solve problems and relate ideas to obtain pleasure and low anxiety. However, the comments of the lecturers earlier suggest that this pleasure is more the security of operating in a system set up to teach the students procedures which can be successfully tested than in developing flexible new skills appropriate for the changing modern world.

Student comments

The following selected comments written by the students in the final questionnaire bring to light several factors that could explain their changes in attitudes. In the perception of mathematics for instance, about a third (32%) reported that the regular mathematics did not allow them to think in a problem-solving manner:

> Since following the course I know mathematics is about solving problems. But whatever mathematics I am doing now doesn't allow me to do all those things. They are just more things to be remembered.
> male, year 5

> I believed mathematics is useful in that it helps me to think. Having said that it is hard to say how I can do this with the maths I am doing. Most of the questions given can be solved by applying directly the procedures we had just learned. There is nothing to think about.
> female, year 3

They saw that their mathematical training is rather rigid. They felt that their lecturers laid too much emphasis on content, and on unchallenging work:

> At the moment I am finding difficulty with maths because I am just not enjoying it. Too much emphasis is put on getting the right answer and not on method and understanding.
> female, year 4

> The mathematical atmosphere here is very bad; there is little discussion and it provides no encouragement to do maths. The content is emphasised over everything else. We are crammed full of lots of bland mathematical abstract theory.
> male, year 3

Some emphasise the way in which the lecturers move fast to complete the content:

> I did not enjoy most of the maths courses—too dependent on the lecturers. I don't find the way most of them teach particularly inspiring. We find ourselves hurrying through to keep up. There is no time to think about the mathematics we are doing.
> male, year 3

Some appreciate their knowledge in problem solving, suggesting it helps them to learn their mathematics and solve problems more effectively:

> The problem solving techniques help me come to terms with the abstract nature of the maths I am doing. I try to connect the ideas together and talk about them with my friends. It is not that easy though. But I felt all the effort worth it when I am able to do so.
> male, year 3

> I find the problem solving knowledge very useful in helping me understand the whys and the hows of advanced mathematics. It is much more satisfying than rote-learning. Furthermore it is actually easier to remember something that you understand.
> female, year 4

There are some who have minor reservations on their problem solving experience. But they believe it is necessary to have a positive attitude:

> The main disadvantage is time. It would take several hours maybe days to understand each new concept. Under the current circumstances we are finding ourselves rapidly hurrying to keep up. Sometime we were too bogged down in the technical details and we end up purely taking down the notes without even concentrating. This really defeats the problem-solving techniques. ... But I think with further support from good teaching as well as tailoring the courses to suit the needs of the students the situation can be improved.
> male, year 5

Summary

Although lecturers prefer students to have a range of positive attitudes to mathematics, they expect the reality to be different. They prefer students to see mathematics as solving problems, making sense, with students working hard, able to relate ideas without needing to learn through memory, having confidence, deriving pleasure, with low anxiety and fear, ready to try a new approach and unwilling to give up easily on difficult problems. On the other hand, they expect them to see mathematics as abstract, failing to understand it quickly, not making sense, working hard to learn facts and procedures through memory, unable to relate ideas, with less confidence, obtaining less pleasure, working only to get through the course, with anxiety, fear, seeking only correct answers, and ready to give up when things get difficult.

By assigning a "desired direction of change" in the direction from what lecturers expect to what they prefer, it transpires that when doing a problem-solving course almost all the changes are in the desired direction and when returning to mathematics lectures, almost all the changes are in the reverse direction.

The findings show that the lecturers have little confidence in the students' ability to think mathematically and teach them accordingly. The students acquiesce to this approach, and set their sights on the lower target of learning procedurally to be successful in routine tasks. In this there is a widespread sense of pleasure although, after the problem-solving course, opinions expressed suggest concern that that the quantity and difficulty of the mathematics gives them little room for creative thinking.

Teaching problem-solving skills is not part of the lecturers' previous experience, consequently the lack of experience and the perceived difficulty of changing a formal system with so much content to be learned are severe deterrents to change. However, given the fact that problem-solving causes "positive changes in attitude" which are largely reversed in the standard course with its more difficult mathematical content, it is appropriate to pose the question:

> Given such a situation, do professors wish to continue to get what they expect, or do they want to change to attempt to get what they prefer?

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WHAT ARE THE KEY FACTORS FOR MATHEMATICS TEACHERS TO CHANGE?

Erkki Pehkonen, Dept Teacher Education, University of Helsinki, Finland

Summary: In the research realized during the spring of 1994 together with prof. Günter Törner (University of Duisburg), our purpose was to find answers to the question: What have been the key factors causing a discontinuity in teacher change? Our test subjects were experienced German teachers (N=13). We used two methods to gather the data: a brief questionnaire and interviews. Through the interviews, we tried to follow the teachers' memory pictures of their change, and decided to use the theme interview for the methodology. During the interviews, a total of 49 statements about the change were mentioned. We could compress these statements into fifteen change factors. The most referred change factors were, as follows: changes in society, experiences with pupils in school, experiences with the school administrations.

It is imperative that any research into teaching and learning, within a framework of constructivism (e.g. Davis & al. 1990; Ahtee & Pehkonen 1994), should take into account the teachers' and pupils' mathematical beliefs and conceptions if we are trying to completely understand their behavior. Already in the beginning of the 1980s, we had evidence that different philosophies (or belief systems) of mathematics teaching lead to different teaching practices (e.g. Lerman 1983).

The focus of this paper is to reveal the factors which teachers have experienced as crucial for their change, the so-called change factors.

The design of the research

During his career, each teacher changes with new experiences all the time. Typically, the change is continuous, since beliefs do not change radically; they evolve through extensive, extended experience. But every now and then there are some bigger steps, points of discontinuity. In this research work, we are interested in these discontinuities. In the research work realized during the spring of 1994 together with prof. Günter Törner (University of Duisburg, Germany), the focus of our research work was to find answers to the question: What have been the key factors causing discontinuities? We are especially asking about the teachers' own recollections of such key experiences.

Test subjects. Firstly we will provide a very short description of the German school system that the interviewed teacher are apart of. After four years of elementary school, pupils face the following four options: to attend the *Hauptschule* with graduation after the completion of the tenth grade; to opt for the *Realschule* (which originally prepared its pupils mainly for the service industry) and graduate after the tenth grade; to go to the *Gymnasium* (academic high school) and graduate after a total of twelve or thirteen years, depending on the State (*Bundesland*); or to spend the same number of years in a *Gesamtschule* (composite high school).

We interviewed 13 experienced German middle school teachers in the spring of 1994. The teachers were expected to have had at least 10 years of teaching experience. Furthermore, they were expected to be innovative in their teaching, at least according to the school administrators who provided us with their addresses. All the thirteen teachers interviewed were from the Ruhr Area (in northwest Germany): five of them were from *Gymnasiums*, two from *Realschules*, one from a *Hauptschule*, and five from *Gesamtschules*. From the *Realschule* and the *Hauptschule*, it was not easy to find teachers who were prepared to be interviewed. Only three teachers were female, all the others were male teachers.

Indicators. To gather the data, we used two methods: a brief questionnaire and interviews. The questionnaire contained thirteen statements about teaching principles in school mathematics. These thirteen aspects emerged as the result of a factor analysis in another study on teachers' conceptions, based on questionnaire results (Lepmann & Pehkonen 1995).
In the interview, we decided to use a theme interview methodology (e.g. Lincoln & Guba 1985) in which many questions are associated with the research problem. For this research, we generated four main questions, as follows: (1) Tell your "history" as a mathematics teacher. (2) How did you teach in the very beginning? (3) How do you teach today? (4) Can you name some factors which might have had an influence on changing you?
The teachers who answered these questions were assisted with some additional questions according to the situation, until we thought that we had extracted the answers to the questions. For example, when discussing the change factors, we might have followed up the questions by using the list of possible sources for the perturbance given by Shaw & al. (1991): pupils, colleagues, parents, administrators, teacher-educators, books, articles, and self-reflection. At the end of each interview, there was an additional question about the questionnaire, as follows: (5) Would you express your opinion

on the questionnaire? Perhaps, you would like to comment further on some of your responses.

Practical realization. The questionnaire was mailed beforehand to the teachers, and they were asked to fill it in. The plan was to help teachers to reflect on their own teaching, and thus partly to structure the interview. Thus, the information received through the questionnaire contributed some parallel aspects concerning the teachers' conceptions. The completed questionnaires were collected after the interviews.

The interviews formed the main source of information. The length of the interview was 40–60 minutes as a rule. Both researchers were present at each interview, in order to have two different viewpoints on the situation. And each interview was discussed thoroughly on the same day. In addition, all the interviews were recorded on video.

Methodology. Using a questionnaire methodology, researchers usually remain on the surface level of beliefs. With interviews and observations, an attempt may be made to go deeper, as well as to find out what the unconscious beliefs are which lie behind the explicated conceptions. Since the structured interview often remains almost on the same level as a good questionnaire, the interviews here were realized applying the methods of naturalistic inquiry in a form of the theme interview (Lincoln & Guba 1985).

We had four main questions which we showed to the teachers beforehand and which formed the core of the discussion. During the interview, we asked more questions if we felt that we had not yet extracted "all the answers" to our main question. The narrative mode of interviews encouraged the teachers to reflect on their past experiences and on the feelings associated with them.

Some results

The results of using the questionnaire supported the main inquiry method: the theme interview. Therefore, we will concentrate here on the results of the interviews. A larger description of the research results (Pehkonen & Törner 1994) will be published in some periodical soon.

Evaluation of the data obtained. The information obtained (interviews and questionnaire) was worked out in the form of a teacher's mathematics-related *snapshot*. The information from the interview of each teacher was written on one page, including the following components: time and place of interview, position, teaching experience, mathematical world view (today), own view of personal change, change factors, and comments on the questionnaire. Each teacher received his "snapshot" by mail for reviewing, and

had two weeks time to react, if he thought that our interpretation was not valid. The teachers were satisfied with our interpretation about their mathematical conceptions, except one teacher. He wanted to make a small addition in one change factor of his view.

During the interviews according to our interpretations, a total of 49 statements emerged about the change. We classified these statements into fifteen different *change factors*. Furthermore, these fifteen factors could be classified into four groups: Experiences as a teacher with individuals (1) pupils in school, (2) own children at home, (3) children of relatives, (4) pupils' parents. Experiences as a teacher with institutions and authorities (5) other school forms, (6) the school administrations, (7) as a class teacher, (8) cooperative teaching with colleagues, (9) changes in society. Experiences as a learner with individuals (10) excellent teacher-tutor, (11) a working group of voluntary teachers. Experiences as a learner with institutions (12) inservice courses, (13) further studies at the university, (14) mathematics education conferences, (15) literature.

The most referred one was factor (9): changes in society; there were altogether ten statements. The other much referred one was factor (1): experiences with pupils in school, with eight statements. And the third factor which was also much referred to was factor (6): experiences with the school administrations, with six statements. The rest of the factors were mentioned by 1–3 teachers. A summary is given in Table 1.

factor	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
Σ	8	3	1	1	3	6	1	2	10	2	3	3	2	1	3

Table 1. The distribution of statements on each factor.

Each factor caused the teacher to begin to reflect critically on his own teaching. Here, we will comment briefly on each of the three most referred factors.

Factor (9). Nearly every teacher reported a drastic change within society, influencing situations within families. The changing society also includes changes in the organization of school (e.g. large classes, restricted financial resources) which have an impact on mathematics teaching. Some keywords were: the crisis of the family in our western society, materialism, TV consumption, a lack of interest in school achievement, deficiency in the potential to concencrate, computerization leading to the lack of ability to communicate, the "electronization" of children's rooms, the lack of pro-

fessional perspectives, deficiency in criticisms, manipulation by the media, and politization of schools.

Factor (1). This factor is one of the most important impact parameters. In the words of Mr. BR: *"It took me nearly an hour to explain the multiplication of fractions: numerator times numerator, denominator times denominator by using sophisticated arguments about operators. Today I am ashamed. After that lesson, a boy who was having much trouble with math adressed me: "Why didn't you tell us the rule straight away at the beginning of the lesson?"* However, teachers do not only report changes concerning pupils as mathematics learners, but also point out the individual and personal dimensions of their pupils as part of a complex social environment.

Factor (6). Many teachers claim that the situation of administrative regulations has become worse in the last five years. The average number of pupils per class has risen although problems with the pupils have emerged. Mr. HO had the following experience: *"It is no longer possible to organize a fruitful inner differentiation when there are more than 30 students in a class..."*

From all 49 given statements about change, the test persons referred to themselves as teachers in 72 % of the answers and as learner only in 28 % of the answers. Some of the interviewed teachers first stated that they have not changed, but were teaching using the same styles as in the beginning of their careers, e.g. Ms. DY and Mr. WI. However, during the course of the interview, they found some factors which might have influenced them.

<u>Key experiences for changing conceptions</u>. Here we will elaborate the obtained fifteen change factors more. In many factors, the "always present" influence of the society is clear. Besides the factor (9), this may be seen also from the factors (1), (2), (4), (5) and (7).

In another classification of the change factors, we will focus on one side on <u>positive factors</u> which will help teachers in change process, and on the other side on <u>negative factors</u> which could form an obstacle for change. Some factors could effect in both directions: positive and negative, depending on the circumstances. The teachers have given principally only positive factors besides item (6). Only the factor (4) was discussed in the negative sense. Further, one teacher mentioned that changes in society have lead to some frustrations rechanging his teaching to a more traditional one. For example, one could be compelled to work cooperatively, although he would like to work independently in his own classroom (8). We believe that here research has to answer some more questions.

Some factors have caused in the teachers a change in the view point, e.g. the factor (2). One of the biggest changes happened, when the teacher was compelled to identify himself with the teacher of his own children or with that of his relative children, since then the personal affectedness was the highest.

It seems to that some changes in a teacher's behavior may be stimulated when the teacher is forced to look from outside on another mathematics teacher (or himself) resp. mathematics as a subject. Then he often is willing to reflect critically his own appearance as a teacher as well as the relevance of mathematics he is teaching.

Reflections of that type may lead to the assumption that mathematics, although the discipline is a fundamental school subject, is limited with respect to the individual development of a pupil (the factor (7)). Although this was reported by the teachers, we don't know the effect of the mathematical competency. Whether one (and which one) or both of the extreme positions is stimulating this perception: a high qualification at university level or a restricted education in mathematics.

In addition to these observations, the question how the teacher understands his role as a teacher may lead to different solutions in instruction: If the teacher is considering himself in the first place as a mathematics instructor, he watches the teaching situations from the view point of mathematics. This is in many cases very different from a pupil's view point which comes in question when the teacher sees himself mainly as an educator, and mathematics is only the content of the study.

Discussion

With our research method – a combination of a questionnaire and an interview, we could see the limitations of the questionnaire method, too. Almost all teachers were giving strong critiques on some statements in the questionnaire, and asking what we wanted to reach with them; and the point was that they were not always on the same statements.

Of course, we are aware that our own conceptions on mathematics teaching are setting the natural limits for our investigations. Nevertheless, the research described here has enlargened our knowledge horizon. We were not aware of all of the mentioned aspects before our interviews. As a measure of validity for our interpretations, one could use the satisfaction of the teachers to our interpretations. Only one test person from thirteen wanted to make one small completion in the view of him we had written.

Our observations during the interviews give support to the results of Bottino & al. (1991) that teachers' choices seem to be many times more affected by pressure from their colleagues in successive school years than by educational considerations. Another research results which are coherent with ours is the work realized by Shealy, Arvold, Zheng and Cooney. In their paper describing the evolution of student teachers' beliefs, they commented that "the greatest effects were interaction with faculty, graduate students, and peers, open-ended problem solving, and his field experiences" (Shealy & al. 1993, 227).

As a result, we have also an open question: Would the high mathematical competency block or promote a teacher's activity? According to our results, the problem might lie in the question, since mathematical competency seems to play an ambivalent role on a pupil-oriented attitude. Our interviews provide us with examples in both directions. A couple of other questions occured from our research: International comparisons in this point could be of interest. Are these change factors culture bound or something universal? Furthermore, it might be interesting to know how big a percentage of all teachers are as far in the change process as most of our test persons. And what will a random sample of teachers respond to the statements on change factors?

Implications for in-service education. In teacher in-service education, we are just trying to reach a change, i.e. to help teachers to grow. Therefore, we aim also with this research to answers for the question "Which kind of teacher in-service education would be optimal, in order to have a change in participants?" In this sense, our results are serving also the teacher in-service system.

Our study had made some school authorities (e.g. in Düsseldorf) sensitive on the meaning of teachers' belief systems for their teaching which have had some positive feedback in the form of teacher in-service courses. At the same time, it has revealed that teaching in many cases could be very surficial, concentrating on rote learning of some procedures and techniques.

For teachers change, there seem to exist at least two effective strategies to follow: change of roles, and change of viewpoint. The change of roles means that the teacher is forced to identify himself with a student, e.g. in a simulation group. Such a situation lets the teacher an opportunity to observe himself from outside. This could as a consequence cause a perturbance in his thinking which is a prerequisite for change (Shaw & al. 1991).

The change of view point could be reached e.g. through letting the teacher to deep interview a pupil, in order to try to understand his mathe-

matical (and broader in general) thinking. Such a task may allow the teacher to accept a pupil as an individual through seeing in him also other aspects than only his mathematical abilities or a lack of them. Thus, he learns to see also mathematics from the pupil's point of view, and understands that mathematics is not, perhaps for all people, the most important subject on earth – nevertheless it is a very challenging one. Therefore, his view of school mathematics is changing. As a consequence, he is able to realize "nonstandard" reflections on mathematics and doing mathematics.

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TEACHERS' AWARENESS OF THE PROCESS OF CHANGE

Vânia Maria Pereira dos Santos   Lilian Nasser

Instituto de Matemática - Universidade Federal do Rio de Janeiro

This paper reports the context of a staff teacher development project and examines the awareness of the process of change experienced by three teachers who have participated in both a continuous teacher training and a teacher enhancement program. Having in mind the aim of investigating if the staff development furnished by this project is leading to professional development, we examined, analyzed and categorized the data collected in 1994. The analysis addresses the three teachers' perceived changes in terms of professional growth, teacher autonomy and metacognitive awareness about mathematics teaching and learning based on the work of Cooney (1994), Kagan (1992) and Santos (1993). Although this growth demands time for teachers to mature their views, willingness and courage to engage into innovations, this work suggests that these three teachers have progressed a lot toward professional development.

In many countries, low mathematics performance of students is usually associated with poor teaching. Nations around the world are demanding better teachers at all levels, but yet too little effort has been taken to understand what prospective and practicing teachers think about doing, learning, teaching and assessing mathematics and how these conceptions can affect both preservice and inservice teacher preparation as well as teaching practice (Dossey, Dossey & Parmantic, 1990). Formal education involves complex interaction between teachers and students who are influencing and being influenced by each other in terms of both cognitive and noncognitive factors. Teachers' beliefs about mathematics and mathematics pedagogy are inherent in the human endeavor of teaching. However, one may question when and how these beliefs appear and if they can be challenged and/or changed.

Thompson (1992) summarizes her review of the literature pointing out that belief systems are dynamic, permeable mental pictures susceptible to change in light of experience, and that the relationship between teaching practice and beliefs is not a simple linear causal-effect, but is a dialectical one. Hoyles (1992) proposes the notion of situated beliefs based on the fact that "teachers reconstruct their beliefs whilst interacting with any innovation . . . [and] all beliefs are, to a certain extent, constructed in settings" (p. 282). Ponte (1994) draws our attention to the fact that the classroom environment and social, educational, and personal constraints also contribute to shape teachers' beliefs, which favors a dialectical perspective of the association between beliefs and practices.

In this paper we analyze the process of change experienced by three mathematics teachers who have been engaged in a project of teacher development. We also examine their awareness about changes in both their pedagogical practices and views about mathematics teaching and learning.

Context of the Teacher Education Project

The Fundão Project (FP project) is an ongoing staff development at the Mathematics Institute at Universidade Federal do Rio de Janeiro initiated in 1980. The FP project acts in three directions: teacher training, research and mathematics teaching at university level. The continuous teacher training component involves university professors, secondary teachers and undergraduate students. The main concerns of this component are to improve the preservice mathematics teacher education,

to provide continuous teacher enhancement, to develop a collaborative effort between university and classroom teachers in order to improve the process of mathematics teaching and learning, and to call attention to the importance of the teacher as an educator. The project work and the environment in which it takes place provide a supportive context leading both to the personal recognition of teachers and to the realization of the importance of their role as mathematics teachers.

Throughout the mutual collaboration between the three groups, the teacher training component benefits from the experience of school teachers and, at the same time, contributes positively for the growth and change in teachers' attitude. The participation of undergraduate students in the project (even in their first year of university course) contributes to their teacher education program through the contact with classroom situations, the discussion about problems of mathematics teaching and the engagement in research teaching experiments. Furthermore, they give valuable ideas when planning and designing school activities because their way of thinking and speech style is closer to the pupils' population with whom the teachers work. In sum, since the beginning of the project the idea was to have undergraduate students, teachers and university professors sharing their expertise and knowledge (Nunes, 1993). Actually, it is a collaborative project of action research where the whole group is involved in identifying important learning and teaching problems and searching together for workable solutions. In other words, it is on-site inquiry aimed at problem resolution which tends to motivate school teachers to become practitioner researchers (Clouthier & Shandola, 1993; Raymond, 1994).

In addition to this component of the FP project, there is a systematic graduate program for mathematics teachers (a two-year enhancement program previous to the master). Through this program the teachers get the opportunity to rethink and reflect about their views, beliefs, conceptions and attitudes toward mathematics teaching and learning. The program is arranged around three issues: mathematical content, methodology for mathematics teaching, and mathematics education. The teacher-students have to enroll at least in respectively four, one and two courses from each issue before carrying on a small research monograph. During these two years, the teacher-students use their own school classrooms as an experimental laboratory to investigate the possibilities of innovating in their mathematics teaching. It is also the first step for becoming a researcher since they are required to systematize their pedagogical planning, write about their classroom investigations, report their observations and trials, and discuss the whole process of these innovations with their peers.

The components of continuous teacher training and graduate teacher enhancement program provide to practicing teachers a full range of opportunities to develop professional autonomy, acquire independent thinking when facing classroom problems, and become adaptive agents that are able to try out innovations and to reflect-in-action (as Schön, 1983 advocates) about the positive and negative aspects of this move. According to Cooney (1994), in teacher education programs, teachers need to have experiences which help them to develop both the conceptual and the pedagogical bases. He suggests that "teacher education programs ought to have features that —

- enable teachers to develop a knowledge of mathematics that permits the teaching of mathematics from a constructivist perspective;

- offer occasions for teachers to reflect on their own experiences as learners of mathematics;
- provide contexts in which teachers develop expertise in identifying and analyzing the constraints they face in teaching and how they can deal with those constraints;
- furnish contexts in which teachers gain experience in assessing a student's understanding of mathematics;
- afford opportunities for teachers to translate their knowledge of mathematics into viable teaching strategies." (p. 16)

All these features are present in the FP project as a natural consequence of the dynamics adopted, i. e., discussing and theorizing about mathematical learning and teaching problems and experimenting ideas in their school classrooms. Furthermore, the teachers engaged in the project are stimulated to share and disseminate the innovative mathematical ideas developed in the project in their school context to their peers and in symposiums, workshops and conferences on mathematics education. All these situations lead teachers to acquire both autonomy (Castle & Aichele, 1994) and mathematical pedagogical power (Cooney, 1994), and develop their metacognitive awareness about teaching in the sense defined by Santos (1993).

It is important to notice that "teachers' conceptions are not easily altered, and that one should not expect noteworthy changes to come about over the period of a single training course." (Thompson, 1992, p. 139) In addition, we should observe how teachers' beliefs and conceptions match and/or are in conflict with their teaching practices and behaviors. Teacher training programs can influence and help teachers to solve these dilemmas if the programs provide them with reasonable time and a suitable setting allowing them to think, reflect, try out innovations, and decide about changing or not their conceptions and practices. Based on the evidence from previous research studies on teacher education (Cooney, 1994; Hoyles, 1992; Ponte, 1994; Thompson, 1992), the FP project works so that each participant (either undergraduate student or practicing teacher) stay regularly linked with the project for at least two years. As a matter of fact, there are teachers engaged in the project for about ten years.

A Look at Some Instances of Teachers' Changes

In 1994, eighteen undergraduate mathematics majors, nineteen practicing teachers and five university professors took part in the teacher training component. The participation of practicing teachers and undergraduate students in this component of the FP project and the development of their metacognitive awareness about the complexity of the teaching-learning process led some of them to enroll in graduate teacher programs (enhancement and/or master level). This year there were twenty five teacher-students engaged in the graduate teacher enhancement program of the FP project . Data for this investigation were collected in March and December of 1994, consisting of:

- undergraduate students and teachers' written reports about the activities developed in the continuous teacher training project both at the university setting and at the school classrooms;
- undergraduate students, teachers and teacher-students' written comments about mathematics education articles and research studies discussed;

- undergraduate students, teachers and teacher-students' written reports with a critical analysis of the influence of either the teacher training or the teacher enhancement components in their views, beliefs and conceptions about mathematics and the mathematics teaching profession;
- teachers and teacher-students' written reports pointing out their perceived views about the five components suggested by Kagan (1992) as essential for defining teachers' professional growth. These components include: metacognitive knowledge about pupils and classrooms; teachers' expectations, beliefs and knowledge about pupils; their self-image as teachers, shifting the focus of attention from self to the design of instruction and then to pupil learning; development of standard classroom procedures; and growth in problem solving skills, becoming able to relate and connect topics of the mathematics curriculum.

Partial analysis of these documents, describing categories of the professional growth of students and teachers involved in the investigation, was reported by Nasser and Santos (1994).

Having in mind the aim of investigating if the staff development furnished by the FP project is leading to professional development, teacher autonomy and metacognitive awareness, we examined, analyzed and categorized the data collected during this year. Our analysis was based on suggestions given by Kagan (1992), Santos (1993) and Cooney (1994). In this paper we focus our attention on the changes pointed out by some teachers during the whole year of 1994. Due to space limitations, we only address the cases of three teachers who have participated in the teacher training and are now enrolled in the enhancement program.

The Three Teachers

Margarida is a beginning teacher facing the dilemmas of the initial career who worries about her awareness to overcome these problems; Susana is an experienced teacher who has been searching for ways of improving her teaching in the last decade; and Anne Marie is a serious and responsible teacher who tries to innovate but needs continuous support from her peers at school or at the university to keep on with her pedagogical trials. Margarida and Anne Marie have taken part in the teacher training as undergraduate students for about two years some time ago, and they returned in 1993 to the University looking for the support of the teaching enhancement program. On the other hand, Susana has participated in the teacher training project as a teacher for two years, and also engaged in the enhancement program in 1993.

Margarida

In March of 1994, she reported the internal conflicts of her professed beliefs about teaching style and teaching approaches to mathematics with her actual teaching behavior in class. She was facing, as a beginning teacher, the dilemma of being attached to the mathematical content while willing to innovate without knowing exactly how, but afraid of losing her teaching jobs in private schools in Rio:

> *One of the most important factors experienced in the continuous teacher training component was the emphasis given to spreading the focus of teaching, in the sense that the teacher is not the only owner of the knowledge. . . but he/she is only a vertex of an*

imaginary triangle formed by the teacher, the learner and the knowledge. . . . When beginning to teach in 1993, I observed that although having been influenced by the variety of innovative activities [from the teacher training], I felt a big difference when facing the reality of working in private schools. The question of the teacher being attached to the programmatic content is too hard and heavy for a beginning teacher. . . . My theoretical attitude was one, but my actual teaching practice was another one, quite different which was making me feel sometimes anxious. . . . My first lessons were the worst ones, because I repeated exactly the same methods used by my previous teachers, which I have always rejected. This gave me a terrible sensation because nothing that I was doing was pleasing me. . . . That situation was letting me anxious, and I felt like a hypocrite, preaching some ideas and acting in a different way that was divergent from my own thinking about teaching. During this time, I have even thought in quitting the teaching career. . . .

In December 1994, when reflecting back about the influences of participation in both components of the FP project, we can perceive the change in her speech about teaching and problems faced in schools. At this moment, she was aware that her teaching attitude and behavior has been affected and influenced the most by the participation in the teacher enhancement program:

. . . . It was at that time, that the discipline of "problem solving" [one of the courses from the program] "fell over my head," and soon I perceived that this discipline was going to be of "great impact;" and it's good that this happened, because that's exactly what I was needing: a shock treatment to awake me from the stillness I was locked in. The approach of this course, in particular, shook me up and changed my way of seeing education. With the assignments and the links with the classroom practice (at last we applied something in our classes!), this problem solving course led me to discover another side of the lessons in which the teacher interacts much more with the pupils, guiding them in the construction of their knowledge. . . . All this has shown me that the application of innovative proposals in classrooms may be successful, being enough for this a certain amount of willingness and courage from the teacher.

When looking back to the entire teacher enhancement program, even though realizing the importance of the mathematics content disciplines, Margarida acknowledges the crucial role played by the mathematics education and methodology for mathematics teaching disciplines in developing her self-image as a creative professional teacher being able to integrate different mathematics topics and to generate interesting activities. Margarida says "*influenced by the three disciplines of mathematics education I tried to do a more dignifying work as a teacher which is coherent with my ideals and my preaching about mathematics teaching. . . This is my second year teaching and I could leave behind a great deal of insecurity.*" Another evidence of her professional growth is the fact that she had the courage to submit a communication about her teaching experiment with the study of trigonometry to the National Meeting on Mathematics Education that will be held in July 1995 in the Northeast part of Brazil.

Anne Marie

She is a teacher with some experience who has already overcome the difficulties of the first two years of teaching. Differently from Margarida who is very metacognitively-aware of her teaching behavior and constraints for implementing changes, Anne Marie is more dependent on the approval from her peers and creates obstacles for the success of innovations. This happens because she works in two private schools having different educational philosophies. One is very

traditional while the other is more open to innovative methodologies. This need of peers' support and interest is noticed in her speech:

> *I try to pay attention to everything that can be directly connected to the learning-teaching process, but I need to have a larger number of people in each school involved in the production of activities and/or research. . . . Nevertheless, I feel that as my level of information increases, my level of concerns increases too, I would like to change experiences and create things more often but several times I face the reality of working with colleagues with low interest in both teaching and change.*

Although Anne Marie has participated for two years in the teacher training as an undergraduate, and she has been enrolled since 1993 in the teacher enhancement program, she seems to have gathered information without processing it. Only now, when conducting the final investigation of the enhancement program, her awareness of her teaching limitations is coming to a conscious level.

> *I see that to become an excellent teacher I will need to have a researcher's attitude, that is to reflect about each lesson given to my students, thinking about their reactions and performances. But the attention should not be only focused on the teacher; it is essential to have complete knowledge of the mathematics content in order to better explore it, and a deep knowledge about the learner in order to understand the learning process. . . . Today I believe to have a better consciousness of my role as an educator. I perceive more clearly the importance of diversifying the lesson styles, of innovating in the way of questioning, of stimulating discoveries, of promoting learning transfers, and of taking advantage from the pupils' mistakes. But this more dynamic attitude is the result of various factors: my professional experience, and moments experienced in the enhancement courses and/or workshops.*

Susana

Susana is an experienced teacher who works with two school realities: middle school pupils from public schools and preservice teachers at high school level from a Center of Elementary Teacher Training. Since the beginning of her career, she has searched for original ideas and materials, and inservice teacher training in order to improve her teaching expertise. Susana was aware of her teaching difficulties but had to keep looking for different ways to solve school problems, as noticed below:

> *Initially in my teaching career, I knew there were mistakes in the educational system, i. e., the pupils didn't learn and I noticed that sometimes I could not communicate with them. I couldn't identify the factors that were causing this situation and I didn't know even which strategies to use in order to modify this reality. . . . While I was getting to know new methodologies and becoming aware of the phases and obstacles to overcome, there wasn't a fixed classroom routine any more.*

Susana had already acquired a lot of teaching experience when she came to work as a practicing teacher in the continuous teacher training program. After two years of participation, she left this inservice teacher training in order to write with some colleagues a mathematics textbook for preservice elementary teachers at high school level. At this point in her career, it seemed as if she had nothing more to learn, but when she engaged in the enhancement program, Susana realized that she still had a long way to go:

The enhancement program allowed me both to deepen and evaluate my knowledge as well as improving at the same time my students' and my metacognitive skills in a constructivist approach. . . . The main change that occurred with me was the incorporation in my teaching routine of: cooperative work, valuation of the language when interpreting mathematical ideas, use of concept maps accompanied of written explanations, and group assessment. . . . I found out that I still have a lot to discover and I feel that all the disciplines of mathematics, mathematics education and methodology for teaching in the program are adding up to my knowledge. I try out theories, adopt a constructivist attitude most of the time, but realize that there isn't a theory that is sufficient to solve alone the problems of the teaching-learning process and I take advantage of the theory I find more suitable in each moment.

Looking at Susana's discourse, we can observe that by incorporating a new repertoire of routines in her teaching practice and by reflecting about the use of several learning theories, Susana is moving toward a scientific approach to teaching. Which, according to Cooney (1994), "honors observing students; hypothesizing about, and examining the effects of, various teaching strategies; and reformulating hypothesis about students' learning." (p. 19)

Discussion

As we said before, changes in teachers' attitudes, beliefs, and practices need time and supportive context to take place. Maybe, this explains why the main acknowledged changes in these three teachers occurred only when they returned to the university to enroll in a teacher enhancement program. They needed time to mature some of the innovative teaching ideas shared in the teacher training component before they were able to develop their metacognitive awareness about mathematics and its pedagogy. Only then, they could acquire autonomy to try out innovations and to incorporate new classroom routines as problem solving, group work, and a teaching practice focusing more on questioning, leading to a constructivist approach. It was crucial for them to have experienced group work as teacher-students before implementing it in their classes. The three teachers explicitly state that the most influential courses from the enhancement program were the three disciplines dealing with mathematics education and methodology. They explain that this occurred because they had the opportunity to develop small pieces of teaching experiments in their schools, thus becoming researcher practitioners.

Throughout the enhancement program, Margarida was able to improve her metacognitive awareness of teaching concerning her strengths and limitations, and was able to find out ways to cope with her conflicts and professional dilemmas. To some extent, Margarida and Anne Marie followed a similar path of change concerning the shift of their attention from the self to the content and, then, to the learning process as suggested by Kagan (1992). The reading and discussions about mathematics education articles were fundamental to provide them with a theoretical basis for professional growth. All of them also acknowledge the importance of experimenting different ways of collecting information about pupils' learning and difficulties for becoming investigators in their own classrooms (e.g., interviews and/or group problem-solving). The act of writing in a more systematic way the results of these investigations as well as reporting the successes, difficulties and failures contributed to develop the awareness of their strengths and limitations and to bring to a conscious level the complexity of trying innovations. This helped them to become

autonomous teachers as described by Castle and Aichele (1994): "Autonomous teachers are self-directed learners who question, study, and search for answers from a need to know. . . . Autonomous teachers construct personally meaningful professional knowledge resistant to education fads or external mandates. . . . and are more confident in what they know." (p.7). In certain sense, Margarida and Susana have developed this kind of autonomy, while Anne Marie is still in the way. As a matter of fact, from the three teachers, Susana has acquired an attitude of a life-long learner who believes that pupils and teachers are always engaged in a learning process, as she concludes her report : "from new methods of assessment and reporting my investigations, . . . I discovered that we [teachers and students] learn how to learn".

The process of change reported in this work is not a closing point, but must be continued. We must keep on investigating if the perceived changes in the beliefs, attitudes and behaviors of these teachers will remain and/or improve, and if they really affect mathematics learning in the classroom environment. In order to assure that the enhancement program actually promoted the changes suggested by this work, we intend to carry on a closer investigation in which we rely not only on the teachers' reports and narratives but also on observations of their lessons, analysis of classroom tasks, and interviews with the three teachers and some of their students. Only then, we will be able to assert that they have become autonomous teachers.

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A COLLEGE INSTRUCTOR'S ATTEMPT TO IMPLEMENT MATHEMATICAL PROBLEM SOLVING INSTRUCTION

Manuel Santos T
CINVESTAV-IPN, México

Problem solving has become an important part of mathematical instruction in the last 25 years. This study focuses on analyzing the role of a college instructor who tries to implement problem solving instruction in his regular calculus class. Results show that even when the instructor openly supported the principles of this approach, there was no consistency in the implementation of those principles in classroom activities. This suggests that it takes time for instructors to conceptualize and accept significant changes in their actual practice.

Introduction

Problem solving has been an important part of the mathematical education agenda in the last 25 years. Lester (1994) pointed out that there has been a lot of progress in understanding how students solve mathematical problems, but he also suggested that there is a need to continue doing research that provides more directions in the implementation of problem solving activities in the classroom. One of the issues that Lester identified as being crucial in the implementation of problem solving approach is related to the role of the teacher during the development of course. "In my view, attention to the teacher's role should be the single most important item on any problem-solving research agenda" (p. 672).

The present study investigates the effects of attempting to provide mathematics instruction based on problem solving in a calculus class at the college level. The course was a regular course taught during one term. It is suggested that this type of study will help instructors to be aware of the potential and limitations when trying to implement some strategies related to problem solving instruction. The study focuses on the analysis of the instructor's views and behaviors observed during the course.

Background to the Study

Mathematical problem solving involves a view of mathematics in which it is important tc find the meaning and connections of mathematical ideas. It also emphasizes that doing mathematics is a social activity in which people interact during the process of understanding or solving mathematical problems. As a consequence, some essential learning activities associated with this approach involved the use of some nonroutine problems, the use of small group interactions, and the consideration of problem-task assignments. Here, the role of the instructor becomes crucial not only in the selection of what activities to

implement but also to what extent those activities appear in the actual practice. Therefore, it is important to document the instructor' behavior while intending to provide a problem solving environment in the development of the class.

Thompson (1988) suggested that instructors' beliefs about mathematics and problem solving influence the way that they conceptualize and implement learning activities in the classroom. The need for documenting the directions of instruction was pointed out by Thompson when she stated:

> reports of instructional studies in problem solving have generally lacked good descriptions of what actually happened in the classroom (except for those in which programmed instructional booklets were used) and have failed to assess the direct effectiveness of instruction (p. 232).

Schoenfeld (1992) stated that learning involves the active process of constructing interpretations of what one sees, and what the student perceives may not be what the teacher intended. Therefore, it becomes important to discuss what the instructor did during the course and to interpret the results in the context of the problem solving approach.

Methods and Procedures and Frame of Analysis

Miles and Huberman (1984) discussed advantages in the use of qualitative data. They expressed the opinion that:

> with qualitative data, one can preserve chronological flow, assess local causality, and derive fruitful explanations. Serendipitous findings and new theoretical integrations can appear. Finally, qualitative findings have a certain undeniability that is often far more convincing to a reader than pages of numbers (p. 22).

The review of related studies suggested that a design which involves gathering information via extensive conversation previous to and during the development of the course, and an interview with the instructor at the end of the course could provide grounds to analyze the direction of the course. In addition, the researcher observed the development of the course and maintained direct interaction with the instructor and recorded class observations. The main interest was to analyze to what extent the activities implemented during the course were consistent with the problem solving ideas discussed with the instructor previously and during the course.

The framework utilized to analyze the information is a modified version of Thompson' ideas on how to categorize teachers' beliefs about mathematics and teaching. The main categories used to analyze the instructors' behaviors included aspects related to the nature of mathematics, the class interaction, and the evaluation of the students work. The frame resembles aspects related to

what mathematics is, what the role of the instructor (students) is, and what it means to learn mathematics, that Thompson identified as important components to analyze teachers' behaviors. Thus, the information gathered via interview, class observations, and fields notes were organized and analyzed by following those categories. t is important to mention that the final analysis and results has been shared with other mathematical instructors and they have identified some of the finding on their own practice. This part enhances the validity of the results in this type of qualitative studies (Marton, 1988).

Limiting the Context: Analysis and Discussion of Results

To analyze the direction of the course and its consistency with the original plan, an interview with the instructor was conducted a the end of the course. Some of the questions address specific issues discussed with the instructor during the planning period of the course; others were inspired by the class observations carried out by the researcher during the development of the course. The analysis will address issues related to the nature of mathematics, the conception of learning, and the students' problem solving evaluation.

The Instructor's Conceptualization of Mathematics.

Schoenfeld (1987) pointed out that the way teachers conceptualize mathematics permeates the classroom activities that are implemented during instruction. He described his own experience as a student in which the instructor could not remember the binomial formula and showed the students how to figure it out. Learning activities that relate the sense of studying mathematical relationships are different from those in which the instructor gives only rules for solving problems. Schoenfeld (1987) stated:

> the important thing in mathematics is seeing the connections, seeing what makes things tick and how they fit together. Doing mathematics is putting together the connections, making sense of the structure. Writing down the results - the formal statements that codify your understanding - is the end product, rather than the starting place (p. 28).

Thompson (1988) pointed out that teaching is a human activity that involves experience, taste, and judgment. She stated that "in my view teaching is an activity that cannot be prescribed; it cannot be reduced to a sequence of predetermined steps to be learned as one learns, say, an algorithm" (p. 234). Therefore, there is room for the instructor to make instructional decisions that he or she considers suitable at a particular moment during the class. Hence, exploring the way the instructor thinks of mathematics and problem solving and

analyzing the activities that he implemented during the study could help to document the type of instruction that he provided during the study.

There are indications suggesting that the instructor endorsed a view of mathematics that emphasizes the conceptual part of mathematics. For example, to the question "which aspects of mathematics would you mention in responding to the question 'what is mathematics?' " (in the interview carried out at the end of the course), he responded:

> ....I [would] probably start by turning the question around and say what is not mathematics, in an effort to try to immediately broaden the questioner's perspective, and in response to that, I would say it is not a matter of going through a few predetermined steps and ending up with an answer. I am not sure that ... a clear description of what is mathematics is that easy, certainly not to those who would be asking it. However, I would include the *uncertain nature of mathematical fact*, call it. I would certainly include mention of how mathematical fact evolves, how it is changed ... with such examples as non-Euclidean geometry because it's relatively accessible, or perhaps more accessible, might be baseball mathematics, where our usual addition of fractions is thrown out the window with very good purpose. I would certainly include mention of attempts to understand or explain and try to distinguish that from absolute truth and discourage suggestion of absolute truth. One would have to include some mention of skills. There is no doubt about it, that one cannot do mathematics without a certain collection of skills and for the most part in undergraduate instruction that's the extent of [the] focus. However, I would want to go well beyond that and include notions of generalizations of patterns. I would like to emphasize the difference between various levels of mathematical activity: skill level, conceptualization level, validity level.

In his response, the instructor differentiated the mechanical approach to mathematics instruction, that is, the identification of a determined sequence of steps (rules) in order to understand content from the approach in which there is room for discussion, speculation, and criticism. His view of the nature of mathematics suggested that mathematics is a subject growing constantly and that there is not absolute truth.

The instructor is firm in his position on the teaching of mathematics at the undergraduate level. He considers the teaching of basic skills as important; he also believes that formal mathematics should not be the focus of undergraduate teaching. At the end of the study, he stated:

> .....I'm beginning to question more and more the utility in formal pursuits. It's only...*formal mathematics is only useful if you have some reference points to evaluate its utility* (my emphasis), and so only if you have some sort of understanding of perhaps some of the aberrations of, call it, intuitive mathematics; let's face it that it's really weird formulas just come in, and they formulate their theories after the fact. So no, I don't think that there is that

much utility for it especially in first year calculus.... However, I do think that leaves lots and lots of room for many other valid mathematical activities and in particular conceptualization and applications of concepts in different settings; they're much much more universal than applications of algorithms.

Learning: The Interactions Between the Instructor and the Students

It is important to relate the activities that the instructor implemented during the problem solving instruction to his views of mathematics. For example, he relied on several examples to introduce each day's content; the students normally spent some time reflecting on the examples, but the instructor was always ready to answer any question from the students without exploring the students' difficulties. The instructor at one point formalized the definitions or theorems discussed during the class; often he demonstrated some of the theorems. This type of instruction occurred more often when the concept of derivative was used to present some of the formulae for obtaining derivatives. For example, all the rules for operating with derivatives (addition, multiplication, and division) were demonstrated by the instructor in one class. One student who probably did not follow the demonstration asked whether these types of proofs were going to be on the final exam; the instructor who might have known the purpose of this question responded, "No, but you have to know them".... Other students asked during the same class why the derivative of the product was not the product of the derivatives. The instructor responded with a formal demonstration of the expression that characterizes such a product. The researcher observed in this session that some students were experiencing some difficulties in understanding those demonstrations and that the instructor did not follow up some of the students' concerns. This type of intervention by the instructor sometimes happened in the course. In addition, when the students asked some questions regarding the development of the proof, the instructor only repeated the proof without exploring the students' difficulties. It seemed that even though the instructor was aware of the students' difficulties, he did not address these issues directly during the instruction; perhaps, this was because of the limited time for covering the content.

It was clear that the model of mathematics that the instructor portrayed in his teaching presented mathematics as a well organized subject. For example, the instructor often introduced and presented the use of algorithms and rules in a sequential manner. For example, during the class several examples that involved the chain rule for determining the derivative of function were discussed. The strategy in attacking this problem was to apply the chain rule

and the formulae for the derivative involved in the expressions. The students in the assignments in which they had to use the chain rule calculated derivatives without analyzing whether or not the functions were differentiable at the corresponding points. Thus, the message given to the students was that mathematics always works and that it is important to use the right approach to solve problems.

The importance of engaging students in practicing basic skills is shown in an analogy which the instructor made between learning mathematics and playing golf. At the end of the course, he stated:

> .... I might give an analogy here. If you are learning to play golf, you first of all learn how to swing a golf club and have it hit the ball, and only when you get the ball flying *you spend a lot of time on the practice* 'till you're getting the ball flying and yet that's not the extent of what you do when you play golf, but that's an important part when you play golf. I agree that the emphasis we show on examinations is somewhat unfortunate and *yet it's practice.*

The instructor's ideas about the use of problem solving were linked to the way that he conceptualized mathematics. He maintains that doing mathematics is a social activity and that this aspect should be integrated into the problem solving instruction. He thinks that this aspect can be promoted by asking the students to work on the assignments together, asking the students to work in small groups during the class, and by discussing examples that show the application of mathematics in various contexts. However, in his actual class, there were few examples in which the students had the opportunity to defend their means of understanding or solving problems.

Even when the instructor recognized the importance of discussing nonroutine problems in the classroom, he also recognized that the actual conditions of the college limit the use of problem solving. For example, he mentioned that the extent of the curriculum, the size of the class, and the testing practices are major concerns that impede trying new instructional approaches. He recognized that to discuss nonroutine problems on a daily basis during instruction takes time and that there is a risk of not covering the proposed curriculum. In addition, the final exams (designed by the department) normally include only routine exercises for which the students have to be prepared. Regarding the number of students, the instructor at the end of the course stated:

> Having class sizes of 38, 39 students ... it's difficult to do anything but present the traditional lecture format. And I think it's quite clear that traditional lecture formats are not very efficient learning tools, especially

> for today's student who doesn't engage in that way any more. They have to engage in other ways.

Evaluation of the Students' Work

Although the instructor recognized the utility of discussing nonroutine problems, he also suggested that the students' examinations should not include these types of problems. During the interview at the end of the course, he stated:

> I think that there is a good deal of ... discussion of nonroutine problems in that it can emphasize how concepts can be applied, it can promote social interaction, it can promote the thinking aspect, the critical analysis aspect of problem solving, and these are all things that are useful in order to learn. I agree that these things are not particularly easily tested and so this is not what comes up in examinations; however, I think that by pursuing them one can improve performance on examinations by knowing the stuff better by applying those things to it.

He went on to say that he provided some coaching to the students on how to write exams in which they have to work quickly to solve 12 routine exercises in about two hours. The students knew that even when they were asked to work on some nonroutine problems in the assignments and during class instructions, these problems could not be part of the final examinations. All the students were concerned about the final exam, and they constantly asked for the correct and most efficient procedures in order to do well in that exam. They often ignored exploring the problem in more general domains or looking for other approaches. They knew that these types of activities are never included in the final examination.

Although the instructor agreed to consider nonroutine problems in the assignments and class discussions, he rarely checked the students' progress in solving the assignments. The researcher, who was in charge of marking the assignments and giving written feedback to the students, periodically reported to the instructor and the students the students' strengths and difficulties. This report was always judged by the instructor to be satisfactory.

Conclusions

Research in mathematical problem solving has suggested that it is important that mathematical classes provide an environment in which the students have opportunities to develop and apply diverse strategies to understand and solve mathematical problems. Results from this study suggest that being aware of the main principles of this approach is an important step on the part of the instructor. However, it is important to develop a mathematical

community among instructors (as a group) in order to develop nonroutine problems and give the support needed to implement problem solving activities. For example, it is necessary to consider learning situations in which the students openly discuss and challenge their ideas. In addition, it is important to value the students communication of mathematical ideas. These classroom activities challenge some views that identify mathematics as an fixed discipline and may produce some conflict in the classroom. Nevertheless, if students get encouraged to participate and to value the interaction with other students, then they may see that what it counts in studying mathematics is the search for meaning and not only to master different procedures. Thus, the process of assimilating and implementing problem solving strategies in mathematics instruction should be seen as an ongoing process in which there is always room for improvement and adjustment. Here, there is indication that instructors need support not only from other colleagues, but also from researchers and institutions to overcome aspects related to the coverage of material and students' evaluation..

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DESIGNING COMPUTER LEARNING ENVIRONMENTS BASED ON THE THEORY OF REALISTIC MATHEMATICS EDUCATION

Janet Bowers, Vanderbilt University

The belief that computers can qualitatively change the nature of work has been well documented (e.g., Norman, 1988). The implication for education is that the computer has the potential to serve as a medium for helping students to learn by providing an environment in which they can model and reconceptualize their actions. This view of learning, which is consistent with the tenets of constructivism, guides the development of the Realistic Mathematics Education (RME) program. This paper reviews the three principles of RME, mathematizing, reinvention, and didactic phenomenology, and contrasts them with traditional views of instructional design in order to derive a set of computer design heuristics that are compatible with constructivism.

Most people who use computers to facilitate their writing process, organize ideas, or model mathematical problems have experienced the ways in which interacting with a computer can qualitatively change the nature of the work that can be done (Norman, 1988; Dorfler, in press; Pea, 1994). Word processors, data bases, and spreadsheets are examples of open-ended computer applications that enable users to create and modify the working environment within which they organize their thinking strategies. This definition of work is consistent with the Freudenthal Institute's conception of learning. According to Freudenthal, "[Mathematics is] an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter" (Freudenthal, in Gravemeijer, 1994, p. 21). This organizing activity, called "mathematizing," is one of three basic assumptions of the theory of Realistic Mathematics Education developed at the Freudenthal Institute. This paper examines these three assumptions, mathematizing and modeling, didactic phenomenology, and student-developed models, and contrasts them with the basic assumptions of standard approaches to instructional design (ID). The goal is to derive design principles for computer learning environments that can give rise to opportunities for students to reflect on and ultimately reorganize their current ways of knowing.

Instructional Design Theory and the Development of Computer Learning Environments

For the past several decades, the practice of designing instructional materials (and educational software) has been guided by the field of (ID). This approach is based on the belief that students can achieve goals over time by mastering successive learning units (chunks defined by domain experts) that are delivered through high-quality instruction (Gagné and Dick, 1983). This approach, which reflects a "top-down" view of design, is thought to be inconsistent with constructivist theories of learning. In essence, the argument states that instructional activities that are consistent with constructivism cannot be created with a top-down behavioral approach because ID does not take into account the view of the students as active constructors of their own ways of knowing. In response to this argument, the proponents of ID state that "While ISTG [Instructional

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Systems Technology Group] has a well-documented methodology, it is not clear how a constructivist would go about carrying out these steps How does one select relevant problems? By job analysis?...If we do a task analysis, are we going to choose irrelevant tasks?" (Merrill, 1991, p. 50). Merrill suggests that while these theories might offer some general assumptions regarding how children learn, they do not offer a systematic way of developing materials.

One response to this challenge is offered by the Freudenthal Institute's approach to instructional design. The philosophy that underpins the Realistic Mathematics Education program (RME) combines instructional design with developmental research in a cyclic process. The basic tenet of this approach maintains that (math) learning is a human endeavor that can be accomplished through mental effort (Freudenthal, cited in Gravemeijer, 1994). This orientation offers an alternative to the top-down approach because it centers on creating instructional sequences that engage students in problem-solving activities which are intended to lead to increasingly sophisticated student-generated strategies. This focus on students' actions, models, and interpretations illustrates the "bottom-up" nature of this approach (Gravemeijer, 1993-a).

By coordinating this cognitive perspective with a social perspective in which children are viewed as active members of a community of learners, software designers can begin to form a holistic picture of students' learning-in-action. As Laurel (1993) points out, the process of designing computer systems should begin with an analysis of *what the users are trying to do*, rather than what the screen should look like. This approach assumes that *what the learner is trying to do* is engage in discourse, negotiate understandings of his or her activities, and reflect on those activities. In taking account both students' cognitive activity and their social obligations, the designer can begin to develop an environment and envision potential implementation schemes. This process will be elucidated by examining the basic assumptions of Freudenthal's approach to instructional design.

Design Issues Based on the Tenets of RME

The three basic tenets of RME form a strong core that reflect beliefs about mathematics, about teaching, and about mathematics education itself (Gravemeijer, 1994). These core assumptions constitute a rare example of a unified set of design heuristics that are consistent with the tenets of constructivism (Cobb, 1994) and hence can serve to guide the development of computer learning environments.

Principle #1: Progressive mathematization and guided reinvention.

The first principle maintains that students learn mathematics by reflecting on their own actions through a process of progressive mathematization. Mathematization is defined as the process by which learners organize their mathematical activity to transform a context problem into a mathematical interpretation. According to Gravemeijer (1994), this process can be conducted in two directions: horizontally and vertically. Horizontal mathematization occurs as students create models of their mathematical activity in various contexts. Vertical mathematization refers to the

process by which students reify these models through progressive mathematization. Gravemeijer (1993-b) outlines several characteristics of this progressive mathematization process. They can be grouped into two categories: developmental aspects and social aspects. The developmental characteristics include the central role of context problems, and the attention that must be placed on the development of situation models as bridges between reality and more abstract mathematics. The social aspects include the interactive character of the learning process, and the roles and obligations that have been established in the social microculture.

The question of how to develop context activities that support these modeling activities is addressed by the principle of guided reinvention. The term "reinvention" is offered as an alternative to the notion of sequencing that is prominent in instructional design theories. The reinvention principle suggests that learning should follow a path that enables the student to reinvent mathematical concepts for themselves as they are guided along a potential learning route. The critical difference between this approach and traditional ID theories is that prospective and potentially-revisable routes are mapped out based on the designer's knowledge of students' actions and of the history of mathematics, rather than from the designer's own conception of how the task should be approached. For developers, these two sources of guidance (an historical account of mathematics and observations of prior students' interpretations) can inform the planning process by suggesting possible learning routes to be included in the computer environment. This does not suggest that development should be limited to a linear procedure. RME's cyclic approach of design, implementation, and revision is critical to the design of computer environments as well because even the most well-intentioned programs elicit unanticipated interpretations. These unexpected uses should be channeled back to revise the program and also serve as insights into the nature of the students' actions with the computer.

One guiding heuristic that has evolved from the principles of mathematization and reinvention is that the starting point of activities must always be experientially real for the students. This does not imply that the students need to be actually familiar with the context in a physical sense, but that the problem and context representations must be understandable (or make sense) so that the students' associated actions can be personally meaningful. According to Gravemeijer (1994),

> *Reality is understood as a mixture of interpretation and of sensual experience. This implies that mathematics too can become part of one's reality. Reality and what one counts as common sense is not static but grows under the influence of the learning process of the person in question. (p. 94).*

The notion of 'reality' in relation to computer environments is highly provocative because it can include, from a semiotic perspective, representations which may or may not have real-world referents (Laurel, 1993). For example, interacting with an animated rabbit or cartoon garden that grows flowers in specific patterns may not reflect a real-world scenario. It can however support the development of strong imagery that is central to the mathematization process. The issue for the

computer developer is to enable the students to act in the environment in a way that is personally meaningful regardless of whether these interactions mirror real-world referents. The intent is to engage users interactively in situations in which they reify their activities. Such an approach circumvents the issue of dualism that often confounds arguments regarding representations (Cobb, in press).

Applying the principle of mathematization to the design of computer systems implies that the environment must allow students to model a problem in multiple ways such as through graphics, symbols, or icons. For example, students using the *Geometer's Sketchpad* (Jackiw, 1991) can first create models of their activities using the drawing tools (horizontal mathematization), and then use their local sketches to develop scripts for their more formal solution heuristics (vertical mathematization). This scripting process, which enables students to create, save, and apply geometric constructions to different sets of "givens," may facilitate the process of vertical mathematization by providing an environment in which students can interact with the agents of the program to form increasingly sophisticated models. Rather than serving as an intelligent tutor, the computer simply provides a medium in which the student's actions and intentions can be realized. It also serves as a social mediator enabling groups of students to discuss their various hypotheses and share constructions over the network.

Principle #2: Didactic Phenomenology.

Gravemeijer terms the second assumption "didactic phenomenology." This principle is based on Freudenthal's notion that learning occurs as students create mental mathematical objects by engaging in mathematizing. In the classroom, didactical phenomenology refers to the teacher's sense of how materials fit into the larger goals of the instructional program and how the students might gradually transform their initially informal activity into increasingly abstract yet personally meaningful activity. By analyzing the micro-didactics of the learning in-situ, the teacher can be aware of how to capitalize on the students' models to support level-raising and the gradual formalization of activity.

Although the designer is not a direct participant in the classroom microculture, the "programmer's voice" plays a role in constituting this mathematization within the educational context (Griffin, Belyaeva, Soldatova, & Velikhov-Hamburg Collective, 1993). This role is to create an environment that can lead to vertical mathematization by *supporting* the students' development of rich imagery. The emphasis on the word "supporting" indicates that students cannot be *given* imagery. Each student creates his or her own mental imagery by reifying his or her own actions within the computer environment. One aspect of a supportive environment is the use of familiar images. Pictures of real-world objects can be used to illustrate potential actions based on the user's prior knowledge of the object's physical properties. One disadvantage of using familiar pictures is that if the properties of the computer-based objects vary too widely from the user's expectations, they could become confusing. A second disadvantage is that images represent

objects rather than actions. In response to this, Laurel (1993) suggests that developers focus on supporting activity and representing this activity in the environment.

The notion of representing action suggests that if developers create a virtual world that elicits reflection on prior actions and makes alternative actions possible, students would be able to progressively refine their strategies in sophisticated ways. For designers, the critical issue is to note that these progressions should be initiated and planned by the students, rather than being imposed by the program. This does not suggest however that students will naturally refine their strategies. In contrast, this process occurs within the classroom microculture. For example, as the students and teacher mutually negotiate the sociomathematical norms, efficiency might emerge as an implicit criteria for discussing solutions (Cobb & Yackel, 1993).

The multiplication microworld developed by van Galen (van Galen & Gravemeijer, 1988) illustrates how didactic phenomenology can inform the design of a computer environment. This program was designed to help students develop increasingly sophisticated strategies for multiplication by providing pictures (such as flowers) in varying array formations. The intent was that students would begin by counting all of the flowers, but slowly develop more sophisticated counting strategies such as counting the number of flowers in each row, and eventually curtailing this process by simply multiplying this number by the number of rows. When discussing observations of students using this program, van Galen and Gravemeijer (1988) note, "We saw examples of children exchanging inefficient strategies for more efficient ones" (p. 5). By conducting research on how the children act in the computer environment and how the social environment affects their actions, the research cycle continues to feed back to inform the design of new software, materials and implementation scenarios.

There are some trade-offs involved in designing computer environments that support level-raising strategies. One such decision is the debate between being aided by tools to accomplish tasks at the expense of actually performing the physical actions. Pea's (1994) solution is to consider the computer and user(s) as one system. From this perspective, the work that can be accomplished by the whole system should be considered viable, rather than trying to account for individual contributions. This is consistent with a more realistic view of how people interact with tools in general. A second trade-off is the prudent use of constraints. There may be a fine line between focusing the users' creative efforts and actually "funneling" their actions.

Principle #3: Self-Developed Models

This third principle discusses how self-developed models serve to bridge the gap between informal and formal knowledge (Gravemeijer, 1994). Assuming a non-representational view of learning, this principle holds that students develop their own models for their initially informal activities. Slowly, through the encouragement of the teacher, students' models *of* their informal activities become models *for* more formal mathematical strategies (Gravemeijer, 1994). This

approach varies widely from a representational view which suggests that students use pre-existing models that contain knowledge in their structure.

This distinction can be illustrated by considering two different microworlds, The Thinker Tools, a physics microworld designed to model various properties of motion (White, 1993), and van Galen's multiplication program discussed above. The instructional objective of the Thinker Tools microworld is "... to introduce simplified conceptual models in the initial stages of learning and then to progress gradually to more complex models" (White, p. 4). In this way, the expert-novice dichotomy is brought to the fore and distinctions are made à priori. That is, the developers determine what simplified models (based on simplified expert models) the students will use initially and then provide more complex models that the students formalize into a set of laws. The Thinker Tools program could be described as "top down" because it begins with a goal of teaching particular concepts via previously constructed representations and specific algorithms for solving problems. Further, prior knowledge is viewed as a stumbling block, something that has to be overcome. White writes, "The thesis is that acquisition of ... knowledge overcomes misconceptions and fosters an understanding of physics and scientific inquiry that older students taught with traditional methods appear to lack" (p. 3).

In contrast to the above emphasis on pre-formed models, programs that are consistent with the RME approach take students' actions as their starting points. Thus, prior knowledge and actions are considered not as residual artifacts that can be progressively eliminated by the discovery of more sophisticated rules, but as "the very essence of cognitive creation" (Kieren, 1993, p. 2). For example, van Galen's multiplication microworld was developed in concert with research indicating that students often progress beyond counting solutions by using thinking or derived fact strategies such as products of doubles and multiples of five and ten. This "bottom up" approach to learning conceptualizes the learner as an active agent constructing his or her own strategies.

In summary, the distinction between the activities incorporated in traditional ID approaches versus those that are consistent with the RME approach can be viewed as a difference in underlying assumptions about students' abilities to generate and develop their own models. While the Thinker Tools is based on a careful analysis of how experts think about and represent established concepts in physics, the Freudenthal Institute's multiplication program is based on a careful analysis of how students' own strategies evolve. While both microworlds enable students to view representations in multiple forms, the ability to develop and modify one's own models and strategies represents a more holistic approach that is consistent with the tenets of constructivism. For software developers, the implication is that student-generated models offer a crucial starting point for conceptualizing how to incorporate potential actions into the computer environment.

Concluding Remarks: Design Based on an Integrated view of the Computer-User System

This interpretation of RME's fundamental assumptions leads to a view of learning as both socially situated and integrally related to the physical artifacts (models) students develop and use. For developers, this suggests a perspective in which the student and computer are viewed as an

integrated system. This view is also consistent with Pea's (1994) contention that knowledge is socially constructed and distributed across minds, persons, and symbolic and physical environments. Given this perspective, software developers can capitalize on the reciprocal relationship between the learner and his or her tools in order to develop computer environments that support the processes of mathematization and level raising. This holistic view of learning-in-action combined with the core principles of RME offers design heuristics for software developers interested in creating environments that enable users to distribute their intelligence (Pea, 1994) in order to organize and reflect on their activities.

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PASSIVE AND ACTIVE GRAPHING : A STUDY OF TWO LEARNING SEQUENCES

Dave Pratt

Mathematics Education Research Centre, University of Warwick

Abstract

This paper reports on the graphing work of children, aged 8 and 9 years, who have immediate and continuous access to portable computers across the whole curriculum.They have been using their computers to generate graphs and charts from experimental data. The unit of analysis is a learning sequence in which the progress of a small group of children on a specific coherent task was recorded over a period of several weeks. The paper describes two such learning sequences to illustrate two types of graphing, which can occur in computer-rich environments. In one sequence, the children collected data after which they explored the graphing facilities on the computer whereas in the other learning sequence graphing is used iteratively as an integral part of the ongoing task.

Introduction

Perhaps the overriding characteristic which distinguishes contemporary living from that as recent as fifty years ago is the central importance of information. In particular, there is a great emphasis placed upon the presentation of data through images, such as graphs and charts, as a means of informing or persuading. There is a tendency to believe that such images are transparent in the sense that the reader will gain immediate understanding of their message (see Dreyfus and Eisenberg (1990) for a longer discussion of this issue). There has been much interest in the use of computers to help children to develop their understanding of graphs. However, a growing body of literature suggests that there is considerable complexity in the cognitive demands of such an approach.

Some of this research has focused upon the misconceptions and illusions that can occur in computer-based environments (see, for example, Yerushalmy 1981 and Goldenberg 1987). Other research has looked upon the support offered by linking the graph-plotting computer directly to an experiment under the learner's control. The evidence (from, for example, Nachmias & Linn 1987, Mokross & Tinker 1987, Brasell 1987) suggests that this data-logging approach helps the child to interpret the meaning of the graphs as they can make direct connections between their actions in the experiment and the feedback generated in graphical form on the computer.

We have previously proposed a pedagogic approach (Pratt 1994), termed active graphing, which offers children a computer-based environment which seems to support the acquisition of fresh insights into the nature of graphs and graphing. This paper develops these ideas further by reporting in some detail two learning sequences, which illustrate contrasting ways in which spreadsheet generated graphs have been used in the classroom.

Methodology

We analyse the graphing work of two groups of children, aged 8 and 9 years, data which was collected by direct observation in the classroom over several weeks as part of the ongoing research in

the Primary Laptop Project[1] in the UK. The observer kept field notes which were later refined by discussion within the project team, including the teacher. The unit of analysis is a learning sequence. We recorded over a period of several weeks the work of a small group of children on a coherent and specific task. Since the children would, from time to time, move away from this particular task to carry out other work, the learning sequence was not continuous but the researchers were in a position to continue monitoring so that, when the children returned to the task in question, detailed observation of the sequence could be continued.

Joy and Shelly's learning sequence arose out of an activity in which the children were asked to consider how they might predict their own adult heights. This resulted in the need to collect and analyse various body measurements of all the children in the class. Andrew, Ben and Sam's learning sequence arose when the children were shown how to make simple paper spinners and asked how the design of the spinner might affect its flight. In particular, they tried to design a spinner which stayed in the air for as long a time as possible.

Results

Learning Sequence 1 : Joy and Shelly

The teacher had asked the class how they might predict their own adult heights and a class decision had been made to collect data about their own body measurements. Small groups of children entered their data into a spreadsheet and this was checked and collated by one group. The class was shown how to use the spreadsheet to generate graphs and charts which they were encouraged to investigate. Shelly and Joy were two quite bright girls, though not in the teacher's assessment mathematically exceptional. Shelly and Joy began to create many different graphs and charts, usually based on the whole set or a large subset of the body data. The two girls were clearly enjoying the process of generating appealing pictures on the screen and had managed to create about six or seven graphs in just a few minutes. They had dismissed some as boring and praised others as interesting. After about half an hour, they commented:

Joy: "That's fun exploring graphs." Shelly: "Yes, because you can draw any graph that you like."

At one point, they became particularly interested in one chart (Fig. 1), generated from the whole set of data. They seemed to have settled on this as the *best* graph so far. At this point, the researcher questioned the girls about their understanding of this chart. Shelly and Joy seemed to think that each bar corresponded to one person and one part of the body.

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1 *The Primary Laptop Project is studying the effects on young children's mathematical learning when they have constant and immediate access to portable computers. The computers are seen as part of a complex working environment, where many aspects integrate to support the children's learning. The project has just completed its third phase in which children of ages ranging from 8 to 12 took part over a period of one academic year. Two researchers worked on the project full-time for the year in cooperation with the normal class teachers.*

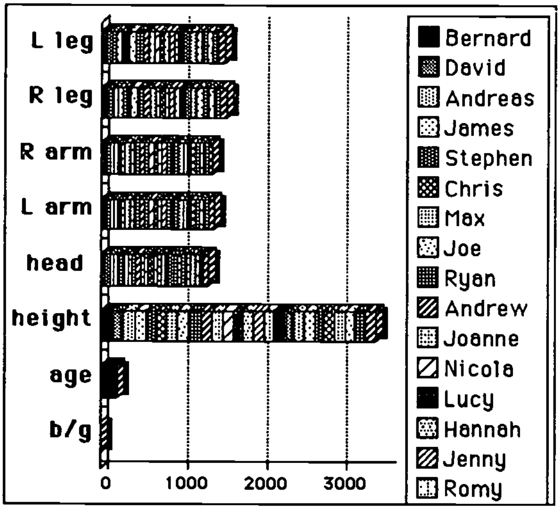

Fig 1 : Joy and Shelly's graph, chosen on aesthetic criteria

So the first bar showed Bernard's left leg, the second bar showed David's right leg, followed by a bar for Andreas's right arm and so on. The two girls were making connections between the ordering of the bars (labelled by the parts of the body) and the ordering of the key (labelled by the children's names). In the extract below, the researcher probed further by encouraging them to face the bugs in their explanations and, when necessary, by focusing attention on some revealing aspect of the chart:

Researcher	*Shelly*
So where is Bernard's right leg or Andreas's for that matter?"	Mmm.
Why are the bars made up of lots of different shading?	Each bar is everybody's.
So how is each bar made up?	Ryan's is at the end because he is the tallest.
At this stage, Shelly had recognised that each bar contains information about every child but she was now ordering the bars according to the heights of the children.	
So who is the shortest?	Bernard.
Is that right?	No.
Shelly knew that Bernard was actually quite a tall boy.	
So what's happening? How are the bars made?	*Silence*
Are all the bits of the right leg bar the same width?	Oh no.
Why is that then?	...because some people have thin legs.

Shelly made a connection between the appearance of the bars and an attribute of people not contained in the dataset, a phenomenon referred to elsewhere as *over-interpretation* (Donnelly & Welford 1989). It is clear that the discussion has prompted Shelly to try to interpret the graph rather than treat it merely as an aesthetically pleasing picture. However, the connection that she was making was not shaped by a knowledge of the conventions of such a graph.

Learning Sequence 2 : Andrew, Ben and Sam

In this sequence, we describe the work of three boys (age 8 and 9), Andrew, Ben and Sam, who, like Shelly and Joy, were bright children but not exceptional and not, in the teacher's view, the most mathematically able in the class. They had been shown how to make paper spinners and were

exploring alterations to the design which might affect the flight of the spinner. They had decided to focus on how the time of flight was affected by changing the spinner's wing-length. In particular, they wanted to find out which wing-length would maximise the time in the air. Their method of working was to make a paper spinner, measure the wing span, time its flight and then immediately enter the data into their spreadsheet. The teacher had spent some time encouraging them to generate scatter-graphs at regular intervals in order to see how their experiment was progressing.

After testing four spinners of various wing lengths, they generated their first scatter-graph. The boys were able to relate each cross with the corresponding piece of data in the table. The researcher decided to probe further into their understanding of the graph (R stands for researcher, A for Andrew, S for Sam and B for Ben).

R: "It's early days, but can you see any patterns yet?" A: "Up, down, up, down."

They were focusing at this stage on the specific data points but they were able to use the information to determine missing data.

R: "What other results do you need?" S: "1..5..7..8..9..10."

Andrew, Ben and Sam set about making more spinners. After seven results, they decided to generate another scatter-graph. When interviewed, it was clear that their thinking was still focused on the individual points rather than seeing any more general pattern. However, when they were asked to

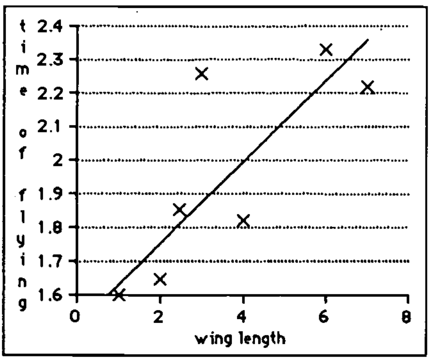

Fig 2 :Snapshot A - An intervention focused Andrew, Ben and Sam's attention onto the trend in the data.

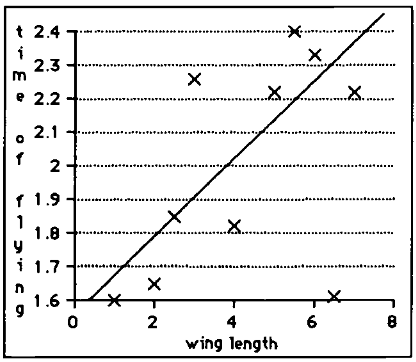

Fig 3: Snapshot B - Ben interpreted the line and Andrew recognised a point which does not fit

show the trend or pattern in the crosses using the computer's drawing tools, they dropped the line over the points and moved it around a little before settling on the position shown (Fig 2). It is likely that the line was seen by the boys as a representation of the pattern in the crosses rather than as a statement about the relationship between the wing-length and the time of flight. After collecting three more

results, they generated another graph (Fig 3) and again there was some debate about where to put the line. When the researcher probed further, a shift in their understanding seemed to have taken place.

R: "What can you say about the pattern?" B: "The longer wing stay longer in the air."

A: "Apart from that one."

Andrew was pointing to the cross representing a wing-length of 6.5 cm.

This was a break-through since it was the first evidence of the boys gaining insight into their experiment by interpreting the graph and/or data. Andrew confirmed his appreciation of Ben's assertion by the way that he was able to identify a misfit. The boys decided to check that piece of data and collect some more.

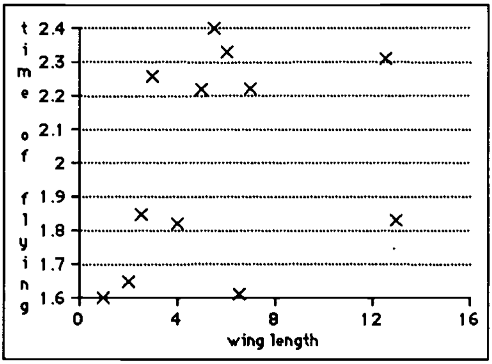

Fig 4 : Snapshot C - More and more data seemed to conflict with the boys' earlier conjecture.

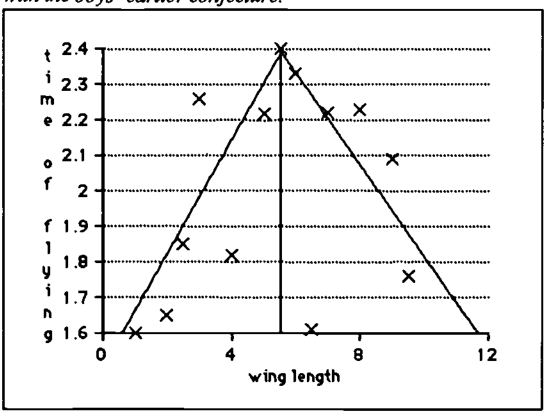

Fig 5 : Snapshot D - Andrew, Ben and Sam recognised a non-linear relationship between the two variables

Andrew, Ben and Sam began to make some very long winged spinners. Along the way, they entered one piece of data incorrectly when a spinner of wing-length 12.5cm was entered as 122.5cm. The error went unnoticed until they drew their next scatter-graph. The feedback from the graph was so different from previous graphs that they were prompted to look back at their results and they identified the mistake.

Later, they began to make their spinners in a different way. Although they did not realise it at the time, their results were being affected by this change. Again it was the feedback from the scatter-graph that alerted them to the possibility that something strange was happening. To make sense of the graph, they had to think back to what they had been doing in their experiment.

As they continued to collect more data, Andrew, Sam and Ben were disconcerted that the new entries seemed to confuse rather than clarify. They were getting more and more exceptions to Ben's earlier

assertion (Fig 4). It never at this stage occurred to them that the relationship might be other than linear. However, after a little more data collection (Fig 5), Sam, in a moment of inspiration, said, "It's an up and then down pattern!" and waved his hands around.

R: "Do you remember when you put the line over the crosses yesterday to show the pattern? Could you do that again?" *Andrew placed a line over the first set of points.*

R: "What happens then? Grab another line to show the rest of the pattern."*Andrew then placed a line over the remaining crosses.*

R: "What would be the best wing-length to use?"*Ben traced his finger over the lines before replying:*

B: "5.5"*and drawing in the vertical line.*

Discussion

These two learning sequences were selected because they seemed to typify two contrasting uses of the graphing facilities available in modern spreadsheet software.

1. Passive Graphing

Conventionally in UK schools, children use a graph to display the results at the end of an experiment; the children come to see the graph as a presentational tool. The emphasis is placed on making the graph look attractive. We refer to this style of graphing as *passive* (Fig 6).

It is important to note that the children's need to draw a graph is motivated by the production of a display. The source of this need is, we think, fundamentally important since it shapes the child's vision of what the graphing activity is about and in turn drives the child's view of which characteristics of the graph are significant. Joy and Shelly's graphing explorations had been motivated by the teacher's exhortation to investigate the graphing facilities on the computers. Their attention was very much focused on the aesthetic aspects of the various graphs that they were so easily able to generate. The presentational aspects of the graphs were foremost in their thinking so that, although they were engaged in a lively interaction with the computer, we would nevertheless classify this style of graphing as essentially *passive*. The interaction was not one in which the use of the graph was furthering their experiment.

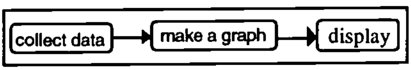

Fig 6: Passive Graphing evident when the presentational use of a graph is emphasised

Joy and Shelly were making what we call *pseudo-mathematical* connections. We use this term to describe a process in which children use the computer to generate objects thus giving the appearance that they are engaged in mathematical activitiy and yet, on closer inspection, it becomes apparent that there is little substance to this illusion.

However, we do not mean to infer that such learning sequences were valueless. Indeed, it is quite possible that these early affective responses might contribute to later success. In any case, we would be very surprised if the children did not need to go through this process in order to test out what the computer was capable of before settling down to more focused work. These experiences allowed the children to gain a familiarity with the technology and some notion of the range of possibilities.

By being given the freedom to investigate the graphing facilities within the spreadsheet, Joy and Shelly made connections with previous experiences of pictures and graphs rather than with the formal conventions as understood by the teacher (cf *the play paradox* in Noss & Hoyles, 1992). The pedagogic question becomes one of how to offer children experiences where they can use graphing as an interpretative instrument and so encourage them to make new connections.

2. Active Graphing

In response to observing such experiences, the teacher asked Ben, Sam and Andrew to generate scatter graphs on a regular basis (perhaps every three or four pieces of data) and to use this information to help them decide on the next action to be taken in the experiment. Fig 7 gives a crude description of the process, which we refer to as *active graphing*. In the *Active Graphing* approach, the children are encouraged to generate a graph after only a few pieces of data have been collected. By studying the graph and the tabulated data, they are expected to try to decide what to do next in their experiment. Further data is collected and more graphs are generated. At each stage, the children are encouraged to pause and reflect upon what this tells them about their experiment. Eventually, a point is reached when it is felt that enough data has been collected to draw some conclusions.

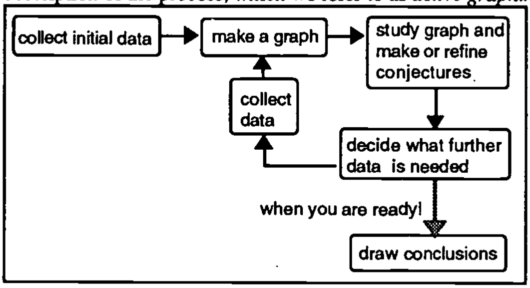

Fig 7 : Active Graphing stressing interpretng a graph

We are struck by Nemirovsky's (1991) method in which, rather than seeing the children's efforts as misconceptions, which often lack power of explanation, he reports positively on the connections that children make. He notes how the children that he observed made two types of connection between graphs generated automatically on a computer as the children manipulated toy cars. He uses the term syntactical translation to describe occasions when the students linked features of one graph with features of another. In an active graphing approach, we too observed children making syntactical translations between different modalities; for example, numbers in the spreadsheet were often connected with points on the graph (and vice versa). The term, semantic translation is reserved for situations where the children made connections between the meaning of the graph or the numerical da and the experiment itself. We observed Andrew, Ben and Sam interpreting graphs in terms of the experiment such as when they recognised the sloping line as indicating that the longer wings stayed longer in the air and later when they made the even more sophisticated non-linear connection. When the children made predictions about their experiment based on the graph or data, they were making semantic translations in the opposite direction.

Nemirovsky sees *solving* a problem as negotiating between two or more apparently conflicting versions of the truth as presented by different modalities. We find this a particularly helpful way of

viewing the interactions of the children as they move between and around the modalities offered by the active graphing approach (cf Hoyles and Sutherland (1989) when referring to two categories of children's problem solving with Logo;*working at a syntactical level* and *making sense of*).

When we talk about the children making mathematical, as opposed to *pseudo-mathematical* connections, we refer to both semantic and syntactical translations. However, we would conjecture that *active graphing* may promote another semantic connection, related to the purpose of graphing. In the *active graphing* approach, the children are using graphs as a meaningful and relevant tool. A child who sees graphing as an analytical interpretative instrument has made a powerful *mathematical* connection which has fundamentally widened that child's grasp of the utility of graphing.

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BETWEEN EQUATIONS AND SOLUTIONS: AN ODYSSEY IN 3D.

Michal Yerushalmy Merav Bohr

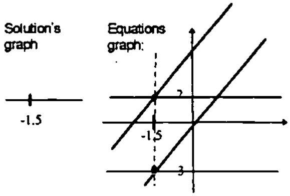
University of Haifa, Israel

Comparison of two functions, is a way to describe an equation. This approach is the one we use through an innovative algebra curriculum in which the function is the central concept. This approach calls for 2D graphic representation of equations in a single variable, and can be generalized using 3D representation of equations in two variables. We had interviewed 5 pairs of 10th graders, who studied with an experimental algebra curriculum. During 15 hours of interviews we explored how they interpreted equations in two variables and their solution sets both, analytically and graphically. The students had not seen this sort of problem before. Throughout their actions we realized that their deep understanding of functions and comparisons in a single variable not just helped them to view 3D graphs but more important, it helped them to relate to, solve and explain problems that are normally the subject for rote learning and memorization.

For the most part, solving equations and inequalities is taught as a set of seemingly arbitrary rules that govern allowed and disallowed actions. Moreover, the equations learnt at school mathematics are these that can be solved analytically, while the infinite number of types of equations that can only be solved numerically is hardly introduced. The message of the traditional algebra curriculum is that all equations should have an algorithm that collapse the equation into a number and that mastering these algorithms is the important knowledge required. It is at least equally important to investigate comparisons, to produce equivalent equations by manipulating the expressions and to analyze the processes they represent (Chazan 1993). Choosing the function to be the central concept around which the algebra curriculum should be organized (Yerushalmy and Schwartz 1993), and the availability of graphic technology, make it easier, and even attractive to describe any equation (or inequality) as a comparison between two functions. Those who consistently tried this approach with their students agreed that it present substantial and clear organization to the algebra curriculum, and open an arena for exploration of the essence of functions, equations and solutions. However, some, rightly so, questioned the consistency and the generalability of the approach while moving to multi variables relations Before proceeding we will illustrate the use of the representation <u>in one variable</u>:

Equivalent equations | Solution's graph | Equations graph:

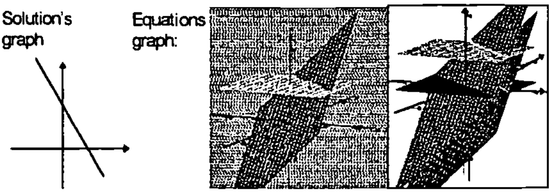
$2x+5=2$
$2x=-3$



and in two variables:

Equivalent equations | Solution's graph | Equations graph:

$2x+y=2$
$2x+y-2=0$



The visual limitations are clear: such representation of two variables is about all one can do graphically. Furthermore, previous research of the understanding of 3D shapes is discouraging (Osta 1987) and the graphic technology, while colorful and flexible than concrete drawings, is still a 2D environment and we should not assume that what seems to be successful approach in 2D representation would prove itself in 3D representation. However, we were eager to explore how would students who educated to look at any equation in a single variable as a comparison of two functions and who were never formally learnt about functions of two variables would think about equations and their solution sets in two variables. Specifically we question the possible views of equation in two variables, the interpretation and explanation of the 3D representation of functions and equations and the implications on the ability to distinct between solutions, functions and equations. Our study is based on 15 hours of interviews with five pairs of 10th graders who learnt their algebra/precalculus course through inquiry guided by special materials which organize the curriculum around the concept of function and make intensive use of multi representation graphic technology. The classes were closely observed during 4 months prior to the experiment. The students, four of were of the strongest in their classes and the rest at the average and above level, volunteered to participate. Each pair interviewed for about three hours and it included a half hour intervention to teach the representation of functions of two variables, using Lego blocks,

paper and pencil and software to graph functions of two variables. During the interview the students worked on the following task: *Describe the solution set of each of the following equations*: $2^x = x^2$; $\frac{x^2}{4}+\frac{y^2}{9}=1$; $x^2y+xy^2=1$. The first equation in a single variable requires comparisons of the graphs of the two functions and was posed to view whether students do view an equation as a comparison. The second can be described as two explicit functions and the third cannot be written as an explicit function of a single variable and solution by comparison of two functions. The interviews were videotaped and transcribed. Here we will shortly describe the conflict students ran into while analyzing equation in two variables using single variable knowledge, their suggestions for representations and their use of the 3D representation to make sense of other equations in two variables. We will provide data from work of one pair.

1. The conflict $\frac{x^2}{4}+\frac{y^2}{9}=1$*:* This part went for about 20 minutes. Students were not familiar with analytic geometry and the assumed background was manipulations skills, experience with functions and comparisons in a single variable and familiarity with graphic technology. The given equation can be manipulated to be written as two explicit functions in a single variable and we expected that students will try this approach first. While the manipulations are not complex, the conceptual shift from a comparison of the type $f(x,y)=g(x,y)$ into two functions $h(x)=\pm\sqrt{\ldots..}$ is not trivial. It requires the understanding that in general any comparison of functions in ***n*** variables could be equivalent to a function in ***(n-1)*** variables but not always. It also requires to read the equation sign in the comparison and in the function as two different objects: the first is symmetrically bi-directional and the later is not. The analysis of the data showed that the problem was an appropriate example to a task that can be carried on successfully without understanding but at the same time opens the stage to deep dilemmas: We will start with a short episode from the interview with Ron and Ed[1] .

Ed: We'll change the equation so y equals something...

[1] Michal and Merav are the authors and the interviewers. Students' names were changed to prevent identification.

Ron: It will result $2y = \sqrt{36 - 9x^2}$ *. This will not help us!*

Ed: I will try to factor both sides and produce two comparisons:

$(2y)(2y) = (6 - 3x)(6 + 3x)$. It (factoring and comparing) did not lead them anywhere.

*Ed: I need to find out when do these two sides are equal; it is easiest with zero. Substituting **0** for y will allow me to find for which values of **x** the right side will be zero as well. That would be the values for which the comparison exist.*

Even though they manipulated the equation and separated the y they continue to relate to it as a comparison and not as to a function of **x**. *Ron: When* $y = 0$ then $x = \pm 2$.

They sketched the parabola $f(x) = 9 - \frac{9}{4}x^2$ and marked the two roots...

Ron: ...but we assumed that y=0!

Ed: Right, so we actually found two points.

Michal: Am I allowed to assume that both sides could also be equal to a 100?

Ed: I can substitute any number for y.

Michal: So then what would you say about the solution set?

A long pause seems to sign that they have no idea how to proceed.

Ron: Lets start to think... There are two variables in here...

Ed: Two variables in one equation.

Michal: what do you mean by that?

Ed: That is something that I have some difficulties to solve..... I can't, it is depended on two things and I can't compare it to something else.

This last statement of Ed clarifies his confusion about the object they deal with: if it is a function then the y values determined by x but here y is an independent variable. But, if it is an equation what are the compared objects? Ed seeks '*something else*' which he could not find. We will later observe how this conflict developed into an invention.

2. An invention of the representation: While planning the experiment we were uncertain if at all to provide and for what purposes software. We provided the 2D plotter because it was the everyday tool that the students were used to have for solving and manipulating equations. In effect, all students tried to plot the manipulated equation in one form or the other but only some of them were able to analyze the problematic representation. For

some of them the inability to type the variable y into the software was a discouraging feedback while for others it was the first indication that something is very different here. We used this opportunity to move into the next stage of the interview:

Merav: Would I offer you something that could help you solve this, what would you take?

Ed: That the software would also accept the y.

Michal: Assuming that this software could accept the y, how would it appear?

Ron: Then I would need 3 equations[2].

Ed: Wait a second...I'm talking here about two totally different axes systems.

Ron: Exactly, two coordinate systems.

Michal: Can you draw it?

*Ron: I have to see both **x** and y because the **x** solves the equation in one way in one system and the y also...the y could get a single value solution, it isn't an ordered pair but in a different system!*

Michal: So in what system?

Ed: Oh, something 3-dimensional!

*Ed: I would have an **x** axis and a y axis that goes in (points towards the inside of the screen), I'm not sure. So each of them is an independent variable but where is the independent variable that they are both dependent in?*

They presented the need for 3 dimensional representation did not spring out of visual considerations of the functions rather by analytical search for a representation that would satisfy their need to express the comparison in two independent variables. In order to test their ability to view 3D graphs we asked to describe a simple function $f(x,y) = x + y$ and provided an addition table on a paper. All that did not help them to think about the shape; it was only when we suggested them a Lego plane and blocks that they started to talk about shapes:

Ed: for example for (1,1), when it is (1,1) so I get here 2 (puts two Lego Blocks one on top of the other on the point (1,1))

[2] Ron might have meant 3 dimensions and not equations or 3 equations: solving for x, solving for y or for the set of x,y.

Ron: Exactly, because building the height...as if we are building an axes system this is three dimensional! Ed put three Blocks at the (1,2) coordinate.

Ed: Should we keep on building? Shouldn't we look here (at the printed addition table) and find out about its shape?

At that point Ed started to view the addition table to the 3D shape and soon he was ready to conjecture about the resulted shape of the function analyzing the differences at the diagonal, the columns and the rows. They concluded that a shape that changes in a constant rate in any dimension should be a plane. It seems clear that the graphic representation nourished from the experience and understanding of functions as an input-output process, from the understanding of the dependency between variables and from their experience in analyzing the function graph's behavior (such as analysis of rate of change or finite differences). Following a short 'Lego session' we introduced the 3D software to the students.

3. Now again comparisons are an option

Once (following the intervention) the students were familiar with the new representation and the technicality of the 3D software we discussed the third task: $x^2y + y^2x = 1$

Ed: I can start by factoring ***x*** *and then by* ***y.***

Ron: It (looking at the product of the two functions) could simplify things for us.

Ed: Maybe we will simplify it well enough...no, we'll factor by ***xy*** *(in one step)*

Their attempt to view the function as a product of two functions was a surprising strategy for us. All we expected at this stage was the use of the 3D software for plotting the two functions. However, since all students learnt to construct functions and analyze their properties using operations between functions (in a single variable) it made a lot of sense for them to generalize this method in the context of two variables' function.

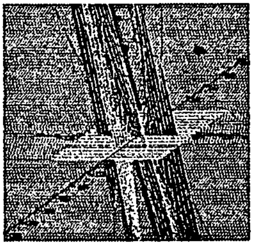
Ed: Now, what about ***x+y****? we know how it looks like...and* ***xy*** *is....let's work for a minute on a multiplication table...*

Ed then built a two dimensional table of values as a tool to draw from about the shape. We suspect that the habit to describe each function's behavior (without any formal knowledge of calculus) using tables of values and finite differences and their representation through out the curriculum enabled him to use this numeric tool.

*Ed: The diagonals are the important part.... Maybe it is not the central idea here but I can see that actually there is a sequence of whole squares. The next one will be 36... So the graph looks ....(*shows with his hands a concave increased curve)

Ron: The slope increases more and more while x increases

They continued with predictions about the product of the two functions (the sum and the product) but soon asked to view it with the software.



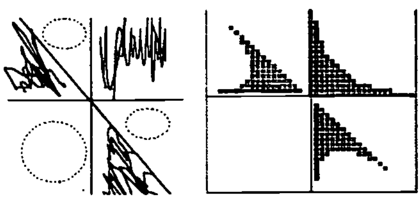
The discussion then turned to the analysis of the very complex graph at the light of the product of the two functions. They talked about symmetry and about viewing properties of each of the components' functions. Finally we asked: *Where is the solution set?*

Ron: All the points that...that touch...

Ed: .. touch the flat thing ...

Michal: Would you draw on the paper what you see on the screen?

The left figure is their prediction. The figure on the right produced by the software.



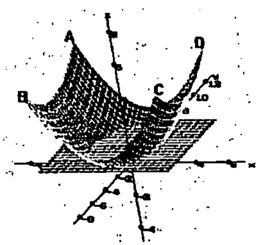
Comparison of two surfaces also motivated a discussion about an <u>empty solution set</u>. It provides another opportunity to view their parallel multi-representation judgment:

Michal: What about the solution set of $x^2 + y^2 = -1$?

Ed: They don't have intersection points.

Ron: Oh! Algebraically they don't have intersection points either, because (points at the equation $x^2 + y^2 = -1$*),..because it can't be, because each one of them is a power and there it can not be a negative number.*

Ron: it would elevate above ,..if the function $f(x,y) = x^2 + y^2$ *is hooked then the flat thing should go down by one.*



Ed: We could use two strategies; Our previous strategy in which we raised the paraboloid...or we could lower down the flat thing and then again attain no solutions.

Discussion

The combination of the objective difficulties of viewing 3D and the complex concepts we deal with caused us a very shaky take off which later turned to an odyssey. The problems we posed, regularly assumed to be non interesting problems, problems that are subject to rote learning and motivate memorization rather than thinking proved to be an arena to sophisticated mathematical discussion. The 3D graphic software was an essential tool but completely insufficient; the deep understanding of functions, of relations as objects created by comparisons, of equivalent equations, and the ability to maneuver between the various representations: numeric, analytic and graphic made this experiment to be an intellectual experience. We are convinced that a learning sequence that equipped learners to invent methods of representation and analysis as reported here is a valuable and important one to consider . We acknowledge the difficulties the representation may cause and that the current technology still does not provide the ultimate solution to the problem. We suggest that students' views and uses of representations mentioned here (and other aspects that will be reported elsewhere) will provide a base for rethinking the algebra curriculum.

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UNDERSTANDING AND OPERATING WITH INTEGERS: DIFFICULTIES AND OBSTACLES

Rute Elizabete Borba
Universidade Federal de Pernambuco

Abstract

Difficulties and obstacles of different nature —epistemological and ontogenetical —have been observed in the operations, conceptualization and understanding of integers. These have been enumerated by many authors that analyzed the historical evolution of this concept (Glaeser, 1985; Boyer, 1985; Nagel, 1979) and also by empirical observations of children solving situations that involve relative numbers ((Mukhopadhyay, Peled & Resnick, 1989). The present study sought to establish parallels between these obstacles and difficulties described and those observed in an empirical study with 96 children of Recife. Performances before a period of instruction and immediately after training addition and subtraction with integers were observed. Didatical difficulties were also analyzed in this report.

Resumo

Dificuldades e obstáculos de diferente natureza —epistemológicas e ontogenéticas —têm sido observadas na operacionalização, conceitualização e compreensão dos números relativos. Estes têm sido enumeradas por autores que analisaram a evolução histórica deste conceito (Glaeser, 1985; Boyer, 1985; Nagel, 1979) e também a partir de observações empíricas de crianças resolvendo situações com números inteiros (Mukhopadhyay, Peled & Resnick, 1989). O presente estudo buscou estabelecer um paralelo entre alguns destes obstáculos e dificuldades descritos e aquelas observadas em um estudo experimental realizado com 96 crianças da cidade do Recife. Foram observados e analisados os desempenhos anteriores a um período de instrução com números relativos e imediatamente após o treino em adição e subtração neste campo numérico. Análises de dificuldades de natureza didática também foram efetuadas neste relatório.

This report refers to an empirical study (Borba, 1993) with 64 fourth grade and 32 sixth grade students. The fourth grade children were divided in groups accordingly to different modes of instruction. These modes differed in the way the sign rules were dealt with and if students were explicitly trained with integers or natural numbers. The students performances were observed in pretests and after a period of instruction, on adding and subtracting using diagrams (fourth graders) or simply using sign rules (sixth graders), posttests were applied. These tests had four parts: numerical equations with integers, situations involving profits and losses, temperature situations and interpretation of numerical equations. The training was given in classrooms with twelve students or more each. This format of study was chosen trying to reproduce situations more similar to those encountered in usual classrooms which may contribute to analyze the adequacy of procedures, problem situations presented and the forms of representation used in formal teaching. The fourth grade students were instructed to analyze situations of profits and losses supported by auxiliary forms of representation: diagrams and the number line. The diagrams used were those proposed by Vergnaud (1982) on which a transformation links two static relationships. An example of the situations proposed was: *Célia is owing Cr$ 42,00 to the bank. She deposited Cr$ 64,00 in her account. What is now her situation?* A correspondent diagram for this situation is:

$$-42 \xrightarrow{+64} ?$$

Vergnaud justifies the use of diagrams arguing that equations and equalities are not used by children to represent relevant relationships that are present in the problems proposed but that they use these representations to recall the sequence of numerical operations necessary to find the results. Diagrams can, therefore, be considered a different kind of equation. It contains additional information specified (measures, states, transformations, or relationships).

The number line was in this present study used as an auxiliary form of verifying the correctness of answers.

Some of the results presented in this study will be analyzed through the establishment of parallels between epistemological difficulties and children performance.

Glaeser (1985) presents six obstacles, evidenced by the examination of mathematical classics, in the development of the understanding of relative numbers. Difficulties and obstacles must be initially distinguished. Obstacles here will be used in the sense purposed by Bachelard (1967) as resistance of conceptions that do not permit advances in knowledge. These differ from difficulties that will here be seen as poor performances demonstrated by subjects while solving problems in a much more superficial sense. The obstacles presented by Glaeser are: (1) incapacity to deal with isolated negative quantities; (2) difficulties in giving sense to isolated negative quantities; (3) difficulties in unifying the number line; (4) ambiguity in the understanding of the two zeros: absolute zero and zero as an origin; (5) difficulties in getting loose of the concrete sense of numbers and (6) desire to use unifying models for both additive and multiplicative fields.

The first two obstacles presented are in fact difficulties in the sense that they demonstrate mere incapacity in dealing with negative quantities and that did not necessarily hinder the advance in the understanding of relative numbers.

The difficulties in unifying the number line is an obstacle that directly impedes the understanding of relative numbers. This obstacle exists when the subjects do not differentiate qualitatively negative and positive quantities or when they simply conceive the number line as a juxtaposition of two opposite semi-lines or even by not considerating both the dynamic and static character of numbers.

Mukhopadyay, Peled and Resnick (1989) investigated how children represent negative numbers before formal instruction. First, third, fifth, seventh and ninth grade students were interviewed while solving equations that involved relative numbers. The authors concluded that the children used quite abstract models and that practically did not exist models amongst the youngest ones. The older children demonstrated progressive development on the understanding of negatives. They initially had a Divided Number Line Model manipulating positive and negative numbers in an isolated way. Progressively they achieved a Continuous Number Line Model that treated both positive and negative numbers as coherent ordered entities.

Results of the performances presented on the first part of the pretest, of this study being reported, showed significant differences between the two grades involved ($F = 8{,}97$; $p < .0009$). Table 1 shows the performance presented by the fourth and sixth grade students on some of the equations presented in the first part of the pretest.

Table 1

Percentage of Correct Answers in the Pretest

Equation presented to child	Fourth grade students	Sixth grade students
+ 400 + (+200)	62,9	87,5
- 400 + (- 200)	1,7	31,3
- 400 + (+200)	2,2	6,3
+400 - (+200)	29,5	40,6
+200 - (-200)	13,2	12,5
- 200 - (- 400)	48,6	28,1
- 400 - (+200)	0,0	18,8

As may be observed the fourth graders presented initially greater difficulties in solving these equations than the sixth graders. It is necessary to point out that these results were obtained before formal instruction on integers was initiated. One of the aspects analyzed is that the sixth graders performed better when signs involved were of the same kind - both positive or both negative. This possibly denotes the use of a Divided Number Line Model and these students may have accepted operating with negative numbers just as they had been doing with natural numbers even before understanding what these "new" kind of numbers meant.

It has been observed by Vergnaud that children meet problems involving directed numbers long before their formal instruction to this conceptual field. He argues that there are six distinct categories of relationships present in the field of additive structures, *i.e.*, problems involving additions or subtractions or both of these operations. These categories are: (1) composition of two measures, (2) a transformation links two measures, (3) a static relationship links two measures, (4) composition of two transformations, (5) a transformation links two static relationships and (6) composition of two static relationships. Vergnaud stresses that time transformations and static relationships are not adequately represented by natural numbers. Natural numbers are adequate only for the relationships involved in the first category. The other categories involve elements that should be represented by directed numbers. These categories are, however, present in many problems proposed to children before they begin to learn about integers. There seems to be a great discrepancy between the structure of problems taught at school and the mathematical concepts that are aimed to be learned. Children have also contact with everyday problems that involve integers such as inverting rotations and compensating gains and losses in games. This may, to some extent, justify that students have previous formal contacts with integers.

The third obstacle presented by Glaeser that refers to the historical difficulties in unifying the number line has straight connections with the fourth one - ambiguity in the understanding of the two zeros. The understanding of relative numbers begin when the child perceives that if negatives are smaller than positives there must be a point where they are originated. This gives the child a new meaning for the zero that is not only the absence of quantity but also an origin.

The second and third parts of the pretest of the present study referred to situations involving profits and losses and temperature situations. Analyzing the differences of these distinct situations better performance was observed with situations of profits and losses. Significant differences were observed when the four conditions (types) of testing were considered ($F_{(3,273)} = 30.66$; $p< .0009$). These differences may be better observed on Table 2 where a sample of the formal representations of the problems proposed is presented. These observations seem to indicate that most students did not realize the necessity to indicate the different nature of their answers. They were not expected to present their answers in a conventional form (with signed numbers, for example) but to indicate clearly that their results were credits or debts or that they represented temperatures above or below zero.

Table 2

Percentage of correct answers in the different situations of the pretest

Formal representation of the problem presented	Fonrth graders	Six graders
	Problems of profits and losses	
(-48) - (-76) = (+28)	**50,1**	**65,6**
(-47) - (+21) = (-68)	**2,1**	**3,1**
	Problems of temperature	
(-13) - (-25) = (+12)	**37,3**	**28,1**
(-17) - (+12) = (-29)	**12,3**	**34,4**

Kobayashi (1988) alerts that enough attention should be paid to the difference of meanings among negative quantities. The first aspect is the case where the zero is determined *a priori* naturally. In these cases the origin is not set artificially. The examples given by the author are: electricity charges and property and debt. The second aspect evidenced by Kobayashi is the case when the origin is determined arbitrarily. The position on a line, measuring temperatures, elevation of the water level in a river, and *Anno Domini* and before Christ are examples presented by the author for this case. Some of these origins are unchangeable because they have been used widely and historically in our society . The third aspect is the case when quantities are determined according to the direction in which the quantities are changed, increased or decreased. Examples

are differentials of temperature, water level, volume etc. In the present study two different kinds of origin are therefore involved. A natural one - as in the case of profits and losses - and an artificial one - the measurements of temperature. The students, in both grades, showed difficulties when negative values were involved but strange enough the better performances were observed on the temperature situations. Brazil is a tropical country where very little variation is observed on temperature measurements and only very few cities occasionally have marked temperatures below zero. What possibly made students obtain correct answers in temperature situations were the terms presented when the problems were proposed on which direct reference to temperatures below zero were made. Even though the problems involving profits and losses had a very natural origin only very few students seemed initially to feel the necessity of making clear that the answers obtained represented debts.

While interpreting numerical expressions involving integers children initially could not explain equations that were representing negative quantities and significant differences were observed between the pre and posttest ($p = 0{,}0007$). While interpreting these equations on the posttests students preferred situations involving debts and credits. Only two children used temperature as basis for interpretation on the posttest. To understand integers children seem to initially need to support their understanding on physical models (such as measuring temperatures, positions on a line) or social models (positive and negative accounts related to money or to results in games). These models closely related to the child's everyday experiences may be used as a starting point in the understanding of integers but this complete comprehension depends on the ability to abstract the invariant of these situations. This abstraction was not evidenced in the posttest of the present study because the groups trained with situations of debts and credits did not seem to transfer their previous knowledge to new situations - measurement of temperatures.

The greatest difficulty presented while interpreting equations with relative numbers was on expressions of the type **(+a) - (-b)**. Children's conception of subtraction as an operation that decreases quantities hinder them to accept initially that the final result in this case is greater than the one started with. This case is also very confusing to children because two minus signs are involved. Carraher (1990) presents three types of meaning for the minus sign that provide categories for the classification of everyday situations involving negative numbers: (a) the minus sign and the operation of subtraction, (b) the minus sign as a mark intimately connected with the magnitude that follows it and (c) the minus sign as a representation of inversion. In this last case, one can mentally reconstruct the starting point by inverting the operations. The results of experiments related by this author comparing subjects performances in written and oral situations indicate that what has often been viewed as a conceptual difficulty may, in fact, be a difficulty introduced by the notation conventions that are usually used in school. These results were to some extent confirmed by the present study where differences in performances were presented accordingly to the different type of situations involving integers. On formal expressions difficulties with subtraction were emphasized. On problems with everyday situations, particularly debts and credits, children's difficulties with this operation were less evident. The interviews with these

children seem to indicate that difficulties are directly related to representational systems and not necessarily demonstrate lack of comprehension on integers.

The final comment to be made refers to the different ways the children were instructed on integers. The fourth grade students were taught using situations of profits and losses and the sixth graders were instructed in using the sign rules. Both groups presented significant differences on the posttests when compared to the performances presented on pretests but a qualitative analysis showed that the sixth graders many times found correct answers using incorrect or incomplete procedures. Many "false" rules were created by these students and some of them succeeded in obtaining correct answers. The fourth graders, taught with diagrams, when interviewed, presented well established explanations for the problems that involved additions but were still confused with subtraction problems despite the use of these auxiliary forms of representation. Probably more time dedicated in instructional programs with this operation is necessary.

Many difficulties encountered by children can be more easily detected and also instructional programs are more efficient in some cases than in other ones. Some obstacles need to be considered in very special ways and more understanding of these profound difficulties may be very useful in elaborating more efficient instructional programs.

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GAMES FOR INTEGERS: CONCEPTUAL OR SEMANTIC FIELDS?

Antonio Carlos Carrera de Souza [1]
Antonio Luis Mometti [2]
Helena Alessandra Scavazza[2]
Roberto Ribeiro Baldino [1]

ABSTRACT

By the ***sign rule problem*** we understand four questions into which we cast Glaeser's historical survey (Glaeser, 1981) and Brousseau's epistemological remarks (Brousseau 1983) about integers: *How to take the bigger from the smaller? How to subtract a negative? Why minus times minus equals plus? What does it mean minus-times something?* In the paper we present a didactical strategy to solve this problem, based on Baudrillard's conception of *game* (Baudrillard, 1979) and on the theory of Conceptual Fields (**CF**) (Vergnaud, 1990). We report some experimental results and discuss them from the point of view of the theory of Semantic Fields (**SF**) (Lins, 1994).

Difficulties about integers are quite old. In his historical survey, Glaeser [1981] describes perplexities of famous mathematicians of the past about the sign rule. The proof that we know today was first given by Haenkel in 1867 in a text about complex variables. We know that it is useless as an explanation for convincing a 13-years old student. Integers have scarcely been dealt with in recent literature. Among 56 research reports presented in PME XVIII, only one explicitly concerns integers [Lytle, 1994]. The sign rule remains a major problem for the teacher.

Works about integers generally display a profusion of suggestions for addition but are insufficient about multiplication. Glaeser [1981] points out this insufficiency in Freudenthal [1973]. "The reading of pages 279/281 does not even suggest that he has realized the astonishing phenomenon studied here" [p. 305]. Freudenthal [1983] offers three simultaneous approaches to the sign rule problem. The first insists on the necessity of permanence of distributive and commutative laws $(-3)\times 4=4\times(-3)$ [p. 434]. This leads to the usual difficulties: students keep asking: ***what does it mean minus-three times something? Less than zero time it?*** The second approach is extension of linear transformations according to what he names the "geometrical-algebraical permanence principle" [ib. id., p. 444]. This raises the problematic relation between discrete-numerical and continuous-geometrical domains. The third approach is simply teaching rules, among which $(-a).(-b)\rightarrow a.b$. Freudenthal asks for "the most simple and effective way to programme the learner with (...) six rules. It is almost nothing compared with the rules a child must

[1] Advisor of the Action-Reserch Group (GPA) of the Graduate Program in Mathematical Education. UNESP, Rio Claro, SP, Brazil.
[2] Senior student in the Mathematics Pre-serviceTeaching Program. UNESP, Rio Claro, SP, Brazil.

learn in order do master a column arithmetic" [sic. p. 457]. When the student comes to the point of asking why minus times minus equals plus it is already too late[3]: he has learned a solution without knowing the problem and is "fed up" with rules.

We thought of anticipating the solution to the sign rule problem as *theorems in action*, according to CF. Our idea was that the roles should be exchanged: the teacher should be the one to ask and the student the one to answer why *minus times minus makes plus*. The didactical strategy should lead the student to provide his own explanation to facts that he should consider as evident: "(theorems in action) are associate with a feeling of obviousness: they are (...) taken as obvious properties of situations" [Vergnaud, 1982, p. 36].

This research was aimed at testing the didactical validity of a certain conception of **game** to solve the sign rule problem. This conception sharply distinguishes *game* from *activity*. At the moment of the game, nothing else counts but following a rule. "The game is the vertigo of the rule. By choosing a rule we suspend the law. The obligation that the game creates is of the order of a chalenge" [Baudrillard, 1979, p. 151]. In particular, no interruptions for registering results or making connections with the sylabus should be admitted. In designing *games for integers* we have been guided by pedagogical beliefs that are best stated as answers to two questions. Question 1: "How can we make theorems become theorems in action?" [Vergnaud, 1982, p. 36]. Our answer was: By engaging the student in *games* where the use of theorems in action leads to better playing strategies. Question 2: "How can we make theorems in action become theorems?" [ib. id.] Our answer was: By introducing adequate work-sheet activities based on the game, after it has been finished.

THE DIDACTICAL PROBLEMS AND THEIR SOLUTIONS

Glaeser's [1981] historical account of epistemological obstacles in the development of integers was made more precise by Brousseau [1983]. We cast these works into four didactical problems:

P1. How to take the bigger from the smaller? 3-5=...
P2. How to subtract a negative? -(-3)=...
P3. Why minus times minus equals plus? (-2)(-3)=...
P4. What does it mean *minus ... times something*? (-3)× ...?

What do we mean by solutions to these problems? Our premisse has been that **integers are operators on signed naturals** and operations between integers are operations with such

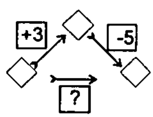
[3] This statement should be taken as rethorical but we believe that it can be justified.

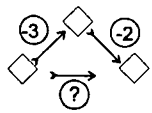
operators. In a state-operator conception, using □ for additive, ○ for multiplicative operators and ◇ for their states, our two basic diagrams are:

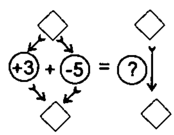
$$+2 \xrightarrow{\boxed{-5}} -3 \qquad\qquad -2 \xrightarrow{(-3)} +6$$

We emphasize that the ± signs before the operator numbers do not have the same meaning as the ± signs before the state numbers. This is easily seen by noting that we can use a red/blue code for the operators and a ± code for their states or vice versa. Denoting the multiplicative operators by f_a , the second diagram represents the transformation of one state into another: $f_{-3}(-2) = +6$. "To multiply a rational integer by $-a$ is to multiply by a and change the sign of the product" [Papy, 1968, p. 334]. We argue that this is not yet the solution of *minus times minus*. P3 is solved when the subjects perform the composition of multiplicative operators: $f_{-3} \circ f_{-2} = f_{+6}$ which is quite different. In our games we have introduced $f_{-3}(-2) = +6$ as a rule and expected that $f_{-3} \circ f_{-2} = f_{+6}$ would come out as a theorem in action. Note that in both cases the calculation $(-3)\times(-2) = +6$ is the same.

By solutions of the four didactical problems (theorems) we mean the following four theorems in action. By a solution of P2 we mean the action of removing a debt by increasing the amount of money in a gain/debts model. By solutions of P1, P3 and P4 we mean the actions of completing the diagrams below without resorting to states.

P1: 

P3: 

P4: 

THE DIDACTICAL STRATEGY: THE GAMES

According to Brousseau's conception of *learning* we cannot hope to solve these problems in one stroke: "learning is the result of experimentation of successive conceptions, temporarily and relatively good that have to be successively rejected or retaken in a truly new genesis at each time" [Brousseau, 1983, p. 171]. In order to provide conditions for experimentation of conceptions, we designed three games based on additive and multiplicative machines [Dienes, 1969A, 1969B].

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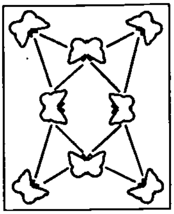
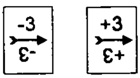
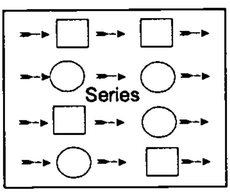


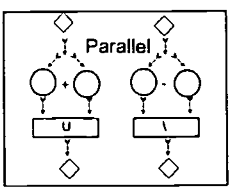
G1. This is an additive state-operator game. The kit consists of a board with a stamped network, cards representing additive operators to be placed on the network's branches, and one-colored beads, representing the operator states, to be placed on the network's knots (butterflies). Two signs are associated to card numbers: an operative sign (an arrow) and a predicative sign (+/-). Operator states <u>are</u> <u>not</u> signed. The objective of the game is to place the cards between knots so as to close commutative circuits. In the <u>advanced</u> <u>version</u>, players should develop schemes for composing additive operators <u>without</u> resorting to the beads. The expected strategy is direct composition of operators, thus solving P1.



We denote this strategy by **G1A**. In applying this game we have observed the emergence of a strategy consisting in mentally keeping track of the states by memorizing the number of beads in one fixed knot, generally the first one that was filled. We denote this strategy by **G1B**. A puzzle occurs when a card is to be placed in such a way as to close two circuits simultaneously and, due to a previous unnoticed mistake, the numbers assigned to it from each of the circuits do not match. We shall refer to this situation as **G1C**. In some cases the number of beads on a knot is not sufficient to allow for the subtraction determined by the card that a player wants to place. It is expected that they decide to increase the number of beads on all previously filled knots. We refer to this solution as **G1D**.

G2. This is a real estate sales game with "red money" representig debts. This game introduces <u>singed</u> <u>operator</u> <u>states</u>. Instruction cards may ask a player to "*remove a $10 red bill*" from the stock of a partner who happens to have no debts. This situation instals the *neutralization of opposites* that solves P2 [Lytle, 1994]. We call it **G2A**.





G3. This is a pawn-track game. The position number to which the pawn has to be moved is obtained from the introduction of its present position number into the entry of a connection of additive and multiplicative *machines*. The track numbers are coded +/-. The operator numbers to be placed in the machines are obtained by throwing two dice with face numbers also coded +/-. To negative multiplicative operators is assigned the property of reversing the sign of the states upon which they act. The player may choose the machine connection best suited for his move. The objective is to put pawns on the positions numbered, say, 30. In the first match only series connections may be used; in the second one, only parallel connections. We define **G3A** as the strategy consisting in operating with the dices' numbers before trying the series connection of multiplicative operators

→○→ ○→. This is the solution to P3. Moving his pawn to a position whose number does not divide 30 indicates that the player has not solved P4. When the player avoids this trap we say that he has developped **G3B**.

THE SITUATIONS

Informal experiments and adjustments were made so as to fit the kits to the *vertigo of the rule* conception of game. Then a systematic teaching experience was made with G1 and observations were made with G1 and G2.

S1. Two of us applied the G1 and subsequent work-sheets in two out of six weekly classes for 5th graders (11 years-old) during 7 weeks in a public school of a relatively rich town of São Paulo State, Brazil. Officially the study of integers should only begin at 12 but the school administration raised no objections. The class was being ruled by one of us. Each meeting lasted for 1.5 hours. The class was divided into 9 groups of 4 students. Ech group received a complete G1 kit. First, beads were used on the knots (butterflies). Then we asked the children to leave them out (advanced game version). Special care was taken not to hint how to perform direct composition of additive operators. Promotional grades were set on criteria of *presence*, *participation* and *performance*. A contest was organized, awarding medals to the three first places but not counting for grades. The last three meetings were dedicated to group-work on four work-sheets. These activities started with problems of completing a circuit that reproduced part of the network board and ended with problems of replacing a serires of cards by a single one in the absence os any butterfly drawings. We asked for solutions of the problems but suggestions were limited to: *do as you did in the game*.

S2. One of us played G1 in two occasions, with 7 groups of mathematics high school teachers (**S2A**) and with 9 groups of college teachers and senior students (**S2B**)[4]. The previously accorded presentation was the following. If the G1A strategy did not occur we would ask the groups for a *direct method* without insisting (S2A) or increasingly insisting (S2B). If G1C occured, we would say: *go back to beads*. As for G1D, we would tell the solution if necessary in order to concentrate on G1A. S2A also played G3. The accorded presentation was to first play the series version and worn the players before starting the parallel version: *there is a pitfall in this game*. As they fell into the trap we would remark: *your are already trapped. Find out why.*

THE RESULTS

R1. Two of the four contest finalists were sutudents considered "weak" according to teachers' general oppinion. The nastiest boy spontaneously worked as tutor of other groups. Only one case

[4] UNIJUÍ, Ijuí (RS), November 11; Un.G., Guarulhos (SP), November 24, 1994.

of cheating was observed. G1D caused no major problems; children promptly suggested the increase of the number of beads. They realized that there was some arbitrariness in this number: *you could have started (the game) with more (beads).* Nevertheless they did not develop G1A, not even the winners. G1B was used all the time, inclusive on the work-sheet situation. Consequently G1C never happened. Some students resorted to writing numbers for the states on the work-sheets and then quickly erasing them while asking: *can't we play any more?* When they faced the no-butterflies problems they became lost. At this point our suggestion: *do as in the game*, became appaerently meaningless. One group that had had poor performance in the game, started developing *ad hoc* strategies, which only succeeded when the cards had the same sign. The others kept asking for help. Nevertheless, considerable ability of mental calculation was developed.

R2. None of the S2B groups and only one of the S2A developed G1A, precisely the one where a member could cite Dienes's page numbers by heart. About half of the groups ran into G1C and one third required explicit solutions for G1D. All groups faced with the demand for a direct scheme of composition of additive operators, started by reinforcing G1B. Faced with increasingly stronger demands for direct methods, four of the S2B groups succeeded, three succeeded after some feed-back information, two gave up and left the room. Three of the groups which later succeeded, chose a minimal-value state as reference for G1B. All S2A groups playing G3 fell into the G3B pitfall. Pointing out: *you are already trapped*, apparently produced little or no effects: palyers continued trying to reach 30 from positions such as 12. Only after repeated being told that their efforts would be useless did they produce an explanation and, in some cases, reset the game. One player praised G3 as excellent but actually rejected it as a teaching instrument: *I am going to play it with my best students outside the classroom and I will introduce some random instruction cards, so that they can get out of the trap.* Our data about G1A are not conclusive.

DISCUSSION

Results in R1 and R2 did not confirm our first pedagogical belief that theorems could become theorems in action by engaging the student in games, conceived as the *vertigo of the rule*; hence the second belief could not be checked. As foreseen by Vergnaud [1990, p. 152] and Dienes [1969A, p. 9], G1A turned out to be much more difficult than G1B. Why? There is no doubt that the different degrees of difficulty can be explained in CF in terms of complexity of the mathematical structures involved in these two problems. But we still have to explain the different outcomes of the two situations S2A and S2B, dealing with the same (G1A) problem. Mathematics teachers and senior students certainly knew P1-P4 as theorems, but they only put P1 into action

in S2B and not in S2A. Nevertheless, the outcome of S2B indicates that they had the means of having done so.

The issue is not why G1A is more difficult than G1B. What we have to find out is why the difficulty of G1A was overcome in S2B and not in S2A. This is a central issue for teaching: how to overcome the difficulty imposed by a given *complex mathematical structure*? This question may be rephrased as: how do mathematicians overcome such difficulties? More precisely, how did they produce such *complex structures*? It seems difficult to answer this question in CF since this theory takes mathematical structures for granted: CS "favours (...) models that accord an essential role to the very mathematical concepts" [Vergnaud, 1990, p. 146]. We could evoke the *affective and dramatic dimension of the situation* but the discussion of Walkerdine [1988] shows that this would be just an attempt to *graft context* into CF. A new approach is needed.

In SF, that is, from the point of view of production of meaning, S1 and S2A are completely different from S2B. Meaning is produced in the *dialectics of the subject and the other* [Lacan, 1973, p. 239]. The motor of this dialectics is a certain demand of the other, before which the subject talks and acts. In S2B the demand required an immediate verbal answer: "mistakes" were promptly available as raw material for meaning production. The social agent in charge of the room hailed with a promised *yes* for an answer. Since he was recognized as an *invited lecturer*, this *yes* was perceived as stemming from a bigger legitimating cultural instance. Therefore desire and fantasy were directly melt with the *power of mathematical discourse*. On the other hand, in S1 and S2A the meaning of a winning strategy would only come in the aftermath, as an *external reference*: a medal. "The focus of their fantasy upon the external reference called up by the practice prohibits the possibility of their driving enjoyment from the power of mathematical discourse" [Walkerdine, 1988, p. 195].[5]

Distinction of the G1A and G1B problems can also be made in SF. We argue that justifying composition of additive operators by resorting to operator states, or by doing away with them, are different *modes of producing meaning*. These justifications belong to different *semantic fields* [Lins, 1994, p. 185]. However it is hard to explain the different degrees of difficulty of G1A and G1B in SF without resorting to some idea of structure.

Briefly, SF indicates the condition that a game situation should satisfy in order to be didactically effective: the implication of the mathematical discourse with the subject's fantasy

[5] This argument also explains why in S2, only upon insistence was G3B solved, and in S1 only in the work-sheet situation was G1A first considered. Non-systematic experiences with G2 indicate that children do not take very long in order to solve G2A by borrowing some money from the bank. Indeed, the demand for a solution of G2A is explicit in the rules of the game.

and object of desire. In order to explain how this condition is present in well succeeded game didactical strategies such as in Giménez [1993], we have to consider how the promised *yes* (the *transfer*, in lacanian terms) is delt with in a *socially and culturally meaningfull context*. The language paradigm is rapidly becomming a central issue in Mathematical Education [Rogerson, 1994]. Further applications of G1-G3 should take such an issue into consideration.

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Cx. P. 474
13500-970- Rio Claro SP
Brazil
Tel/Fax: 55-195-34-0123
baldino@rcb000.uesp.ansp.br

Cognitive Science and Mathematics Education: A Non-Objectivist View

Laurie D. Edwards
University of California at Santa Cruz

Rafael E. Núñez
Stanford University

The focus of this paper is on new paradigms in cognitive science, and on the relationship between these paradigms and mathematics education research. We will contrast what is generally understood as "cognitive science" with emerging alternative paradigms in the discipline. The main distinguishing feature of these new paradigms is a view of mind that does not separate cognition from biology nor from culture. We propose that this view of the mind has important implications for research and practice in mathematics education, and we offer examples from research in support of this thesis.

Introduction

The focus of this paper is on new paradigms in cognitive science, and on the relationship between these paradigms and mathematics education research. These new views of cognition entail a reconsideration of the idea that knowledge, including mathematical knowledge, is objective, abstract, rational and context-free. The most important entailment of emerging paradigms in cognitive science is a stance which holds that it is theoretically and empirically inadequate to consider knowledge independently from the knower. Although this idea is by no means a new one, either to the mathematics education community or to scholars of cognition more broadly (c.f., Cobb, 1994; Cobb, Yackel & Wood (1994), Lave & Wenger (1991), Rogoff (1990), Sfard (1994)), we believe that more can be done to delineate the implications of this concept. The issues of objectivism and non-objectivism invite a deeper reflection on basic assumptions of mainstream cognitive science, and on the implications of these assumptions for research and practice in mathematics education. We raise, for example, questions like the following: If knowledge depends on the knower, what does it mean to explain cognition as information processing, or as symbol manipulation? What can it mean to describe a learner as "possessing" certain mental representations of a mathematical concept?

Underlying these questions is an implicit presupposition that information exists prior to the knower, and that mathematics itself has an objective reality independent of the learner—mathematical objects exist "out there somewhere," waiting to be "re"-"presented" in the mind of the subject. Learning in mathematics, in this view, consists in accurately representing and manipulating the objects of mathematics.

The goal of this paper is to present an alternative non-objectivist perspective which we claim is more adequate for understanding and explaining mathematical cognition. We will describe this perspective in theoretical terms, and will illustrate it through the interpretation of research results taken from work on children's understanding of infinity and transformation geometry.

Theoretical and philosophical frameworks

One of the central goals of mathematics education research is to understand the thinking involved in doing and learning mathematics. In addressing this goal, it is important to make use of resources outside the field of mathematics education, and even outside the discipline of psychology. The field of cognitive science constitutes one such resource. Unfortunately, the term "cognition", and cognitive science in general are often understood to refer to a particular theoretical approach focused on individual reasoning, often explained in computational terms. The result of this interpretation has been that many mathematics educators, especially those concerned with social and cultural factors, have overlooked the potential contribution of cognitive science as the scientific study of knowledge.

There is a widespread belief that in explaining human cognition it is necessary to refer to mental representations and information-processing, and "to posit a level of analysis wholly separate from the biological or neurological, on the one hand, and the sociological or cultural, on the other " (Gardner, 1985, p. 6). Moreover, there is a strong belief that central to any understanding of the human mind is the idea of computation. Gardner, among others, expresses this view, when he states, "Not only are computers indispensable for carrying out studies of various sorts, but, more crucially, the computer also serves as the most viable model of how the human mind functions" (loc. cit.). Herbert Simon, identified as founder of the discipline, describes cognitive science as "the study of intelligence and intelligent systems, with particular reference to intelligent behavior as computation" (Simon & Kaplan, 1989, p. 1).

Mainstream cognitive science approaches that are consistent with this view include symbol-processing models (cognitivism; c.f. the work of Simon, Newell, Pylyshyn, Fodor) and parallel-distributed processing models (connectionism; c.f., Rumelhart, Smolensky and others). However, with Gardner, we define cognitive science more broadly, as a multidisciplinary, "empirically based effort to answer long-standing epistemological questions ..concerned with the nature of knowledge" (loc. cit.). Cognitive science is, most simply, a science whose subject is cognition; an inquiry into knowing which utilizes the scientific method. In this view, there is nothing that implies that a cognitive scientist must endorse the more restrictive, computationally oriented framework mentioned above. The central paradigms in cognitive science today, cognitivism and connectionism, thus do not define cognitive science, rather they constitute only two approaches that happen to be popular today.

The fundamental problem with these approaches is, first, that they are at heart objectivist, and second, they are focused at an individual level of analysis. By "objectivist" we mean a philosophical tradition that:

> .. assumes a fixed and determinate mind-independent reality, with arbitrary symbols that get meaning by mapping directly onto that objective reality. Reasoning is a rule-governed manipulation of these symbols that gives us objective knowledge, when it functions correctly. (Johnson, 1987, xxi-xxii)

That is, the world, including the mathematical world, is made up of objects external to and independent of the knower (c.f. White, 1956). Mainstream cognitive science paradigms for the most part take this view as foundational, and devote their efforts to building models of cognition that can account for how humans succeed and fail at representing the world, and at processing information from it. This view is very different from the foundational assumptions of several emerging paradigms in cognitive science: we

will describe briefly the central elements of a non-objectivist approach to cognitive science before turning more specifically to its relevance to mathematics education.

Non-objectivist cognitive science

Our view of cognition is based on work by non-objectivist cognitive scientists, active in several intersecting domains. These researchers have in common an approach to cognition which explicitly includes both the biological and the social. Central figures in this approach include Lakoff and Johnson from linguistics (Johnson, 1987; Lakoff, 1987), Maturana and Varela, working in theoretical biology (Maturana & Varela, 1987), and Rosch from cognitive psychology (Varela, Thompson & Rosch (1991); also summarized in Lakoff, 1987). These scientists see cognition as a biological, embodied phenomenon which is realized via a process of co-determination between the organism and the medium in which it exists. Rather than positing a passive observer taking in a pre-determined reality, a non-objectivist cognitive science holds that reality is constructed by the observer, based on culturally determined categories as well as individual bodily experience. As Lakoff states, in describing what he calls embodied concepts, "A concept is embodied when its content or other properties are motivated by bodily or social experience.. Embodiment thus provides a *nonarbitrary* link between cognition and experience" (Lakoff, 1987, p. 154).

This paradigm is non-objectivist because it rejects the necessity of an objective world that is already "cut up into" objects with pre-existing properties. Instead, as Putnam states, " 'Objects' do not exist independently of conceptual schemes. We cut up the world into objects when we introduce one or another scheme of description. (Putnam, 1981, p. 52, quoted in Sfard & Thompson, 1994, p. 14). A non-objectivist perspective allows us to consider cognitive activity beyond the level of the individual, since conceptual schemes are in large measure socially constructed, and also to acknowledge the role of biology in cognition (Johnson, 1987; Lakoff, 1987; Maturana & Varela, 1987; Varela, Thompson & Rosch; 1993) Within a non-objectivist cognitive science, the knower and the known are codetermined, as are the learner and what is learned. Thus, cognition is about enacting or bringing forth adaptive and effective behavior, not about acquiring information or representing objects in an external world.

Mathematics and mathematics education

Traditional views in the philosophy of mathematics (platonism, formalism, constructivism) consider the existence of mathematical objects (or formulas), in various degrees and forms, as being independent of human understanding (Davis and Hersh, 1981; Kitcher, 1984; White, 1956). White, discussing this issue at length, quotes Hardy as stating: "I believe that mathematical reality lies outside us, and that our function is to discover or *observe* it" (White, 1956, p. 2394) Davis and Hersh note that, although many contemporary mathematicians might disavow this statement, in their daily work, they act as if it were true: "The typical working mathematician is a Platonist on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of

this reality, he finds it easiest to pretend that he doesn't believe in it after all" (Davis & Hersh, 1981, p. 321).

Clearly, within the mathematics education community, such a view no longer holds great currency. Research carried out with children in classrooms or homes or in the streets cannot pretend that mathematical knowledge is an abstract, disembodied, asocial entity . However, we believe that in many cases, elements consistent with mainstream cognitive science persist even within research that acknowledges the socially-constructed nature of mathematics. These often subtle ways of thinking and talking about cognition can be characterized as a kind of residual objectivism.

As a generic example, a piece of research may examine in detail the social interactions, discourse practices, or cultural factors involved in students' learning within a particular area of mathematics, be it fractions or functions. But, when discussing what the student knows *as an individual*, in describing what the learner takes away from the social situation, the framework often shifts, and mathematical knowledge is described in terms of mental representations and the storage/retrieval/processing of information. For example, researchers talk of students "acquiring" a concept, or "building an internal representation" of a mathematical entity. This view of mathematical knowledge is objectivist, in that it presupposes the very existence of an independent mathematical reality which contains objects to be represented and information to be processed.

Furthermore, this view may reflect an objectivist stance toward mental constructions themselves. Instead of using the terms *mental representations* or *mathematical objects* as theoretical tools to further understanding, which can be more or less adequate or useful (see Sfard's comments on mathematical objects in Sfard & Thompson, 1994), in some cases the researchers' goal seems to be to discover and describe mental structures which are presumed to actually exist. Statements like the following demonstrate this objectivism concerning mental representations: "I am deeply supportive of perspectives that challenge the presumed connection between our introspections about our knowledge and the *actual underlying representations* " (emphasis added; Kirshner, quoted in Sfard & Thompson, 1994, p. 5).

It may very well be that within mathematics education, we find the equivalent of Davis and Hersh's weekday Platonists and Sunday formalists. We may declare that the mathematical or conceptual objects we talk about are only theoretical constructs, with no existence independent of the theorizing individual or community, yet find it easier to utilize the objectivist vocabulary of mainstream cognitive science when describing the results of our research into mathematical learning.

<u>Thesis</u>

Returning to the issues with which we opened this discussion, we propose the following thesis: Objectivism is not a necessary element in cognitive science or in mathematics education research. Cognitive science does not have to consider either knowledge or its components as existing objectively (as is done in the information-processing framework and in discussions of mental representations). Furthermore, we claim that non-objectivist cognitive science of a particular kind is more useful and adequate than objectivist cognitive science for mathematics education research and practice. In support of

this thesis, we will present specific examples from each our research programs, examples in which a non-objectivist interpretation is applied to selected findings.

Examples from research

Children's understandings of infinity

Our first example comes from a study of children's ideas about infinity carried out by the second author (Núñez, 1993a, 1993b, 1994). The research methodology was the clinical interview, focused on Zeno's paradox. The researcher presented the paradox to children from 7 to 14 years of age, using a statement like the following: "Imagine that we are asked to go from a point A to a point B, but we are told to do so by following a rule which says: first go half of the way, then half of what remains, then half of what remains, and so on. Do we ever reach point B?" Zeno's paradox arises because, if this rule is followed, then we never reach point B, yet this is contrary to our experience, in which a journey always reaches a destination.

In a series of experiments described elsewhere (Núñez, 1993b, 1994), the author found a developmental, age-related difference in the responses to the paradox. Younger children often altered the conditions of the situation such that the problem would be solved. For example, Ban (who is 7 years, 9 months old), eventually resolved the paradox by changing the destination of the "journey:"

> Ban +(7;9): We get closer .. but if we go half of the way there is just a step left, and if we go the (other) half, there is just one half left, .. and we go that half and we arrive .. (thinks and re-analyzes with his fingers near the edge of the table),. *or maybe there is a small piece left before the arrival and we decide that that's the place where we have to stop.* ..We put the arrival before the place where we want to go, so then we are sure that we arrive.

The opinions of students older than 10 or 11 were very labile and counterintuitive. They seemed to realize the difficulty of resolving the paradox. Their answers were full of doubts and hesitations:

> Mal -(14;8): Does it arrive? (thinks, whispers) .. I think we don't arrive exactly, maybe about a millimeter away. I think that we will arrive, but that we are not going to arrive immediately, .. if we have every time, .. well, I don't know, I think that it will arrive exactly at the point, and I think that it will not arrive exactly. I have two opinions. I don't know whether it will be exactly or at about a millimeter away.

According to our theoretical stance, mathematics is a language shaped historically and based on consensus. In Zeno's paradox, agreeing to respect the conditions of the problem (that is, not changing the "givens" of the situation) takes place within this consensus. The consensus in turn, depends, in part, on the biological structure of the subjects (e.g., their level of neurobiological development). This issue is essential because the conceptual world that emerges from the cognitive activity of young children is based on a consensus which is different from ours (because their neurobiology, their language, their embodied cognition is different). Thus, what we call rigor in mathematics has a different meaning for students of different ages, and younger students take a different attitude than older ones about the possibility of changing the conditions of Zeno's paradox.

The children's unnoticed alteration of the conditions of the paradox takes place in a different domain of consensus which we happen to not share. As a further example, we have found that before age 12, some students show a striking ontological difference between two "kinds" of infinity. A 10–year–old, Yak, answered a question about his use of the terms "bigger" and "smaller" infinities:

> RN: If that is the "bigger" infinity, as you say, when you say "smaller" infinity, is it the same thing but in the other sense?
> Y: Yes, and there is only one difference. At a certain moment it becomes so small that we can not even know where it is.
> RN: So what is the difference then, between the "bigger" infinity and the "smaller" infinity?
> Y: *At a certain moment, when we are in the smaller infinity, it stops, whereas in the bigger infinity it could continue until .. infinity*

In other words, there are endless infinities and infinities which stop, both being infinities! The idea of an infinity which stops contradicts our very notion of infinity, as mathematically-experienced adults. Children's notions about infinity also change, depending on individual characteristics and on the context provided for the paradox. Ten-year–old high performers, and 12- and 14-year–old average students gave different answers with different arguments to isomorphic situations in which the context had been changed (e.g., the distance to be covered in the journey).

Do these alternative notions represent misconceptions about what infinity means? We don't think so. They are part of a shaping of consensual space based on bodily grounded experiences, just as the idea of "indivisibles" of the plane were for Archimedes or Cavalieri.

"Conceptual bugs" in transformation geometry

The next example is taken from a qualitative research study by the second author, and focused on the learning of twelve middle-school students who spent about 7 hours working in pairs with a computer microworld for transformation geometry (Edwards, 1990,1991). This microworld was made up of Logo procedures corresponding to rigid transformations of the plane (translation, reflection and rotation), and included a game in which the transformations were used to superimpose two congruent shapes.

One research result concerned certain expectations the students demonstrated about the rotation transformation. The rotation command takes three inputs, for example "ROTATE 10 20 45." The first two inputs specify a fixed point about which the entire plane rotates (in this case, the point (10, 20), and the third input specifies the amount of the rotation (here, 45 degrees clockwise). About one third of the students who used either a pilot or final version of the microworld initially expected the rotation command to work differently than it actually did. Instead of envisioning a whole plane rotation, these students thought that the ROTATE command would first move the shape *to* the center point, and then turn it (Edwards, 1991).

The students quickly realized that this was not how the microworld functioned, and were able to adapt and effectively use the whole-plane version of rotation as implemented on the computer. What is of interest to our thesis, however, is not the actual performance of the students with the microworld, but rather the interpretations that are possible concerning the students' initial ideas about rotation.

One possible interpretation, which the author made when first analyzing the data, is fundamentally objectivist. Under this view, the students had a "misconception" about the nature of rotation. The mathematically correct version of rotation, involving a single movement or mapping of the whole plane, was instantiated in the computer microworld. Any alternative notions that students held about rotations were simply incorrect, based on a lack of understanding of rotation as a mathematical entity. This misunderstanding was referred to as a "conceptual bug" by the author in early written analyses (e.g., Edwards, 1990). An objectivist interpretation suggests that we can "correct" such misconceptions simply by providing accurate representations of mathematics as it actually exists.

Under a non-objectivist interpretation, however, we would not describe the students' initial conceptualizations of rotation as "misconceptions" (cf., Smith, diSessa & Roschelle, 1993/4). Instead, the students' understanding of rotation would be seen as an appropriate, adaptive expectation arising from their own embodied experience in the world. In the everyday world, when something turns, it almost always turns around a center point which is *part* of the object itself. When a child turns around, she turns around the central axis of her own body; if she plays with wheels or tops or other turning objects, these objects turn around their own centers. It is rare to experience an object that turns at a distance, from a center point in an arbitrary location. Yet this is the kind of rotation which is instantiated in the microworld (Edwards & Zazkis, 1993).

The students' initial understanding of rotation differs from the more general, whole-plane notion held by the mathematician. Some might argue that this is simply a matter of convention, of agreeing upon a way to describe rotations (that is, mathematicians may choose to describe rotation as a whole-plane transformation, but they could just as well adopt the students' convention of first sliding and then turning). But this assumes a single mathematical entity, rotation, which each community describes differently—an objectivist view. Under a non-objectivist view, each understanding of rotation is completely different, since each derives from a different set of experiences, from a different practice. Each is couched in a vocabulary that the knower has found most useful; furthermore, in the mathematics community at least, this vocabulary is shared and formal.

In order to modify the students' initial, perfectly adaptive understanding of rotation (or any other mathematical notion), it is not enough to provide the "mathematically-correct" version. Instead, we must create situations in which the new mathematics more adaptive than the old, and provide social support for students as they learn a new way of talking about what they are doing. Computer microworlds can provide problem-solving situations and games that require the use of new mathematical objects and operations. Teaching with an emphasis on communication can provide the necessary social support.

Discussion:

From a non-objectivist position, mathematics is conceived as totally dependent on human beings— it emerges in the interaction of biological beings as they evolve within and adapt to their medium, and is therefore contingent on the very nature of the interwoven process of embodied concepts and social interactions. This view rejects an objectivism which reifies either mathematics itself or the constructs we create to help us tell the story of mathematical learning. We believe that research and practice in

mathematics education cannot help but be enriched by drawing from the new perspectives provided by non-objectivist approaches to cognitive science, which include an increased and more sensitive emphasis on the bodily and social bases of thinking, learning and knowing.

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OVERCOMING LIMITS OF SOFTWARE TOOLS: A STUDENT'S SOLUTION FOR A PROBLEM INVOLVING TRANSFORMATION OF FUNCTIONS[1]

MARCELO C. BORBA
Graduate Program of Mathematics Education
Mathematics Department
UNESP-Rio Claro
(State University of Sao Paulo at Rio Claro, Brazil)

ABSTRACT

This paper describes how a student, confronted with discrepant results while using a multi-representational software program, solves this discrepancy by mentally adding features to the software design. The discrepancy occurred while the student was working with transformations of functions. A model for understanding in multi-representational environments is refined in order to incorporate the data discussed in this paper. Students' reasoning can inspire changes in software design.

INTRODUCTION

In recent years much emphasis has been given to the role of software tools as a mediator of students' knowledge and as a way of enhancing students' learning (Borba & Confrey, 1992; Confrey, 1994; Beare, 1994; , Edwards, 1994; Kieran & Hillel, 1990; Monaghan et al. 1994, Nemirovsky, 1994; Noss et al, 1994; Schwarz, 1994). On the other hand, not very much has been said about how students have been "substantially" changing the design of software in order to adjust it to the mathematical ideas they are engaged in. In this paper, a case study is presented in which a student overcomes the limits of a software design in order to solve a problem involving transformations of functions. The importance of this kind of student input for software development and for mathematics education in general will be discussed.

THE RESEARCH

This particular research was part of a larger project[2] on the teaching and learning of functions through contextual problems and transformations of functions using a multi-representational software. The overall pedagogical framework (Borba, 1993; Confrey & Smith, 1991; Confrey, 1994) includes a critique of the dominant role of algebra in mathematics education. Transformations of functions, in particular, are usually dealt as effects on the graph resulting from changes in the coefficients of an algebraic expression. Borba (1993) have argued that this is just one approach among several others, especially with the facility provided by multi-representational software, such as Function Probe

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[2] The research reported in this paper was developed as a part of more encompassing research developed by the Mathematics Education Research Group at Cornell University directed by Dr. Jere Confrey.

(Confrey, 1991). It had been previously discussed how Doug, a high school student, have dealt with transformations of functions in our research (Borba & Confrey, 1992; Borba, 1994). In this paper it will be discussed how Ron, another 16 year old high school student from Ithaca, New York, U.S.A., dealt with this theme. Ron's performance can also be described by the model for students' understanding in a multiple representation environment (Borba, 1993, 1994), a model in which understanding is seen as a process in which findings in one representation are justified based on arguments developed on another representation.

Function Probe (FP) is a multi-representational software which allows, among other features, a graph to be transformed into another through direct actions on the graph using translation, stretch or reflection icons. FP allows both the reflection line and anchor line for the stretch to be moved. Therefore, a graph can be reflected on or stretched from any line parallel to the y or x axis. Points can be sampled from a graph allowing a transformation in a continuous graph to be studied as actions on discrete points. These samples can be sent to the table where they will be stored in "x" and "y" columns. Any pair of columns can be sent back to the graph window as points and columns of numbers can be altered in the table window. In this study, three students were interviewed in a teaching experiment fashion (Cobb & Steffe, 1983) over approximately eight two-hour sessions. Students were introduced to the features of the software through three tutorials and then given tasks involving absolute value, quadratic, and step functions.

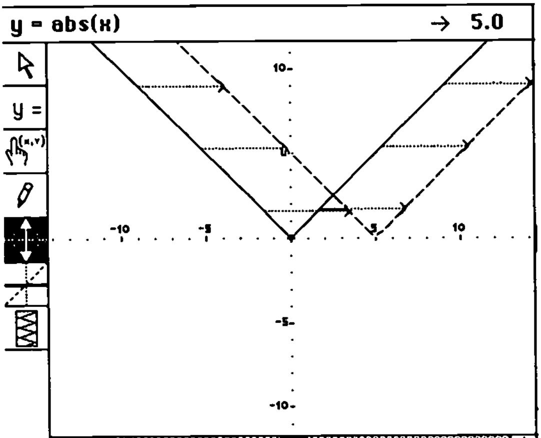
The tasks were based on a model which was an alternative to the algebra-dominated approach. In the first part of the model, the focus was entirely visual. Using the transformations icons, subjects attempted to transform graph A into graph B. In the second part, a numeric component was added as students were asked to predict what would happen to the coordinate values of a given point of a graph when the graph was transformed. Finally, students were asked to explore the relationships between actions in the graph and coefficients of algebraic expressions. The interviewer was the author of this paper.

RESULTS

The Problematic

Ron proceeded through the visual part of the model quickly and with few obstacles. He was able to predict and explain the relationship between the visual and the numeric values of the points easily and without hesitation. As he sought to create algebraic descriptions of the transformations, he generated what was labeled "covariational equations" (Borba, 1993, 1994). These equations express in algebraic terms a transformation in the table world. For instance, in Fig. 1 we can see a horizontal translation to the right by 5 units which can be expressed by the covariational equation $x' = x + 5$ and $y' = y$ where (x', y') are the coordinates of the translated set of points and (x, y)

are the original set of points. These covariational equations express a type of algebra which was labeled "table algebra" (Borba, 1993) due to its close connection to the table micro-world.



x	y	x'=x+5	y'=y
-5.0	5.0	0.0	5.0
-4.0	4.0	1.0	4.0
-3.0	3.0	2.0	3.0
-2.0	2.0	3.0	2.0
-1.0	1.0	4.0	1.0
0.0	0.0	5.0	0.0
1.0	1.0	6.0	1.0
2.0	2.0	7.0	2.0
3.0	3.0	8.0	3.0
4.0	4.0	9.0	4.0
5.0	5.0	10.0	5.0

Fig. 1. On the top part of this figure the graph is being horizontally translated by 5 units. Points have been sampled (not shown in the figure) and sent to the table window on the bottom part of the figure. Also shown on the bottom part are points of the corresponding covariational equations.

As Ron was involved in the third part of the teaching experiment - studying the relationship between graphical transformations and changes in the coefficients - he ran into a problem similar to one which often happens in the classroom when computers are not used. Ron used the iconic facilities of FP - which allow him to drag a graph - to horizontally translate by 5 units the graph of $y=x^2+3x+5$. Having the model of $y=ax^2+bx+c$ in mind, Ron could notice, looking at the screen, that in the translated graph $c=15$. Despite his finding, he decided to check it, using paper and pencil and his previous

results from the analysis of covariational equations. He claimed: " . . .*I know an easy way to find out what c will be . . . Just substitute . . .* do the formula that I did [the covariational equation discussed before]. y. . .so x, x^2 + . . . I think this will work . . . 3x + 5, and the thing . . . transformation was to the right by . . . It's $(x+5)^2 + 3(x+5) + 5$.". He wrote down what he thought was going to be an algebraic formula for a horizontal translation by 5 to the right: "(x,x^2+3x+5) --> $[(x,]\ (x+5)^2 + 3(x+5) + 5$" . Next he developed the above expression until he reached "$(x+5, x^2+13x+45)$". Ron was then confronted with a discrepancy in the results: he had found using the icons that a horizontal translation by 5 to the right would make c= 15 and now, using his algebraic algorithm, he predicted c=45.

To make things more complicated, the feature of FP which allows the equation of the transformed function to be shown was turned on and showed y = (x-5)^2 + 3(x-5) + 5. Besides the discrepancy of values for "c", Ron now had a new problem: how to explain that a horizontal translation to the right resulted in a change in a "minus sign" in the FP display. Moreover, as he put y = (x-5)^2 + 3(x-5) + 5 in the $y = ax^2+bx+c$ format, he was reassured that c=15.

The roots of Ron's latest puzzle can be found in his investigation in the table world. As he was investigating the relationship between transformations in the graph and changes in the coordinates of transformed points, he generated covariational equations in which there was no contradiction between the "plus sign" and a horizontal translation to the right, since x'=x+5, y'=y could express such a transformation.

In trying to solve this problem, Ron developed three solutions: a visual one, an algebraic one and a visual-algebraic one in which he "transformed" FP. In this paper, the focus will be on the last solution proposed by him[3]. In the first two solutions, Ron found explanations for the discrepancy between the different values he had found for "c" but he could not find explanation for the discrepancy between the movement to the right and the minus sign. He in one instance found an explanation for the movement to the right and on another find reason for the minus sign, but there was no explanation yet for a coordination of both events.

The Visual-algebraic solution: changing Function Probe

On the first two solutions for this problem, Ron was still working with the equation which was originally involved in this problem: $y = x^2 + 3x + 5$. When he started working on the third solution, he was already working with equations in a different format: y=Af(Bx+C)+D. This format was suggested by the interviewer since it keeps separate the vertical transformations from the horizontal transformations and, although it was suggested from the beginning, Ron started working with it only at this point of the teaching experiment.

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[3]See Borba (1993) and Borba & Confrey (1993) for a discussion of the first two solutions.

After some preliminary investigation with the model $y = Af(Bx+C)+D$, Ron was focusing on the understanding of C, working with the family of functions $y = (x+C)^2$ and keeping
A=B=1 and D=0. In this investigation, he had the first breakthrough of what would become his new solution to an old discrepancy:

> Wait, wait, I just want to change it back to $+5^2$ [$(x+5)^2$]. That's all. So now *if you imagine this x-axis shifting over completely by 5.* Shifted +5, okay? . . . Now, and change all the values back to what they should be okay? And then over here at -5 is going to be where the y-axis is going to be -5 is going to be, see what I'm trying to say?

Ron was building an argument in which he was seeing a horizontal translation as a result of an action on the x-axis instead of seeing it as a shift of the graph. As the interviewer realized the nature of Ron's argument, he asked Ron if that could explain why, when C had a "+5 change", the graph would move to the left, bringing back the original problematic in a new scenario. Ron's response was quite telling: "Um. . . yes. If you do . . . if you think about it right. Let me think about it. Yeah. See if you pull this [the x-axis] to the right, -5 is getting to end up at the zero, and in the distance will be 0 at -5, see?." Ron's articulation suggests that he was understanding "+5" as a movement of the x-axis to the right. The x-axis would be pulled back to the right because "[b]y *being a function, it pushes it all back the way it was supposed to be* . . . so then this whole thing slides back and then it's plotted where it is now . . . ".

A way to interpret Ron's reasoning is by thinking of a metaphor of "double rubber sheets" proposed by Borba (1993) as a refinement to the "rubber sheet metaphor" (Goldenberg & Kliman, 1990). In the double rubber sheet metaphor there is a back rubber sheet with the Cartesian axes and a front rubber sheet with the curve on it. Previously, I had chosen to think of transformations as impacting on the front rubber sheet while the back rubber sheet remained permanent. The same kind of reasoning is embedded in the design of Function Probe. Ron, however, was thinking of the horizontal translation as being an action on the back rubber (x-axis) sheet, followed by an adjustment which would push both rubber sheets so that the origin would be back to its original position. The "original position" can be understood by thinking of an observer who is standing still looking at the origin before the graph has been transformed. In this sense, Ron could coordinate his action with the plus or minus sign in the equation written in $y = f(x)$ form and with the final "visual" result of the graph.

After this point, Ron used the double rubber sheet metaphor to extend his idea to stretches and also to all the coefficients of the model $y=Af(Bx+C)+D$ as shown in the next two excerpts:

> B and C. . . Anything inside the function, stretches and pulls and shrinks it this way or moves it this way [horizontally]. Anything outside the function stretches or pulls it this way [vertically], so now only one rubber sheet changes . . .

. . .

Because B and C [in y=Af(Bx+C)+D] are inside the function. That means you haven't done the function yet, so you're still working on the x-axis. When you're outside the function, you've already done [with] the function and you're working on the y-axis now - directly with the y-axis.

In the first excerpt, Ron tries to explain how he visualizes his model: he noted that he could see B and C as connected to action on the x-axis, because they are "inside the function", which can be understood as the coefficients which "act" on the variable "x" before prototypic functions (Confrey & Smith, 1991) such as $y=x^2$, $y=|x|$ and $y=[x]$ are "applied" to the variable; in contrast, anything outside the function, A and D, are going to be in the front rubber sheet where the parabola or the line is. Based on this argument, he concludes that the coefficients "inside the function" are linked to horizontal actions, whereas the other ones are connected to vertical actions, as he explains in the second excerpt. Ron's conclusion is coherent with the model y=Af(Bx+C)+D, which separates vertical actions from horizontal actions.

Ron added thread to his metaphor to represent any curve that was being graphed, made the front rubber sheet a "sticky rubber sheet" to glue the thread, and used "pins" to attach the two rubber sheets together when necessary and so on. He ran into some trouble to adjust what he was visualizing and the metaphor he was using to express what he had in mind and the coefficients. But after some time of interaction with the interviewer and with the ideas embedded in the design of FP, Ron came to what he called an "overall generalization":

R: Here's what you do, okay? Here's the overall generalization. You have 2 rubber sheets, one of them is your graph. The clear one is your ... the other one is your function.
I: Yeah, the one on the bottom is the axes.
R: Is the axes.
I: Okay. . .
R: So anything that is inside the parameters of the functions means you change the graph before you do the function. That means that you're . . . that means that B and C are a stretch and a contraction [he probably got confused and said contraction instead of translation] before you do your function, okay?
I: So you're going to do that on the graph or the axis?
R: On the graph. Or not on . . . on the axes, not on the function, not on the function.
. . .
R: . . . Then when everything is all stretched, you pin it down and you slap our function on top of it, then let everything bounce back the way it should be. That means that when you shift it to the right 5, plop it on, when you bounce it back, it will move it left 5, okay? If you look at your function, shift this over, plop it on, and move the whole thing . . .
. . .

R: Plop that on, move it over, okay? Now when you change A and D, what happens is you take your clear one, keep this where it is and you stretch your clear one up this way. Then, when your clear one's all stretched, you plop it on.

Ron was able to expand the notion of change in the axes, incorporating vertical and horizontal translations and stretches. Moreover, he constructed his version of the double sheet metaphor, improving it, and was able to coordinate the numerical changes in the coefficients with the transformations on the graph.

DISCUSSION

This case study contributes to a growing literature (e.g. Proceedings of PME-94, PME-92) suggesting that working in computer environments can be something quite different than working in other environments. Constant and fast feed-back, easy use of several representations at the same time and flexibility in students inquiry have been some of the features listed to support computer use in mathematics education. On the other hand, most classrooms are not equipped with computers; and it can be argued that problems solved with the help of computers are not relevant for the general classroom.

This potential contradiction does not occur in the case discussed in this paper since the problem Ron dealt with - coordinating a horizontal translation of a graph with the changes in the coefficients - takes place in classrooms without computers, too. In addition, although the three solutions presented by him were mediated by FP, the ideas developed by Ron can be implemented by students and teachers who have no access to computers. The fact that Ron's solution can have impact on every classroom adds importance to his findings.

The literature has emphasized either the role of the software in shaping the thought of the student or, often times, the construction by the student as if it had happened without the strong support of the design embedded in every computer software. I have proposed the idea of an "intershaping relationship" (Borba, 1993) as a way of giving equal emphasis to the influence of a software design on a student and the role of the student in not only appropriating this design but actually shaping or mentally enhancing a software as Ron did in the case discussed in this paper.

Ron's solution to the problem can be framed by the model of students' understanding discussed in Borba (1994), since the problem emerged as a discrepancy between results obtained in different representations, and he found in the graph representation an explanation for an algebra-related problem. It should be noted however, that Ron's solution demands a refinement of the model, since in order to coordinate the "minus sign" with the horizontal translation to the right, he had to mentally change the structure of the graph representation of FP. He had to build his own notion of translation, overcoming the limits of FP, which is a very rich software with powerful transformations icons, but which cannot allow for transformations in the axis in a dynamic manner. The

model (Borba, 1994) should emphasize, therefore, the role of students' changes in multi-representational software as they justify a result found in another representation. Close attention to students' thoughts, such as Ron's could also result in relevant suggestion for future software design

Finally it should also be noted that the pedagogical approach of teaching functions through transformations, exemplified by the original way Ron approached this problem, emphasizes the notion of family of functions and the focus is on the coefficients and not on the variables. Such an approach combined with others which focused on the notion of variable could help students to reach the function sense proposed by Eisenberg & Dreyfus (1991).

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Graphs that Go Backwards

Tracy Noble and Ricardo Nemirovsky[1]
TERC, Cambridge, MA, USA

This paper reports on results of interviews with high school students using a motion detector to create graphs of velocity vs. position. We are interested in the possibilities offered by non-temporal and non-functional graphs, such as velocity vs. position graphs, for exploring graphing and motion from a different perspective. In the transcript and analysis of selected Episodes of an interview with the student Noam, we trace the development of his ideas about the relation between the shape of the graph and the path of his motion with the car, as well as his use of language and gestures in explaining the responsiveness of the graph to his actions.

Introduction

Much of the literature on students' use of motion sensors to create graphs has focused on graphs of some quantity, such as distance or velocity, over time (Thornton and Sokoloff, 1989; Nemirovsky and Monk, 1994). Temporal graphs (of some quantity vs. time) have a common logic: the graph continually progresses forward, just like time. Doubling back is not allowed. Thus, temporal graphs are all graphs of functions: for a given time (or x-value), there is only one corresponding distance or velocity value (y-value) on the graph. Given this relationship to functions, graphs of some quantity, such as distance, vs. time are useful for exploring functions; for instance, linear functions can be explored by graphing distance vs. time for motion at a constant speed.

Given the value of making temporal graphs in learning about functions and motion, why would one ever choose to venture outside this realm? There are, of course, many other types of interesting graphs to explore with students (Janvier, 1978; Bell, Brekke and Swann, 1987a, b, &c). In dynamical systems modeling, for instance, graphs of velocity vs. position, which are examples of phase space graphs, can be very helpful for understanding a system's behavior (Janvier, 1978; Nemirovsky, 1993; Tufillaro, Abbott, & Reilly, 1992). In phase space, the state of a system is well-defined by a single point in the space, and the space is often many-dimensional. However, for a single, solid object whose state is defined only by its position and velocity, the object's phase space is its velocity vs. position graph. These graphs can often elucidate different features of the behavior of a system than temporal graphs do. For example, in an oscillatory system, such as a spring undergoing simple harmonic motion, the trajectory in velocity vs. position space will trace out an ellipse or a circle, and will keep re-tracing the same path until the motion slows down and the trajectory spirals in towards the center of the ellipse. Alternatively, a graph of position vs. time for the same system would be sinusoidal, and the damping would be visible as a decreasing amplitude of the sinusoidal curve. Given the alternative way of looking at these and other dynamical systems that velocity vs. position graphs offer, we conjectured that students may also find valuable insights into motion and graphing through work with these graphs.

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This paper will follow the work of a high school student named Noam as he tries to understand the velocity vs. position graphs he produces by moving a hand-held toy car in front of a motion detector. Noam begins his work with the graphs by describing the graph as if it corresponds to a trail left by the car, or a trace of the car's motion. During the course of the episodes, Noam slowly constructs a sense of the meaning of the graph that is more like that used by experienced graph-users. Some authors have described students' initial ways of thinking about graphs, some of which closely resemble Noam's, as misconceptions (for instance, the "graph-as-picture" misconception) which must be removed and replaced with correct conceptions (for a brief review, see Clement, 1989). We argue, instead, along with Smith, diSessa, & Roschelle (1993), that students' processes of learning often can be better described as a process of adapting and refining initial ways of thinking as they experience the responsiveness of the graph to their actions (Monk and Nemirovsky, 1994; Nemirovsky, 1994). We suggest that as Noam learns more and more about the responsiveness of the graph in the course of the Episodes described herein, he does not abandon his initial way of thinking about the graph and adopt some new way. Instead, Noam subtly changes and refines his thinking, eventually coming to think of the graph in a way that would not only be considered "correct" by experienced graph users, but that also retains many features of his initial thinking that turn out to be powerful for understanding velocity vs. position space graphs.

Other themes that arise from the analysis of the transcript described in this paper are Noam's use of gestures and his use of language. Noam's gestures provide a way for him to talk simultaneously about the car's motion and the resulting graph, with the use of a single gesture. We have also analyzed Noam's use of language, tracing his use of the word "straight" throughout these Episodes. We have found that his use of the word "straight" does not follow some linear progression from "incorrect" usage to a more mathematically "correct" usage. Instead, Noam uses the word "straight" flexibly, to describe a number of different concepts; it becomes a tool for him to use to explore new ideas, not just the word for an non-curved line.

Methodology

In this study, five students were interviewed for five hour-long sessions each (except the first student, who was interviewed three times) using individual teaching experiments (Cobb and Steffe, 1983). The interviewer posed some pre-determined problems to the students, but also aided the students in exploring questions of their own whenever possible. The videotape of each interview was analyzed in order to plan for the next interview.

The first interview with each student began with playing a game in which the student and the interviewer, Tracy Noble, made representations on paper of the toy car's motion on the table for the other person to use to act out the intended motion. At some point during each student's first or second interview, Tracy and the student began using the motion detector to make graphs of the car's motion, beginning always with velocity vs. position graphs, and sometimes moving on to temporal graphs as well. The motion detector senses the distance away of the nearest object in its path, and the software (MacMotion™) can use this information, gathered over time, to compute the velocity of the object as well.

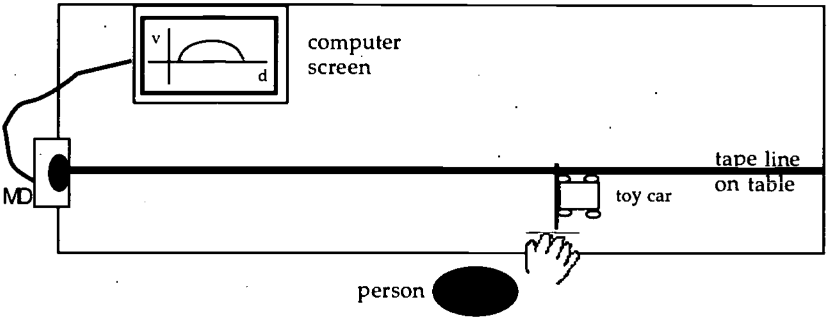


Figure 1

Before beginning work with the motion detector, the students were only given the minimal information that they needed to begin making graphs: that is, they were told that the small plastic box will detect their motion, and that they should move in front of it to make a graph on the computer screen. Some might argue that time might have been saved in this process by simply "explaining" to Noam the meaning of the velocity and position axes ahead of time, and that all of the work he did in these episodes was unnecessary. However, the graphs on the computer screen all had axes labels, unit labels, and scales on the axes, so that if that information is relevant to the student, he can read it off the computer screen. We would argue that, instead, that no explanation can replace personal exploration of the tool and the meanings of its responsiveness to the environment.

Noam

At the time of this interview, Noam was a high school senior at a Boston-area vocational and technical high school that is connected to a traditional high school. Noam had been primarily in the vocational track, which meant that for his senior year he spent every other week working in a graphic arts shop, and the intervening weeks in academic classes. Noam had taken two years of math: arithmetic and a pre-algebra/algebra class, and one year of general science at the time of the interview. He was at the time of the interview enrolled in a physics class.

Transcript

The following Episodes occur about halfway into Noam's first interview, when Tracy introduces Noam to the motion detector for the first time. The four Episodes last a total of about 10 minutes, and they occur in immediate succession. [Notation:"..." indicates the voice trailing off, and ". . . " indicates deleted transcript.]

Episode 1-- What is the Graph?

Episode 1 begins when Tracy introduces the motion detector to Noam for the first time, and asks him to move the toy car in front of it. He had not worked with a motion detector before.

Noam moves the car at a slow, relatively constant speed, along the path below:

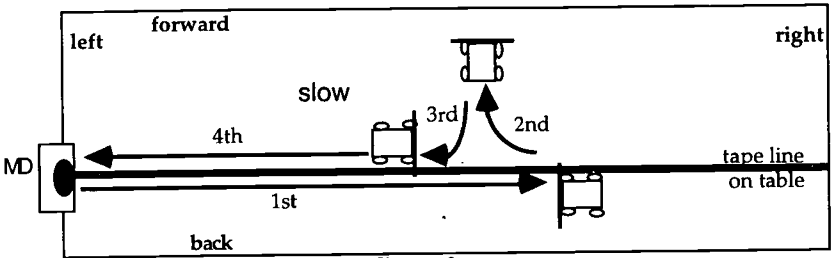
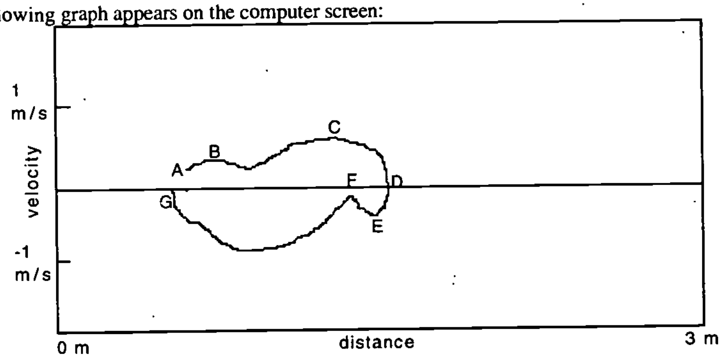


Figure 2

The following graph appears on the computer screen:



Graph 1

Tracy asks Noam to "walk her through" the graph.

> *Noam: Okay. (pause) Where did I start [moving the mouse on Graph 1 between A and D]? I started from this [A] It was - I started from here [A]? [Tracy: I think you started from here [A]. . . .] How come it's [Graph 1] not going in a straight line [runs finger across Graph 1 in a horizontal line]...[points to table on which he had moved the car] like I did? [Tracy: In a straight line like you did...] Okay. Either way [dismissing the topic of the straightness]. Okay. I started coming here [running the cursor from B to C along Graph 1]. It seems like I turned...[running cursor downward from C to E on Graph 1],backed up [moving cursor up to F],and started going over here [moving cursor from F back along the x-axis to G].*

Tracy asks Noam about what he had expected, and he explains why he had expected the graph to have a straight line on top by acting out the motion he believes he made with the car. He has been influenced by the correspondence he sees between the graph and the motion, and as a result forward and back are flipped from the original (Imagine flipping Figure 2 over the tape line).

Notes on Episode 1: In trying to make a correspondence between the graph and the motion he made, Noam asks why the graph is "not going in a straight line like I did," because for motion left to right along a straight path, Noam expects a straight line on the graph, not the bumpy one he sees from A to C [Graph 1]. Noam uses "straight" here to refer to the quality he notices in his path with the car: its lack of curvature. This is the first example we see of Noam expecting the graph on the computer screen to look like the path he took with the car. Another example occurs when Noam analyzes his turning motions, associating the 2nd and 3rd motions in Figure 2, with the curves C to E, and E to F,

respectively (on Graph 1): "It seems like I turned...[running cursor downward from C to E on Graph 1] backed up [moving cursor up to F]." Noam may associate these portions of the graph with turns because these parts of the graph have a curved shape (See Graph 1), or because these lines move vertically on the graph, which he may connect with forward and back motion on the table (See Figure 2). It appears that Noam expects a direct correspondence between the path taken by the car and the graph produced, almost as if the graph were a trail left by the car.

Episode 2 -- Stopping and the Graph

Immediately after Episode 1, Tracy suggests that Noam make sure the card at the front of the car is always facing the motion detector, so that the motion detector can read the position of the car. Noam moves the car as shown below:

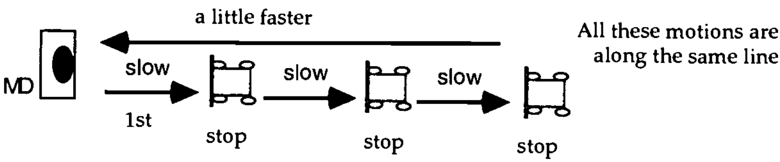
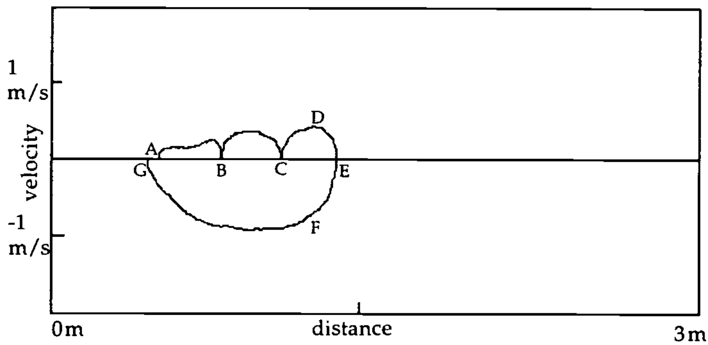


Figure 3

This produces the following graph:



Graph 2

Noam talks through this graph, saying he started at A, and stopped at B, C, and E. Then Tracy asks him how he can tell from the graph which places he stopped at, and Noam responds below:.

> *Noam: This here. [points at Graph 2, at B or C] [Tracy: That there?] Indicates stop [gestures a downward curve with his hand, like the graph shape at B or C:], because it [graph near B or C] doesn't* go *like that one [points at Graph 2, from E to G, and runs his finger from Left to Right to Left across the curve]. [Tracy: Okay.] It's [graph near B or C] not a straight line [gestures along a straight line in the air with his hand]. Even though that one's [from E to G] not straight but...*

Notes on Episode 2: Noam's description of Graph 2 includes a new feature: stopping, and Noam has associated stopping with the area of the graph near points B, C, and E. Given the trip with the car, which did not involve turning, the places where the graph curved downward to the x-axis took on a

new meaning for Noam: "[graph near B or C] doesn't *go* like that one [points at Graph 2, from E to G]." Thus, there is a sense in which the graph shape from E to G (Graph 2) can "go," as opposed to the graph shapes at B and C. This is closely related to the experience of drawing this graph (Graph 2), or following the line with one's finger. Tracing the graph near the points B and C requires an actual stop of the finger, so that line "doesn't go," whereas the stretch from E to G can be drawn in one continuous stroke, so that this line does "go".

Another feature of this Episode is that Noam's gestures link the graph he has made (Graph 2) with the act of drawing or tracing the graph. When Noam says that the graph at point B or C "Indicates stop [gestures a downward curve with his hand, like the graph shape at B or C:]," his gesture indicates both the shape of the downward curve of the graph, and the abrupt stop of the car on the table that produced the shape. In this Episode, we also see an example of Noam's changing use of the word "straight." Noam uses the word "straight" to describes the difference, on Graph 2, between the graph near B and C and the graph between E and G when he says that the graph near B or C is "not a straight line . . . Even though that one's [from E to G] not straight but...". Noam is apparently now using "straight" to describe his sense that the graph at B or C is interrupted, not smooth, as opposed to the uninterrupted and smooth graph from E to G. Even though this use of the word is unconventional, when Noam stretches its usage, he is able to talk about this smooth quality of lines, which he may have had no other specific language to describe.

Episode 3 -- A Stop or a Turn?

Immediately after Episode 2, Tracy and Noam are discussing Graph 2 (See Graph 2 in Episode 2) when Noam describes a rule he believes: when any object moves right and then left, it must stop in between those two motions. He illustrates this point with gestures which decompose forward and back motion into pieces. Then he describes how Graph 2 relates to this rule:

Noam: But that [Graph 2] doesn't show that [the stop between left and right motion]. It just shows here [running hand horizontally along computer screen], turning around [runs hand along downward curve on computer screen, curving to the left, then continues moving to the left, horizontally, into space]. . .

Tracy asks Noam to clarify what he sees as the difference between points B and E on Graph 2, and he answers as follows:

Noam: This one [Graph 2, B] stops and starts - starts all over again [traces part of Graph 2 at B]. . . Oh, maybe that [Graph 2, area of E] should stay like that because, uh, I'm stopping [moves hand to the Right, stops it], but then I continue on the same road [moves hand back to the Left]. . . Back - just back [moves cursor from ~E to ~G again]. . . 'Cause the other stops don't go back. . . Stop [stop cursor at C] and go in the same direction [to the Right]. And over here [points with cursor at E], I stopped and went to the opposite direction [moves cursor to the Left].

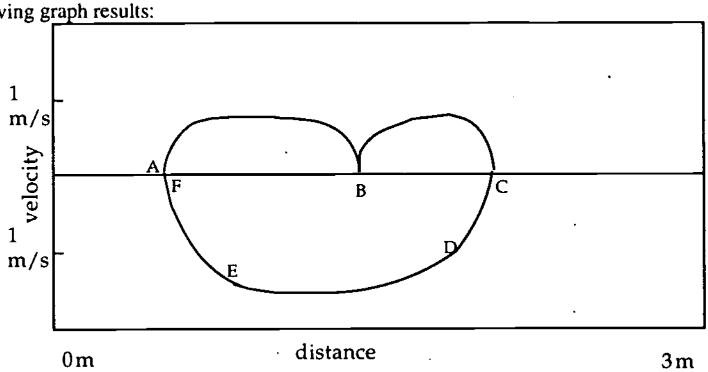
Notes on Episode 3: In this Episode, we find that Noam's way of understanding certain curves as indicating turns and his newer association of some curves with stops has caused a conflict centered on the point E of Graph 2. Noam is certain that there must be a stop between his right and left motions (see Figure 3), and thus that he must have stopped at point E. However Noam also associates curves with turns of the car, and he is concerned that Graph 2 at point E only indicates "turning around," and doesn't show the stop. While Noam is analyzing Graph 2, he comes upon the realization that perhaps

the shape of Graph 2 at point E makes sense, "because, uh, I'm stopping [moves hand to the Right, stops it], but then I continue on the same road [moves hand back to the Left]. . . Back - just back [moves cursor from E to G]." Noam has discovered that there is something about point E that makes it look different from the stopping places at B or C: at point E he stops and comes back to the left again, running over "the same road" a second time, whereas for the other stops at B and C, he stops and then goes "in the same direction [to the Right]." Noam has found that the point E is both a stop and a turn, because he does stop there, but, unlike the stops and B and C, he changes direction at point E, a kind of turning around, and moves back to the left.

Episode 4 -- Slow and Fast

Immediately after Episode 3, Tracy suggests that Noam try doing another motion, and he does a motion involving another turn off to the side and a crossing of the line of tape on the table, reminiscent of his first motion in Figure 2. While talking through this graph, Noam associates turning on the graph with turning the car, and crossing the axis with crossing the road, once again speaking of the graph as if it were a trail left by the car.

Then Noam tries another motion: a linear motion in which he moves the car to the right slowly, stops, moves it further to the right slowly, stops, and moves the car back to the left very fast. The following graph results:



Graph 3

Tracy begins to ask Noam about this trip, and he describes it as follows:

Noam: Okay, so I can tell right now, that the wider it [moves his finger up and down across the vertical width of Graph 3] is -the line is off, that shows it the speed. Okay. [Tracy: Okay.] Just by looking at this one I can tell. Okay. I came here [moves cursor from A to B, along x-axis]. Stopped [B]. And then continued here [moves cursor from B to C, along x-axis]. Stopped [C]. And I rushed back [moves cursor back to the left]. So, this [indicates with cursor the depth of Graph 3 from x-axis to D]- so - what are these numbers supposed to be [pointing to the numbers along the y-axis on the graph]? Velocity? [Tracy: Yeah.] So it shows here - it shows here that it's [i. e., speed of "rushing back" is] over one velocity.

Notes on Episode 4: In the second trip Noam takes in this Episode, we see for the first time Noam making a purposeful distinction between the speeds of two different parts of a trip, so that the part of the graph below the x-axis has a much larger vertical extent (is "wider") than the part above the

x-axis. The new salience of the vertical "width" of the graph causes Noam to ask what the vertical axis label is, and even try to associate the vertical extent of the lower part of the graph with a value of velocity: "it shows here that it's [i. e., speed of 'rushing back' is] over one velocity." Because previous trips have involved turns and backward and forward motions perpendicular to the tape line, or have involved moving at a constant speed, the vertical extent of the graph hadn't been a useful feature for distinguishing different parts of the graph.

Discussion

These Episodes illustrate a process in which Noam adapts and refines his initial way of understanding velocity vs. position graphs to account for the new types of responsiveness he notices when he makes each new graph. In Episode 1, Noam's way of understanding the graph involved thinking of it as a trail left by the car; then in Episode 2 he made a linear motion with several stops, and located the stops on the graph, using the method from Episode 1 of traveling along the graph as the car travels along its path. However, instead of looking for the actual path of the car in the trajectory of the graph, Noam looks to it this time for the feeling or sense of the motion, that is, whether a line seems to "go" or not. Then in Episode 3, Noam's original thinking of the curves of the graph as turns allows him to see that the stop at the point E is different from the other stops, and that point E actually represents both a stop and a turn. Finally in Episode 4, Noam gives a highly detailed account of the trip he made to produce Graph 3, as one might expect from an experienced graph-user. Even while continually adapting his thinking, Noam's original way of thinking about the graph provides him with important resources that he draws on to interpret each new graph he makes.

In Noam's use of the word "straight" throughout these Episodes, we note that the word is useful to Noam, even when he is not making progress toward "correct" mathematical usage. The search for correct word-meanings or accurate gestures does not drive the process of inquiry. Instead, Noam's words and gestures are both used flexibly, as tools in the service of his interpretation of the responsiveness of the graph.

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THE GRAPHICAL, THE ALGEBRAIC AND THEIR RELATION-THE NOTION OF SLOPE

Shakre Rasslan and Shlomo Vinner
The Hebrew university of Jerusalem, Israel

ABSTRACT

Some aspects of the slope concept were examined in 174 Arab and Jewish Israeli students. One of the research questions was whether students realize that the slope is an algebraic invariance of the line and therefore does not depend on the coordinate system in which the line is drawn. Another question was whether the students are able to move back and forth from the algebraic aspect to the graphical aspect of slope in order to perform simple tasks. The results show that between 1/2 to 2/3 of our sample can be considered as "illiterate" in certain contexts of the slope concept.

§1. The Notion of Slope and the Research Problem. The slope is a central concept in the chapter about linear functions and their graphs. Recently, some of its cognitive aspects got a special attention in several studies, either in purely mathematical contexts (Moschkovich et al, 1993; Mullis et al, 1991) or in "real world" contexts (Clement, 1989; Goldenberg, 1988; Nemirovsky, 1994). This paper will deal with the notion in purely mathematical contexts.

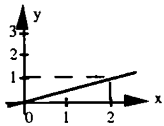
In many countries (also in Israel) the chapter on linear functions is taught already in the ninth grade. The topic is mentioned again and again in high school courses and elementary college courses (pre-calculus and calculus). In the common mathematical textbooks one can find definitions like the following: 1. If (x_1, y_1) and (x_2, y_2) are two points on a straight line then its slope is $(y_2 - y_1)/(x_2 - x_1)$. (See, for instance, Leithold, 1981, pp.30-31.) 2. If $y = ax + b$ is the equation of a straight line then the number a, which is the coefficient of x, is called the slope of the line. (Lang, 1973, p.28.) These definitions are absolutely algebraic. Nevertheless, the word "slope" has a potential of a geometrical interpretation because of its everyday meaning. In a previous study (Ben David, 1986), where close observations were made on 10-th grade class, it was found 7 students out of 33, when asked the slope of a straight line in a context of algebra course , responded that it was the angle between the straight line and the x-axis. This misconception was not reported by Clement (1989), although similar misconceptions were mentioned ("height for slope" and "slope for height," p.85). We were curious to see whether the "angle for slope" misconception will be found in a bigger and more heterogeneous sample. We wanted, as well, to examine whether students realize that the slope is an algebraic notion which does not depend on the graphical representation of the line equation and we were also curious to examine additional aspects of the slope notion which are directly related to some general mathematical abilities: the ability to pass from the graphical aspect

to the algebraical aspect of a concept, the ability to pass from the algebraic aspect to the graphical aspect of a concept and the ability to pass from the algebraical to the graphical - back and forth. The mathematical education assumption about concepts which have algebraic and graphical aspects is (or, at least, should be) that students should be able to cope with simple tasks which involve passing from one aspect to another.

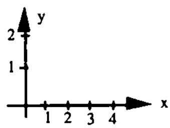
§2. **Method.** Our sample included 3 classes of Israeli Arabic students and 3 classes of Israeli Jewish students, 10-th, 11-th and 12-th graders. The total number of students was 174.

In order to answer our research questions we compiled a 7-question questionnaire which we administered to the above sample. The questions were:

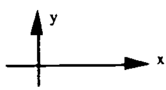
1. In the following coordinate system (please notice that the x-axis and the y-axis have different scales) a straight line is drawn. What is the equation of this line?



2. Draw the straight line the equation of which you have found in the previous question in the following coordinate system.



3. What is the slope of the straight line in the first question?
4. What is the slope of the straight line you have drawn in the second question?
5. Which of the following statements is a true statement? Explain your answer!
   a) The slope of one of the above lines is greater than the slope of the other one.
   b) The two slopes are equal.
6. In the following coordinate system draw a straight line whose slope is negative.



7. It is given that the points $(1, y_1)$ and $(3, y_2)$ are on this line. It is given that the slope of a certain line whose equation is $y = ax + b$ is negative. It is also given that the points $(1, y_1)$ and $(3, y_2)$ are on this line. Which of the following statements is true? Explain your answer! a) $y_1 < y_2$ b) $y_1 > y_2$ c) $y_1 = y_2$ d) It is impossible to know because a and b are not given.

The first five questions were meant to examine whether the student is able to pass from a given line

in a coordinate system to its equation, whether he or she are able to identify two different lines in two different coordinate systems as representing the same algebraic entity and whether he or she are able to overcome a supposedly natural tendency to relate to the technical term "slope" its everyday meaning. Question 6 was meant to examine whether it is clear to the student that a negative slope means an obtuse angle with the x-axis. In other words, we wanted to examine whether there is a graphical association in the student's mind related to the term "negative slope" and whether this association is a correct one. In order to answer question 7 the student has to relate to the picture corresponding to the term "negative slope", to realize that the point $(3, y_2)$ is at the right side of the point $(1, y_1)$ and therefore it is lower than the point $(1, y_1)$. Since the y of a lower point is smaller than the y of a higher point then $y_1 > y_2$. Thus, moving back and forth from the algebraic to the graphical is crucial for carrying out the task of question 7.

We consider all the above questions as simple and elementary. No special preparation is required in order to answer these questions. Only a true understanding of the basic mathematical language is needed. Such understanding of the mathematical language is, so we believe, a necessary working assumption for every mathematics teacher in the 10-th grade and above. Otherwise, it is like teaching short stories by O. Henry in an English as second language class to students who do not have a minimal English vocabulary. Question 7 might seem to the reader a harder question than the other six. We consider it as a "simple complex" question. It is simple in the sense that the associations needed in order to solve it are immediate and straight forward. No tricks are involved. It is complex because it calls for elements which are not mentioned explicitly in the question and it requires more than one step. Calling for elements which are not mentioned directly in the question is necessary for every non-trivial task. Reading comprehension tasks which do not have these characteristics cannot examine any high level thought. Again, by saying "high level thought" we mean only one level above the elementary level, nothing higher. As to the more than one step which is needed in this case we would like to add that the moment one step is made the next step, somehow, follows directly from the previous one. It is a process similar to what Davis (1984) calls a "visually moderated sequence."

§3. Results and Analysis.

Question 1

Category I: A correct answer (62%, N=174)

Category II: A wrong answer which is a result of computational mistakes (3%).

Category III: A wrong answer resulting from a reconstruction failure of the slope formula (5%).

Category IV : A wrong answer which is a result of a defective algebraic thinking (19%)

Examples: 1. *The point is (1,2). If we substitute it the equation line we will get: 2 = (1/2)*1+ b Therefore b = 1.5 and the line equation is y = .5x + 1.5* (There are two elements in this answer that

we would like to point at. First, the student does not know to pass from the point in the graph to its algebraic representation. Second, the student decides that the slope of the line is 1/2 without any attempt to justify it. Perhaps he guessed it, perhaps he got it from his neighbor's questionnaire.)
2. *The line equation is* $y = ax + b$, $1 = 2x + b$, $1 = 2x + 0$, $x = 0$ *and, therefore the line equation is* $y = (1/2)x + (1/2)$ (The roles of the letters are not clear at all to the student and because of that his substitutions are totally wrong. He writes a = 2 instead of x = 2. In addition to that he assumes, without giving it any justification, that b = 0. From the equality x = 1/2 he infers that a = 1/2 and even more surprising, b = 1/2 which contradicts what he wrote earlier in an implicit way: a = 2, b = 0.)

Category V : Meaningless answers (9%) Examples: 1. *The equation of the line is determined by the angle.* 2. *The equation of the line is* $y = 2x + 1$ 3. *From the angle we see that the slope is 1. Therefore the equation of the line is* $y - 1 = 1*(2 - x)$, $y = -x + 3$

Question 2

Category I : The student drew the line of the first question in the coordinate system given in the second question (69%). In this category two solution strategies were used. The first strategy used the equation y = (1/2)x as a starting point. By means of this equation two or three points of the line were calculated and marked in the given coordinate system and then the line through them was drawn. In the second strategy the task was considered as a map drawing task. A geometrical shape in one map should be drawn in another map which has a different scale. In this particular case it is sufficient to identify the location of two points of the geometrical shape in order to draw all of it. Since it was impossible to distinguish by means of the questionnaire between students who used the first strategy and those who used the second strategy we included all of them in the same category. We have no doubt that there were students who used the first strategy because there were three points marked on the line they drew. We also have no doubt that there were students who used the second strategy because they got an incorrect equation in question 1 whereas their drawing to question 2 was correct. The second strategy is considered by us more sophisticated than the first one because it uses elements which are specific to the context of the question and thus enables a shortcut. The first strategy just uses the common algorithm for a line drawing as if there was no question 1 in the questionnaire.

Category III : Answers in which an incorrect line was drawn (23%). Among these answers there were lines which did not pass through the origin or lines which did not pass through (2,1). Usually it happened because the line equation which the students got in question 1 was wrong and the students used the first strategy in category I. The students who used this strategy should not be blamed for using an unsophisticated strategy for the given context. They should be blamed for not being aware of the contradiction which was obtained. Focusing on one technical detail while losing the connection to the entire context is, unfortunately, characteristic to many situations of problem solving in mathematics education. This phenomenon is even more salient in the next

category.

Category III : Answers in which a curve was drawn (3%).

5% of the students did not answer this question at all.

Question 3

Category I : The slope is defined either as the coefficient of x in the line equation $y = ax + b$ or as $(y_2 - y_1)/(x_2 - x_1)$ where (x_1, y_1) and (x_2, y_2) are two points on the line (71%). In this category we included also students who gave an incorrect equation for the line. Also here, as well as with the first strategy in category I, question 2, we want to say that although there is nothing wrong with the application of the formula $(y_2 - y_1)/(x_2 - x_1)$ it indicates that the student is not aware of the fact that the calculation leads to the coefficient of x in the line equation, which is the slope he was asked about. 38% of the students are not aware of it.

Category II : The slope of the line is defined as the derivative of $y = ax + b$ (2%). Although this category is quite small we wanted to keep it as a separate one because it reflects a common tendency of thought: a property implied by the original definition takes its place and the original definition is forgotten. (Similarly, in the case of increasing function there are students who claim that an increasing function is a function whose derivative is positive.)

Category III : A failure in the application of the formula $(y_2 - y_1)/(x_2 - x_1)$ (9%). The impression made by these answers was that the students used distorted formulas like $(y_2 - y_1)/(x_1 - x_2)$, $(x_2 - x_1)/(y_2 - y_1)$ etc. or they failed in the number substitution.

Category IV : The slope of the line is defined as its equation or part of it (6%). Examples:
1. *The slope is* $y = (1/2)x$ 2. *The slope is* $(1/2)x$ (The students probably remember that the slope is connected somehow to the line equation but they cannot dissociate from the equation the relevant part.)

Category V : Vague answers in which the student uses the notion "slope" in a meaningless way (6%) Examples: 1. *The slope* $a = 0$ *because the line passes through the origin.* (This is a distortion of the rule: If the line passes through the origin then $b = 0$ in the equation $y = ax + b$.)

2. The slope is $\Delta y/\Delta x$ (The student repeats the general definition although he was asked to give the slope of a specific line.)

5% of the students did not answer the question at all in spite of the fact that they gave an equation in their answer to question 1.

Question 4

The categories of this question were supposed to be the same as in question 3. This was true in the majority of the cases. Howevever, 9% of the students who answered question 3 correctly gave a wrong answer to question 4. This was a result of the fact that the students used once again the expression $(y_2 - y_1)/(x_2 - x_1)$ in order to calculate the slope of the line. In the second time they

chose two points on the line they drew in question 2. These points were not identical with the points they chose in question 3 and some computational mistakes which led to the wrong answer were made. The students were, probably, not aware of the contradiction in the same way as in category II and category III of question 2.

<u>Question 5</u>

Category I : Answers which claim that the slopes of the two lines are equal and also explain it correctly (65%).

Category II : Answers which claim that the slopes are equal based on wrong explanation (5%). The wrong explanations are like: 1. *The two lines are parallel therefore their slopes are equal.* 2. *The two slopes are equal because the lines do not intersect.* 3. *$a = 0$ in both lines.*

Category III : Answers with no explanation which claim that the slopes are equal (8%).

Category IV : Answers which claim that the slopes of the two lines are not equal with or without explanations and also meaningless answers (19%). Examples: 1. *The slope of the first line is 1/2 and the slope of the second one is 2.* 2. *It is impossible to determine what the right answer is because the line equation is not given.* 3. *The first answer is the correct answer because when the slopes are equal then the lines are either identical or perpendicular.* 4. *The first answer is correct because the scales are different.* 5. *The first answer is correct because the angles are different.*

3% did not answer the question. From the above categorization it turns out that only about 65% of the students we can claim for sure that they know the concept of slope. About other 8% (category III) we cannot claim it but we also cannot claim the opposite. About all the rest (categories II, IV and those who did not answer), which are 27% of our sample we can claim that the concept of slope is not clear to them. 3% (some students in category IV) tied the concept of slope with the angle that the line forms with the x-axis. This was in contrast with our expectations and we will discuss it later on. The claim that between 27% and 35% do not have basic knowledge about such an elementary concept is potentially traumatic. Just remember the strong reaction of the mathematics education community in the USA to the fact that 1/3 of the students in a calculus course at a certain university failed in the students - professors task ("There are six times as many students as professors at this university," Rosnick and Clement, 1980)

<u>Question 6</u>

Category I : The student drew a line with a negative slope (84%).

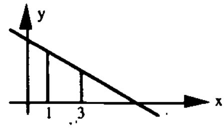
Category II : The student drew a line whose slope is positive but the drawing contains some elements which can be associated with the notion "negative." (b is negative, x is negative or y is negative) (8%)

8% of the students did not answer the question.

<u>Question 7</u>

Category I : The student chose the correct answer $y_1 > y_2$ and also explained correctly his or her

choice (37%). Examples: 1. *$a < 0$ means when x increases y decreases.* 2. *The function decreases.* . 3. *According to the drawing:*



When we analyzed the task in question 7 (see §2) we suggested that the student is supposed to pass from the algebraic aspect to the graphical aspect and back. This is quite obvious in example 3 above but it is not so obvious in the first two examples. We assume that the claims about the decreasing function are implicitly associated with appropriate mental pictures and therefore, in fact, the students passed from the algebraic aspect to the graphical aspect and back.

Category II : The student chose the correct answer $y_1 > y_2$ but did not explain (9%).

Category III : The student chose a wrong answer (with or without an explanation) or chose the right answer but gave a wrong explanation (51%).

3% of the students did not answer the question.

When summarizing the categories of question 7 it turns out that between 51% to 60% are incapable of carrying out, what we called in §2, a "simple complex" task. This is another example of the inadeqaute level of mathematical thought that so many students demonstrate. If we compare this to our comment at the end of our analysis of question 5, relating our results to those of Rosnick and Clement (1980), we may discover the same features. When a problem which is a little bit more complex, but still simple, is presented to the students (there it was: "At Mindy's restaurant, for every four people who ordered cheese cake, there were five people who ordered strudel.") the number of those who fail is doubled. The percentage of "illiteracy" in our case is between 1/2 and 2/3. (Note that in the above comment and in similar comments, we do not intend to be judgmental toward the students. We only discuss the quality of mathematics learning and teaching in the current situation versus the expectations of the mathematical education researcher community (as expressed for instance in the Curriculum and Evaluation Standards, 1989.) A similar comment about a "slope task" was made in Moschkovich et al, 1993, p. 70)

As to the "angle for slope" misconception that we were looking for, it has not been found. When we looked for an explanation to the fact that it appeared in Ben David's sample (1986) we were told that its students had had in their junior high period a course in "technical drawing." This was not a mathematics course but during the course the students drew a lot of straight lines and they related to the angle between those lines and the horizontal direction. Thus, in a spontaneous way, a noticeable number of students associated in their mind the word "slope" with a picture of an angle. This association remained dominant also after the technical notion of slope was introduced to the students. On the other hand, the students in our sample got only the technical definition of slope and no attempt was made to connect it to the angle between the line and the x-axis. The students

themselves, who are, usually, quite indifferent to mathematics, did not bother to make any connections between the everyday meaning of the word "slope" and its technical meaning. This situation presents a dilemma to the teacher: should he or she attempt to tie the technical meaning of a notion to its everyday meaning (in case there is such a meaning) or should they avoid it. If they do it they may raise the danger of misconceptions. If they do not, misconceptions may be avoided but the students might remain detached from the mathematical notions, feeling that they have nothing to do with everyday life. As with any serious educational dilemma it is impossible to advise what to do. The decision, apparently, depends on the teacher, on the students and on the particular context of the educational events.

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VISUALISING QUADRATIC FUNCTIONS: A STUDY OF THIRTEEN-YEAR-OLD GIRLS LEARNING MATHEMATICS WITH GRAPHIC CALCULATORS.

Teresa Smart
School of Teaching Studies, University of North London, UK.

Abstract: The graphic calculator provides the mathematics learner with a powerful tool that is relatively cheap and portable. Previous work has shown that girls in particular respond well to using technology that is both personal and private. This paper reports on work that took place in a lower secondary school mathematics classroom. The pupils (all girls) had free access to graphic calculators during their mathematics lessons. Prior to this, the girls had not used a graph plotting calculator or software. As a result of the experience, the girls started to develop a robust visual image of many algebraic functions. They were happy to describe how changing the value of a coefficient would lead to a transformation of the resulting graph. They discussed graphical methods to solve algebraic problems, often illustrating their ideas by using their hands to draw diagrams in the air.

INTRODUCTION

For students in the advanced mathematics classroom, the use of graph-drawing software has been shown to increase the ability to visualise and investigate algebraic functions. With cheaper graphic calculators these facilities are now more widely accessible, although projects to integrate graphic calculators into the 11-16 classroom are still relatively new. Little research has been done with this younger group, and this initial study shows a similar impact on girls in a lower secondary school.

BACKGROUND TO THE STUDY.

The importance of computer graph drawing software for the mathematics classroom has been well researched. (Tall 1989; Leinhardt, Zaslavsky et al. 1990; Duren 1991) As Duren noted "the availability and access to graphics software for the secondary school mathematics curriculum has provided two powerful learning modalities for students: visualisations and investigations." (Duren 1991 : 23) In particular the benefits of the use of computer software on a student's understanding of the functions (Breidenbach, Dubinsky et al. 1992) and in developing a visual approach to transformations and graphs (Bloom, Comber et al. 1986; Dugdale, Wagner et al. 1992) have been demonstrated. Now the availability of graphic calculators has made this graph drawing software widely accessible and provided a powerful resource for the mathematics classroom.

As Kenneth Ruthven, the co-ordinator of a project funded by NCET (National Council for Education Technology, UK) to investigate the use of the graphic calculator in the post 16, Advanced level (A–level) classroom wrote: "The impact on the project students of unrestricted access to graphic calculators, either as a personal or classroom resource, was impressive." (Ruthven 1992 : 5) The NCET project found that students who had used the graphic calculator in their mathematics lessons did significantly better when tested than a control group who had had no access. (Ruthven 1990) The research took place in the A–level classroom, but as the price of the graphic calculators comes down, then schools have started to buy them for younger pupils' use.

BACKGROUND TO THE CASE STUDY CLASS

This paper is based on a case study of a class of 13 - 14 year old girls observed during their mathematics lessons over a 13-week term. The pupils were from a girls' school situated in a deprived area of London (UK). It is popular and over-subscribed, but there were growing concerns that the girls were underachieving in mathematics. One of the steps taken by the school to try to raise pupil's achievement was to investigate the benefits of integrating technology into the mathematics classroom. As a first stage the school acquired a set of 35 graphic calculators and set out to ensure that as many pupils as possible had an opportunity to become familiar with the calculators.

At the same time, a decision was made to give one class unrestricted access to the calculators for every one of their mathematics lessons, and to study the results. In this class, the calculators became a form of personal technology – they could be picked up and used whenever needed.

I chose to work in a girls' school because I was aware of gender issue surrounding the use of computers in the secondary classroom. (Culley 1988; Hoyles 1988; Elkjaer 1992; Cole, Conlon et al. 1994) Previous work with graphic calculators together with the results of the NCET project had shown that girls work well with this form of technology. An explanation offered for this is that the girls value a form of technology that is personal, enabling them to be less anxious about making mistakes on a private screen rather than a public computer monitor. (Smart 1992) Also, in this study we took note of research showing that all pupils, and particularly girls, benefit from collaborative work with a computer. (Johnson 1985; Underwood and McCaffrey 1990; Underwood 1994) One emphasis of our work with the graphic calculator was to encourage the girls to share their ideas and discuss their findings.

THE STUDY

My first aim, working closely with the class teacher, was to investigate whether it was possible to integrate the graphic calculator into the mathematics curriculum. The school followed a text book scheme. It was important that this class - who were in the top third ability range for their year - did not fall behind the parallel group who were not using the calculators. The teacher wished to keep up to date with the text books, bringing in the calculator when beneficial. My role was as an observer and more "expert" user of the calculator.

The second aim of the study was to observe the influence of unrestricted access to technology on the girls' mathematics learning. I hoped to find, as Goldenberg noted, that: "Proper use of visual imagery gave students new depth and clarity in thinking about old problems. The mathematical richness in linking graphical and symbolic representations of functions also gave students opportunities to pose and explore new problems." (Goldenberg 1988 : 136)

Our third, more detailed goal, was to see if the graphic calculators would help the pupils to develop:

- a more investigative approach to their learning of mathematics, using the calculator to predict, test and generalise;
- a graphical approach to solving problems; and
- an interest in writing and talking about the mathematics they learned through using the calculator.

Through this, the teacher wanted to create an atmosphere of enjoyment and exploration of mathematics in the class with the girls discussing and directing their own mathematics learning.

EXTENDED PROJECT WORK ON PARABOLAS.

In this paper I concentrate on only one part of the case study. Four weeks into the term, the girls were all familiar with the facilities of the calculator. Already we could see that access to the calculators had encouraged them to extend problems posed in the class beyond what was expected in the text book scheme. The teacher decided that so much mathematics was developing that it was preferable to use normal class time to follow through and extend any problems. The pupils then kept up to date and completed the allocated chapters for homework or during "catch up" lessons. At the same time, we felt that the pupils were confronting new ideas without time for consolidation. Things were going too fast. We decided to ask the pupils to work on a piece of extended project work. Their task, over the next four weeks, was to explore the graphs of quadratic functions. They could explore using either graph paper or the calculator or both. Some time was made available during the normal class periods and extra workshops were organised during the lunch hour when the calculators and teacher help would be available.

For their extended project, pupils were asked to explore the graphs of the following functions

$y = ax^2$ when a is positive and when a is negative

$y = x^2 + b$ and $y = x^2 - b$

$y = (x + c)^2$ when c is positive or negative

The first thing they did was make a plan (something they had learned to do in geography and not mathematics). We wanted to encourage them to investigate the functions by making and testing predictions rather than by drawing one graph after another. The ease with which the picture of a graph could be produced and compared with its previous state encouraged them to work in an explorative way.

On completion of the project, the girls were required to hand in a written report of their explorations. This was assessed. Later we asked the girls to discuss their findings and their feelings about using the graphic calculator. We made a video of groups of 5 or 6 girls, discussing their work. Talking about mathematics was not something these girls had much experience of. They were more accustomed to answering the teacher's questions. To help the discussion, each group was provided

with a set of card with questions. Each pupil was asked to pick up a card and talk about the question. The other pupils could then add to the discussion or choose another question.

Below I illustrate the progress they made with the extended project work on parabolas using the girls' own words written and spoken.

They had very little experience of writing and talking about mathematics. They were encouraged to make a plan. They felt this plan helped. As Shazia said: "It helped me because it put the work into stages. I didn't have to do everything at once. It was good because I knew what to do after, so the plan was quite useful." Angie agreed: "yea, with my plan, I did it differently. I planned it in stages, like if I was doing $y=ax^2$, I planned that, then I planned for $y= x^2+b$. So I found that easier. I thought it would be easier than making one big one."

MAKING AND TESTING PREDICTIONS

At every stage the pupils were making and testing predictions. Shafa gave as steps in her plan:

" – Predict what the graphs would turn out like, then say if you were right or wrong

– See if the graphs have anything in common

– Write some conclusions on what you have found out."

Angie described in her written report her predictions for the graphs of $y=ax^2$ when a is positive and then negative: "I was right in my prediction that the centre of the graph will always be 0. I was also right in saying that as a gets bigger the curve will become steeper and closer to the y–axis. When a is negative the U shape is formed upside down – and as a gets bigger the curve got steeper too and I got that right too." Her partner Amina made a different prediction. She said: "About $y=ax^2$ with the positive one [a]. I found it was a curve and as a got larger it went closer to the vertical axes so my prediction for the negative a is that when well the curve will be upside down but when the a got larger it would move away from the vertical axes and would be less steep. But I found out imy prediction was wrong. and it was exactly the same as the positive one but upside down."

When exploring the graph of $y=(x+c)^2$ nearly all the pupils were surprised at what they saw, having made false predictions. As Shahanaz wrote: "My prediction was partly wrong because I predicted that whatever $+c$ was then the graph would go through the value of x = it [Here Shanaz is using "it" for $+c$, generalising using a mixture of words and symbols]. But instead I found that the curve would go through the negative side of the x-axis, through whatever c was e.g.. $y=(x+5)^2$ it would go through –5. But when we looked at $y=(x-c)^2$ then our prediction was right whatever $+c$ was it would go through $-c$ so in the case of $(x-3)^2$ it would go through minus -3 which is +3."

Shafa found the same but she explored further. She wanted to understand why this curve went through the opposite side. She wrote in her report:

"I was so surprised when it [$y=(x+3)^2$] turned out the other way, so I did the equation

$$y=(x+3)^2=0$$
$$\Rightarrow x+3=0$$
$$\Rightarrow x=-3$$

so the solution was $x = -3$ "

VERIFYING RESULTS PREVIOUSLY LEARNED, WITH PAPER AND PENCIL, IN CLASS.

In using the calculator, the pupils discovered rules they had learned in a symbolic form in the classroom but did not necessarily feel confident with. For example, in exploring the graphs of $y=x^2 + b$ or $y=x^2 - b$ many of the girls found for themselves a visual image of the rule: "a minus followed by a minus is a plus". Shibley said:

"I did these two graphs and I discovered that if you take $y=x^2+ b$ and $y=x^2$ *minus a negative number*, it is the same."

I asked her if she was expecting this result

"When I predicted what the graph of $y=x^2 - - 8$ would look like, I thought the whole [positive] numbers were supposed to be at the top of the screen. I thought the minus was going to be at the bottom. After I realised, that a minus and a minus is a plus."

Shakila went further. She explored the effect of putting two quadratic functions together. She drew the graphs of functions of the form: $y=(x+c)^2 \mp x^2 \mp b$

She predicted that the graphs would be of the same shape as all the other graphs (parabolas), and the b and the c would only affect the position. She found that she didn't always get a parabola on her screen but sometimes a straight line and sometimes a parabola steeper than the expected graph of $y=x^2$. She was interested enough to work out why this happens setting out to simplify the algebraic expression. She wrote as her conclusion:

"This never had a lot to do with parabolas but I also learnt something, for I found out why I got a line graph. I had to do research to be able to understand or otherwise I didn't have a clue of why I got line graphs."

Goldenberg noted that: "if the connection between the analytical representation of the function and its graphical representation is perceived as magical or arbitrary, the two representations cannot inform each other." (Goldenberg 1988 : 153) Shakila when surprised with the graphical representation she found on her screen, and did the research going back to examine the symbolic form.

Other pupils however, just accepted the graph produced on the calculator without questioning its validity. As Leinhardt warned: "In computer based instruction, ... the graph the machine produces is

unquestionable. A teacher should be aware of the 'magic' effect this may have on students." (Leinhardt, Zaslavsky et al. 1990 :7) This "magic" led to several of the pupils feeling convinced that there was something wrong with the graph of $y=(x+c)^2$ when the value of c lay outside the range of the calculator. It was no longer a parabola, and as Janet said, "well, they weren't what I thought they were going to be, because the graphs before when I had plotted $y=ax^2$ and $y=x^2+b$, they were all the same - they were all U shapes - but with these ones with say $y=(x+4)^2$ the curve was not the same so I couldn't make predictions". Janet later, after discussion with the teacher, readjusted her mental image of the graphs of $y=(x+c)^2$. She said in her group discussion several weeks later: "Well the graphs are all the same shape but they appeared different shapes because we did the axes different and the graphs went off the screen so you have to add it on yourself in your mind." Other girls had greater difficulty. When they saw an unexpected feature in the graphs they made up a set of rules to interpret the feature, falling into the danger that Goldenberg had warned about. "Our earliest experiments showed that students often made significant misinterpretations of what they saw in graphic representations of function. Left alone to experiment, they could induce rules that were misleading or downright wrong." (Goldenberg 1988 :137) Several of our students believed when the graph was translated up or down the screen as in the case of $y=x^2- 6$ and $y=x^2+6$ then, as Dimpey said: "the top parabola is narrower and you see less of it and the bottom graph is wider and less steep. The graphs in this group do not have the same shape." This was agreed to by all the other girls in her group. These girls needed another experience besides the calculator. This could be provided by moving an acetate sheet with the graph of $y=x^2$ up and down the calculator screen, but they also needed more time exploring the graphs and the equations by hand.

WERE THEY CONFIDENT WITH THEIR GENERALISATIONS?

As Shahanaz wrote:

> "We decided to take the last two equations [$y=(x+c)^2$ and $y=(x-c)^2$] a step further by seeing what will happen with $y=(x-c)^2 - d$. We predicted that if we did $-d$ it [$y=(x-c)^2$]would go down and if we did $+d$ it would go up (below x-axis above x-axis)."

When they were asked if they would know how to investigate the graphs of $y=x^3$ and $y=ax^3 + b$, and if they could predict what the a and the b would do to the graph, Shafa, Shakila, Shahanaz and Henna felt confident enough to discuss what they would do. They use their hands to show the shapes and the movements up and down the screen. Below is what they said:

Shafa: Well do the same as this [$y=ax^2$] and write down some sort of prediction - like what you would think the graph would turn out to be and write down all the equations and then try to learn how to do cubed on the calculator. We would follow the plan here.

Henna draws an S shape on her paper.

Shakila: Wasn't it one of those S type graphs like that? [she points to Henna].

Shafa: The a would move it to the side or maybe go up or something

Shahanaz: It would spread it out; it would make it steeper and wider.

Shafa: Oh yea, when a is positive, the higher it gets the steeper it [the graph] is.

Henna: The $+b$, well the $+b$ that [pointing to the stationary point of the graph] would start going up as well.

The girls know exactly how they would explore other functions and transformation, and were quite confident that they could use the graphic calculator to do this.

DISCUSSION.

All the girls in the class handed in an extended piece of work exploring all the cases and produced work that was considered by the teacher to be of a higher level than what she would predict for them. Several of the 30 girls extended the work to consider combined transformations, such as predicting the shape and position of the graph of $y=(x+3)^2 - 4$. or $y=2x^2+ 6$. They found that the work with the graphic calculate was "quicker and more accurate" and as Natasha said: "the calculator draws the graphs on top of each other so it is easier to see patterns." One of the strongest pictures gained from the study is that of the pupils' talk about <u>seeing</u> the pattern, <u>seeing</u> that a "minus and a minus is a plus", predicting what the graph of a combined function would <u>look like</u>. One of the aims of work with a graph plotting facility is to encourage the learner to move freely and easily between the pictorial and the symbolic representation of any function. These girls achieved this ease of movement. They developed strong powers of visualisation. When a new problem was posed they tended to adopt a visual strategy to solve it.

All the girls were able to talk fluently and articulately about their work with parabolas. This was impressive as the majority of the pupils in the class do not have English as their first language or the language that they use at home with their families. Their school recognised the importance of language development through the curriculum, and I saw the graphic calculator as providing an extra stimulus for meaningful talk. As a group of mathematics teachers working with bilingual pupils noted: "The main aim of the maths department is to provide interesting lessons for all students, in order to develop their mathematical skills and knowledge. A central way of achieving understanding of mathematics is by talking, reading and writing about it. In order to do this we must provide students with the appropriate mathematics vocabulary and the appropriate stimulus for the use of language to take place." (Cox, Gammon et al. 1993 :9) I believe that the ease in which the calculator produces a visual image of the function, and the need to retain a picture of this image, pushes the pupils into talking and describing, and hence using "appropriate mathematical language".

My observation finished at the end of the term but the pupils' use of the calculator did not. When I returned several weeks later, the class was working on a textbook chapter on linear inequalities. They extended their work to include higher order inequalities. When I asked about the solution to $7 - 3x^2 > 0$, one of the girls replied. "Well the graph is upside down because it's minus and it's up there

because it's +7 so it will be all that inside the curve." This girls and many of the others felt confident with questions that often cause difficulties for more mathematically sophisticated students because of their well developed powers of visua!isation.

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