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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following research papers: "Calculators in Primary Education" (van den Brink); "Microworlds and Van Hiele Levels" (Dreyfus, Thompson); "Dynamical Mazes and the Ob-Serving Computer" (Harmegnies, Lowenthal); "Iterative and Recursive Modes of Thinking in Mathematical Problem Solving Processes" (Hausmann); "Computer Modelling of Mathematical Problem Solving Processes as an Instrument of Competence Analysis" (Holland, Reitz); "Children Learning Mathematics--Insights from Within a LOGO Environment" (Hoyles, Sutherland); "The Construction Process of an Iteration by Middle-School Pupils (Laborde, Mejias); "Pegboard and Basis for Programmation--In 5- and 6-Year Olds" (Lowenthal); "Self-developing Strategies with a Calculator Game" (Meissner); "The Effects of CAI on Affective Variables in Mathematics" (Mevarech); "Acquisition of Number-Space Relationships: Using Educational and Research Programs" (Rogalski); "Learning Programming Constructing the Concept of Variable by Beginning Students" (Samurcay); "De Baas over de Computer" (Schoemaker); "The Use of Calculators to Develop Reasoning Processes and Achievement in Problem Solving" (Southwell); "Microcomputer-assisted Sophistication in the Use of Mathematical Knowledge" (Taizi, Zehavi); "Using Computer Graphics as Generic Organizers for the Concept Image of Differentiation" (Tall); "Maths for All? Just Have a Look" (Dekker); "A Study of Mathematics Education in Classroom Situation--A Methodological Research" (Hirabayashi); "Episodic Analysis of a Mathematics Lesson" (Romberg); "What Problem is the Child Solving?" (Escarabajal); "Comparison of Models Aimed at Teaching Signed Integers" (Janvier); "The Equation-Solving Errors of Novice and Intermediate Algebra Students" (Kieran); "Children's Informal Conceptions of Integer Arithmetic" (Murray); "Obstructions to the Acquisition of Elemental Algebraic Concepts and Teaching Strategies" (Filloy, Rojano); "Visualizing Rectangular Solids Made of Small Cubes: Analyzing and Effecting Students Performance" (Ben-Chaim, Lappan, Houang); "The Teaching of Reflection in France and in Japan" (Denijs); "Geometry Doesn't Fit in the Book" (Goddijn, Kindt); "Middle School Pupils Conceptions about Reflections According to a Task of Construction" (Grenier); "Geometric Concepts of Portuguese Preservice Primary Teachers" (Matos); "A Preliminary Investigation of the Transition between Two Levels of Intellectual Functioning" (Collis); "The Development of Abstract Reasoning--Results from a Large Scale Mathematics Study in Australia and New

Zealand" (Ellerton); "Pupil Cognitive Ability Levels Compared with Curricular Demands When Six-to Eight-Year Olds are Taught Arithmetic Operations" (Harrison, Bye, Schroeder); "The Design of a National Pre-Vocational and Vocational Numeracy Scheme: The Relationship of Psychology to Curriculum Development" (Rees, Barr, Gould, Leary); "Experimental Study of Pupils' Treatment of Refutations in a Geometrical Context" (Balacheff); "Problem Solving Processes of Gifted Students" (Ballew, Cunningham, Sexton); "Abstraction and Activity" (Dorfler); "Toward Understanding Mathematical Thinking" (Eisenberg, Dreijfus); "The Description of Errors by Models of Cognitive Science" (Hasemann); "Graphic Tool for Calculus" (Keita); "Affective Influence on Mathematical Problem Solving" (McLeod); "Identifying and Disseminating the Traits of 'Natural' Problem Solvers" (Peck); "Different Modes of Thought in Stochastic Thinking" (Scholz); "Disregarding the Hierarchy in a Behavioral Program as an Error Type in Pupils' Computation Tasks" (Becker); "The Use of Models in the Construction of Numeration: The Evolution of Representations Towards an Efficient Symbolism" (Bednarz, Janvier); "The Decimal System: Understand and Notation" (Carraher); "Working with Simple Word Problems in Early Mathematics Instruction" (de Corte, Verschaffel); "The Development of Number Concepts in Low Attainers in Mathematics Aged Seven to Nine Years" (Denvir); "Teaching Multi-Digit Addition and Subtraction" (Fuson); "Teaching Counting On for Addition and Counting Up for Subtraction" (Hall, Fuson, Willis); "The Development of Spatial Imagination Abilities and Contextualisation Strategies: Models Based on the Teaching of Fractions" (Figueras, Filloy, Valdemoros); "Changes in Decimal Reasoning" (Nesher, Peled); "Selected Results from the Rational Number Project" (Post, Behr, Lesh, Wachsmuth); "Cognitive Processes and Products in Proportional Reasoning" (Reiss, Behr, Lesh, Post); "The Concepts of Repeating and Non-Repeating Decimals at the Senior High Level" (Vinner, Kidron); "Inconsistent Student Behavior in Applicational Situations of Mathematics" (Wachsmuth); "Misconceptions in Graphing" (Clement); "The Mathematics of Psychological Education" (Greer); "The Acquisition of Basic Multiplication Skills" (Heege); "Estimation: Practice and Process" (Markovits, Hershkowits, Bruckheimer); "A Transfer Test for Mathematics Containing Items with Cumulative Hints" (Meijer, Perrenet, Zeillemaker, de Leeuw, Groen, Kok, van Blokland-Vogelesang); "Geometrical and Numerical Strategies in Students' Functional Reasoning" (Ponte); "Development of Audio-Visual Testing for the Improvement of Teaching-Learning Process in Mathematics Education" (Yanagimoto); "Errors, Expectation and the Teacher" (Bouhadana, Zehavi, Bruckheimer); "The Relative Importance Given by the Teacher to the Students' Answer and Reasoning" (Dionne); "Confidence, Competence and Inservice Education for Mathematics Teachers" (Fresko, Ben-Chaim); "Activities with Teachers Based on Cognitive Research" (Hershkowitz, Vinner); "PMP, A Progress Report" (Skemp); "The Evaluation of Two Types of Knowledge at Different Points in Time After Study" (Allwood); "A Constructivist vs. a Formalist Approach in the Teaching of the Even-Odd Number Concept at the Elementary Level" (Herscovics, Bergeron); "The Sequencing of Activities for the Teaching of the Even-Odd Number Notion by Primary Schoolteachers Without Their Prior Analysis of this Concept" (Bergeron, Herscovics); "The Sequencing of Activities for the Teaching of the Even-Odd Number Notion by Primary Schoolteachers Following Their Prior Analysis of this Concept" (Nantais, Bergeron, Herscovics); "Towards a Frame-work for Constructivist Instruction" (Confrey); "Representation and Deduction" (von Glasersfeld); "Educational and Psychological Considerations in the Development of Synthetic Geometry Curriculum for Israeli Low Ability Students" (Hoz); "Passive versus Active Use of Knowledge" (Sackur-Grisvard); "The Teaching of Infinity" (Tirosh, Fischbein, Dor). (MKR)

# Proceedings of the Ninth International Conference for the Psychology of Mathematics Education

# Proceedings of the Ninth International Conference for the Psychology of Mathematics Education

Volume 1: Individual Contributions

Edited by Leen Streefland,
Researchgroup on Mathematics Education and
Educational Computercentre (OW & OC),
Subfaculty of Mathematics,
State University of Utrecht, The Netherlands.

**PME**
International Group for the Psychology of Mathematics Education
Noordwijkerhout, July 22nd-July 29th

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## Preface

The International Group for the Psychology of Mathematics Education (PME) was founded in 1976 at the third International Congress for Mathematics Education in Karlsruhe, in order to promote international contacts and the exchange of scientific information in the psychology of mathematics education. The objective of the Group is to further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof. Thusfar PME conferences were held in Utrecht (1977), Osnabrück (1978), Warwick (1979), Berkeley (1980), Grenoble (1981), Antwerp (1982), Shoresh, Israel (1983) and Sydney (1984).
The ninth conference will take place at Leeuwenhorst Congres Center in Noordwijkerhout near Leijde and The Hague, from July 22nd to July 26th, 1985.
The scientific program includes plenary lectures on topics of interest to the group as a whole and both working group sessions on special topics as well as working group sessions with presentations of individual contributions followed by discussions.
The papers with the individual contributions are collected in these proceedings. The plenary addresses have been published in a separate volume.
The individual contributions have been classified under the following headings.

1. Computers
2. Classroom observations
3. Directed numbers and algebra
4. Geometry
5. Long term concept and reasoning development
6. Mathematical thinking
7. Number concept and basic arithmetic
8. Rational numbers and Decimals
9. Students strategies and conceptions
10. Teacher behavior and attitude
11. Theory of instructional organisation

Because of overlapping, contributions within the category 'Computers' will also be assigned to one of the remaining category-numbers between brackets.
The order in which the papers will be presented at the conference will not necessarily be identical with the order in which they appear in this volume. Complete details will be given in the conference program.
In order to locate a particular contribution you may use the table

of contents at the beginning of the volume, or the list of contributors at the end.

The Conference Committee: F. van der Blij, F.J. van den Brink, B. Dekker-Heijman, J. van Dormolen, H. Freudenthal, L. Streefland, A. Treffers, assisted by the International Program Committee, consisting of:
A. Vermandel (Belgium), L. Burton (UK), H. Freudenthal (The Netherlands), K.B. Koster (The Netherlands), H. Meissner (West Germany), L. Streefland (The Netherlands) and G. Vergnaud (France).

# TABLE OF CONTENTS (76)

## 1 Computers (16)



## 2 Classroom Observations (3)

3 Directed Numbers and Algebra (5)



4 Geometry (5)



5 Long Term Concept and Reasoning Development (4)



6 Mathematical Thinking (9) 8

## 9 Students Concepts and Strategies (7)



## 10 Teacher Behavior and Attitude (5)



## 11 Theory of Instructional Organisation (9)

# CALCULATORS IN PRIMARY EDUCATION

*Jan van den Brink* (OW & OC)

## 1. *Summary*

A report is given of research into the use of calculators in primary school, grade 4 to 6 (pupil's age: 10 to 13).
The results are compared with those of previous investigations in kindergarten and grade 1 to 3.
The observations were taken from teacher's perspective in actual class situations.
As arithmetical subject is reported: the different procedures of different calculators by which various results can be given for one and the same sum.
The problem comes to a crisis only when a context is at issue. The algorithms of calculators are subjected to modifications by children. Finally four phases in the learning of arithmetical procedures with calculators are proposed (isolation, accepting the divergences, crisis or confrontation, restoration or construction)

## 2. *Introduction: Earlier research*

In our previous investigations into the use of the pocket calculator by kindergarteners and primary school children we distinguished three phases:

I *Children's ideas and activities*

This concerns an investigation into children's ideas about the calculator and how they handle the machine. The conversations were on a one-to-one basis.

II *Dialogues*

Investigations into the use of the calculator by groups of two children

III *Class*

Investigations into the use of the calculator in actual class situations.

A number of conclusions have been drawn from this earlier research in kindergarten and primary school grades 1 to 3. (see Van den Brink, 1984, p. 464, 465):

a) Without knowledge of arithmetic the calculator cannot be used in a meaningful way. Knowledge of arithmetic is a fundamental necessity.
b) The calculator has many facilities which are related unexpectedly to traditional arithmetic.
c) The calculator sometimes helps expand arithmetic concepts.
d) On the other hand, there is the danger that children will get stuck in certain simple procedures on the calculator.
e) The calculator has its own limitations with regard to other arithmetic models and notations.

f) The students' thought processes while doing arithmetic can be verified numerically on the machine.

g) Wild experimentation is characteristic for the first use of calculators.

h) "The calculator cannot make mistakes" is the opinion of young children.

It is striking that these conclusions, drawn from research with *young* children in the lower primary school grades, are also true for *older* children in grade 4, 5 and 6.

3. *Present research*

The present research refers primarily to the middle and upper primary school grades (4, 5 and 6).

In the previous research, a lower limitation was determined: children must be able to do arithmetic before they can deal with the calculator (conclusion a) ).

We now attempt, beginning with a given (arithmetic) subject in the 6th grade, to determine the 'greatest lower limitation' for that subject.

The researcher teaches first the 6th grade, then the 5th and, finally, the 4th and makes note of the children's reactions by means of a hand-held video camera, which he carries under his arm. This method of observation and registration I call *'teacher's perspective'*.

4. *Example of a research situation: Counting on differences*

A recent subject is the introduction of *different calculators at the same time*. Authors of a Dutch arithmetic method are of the opinion that it is didactically better for *all* children to use *one* type of machine, rather than a variety of types. (This standpoint is sometimes also held in secondary education for financial reasons.)

Just the opposite proved to be the case:

In the first lesson the children were asked to bring a calculator from home. The machines were examined to see how they differed. The curious thing was that, not only did they differ in terms of keys and notation, but that various machines could give various results for one and the same sum .... (e.g. 4x5-4x5 = 80, according to many calculators)

5. *Some tendencies*

a. It was fascinating, that the children simply would show *no surprise* in the beginning at this phenomenon of different answers for one and the same sum. 'The calculator cannot make mistakes', was their opinion according to former conclusion h) And they explained the way in which various machines counted to get that different result. Each of these algorithms — even nonsensical — was of equal

value for the children.

b. The differences only became a problem within *all sorts of contexts* and a new question had been born: how to manage the calculators in order they fit in with our normal arithmetic?
   In other words: how to change the algorithm of an *authority* (the calculator)?
c. The *authority's* way of doing arithmetic was always regarded as being *infallible*. The calculator can cause this image to totter: for it were not the students who had to change their own way of calculating, but the authority.

Summarising we can distinguish four phases in the learning of arithmetical procedures.

I *Isolated procedures* (and contexts)
The operations of different machines are isolated by the pupils from each other and from ordinary arithmetic.

II *Mere ackowledgement of divergences* (accepting).
In this phase the divergence in calculating procedures is mere acknowledged.
The pupil considers different manners as equally rightful: they are different ways to act - there is no preference.
There is no reason to be amazed about the divergences.

III *Crises* (confrontation)
In order to leave the phase of mere acknowledgement the calculating procedures must appear in a *context*. The crisis between two bare different procedures arises only when a context is at issue.

IV *Restoration* (construction)
Most often the child's procedure is rejected or modified, because the teacher's algorithm is sacrosanct.
With the calculator, however, the algorithm of an authority (the calculator) is subjected to modifications.
In this way the calculator provides the pupils with a fresh view on the subject "arithmetic".

6. *Report*

The research by the OW & OC into calculators attemps to impart all sorts of – and sometimes deeply seated – arithmetic concepts starting from similar 'natural' school situations (for example: there are simply a lot of different machines in the classroom)

A report will be made on this matter.

## 7. *References*

Brink, F.Jan van den, *Integration of the calculator into children's education*, Proceedings of the eight international conferences for the PME 1984, Australia, pp. 463-466.

# MICROWORLDS AND VAN HIELE LEVELS

Tommy Dreyfus[1] and Patrick W. Thompson[2]
San Diego State University, USA

*Two sixth graders were observed as they used a computer microworld in which integers are represented as transformations. It is described how the students (re)construct the mathematics as they progress from one van Hiele level to the next.*

Freudenthal [1972] has distinguished ready-made mathematics from mathematics as an activity. He made it clear that the active performance of mathematics is essential in the learner's progress from one van Hiele level to the next. The relationship among van Hiele levels is that the organizing activity of any level becomes the object of analysis on the next higher level. The question arises as to what settings would facilitate progress from one level to the next.

It has been argued elsewhere [Dreyfus, 1984] that computer environments can be consonant with Freudenthal's theory. Such an environment, or microworld, should give the student the opportunity to construct *mathematical* knowledge. Thompson [1985] has given a detailed analysis of the principles according to which a microworld should be built. In the present paper, one such microworld will be analyzed with respect to its incorporation of van Hiele levels. Two students will be followed as they progress through these levels.

The INTEGERS microworld [Thompson, 1984] is a learning environment for the additive group of integers. It is built in such a way that it can be used by students of elementary arithemetic as well as by students of group theory. This has been achieved by adhering to the following principles:

1. There must be a strong distinction between the curriculum and the environment upon which the students act (and the metaphor within which they reason). The software presents the environment and metaphor, and incorporates the mathematical structure within a model. Instructions and questions for the student are given in print. This allows teachers to adapt the curriculum to the purposes of instruction as well as to the levels of the students. Thence, continued use of this model is apt to contribute to the integration, in students' minds, of the various levels at which the concept can be examined.[3]
2. The representation of a mathematical structure by a model must be such that the model incorporates the structure in detail, yet must be easy to use even at relatively low levels where the full power of the model is not needed. The presentation of

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[3] For example, INTEGERS can be used in Grade 1 to teach addition, in Grade 6 to teach negative numbers, in Grade 8 to teach variables and operations, and in Grade 12 to teach the Abelian group structure of integers.

mathematical content should make aspects of its structure accessible to students regardless of their level of thinking.

## The Environment for Integers

In the case of INTEGERS, these principles were expressed by presenting an integer as a transformation of position, and operations upon integers as compositions of transformations. More precisely, an integer is reflected in INTEGERS by having a turtle walk left and right on a number line in response to one or more numbers being entered. The response of the turtle to a number being entered is governed by these rules:

*number* The turtle walks *number* turtle-steps in its current direction.

*–number* The turtle turns around, does *number*, and then turns back around.

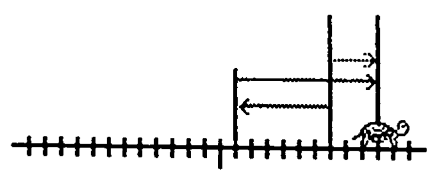
The turtle's grammar for numbers is:

i. A whole number is a number.

ii. The negative of a number is a number.

iii. Two or more numbers entered simultaneously is a number *(the net effect of executing them consecutively)*.

Figures 1 and 2 illustrate the turtle's grammar. The readers should note the grammar's recursive structure. The combination **–60 90** is a number (by iii.), and hence **–[–60 90]** is a number (by ii.). Figure 3 explains the grammatical structure of Figure 2.



LAST COMMAND: -60 90

COMMAND:

POS: 100 NET: 30

Figure 1. The effect of entering **–60 90**. The tall vertical line on the left shows the turtle's beginning position. The tall vertical line on the right shows its ending position. The intermediate (shorter) vertical line shows where the turtle finished executing **–60** and began executing **90**. The turtle did **–60** (lower heavy arrow) and then did **90** (upper heavy arrow), which resulted in a net effect of **30** (top arrow). POS tells the turtle's current position. NET tells the net displacement of the turtle's position caused by the last-entered command.

INTEGERS has been used by a student assistant in tutoring two sixth graders, whom we will call Kim and Lucy. They met for eleven sessions of approximately 40 minutes each. the sessions were audiotaped and transcribed. These data have been analyzed from several points of view: (1) progress through several van Hiele levels, (2) the formation of the concept of

composition of integers, and (3) the formation of the concept of negation. In this paper we concentrate on the first two analyses. The formation of concepts of negation will be discussed in a separate paper [Thompson & Dreyfus, in preparation].

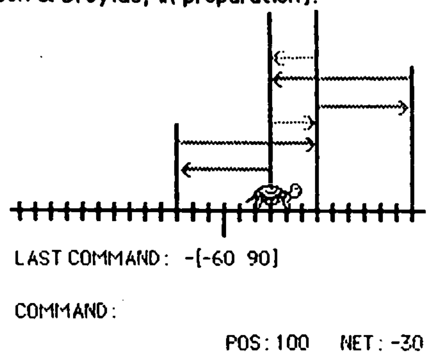



Figure 2. The effect of entering **-[-60 90]** after having entered **-60 90**. The right tall vertical line shows the turtle's position when it began executing **-[-60 90]**. The upper three arrows document the turtle's itinerary in response to **-[-60 90]**.

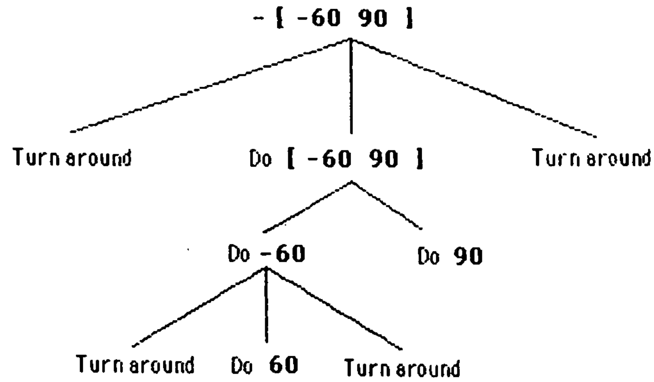



Figure 3. The method by which INTEGERS parsed the expression **-[-60 90]** in Figure 2. Read the parse tree thus: To do **-[-60 90]**, turn around, do **[-60 90]**, and then turn back around. To do **[-60 90]**, do **-60** and then do **90**. To do **-60**, turn around, do **60**, and then turn back around.

## Kim's and Lucy's Progress

### *Level 0*

The lowest van Hiele level (Level 0) is the distinction between position and transformation of position. Typically, a student on this level would be asked, for example, to place the turtle at 30 and to make it walk to any of 90, 10, or −80. Variations on this problem are: (1) give the starting position and the transformation; ask where the turtle will end, and (2) give the transformation and the ending position; ask where the turtle began. The objects of attention at this level are the states (positions) of the turtle. The organizing activity is the determination of relationships among beginning position, ending position, and transformation.

While at Level 0, Kim and Lucy showed a persistent confusion between integer (transformation) and position. Their confusion is illustrated in the following excerpt.

(Turtle is at position 30.)
INT: What would happen if you just put in thirty.
LUCY: Would it just stay there?
INT: Try it. (Lucy enters 30 and presses RETURN).
LUCY: Oh yeah, it does that line.
INT: And what did the turtle do?
LUCY: It—then it goes thirty.
INT: Okay, try it again. Put in thirty, but don't press RETURN. (Lucy types 30) Where is the turtle going to end up when you press RETURN?
LUCY: Over there (points to the position 30).
INT: Go ahead and press RETURN.
LUCY: (Presses RETURN) Oh, it'll go another thirty spaces.

It took Kim and Lucy till the third session before they had created a reliable distinction between position and transformation. Even then, it was not uncommon for them to lapse back into confusion when questions got intricate or required a new focus. That is to say, the separation of Level 0 and Level 1 was not pure, as both Lucy and Kim would at times behave as if their thinking was Level 0 while at the same time they were working more or less successfully at Level 1. For this reason we use van Hiele levels more as descriptions of performance than as statements of competence. As statements of competence, one could say only that a child *can* think at a given level, but not that he or she *will* think at that level given the capability to do so.

### *Level 1*

While the objects of thought at Level 1 are transformations, the students' activities are organized by their comparison of different transformations and by problems that emphasize the independence of position and transformation. The extent to which students formalize an integer transformation as an equivalence class of position pairs is open to question. The question of formalization arises at each level of our analysis. It will be treated more fully in the discussion of Level 2.

Kim and Lucy progressed fairly quickly through Level 1 and seemed to have little trouble conceptualizing a turtle movement as a transformation from one state to another. Since a turtle movement is accomplished by entering an integer numeral, we surmised that they had conceptualized integers as transformations. Quite surprisingly, this was true for negative as well as positive integers. They quickly apprehended that a minus sign in front of a number causes two turns of the turtle—one before and one after its walk.

### *Level 2*

At Level 2, integer transformations are the objects of thought. Transformations can be composed (integer addition) and/or inverted (negation). The students' activities were organized by composing and inverting integers and integer expressions. The following excerpt

from the fifth meeting is representative of Kim's and Lucy's activities at Level 2. The new level of discourse is suggested by the introduction of the term "net effect," which means the net displacement caused by the composition of two or more integers.

INT: (Enters **20 −30**; the turtle walks accordingly) What is the net effect? What is that little white dotted arrow?
LUCY: Ten.
KIM: No, see, it's fifty, because it . . .
INT: I wanted to know . . .
KIM: Because like, it would be going . . .
INT: If I was sitting right here at the end of this arrow and I wanted to move the same amount as that arrow, what number could I put in?
KIM: Fifty, no . . .
LUCY: Yeah, negative fifty.
INT: Negative fifty?
KIM: I doubt it.
LUCY: Well, if I was facing . . .
INT: Which way is the turtle facing? How do we make it turn around?
KIM: Negative.
INT: Negative. Right. Okay, I want to make an arrow just like this (net effect arrow).
LUCY: Oh!! Negative ten.
INT: Negative ten. Try it.
LUCY: (Enters **−10**; the turtle walks accordingly)
INT: Are those two arrows the same?
LUCY: Yeah.
INT: So, what is the net effect of negative thirty twenty?
KIM: Fifty.
LUCY: Fifty.

It turned out, in this instance, that by "net" the students meant "total," as in "total number of turtle-steps walked." We contend that the driving force behind the attribution of "total steps" to "net" was Kim's and Lucy's conception of number. Minus signs were not part of numerals; rather, they qualified numerals. Similarly, the direction of a transformation was not part of the transformation; rather, it qualified the transformation. To add two of *their* transformations, one would add the number of turtle steps in each. A direct analogy is college students' common mistake of adding two vectors' magnitudes to determine the magnitude of their sum.

The above dialogue is typical of the kinds of discussions that are elicited by the didactical situation of two or three students, with or without a tutor, engaged in solving a problem and having the opportunity to make inquiries by means of a microworld. As illustrated, inquiries do not guarantee insight, but they provide opportunities for insight.

Kim and Lucy proceeded to understand what happens in concrete "turtle actions" in response to the entry of two integers. Over the next four sessions they became more proficient in constructing these descriptions, and they became able to handle a wide variety of cases, such as **−70 −40** and **−90 50**. But this observation is of little interest in regard to our focus on their understanding of the structural properties of integer addition (composition of transformations). Did they, in fact, grasp that two transformations, applied one after the other, are equivalent to one appropriately chosen transformation? Did they formalize the notion of net

effect by understanding that "net effect" denotes a *resulting* transformation? The answer was *no* to both questions. The following excerpt illustrates why we say this.

INT: I'm going to put in twenty ten (enters **20 10**). Now, what is going to happen?
LUCY: It'll go thirty.
INT: It'll go thirty.
LUCY: Thirty, without having to turn around and stuff.
INT: It'll go . . .
KIM: Twenty and then it'll go ten.
INT: The net effect is thirty?
KIM: Thirty. Yeah.
INT: (Presses RETURN; turtle moves accordingly) Right. So he's moved thirty. Now, if I put in a negative twenty ten (types **–[20 10]**), what is he going to do?
KIM: Face the other way, and then go twenty, and then go on and go ten, and then turn around.
INT: So, what's the net effect going to be?
KIM: Negative . . . (pause) ten.
INT: Negative ten?
KIM: Mmm-hmmm (yes).
INT: Okay, let's look at this one more time. Here we have twenty ten (points to arrow diagram), and what did we say the net effect of that was?
LUCY: Thirty.
KIM: Thirty.
INT: Okay. Now here we have the *negative* of twenty ten.
LUCY: Negative . . . (pause) negative thirty?
INT: Press RETURN and see what happens.

Kim and Lucy afterward expressed their feeling that they had understood the point of the question, so the interviewer gave them **–[40 –20]** and asked them to predict its net effect. There answers were:

LUCY: Twenty, . . . no, negative twenty, . . . no . . .

KIM: Negative sixty?

Clearly, they were still uncertain. It must be concluded that they had not yet formalized the net effect of two integers as being a single integer. It was only at the end of the eighth session that a question about the net effect of **–[–50 30]** immediately brought the answer "negative negative twenty." This may be taken as an indication that by then an abstract formalization of the composition of integer transformations had taken place.

### *Level 3*

At Level 3, composition and inversion of integers and integer expressions become the objects of attention, and their structural properties are emphasized. Kim and Lucy did not reach Level 3.

We argue elsewhere [Thompson & Dreyfus, in preparation] that Level 3 in regard to integers may be Level 0 in regard to algebra. We cannot address that issue here.

## Conclusion

It may be hypothesized that abstractions made in the context of a powerful and consistent model and through explorations of that model are more stable than rules for operating

upon integers that have been learned by rote. At this time, no data exists to support that hypothesis.

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# DYNAMICAL MAZES AND THE OB-SERVING COMPUTER

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LOWENTHAL F.

ABSTRACT

LOWENTHAL (1978) has defined what is now known as "Non-Verbal Communication Devices" (NVCD). Researches concerning the use of NVCDs by several kinds of subjects show the importance of this kind of approach (LOWENTHAL, 1980, 1984). A new research trend has been recently described (HARMEGNIES and LOWENTHAL, 1984) which enables us to extend previous research and to study aspects of NVCDs which have not been studied before. On the one hand we use computerized NVCDs with children : the computer gives (or "serves") information ; on the other hand we focus on new research topics with could not be studied without having recourse to informatics : the computer "observes" children.

## 1. Introduction.

In a recent paper (HARMEGNIES, LOWENTHAL, 1984), we have described a new framework for the study of NVCDs. Another paper describes the device conceived to carry out our first experimental investigations within this new research trend (LOWENTHAL, HARMEGNIES, in print). In this presentation, we will analyze and discuss the first experimental results obtained by means of this device.

## 2. The objects of study.

Our intention is to study the strategies subjects develop in order to solve problems presented by means of NVCDs. Considering subjects as black boxes, we could say that we are interested both in their output - and in their input - strategies : on the one hand, we want to study the productions of our subjects, i.e., the step-by-step constructions of final solutions (output-strategies) ; on the other hand, we try to study the ways in which subjects get hold of the information they need (input-strategies). For this purpose, we have been led to conceive a special computerized setting which allows us to collect the relevant data. In this paper, we will only consider data involved in input-strategies.

## 3. Experimental setting.

We ask each subject to reproduce on a base-board, using the bricks described by COHORS-FRESENBORG (1978), a diagram which is "hidden" on a television screen controlled by a computer. In fact, the experimenter provides the subject with a television screen and a box furnished with 25 buttons. On the screen, the subject sees a big rectangle subdivided by a grid into 25 small rectangles. There is a reproduction of these rectangles and of this grid on top of the button box : there is a button in the centre

of each small rectangle. The grid on the screen divides the diagram into 25 zones. Each preasure on a button provokes, in the corresponding zone of the screen, the apparition of the part of the diagram which the subject wishes to observe. This image remains present on the screen as long as the subject keeps pushing the corresponding button. Two images occupying two different zones cannot appear simultaneously on the screen.

In order to record the production behaviours of our subjects, we videotape all their problem-solving activities. Moreover, whenever a zone of the diagram is observed, the computer identifies it and keeps track of the event and of the time counter value when the image appeared and when the image disappeared (the time counter beeing put on zero when the subject started working). A synchronizing signal generated by the computer is also recorded on the videotape. At the end of the execution, all the data concerning the perceptive activity were stocked on a disk for further processing.

4. Experimental device.

Two groups of subjects were used in this experiment : a group of "trained subjects" consisting of children familiar with NVCDs and a group of "untrained subjects" composed of children unfamiliar with NVCDs. All the subjects were enrolled in upper primary school classes. The experimental treatment was the same in both groups. First of all, the subject was given two easy NVCD training problems, so that he had an opportunity to become familiar with the device used. Test problem 1 was given to the child immediately after this training session. It was immediately followed by test problem 2, which is more difficult than test problem 1.

5. Results.

5.1. The variables.

In this presentation, we will only focus on global results, intended to give a first general description of the ways subjects get the information they need to solve the test problems.

These global results will be expressed in terms of four variables of which the values will be measured for each subject in each test. These variables are :

1) The Number of Observations (NO), i.e., the total number of time buttons are depressed ; that is to say : the number of time zones of the diagram observed by the subject.
2) The Total execution Time (TT), i.e., the total time elapsed between the beginning and the end of a problem solving session.
3) The Observation time (OT), i.e., the part of TT during which the subject has observed zones of the diagram.
4) The Relative Observation Time (ROT) is defined as 100 OT/TT and indicates the relative importance devoted by the subject to the collection of information.

As we cannot, in this brief paper, reproduce tables for all individual results, we have grouped in table 1 the means and standard deviations of the four variables in each group for each test.

| | | Untrained group (N = 10) | | Trained group (N = 14) | |
|---|---|---|---|---|---|
| Test 1 | NO | 136.2 | (37.78) | 140.64 | (120.46) |
| | TT | 790.3" | (397.24") | 727.71" | (620.83") |
| | OT | 262.4" | (161.76") | 254.43" | (292.44") |
| | ROT | 34.24 | (14.91) | 32.06 | (11.75) |
| Test 2 | NO | 201.2 | (48.25) | 150.86 | (67.36) |
| | TT | 1066.8" | (389.92") | 823.21" | (345.51") |
| | OT | 327.7" | (103.74") | 242.93" | (167.26") |
| | ROT | 33.94 | (14.64) | 28.28 | (11.77) |

*Table 1 : Means (underlined) and standard deviations of NO, TT, OT and ROT in each group for each test.*

5.2. The statistical treatment.

For each of the four variables, the statistical treatment will consist first of comparing results to test 1 and test 2 in each group (intertest approach). This will be done, on the one hand, by means of a student's t test for matched samples and, on the other hand by means of Bravais-Pearson's cross correlation coefficient. An intergroup approach will afterwards be developped by computing for each test problem, a student's t statistic suited to the comparison of two means drawn from independant samples.

5.3. Analysis of the results.

5.3.1. Number of observations (NO).

The mean NO is 155, i.e., subjects need, on the average, 155 observations of zones of the hidden diagram in order to solve the problems they are confronted with.

In both groups, average NO increases from test 1 to test 2. Those differences are nevertheless significant in the untrained group only (untrained group : $M_1$ = 136.2, $M_2$ = 201.2, t = 3.95, p = .004 ; trained group : $M_1$ = 140.67, $M_2$ = 150.86, t = .47, p = .65). The cross-correlation coefficient between test 1 and test 2 is not significant in the untrained group (r = .36, p = .30) although it is very significant in the trained group (r = .80, p = .0009).

(1) The student's t test for independent samples had to be replaced here by the Mann-Witney U test for independent samples of rankable scores, because of the too great between-variances F ratio.

The differences between the means of the groups are not significant for test 1 ($M_1$ = 136.2, $M_2$ = 140.64, Mann-Witney's U = 64.5, p = .10 (1)) but weakly significant for test 2 : the average NO seems greater in the untrained group ($M_1$ = 201.2., $M_2$ = 150.86, t = 1.93, p = .06).

5.3.2. Total execution time (TT).

On the average, subjects spend about 14 minutes (839 sec.) solving one problem. In both groups, average TT are longer for test 2 than for test 1. Nevertheless, this increase is only significant in the untrained group ($M_1$ = 790.3, $M_2$ = 1066.8, t = 2.36, p = .04), while it is far from being significant in the trained one ($M_1$ = 727.71, $M_2$ = 823.21, t = .89, p = .61). The correlation between both tests is moreover weakly significant in the untrained group (r = .60, p = .06) and very highly significant for the trained subjects (r = .83, p = .0004).

The differences of means between groups are very far from being significant at test problem 1 ($M_1$ = 790.3, $M_2$ = 727.71, t = .29, p = .77), but nearly significant at test 2: the average execution time tends to be longer in the untrained group ($M_1$ = 1066.8, $M_2$ = 823, 21, t = 1.54, p = .12).

5.3.3. Observation time (OT).

The average time subjects spend observing the diagram is about 4 minutes and a half (267 sec.).

In the untrained group, the mean OT increases from test 1 to test 2 ($M_1$ = 262.4, $M_2$ = 327.7) ; this difference, althouth nearly significant (t = 1.52, p = .16), should nevertheless be considered with prudence.

On the contrary, in the trained group one can observe a slight decrease of mean OT ($M_1$ = 254.43, $M_2$ = 242.93), but this one is not significant at all (t = .28, p = .78). The interest correlations are weakly significant in the untrained group (r = .61, p = .06) and very highly significant in the trained group ( r = .93, p = .00001).

The differences of means between the two groups are not significant in test 1 ($M_1$ = 262.4, $M_2$ = 254.43, t = .08, p = .94) and nearly significant in test 2 ($M_1$ = 327.7, $M_2$ = 242.93, t = 1.47, p = .16) : the trained subjects tend to spend less time observing the diagram of test 2 than the untrained ones.

5.3.4. Relative observation time (ROT).

The general mean of ROT is 31.8 : the subjects spend more or less 30% of the total execution time to scan zones of the diagram. This variable is very valuable, since it establishes a relationship between OT and TT ; this relationship is independent of TT, which shows a great variability among subjects. ROT is moreover very useful because it reveals the global structure of the subjects' cognitive strategies in solving the problem: while the other variables give only quantitative information, ROT gives qualitative information.

In both group, one can note a slight decrease of ROT from test 1 to test 2, but these differences are not significant (untrained : t = .067, p = .94 ; trained : t = 1.49, p = .15). The correlation coefficients are weakly significant in the untrained group (r = .58, p = .076) and very significant in the trained group (r = .79, p = .0011).

The differences of means between groups are significant neither for test 1 (t = .34, p = .74=, nor for test 2 (t = 1.004, p = .33).

6. Conclusion.

From a quantitative point of view we can note that :

- In the trained group, average NO, TT and OT are the same for test problem 1 and test problem 2 : trained subjects do not need more information for test 2 than for test 1, although test problem 2 is more difficult to solve. Moreover, since the correlation coefficients are very significant, one can conclude that most of the subjects keep the same input-strategies for both problems.
- On the contrary, in the untrained group, the average NO, TT and OT are significantly different. In fact, these differences consist of an increase of the mean values at test 2 : we shall conclude from this that, as a general rule, the untrained subjects need to collect a larger amount of information to solve problem 2 than they need for problem 1. Furthermore, since the correlation coefficients are weakly or not significant, it seems that subjects change their input-strategies when they are confronted with test 2.
- As test 1 is rather easy, one can assume that subjects spontaneously develop well-suited input-strategies : this hypothesis seems to be confirmed by the not significant tests between groups for test problem 1. Conversely, the (nearly) significant superiority of average NO, TT and OT in the untrained group at test 2 suggests that the input-strategies developped by trained subjects are better than those developped by untrained subjects.

From a qualitative point of view we can note that :

- Although ROT exhibits no intergroup global variations, in the trained group, the average ROT is nevertheless smaller for problem 2 than for problem 1. Although nearly significant only (p = .15), this difference indicates that, in the trained group, the input-strategies tend to improve when subjects are confronted with a more difficult problem.

Thus, trained subjects don't only develop better input-strategies : when faced with more difficult problems, they seem to be able to qualitatively improve their strategies, while untrained subjects just stamp about.

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# ITERATIVE AND RECURSIVE MODES OF THINKING IN MATHEMATICAL PROBLEM SOLVING PROCESSES

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## SUMMARY

Iterative and recursive methods of problem solving are important in mathematics and computer sciences as well. It is not easy for students to acquire these concepts in mathematics instruction. The use of the micro-computer in the class-room might give additional practice with iterative and recursive structures. In a suitable programming environment students can generate their own procedures which may reinforce understanding.

## THE CONCEPTS OF ITERATION AND RECURSION IN MATHEMATICS EDUCATION

The concepts of iteration and recursion play a crucial role in mathematical problem solving processes. It is very important that students be familiarized with these strategies, which make a successful handling of various mathematical problems possible.

In many cases, iteration seems to be the initial concept. Its characteristic quality is the repeated execution of a particular notion. A sequence of terms is produced so that $a_n$ and $a_{n+1}$ are determined from their predecessors in the same way. Repeating the same operation may enable the problem solver to recognize relations between the elements of a solution, or it may help him or her develop suitable ideas for the proof of a mathematical theorem.

A typical example for the iterative mode of thinking can be demonstrated by the solution of the following problem:

> A plane P is dissected by a number n of straight lines. Every pair of distinct lines are incident with a unique common point. What is the number of areas in which the plane is split up by the lines?

Solving this problem iteratively is a kind of a systematic trial and error approach. Drawing one straight line generates two areas. The second line leads to four distinct areas, adding a third line produces seven areas. If this process is continued, the student will probably realize a regularity in this sequence of numbers.

As in other problems, in this example, the iterative approach may prefigure the elements of the solution, and may exhibit relations between those elements. Moreover, it may help the problem solver work out ideas for a generalization, and even for a formulation of a mathematical theorem.

A closed formula for the number of areas makes use of a recursive description:

$$a_1 = 2 \; ; \quad a_{n+1} = a_n + (n+1) \qquad (n \in N) \; .$$

This formula can also be generated, when the student tries to analyze the problem from the beginning in a recursive mode. Using recursion means working on a more abstract and formal level. The problem is reduced to an isomorphic one with simplified conditions. The problem solver has to look at the differences between two adjacent terms of the sequence. So, for example, he or she may compare the situation for n=5 and n=4, and deduce the general rule from this special comparison. If one line is removed, only the rate of change in the number of areas is of interest.

This recursive mode of thinking will give rise to the above formula almost immediately, which is a short, elegant, and significant formulation of the problem's solution.

Recursion is a very important and powerful tool in mathematical problem solving processes. It is used in order to get a closed definition of a specific sequence of numbers. Moreover, recursive thinking is the basis for the proof of a theorem by induction, and constructions may also have recursive roots (MÜLLER & WITTMANN 1978[2]). The application of recursive methods may be initialized by iterative processes. If a recursive formulation of a solution is prepared in this way, iteration can be regarded as an empirical strategy which is direct and local. Special cases of the problem are seperated and worked out. This may help the problem solver in the process of finding a general solution. A recursive definition can be proved by induction but may be found intuitively by iterative construction of a sequence (COHORS-FRESENBORG 1979).

But iterative processes must not necessarily precede recursive formulations, as several investigations show. OERTER (1971) describes problem solving processes where the subjects had to cope with the

TOWERS OF HANOI. The problem solvers needed some time until they got acquainted with the problem and its structure. Then, some of them directly realized the optimal strategy for the solution, which is recursive. Especially, iterative modes of thinking were not always recognizable.

An investigation by ANZAI & UESATO (1982) produces a similar result. Using recursive formulations for mathematical functions, they discovered that the understanding of these formulations was facilitated if the subjects understood iterative structures. But their investigation also shows, that there are problem solvers who are able to work with recursive definitions without preceding knowledge of iteration.

The concept of iteration is explicitly or implicitly part of the mathematical instruction in all grades, whereas the use of recursive methods is usually restricted to the upper grades of secondary schools. At this point the concept of recursion is often hard for the students to understand. There are a lot of reasons for these difficulties, but surely one important reason is the fact, that pupils have not had sufficient opportunity to practice this methods at different levels. The principle of a spiral approach to mathematical concepts, which is widely accepted, presupposes an early and simultaneous teaching of iteration and recursion. Elementary problems in this respect should be experienced by the students.

## ITERATIVE AND RECURSIVE CONTROL STRUCTURES

The concepts of iteration and recursion are also important tools in the computer sciences. They are powerful control issues in many programming languages which permit the repeated processing of a specific part of a computer program. They provide efficiency in formulating a problem's solution in a computer code, and using memory economically as well.

These two types of control structures represent two different modes of processing. The concept of iteration thus stands for looping back in the execution of the algorithm. Parts of a computer program are done again by returning to a fixed line. On the other hand, recursion means that the procedure calls a copy of itself; the definition of the algorithm is using itself as a sub-algorithm. While executing the procedure

the inputs have to be simplified in every call. So, if there is an adequate stop condition, the program will terminate in a finite number of steps. Calling a procedure is passing the control on to it. The control is returned to the calling procedure after the job is done. A recursive procedure particularly works on various levels at once.

For students, recursive methods in mathematics are not easy to understand. Working with a computer produces similar difficulties. An investigation of KURLAND & PEA (1983) reveals some reasons for misconceptions in the understanding of recursive procedures. The investigation was performed with children between 8 and 12 years of age, who were fairly familiar with LOGO. The children could predict the effect of a procedure using last-line recursion but failed when coping with programs which required embedded recursion.

These difficulties also arose in a test with adult programming novices at the Pädagogische Hochschule Karlsruhe. The students were supposed to analyze two similar procedures. They had been working in a LOGO environment for some weeks when they passed the test.

The interpretation of a procedure using embedded recursion especially was influenced by misconceptions on the role of a recursive call. The students' mental model is in many cases the same as for iteration, that is, the looping back of the program to the beginning. While this model works for some last-line recursive procedures, it will not lead to correct prediction in the case of embedded recursion. There were three typical misconceptions:

(i) The students had learned, that a recursive line in a LOGO procedure calls a copy of itself. But this knowledge seemed to be more passive than active. They did not accept the difference between this model and the looping model.

(ii) The students sometimes ignored lines in a program which did not fit in their pattern of anticipation.

(iii) The local quality of the STOP command in LOGO seemed to be difficult. The students were misled by the conception, that STOP meant an immediate break in the program. They did not realize that the control was returned to the calling procedure by STOP.

All these misinterpretations are mainly caused by problems with conceptualizing the operation of the procedure on various levels. In their mental model the students are influenced by sequential ways of thinking on just one level. They are able to conceive structures which

repeat certain commands, whereas it is not at all easy to accept the calling procedure to wait and see, that control will finally come back to this calling procedure, perhaps after a lot of excursions on different levels.

## COMPUTERS IN MATHEMATICAL EDUCATION

The problems which have been described here for computer sciences are very similar to those in mathematical instruction. Modes of thinking which are familiar to the students from their everyday lives, and iteration is such a mode of thinking, have to be changed in order to provide an understanding of recursive structures. The student is supposed to use the concept of recursion and to create recursive productions actively. So he or she has to become acquainted with the relevant concepts and techniques, and is required to learn many procedures which do not only apply to a specific set of problems. Working with the micro-computer in mathematical instruction may be helpful. Though there are corresponding difficulties in computer sciences, the combination of both aspects can promote a better understanding of mathematical concepts and computer science concepts as well.

For example, recursive definitions of sequences sometimes seem to initiate circular processes. With the aid of a suitable programming environment, students are able to realize that such a process will terminate.

BIERMANN E.A. (1977) demonstrate an approach to the TOWER OF HANOI problem. A first result in their process is the recursive formulation of the solution. Let m(n) be the number of moves which have to be performed in order to create a tower of n disks. Then the definition

$$m(1) = 1 \quad ; \quad m(n+1) = m(n) + 1 + m(n) \qquad n \in N$$

may be generated by using iterative and/or recursive modes of thinking. The authors state that the formula is not too much of a help when students have to operate with large numbers. But the structure of the problem is pointed out by this definition. With the aid of the computer the numerical solution can be found immediately. The definition may easily be translated in a LOGO procedure, which will output the number of moves.

```
TO TOWER :N
   IF :N = 1 OUTPUT 1
   OUTPUT 1 + 2 * TOWER :N - 1
END
```

So the computer can do the fatiguing job of calculating. For the students this provides important experiences. They can recognize that the recursive structure of the problem's solution leads directly to its solution, and they may generate the idea for the formula

$$m(n) = 2^n - 1 \qquad (n \in N).$$

Working out examples like this one, and working with a computer in general, can clarify the differences between iterative and recursive concepts. Moreover, it may serve as an aid to dealing with these concepts on an elementary level. So the computer may encourage the development of functional mental models in mathematics. Children can write their own programming ideas and can test whether they will work or not.

Computers in the class-room might give the opportunity for practicing iterative and recursive modes of thinking at lower levels and early in secondary school education. Converting an iterative into a recursive mode of thinking is important in order to embed a lot of isolated problems in more general structures.

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# COMPUTER MODELLING OF MATHEMATICAL PROBLEM-SOLVING PROCESSES AS AN INSTRUMENT OF COMPETENCE ANALYSIS

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## PRELIMINARY REMARKS

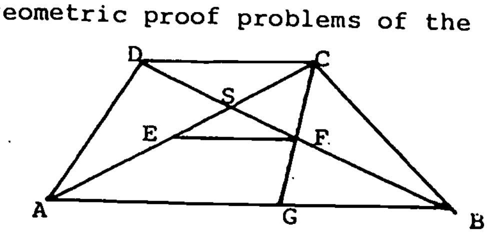
This paper aims at the computer modelling of solving mathematical problem classes under the aspect of rational task analysis (Resnick /Ford 1981). The objective in developing a computer program from this point of view is to gain detailed information concerning the knowledge and skills which a student needs to solve the problems of the class. The rendered information may then be used in a tutoring program for students. A computer program which serves this educational function has at least to meat the following two preconditions:
(a) The internal information processing is transparent and interpretable as problem solving process of a human problem solver. (The running computer should deliver a protocol of "loud thinking" which could stem from a human problem solver.)
(b) The computer program is structured as a hierarchy of procedures which is interpretable as a learning hierarchy.

A computer program for geometric calculation problems which only models the aspect of chaining backwards has been presented on the PME 7 in Israel (Holland 1983). Meanwhile a computer model has been developed and implemented in MICRO PROLOG on an APPLE-2e which solves geometric proof-problems. The methods used are similar to those of H.Gelernter's "Geometry Theorem Proving Mashine" (Gelernter 1963, Bundy 1983). We present this computer model in the following as an example of our general objective.

## A COMPUTER MODEL FOR SOLVING GEOMETRIC PROOF PROBLEMS

The model solves geometric proof problems of the following kind:

Given: $AB \parallel DC$
$AE=EC$
$BF=FD$
Prove: $EF \parallel AB$



```
ZIEL: (par (E F) (A B))
KONNEKTOREN:
(1 MLDR (s-gl (E A) (E C)) (s-gl (F G) (F C)))
(1 P-TRANS (par (E F) (C D)) (par (C D) (A B)))
NEUZIELE: ((s-gl (F G) (F C)))

ZIEL: (s-gl (F G) (F C))
KONNEKTOREN:
(1 DKS (dr-gl (F G B) (F C D)))
NEUZIELE: ((dr-gl (F G B) (F C D)))

ZIEL: (dr-gl (F G B) (F C D))
KONNEKTOREN:
(2 SWS (s-gl (G B)(C D))(s-gl (B F)(D F))(w-gl (G B F)(C D F)))
(2 WSW (s-gl (B F)(D F))(w-gl (G B F)(C D F))(w-gl (B F G)(D F C)))
(3 WSW (s-gl (G B)(C D))(w-gl (F B G)(F C D))(w-gl (G B F)(C D F)))
NEUZIELE: ((s-gl (G B) (C D)) (w-gl (G B F) (C D F)))

ZIEL: (s-gl (G B) (C D))
SACKGASSE
NEUZIELE: ((w-gl (G B F) (C D F) ((w-gl (B F G) (D F C)))

ZIEL: (w-gl (G B F) (C D F))
KONNEKTOREN:
(0 WEWI (par (G B) (C D)))

*** (w-gl (G B F) (C D F)) folgt aus ((par (G B) (C D))) wegen WEWI

ZIEL: (w-gl (B F G) (D F C))
KONNEKTOREN:
(0 SCHW)

*** (w-gl (B F G) (D F C)) folgt aus () wegen SCHW
*** (dr-gl (F G B) (F C D)) folgt aus ((s-gl (B F) (D F))
                                      ((w-gl (G B F) (C D F))
                                      ((w-gl (B F G) (D F C)) wegen DKS
*** (par (E F) (A B)) folgt aus ((s-gl (E A) (E C))
                                   ((s-gl (F G) (F C)) wegen MLDR
```

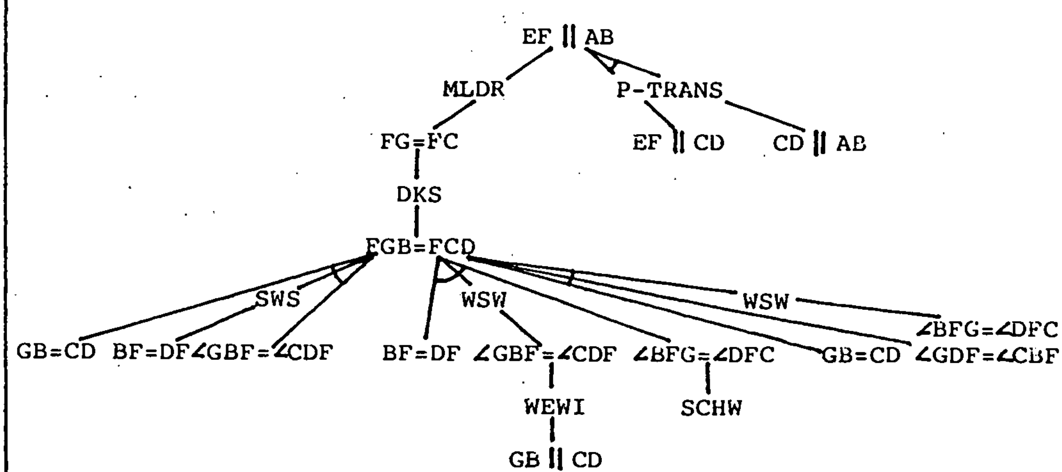



Figure 1: Protocol and AND/OR tree

The non-metric premises of the proof problem are implicitely given to the students by the figure. In the computer model they are represented by a special predicate "pointrow". For example the list (A E S C) is a pointrow. The computer model finds the proof by chaining backwards. A shortened protocol of the problem solving process and the corresponding AND/OR tree are shown in Figure 1. Knowledge and skills which are necessary to solve the problems of the problem class are represented in three modules: SAETZE, ANWENDUNG and STRATEGIE.

In the module SAETZE each admitted theorem is represented as a list to the predicate SATZ. As an example Figure 2 shows the theorem "MLDR" which is applied in the first backward step (Figure 1).

```
((SATZ MLDR                        (/* name of the theorem)
   (par (U V) (X Y))               (/* conclusion)
   ((mldr-fig (Z X Y U V)))        (/* list of non-metric premises.)
   ((s-gl (U X) (U Z))             (/* list of metric premises)
   (s-gl (V Y) (V Z)))

((mldr-fig (Z X Y U V))
   (dreieck X Y Z)
   (zwischen X U Z)
   (zwischen Y V Z)
```

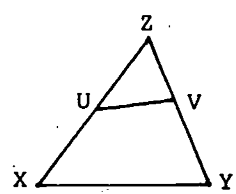



Figure 2

The module ANWENDUNG models skills which are needed to test a theorem for its applicability in a step of chaining backwards (b-step). We call a theorem b-applicable, if the following test conditions hold:

(1) The conclusion of the theorem matches with the actual goal (the "ZIEL" in Figure 1).

(2) The non-metric premises of the theorem may be matched with the non-metric premises of the task.

(3) As a consequence of (1) and (2) all metric premises of the theorem are instantiated (i.e. all variables are bound by constants).

(4) The instantiated metric premises are true sentences. (Otherwise futile efforts would be spent in trying to prove sentences which are in fact false.)

(5) No one of the instantiated premises is an ancestor of the actual goal in the AND/OR tree. (A necessary condition to avoid circularity).

We illustrate the test in applying the theorem MLDR to the goal: (par (E F) (A B)) (first step in Figure 1).

(1) The goal matches with the conclusion (par (X Y) (U V)) of the theorem and renders the substitution: A/X, B/Y, E/U and F/V.

(2) The match is successful and finds mldr-figure(S A B E F) with the additional substiution: S/Z.

(3) The metric premises of the theorem are instantiated to (s-gl (E A)(E S)) and (s-gl(F B)(F S))

(4) Unfortunately both sentences are false.

A new attempt leads in steps (1) to (4) to the discovery of (mldr-fig C A G E F) and the instantiated premises (s-gl (E A)(E C)) and (s-gl (F G)(F C)). As one may judge from a well drawn figure, in this case both are apparently true sentences. Since condition (5) is necessarily true for the root of the AND/OR tree the theorem MLDR is b-applicable to the goal sentence (par (E F)(A B)) and the ouput of the procedure is the "connector"

(1 MLDR (s-gl (A E) (E C))(s-gl (F G)(F C)))   (c.f. Figure 1).

The module STRATEGIE represents the strategic skills which are needed to perform the backward chaining. This is done by a depth-first backtracking algorithm. As a crucial point it should be noticed that each node in the AND/OR tree is fully developed befor a new goal sentence is selected for the next step. That is, for each goal sentence successively all admitted theorems are tested for b-applicability and accordingly all connectors are determined which have the goal sentence as a consequence. This permits the evaluation of each connector and the choice of a connector which seems to be promising for the further search process. In the present state of our model the evaluation function is still very simple: Each connector is evaluated with the number of those of his members which are not yet "solved". (A sentence is called "solved" if it belongs to the metric premises of the task or has been proved meanwhile.)

## CONSEQUENCES FOR A PROBLEM SOLVING TRAINING

What does the computer model teach us for a problem solving training ? Corresponding to the three modules SAETZE, ANWENDUNG and STRATEGIE of the computer model we can discriminate between:

(a) Knowledge of the set of admitted theorems,
(b) Skills for applying a theorem to a goal sentence,
(c) Strategic skills to perform the process of chaining backwards.

To (a): The knowledge of those theorems which are admitted as proving-tools for the special problem class is acquired by simple rote learning. At the beginning of the training the theorems should be presented to the students listed on a sheet of paper and ranged in an order which supports the application in a b-step.

To (b): As the analysis of the modul ANWENDUNG has shown, the procedure of applying a given theorem to a given goal statement in a b-step is a rather complicated process. The student has to perform the test for b-applicability in a similar way as the computer model does this job. Some of the sub-procedures which are involved should be trained in isolation. This concerns particularly the satisfaction of the non-metric premises in step (2), where the student has to detect some special pattern in the given figure, for example a pair of congruent triangles, a pair of alternate angles or a mldr-figure (Figure 3).

To (c): In contrast to the simple method of chaining forwards, which probably does'nt need any special training, the strategy of chaining backwards is an intricate process, especially if backracking is necessary. Before using the method of chaining backwards in geometric proof problems, the students should be trained in this method with geometric calculation problems, which are altogether more easy to solve than proof problems. As own investigations indicate, a training in the method of chaining backwards with calculation problems is successful with the upper third of 14- and 15 graders (HOLLAND 1983). As mentioned before, the computer model uses a strategy which, in developing the actual goal sentence, tests all admitted theorems for b-applicability at once. The resulting connectors are evaluated and the most promising connector is choosen for further development. The remaining connectors are disposable if backtracking is necessary. This search strategy for chaining backwards is in full accordance with a "general problem solving method" which L.Landa has propagated as a result of extensive case studies with students who had to solve similar geometric proof problems (LANDA 1976).

We conclude with a short discussion of the adequacy of the computer model. It is evident that one can construct other models which are equivalent to our model because they solve problems of the same problem-class, but which use different representations and methods. But how is adequacy of a model to be judged? The cognitive psychologist is interested in a model which mirrors the actual problem solving behavior of the human problem solver with the highest possible degree of accuracy and which takes individual differences into account (GREENO 1978). On the other hand the educational psychologist is more interested in a model which exhibits the knowledge and skills which he judges as essentials for the teaching process. From the latter point of view it is no deficiency that our model uses a strict strategy of chaining backwards, although evidently no expert or student would use this method without some inserted phases of chaining forwards.

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# CHILDREN LEARNING MATHEMATICS - INSIGHTS FROM WITHIN A LOGO ENVIRONMENT

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This paper is concerned with how the use of Logo within the secondary school mathematics curriculum can allow the teacher to gain insight into pupils' mathematical reality i.e. the way mathematics is learned, how it is perceived and the situational factors embedded within the practices of mathematical activity which affect its subsequent use. Logo is a computer programming language derived from the LISP family. It was designed by Feurzig, Papert et al (1969) to provide a mathematical environment accesible to children of all ages and abilities. The procedural and extensible nature of Logo encourages problem solving activity, in particular the breaking down of problems into parts and the use of part solutions as building blocks within alternative structures.

The Logo Maths Project (Hoyles, Sutherland & Evans 1985a) which commenced in September 1983 is a three year longitudinal study with children who were 11/12 years old at the beginning of the period of research. The project is monitoring and evaluating how Logo can be used within mathematics classrooms which have adopted a 'pupil centred' approach to learning. Since one of the objectives of the research is to monitor the effects of collaborative learning the pupils work in pairs at the computer. The computers are placed in the corner of the classroom and pupils take turns to work at the computer during their 'normal' mathematics lessons. The researchers act as participant observers in the classroom. They are therefore able to study influences of classroom context and management and the spread of ideas between the Logo and non Logo work. Working continually in the school in this way together with the administration of occasional structured tasks enables the researchers to experience the dynamics of the pupil learning and they are thus better able to understand and interpret it . Systematic data is being collected throughout the three years of the project for four pairs of case study pupils (one boy pair, one girl pair and two mixed pairs). The data includes recordings of the pupils' Logo work, all the spoken language of the pupils while working with Logo (a video recorder is connected between the computer and the monitor), the researchers interventions and a record of all the other mathematical work undertaken by the pupils. All the video data is transcribed and this together with pupil and teacher interview data provides an extremely powerful data base from which to analyse the complex three-way interactions between the pupils and the computer. Since September 1984 in addition to this case study work, the research has been extended into ten further classrooms where data is being collected by questionnaire, teacher

and pupil interview, structured tasks and task-based interviews.

Mathematics educators are becoming aware that pupils do try to make sense of the mathematics presented to them and although when questioned the child's response is perhaps wrong from the perspective of the teacher it is rarely random or illogical from the perspective of the child. 'Errors are not understood as mere failures of pupils but rather as symptoms of the nature of the conceptions which underlie mathematical activity.' (Balacheff 1984 p36). In addition despite exposure to formal methods children tend to solve mathematical problems in the informal ways which work for them in specific cases (Booth 1984). In order to help the pupil, the teacher needs therefore to gain insight into the child's intuitive conceptualisations, to work with them and adapt them to a more formal conceptual framework. The problem for the mathematics teacher is that only the product of pupil's work is available and the process of solution and way of thinking is not necessarily clear. Talking to the pupils elicits more information about their conceptualisations but often only a very incomplete picture can be obtained. It is the contention of this paper that Logo programming activity within the mathematics classroom can render pupils' intuitive mathematical conjectures and strategies more accesible to both the pupils themselves and to the teacher. The pupils' planning, rough work,(1) methods of recording and construction of Logo programs all provide a powerful means of illuminating both how the problem is perceived and how it is to be solved.

The following provides a framework for discussing some of the insights into children's learning of mathematics which have already emerged during the Logo Maths Project. (2)

1 **Fragmented domains of knowledge.**
2 **'At Homeness' (3) with decimals.**
3 **Contextual influences on pupil conceptions.**
4 **Restricted learning environments.**

**1 <u>'Fragmented domains of knowledge'</u>**

The longitudinal transcript data has made it possible to trace the development of the knowledge base of the case study pupils over an extended period of time. Observations support the notion that knowledge is fragmented into 'subjective domains of experience' (Bauersfeld, 1984) and that specific concepts must be re-experienced and relearned in different contexts before a synthesis can take place.

An example of this is given by John and Panos's developing understanding of 360 degrees as a total turn over a period of seven months. (Hoyles, Sutherland & Evans 1985b). Before starting the Logo activity Panos and John successfully completed work from a mathematics booklet which 'covered' 360 degrees being the angle around the point. In their first session they recalled with assistance that there were 360 degrees 'in a circle' and were able to relate this to the drawing of a complete circle. However in their next two sessions when attempting to draw a complete circle they consistently used an experimental trial and error approach. They then produced a rotated pattern of small circles with a turning angle of 2.5 and 50 REPEATS. By counting the image on the screen they worked out that they needed 16 REPEATS when the turning angle was 22.5 but they did not relate these two numbers to 360 degrees. Subsequently, they produced other rotated patterns with their small circles by using a halving and doubling strategy on the appropriate numbers of an existing complete rotated pattern. John however wanted to use a global strategy of "the smaller the gap the more circles you'd need" to produce the complete rotated patterns. In order to provoke Panos and John into discovering the relationship between the number of repeats and the angle turned in a rotated pattern we grouped them with two girls and gave them a structured task (figure 1). Panos initially extended his halving and doubling strategy into the use of proportion.

"I'm thinking how to do it for 18...it's not easy...24...if 6 is 60...I think it is 20... 60 divided by 3 is..."

In order to encourage Panos to develop a global idea of the task rather than rely on his step by step 'scaling' strategy all their written records were hidden. (These had listed all the commands for the complete patterns which they had already discovered). We also suggested that they try to find the turning angle for 20 REPEATS. Lucy who was in the group suddenly had a flash of insight.

(Lucy) "I think it is 28...because 20 into 360 is 28."

They tried this but because of the incorrect arithmetic their conjecture was not supported. Panos however checked the arithmetic and said:-

(Panos) "360 divided by 20 is 18 not 28."

They tried this and were delighted with their result. By the end of the session both Panos and John were able to use the relationship between the number of REPEATS, the turn and 360 and were able to articulate the relationship.

In their following session they extended their 'rule' to produce regular polygons and eventually used it to write a generalised regular polygon procedure. Several weeks later they were still able to use this 'knowledge' as part of a larger project. We realised when

we questioned Panos that he did not perceive that there had been any development in his knowledge. We asked him why he had used 360 when calculating an angle needed for a rotated pattern:-

(Panos) "Because that's the number round a point...I knew it...they used to teach us that in our old school...it's the only thing they did teach us...miss...they used to go on and on about it".

Studying John and Panos's developing understanding of 360 degrees as a total turn has highlighted the lack of initial transfer from a 'rote' understanding of degrees as the sum of the angles round a point and an ability to use this in a static context, to the dynamic context of 360 degrees as a total turn.

## 2 'At homeness' with decimals.

We have found many instances of pupils' resistance to the use of their 'formal' knowledge of decimals in the Logo context and inadequacies in their understanding of, for example, the order within decimals. However meaningful projects have provoked the context for building up confidence in these numbers and for ordering them appropriately.

In her fourth session Janet felt that decimals were 'peculiar' and was loathe to use them in her Logo work. Sally her partner had calculated that 32.8 was the turning angle needed to draw an eleven sided regular polygon but Janet wanted to change the goal:

(Janet) "Why don't we do the 12..........it won't have a point..."

Later when drawing a clown's face the pair wanted to place the nose in the middle of the face which required a move of 'half 15'

(Sally) "Now do backwards 7.5"

(Janet) "Forget about .5 it's silly"

During their second year of using Logo however Sally and Janet have been given a structured task which involved using an input to scale all the distance commands in a design. This task provoked the use of decimals as input in a meaningful context and Janet was confidently able to use them correctly.

Mary and Nadia had been working on a project of drawing a solar system (figure 2) for 10 sessions over a period of 30 weeks. After drawing all their planets they wanted to design a tiny star. They tried to design this in direct drive but because their design was so small the screen output was not clear enough for them to check the angles. Mary asked us if they could design a larger star and then 'shrink' it. When they had successfully designed their 'large' star in direct drive they were shown how to write a procedure for

their star with a distance scale factor as input (figure 2). They then started to 'make sense of' the input to their star procedure. Melanie discovered that multiplying distances in her star by 0.5 would make it smaller. She then predicted that multiplying by 1.5 would also make it smaller because "1.5 is not a whole number". Concrete experimentation on the screen led her to realise that her conjecture was wrong.

## 3 Contextual influences on pupil concepts

From studying the transcripts many instances have occured in which the context of the first contact of an idea has very obviously influenced its conceptualisation. Discriminations in terms of the characteristics of a new concept are made according to the demands of the context in which it is experienced. (See also Davis 1984 p363) In Logo programming 'incorrect' invariants within a concept become quickly apparent as computer response provokes the pupils to reflect upon their original abstractions and modify them. In other contexts original conceptions are more likely to persist.

Ray and Helena first used an input of 100 in the REPEAT command in the context of drawing a circle (REPEAT 100 [FD 7 RT 7]. For many sessions they associated this integer input of 100 with the REPEAT command until a project of filling in a square provoked them into trying a different input and finding it worked!

Joanne first used the REPEAT command within a procedure in order to draw a rotated hexagon. She persisted for many sessions with the idea that this was the only way it could be used. When this misunderstanding became apparent, interventions were made to try to help her see the REPEAT command could be used in direct drive. However tracing through her work seems to suggest that Joanne is still not confident about this. She tends to try out a series of commands in direct drive without using the REPEAT command and then collects together appropriate terms with the REPEAT command later in the editor.

In their 18th session after considerable experimentation Jude and Linda had discovered that six REPEAT turns of 60 degrees were needed in order to draw a regular hexagon. This discovery led Linda to conjecture.

(Linda) "Oh that's excellent..so what it is is REPEAT 60 for 6 times...so for an octagon it will probably be REPEAT 80 and for a pentagon REPEAT 50.......try a pentagon next...I'm feeling really brainy now..."(4)

This was a 'classic' case of over generalisation from a single instance. It was however quickly remedied in the computer context as when they typed:-

REPEAT 5 FD 50 AND RT 50

their conjecture was obviously incorrect.

### 4 Restricted learning environments

Pupils frequently and spontaneously create a restricted learning environment in their Logo work by using a small domain of numbers as inputs to computer commands. The importance of working in this restricted environment should not be underestimated in that it may allow the pair to build up their confidence and feelings of control, yet be flexible enough to allow a considerable amount of creative exploration. (see also Turkle 1984). It is however important to be aware that the process by which goals are constructed within these environments may not be apparent to the pupils and that the tools available within these environments define what can be done in ways that are not necessarily in the control of the pupils.

> Throughout the whole of their first year of programming in Logo,Linda and Jude restricted their planned angle input to multiples of 45 which were less than or equal to 180. This strategy for angle input influenced the shapes they chose to produce i.e. squares, cubes, rectangular letters and enabled them to cope easily with parallel lines or symmetrical slanting lines. At the beginning of their second year it was suggested that they draw some regular polygon shapes. In order to draw a hexagon they initially tried FD 40 RT 45 repeated 6 times and produced an incomplete octagon. This provoked considerable trial and error and experimentation until eventually they thought to use a turn of 60 and produced the regular hexagon. When asked why they had always used multiples of 45 for angle input before Linda replied
>
> (Linda) "Dunno....cos we didn't want to explore any different angles I suppose".
>
> After they had achieved a hexagon they attempted to draw an octagon. They had achieved this many times before using their 45 degree strategy but had obviously not reflected on its construction. To use 45 degrees was not now their first inclination and they tried to use '6 sides for 60 degrees' so '8 sides for 80 degrees' as described earlier in section3.

### CONCLUSION

Insights into pupil mathematical thinking gained through observation of their 'spontaneous' Logo programming can be used to increase the teacher's awareness of the pupil strategies and potential. In addition the teacher can use structured tasks in Logo to gain insights into pupil understanding of specific mathematical topics (e.g. decimals, negative numbers). With these experiences,we maintain that the teacher is better able to decide what activities are necessary to help the pupils move from intuitive to more formal mathematical frameworks in both Logo and non-Logo contexts. In addition observation of the pupils' collaborative work at the computer may also provoke the teacher to look more carefully at the role of discussion in representing and formalising a task.

The important influence of the atmosphere in the mathematics classroom must not be underestimated. It is our contention that the potential of Logo will only be realised if the pupils are able to work at the computer within a non-competitive environment in which they have the confidence to try things out for themselves in an experimental manner. This type of mathematical environment encourages pupils to be actively involved in their own learning. We have found that pupils who have passively copied procedures from other pupils or from a handbook because they 'needed' an impressive outcome have not been provoked to reflect on these procedures and have learned little or nothing from the activity. Similarly we have observed that intervening to 'tell' pupils a rule or an idea has had little or no effect until the context of the project has been such that the pupils themselves have seen the need for these ideas. It would appear that in such contexts the mere act of copying or being told induces both a focus on product and a passivity in the pupils which is destructive to real learning. Within an atmosphere which encourages active reflection however, pupils have been able to devise and overcome challenges within their own projects in ways which we would not have been able to predict. The influence of Logo on the teacher's perception of their pupils' mathematical 'ability' is being investigated in our continuing research. This research continues until August 1986.

Footnotes

1. Pupils seem quite naturally to make rough plans and notes during their Logo activity. It is almost as if the 'neat' product on the screen frees them to use pencil and paper for rough work in contrast to their other mathematical activity in which they tend to be 'conditioned' to produce 'neat' work.

2. It is recognised that many of these intuitions are already well known to mathematics educators. The point made here is that the Logo environment makes such pupil conceptions more public and open to scrutiny and in ways which are not threatening to pupil self concept.

3. Reference Cockcroft 1982 para 39, for description of 'at homeness' with numbers.

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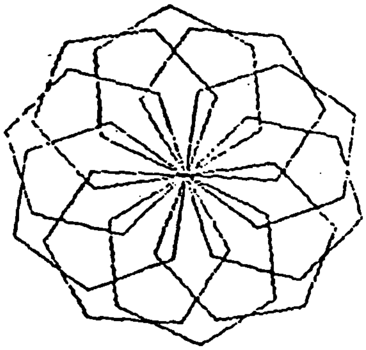
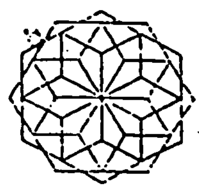
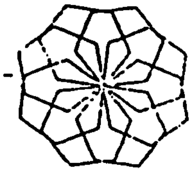
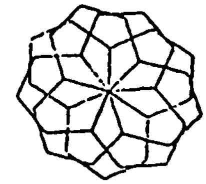
Figure 1.









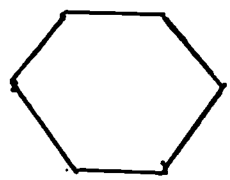


# ROTATED PATTERNS

## TO FIND THE SECRET OF THE COMPLETE PATTERN

For this task you will be divided into groups of four.

First we want you to write a procedure to draw a HEXAGON.



```
TO HEX
REPEAT 6 [FD 50 RT 60]
END
```

Then rotate the HEXAGON to make a pattern - use the REPEAT command.

We want you to find out HOW MANY REPEATS YOU NEED to complete the pattern.

Write down what you've found out.

Using the same HEXAGON make a different rotated pattern.

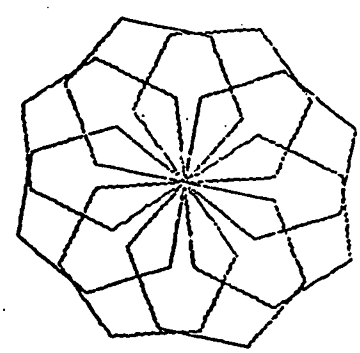
How many REPEATs did you need to complete this pattern?
Write it down.

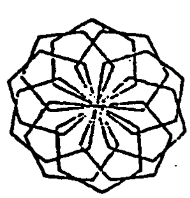
Try some more!

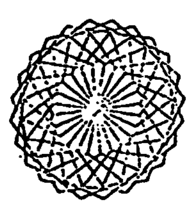
Test out your ideas by making lots of completed patterns.

REPEAT $\boxed{?}$ [HEX RT $\boxed{?}$ ]

What is the connection between these numbers.







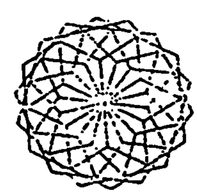
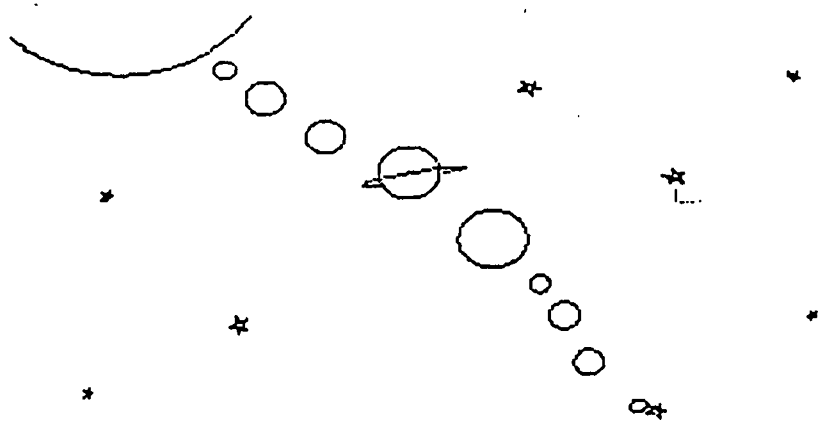


**Figure 1.**



```
STAR 'SHRINK
LT 75
FD MUL :SHRINK 40
RT 150
FD MUL :SHRINK 40
RT 150
FD MUL :SHRINK 35
RT 135
FD MUL :SHRINK 30
RT 135
FD MUL :SHRINK 35
```

**Figure 2.**

# The construction process of an iteration by middle-school pupils : an experimental approach.

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**Abstract :**
Iteration is a fundamental concept in computer sciences which allows the construction of algorithms requirig the repetition of a large number of a prior unspecified number of identical actions. The object of the present paper is to study the construction process of an iteration by middle-school pupils (age 11-16) who have never been taught programming. The pupils were placed in an experimental situation where they were confronted with the task of writing a programme for an imaginary robot in order to find a solution to a problem specifically based on iteration. The analysis of the observed strategies is carried out with reference to iteration, to the set problem, and to the experimental situation.

A central feature of the programming languages known as "procedural" is their iterative structures, as they enable the repetition of either a large number, or an "a priori" undefined number of identical sequences of actions.The object of the present paper is the study of the construction processes of an iteration by middle school children (age 11-16) who have never been taught programming.

The experimental design that was adopted consisted in placing the pupils in a problem situation for which the writing of an algorithm containing an iteration was an efficient tool for solution, and thereby setting up favourable conditions for encouraging the occurrence of what might be called "an artificial microgenesis" of iteration.

The choice of the problem situation and the experimental context were based both on "a priori" analysis of the notion of iteration, and the way that this is expressed in a programme.

## I- ANALYSIS OF THE NOTION OF ITERATION AND ITS EXPRESSION IN A PROGRAMME.

Iteration provides an appropriate and economical solution essentially to two sorts of problems : the execution of a large and predefined number of identical sequences of actions, and the execution of repeated identical sequences of actions, the number of which has not been predefined, but

which have a stop condition.
The construction of an iteration involves different operations, the most important of which are the following:

- the identification of a set of identical sequences of actions to be repeated (the body of the iteration).
- the recognition of this set of sequences of actions as an object to which a repetition operator applies.
- the control of the repetition through the insertion of a condition ensuring that execution terminates after a certain finite number of steps (the stop condition). At each step of the iteration the condition must be evaluated, this requires the updating of the values of the variables on which it depends.

The expression of iteration in a programme therefore requires the use of "repetition markers", the definition of the iteration body, and the indication of the condition.

As can be seen, the construction of an iteration raises problems, the origin of which, stems as much from the complexity of iterative structures as from the constraints of programming languages in which it is expressed. One of the fundamental hypotheses of this study consists in considering **that algorithmics problems and the problems of expression in a programming languages are linked.** The conception of an algorithm depends on the means that are available for its expression, and these means depend on the communicative context. The programme is designed so that an operator (either a person or a machine) carries out a given set of instructions. The analysis that we make of the elaboration of a programme by the pupils consequently takes into account:

- their conceptions both of the iteration as an algorithmic structure and of the functioning of the system carrying out the programme (Rouchier, Samurçay, 1984)
- their knowledge of the communication code.

## II - THE CHOICE OF THE PROBLEM AND THE PROGRAMMING CONTEXT

### II-1 The choice of the problem

The experimental approach consisted in asking pupils, working in pairs to write a programme for an imaginary robot designed to make it solve a problem. The problem that was selected consisted in finding out the sum of the numbers written on balls which the robot randomly selects, one by one, from a box. Two modes, in which iteration provides an optimal

solution, were selected (for a full description see the addendum, below).

Mode 1 : the robot must find the sum of the numbers marked on all the balls in the box, the number of which is "a priori" unknown.

Mode 2 : the number of balls selected must be exactly 150.

The pupils are faced with the task of writing an algorithm requiring the iteration of a familiar operation : that is to say, addition. This is a problem which the pupils know how to solve by "traditional means". The problem for them lies essentially in the organisation and the expression of the actions to be carried out, and in particular, in the construction of the iteration. The choice to randomize the numbers to be added means that the pupils do not carry out the addition separately from the writing of the algorithm, the latter being only used to write the result that has been found and becoming completely separate from the repeated adding process.

## II-2 Variations in robot competence

The same problem-type was given to the same pair in two sessions, in which the competences of the robot were different. During the first session, the robot could use a calculator to add two numbers. During the second session, the robot had at its disposal two "magic slates", on which it could write, with the limitation that only one number at a time could be written on a slate, however the robot was capable of adding two numbers written on different slates.

In both sessions the robot was capable of selecting a ball from the box, reading the number on it, and either typing it out on the calculator (1st session) or writing it out on the slate (2nd session).

The two sessions lasted approximately one hour each and occured with one-week interval. The material (box, numbered balls, calculator, magic slates) was supplied to the pupils. During the first session the calculator masked the management difficulties of the addition variables. The second session introduced further problems :

- managing three numbers (the two numbers to be added and their sum) with only two memories.
- initializing the iteration differently to the way that it functions in normal running conditions.

## II-3 An interactive and communicative situation

The experimental situation is an interactive and communicative situation

which makes two types of interaction possible :

- interaction with the experimenter who simulates the execution of the programme once the pupils say that they have finished. Such a validation of the program is more informative for the pupils than machine validation, where error messages are usually reduced to a minimum. Moreover, simulation makes clear the relations between actions and statements which are generally veiled within the working of the computer.

- interactions between pupils; pupils worked in pairs to elaborate a single programme. The advantage of this approach is that it provides a corpus consisting of exteriorizations of mental processes during the verbal exchanges between the partners. However, it further has the great advantage of tending to facilitate an evolution in problem solving. This is due to the disagreements and the conflicts which may encourage both partners to perceive the existence of other points of view and thus to see their own productions more objectively, thereby moving towards a new analysis of the problem (Balacheff, 1981, Guillerault and Laborde, 1981)

### II-4 Communication with the robot.

To communicate to the robot a very rudimentary language was designed, a language adapted both to the problem and to our aim, that is to say, the study of the construction of an iteration. This language can be found in the addendum.
The study was carry out on 14 pairs of pupils aged between 11 and 15. 6 pairs functioned in mode 1 and 8 pairs functioned in mode 2.

## III - MAIN RESULTS OF THE STUDY

In certain points the results correspond to studies carried out by others (Soloway,1982, Rogalski, 1985, Samurçay, 1985).

### III-1 Iteration construction.

Iteration can really be said to be the result of a construction elaborated after pupils have confronted with several difficulties and have written several intermediary programmes which have been tested by simulation.
There are three stages in this construction:

1- The juxtaposition of identical sequences of simple instructions.

2- Once the inefficiency of writing a programme repeating the same instructions several times have been recognised, the pupils mention the idea of repetition by using a repetition marker, but the body of iteration is not brought out.

3- Body is identified. The necessity of defining it is perceived (thanks mainly to the experimenter's simulations). The presence of the stop condition and its place in the content of the loop may still pose problems for certains pupils.

III-2 The stop condition.

The stop condition is generally forgotten in the beginning stages. Awareness of its necessity is triggered off for certain students during the simulation of the experimenter. The appropriate location of this condition within the programme is a further problem to solve as it requires anticipation with respect to the development of the programme, which is only obtained with sufficient accuracy after several simulations. One type of initial behaviour consisted in locating the stop condition outside the content, immediately before the end of the programme as it is this condition that stops the programme. There is a confusion between the function of the condition and the condition itself.
This is the spontaneus behaviour of a speaker in a any form of communication which consists in speaking to the addressee as if he had the same competences and the same knowledge. As the pupil carries out the actions himself he does not feel the need to check the state of the box each time a ball is withdrawn.

III-3 Reversion to spontaneous procedures when the problem becomes more complex.

The second session introduced new problems; the initialization of the iteration and the management of the memories. When faced with these difficulties, and after being confronted with the failure of a first programme transposed directly from the first session, the pupils reverted to their first way of writing a programme without iteration. It was only later that they reconstructed an iteration with reference to the new data.

III-4 The simulations.

The simulations play an important role in the dynamics of the problem-solving. They can validate programmes, or invalidate them. They make the pupils aware of certain things (for example the stop condition). But the effect of simulations carried out by the experimenter is more important than the effect of those done by the pupils. The pupils are unable to read their programmes without taking into account the meaning that they have given to it during its elaboration. Validation carried out by

experimenter simulation is more efficient because it is less subjective.

## ADDENDA

### The two problem modes

**Mode 1**

There is a box containing numbered balls. It is not known how many balls there are in the box.The aim is to find out the sum of all the numbers written on the balls of the box. Orders are given to a robot to calculate this sum. The robot is controlled by a keyboard. A list of the names of the keys and their meanings is supplied.

WRITE DOWN IN AGREEMENT WITH EACH OTHER THE LIST OF KEYS YOU MUST PRESS SO THAT THE ROBOT CALCULATES THE SUM OF THE NUMBERS WRITTEN ON THE BALLS.

**Mode 2**

There is a box containing 1000 balls numbered from 1 to 1000. The aim is to find the sum of the numbers written on 150 balls. Instructions are given to a robot to calculate the sum of the numbers written on 150 balls which it selects randomly from the box.The robot is controlled by a keyboard. A list of the names of the keys and their meanings is supplied.

WRITE DOWN, IN AGREEMENT WITH EACH OTHER, THE LIST OF KEYS THAT YOU MUST PRESS SO THAT THE ROBOT CALCULATES THE SUM OF THE 150 NUMBERS WRITTEN ON THE BALLS

**The language of the robot**

It consists of a set of terms which are mnemonics each one of which is linked to one of the operative competences of the robot. The robot control instructions were the following:

- MARCHE : Switch on the robot.
- ARRET : Switch off the robot.
- PUB : Take a ball.
- LIST : Read the number which is written on the ball that has just been taken.
- COMPT: Add 1 each time COMPT key is pressed.
- CONT : Continue the instructions which follow CONT until further order.
- REC : Begin again instructions which follow REC until further order.
- REP : Repeat instructions which follow REP until further order.
- FINCONT : Terminate list of instructions to be continued.
- FINREC : End of the list of orders to be began again.
- FINREP : End of list of orders to be repeated.
- TEST : Test whether the box is empty. If it is, stop programme execution but remain switched on. If box is not empty continue as before.
- TEST N° : Follow TEST N° by a number to check whether COMPT has the same number. If it is does then stop, if it does not then continue.

**1st. session with calculator.**

- TAPE : Print on the calculator number that has just been read.
- TAPE+ : Actuate key "+" on the calculator.
- TAPE= : Actuate key "=" on the calculator in order to find answer.

**2nd. session with the magic slates.**

- EC1 : Write on slate n°1.
- EFF1 : Wape out slate n°1.
- ADD12 : Add numbers on slate n°1 to those on slate n°2.
- EC2 : Write on slate n°2
- EFF2 : Wape out slate n°2

## Examples of the development of pairs

1st. Session. (Patrick and Morad)
After a short stage of juxtaposition of instructions Patrick and Morad realize that the sequence PUB LIS TAPE TAPE+ has to be repeated. Then they write the following programme :
MARCHE PUB LIS TAPE TAPE+ REP FINREP TAPE= ARRET

The repetition marker is mentioned but the body of iteration is not constructed. The experimenter asks the pupils to simulate their programme but the simulation is carried out according not to the real programme but to the programme pupils think to have written. The programme is proposed to the simulation of the experimenter which shows that the robot cannot do anything between REP and FINREP. The following programme is written without stop condition :

MARCHE PUB LIS TAPE TAPE+ REP PUB LIS TAPE TAPE+ FINREP TAPE= ARRET

The simulation of the experimenter points out that the programme loops. Patrick expresses his surprise : "we have written FINREP". The experimenter explains the function of FINREP. Some other programmes are written without REP or FINREP because these instructions seem to be the source of troubles. The same programme as earlier is written and proposed to the simulation of the experimenter. It loops again but now Patrick becomes aware of the necessity to use the instruction TEST for terminating the loop. The final programme is written :
MARCHE PUB LIS TAPE REP TAPE+ PUB LIS TAPE TEST FINREP TAPE= ARRET

2nd. Session. (Ariel and Pierre Yves)
The first programme is a direct transcription of the last programme of the first session :
MARCHE REP PUB COMPT LIS EC1 PUB COMPT LIS EC2 ADD12 EFF1 EFF2 EC1 TEST150 FINREP ARRET

The simulation carried out by the pupils shows them that this programme cannot function. They themselves simulate what they should do and thus become aware of the necessity of an initialization which leads them to the elaboration of a second programme :
MARCHE PUB COMPT LIS EC1 REP PUB COMPT LIS EC2 ADD12 EFF1 EFF2 EC1 TEST150 FINREP ARRET

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# PEGBOARD AS BASIS FOR PROGRAMMATION - IN 5- AND 6- YEAR OLDS -

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Computers play an increasing role in our civilisation : this includes an increasing role in education. Authors such as PAPERT (1980) claim that some programming language (i.e. LOGO) used with a computer favours the cognitive development ; we claim that the use of a simple non-verbal communication device (N.V.C.D.) makes it possible a) to introduce **WITHOUT THE USE OF COMPUTERS** the first elements of such a programming language in 5-year olds ; and b) to observe and analyze each step in the reactions of very young children when they are confronted to problems concerning a programming language.

During a first phase in our research we chose an adapted N.V.C.D. and we tested this N.V.C.D. with some normal and handicapped children : this enabled us to organize a hierarchy of exercises in order to teach the first elements of programmation while observing the verbal and non-verbal behaviour of young subjects. During a second phase, we used this sequence of exercises with a group of 5-year olds. The object of our presentation is triple a) describe the N.V.C.D. used, b) describe the sequence of exercises used and c) describe and discuss the observations made concerning verbal and non-verbal problem-solving behaviours in 5- and 6-year olds. The results concerning handicapped children will also be mentioned and discussed.

## 1. Device used.

The device itself consists of a white plastic board furnished with holes. In these holes one can put coloured plastic nails. The pegs are defined by two variables : their colour and the shape of the head. There are seven colours : yellow, green, red, orange, pink, light blue and dark blue ; the heads can be squares or quarters of a circle (we call these "triangles" or "points". Using these coloured pegs, children can make a mosaïque. But the pegs can also be used as an introduction to programming languages (SAERENS, 1984) : a triangle is used as "name" for a sequence of squares and a sequence of triangles represents a programme.

## 2. Method.

We used short sequences of square-headed pegs : each sequence received a name represented by a triangular peg and thus became a **subroutine**. A short sequence of such triangular pegs represented a **programme**. One could use this setting to ask three kinds of questions. Firstly, if the subroutines and the programme are given, one can ask a child to produce a long sequence of square pegs by replacing each triangular peg of the programme by its "meaning" (the child has to execute the programme) : i.e. the

child sees on his pegboard the equivalent of the center and the left columns of figure 1. He must build the right-hand part using square pegs. Secondly, using the subroutines and the execution of the programme, one can ask a child to describe (using triangular pegs) the programme which has been used : i.e. the child sees pegs on the parts of his pegboard corresponding to the left and right parts of figure 1. He must build an equivalent of the center part.

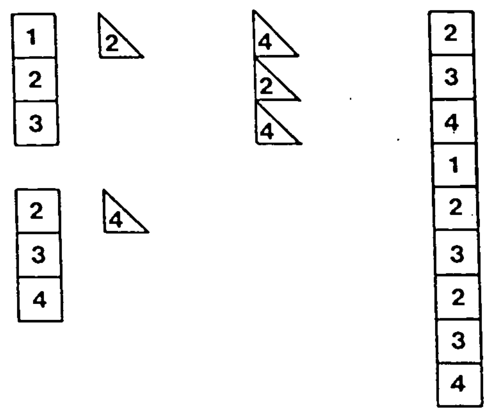
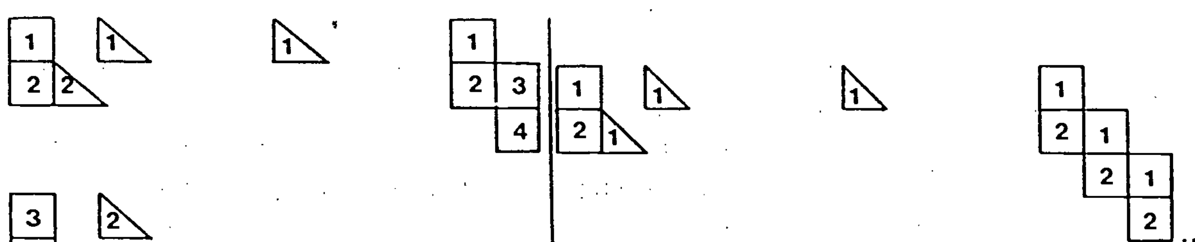


*Figure 1 : On the left two subroutines with their "names" (respectively triangular peg of colour 2 and triangular peg of colour 4), in the center a programme consisting of three names for subroutines, on the right the execution of this programme.*

Thirdly, using the programme and its execution, one can ask a child to define the subroutines using pegs : i.e. the child sees pegs on the parts of his pegboard corresponding to the right and center parts of figure 1. He must reconstitute the left part.



2.a. 2.b.

*Figure 2 : Triangular pegs (i.e. "names") are introduced here as part of the subroutines : this leads to the introduction of recursive processes, finite for figure 2.a and infinite for figure 2.b.*

Moreover, it is possible to introduce triangular pegs inside the subroutines (see figure 2): these "inside" triangular pegs are names of another or the same subroutine. This makes it possible to introduce recursive processes in these exercises (this includes infinite recursion concerning a subroutine containing a peg representing its own name. We also chose to use a special triangular peg (the yellow one) to indicate a change in direction :

instead of keeping the same direction for further constructions the children were told to

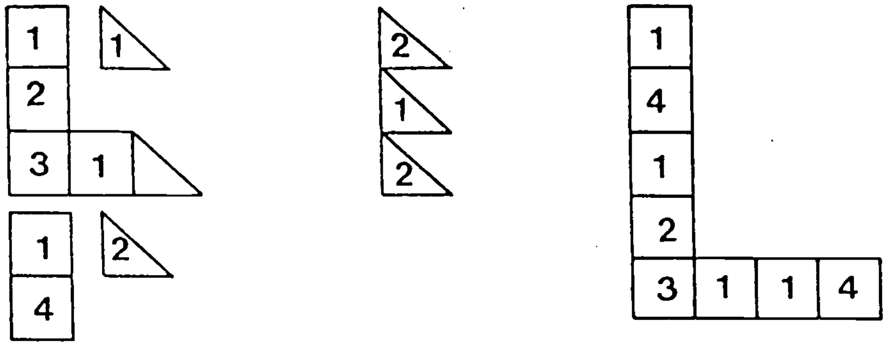


*Figure 3 : Subroutines can contain a special directional triangular peg.*

place their next square peg in the direction shown by the point of the "yellow triangle" (see figure 3). This made it possible to introduce exercises such as a) "place the square pegs corresponding to this programme" (an "L") and b) "find a programme to draw a square, a house".

3. Technique.

We worked with 5- and 6-year olds. They worked by groups of two and all the sessions were videotaped. The exercises were hierarchically organized : there were six main steps which followed an introductory freeplay session. The first exercises introduced the use of "squares" and "triangles" as basis for a programming language : the aim was to execute a programme. Firstly we used one single subroutine containing from 1 to 4 squares ; later, different subroutines were used simultaneously. In the beginning, the triangles of the programme were introduced one after the other, later, they were presented simultaneously to the subject. The second step of the hierarchy was the research of an adequate programme. At first the subject was asked to execute a programme; after the execution, the triangles were removed and the subject had to reconstruct a sequence of triangles adapted to the construction. Later, he had to discover a programme for a construction which was presented to him without execution-phase. The third step was the research of adequate subroutines. This kind of exercises implied the formulation (by the subject) of hypotheses which had to be tested and possibly reformulated by the subject.

Further steps of the hierarchy were : the introduction of triangles in the definition of a subroutine (as it is done in LOGO), the use of non-linear subroutines (such as an L) and finally the introduction of directional triangles : we used yellow triangles which were firstly placed in the programme between two normal instructions ("absolute reference system") and later inserted in the subroutines ("relative reference system").

For each of these steps, the gradation of exercises was based on : the number of

subroutines (1 to 5), the number of squares contained in each subroutine (1 to 4), the colour of the squares and the order in which they are placed, the relations between subroutines (same first squares, same last squares, the last square of one subroutine is the same as the first square of another one, ...), the number of triangles in the programme and the order of these triangles in the programme.

4. Results.

Results from this preliminary study indicate that all the 14 normal subjects easily learned to execute a programme, about 2/3 were able to reconstruct the programme which had been used while only 1/3 of the pupils were able to solve the other kinds of exercises. After 2 years of training with several N.V.C.D.s, a young aphasic boy aged 7,6 years was confronted with this kind of exercises, (LOWENTHAL and SAERENS, in print). this child was unable to speak and/or understand spoken words. He had been considered as severely mentally handicapped till a treatment using N.V.C.D.s could be initiated when the boy was 5,6 years old. He learned to handle this new kind of problem as he had done for other N.V.C.D.s : by trial and error. His learning process was nevertheless similar to that of normal children and the experimenters observed several mistakes comparable with those done by learning disabled children without aphasia, and by normal children.

The mistakes young children do give important informations as far as their cognitive abilities are concerned. We describe here some of them. During the execution of a programme subjects often simply replace a triangle by a square of the same colour (and not by its "meaning"). When a subroutine starts and ends by the same square, subjects tend to omit one square as shown on figure 4. Subjects might also skip one step of the programme when the same subroutine is used twice consecutively and there can also be construction problems when the order of the triangles in the programme is not the same as the order in which the subroutines are presented.

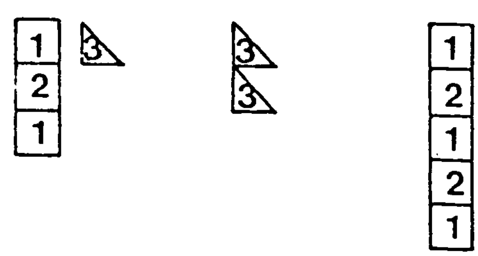



Figure 4 : Construction mistake.

When the subjects must research a programme some of them place in the central column (programme) triangles of the same colour as the squares and in the same order ; others discover the groups of squares associated to the subroutines but place a triangle of the same colour as the first square of the group instead of using its actual "name". Sometimes, they place the triangles they want to use to make a programme in the same order as that used to present the subroutines. The subjects sometimes hesitate when the relationships between the different must research adequate subroutines : many problems are due to the fact that the subjects do not analyze the construction in function of the programme, they look globally at the construction and use the first triangle in the programme as name for a first group of squares, they repeat this for the second triangle, and continue till they have "cut" the construction into small groups which, put together, give exactly this construction. One notices that the number of squares in the programme, and the length of each group equals the total number of squares in the construction divided by the total number of triangles in the programme : a same "name" can be attached to different groups of squares and the subjects do not check what occurs when a triangle is used twice (or more) in the programme. The very simple exercises were the only one to be easily solved by all the subjects. This led us to reorganize the gradation of exercises inside this step of our hierarchy as follows :

I) First column : 1 subroutine with 1 to 4 squares, second column : 3 to 4 triangles, third column : construction.

II) First column : 2 subroutines (1, 2, or 3 squares), second column : 3 - and later 4 - triangles (placed in such a way that the same triangle is not used twice consecutively), third column : construction.

III) First column : 3 subroutines, second column : 4 triangles, third column : construction.

Some of the exercises we used enabled the subject to produce different solutions.

When triangles were introduced in the subroutines, some subjects had problems due to the position of the triangle in the subroutine, some simply replaced this triangle by a square of the same colour and others used only the squares of the subroutine in the construction and did not take into account the existence of this inside-triangle.

The L-shaped subroutines introduced another kind of difficulty : some subjects did not know how to place the squares of a subroutine with respect to the squares of the preceding one.

Finally, the subjects had spatial and orientational problems when using the orientational ("yellow") triangle, mainly when this triangle was placed inside the subroutines (relative reference system).

## 5. Discussion.

Despite all the mistakes we describe here, very young subjects learn, progress and eventually produce without mistake "a programme to construct a square". It is thus obvious that this use of a pegboard enables the experimenter to introduce the first notions of a programming language without a computer. We used this to introduce recursion and a pseudo-LOGO which is completely non-verbal in first grade and in kindergarten (while PAPERT's LOGO is full of verbal elements : "FORWARD", "LEFT", ...). Results concerning an aphasic child are described in LOWENTHAL and SAERENS (in Print).

This device makes it possible to observe, among other elements the spatial organization in young children. Moreover it makes it possible to observe the mistakes made by young children solving logical problems in a highly structured situation. It makes it also possible to introduce non-verbally young subjects, including handicapped subjects, to the first notions of programmation.

All these facts convince us that although the facts reported by PIAGET (1936, 1947) are irrefutable, his interpretation of them is not of the same kind. We much prefer the framework described by BRUNER (1966a, 1966b) and used by HARMEGNIES and LOWENTHAL (1984) in their introduction.

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# Selfdeveloping Strategies with a Calculator Game

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We report upon an investigation with 33 students who played the calculator game Hit-The-Target. On the base of about 1000 guess-and-test sequences we study the question, if students develop mathematical strategies by themselves (without the help of a teacher) and which these strategies are.

## 1. Design

Hit-The-Target is a calculator game where the student in a guess-and-test sequence has to find the second factor for a multiplication problem:

(a) $\boxed{?} \xrightarrow{x28} [961, 983]$

| input | output |
|---|---|
| 12 | 336 |
| 18 | 504 |
| 23 | 644 |
| 28 | 784 |
| 35 | 980 |

$\boxed{?} \xrightarrow{xb} [c_1, c_2]$

Guess a number to multiply by b to get a result between $c_1$ and $c_2$. Record your guess and your result. If your answer did not hit the target interval guess again and continue as before.

The variables b, $c_1$ and $c_2$ of the about 200 problems were chosen according to the age of the students. One of the easiest and one of the most difficult examples may be

$\boxed{?} \xrightarrow{x3} [42, 46]$ resp. $\boxed{?} \xrightarrow{x78} [7474, 7510]$.

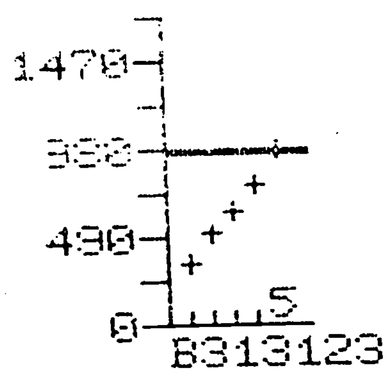
We worked with 24 students in a third grade class (about 9 years old, student code 2 letters) and with 9 students in interviews. The class was familiar with calculators. The students got an introduction of the game by the teacher. He explained the rules without giving hints for strategies and gave five problems as examples which had to be solved in front of the whole class. For each guess another student was asked to give a suggestion. After

that introduction the students got individual worksheets, each with six different problems. After finishing one worksheet the student got the next one. There were three periods for working on the worksheets: 20 minutes after the introduction on Tuesday, 30 minutes on the following Thursday and 40 minutes on the following Monday. The slowest student solved 15 problems, the quickest 44.

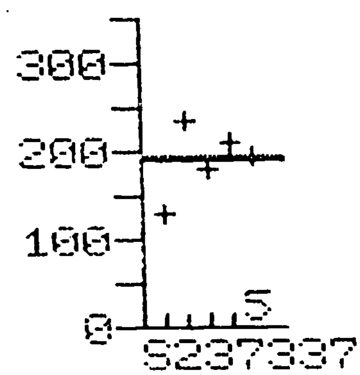
We also worked with 9 students (age 8-12, student code 1 letter + 1 digit = grade) in videotaped individual interviews. The interviewer had to motivate to "think aloud", while the student had to record each guess on a given worksheet. The time for introducing calculator and game varied between 2 minutes and 8 minutes. Each session lasted between 15 and 40 minutes (10 games per session, mostly 2 sessions per week, 50 problems at the most).

The guess-and-test protocols were transformed into graphical descriptions. Each graph displays the target interval, the number of guesses (on the x-axis), a "+" for each guess (calculator output), and a code number:

(b)




(c)




(b) Bettina (B3, grade 3) worked on problem "123":

$\boxed{?} \xrightarrow{x28} [961, 983].$

It was her 13th ("13") problem, she needed 5 trials (see (a)).

(c) Sylvia (S2, grade 2) worked on problem "337":

$\boxed{?} \xrightarrow{x13} [191, 196].$

It was her 37th ("37") problem, she needed 5 trials.

2. Starting Numbers

The first guess (consciously or unconsciously) is an estimation problem. Many students chose multiples of 5 or 10 as starting numbers for products > 1000. Some used their favorite numbers as much as possible (e.g., Lorenz (LT) started 18 of his 28 games with number 15). For most of the problems the starting numbers were too small. Though the problems step by step went harder some students improved in guessing better starting numbers. In some cases the students only saw the sequence of digits but not the order of magnitude. Some notice that immediately, others need one or two more guesses.

An important aspect for the quality of the starting number seems to be the size of the factors and the size of the product. E.g., Jochen (JG) guesses excellent starting numbers when one of the factors is a one-digit-number, but he clearly failes when both factors are bigger than 30.

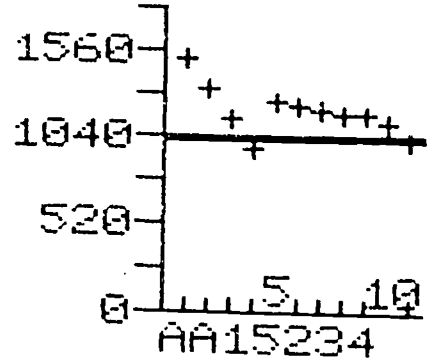
3. Strategies

Three students obviously did not understand the game or could not find appropriate strategies (DB, HH, OM). They often started the next problem without reaching the target interval of the preceding, their trials often seemed to be arbitrary, they used favorite numbers. There are some other students with 1 or 2 wrong solutions (all other problems correct). Most of these mistakes can be explained as reading or pressing mistakes, e.g. factor 73 instead of 63 or interval [2429, 2487] instead of [3429, 3487].

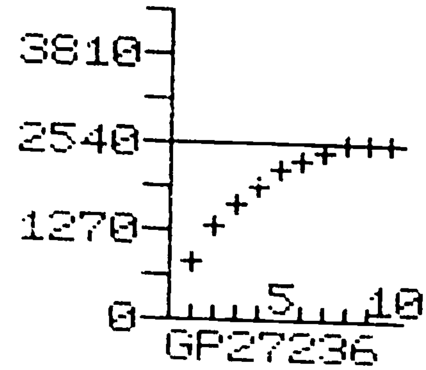
30 of the 33 students learnt how to play Hit-The-Target successfully. About half of the problems were solved by 3 or less guesses. E.g., Rolf (RK) only needed 43 guesses for solving his 23 problems. Students like Rolf estimated and guessed excellently, but their protocols are too short for an analysis.

Therefore we discuss in the following only protocols with more than 3 guesses. In those cases the students first had to elaborate an appropriate strategy, sometimes they had difficulties to concentrate, or the problems were too hard for them. But all these students discovered:

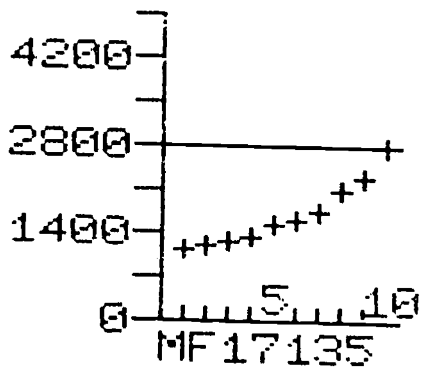
(d)



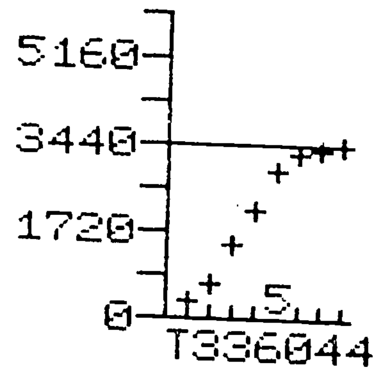

(e)



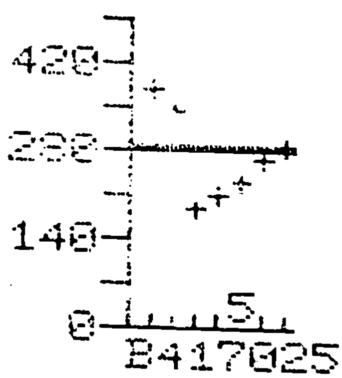

(f)



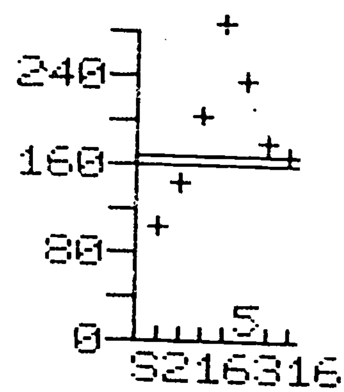

(g)




(h)




(i)




(1) calculator output too big, then take a smaller input,
(2) calculator output too small, then take a bigger input.
Most of the students approached the target interval from one side (see a, b, d, e, f, g):
(3) monotonic approach.
Only a few students tried to "encircle" the target (c):
(4) encircling approach.
We observed different kinds of monotonic approaches:

(5) linear approach: constant steps by tens, by fives by twos or by ones (d, g),
(6) almost linear approach (a, b, h, i),
(7) speed up approach (f, g),
(8) slow down approach (e, g).

There are logical changes of strategies. E.g., Tanja (TP) passed the target by tens and went back by twos (5). Or Jutta (J6) started with big steps (6), went back by fives (5), and turned back again (8). Others notice that their approaching is too slow:

(9) make one big jump (h).

After their jump they continue with their previous strategy (when they did not pass the target) or they choose a new strategy (h).

But there is also a non logical behavior. Some students need more than one estimation at the beginning:

(10) find the right "direction".

We interpret this behavior as "strategy oriented" and not yet "target oriented". After some "wrong" steps they suddenly notice the target and then work "logically".

There are cases where the students pass the target without realizing it for one or two more steps (i). In other cases the students are so surprised having passed the target, that they "forget" their last steps and fall back behind the knowledge they already had (d).

## 4. Summary

The number of protocols is not big enough and the video tapes give not sufficient evidence for significant statements upon developmental effects. But together with observations from other classes and other investigations with the calculator game Hit-The-Target we dare to generalize:

- Many students improved their estimation skills.
- Most of the students discovered and practiced effective procedures without the help of a teacher.

- Many students got a "proportional feeling" for multiplication facts.
- Most of the strategies seem to be unconscious to the students. They use a strategy (proved by the records), but they often cannot explain, why they chose just that number as the next input.

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# THE EFFECTS OF CAI ON AFFECTIVE VARIABLES IN MATHEMATICS

Zemira R. Mevarech

Bar-Ilan University, Israel

In many countries, elementary and high school students are given the opportunity to drill and practice with computers. These programs are generally called Computer Assisted Instruction (CAI). Recent surveys (e.g., Collis, 1984) have shown that 68% of the schools in United States, 70% in Canada, and 98% in Great Britain use computers for instruction. Other countries, such as Israel, Spain, French and West-Germany are in the midst of, or rapidly approaching, implementation of CAI systems in the schools. In most cases, CAI has been applied for drill and practice in mathematics. Because CAI is so widely used, we are led to wonder about its effects on schooling outcomes. How have students responded to CAI? What is the effect of CAI on students' ability to solve different mathematics problems? How has it affected self concept in mathematics? To what extent has it reduced mathematics anxiety? The present study addressed these questions by focusing on the Israeli CAI program, called TOAM, the Hebrew acronym corresponding to Computer Assisted Testing and Practice.

TOAM provides computer assisted testing and practice to each student and reports to the teacher on his or her progress and difficulties. The system was developed at Tel-Aviv Centre for Educational Technology on the basis of the program leased from the Computer Curriculum Corporation

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(CCC) at Stanford University. This system is used in Israel since 1977. The curriculum consists of 14 multiple topics in the forms of item strings, or strands, that cover all topics in elementary school mathematics (grades 1-6) except geometry. Each strand includes a sequence of items varying in the degree of difficulty. The strands are: numbers, horizontal addition, horizontal subtraction, vertical addition, vertical subtraction, equations, measurements, horizontal multiplication, word problems, vertical multiplication, division, fractions, decimal fractions and negative numbers. (For more details see Osin, 1981.)

The CAI program starts with a 12 diagnosis sessions of 10 minutes each and continues with a two 20-minutes weekly practice sessions. During CAI sessions students solve problems from all strands beginning at the level determined during the diagnosis sessions. At every point in time, the difficulty level of the problems is matched to the ability level of the solver. Three attempts are given to solve a problem. Each attempt is followed by a reinforcement statement. Closing a session, the number of problems answered correctly and incorrectly are displayed on the screen. Periodically, the teacher and the principle are provided with summary reports concerning the performance level of individual learners as well as the whole classroom.

<u>How did CAI affect mathematics achievement?</u>

Not surprising, most of the studies on CAI have focused on its effect on mathematics achievement. Recent surveys using meta-analysis techniques showed that these programs tend to have positive effects on

achievement ranging from .33 to .45 Effect Size (Kulik, Bangert and Williams 1983). The Israeli CAI program, TOAM, yielded similar results. Osin (1981) reported a continual increase in mathematics achievement of disadvantaged students exposed to CAI. He showed that children who entered the system far below the national norms in mathematics ended up above or at the norm after using CAI for several years. A follow up analysis of one school using CAI (N=600) demonstrated the pattern of changes in mathematics achievement during the years 1978-1984:

| Grade | 2nd | 3rd | 4th | 5th | 6th |
|---|---|---|---|---|---|
| Average level-1978 | 25 | 28 | [illegible] | 36 | 41 |
| Average level-1984 | 33 | 43 | 51 | 62 | 69 |
| Expected level | 29 | 39 | 49 | 59 | 69 |

In addition to this research, other studies have compared mathematics achievement of CAI and non-CAI learners. Mevarech and Rich (1985) reported that CAI students scored higher than their counterparts who studied mathematics without CAI. Furthermore, Mevarech (in press) indicated that CAI increased mathematics achievement when it supported individualized or conventional instruction. The differences between the CAI and the non-CAI learners were manifested on simple computational problems as well as on word-problems.

<u>How did CAI affect self concept in mathematics?</u>

Self concept refers to the way an individual perceives himself or herself (Shavelson and Bous, 1982). General self concept can be broken into academic and nonacademic components which in turn can be broken into more specific factors such as self concept in mathematics, literature and science. Mathematics self concept (MSC) consists of an

individual's perception of self with respect to mathematics aptitude, mathematics achievement and affection toward mathematics (Mevarech and Rich, 1985). There are several ways MSC is related to mathematics learning. Previous research has shown positive correlations between MSC and achievement. Students at all levels of education who feel confident in learning mathematics achieve at a higher level than students who have negative perceptions about their own mathematics ability. The CAI did not change this relationship. In our studies, the correlation between MSC and mathematics achievement was approximately .30 within CAI classrooms.

While the above studies focused on MSC as a means for explaining variation in mathematics achievement, other researchers have considered MSC as an important schooling outcome regardless of achievement. A comparison of MSC scores of CAI and non-CAI students showed that at least in the elementary schools, at third, fourth and fifth grades, significant differences were found between the two groups. In all cases, CAI students scored higher on all dimensions of MSC questionnaire than their counterparts not using CAI (Mevarech and Rich, 1985; Mevarech, in press).

MSC was also used to explain the underpresentation of females in mathematics classrooms. In general, high school girls tend to participate less and score lower on measures of MSC than boys even when mathematics ability is controlled. Did CAI differently affected boys and girls? Our studies have shown no sex x treatment interaction on measures of MSC. The CAI equally improved achievement and MSC of boys and girls.

Important both within and outside of CAI literature is knowledge of what factors affect MSC and how can they be manipulated. Perceptional psychology presents a model of MSC as a product of accumulated patterns of success/failure. Accordingly, MSC is a function of the amount and intensity of feedback provided by the environment. CAI provided not only immediate feedback to each response, but also a high rate of positive reinforcement resulted from the matching of problem difficulty level and student ability level. Probably, this type of feedback gives the learner the feelings that he or she can succeed in mathematics.

<u>How did CAI affect mathematics anxiety?</u>

Mathematics anxiety involves feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematics problems in a wide variety of ordinary life and academic situation (Richardson and Suinn, 1972). Several recent studies on the nature of mathematics anxiety attempted to build a causal-sequence model of mathematics anxiety and mathematics achievement. These studies assumed that anxiety causes deficits in the information processes which in turn lower mathematics achievement. Lack of knowledge further increases anxiety which causes deficits in information processes and so on.

To what extent, then, can CAI break this causal sequence? We already saw that CAI yielded higher scores on mathematics achievement than conventional instruction. Furthermore, CAI students showed a lower level of mathematics anxiety than non-CAI students. This difference was statistically significant (Mevarech, in press). Yet, a clear

cause-effect relationship between mathematics anxiety and mathematics achievement cannot be induced from these studies. Future research may address this question by manipulating different features of CAI and analyzing their effects on mathematics anxiety and achievement.

Although there are many more aspects of CAI effects that are not discussed here, the data reported pointed to the merit of this issue. Systematic use of theoretical models is the way to further expand the knowledge about the affective domain in CAI. Carefully designed studies may identify the factors that should be manipulated in CAI to produce an improved system that would increase mathematics achievement as well as affective outcomes. There is no question that CAI is a powerful system if only we would know how to take advantage of it.

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# ACQUISITION OF NUMBER-SPACE RELATIONSHIPS: USING EDUCATIONAL AND RESEARCH PROGRAMMS

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Abstract

The author uses a number of programs adapted to the analysis of the strategies applied by students when they have to associate a number to a point on a line (or two numbers to a point in the plane).
The aim is to study the relation between spatial and numerical representations by the students (6th and 7th grades) in one dimensional and two dimenional situations.
The methods consists in playing the following game: reach a target on a screen by giving appropriate number(s). Several programms are used for testing the effects of the variables: informations given to the student, kind of numbers, position of the target (by respect to present informations), dimensionality.
The results show the importance of these factors and the interactions between them; one can notices the great differences between subjects and in particular between girls and boys.

## AIM OF THE STUDY

Constructing operational relationships between number and space is a long and difficult process by children and students. Pupils are taught to assigne a characters, then numbers, to points on a line or in a grid at the primary school, but we know that students have difficulties to deal with scales and cartesian representations of space in secondary school.
We think that the cognitive operations involved in the identification of points in space by numbers are complex. In fact two conceptual fields are in interaction, involving on one hand spatial notions and on the other hand numerical conceptions. How are the conceptions related? We construct (1) a set of specific programms to give answers to the following specific questions:
a)is the notion of interval conserved? A point may be between two others (on a line) in a topological sense and a number may be between two others in the numerical order: are the students able to conserve this relation when associating numbers to points?

---

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b) Is the notion of "order" conserved? a point may be at the left or at the right of another point on the line, a number may be larger or smaller than another: are the students able to do the appropriate association between space-orientation and number-order?

c) Are the students able to evaluate the order of magnitude for a researched number by using available information about the scale and the coordinate of a given point?

d) Does it exist an effect of the characteristics of the numbers themselves on the preceeding relationships? Are the responses of students the same for small numbers (until 11 or 12), for familiar numbers (smaller than 100), large numbers (as are 200, 450..) and decimal numbers (as 0.5, 1.7)?

e) What is the difficulty to conserve these relations for the two coordinates of a point in the cartesian plane?

METHODOLOGY

The method consists in playing a game: reach a target on a line by giving a number, or a target in the plane by giving two numbers. For each tried number (for the line) there is a feedback: "out" appears on the screen if the number is two large, or a trace appears on the line corresponding to the given number.

The number of trials needed to reach the target is recorded, and appears on the screen when the student succeeds. A succession of items is presented to each subject, so that he/she can adapt his/her strategy along the game.

A subprogram records the succession of tried numbers for each item (and each subject): so it is possible to analyse precisely which relations are conserved.

We will give results obtained on 3 programs: two related to a one-dimensional relation space-number and one related to bi-dimensional relation.

The description of the items for each program uses the following notations.

Concerning the informations available to the subject, three situations are used:

DODE: the origine 0 is given, and another point, so that the scale is also given.

DPDE: two points are given (with the corresponding numbers), so that the scale is also given, but the origine has to be infered.

DONE: the origine is given but no other point, so that the scale is unknown while a number is not given corresponding to a point inside the screen. Further trials are called "informed trials" in this situation.

Concerning the topological relations of the target and the known points three cases are possible; we note T for the position of the target, a and b for the points, when the are different to 0, and 0 for the position of the origine.

The topological cases are the following: aTb or 0Tb; Tab; abT or 0bT (when using positive numbers).

The categories for the numbers corresponding to the coordinate(s) of the target are: S (small: 5-12); F(familiar); L(large: 200-450); D(decimal: 0.2-1.7).

The programs for one-dimensional relation consist of 2 blank items with familiar numbers, 4 items DPDE (S,F,L,D) and 4 items DONE (S F,L,D), with the two orders:

first the DPDE items (program CIBUN3) or the DONE items (program CIBUN2).
The program for bi-dimensional relation consists of 1 blank item with familiar numbers, 4 DONE items (with small or familiar numbers) and a DONE test-item. We denote it CIBDEU.

The subjects are students of 6th and 7th grades, in secondary schools (11-13 years old children). For each situation the number of girls and this of boys are approximatively the same.

## RESULTS

The first result is the conservation of interval by the children, in the different cases for one-dimensional relation space-number. The second is the existence of difficulties in relating spatial positions of points and relative size for numbers: half of the students (of each grade) make at least an error in the DONE situation.
On the set of all the items, 18 percent present at least an error for the 6th grade, and 10 percent for the 7th grade students. The effect of school level appears limited. In the bi-dimensional relation tasks the students have more difficulties in the conservation of the order for the two coordinates, and even for the conservation of intervals (the analyses are not yet achieved for all items).

The number of trials needed to reach the target depends strongly of the kind of information which is available. The figure 1 gives the mean number of trials for the two one-dimensional situations and for the situation of CIBDEU, for each grade.

| | CIBUN2 | | CIBUN 3 | | CIBDEU | |
|---|---|---|---|---|---|---|
| | DPDE | DONE | DPDE | DONE | DODE | DONE |
| 6th | 3,5 | 9 | 3,6 | 7,8 | 5,7 | 6,3 |
| 7th | 3,2 | 6,2 | 2,9 | 5,5 | 5 | 7 |
| Total | 3,3 | 7,6 | 3,3 | 6,7 | 5,4 | 6,6 |

fig.1
Mean numbers of trials to reach the target.

In all situations the absence of information about the "scale" gives the task more difficult. (Situations DONE). The difference between the one and the bi-dimensional situations is limited; nevertheless the items of the bi-dimensional situation present small and familiar numbers, with a scale maintained constant: we think it is the reason of the relatively good score.
When information about scale is available, the one-dimensional situations are easier, with a very little difference for the two grades. It is necessary to analyse each item in order to explain these results.

The figure 2 shows the mean number of trials for each item of DONE and DPDE, for the two grades, for the situation CIBUN3: for the DONE situation the whole number of trials is given, then the number of "informed trials" (included the first trial inside the screen) where the students are now in a situation with two informations (so they can infere the scale as in DPDE).

| | DONE | | | | M | DONE: "informed" Trials | | | | M | DPDE | | | | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | S | F | L | D | | S | F | L | D | | S | F | L | D | |
| 6+L | 6,8 | 6,8 | 9,9 | 7,6 | 7,8 | 4,7 | 6,5 | 9,9 | 5,4 | 6,6 | 3,7 | 3,2 | 2,8 | 4,7 | 3,6 |
| 7+L | 4,5 | 5,7 | 6,3 | 5,4 | 5,5 | 3 | 5,4 | 6,3 | 4,1 | 4,7 | 2,8 | 2,2 | 2,4 | 4,1 | 2,9 |
| T | 5,6 | 6,2 | 8,1 | 6,5 | 6,6 | 3,8 | 6 | 8,1 | 4,7 | 5,6 | 3,2 | 2,7 | 2,6 | 4,4 | 3,2 |

fig.2
Mean number of trials on each item of CIBUN3

It appears an important effect of the order of magnitude of the numbers involved in the task in the DONE situation. Two kinds of elements are present in the strategies of the students: one concerns the property of the first tried number, the second concerns the research of the value of the target when informed about scale.
The first tried number lies in the familiar range (as shown by an analyse of the first trials: more as 55 percent are familiar numbers, less as 20 percent are small numbers): so the first trial is almost always in the screen for familiar and large numbers (F and L items); the students needs a mean of 2 or 3 trials to give a "sufficiently small" number for the S and D items. (Let us notice that the unit lies always inside the screen).
The use of an "approximation strategy" to reach the target when informed about the scale explains the order of the mean numbers of trials for the S, F and L items.
The decimal values involved in DONE and DPDE situations produce little more difficulty than the small values for the developpment of this strategie.
The relation between numbers and points in the two-dimensional plan is more complex: the figures 3a and 3b give the mean number of trials for the 6 items of CIBDEU.

| | N | item 1 | DODE 2 | 3 | 4 | 5 | test 6 | Mean (1-6) | Mean (2,3,4,5) |
|---|---|---|---|---|---|---|---|---|---|
| Girls | 24 | 8,7 | 4,9 | 7,9 | 3,3 | 5,4 | 7,3 | 6,2 | 5,3 |
| Boys | 28 | 7,5 | 3,9 | 7,2 | 2,8 | 5,6 | 5,3 | 5,3 | 4,8 |
| Total | 52 | 8,1 | 4,5 | 7,5 | 3 | 5,5 | 6,3 | 5,7 | 5 |

fig. 3a

| | N | item 1 | DODE 2 | 3 | 4 | 5 | test 6 | Mean (1-6) | Mean (2,3,4,5) |
|---|---|---|---|---|---|---|---|---|---|
| Girls | 14 | 4,3 | 4,8 | 7,8 | 3,2 | 5,7 | 8,1 | 5,6 | 5,4 |
| Boys | 13 | 3,5 | 2,7 | 6 | 3,1 | 5,6 | 6 | 4,5 | 4,3 |
| Total | 27 | 3,9 | 3,7 | 6,9 | 3 | 5,6 | 7, | 5 | 4,8 |

fig. 3b
Mean number of trials for CIBDEU, (6th and 7th grades), for girls and boys.

The difference between grades concerns the first trial, and less significantly the TEST item. The 2nd and 4th items which use small numbers are easier than 3rd and 5th items involving familiar but larger numbers.

The comparison between one-dimensional and bi-dimensional situations for DPDE and DODE items (without decimal numbers) shows a specific effect of dimensionality. The The only exception concerns the comparison between the S-item of CIBUN with the topological relation abT and one of the S-items of CIBDEU with relation OTa x OTb : they appear to be of the same order of difficulty; in this case there is an intervention of interval-conservation.
Concerning individual differences, we can notice that boys have better responses than girls in all items (except the easiest one); this difference is particularly important in the DONE situation used as TEST item. There are analogous results in the one-dimensional situations (F and L items), as shown by figure 4 below.

| | First test (DONE) | | | | | Second test (DONE) | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | S | F | L | D | Mean | S | F | L | D | Mean |
| Girls | 2,8 | 6 | 6,7 | 4,1 | 4,9 | 3,5 | 5,4 | 6,9 | 3,5 | 4,8 |
| Boys | 3,2 | 4,8 | 5,9 | 4,1 | 4,5 | 3,4 | 4,9 | 5,8 | 2,9 | 4,3 |
| T | 3 | 5,4 | 6,3 | 4,1 | 4,7 | 3,4 | 5,2 | 6,2 | 3,2 | 4,5 |

fig. 4
Mean number of informed trials for girls and boys for CIBUN3 with a second test 3 months later the first one (7th grade).

The second test gives the same results (except for decimal numbers) as the first: we expected certainly no difficulty due to the existence of a relation between space and number at this grade, but the lack of positive change in strategy is surprising.

## CONCLUSION

The main result is the strong interaction between the spatial and the numerical properties when students have to attach a number to a point (or two numbers in the plane). The topological properties are first conserved; nevertheless the order relation is not ever strictly conserved when the information is limited to the position of origine. The relation between the order of magnitude and the distance did not appear easy to appreciate in the situations used because of the limitation of size of the screen: programs using "microscopes windows" would be necessary for a productive analysis.
Moreever, a very important effect is produced by the numerical field used: small numbers are very well linked with spatial positions: an "approximation strateg " allows the students to reach the target in few trials, even in the bi-dimensional situation.
The informations given in the situations have drastics effects on the task for

familiar and large numbers, in one-dimensional as well as in bi-dimensional situations.

The analysis of the students' behaviour brings elements for an educational use of such programs: first the variation of order of magnitude of numbers is necessary for the constitution of a real relationship between space and number. The use of small numbers may be a way to introduice students to the tasks. The presence of items with large number is necessary to differentiate "approximation strategies" and "proportionality strategies". But this is not sufficient: the tasks allows the students to success without an elaborate "proportionality strategy". The research perspectives are the analysis of situations where the number of trials is strongly limited, with large numbers, so that only the use of proportionality allows the students to success.

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# LEARNING PROGRAMMING: CONSTRUCTING THE CONCEPT OF VARIABLE BY BEGINNING STUDENTS

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**We present a synthesis of our studies on the acquisition of the programming concept of variable by beginning students. Our studies are based on the cognitive and epistemological analysis of this concept and on the analysis of pupils behaviours observed in various didactical situations. We distinguish various types of variable with respect to different criterions and illustrate how the pupils operate differently on them. The subjects are from different levels (from 9 to 16 years old) and programming with LOGO, PASCAL and LSE languages.**

## 1.Introduction

The researches reported here are the parts of didactical experiences whose prime objective was the identification and the analysis of conceptual difficulties encountered by students who are learning programming. In this presentation we focus our attention on the concept of variable and its utilization in problem solving by beginning students. The presentation may allow to introduce also a debate concerning the similarities and differencies between the mathematical and the programming meaning of the notion of variable.

## 2.Analysis of the variable concept

The programming activity covers a very large field of concepts and procedures like variable, assignement, looping, recursion, etc...These concepts and procedures are not on the same level of complexity. Hence, a conceptual hierachy can be establised among them. When we talk about a conceptual hierarchy or a conceptual analysis, it is very important to understand here that this analysis is based on both an analysis of subject's activity and an analysis of the subordinate concepts.

When we examine programming activity we remark easily that the concept of variable is the heart of every problem-situation, no matter how simple it is. Because the solution of a programming problem is not a <u>result</u> but a <u>procedure</u> which has to be represented by the subject as a function operating on data. This representation necessitates to consider the data as variable. From the programming point of view a variable is defined as an address, but this formulation is not sufficient for analyzing this concept in its functional meaning for the subject. In fact when the values of a variable change, its naming and its relationships with the other elements of the program are invariant. This property of invariance appears in particular in the activities of naming variables and control their values in the program.

In the programming activity the subject is confronted with two types of activities about the variables:

- the construction of the meaning and the operations on the variable: naming, declaring, assignment, input and output of data;
- the control of the particular values which will be taken by the variables when running the program. This control intervene on the planification of the control structures like looping, conditional statement and recursion.

These characteristics show that it is not possible to study the variable concept isolately; it is necessary to take into account the set of situations in which programming concept of variable achieves its full meaning. However the concepts with which the concept of variable interacts depend on the nature of programming language used in the activity. For instance, at the initial phase of the programming learning, the meaningfull activities about variables are different according to the PASCAL and LOGO languages. When in PASCAL programming the concept of variable achieves its full meaning in the looping activity which implies also the notion of assignment, in the LOGO programming this meaning can be constructed first with the notion of procedure. This is due to the fact that the looping activity in LOGO makes reference to the recursion which is a very complex notion. Hence the acquisition of the variable concept can not be studied independently from the programming languages and the domain of problems to solve.

Actually we can define at least three types of criterions which allow to distinguish between different types of variable.

We determine the first criterion with respect to the programming languages. The programming languages are all created to solve some specific problems, thus they are defined for manipulating some specific objects with appropriate operations on them. So the programming activities involve important particularities on variables manipulated in different languages. In this study the exemples we give are on the three different languages: PASCAL, LSE (is a french programming language elaborated with didactical intentions in programming) and LOGO. PASCAL and LSE are languages belonging to the same family of imperative languages, they are characterized in particular by the existence of the assignment command. LOGO is a functional language, the assignment command exists but is marginally used, it is not appropriate to the LOGO programming structure. The manner on which the values are assigned to the variables in these two different families of languages introduce important differencies in the treatment of the variable by the subjects.

The second criterion concerns the semantic domain of the problem. Operating on numbers or on characters or on graphic objects has not the same meaning for the subjects. This idea is supported by the hypothesis on the "familiar procedures" used by beginning students in programming activity. For exemple, as we will see above, the first strategies used by beginning students in the numerical domain are very close to an algebraic representation of the problem: this representation has some important effects on the way the subjects operate on the variables. For the moment we distinguish between variables representing:

-numbers;

-character strings;

-graphical objects;

The third criterion concerns the functional status of variables for the subject's mode of action planning. In this context we distinguish two types of variable:

- external variables corresponding to the values controled by the program users; i.e., variables which are explicit inputs and outputs of the problem;
- internal variables corresponding to the values controled by the programmer: i.e., variables which are necessary only for the programmed solution of the problem.

Of course these three criterions are not completely independant but they allow to analyze more finely the subjects behaviours.

For exemple, consider the problem of permutation of two numbers. The variables corresponding to two given numbers are external, the intermediary variable which has to be used to keep one of the values at the moment of change, is internal. In LOGO programming this aspect cannot be observed in a single-procedure progam in which generally all variables are controled by the programmer, but all of the programs involving more than one procedure (i.e., when it is a call of procedures) implie a control on internal variables.

3.Population

The results we use to illustrate each point are obtained in different works in which we have studied the initial phase (approximatively 15-20 hours) of the programming learning. The students were from different levels (8-9; 13-14; 16-17 years old) and worked with different languages as graphic and non graphic LOGO, PASCAL and LSE.

4.Analysis of results

It is obvious that the results we analyse are illustrative, nevertheless they reflect difficulties encountered by pupils in the acquisition of the variable concept.

The first distinction which concerns the programming languages has two important effects:

-the definition of the conceptual field about which the didactic experiences will be organized in the initial phase of the programming concerning the concept of variable. As we point out above, concerning variable concept, the conceptual fields in LOGO and PASCAL are not the same. In PASCAL this the meaningfull field can be defined around the concepts of looping and assignment, in LOGO it is more appropriate to choose the interactions variable/procedure and variable/recursion. The notion of procedure which exists also in PASCAL (in relation with the conepts of local and global variables) is not on the same level of difficulty for the subject with as regard to its homologous in LOGO: its is more difficult to acquire.

-the second effect concerns the nature of objects which can be manipulated with a the given language. It seems to us that although the LOGO as "command language" is more easy to manipulate at the beginning phase, with respect to its objects (graphic or not), the decentration from the semantic content of the object constitutes an obstacle for the acquisition. For exemple the students confuse very often the

procedure with the end product which is a trace.

The evolution of the acquisition of the variable concept in LOGO can be sketch as follow:

-when we introduce procedures with inputs with 8-9 years old children the main difficulties they encountered were operating on the name of variables instead of particular values and naming differently the variables they used in other problems. Children used systematically the same name SIDE ,used for the first time to introduce the concept of variable, to designate different variables.

-we have observed with more older children (13-14 years old) that they have no problem to use variables in construction of the simple procedures when the variables designate the measures of distances and of angles. When the variable represent a number of repetition like in the construction of an arc, some problems appear in the identification of what is varying: distances or number of repetition?

```
TO ARC :N
  REPEAT :N  FD 1 RT 1
  END
```

The variable N represents the number of repetition but also the length of arc.

An other problem observed in this level concerns the confusion between the status of the names of variables and the names of procedures. This conception is observed when children want to define a procedure for the variable name. They probably think that each time they used a name they have to define a procedure without which the computer can not understand the meaning of this name.

Notice that the variables pointed out above are all external. The internal variables intervene in LOGO when there is at least a call of procedure. The difficulties observed with the use of internal variables are:

- use of different variable names for the different calls of the same procedure;
- identification of what is varying in the construction of recursive procedure and the control variable;
- confusion between name and value in the parameter passing between different procedures.

We think that the confusion between objects and procedures which define them is an important specificity of learning programming in LOGO. More detailed researches should to be made to clarify the question.

**The problems related to the treatment of variables in PASCAL programming can be summarized as bellow:**

In PASCAL programming the property of invariance of variables appears, particularly, in the loop struc tring, in which variables are involved with a status of accumulator as below exemple:

sum sum+number

The student has to designate by the same name both the preceding value and the present value which is a function of the former, and has to treat the assignment sign as an asymetric relationship.

When the student has to solve a looping program , he has to consider three types of operations on the variable: update, test and initialization. We hypothesize that

the schemes used by beginners will be based on their previous knowledge and familiar procedures that relate to the specific poblem to be solved. For exemple, it is possible for the students to use their mathematical model of variable. But this conception is insufficient because the mathematical description of a variablee is a static description, i.e., it designates the representative of set

$$x \quad a,b$$

or an unknown in an equation

$$x+2=5$$

The description of a programming variable is more dynamic; the student has to identify the law of transformation of its values, as, for instance

$$n \quad n+1$$

The observations with 16-17 years old college students indicate that (Samurçay,1985a) in the first 10 sessions, they have serious troubles in conversion of their algebraic description into a procedural description. We argue that the algebraic conceptions of variable, equality sign and equation constitute a necessary but an insufficient model on which can be build the programming concepts of variable,assignement and loop-construction.

The observations of the same students in more advanced phase of the learning process (Samurçay, 1985b) show that :

- the students operate more easily on internal variables than on the external variables;

-initialization is a very difficult operation, even for students who realize a high-level strategy in the construction of a loop invariant: the **assignment** statement is more difficult to use than the **read** statement;

-the construction of control variable and the construction of loop invariant are on the same level of difficulty.

**The observation of 16-17 years old students with LSE programming** shows similar results with our results in PASCAL. Moreover we observed in this study that, in the construction of control variable, students manipulate more easily the numerical data that character strings. We observed also an important effect of the familiar procedures on the treatment of variables. For exemple, the function which allows to extract a sub-string from a character string is interpreted by students as a function which not conserves the initial content of variable. The students can represent easily some operations on the initial and the final elements of the string, but have difficulties with operating on the complementary part.

## 4. Conclusion

Our results with different studies on programming learning allow to formulate some conclusions about the nature of difficulties encountered by students in their acquisition of variable concept. Of course the investigations on the domain of programming learning are just beginning and we don't have yet a reasonable number of systematic observations to infer the generality of our results. However we think that these conclusions may constitute a theoretical base to approach the didactical problems related to the programming learning.

The acquisition of the concept of variable is a long process in programming learning, the idea of variable doesn't appear spontaneously with young pupils (8-9 years old), it necessitates a didactical intervention (Hillel, Samurçay, 1985).

For all progamming languages with which we have worked, our results indicate that a correct representation for internal variables is more difficult to construct that for the external variables.

The operations on the variables are not all on the same level of difficulty, operations concerning the control of actions (conditional statement, test variables in looping and in recursion) are very difficult to conceptualize.

The familiar procedures used by subjects at the beginnig stage of learning progamming depend on the problem domain. In the numerical domain the students use most often their algebraic models which may constitute also an obstacles for the construction of programming meanings of the concepts. In the character strings domain the subjects have more difficulties to formulate their familiar procedures for which they don't dispose such a formalized writing system. The familiar procedures associated with the graphical domain are very close to subject's pictural model of the figures: decentration from this model encounters some specific difficulties on which we need new investigations.

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# "DE BAAS OVER DE COMPUTER"

The title means "The master of the computer". It is part of a computer project below average ability seventh grade classes of the (non-comprehensive) Dutch educational system.

The project is being carried out by the sub-department OW & OC of the Mathematics Department of Utrecht State University, whose general task is developmental research in math education.

The project is concerned with questions like the following:

Can we teach these students getting about with the computer while taking themselves initiatives, keeping track of what is going on, and using the computer as their servant who carries out the partial tasks set by them?

This aim is stressed by the name of a publication, appearing in the title of the present article. The publication [1] includes working sheets for the students, information for the teacher, and data on the software belonging to the project.

A second question raised in this project asks:

Is it possible to rouse the interest of teachers of all subjects taught at school for this kind of instruction, and to stimulate their disposition and ability to look in their own area for possibilities of using the computer in a similar way, according to their own view on instruction in their subject area?

*A possibility for computer education*

Among others these questions have led the group OW & OC to a research design of close collaboration with one school of the type described above.

The 7th grade students get 40 computerlessons during one year. The material and the software are developed in preceding years and tested by members of OW & OC at the same school. The teachers coming from quite various subject areas, had followed a course, developed by OW & OC and built upon the students material.

Above the 7th grade computer instruction stops as a separate subject. In the near future one can probably do with less than 40 lessons in the 7th grade as in the course of the years more and more students enter the 7th grade with previous computer experience. It is the intention at this school that from the 8th grade onwards the computer shall be used as a tool in the single subject areas.

Some examples of such a use of the computer will be developed in 1985 in collaboration with the respective teachers of the school.

Sofar no computer instruction has been foreseen for the 9th to 10th grade. No experiments for computer science as an optional subject covering these grades, have been started in our country, although it seems that the present design would prove to be a good preparation for computer science as an optional subject.

Worksheet

*The micro on time*

1. The tape contains
   also a programme to make a
   calendar
   ► Print at ➣
   calendar


2. ► Which day of the week is 29 February 1984
   according to the micro?
   Answer: ...
   ► Which date is the last Friday this year?
   Answer: ...

3. ► Find out with the micro:
   Which day of the week is St. Nicholas* this year?
   Answer ...
   And Christmas?

4. On 10 July 1584 William of Orange was murdered.
   ► Which day of the week was it?
   Answer ....

5. Do you know which day of the week you were born? You can find
   it out with the calendar programme.
   ► It has on ...

*What do they learn in 40 lessons, and what is beyond?*
Students learn getting about with programmes like arithmetic, sorting, calendar. The following example stems from "De baas over de computer".

Notice that after printing the numbers of year and month, the corresponding month calendar appears on the screen.

The teacher instruction belonging to this worksheet is p. 91.
"*The micro on time*
The teacher repeats that the computer recognises the words 'arithmetic' and 'sorting'.
She tells there are more things waiting for you: calendars and various kinds of data.
With the next programme the computer figures out calendars for you.
Remarks on single questions.
Question 2 to 4.
These questions are meant to teach the use of the programme that makes calendars.
The computer understands year numbers printed. At the place 'month' the number of the month must be printed: 1 for January, 6 for July, and so on.
In the beginning this can cause difficulties.
Question 5.


* In our country Santa Claus is celebrated 5 december night.

This question is particularly appreciated by the students. Sometimes they also look for the birthdays of their father and mother.

A piece of observation:

They continue with the calendar. At 'this year' they hesitate; when asked which year it is now, they answer 1984 and print it.

'Does 2 really suffice?' (to indicate the month)

'It does.'

They enjoy the results. The problems of having the computer figure out a calendar went smoothly. When they deal with question 5 Monique also wants to know which day her father was born. They find it for all the four fathers of the group. They also look for their own birthday this year. They enjoy it that the computer can do this job."

The next worksheets of the calendar programme require solving strategies. [2]

For instance at such a question as

*Easter 1984 will be the first*
*Sunday of April; while Whitsunday is*
*always seven weeks later. Which date*
*is Whitsunday?*

One of the students we observed started counting on her fingers the Sundays in April. In order to count further she would call in the calendar of May. A boy in her group said there was no need to.

One month has just four Sundays. No, she insisted, there are months with five Sundays. Properly said the students followed with the computer the same strategy as we do with our pocket agendas.

The calendar programme was followed by one that counts the number of days from one date to another. Students find out their age in days.

At another question they must first calculating themselves before consulting the computer:

*'A year has 365 days. So on one's*
*100th birthday one is 36500 days old.'*

Afterwards they realise that because of the leap-years it must be more. On the micro they got 36526. The micro operates the same way as the travel agencies: an eight days trip means from Saturday night to Saturday morning.

Many students do not grasp it. It becomes clear when the micro reckons 50 days from Easter to Whitsunday (as shows the old word Pentecost, which means 50 in Greek)

The 100-years-question can be answered on various levels. The centuries are not all alike: 1900 was no leap-year but 2000 will be.

Does it make a difference if one's birthday is 29 February?.

Within these two programmes students learn about time reckoning and about the computer. The various programmes together give them a lot of experiences about the computer.

An editor and a text processor has been developed for the students. The 40 lessons don't include programming proper in any programming language, although in a way using text commands might be considered as a kind of programming.

Moreover they learn getting about with a data store the elements of which are being produced by themselves.

*Introduction*

In Dutch we use the name "Weetjesbank". It could be translated as Info. Items Bank. We want pupils (12 - 13 years old), to experience searching for a large amount of more or less similar things, rapidly.

In addition to searching for information in a card file these pupils gain experience in searching by means of a microcomputer. In the first lesson the teacher gives a few examples of info. items: valuable information:

The cathedral spire is 110 meters high.

The Kajagoogoo's ex-singer's name is Limahl.

Each group of four pupils gets a card file with 150 cards. The pupils have to prepare themselves for a contest. The teacher asks a question, for example: "What is the name of the Russion Secret Service?" All groups search industriously for the card with "The Russion Secret Service is called KGB". The group which holds the card up first, gets a point. Pupils start to prepare themselves for the contest. They have to report something about their preparation. They receive written questions such as:

- How did you divide the work?
- What made you get on with it quickly?

On the next worksheet they are directed to a division in categories.

There are questions such as:

- What do you do with info. items that don't seem to belong to anything?
- What do you do if an info. item belongs to more than one category?

Lastly they have to answer again:

- What makes your division so fast?

The team contests comes next. Pupils use different strategies such as: They split up the categories, lay out the cards, each pupil from the group guarding a few categories.

A group lay all cards uncategorised on the table. They divided the territory in four equal pools. Another group divided categories with many cards between the four members of the group. By a pop music info. item all four searched in their own subset. In another group they made four stacks of info. items. Before a new question was asked these pupils literally got into a starting position; then they rushed to the

stacks.

Still another group set the info. items in alphabetical order of keywords. In the previous lesson they had split up the cards alphabetically according to the first words of the sentence. When they realised that they had a whole row with "the"- info. items, they changed their strategy. They had designated a keyword in each info. item, after which they sorted the info. items alphabetically, in accordance with the keywords.

Next lessons they use the computer. They have different tools searching for categories and searching with a string.

This software was specially developed for this purpose.

To our opinion the pedagogical and didactical issues should dominate technical issues. [3] [4]. The computer can do various jobs depending on the programme it is charged with. It is a many-machines machine. The variety of programmes lends the computer various faces. What is being printed can be interpreted in different ways. '2*3' is a problem in the arithmetic programme, a word for the text processor, a wrong yearnumber in the calendar programme.

The many-machines-machine is an example of an image about the computer. During this developmental research we have become more and more convinced that building of images can be very important to get an understanding of how to "steer" the computer. Images as for instance the boss and the many-machines-machine can be presented to students. We have observed situations where students had their own image belonging to how to search with a string in a text. The teacher started a discussion about these images and presented her image Anyhow this image helped because it enabled students to answer questions like: "If we use cola, col, and co as strings, which one finds more sentences in a text?" A usefull preparation for a next question:" I don't know exactly how a name is spelled in specific text, what can I use as a string?" It would be very important if students could develop an attitude of generating images which they could adapt after new experiences.

In this work we present the students problems that can be solved by using different programmes on the computer. They have to decide whether they want to use a certain programme, they start the programme, insert their data, they write down the result, go back to the boss and select another programme that they need on their road to the solution of the problem.

Text processing and data store are building blocks for activities in subject areas where the computer is used.

Examples:

In mother language lessons students learn from their own composition about readability (paragraphs, intermediate headlines).

In geography students can collect data about the regian where they live and store

them among data collected before. Using the computer is only meaningful if appropriate classifications of the data sharpens the picture of the region.
In biology the tree structure of the characteristics plays a role.
Teachers are stimulated to provide ideas with regard to their own subject areas OW & OC will try to develop appropriate software.
It still remains under discussion for which subjects LOGO programming might be useful.

*One school, if appropriately supported, can never come out to be a failure*
Of course the development at one school, with extraordinarily intensive counseling cannot straight forwardly be transferred to schools lacking this special help.
New creations in education require a massive approach like ours. Followers need not undergo the same development.
Our teacher course for school teams can anew be tested by teacher training institutions and adapted to dissemination requirements. At this very moment experiments of retraining school teams are already being conducted at training institions.
Meanwhile the students material of 'De baas over computer' is being used at other schools thanks to the teachers instructions and the availability of the software.
After an intensive developmental process at one school experiences on computer aided instruction in various subjects can be disseminated by means of retraining courses and publications.


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# The Use of Calculators to Develop Reasoning Processes and Achievement in Problem Solving

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Problem solving has become accepted as an important focus of mathematics education at all levels. With this emphasis on the development of reasoning skills and the increasing availability of simple and cheap calculators, it is appropriate to investigate the possible effects that the use of calculators will have in problem solving.

The potential of the calculators as an aid to mathematics instruction has not been fully realised or accepted. This potential needs to be developed to the fullest extent. Using a calculator to relieve the student of routine computational tasks allows the student to focus on such skills as interpretation, analysis and formulation of decisions, all skills which are critical in solving problems.

While the claims made concerning the benefits of using computers seem reasonable, many teachers remain sceptical not only of the benefits of using computers in problem solving but also the desirability of introducing calculators into formal school work at all. The fears expressed include the danger of students becoming so reliant on their calculators that they will lose their proficiency in basic computational skills such as basic number facts and their four operations with whole numbers and decimals.

Research has shown, however, that this is not the case. Wheatley (1980) found that Grade 6 students who had used calculators in their instruction used more problem solving processes and made fewer errors. Shumway et al. (1981) found that the use of calculators increases children's computational power with little instruction. Szetela (1982) on Grades 3, 5-8, found no loss of paper and pencil skills after using calculators during instruction. He also found that students using calculators did not choose correct operations any better than students without calculators but that calculators helped students to compute more accurately.

The related question as to whether the use of calculators makes a difference in problem solving was investigated in an earlier study by Szetela (1980). In this study it was found that the availability of calculators was a critical factor in solving word problems. When students used calculators on problem solving tasks after a period of instruction, they performed significantly better than students who did not use calculators. There were no significant differences in performance when paper and pencil only were used.

As Szetela's study (1980) was with a sample of high ability elementary students, the ability to solve problems may have been fairly highly developed. The purpose of the present study was to compare performance on certain problem solving tasks before and after a period of instruction in the use of a calculator for solving problems using a sample of students of mixed ability. In doing this, it was hoped to determine whether students using calculators tend to attempt more problems, use more correct procedures and achieve more correct solutions than when they do not use calculators. A secondary purpose was to investigate the nature of the problems attempted and solved as well as the nature of the solutions given.

Accordingly, it was hypothesised that:

(i) the use of a calculator during instruction and practice in problem solving skills will increase the subject's ability to solve problems;

(ii) subjects will perform better on vocational type questions than on non-routine type questions; and

(iii) the use of a calculator during instruction will develop confidence in problem solving skills.

## SAMPLE

The sample consisted of the entire Grade 10 students in two Provincial High Schools in Lae, Papua New Guinea. It was considered that within the context of education in Papua New Guinea, Grade 10 is a critical stage in the transition from secondary to tertiary education, and therefore the ability of the students to benefit from the study proposed would be maximised.

Fifty one of the 192 subjects were females attending a day school catering mainly for students from the urban area. The other school is a boys' boarding school with students coming mainly from villages in the district. Despite all students being in Grade 10, the age range was wide. Again, because of the nature of education in Papua New Guinea, ages in each grade vary considerably. Table 1 shows the sample distribution of students who participated in the pre and post tests (n = 206). For the retention test 192 subjects out of the original sample were available.

Table 1

Sample Distribution

| Age in Years | Uncertain | | $14 \le a < 15$ | | $15 \le a \le 16$ | | $16 \le a < 17$ | | $17 \le a < 18$ | | $18 \le a < 19$ | | $19 \le a < 20$ | | $20 \ge a$ | | Total | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | M | F | M | F | M | F | M | F | M | F | M | F | M | F | M | F | M | F |
| School A | 7 | - | 2 | - | 14 | - | 34 | - | 24 | - | 14 | - | 7 | - | 1 | - | 103 | - |
| School B | 7 | 13 | 1 | 3 | 10 | 7 | 18 | 29 | 6 | 3 | 6 | - | - | - | - | - | 48 | 55 |
| Totals | 14 | 13 | 3 | 3 | 24 | 7 | 52 | 29 | 30 | 3 | 20 | - | 7 | - | 1 | - | 151 | 55 |

M.E.C. Technical Report No. 16: B. SOUTHWELL 'Problem Solving With Calculators for Grade 10 Students', 1984.

PROCEDURE

The testing and teaching program consisted of six lessons, including a pre-test and a post-test with four periods of instruction between. The program was established as follows:

Lesson 1: Pre-tests (i) Problems (ii) Basic Number facts

Lesson 2: Introduction to the calculator and its use in computation

Lesson 3: The use of the automatic Constant.
The solution of routine problems

Lesson 4: The use of the memory keys.
The use of an "estimate and check" strategy

Lesson 5: Looking for a pattern strategy

Lesson 6: Post-test.

The pre-test consisted of fifteen problems including both vocational and algebraic content. The post-test consisted of seven of the same questions which were not attempted in the pre-test.

One class in each school completed the post-test without a calculator. These classes were told they did not have to complete the computations if they felt they were going to take too long. In fact, only a very few did not attempt to complete all calculations. Those who had the correct method attempted the calculations.

A retention test was administered eight and six weeks respectively after the post-tests for the two schools. This consisted of six items, three of which were non-routine while the other three were vocational in nature.

RESULTS

(i) Analysis of Question Types on Pre-test

Only 5.2% of the sample answered all four non-routine questions in the Pre-test correctly. Considering the nature of the questions and the background of the subjects, it is interesting to note that 46.9% of them answered two or more questions correctly.

No one was able to answer all the vocational questions correctly and only 19.8% were able to answer any question correctly.

(ii) Analysis of Question Types on Post-test

On the post-test, 91.1% (in contrast to 22.4% on pre-test) of the subjects had two or more non-routine questions correct, ie. they gave at least 50% correct answers.

With the vocational questions, the contrast is even more marked. Whereas no subjects had three correct answers in the pre-test and only 3.1% had two correct answers, on the post-test, 58.9% had three correct responses and 90.6% had two or more correct answers.

(iii) Analysis of Question Types on Retention Test

On non-routine questions, 93% of subjects had two or more questions correct. On the vocational questions, 86% of subjects had all questions correct.

(iv) Preformance on Pre-, Post- and Retention Tests

Table 2 gives the mean mode, median and range for each of the three tests.

Table 2

Measures of Central Tendency on Tests

| | Pre-Test | Post-Test | Retention Test |
|---|---|---|---|
| Mean | 4.453 | 20.432 | 32.401 |
| Mode | 0.000 | 7.000 | 17.000 |
| Median | 4.194 | 15.235 | $\approx$ 30.000 |
| Range | 0 - 25 | 0 - 71 | 0 - 92 |

The definite improvement in performance over the three tests in all measures can be seen. These differences are highly significant (.0001).

(v) <u>Number of Questions Attempted</u>

In the pre-test only 19.8% of the subjects attempted three ore more questions and about 10% did not seriously attempt any questions. In contrast, in the post-test just over a quarter of the subjects attempted all questions. A further increase took place in the retention test in which 68% of the subjects attempted all questions and all subjects attempted at least half the questions.

Table 3 indicates the increases in numbers of correct respnses, the number of questions attempted and the percentage achievement, in relation to the two types of questions.

Table 3

Comparison of Means of Three Tests

| Variable | Pre-Test | Post-Test | Retention Test |
|---|---|---|---|
| Correct Responses Non-routine | 1.432 (35.8%) | 2.833 (70.4%) | 2.661 (88.7%) |
| Correct Responses Vocational | 0.229 (7.6%) | 2.469 (82.3%) | 2.823 (94.1%) |
| Questions Attempted | 1.630 (23.3%) | 5.323 (76.0%) | 5.500 (91.7%) |
| % Achievement | 4.453 | 20.432 | 32.401 |

An analysis of variance indicated that the group of a subject had no effect on the number of questions attempted in the pre-test. A similar pattern exists in relation to the post-test.

(vi) Relationships Between Variables

All except five variables have significant correlations with the percentage achievement on the post-test. These five are the basic division facts, the frequency of correct responses on the non-routine type questions and the frequency of all categories of responses on the vocational questions. These last four do relate to the pre-test performance. As might be expected, the total percentage on the post-test correlates highly with the various categories of non-routine and vocational questions on the post-test.

Mastery of addition, subtraction and multiplication basic facts each correlate with the other three operations and with percentage performance in both the pre-test and post-test. Division, however, correlates with performance on the pre-test but not the post-test.

The sex of the subject correlates highly with the number of questions attempted in the post-test, the number of non-routine questions correct and the percentage score on the post-test but not on the number of vocational questions correct.

(vii) Differences in Performance of Experimental and Control Groups

Tests of significance indicated that the performance of subjects in the experimental and control goups were statistically significant for certain variables. They include the percentage performance on all three tests and the number of questions attempted on the retention test.

## DISCUSSION

### 1. The Number of Questions Attempted

The increase in the number of questions attempted in the pre and post tests is highly significant. While a further increase took place in the retention test, the increase was not statistically significant. These increases do, however, indicate a marked increase in confidence on the part of the subjects. That this increase was equally significant for those subjects not using a calculator in the post-test, indicates that further consideration is needed as to the exact nature of that increase in confidence. Elements such as the use of calculators, the instruction in strategies or the researcher's style of teaching are all possible factors.

2. Achievement

That the intervening lessons had a positive effect on the performance of the subjects is indicated by the highly significant difference between scores on the pre-test and the post-test. The difference between scores on the post-test and retention test is also highly significant. Again, whether this is directly due to the use of calculators to the learning of specific strategies or to a combination of these is not clear. That the classes using calculators for the post-test did little better than those not using them seems to indicate that the learning of the strategies may have been the critical factor.

Another factor may be the relative novelty of the calculator as an aid for solving problems. This is indicated by the difference in achievement in the two schools used for this study. The subjects in the school in which most students were unfamiliar with calculators did slightly better in both pre and post tests than the subjects in the other school where approximately half the students had some knowledge of calculators.

Further contributing evidence to support the contention that the novelty of the calculator played a significant role in the observed improvement is that all except two subjects in the first school recorded improved percentage achievement scores while only 75% of the second school did. Also the classes which reputedly included the more able students on general subjects as judged by their teachers appear to have made the greater gains.

A comparison of the number of correct responses given by subjects on the pre-, post- and retention tests indicates a definite progressive improvement with the greater gain taking place between the pre- and post-tests. This supports the view that the use of the calculator or the intervening lessons had a significant effect on performance.

3. Types of Questions

There is a definite switch in preference for the type of question attempted. In the pre-test the mean number of correct responses on the non-routine questions was 1.432 which corresponds to 35.8%. On the vocational questions, the number of correct responses was only 7.6%. The post-test, however, produced a better percentage of correct responses on vocational than non-routine questions. It could be that the availability of calculators to do the routine calculations which are more prevalent in the vocational questions than in the non-routine ones enabled subjects to respond more accurately in the post-test to those questions.

It seems reasonable to conclude that, with calculators to use, Grade 10 students appear to find problems of a vocational nature less difficult than those which are non-routine in nature. This could be due to the perceived relevance of questions relating to postage, agriculture, etc., to the more routine nature of their solution or to lack of experience in algebra.

4. Sex Differences

There appears to be a strong relationship between the sex of the subject and the number of questions attempted and performance on the post-test. Also sex seems to have a bearing on the number of correct non-routine responses.

That the sex of the subject correlates with aspects of the post-test but not with the pre-test may well be due to growth in confidence engendered by the use of the calculator or the interaction with a significant female role model in the person of the researcher.

5. Retention Test

From an analysis of the actual questions, it seems likely that the subjects found the questions in the retention test slightly easier than those used in the post-test. The added maturity and mathematical experience in a six to eight week period should also be a contributing factor.

## CONCLUSION

While no definitive results can be drawn from this study, there are indications that Grade 10 students improve their problem solving skill through the use of calculators. It is reasonably clear that they find vocational type questions easier than those of a non-routine nature.

The most important result seems to be the significant growth in confidence on the part of almost all students.

The results of this study provide evidence to varying degrees to support the three hypotheses presented.

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# MICROCOMPUTER-ASSISTED SOPHISTICATION IN THE USE OF MATHEMATICAL KNOWLEDGE

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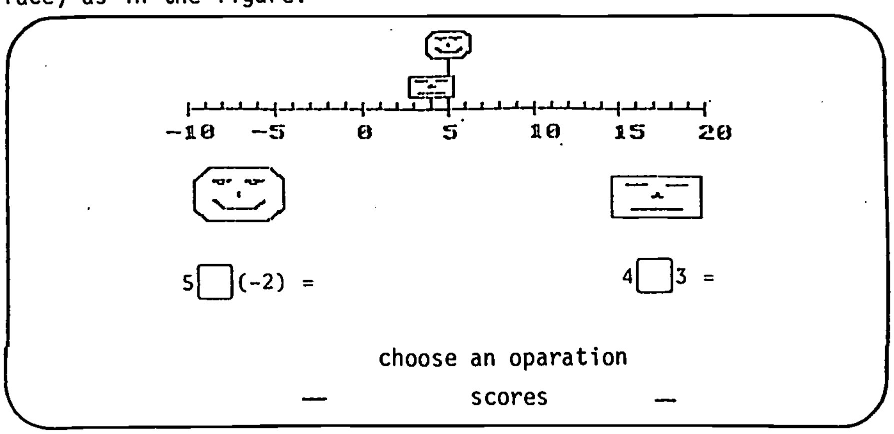
In order to appeal to and to challenge students in the usual heterogeneous class, it is obviously desirable to create learning situations which are flexible, in the sense that students can find in a given situation the cognitive level and demands suited to their needs. Such situations are naturally created by skillful teachers, but the advent of the microcomputer opens up new possibilities in this field. Pierstrup (1984) maintains that powerful learning software encourages the learner to think flexibly. One recommended type of powerful learning software is interesting games, which may offer elements of chance, a sense of risk and the unexpected.

We describe here a package which takes the form of a game that can be used at various cognitive levels. It has built-in challenges to the student, which encourage him to make more sophisticated uses of his mathematical knowledge.

## TWO SIDES TO ZERO

The package deals with arithmetic operations of signed numbers and is intended to be integrated in the teaching of this topic at the grade 6-7 level. It was developed following the ideas of a successful table game (Taizi, 1979). The potential of the microcomputer enabled the creation of a much more powerful educational game.

There are various versions and they are used to create challenges. But even within a single game various cognitive demands are made. We describe the basic version in which a single player (represented by the octagonal face), plays against the computer (the rectangular face) as in the figure.



The player and the computer occupy positions on the number line. In the illustration, the player on 5 stands to the right of the computer on 4. His aim to "leapfrog" the computer, but at the same time, in order to increase his chances of winning, he should try to prevent the computer doing the same to him immediately afterwards. The player moves by choosing one of the four arithmetic operations, and using it to combine the number which represents his current position and another number, randomly selected for him by the computer (-2, in the example). The same applies for the computer and each of them scores a point for leapfrogging the other.

To move intelligently, the player has to analyse the situation. In the example, subtraction not only does not gain him a point, but the computer will. Division is impossible since the game is restricted to integers. Addition will gain him a point, but so will the computer in his turn. The best choice for the player is multiplication, whereby he scores a point and the computer does not.

The philosophy which guided the design of the game is that the student will not be a passive solver of exercises presented to him by the computer. He should play an active role in the setting of the exercise.

The objectives of the courseware include:
- practice in arithmetic operations on signed numbers.
- training in simultaneous considerations of four elements, the positions of the two players, and their randomly selected numbers.
- development of strategies in order to increase chances of winning.

The version described consists of three rounds. Each round finishes when one of the participants accumulates 5 points. The score record at the end of a game of three rounds, almost always shows that the player improves his performance in successive round. The better scores are due to his increased sophistication.

## ANALYSIS OF THE COGNITIVE ACTIVITIES

After considerable experience with students, we identified a hierarchy of the use of the mathematical knowledge, and the related difficulties.

At every stage, the player needs to consider the possible results of four arithmetic operations on two number pairs. All these considerations have an intrinsic mathematical content: the concept of order on the number line, the effect of the operation on the order (e.g. if one adds a negative - it always means moving to the left, if one multiplies by a negative and the position is also negative - it means moving to the right) and the significance of the interval in which the situation occurs (see the example given later).

Difficulties with the required mathematical knowledge are obviously going to reduce the chances of winning - but the computer, by its responses, will help the student to reconsider his strategies and see some of his mistakes. Two types of difficulties were observed: elementary difficulties in actually calculating the results - more mistakes occured, as expected (Kuchemann, 1981), when the position number was negative; and "strategic" difficulties in translating the results into terms of "leapfrogging" and "running away".

Four levels of sophistication were identified:

1) The player considers only the one possibility: whether he can leapfrog the computer.
2) He considers the possibility of both leapfrogging and preventing the computer from doing the same in its immediately following move.
3) He also considers how to be safe at the next stage.
4) In addition to (1, 2, 3) the player also considers achieving a strategic position, from which he increases his chances of leapfrogging the computer at the next turn.

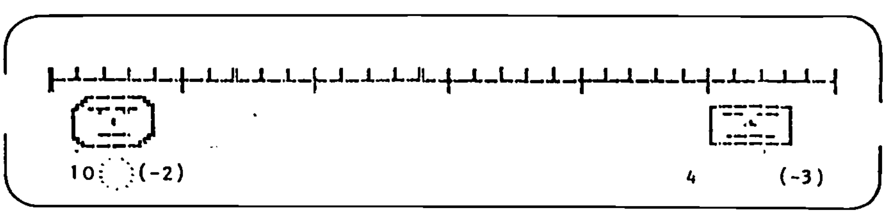
Students who have elementary difficulties start at the first level and progress to the second level. While playing at the second level, they make fewer elementary mistakes. The more advanced students start at the second level and after several games, they move to the third level (a careful defensive approach). Those who play many games may move on to the fourth level. Students who have mastered the arithmetic of integers, quickly move on to more sophisticated versions of the game, which contain challenges at a higher cognitive level.

SUPPLEMENTARY ACTIVITIES

There is no doubt that there are advantages in individual work in the micro lab. However, in this type of work one lacks the fruitful interaction with classmates and the teacher, which may occur in other classroom situations. Thus, the package also includes several worksheets, which invite discussion of the strategies that were developed while playing the games. The teacher can draw student attention to the mathematical ideas mentioned above.

As an example we bring here one item, followed by discussion of its solution.

"In the figure, the numbers on the line are missing.



We know, however, that the student was not sure whether to divide or to subtract. Use this information to assign numbers on the line and to place the players in their positions. (Note that the end-points of the number-line are multiples of 5, and that the student plays first)."

Solution:
We note, first of all, that the number line is 30 units long. Multiplication would take the student to -20, and he would thus both "leapfrog" and "run away". Since he did not choose multiplication, it is reasonable to conclude that -20 is not on the number line.

By division he will leapfrog to -5. From his hesitation we learn that the computer will also leapfrog (to -12). Hence, the interval is from -15 to 15.

Addition and subtraction will both lead to no leapfrog but he will escape the computer. The student considered subtraction, probably because he thought that this is a "better run away"; 12 is further from zero than 8, and has more divisors - it might help at his next turn.

In the presentation we shall demonstrate the different versions of the game and the worksheets, to meet different student needs, and discuss our experience in using the courseware.

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# USING COMPUTER GRAPHICS PROGRAMS AS GENERIC ORGANISERS FOR THE CONCEPT IMAGE OF DIFFERENTIATION

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At the meeting of PME in Israel, a geometric approach to the calculus using computer graphics was outlined [Tall & Sheath 1983]. A current project is investigating the learning and growth of understanding of calculus concepts using this approach. Results to date show that those using the computer have a significantly better idea of the derivative as the global gradient of the graph and are better able to visualise the gradient as a dynamic process.

## THEORETICAL PRINCIPLES

The psychology of mathematics education has traditionally had more to say about the learning of younger children and less about the areas of mathematics that involve more complex principles with older students and mature mathematicians. In the latter case the mathematics syllabus tends to be more logically organised but, as has been shown in [Cornu 1983] and elsewhere, there are significant cognitive difficulties in learning these more advanced concepts. Mathematicians often talk about the need for an "intuitive" approach first, but in [Tall 1985a] I suggest that what is intuitive mathematically (e.g. "geometrically obvious") need not be intuitive psychologically. I propose that we use the term *cognitive* approach for one where the various stages are presented to the learner in a manner appropriate to his current cognitive state.

A cognitive approach does not mean simply breaking the task into a sequence of sub-tasks. The learner will usually get more from a learning situation if he has an understanding of the overall goal and this is not achieved by breaking the task into small pieces and presenting it a piece at a time. Skemp presents intelligent learning as a goal-oriented activity. Dienes, almost in spite of his principle of building up a concept in stages, believes in the "deep end principle". In the theory of "meaningful learning" [Ausubel et al. 1978] describes an "advance organiser" as "introductory material presented in advance of and at a higher level of generality, inclusiveness and abstraction than the learning task, explicitly related both to existing relevant ideas in the cognitive structure and to the learning task". Such organisers, *by definition*, require the learner to have relevant higher level structure than the task itself. The introduction of calculus, in common with other major steps in mathematics, breaks new ground and a different kind of organising principle may be more appropriate.

## GENERIC ORGANISERS

A *generic organiser* is a microworld which enables the learner to manipulate examples of a concept. The term "generic" means that the learner's attention is directed at certain aspects of the examples which embody a more abstract concept. Concrete examples of such organisers include Cuisenaire rods and Dienes blocks. The same principles are embodied in the computer programs in Graphic Calculus [Tall 1985b]: they embody theoretical structures that the user may come to understand through using the generic organiser in specific examples. The existence of such a structure does not guarantee that the user will abstract the general concept. To help the learner get the best out of the system and to avoid developing misconceptions, an external *organising agent* in the shape of guidance from a teacher, a text book, or some other agency is useful to point towards the salient generic features and away from misleading factors. A *generic organisational system* consists of a generic organiser and an organising agent. Using such a system, the learner will still form an idiosyncratic concept image of the concept represented generically. In addition to guidance he may need time for free exploration to iron out the creases in his understanding.

In Graphic Calculus generic organisers in the form of computer programs have been developed for the derivative, the antiderivative, the integral (area), the solution of first order, second order and simultaneous ordinary differential equations.

## GENERIC ORGANISERS FOR THE CONCEPT OF DERIVATIVE

The concept of derivative is introduced globally using two generic organisers. The first is a program to magnify graphs, allowing the student to see what happens when graphs are magnified. Many graphs look "less curved" under high magnification, although guidance will be needed here for straight lines rarely look straight in "high" resolution graphics. The microworld is limited because virtually all the functions that can be typed in have this "locally straight" property and help may be necessary to generate counter-examples such as abs($x^2$-1) or abs(sinx). Even these only fail to be locally straight at isolated points, so further discussion and non-examples are necessary. After using this organiser, the student has the possibility of looking along a graph, magnifying it in his mind's eye, and seeing the gradient vary.

The second organiser is a program which represents the varying gradient. It draws the "practical gradient" of a graph by moving the chord through the points (x,f(x)), (x+c,f(x+c)) along the curve for increasing values of x and fixed c, plotting the gradient as a sequence of points as it goes. The notion of gradient of a curve is given *dynamically* and *globally* in one go, although the final product is a *static* picture of the gradient graph. Thus the concept image of the notion of gradient is generated with both dynamic and static aspects: it is both *process* and *concept*.

RESEARCH

The materials have now been in use in schools at various stages of development for three years, initially for refinement and debugging. This school year began a project to test their use in the classroom. In Britain this must, of necessity, be a limited project, for the syllabus is specified for external examinations and this gives little room for experiment.

The teaching of calculus to sixteen year olds is done in small groups. Two schools provided a total of three experimental classes (each with a single computer) having 12, 15 and 16 students, and five control groups with 9, 11, 15, 16 and 18 students. Within each school every effort was made to cover the same material, one school using a single textbook, the other selecting material from two others. In one school I shared the teaching with the experimental teacher and for the other I provided written guidance to the teachers suggesting how the programs should be used for demonstration and exploration. The teachers kept a log of the work covered, the classroom activities and the use (in the experimental classes) of the computer.

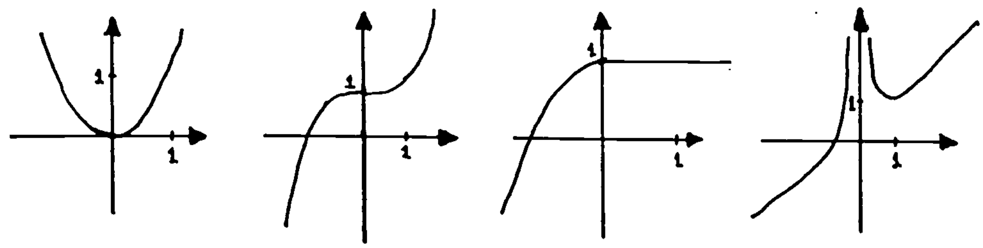
In one school, few of the candidates had received previous exposure to the calculus (3 out of 15 experimental, 1 out of 9 control), in the other the position was reversed (27 out of 28 experimental, 50 out of 60 control). A pre- and post-test included questions on calculating the gradient between points on curved graphs and a simple question requesting the derivatives of $x^4+3x^2$, $\sqrt{x}$, $1/x^2$. On both these questions the experimental group scored higher than the control group. To reverse the bias in later comparisons, the subset of all experimental students with some calculus experience (Ecalc) was compared against the better control students scoring more than half marks on the formal differentiation (Ccalc); Ccalc now scored higher on the formal differentiation pre-test than Ecalc, but not significantly so. To obtain an idea of the effects of the treatment of students of differing abilities, the gradient pre-test question was used to divide experimental and control groups into three subgroups each: high, medium and low.

The post-test, taken at the end of the first exposure to differentiation, maxima and minima, included these same questions. An analysis of covariance showed no significant difference between Ccalc and Ecalc in the improvement of skills of formal differentiation and an analysis of variance between the full experimental and control groups at the end showed no significant difference either, though it should be admitted that the question concerned was not a good test of this ability.

Three questions on geometric notions showed a different story.

SKETCHING A DERIVATIVE

Students were asked to draw the derivatives of the following graphs:



The first and second could be done by guessing the formula for the graph, differentiating formally and drawing the derivative, the third and fourth required the gradient to be seen by eye. On this test, out of 20 marks, the students scored as follows:

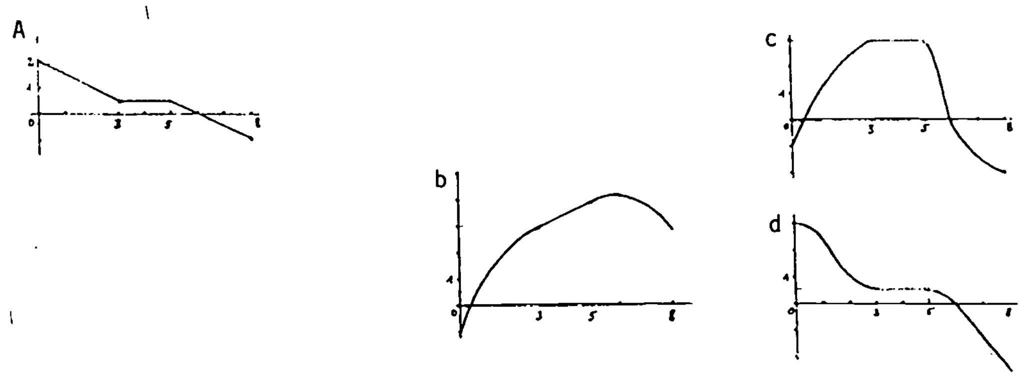
| | no. of students | mean | S.D. |
|---|---|---|---|
| Experimental | 43 | 16·9 | 4·2 |
| Control | 69 | 7·5 | 5·8 |

An analysis of variance of the individual scores using the F-test showed the difference in means to be significant at the 0·1% level. However, the better earlier scores of the experimental group showed that this might be suspect. Comparing the *experimental low* with the *control high* still showed a better performance, significant at the 5% level. Comparing like with like in the high, medium and low groups showed the superiority of the experimental students at the 0·1% in each case. Comparing Ecalc with Ccalc also showed the superiority of the experimental students at the 0·1% level.

The only comparable groups showing similar performances were those subgroups of the control and experimental students who obtained full marks on the pre-test in formal differentiation. Clearly good students develop the notion of global gradient unaided after a time. Giving the same questionnaire to students arriving at university who had not used the computer showed the same level of performance as the high experimental students.

RECOGNISING A DERIVATIVE

Students were shown graph A and told that it is the derivative of one of the graphs b,c,d. Which ? Give reason(s).



32 out of 43 experimental students chose correctly, compared with 20 out of 69 control students (significant at 0·1% using the chi-square test). When reasons were taken into account, 29 out of 43 experimental students were correct, compared to 12 out of 69 control students (also significant at the 0·1% level). Comparing high, medium and low groups showed the experimental students better at all levels, but only the top level had a significant result (0·1%). The task proved more difficult, so the lower success rate and smaller numbers in the lower groups failed to give a statistical difference with the chi-square test. Testing the hypothesis that the experimental group will be better with a one-tail Fisher exact test gives significance at the 5% level for the middle group and 10% for the lowest group. The geometric treatment even benefitted those who had previously done calculus, for the group Ecalc outperformed the Ccalc group at the 1% level.

SPECIFYING A NON-DIFFERENTIABLE FUNCTION

The final task on derivatives was to "give an example of a function defined at x=1, but not differentiable at x=1". 15 out of 43 experimental students were successful against only 1 out of 69 controls (significant at 0·1%). Comparing Ecalc with Ccalc gave the experimental students a superiority at the 1% level. This is only to be expected as the teacher log of lessons showed two out of three experimental groups considered the notion of differentiability in detail, looking closely at "graphs with corners" and other non-differentiable functions. In these two groups 22 out of 27 students attempted the task and 13 gave a satisfactory response.

## CONCLUSIONS

Using generic organisers for the derivative on the computer can enhance the geometric notion of the concept for students of all abilities without seeming to adversely affect formal manipulation.

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# MATHS FOR ALL? JUST HAVE A LOOK

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## Summary

Two classroom observations of two boys working together on two booklets on graphs form the core of this paper. The observations show individual differences in the learning processes of the boys and show in particular where the boys can help each other and where they cannot.

## The research project

In co-operation with Paul Herfs and Jan Terwel and with assistance from Hans Freudenthal I have been working on a research project "Mixed ability groups in mathematics education 12-16". It is one of our purposes to support the mathematics department of the S.L.O., foundation for curriculum development (former I.O.W.O.), which develops new material for children in the age of 12-16. We have partly evaluated this material by direct observations in classrooms of a comprehensive school. In these classes children (age 12, 13) work in heterogeneous small-groups of mixed ability, sex and cultural backgrounds. The first booklet we have evaluated explains the language of graphs, the translation of real situations like the weather into global graphs and vice versa. The second booklet focusses on more local and quantitative aspects of graphs. Each observer is part of a small-group in order to see clearly how the learning process of the group and its individuals develops. We attempt to describe this process as fully as possible. We try to find out how the heterogenity of the group can be used as a stimulus for the learning of mathematics. When a child or a group is blocked in the learning process we look for explanations and try to formulate suggestions to improve the material and the teaching, since our aim, as well as the aim of the teachers and the developers, is to create a mathematics for all.

Interpreting and misinterpreting a graph, an observation

Exercise: Sietske and Marjolein are jogging together, but their way of running is quite different.
Tell something about it.

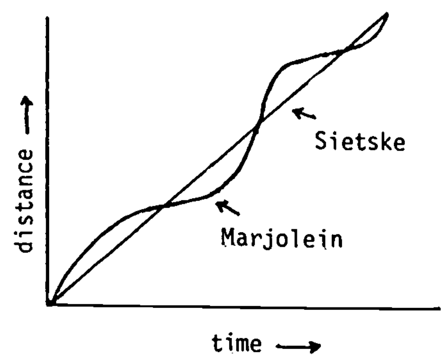


figure 1

"Sietske runs whithout stopping and Marjolein runs fast and then slowly again", says Rik.

"Are you sure?", Jôsé asks, "then why does she make these silly curves?"

"Yes, because look, here she slows down and there she speeds up again", Rik points at the graph. Jôsé objects: "But then you go ..."

"Oh, if you think you know better, ", says Rik, "well, tell me then, she doesn't walk in curves".

"Oh, look at that", says Rik, "he doesn't understand one damned little thing about!"

"That has got nothing to do with it, fatty", Jôsé says.

Rik calmes down and starts to explain:

"Now look, first she walks normal, then he, she slows down ..."

Jôsé: "But it would be better if she went in one straight line. She's quite stupid".

"What stupid", says Rik, "that's the way this graph is made".

"Yes, quite stupid, isn't it", says Jôsé, "imagine me walking all the way in that direction to rest, well, then I'm still tired".

He follows the curve with his fingers.

"You don't understand one damned little thing about it!", says Rik. "Look here, this one walks steady, you know. If she walks let's say ten she keeps on walking ten. Look, and this one starts walking ten and then again she walks five and then she

starts walking ten again and then again five".
Rik sounds very patiently.
"Yes", says Jõsé, "but then why does she make these curves?"
"She makes a curve, she makes a curve, sure", Rik's voice is getting ironical, "look, here she makes a curve into a little street, and then she goes around that street, yes, that's very clever of you".
"A little street?", says Jõsé, "I don't see a little street".
"Oh well", says Rik, "there is a landscape, here there is a tree and then you go around that tree and then, tell me what you mean?"
"I don't mean anything at all, I was just asking why this one walks like that, in a curve".
"Because that goes slowly", says Rik.
Jõsé: "Yes, but then you are not going to walk like that, along that line!"
Rik: "That's why it is a graph, slowly and fast, that's the way this graph is made".

An enormous cognitive conflict. Jõsé can express exactly what he doesn't understand. Rik understands exactly what Jõsé doesn't understand but there is nothing for Rik to reach Jõsé. He even uses numbers to make the graph more concrete. Eventually he joins Jõsé in his misinterpretation as a joke but Jõsé cannot understand this as a joke. Misinterpretation of a distance/timegraph is a very well-known problem in mathematics education. Rik understands the construction behind the graph completely but for Jõsé it may be better if they were asked to construct a distance/timegraph themselves. Understanding is in the making and even in that case there will be enough left for them to discuss.

Sketching and plotting a graph, an observation

Exercise: There is a well-known rule with which you can calculate your braking distance at any speed:
*Take the square of the speed and divide it by 100. Then you have the braking distance in meters.*
*Make a graph from speed 0 upto 150 km/h.*

"You have to calculate it all", Jõsé says grumbling and he starts reading aloud the exercise.
"Let me read that", says Rik, "I'm a professional in reading".
Rik takes over.
Jõsé is calculating the braking distance at 40.

"1600", he says.
"1600?", Rik asks.
"You still have to divide it by 100, then it will be 16", says Jôsé.
"Let's start at the nought", Rik suggests.
"Then you have to draw such a line", Jôsé continues and he moves his hand like this:

Rik hesitates and Jôsé explains:
"40 times 40 is 1600 isn't it, and then you have to divide it again by 100 and then it is 16. You have to draw a graph, so, 40, and then you have to pull a line from zero like this, tsjjjj".
Again he sketches the graph with his hand.
"No, of course not", Rik objects, "you have to put more dots".
"At 80", Jôsé suggests looking at the blackboard. The teacher had taken 80 as an example to explain the meaning of square.
"64", Jôsé says and he puts the dot at the right place.
"Do you have to draw that?", Rik asks.
Jôsé reads aloud again: "Make a graph from speed 0 upto 150 km/h".
"Upto 150", says Rik, "at 150 you too have to put a little dot. You can calculate it all".
The teacher passes by and asks whether they have enough dots to draw the graph.
"Sure", says Jôsé.
"Oh no, I don't think so", Rik objects.
The teacher suggests they calculate some more and she leaves.
Rik and Jôsé start calculating some more points. Especially Rik really gets involved.
"You can put a dot everywhere", Rik cries, "you can do it too at 100".
"Yes well, then you have a line", says Jôsé, "let's do it at 100".
"Well now, you can do it anywhere", Rik leans back.
"100 times 100", Jôsé is calculating.
"You can pull that line very easily", says Rik.
Jôsé: "100 times 100 and then ..."
"Now we have it all", concludes Rik.
"Now we will draw the line", Jôsé suggests and so they do.

This is the first time for Jôsé and Rik that they have to construct a 'square' graph themselves and now it is Jôsé who understands the construction. He almost can sketch the graph right from the formula. In fact he needs only a few points to 'feel' the

form. Rik isn't so sure. He needs more points. Even the step from points to a line is not an obvious one for him. And that's nice, José helps him in calculating and plotting more points and then it is Rik who finally gets this global insight in the form of the graph which José expressed right at the beginning. They both end with a fine graph: quite precise and of good global form.

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# A STUDY OF MATHEMATICS EDUCATION IN CLASS-ROOM SITUATION
## A MATHODOLOGICAL RESEARCH

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(SUMMARY)

Here is illustrated a method of research in mathematics education which is not an isolated procedure but a kind of the system-approach, taking as an example an observation of one hour's class-room activity to learn the concept of 'ratio per unit' in 5th grade. It is the auther's intention to argue that the method like this is one of the most effective which will really contribute to the development of mathematics education.

## 1. INTRODUCTION

Mathematics education is a very complex phenomenon and it is gradually recognized that its research needs a so-called system-approach.[1] The central aim of this approach, I think, is to catch the reality without killing the essence of the phenomenon. We can't access to the reality of such a complex phenomenon by means of an isolated observation of its components separately like an experiment in a laboratory of psychology. We should observe it as a whole in relating systematically each component to others.

Even one lesson of mathematics in a class-room is a living whole and is regarded as a miniature of the whole mathematics education. It has many varieties of problems which could not be observed in isolation, but should be investigated through a system approach. And, how to organize this way of approach is in itself an essential problem of the practical study of mathematics education. In this paper, however, I can't propose the finished progamme of this approach, but only suggest some problems which should be considered in such a way of research in education.

The main problems that I could be aware of are as follows:

1) How to designe mathematics lesson so as to be successful. ----- didactical problem.

2) How to help pupils in their concept-formation. ----- psychological problem.

3) What roll can teaching-aids play in mathematics education?
----- methodological problem.
4) How far adequate is mathematics in solving the real problem?
----- epistemological problem.

2. BRIEF DESCRIPTION OF THE LESSON

Pupils: 5th grade (10 - 11 years old) in Elementary School.
Theme: Ratio per unit
The process of the teaching:

1) Teacher showed a table of a record of basket-ball shooting as follows and asked who the best player is of three children A, B and C.

| number of / child | shooting | success |
|---|---|---|
| A | 4 | 2 |
| B | 10 | 4 |
| C | 20 | 6 |

2) A pupil said "C is the best, because he succeeded 6 times, the highest number of success."

3) Another pupil said "No, C is the worst, if we compare the number of failure instead of success."

4) The third pupil said " We can't say who the best is, because the number of shooting is different to each other."

5) Teacher said " If numbers of shooting are different, any way to make them the same?"

6) Teacher showed them three elastic strings which were 4, 10 and 20 decimeters long with red colored part of 2, 4 and 6 decimeter respectively, and pulled them all to the same length.

7) A pupil said "We had better imagine that all of them shot 20 times each." Most pupils agreed.

8) Teacher: "Any other number of shooting can you imagine?"

9) some pupils mentioned 40 and 60 times.

10) Teacher: "O.K., then how about 100 times?" All pupils agreed. (This might have been the best opportunity to refer to 'percentage' but he left it to the next teaching.)

11) Teacher continued "Then, how about only 1 time?" Children agreed though not so willingly. Thus teacher introduced the concept of 'ratio per unit'.

3. PROBLEMS AND DISCUSSIONS

Though it was but an usual lesson, we could be aware of many problems of various kinds most of which are common to all lessons in mathematics.

1) For the desige of successful lesson.

The lesson to make pupils understand a new mathematical concept would be the most successful, if it starts from the situation seemingly chaotic or contradictory and ends with its clear resolution by using that concept effectively.

In this lesson some pupils measured the skill of shooting a ball by the number of successes and others do it by the number of failures. On one hand the most successes mean the best player, and on the other hand the most failures mean the worst player, but these two criteria are contradictory to each other and C becomes the best and worst player at the same time. It is here for the concept to come out as the useful tool for the resolution of this contradiction.

We have in our country several editions of arithmetic textbook in Elementary School, and according to my investigation they are sorted into two groups in the way of introducing the concept of 'ratio per unit'.

(A) One is using the example like the following:

"City-hall, TV-tower and department-store are 20m, 50m and 30m high respectively. If we take the height of TV-tower as a unit , how much are other buildings?"

(B) The other is using the same example of ball-shooting as above and asking who the best player is.

There is a fundamental difference between these two ways of of introduction. In (A) we think the ratio of magnitudes of the same kind, that is, height to height (extensive ratio). But in (B) we think the ratio of magnitudes of different kind, ratio of numbers of success to numbers of failure (intensive ratio).

I can't say which is better as the introduction to the concept in this earliest stage. But from the didactical point of view, we should notice that the teaching using the example (B) could be more successfully developed than that using (A).

In (A) pupils would only calculate the ratio according to the instruction without knowing why to do so. But in (B) they should re-invent the concept and would be well aware of the roll of this concept as a tool to clarify the problematic situation.

2) To help pupils in their concept-formation.

In the stage 7) of the process of this teaching, we found a pupil who made a curios table like in the right. How can we understand this table?

| | shooting | success |
|---|---|---|
| A | 20 | 18 |
| B | 20 | 14 |
| C | 20 | 6 |

He made this table not by multiplication but by addition to have 20. He could not understand the aim to make 20 as an imagined shooting number. To him the elastic strings could not have any sence as a teaching-learning aids to ratio. Perhaps he has not ample experiences enough to learn it or can't reorganize his own experiences so as to learn it , and it is for him too early to understand the concept through such a process of teaching.

From this observation, we think it would be right that recent psychologists make a distinction between the concept-formation and the concept-attainment. There is a long history of concept-formation in the mind of children, and the concept-attainment is the final scene of this long history, though for some pupils the history might be continued further in the advanced study.

Considering this long process of the concept-formation, we think it would be a good policy to set up appropriate materials to anticipate the important concept that is to be learnt in the later grade. But before doing this, we should collect more data about children's concept-formation, and in this respect we are happy to be informed many things from the phenomenological investigations of Dr. Freudenthal in Hollannds.[2] It would mean a real progress of mathematics education to accumulate more phenomenological facts in mathematical concept-formation in children.

3) Roll of teaching-aids in mathematics teaching.

From this teaching, we could be aware of two problems concern-

ing the roll of teaching aids in mathematics education.

(1) A topic of real situation is itself a kind of teaching-learning aids in non-material level, and adoption of a topic decide the way of teaching and pupil's understanding.

I have already alluded to my investigation of text-books and mentioned that there were two kinds of ways (A) and (B) to introduce the concept of ratio; in (A) the comparison among buildings is used, and in (B) topics are almost the same as that of ours.

I should like to say that (A) is corresponding to the 'formalist' view and (B) is to 'constructivist' view of teaching in terms of Mr.Herscovics and Mr.Bergson.[3] Though originally this distinction was applied to the mathod of teaching, I would say that this is also the distinction of topics the teacher uses. We should look for good topic in order to develop a good teaching in constructivist way, which I also believe to be desierable in mathematics eduction.

(2) In this teaching, we could notice that there are two kinds of teaching aids: the one is to give the experience to understand a concept and the other is to make pupils acquire a concept by organizing their previous experiences. Elastic strings were the aids of the latter kind.

4) An epistemological problem: Adequacy of mathematics.

On the way of this lesson, when teacher showed the data and asked who the player is, an accident has happened which was very much trouble to the teacher who wished to lead the lesson along with his ready-made programme. One pupil stood and answered "We have to divide the number of success by the number of shooting and compare them to each other." Perhaps this pupil was taught it by parents or studied in JUKU, the notorious private school in today's Japan under the examination fever. But anyhow he said all things that was to be learnt in this lesson. In reality the teacher was annoyed with this answer and asked to the pupil not to say any more because he knew far more than others about this topic.

After the lesson, I said to this teacher that he had better say to the pupil "Why don't you multiply the two numbers?", then the pupil would not be proud of his shallow knowledge. In this lesson ratio is adopted as the measure of the skill, but it is radically doubtful if this measure is adequate to show the degree of the skill.

However, this is not the place to talk about adequacy of mathematics, but only to foster the concept of ratio; we should seek for another opportunity to make pupils reflect on this problem.

## 4. FINAL REMARKS

The practical final aim of the research of mathematics education is to help teachers to produce the effective classroom activities of everyday in every school. In reality mathematics education is realized through such daily activities.

As we saw in this paper, even one hour's lesson as a classroom activities have many problems, each of which an attractive theme of the research in mathematics education.They might be investigated in isolation, but in such a way of research we often lose to see the whole image or the unity of mathematics education which is something of a complex organism; we should see the whole relating each component to each other.

Though it was not satisfiable nor successful, I showed here an example of such a study.

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# EPISODIC ANALYSIS OF A MATHEMATICS LESSON

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Abstract

This paper presents an example of the use of "Episodic diagrams" applied to the observational time-on-task data gathered in six classrooms. A comparison between the intended structure of the lesson and what actually occurred is made. From the diagrams, it is apparent that teachers modify lessons. The modifications reflect differing intents of what the lesson was about and how it should proceed.

## INTRODUCTION

At last year's PME meeting in Sydney, I described an approach for the reaggregation and analysis of time based data (Romberg, 1984). The approach was labeled "episodic analysis," and involves creating diagrams which capture the instructional sequence of a lesson. Since then Allan Pitman and I have been reaggregating observational time-on-task data gathered in several classrooms as part of a collaborative three-year longitudinal study on the development of initial addition and subtraction problem-solving skills. The study was conducted by the Mathematics Work Group of the Research and Development Center at the University of Wisconsin (Romberg, Carpenter, & Moser, 1978).

The episodic diagrams are based on a model in which lesson content, teacher planning, and the actions of both teachers and pupils during a lesson were related (Romberg & Pitman, in preparation). All of these four aspects provide sources of information relevant to effective education theorizing and observational analysis of the structure of classrooms and behaviors with them. The model is temporally based, and suggest causal and general correlational linkages between groups of variables.

Close attention is paid to the content of the lesson, particularly through our access to the actual materials used, and information as to whether at any given time a teacher was using, modifying, or replacing parts of it.

## EPISODIC ANALYSIS

This analysis procedure enables us to use the statistical data to produce models of episodes in the daily operations of classrooms. These episodic diagrams recreate significant aspects of each lesson observed, including content, teacher and pupil behaviors.

In order to produce these diagrams, the time-on-task data was recoded in order that variables could be built which reflected teacher and pupil behaviors, classroom structure, and content. Graphical presentation permits inspection for characteristic patterns involving both concurrent and subsequent related sections, and the identifica-

tion of patterns of behaviors and interactions associated with particular types of content, and with teachers and classrooms.

From such diagrams contrasts are possible between the intentions of the teacher; those of the producer of the materials used; and the classroom actuality.

The graphical diagram format developed enables a clear presentation of data derivable from time-on-task analyses, without the difficulty of extraction from dense tables. Thus, time actually used is available from the horizontal axes; use of groupings with whole class, small group and individual work is clearly evidenced; levels and types of interaction between teacher and pupils and among pupils can be gauged not only in an aggregated form, but in context of the manipulation of classroom setting, changing levels of off-task behaviors, and (where such information is available), in relation to the materials and mathematical content of the lesson.

SIX TEACHERS DEAL WITH AN ACTIVITY

Activity part S4D4 presents children with a worksheet of addition and subtraction sentences for completion (Kouba & Moser, 1980). Teachers are instructed to "...not demonstrate how to solve or encourage any particular method of analysis." Children "totally confused or discouraged" are to work in small groups. Discussion by children at the end of the lesson is to present the range of methods used in solving the ten sentences, and to deal with reasonableness of answers.

Figure 1 illustrates the assumed lesson format as derived from the text, and time estimates by discussion with its developers. Figures 2 to 7 present the lesson format for five of the six teachers who used the activity.

If we look initially at the time taken in teaching this part, we note that the expected 40 minutes in fact ranged from 8 to 41 minutes. Moving to the grouping patterns, we observe that only one of the five teachers had a sequence of grouping consistent with expectations of the material producer: Teacher 1. (see Figure 2)

Following through the analysis for this teacher, we can see that the situation is not so straightforward. First, pupil engagement shows us that most observed pupils were off task during individual activity. The off-task behavior suggests that the children could not do the sheet. The on-task engagement following is entirely teacher talking/pupils listening, with a dearth of questioning. Contrast this with the clear direction given teachers in the guide material.

Teacher 2 spent 8 minutes, split over two days, on S4D2 content of which two minutes were spent on directions and the rest on individual seat work, during which the teacher spoke with individuals. Children were largely off-task, and the exercise not pursued. The episodic diagrams are not included for this teacher.

Teacher 3 (Figures 3 and 4) appears to have followed the materials with more fidelity. After a period of seatwork on D2 content commencing partway through the

mathematics session, with the teacher dealing with individuals, the class is brought together for discussion. There are indications of teacher listening as well as speaking. Pupils are engaged in content. The discussion is continued the next day.

The pupils of Teacher 4 (Figure 5) spent the entire 33 minutes on individual seatwork, after an initial directive introduction. There is no discussion of methods in which the class shares. The small group observed is finishing earlier (part D1) work, and does not get to work on part D2 in class.

Teachers 5 and 6 (Figures 6 and 7) seem superficially to have handled the activity in similar ways. An extended introduction is followed by five to eight minutes of seatwork, and the next activity commenced without review. The most significant differences between the two are the teacher/pupil interaction during the large group sequence; while Teacher 6 is seen to dominate with teacher talk, Teacher 5 engages the children in more discussion as evidenced by the increase in teacher listening. Further, following the interactive discussion, the pupils of Teacher 5 tend to be off-task during seatwork; possibly having completed the bulk of the questions already. Those of Teacher 6 complete the items following the discussion.

One thing which becomes clear from inspection of sequences of lessons by teachers is the degree to which flexibility of organization interacts with content and materials. Some teachers work to a steady "large group-individual work-large group" sequence lesson after lesson, followed by up to a week of individual work. Others vary the sequences much more responsively with the material and content.

Even given the pressure which a teacher must experience in a research setting to follow the instructions in a written curriculum, it is clear that considerable modifications of material usage occurs. Modification in the structuring of the lesson is seen to have effects regarding the differing intents of material-developers and of teachers. This has occurred in the case of part D2 of Activity S4. The intent of the writers of this material was to generate cognitive dissonance: hence, the instruction to teachers not to demonstrate solutions. Our discussion of episodic diagrams suggests that children generally were not put in the dissonant position, or if they were, were rescued immediately with teacher dominated instruction.

## SUMMARY

The purpose of this paper is to focus on the usefulness of the episodic diagram as a method of reanalyzing time-on-task data to maximize the richness of its interpretation. We are now in the process of reanalyses for 148 lessons on addition and subtraction (Kouba & Moser, 1979, 1980). It is our expectation that we will be able to categorize lessons and implied pedagogical style of the teachers, in order to comprehensively discuss the use of materials, selection and rejection of content, and the relation of pedagogical style and material intent. In our perusal of several

topics, we are confident that these analyses will give understandings of the complexities of the interactions of teacher, child, and content.

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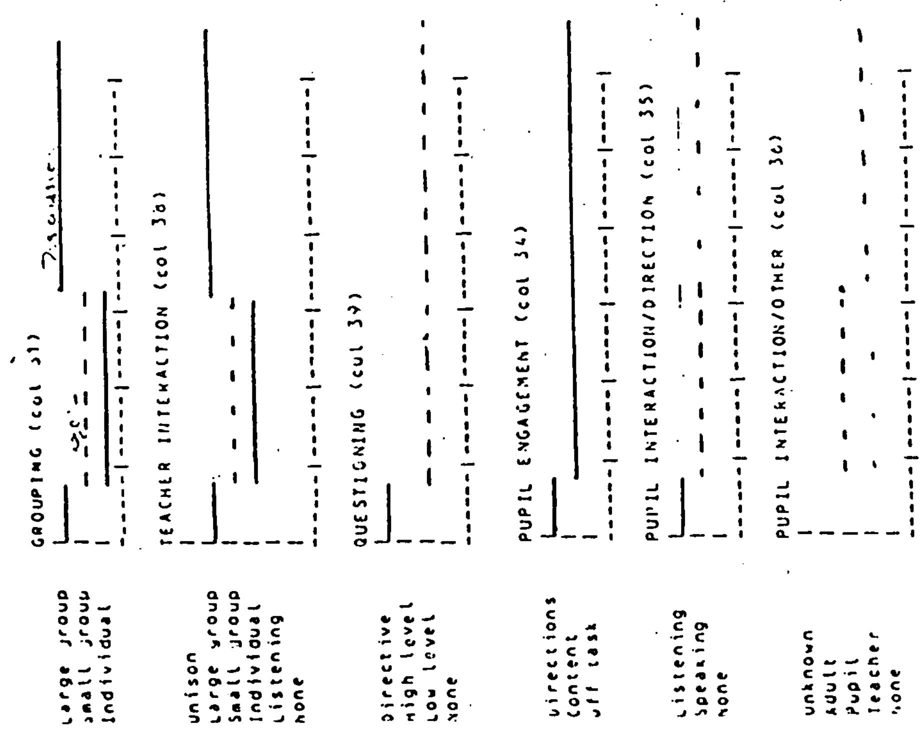



Figure 1. Intended Episodic Diagram for Topic S4, Activity D, Part 2

```
                  GROUPING (col 31)
Large group       |44444             4444444444444444
Small group       |
Individual        |      22222222
                  ------|-----|-----|-----|-----|-----|

                  TEACHER INTERACTION (col 38)
Unison            |
Large group       | 4444          4 444   44444 44
Small group       |
Individual        |          22
Listening         |                             1
None              |0    00000  0 0   000           0
                  ------|-----|-----|-----|-----|-----|

                  QUESTIONING (col 39)
Directive         |                  3     3
High level        |
Low level         |  1        1              11
None              |00 00000000 00000 00000   0000
                  ------|-----|-----|-----|-----|-----|

                  PUPIL ENGAGEMENT (col 34)
Directions        |    2         22
Content           |11 1 111        1 1 1111111111
Off task          |        00000      0
                  ------|-----|-----|-----|-----|-----|

                  PUPIL INTERACTION/DIRECTION (col 35)
Listening         |   22          2 222   22222222
Speaking          |
None              |000  00000000 0    000
                  ------|-----|-----|-----|-----|-----|

                  PUPIL INTERACTION/OTHER (col 36)
Unknown           |
Adult             |
Pupil             |                             2
Teacher           |   11          1 111   11111 111
None              |000  00000000 0    C00
                  ------|-----|-----|-----|-----|-----|
```

Figure 2. Episodic Diagram Observation of Teacher 1, Topic S4, Activity D, Part 2

```
                  Part 1                   Part 2
                  GROUPING (col 31)
Large group       |4444                        4444444444444444
Small group       |
Individual        |    2222222222222222222222
                  ------|-----|--+--|-----|-----|-----|

                  TEACHER INTERACTION (col 38)
Unison            |
Large group       |44  4       4               444      44   44
Small group       |
Individual        |  2         2  22 2  2           22  2
Listening         |                               11      1
None              | 0  000000 000 00  0 00 0   00       0
                  ------|-----|--+--|-----|-----|-----|

                  QUESTIONING (col 39)
Directive         |
High level        |
Low level         |11 1                    1       1 11  1
None              | 0 0000000000000000000000 0000000 0  00 0
                  ------|-----|--+--|-----|-----|-----|

                  PUPIL ENGAGEMENT (col 34)
Directions        |  2   2   2         2      2
Content           |1111 111 1   1111111 1     1 111111111111111
Off task          |          0            000
                  ------|-----|--+--|-----|-----|-----|

                  PUPIL INTERACTION/DIRECTION (col 35)
Listening         |22 22       2               222  2222222 222
Speaking          |          1
None              | 0  0000 0 000000000000000     00       0
                  ------|-----|--+--|-----|-----|-----|

                  PUPIL INTERACTION/OTHER (col 36)
Unknown           |
Adult             |
Pupil             |         2                       22
Teacher           |11 11         1               111    11111 111
None              | 0  0000 0 000000000000000     00       0
                  ------|-----|--+--|-----|-----|-----|
```

Figure 3. Episodic Diagram Observation of Teacher 3, Topic S4, Activity D Part 2

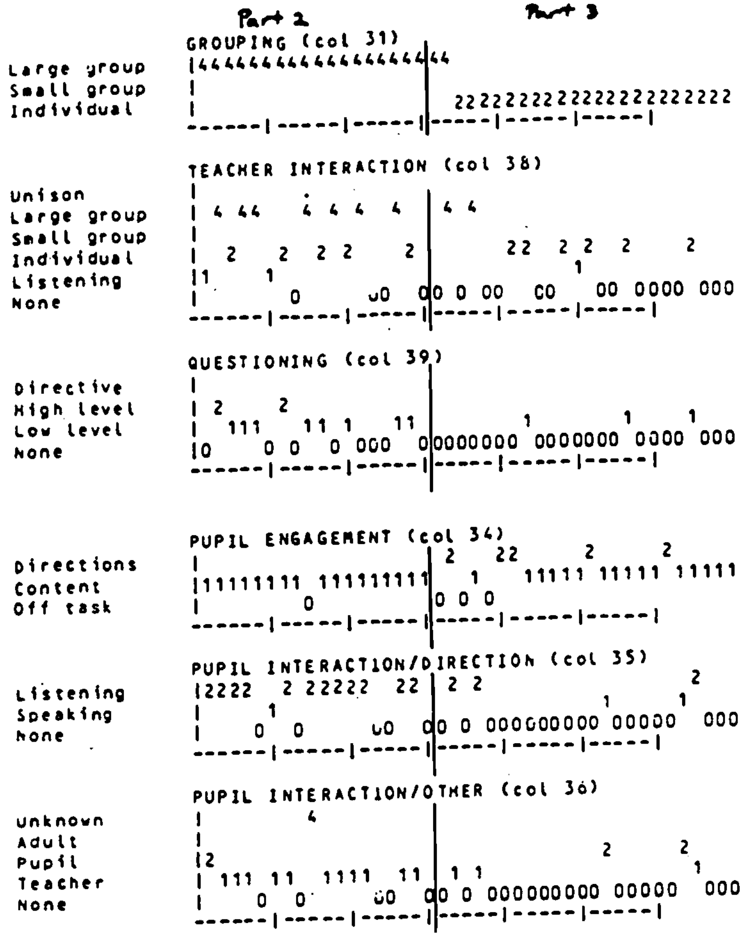



Figure 4. Episodic Diagram Observation of Teacher 3, Topic S4, Activity D, Part 2

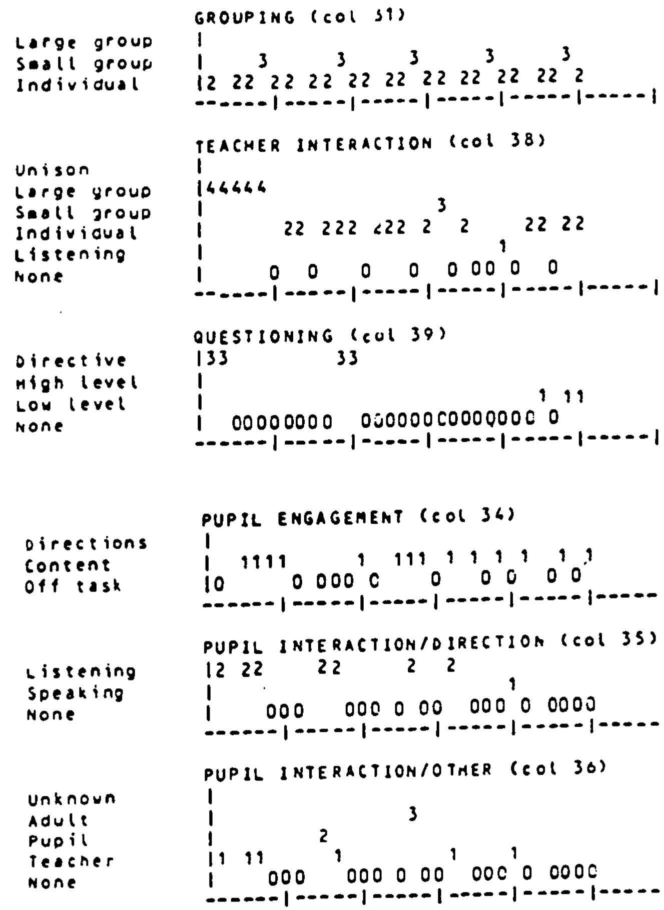
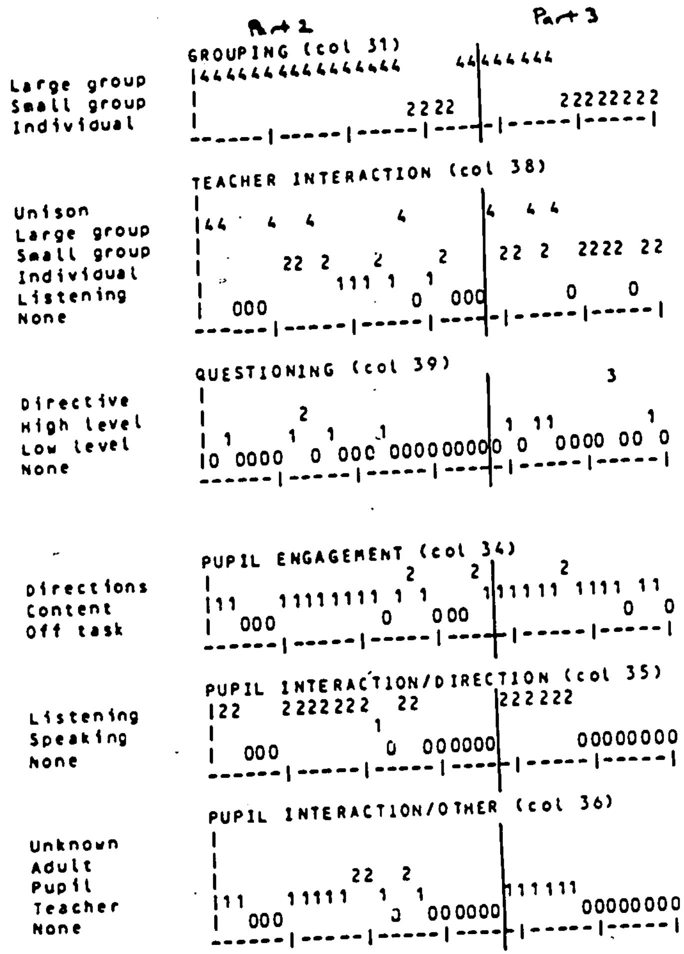



Figure 5. Episodic Diagram Observation of Teacher 4, Topic S4, Activity D, Part 2

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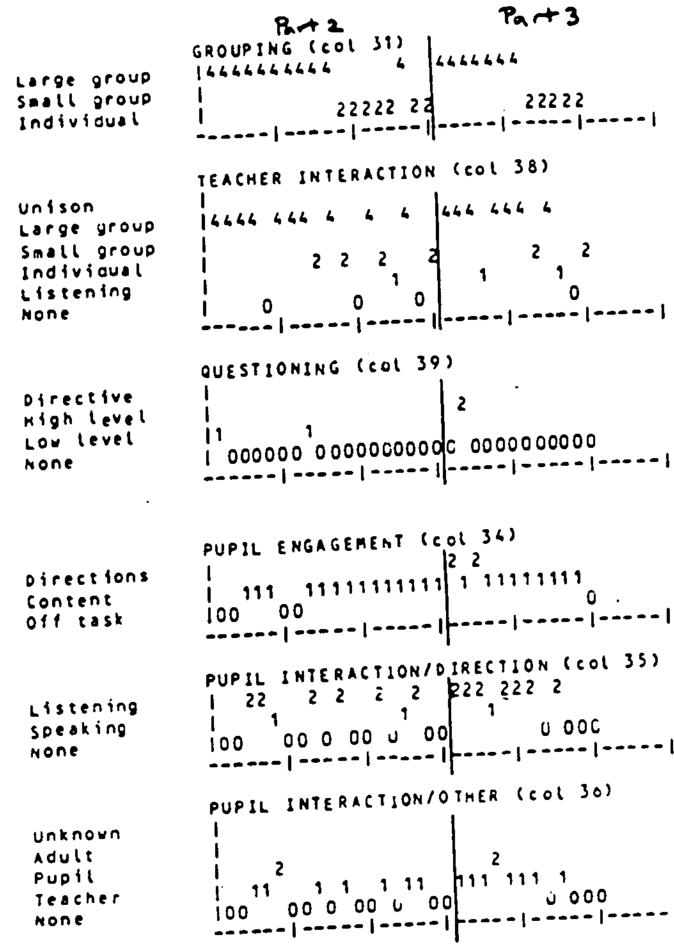


143



Figure 6. Episodic Diagram Observation of Teacher 5, Topic S4, Activity D, Part 2

Figure 7. Episodic Diagram Observation of Teacher 6, Topic S4, Activity D, Part 2

# WHAT PROBLEM IS THE CHILD SOLVING ?

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Some degree of consistency may be set while children are giving wrong answers to elementary arithmetic problems. Incorrect answers may somewhat suggest that the resolved problem is actually different from the initially proposed one. That is to say that the child replaces the given problem by another one which he knows how to solve (cf.the notion of "glissement de sens (slipping-meaning)" in Vergnaud, 1982). Everything happens as if the failure was placed more on the interpretation level of the problem, instead of in its solving procedure, as far as we are able to isolate those two phases.

Understanding a problem is establishing relationships between the elements given in its verbal statement. The concept of "schema" is considered by many authors as an useful tool to describe the necessary knowledge required to build up these relationships (Riley, Greeno & Heller, 1982; Escarabajal, Kaiser, Nguyen-Xuan, Poitrenaud & Richard, 1984). A schema is built up from three kinds of knowledge: logical knowledge, mathematical knowledge, and semantical knowledge. A schema is a set of variables, called "slots". Links are connecting these slots: they can be either relations or operations. Hence, one can suppose that the interpretation of a problem statement, i.e. the process which allows to give a meaning to this statement, consists of an activation of one of the available schemata, followed by an instanciation of the slots of the schema. Therefore, the problem solution can be infered from the relations defined by this schema. Consequently, a schema appears as a kind of "hard structure", that is to say a mold into which the problem statement will be insert.

One of the possible hypotheses to explain incorrect answers relies in supposing that the child, not possessing the appropriated schema, activates one of those he actually possesses, corresponding to an already known category of problems. Therefore, this means that the

Figure 7. Episodic Diagram Observation of Teacher 6, Topic S4, Activity D, Part 2

Figure 6. Episodic Diagram Observation of Teacher 5, Topic S4, Activity D, Part 2

answer, as incorrect it can be regarding the given problem, is correct, concerning another one, misleadingly recognized through the activated schema.

The questions we try to answer to are the followings ones:

1. Can we rationnally assume that the child understands and solves a different problem that the given one?

2. To what extend can we suppose that understanding a word arithmetical problem is nearly activating an already existing schema?

METHOD: 51 children (9-11 yrs old) are given successively two statements of problems (target problem). They are asked to invent for each target problem another one. The instruction given to the children is to invent a problem which, in one hand, tells a different story than the target's one, and, in another hand, implies the same reasonning for finding the solution. This experiment has been conducted during class-times. The children wrote their answer on paper book. Looking-puzzled children were proposed the context of "a boy playing marbles". When they have proposed two invented problems, the children were asked to solve them.

We used eight different target problems all consisting of a "composition of two elementary transformations" (cf. Vergnaud, 1982). The structure of these problems can be represented as follows: $\underbrace{\boxed{e1} \xrightarrow{T1} \boxed{e2} \xrightarrow{T2}}_{T3} \boxed{e3}$. T1 and T2 are the two elementary transformations, and T3 is the composition of T1 and T2; e1, e2, e3 are states. In the present situation, T1 and T3 are always given, while T2 is the question. T1 and T3 are always of opposite algebraical signs.

The eight problems have been constructing by co,bining the following variables: * context of quantity of money or context of quantity of persons in a bus; * presence or absence of the initial state e1; * order of the algebraical signs of the transformations T1 and T2: either +- or -+.

An example of a target problem:

$$\underbrace{\boxed{34} \xrightarrow{+50} \xrightarrow{?}}_{-23}$$

Marc has a saving of 34F. His father gives him 50F. He goes to a book shop and buys a dictionary. Back home he counts his money and notices that his saving have gone down by 23F. How much does the dictionary cost?

HYPOTHESES: The structure of the invented problems send back, in our opinion, the understanding of the target problem. If we suppose that in order to understand the target problem, the child activates a schema, we can predict that:

1. If the schema is an appropriate one, then the structure of the invented problem would be the same as the structure of the target problem. If the activated schema is inappropriate, then the structure of the invented problem would be differente, and specially must be more elementary.

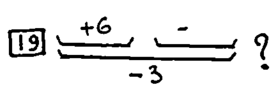
2. The problem solution given by the children must be compatible with the structure of the invented problem.

3. The structure of the two invented problem would must be very the same (identical one with the other).

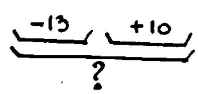
MAIN RESULTS:

<u>About the structures of the product statements</u>. 70 per cent of the invented probems presented different structures from the given target problem's (cf. figure 1). We can arrange the variants in three categories. We illustrate those with statement produced by children.

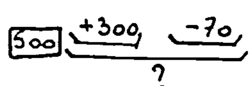
A- Children maintain the composition structure but change the place of the question either on the final state or on the composed transformation:




Pierre has 19 marbles. He wins 6. Then he plays again and loses. At the end of the game he counts his marbles. He notices that he is missing 3 marbles. How many marbles has he got?




This morning, before going to school, Jil counts his marbles. At playtime, he loses 13 of them and that evening on his way back he buys a packet of 10 marbles. How many marbles does he need in order to have the same number of marbles that he had in the beginning?




The till of a big store takes 300F of goods. It already had 500F. It gives back 70F to someone. How much more money does it have than it have than it had before?

B- The children leave out the composed transformation (T3). The most frequent deformation consists of arranging the actions in a temporal succession, hence the information on the component transformation is displaced in the final state:

$\boxed{50} \xrightarrow{-35} \xrightarrow{+?} \boxed{30}$ My brother has 50F in his pocket. He goes to buy a puzzle for 35F. Later, his mother gives him somme money. He counts It and sees that he has 30F. He says that his money gone up. How much money did his mother give him?

$\xrightarrow{+61} \xrightarrow{-?} \boxed{15}$ Marc counted his marbles. In the afternoon his friend gives him 61 more. He risks them and when he gets home he counts that he has 15 left. How many has he lost?

C- Finally, the children forsake the structure of composition in order to substitute it with a structure of one single transformation:

$\boxed{23} \xrightarrow{?} \boxed{18}$ Jean has 23 little cars, he goes to the garden. He comes back home and notices he has 18 left. How many has he lost?

$\boxed{5} \xrightarrow{-3} ?$ A child has 5 sweets. He has 3 missing. How many has he left?

$? \xrightarrow{-10} \boxed{2}$ Pierre plays at marbles. He risks 10 marbles and he loses them and he has more than 2 left. How many marbles has he?

<u>About the compatibility between structure and solution</u>. When the structure of the invented problem is the same than the target one: only 22% of the solution are compatible with the invented problem. The following example is typical: at te statement "Jean has 38 marbles. Playing with a friend he loses 18 marbles. He buys another bag of marbles. He realises that his number has increased of 8 marbles. How many marbles has he bought?" child's solution is "he has bought 28 marbles: 38-18=20; 20+8=28". We immediately can suppose here that most of children did not activate a "composition transformation" schema, but recopied the surface details of the target problem's statement.

When the structure of the invented problem is different of the target one: only 70% of the solutions are compatible with the invented problem. For example with the statement (3) child says: "I can't answer because I don't know the number of marbles that he had in the beginning". We can see here that our hypothesis about activation of a schema for understanding is not sufficient: the understanding of relation's qualitative meaning does not help him to solve.

<u>About the comparaison between the two product statements</u>. It is perhaps with this last observation that we can back up our's misgivings (uncertainty) about an understanding in terms of a schema's activation: only 21% of children make up two problems of identical

structure. If they add an initial state they add it for the two problems, but the subject of the question can be either a state or a transformation. More, 38% of children produce two statements belonging to only one among of the three categories (A,B,C) upper mentionned.

DISCUSSION. It would clearly appear that the majority of children do not possesses the relevent concept of transformation composition. This is not surprising. They all do not reduce the problem to a single transformation's elementary schema. They rather attempt to duplicate "this schema" in a time process. This implies the obligation of an origin. This means that transformation cannot be independant of the states, but remains defined as the modification of a state to produce another state. Here we are according to Gellman's analyse (1978) who says that arithmetic operations are primitively understood as increasing or decreasing quantities. This interpretation is apparently compatible with an interpretation in terms of schemata's activation: we might present this "primitive conception" like a schema, with its network of empty slots organizing the meaning of the statement. However, the variety of built-up statements produced by children drive to a less "determinist" process of understanding. Or, to say the least, rather more "dynamic"...

Wanting to make models of understanding process needs to consider more a construction "on line" of the relationships than a recognition of a formel structure of relationships.

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The structure of the invented problem is the same than the target one:

The invented problem's structure is different from the target one:

No answer, or coding failure:

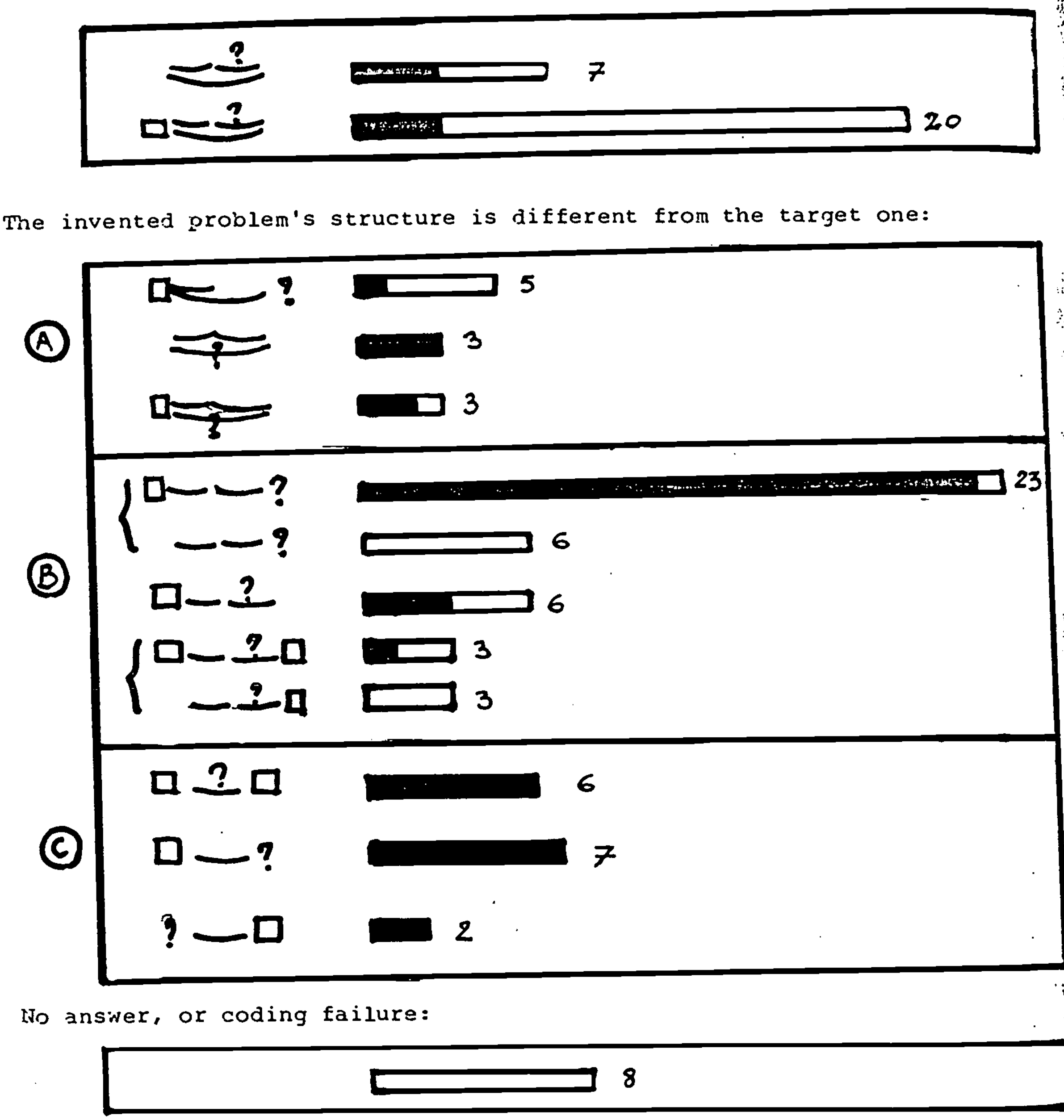


Figure 1: Frequency of invented problems' structures (we have blakened the cases where the further given solution is compatible with the structure).

# COMPARISON OF MODELS AIMED AT TEACHING SIGNED INTEGERS

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We find in text-books several models for teaching signed integers (which we also called generalised integers). In this paper, we attempt to classify those models and question the role and nature of the symbolism introduced. Symbolic models are distinguished from mental image models. A teaching experiment and its results are presented. They show for addition items a significant superiority of a mental image model.

Certain mathematical ideas have natural phenomenological counterpart since they result by a process of abstraction from objects or observable entities. This is the case for integers, fractions and most geometric concepts. We also note that very often relations between those objects have observable meaningful counterparts as example addition, substraction, sharing...

The negative integers are special in the sense that they were devised at first to solve equations of the type x+a=b (a and b integers, a>b). History tells us that they were first considered as "number to substract" long before the number line came as a standard representation. Even with Descartes (see GLAESER, 1981), the negative numbers were a special case and the number line was not "unified". Each half-line was considered separately. There was (and still is) a great epistemological obstacle which had (and still has) to be overcome before numbers become a mere position on a line. Indeed, numbers were naturally associated for centuries with magnitudes (grandeurs, in french). The first implicit model of numbers was any scheme showing a potential collection of units.

For a long time (and perhaps till very recently) negative numbers were entities obeying rules with respect to the usual arithmetic operation and vaguely (at times) associated with debt. There is a gap of a century (1850-1950) in our history of models which we would like to fill soon. So far, we imagine that modern pedagogues, such as Dewey, Montessory or text-book writers inspired by their philosophy, created teaching schemes which would give "concrete" meaning to the negative numbers.

In fact, the views of most authors on the use of models do not seem to converge. On the one hand, many believe that models should be temporary. In other words, as soon as a correct and efficient handling of the abstract objects are achieved, the model should be forgotten. On the other hand, there are other people who believe that a model is irreversibly a model for life since it can brought back into play at any later moment when there is a need for the rules to be "rediscovered".

At any rate, the main idea behind a model is its transferability. With Ahmed Daife (see DAIFE, 1983), we have tried to discover if there were substantially different models of generalised (1) integers. It turned out that we found two sorts of models: the ones related to the number line in which integers are either position or motion. A second set of models is based on the idea of mixing objects opposite in nature (such as protons and electrons, full and empty, black and white objects...) and using couples or pairs to basically represent them. In fact, in each case we have an equivalent class for each integer.

Each model attempts to give meaning to the generalised integers and to at least one operation (the addition). Substraction often requires acrobatics in order to be defined v.g. as in the number line model. Actually, in many models, substraction has no contextual meaning but is presented as the addition of the element opposite in nature.

This classification although very useful for our final analysis is not the one on which the experimental research bear. In fact, we have singled out another criterion which is used to distinguish between models.

Symbolic vs mental image model

The first pedagogical objective of a model for generalised integers is to remove the "ambiguity" of the minus sign. In fact, inside an expression such as (-1)-(+2), the sign - has two different meanings. In defining the new objects (negative integers), one can manage to use the minus sign only with its operation meaning. But this can be done in two ways as we shall see. For symbolic models (as we call them), the symbolism introduced relates only loosely to an initial situation roughly described. For instance, in the coloured number model which we studied, black and red integers are utilised. We shall write nR (resp. nB) for n in red (resp. n in black). In other

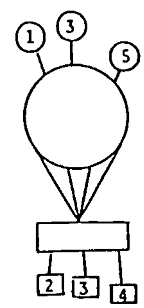
(1) We prefer this name since "directed" presupposes in a way the number line.

models, one finds $2^{\square}$ and $2^{\nabla}$, $\overleftarrow{2}$ and $\overrightarrow{2}$, $\bar{2}$ and $\overset{+}{2}$. One can imagine several similar schemes. Most models in fact involve transitory symbolic artifice. And often, the minus sign is used somewhere around the integer but not in front.

In the coloured number model, a coloured integer is associated with a couple of integers. To $(a,b)$ correspond $(a-b)B$ if $a \geq b$ and $(b-a)R$ if $a \leq b$. Even though couples are assigned situational meaning, the rules of addition are derived from the definition $(a,b)+(c,d)=(a+c,b+d)$ and its colour consequences. As a result, the rules are learnt in terms of colour and have no relation to the initial situation be it a target, a number line or a game played with dice. They are expressed roughly as follows:

1) When you add two numbers of the same colour you make an addition and keep the same colour;
2) When you add two numbers of different colours, you substract the small one from the big one and take the colour of the big one.

Substraction is defined in terms of adding the number of opposite colour. Clearly neither addition nor substraction has a contextual meaning based on mental images or even pictures. That is why such a model was called a symbolic model.




The other model we tested involves a hot-air balloon to which is tied sacks of sand and helium balloons. Those objects are brought regularly by a helicopter. In LUTH (1967), sacks and helium balloon vary in size which determines their effect on the hot-air balloon. A n unit helium balloon raises the dirigible by n meters while a sack of n unit lowers it by n meters.
In the model we experimented all balloons and sacks were of one unit and several were tied at the same time. The symbolism the students developed was b and s for 3 balloons and 2 sacks. Adding means tying an extra element and substracting an element is identified with taking away or removing helium balloons or saks from the dirigible.

The experiment

The teaching involved two groups of pupils aged about 13-14. Group A (76 pupils) were taught generalised integers for 26 lessons with the hot-air balloon model. There were 105 pupils in group B who were initiated to the generalised integers via the coloured number model. Testing involved two more groups C (113 pupils aged 14-15) and D (111 pupils aged 15-16). The experiment was carried out in Casablanca (Morocco) where the national curriculum includes the coloured number model.

The testing included several items of which we shall consider only the arithmetic ones. Test 1 and 2 involving 18 addition and substraction items were written a fortnight after the end of 19 lessons. The 1′ and 2′ included the same items but written with the model symbolims were passed two weeks afterwards. Table 1 at the end presents the results of group A and B.

Results

The arithmetic version

The first hypothesis was: mental image models are better than symbolic models for ensuring a better performance in the arithmetic items. It was as we can see in Table 1 only verified for addition items. The difference in performance is significant at .05 level except for two items. The performance on substraction items is similar for each model. The results of a few items will be examined at the conference.

The model version

For the items expressed with the model symbolism, the results are similar. Group A is significantly better on the addition items for four items out of nine while we observed no difference with the substraction items. The difference of performance with respect to the type of items will be examined at the conference.

Using the model to solve the arithmetic items

Extensive interviewing has shown that for many items group A pupils could use the model to solve arithmetic item while group B pupils could not. The following table giving the overall result for (-6)+(-3) is very illuminating.

GROUP A

| M \ A | S | F | Total |
|---|---|---|---|
| S | 67 | 0 | 67 |
| F | 7 | 1 | 8 |
| Total | 74 | 1 | 75 |

GROUP B

| M \ A | S | F | Total |
|---|---|---|---|
| S | 77 | 2 | 79 |
| F | 25 | 0 | 25 |
| Total | 102 | 2 | 104 |

S: Success F: Failure A: Arithmetic M: Model

Interviews have even shown more. Group A pupils who had made mistakes could correct them by going back to the model while group B pupils would repeat within the model the same mistake.

A few other tables of this kind will be examined at the conference.

## Conclusions

We believe that the balloon model was more efficacious because 1) the integers were related to concrete objects which as part of mental images could be intellectually processed, and 2) generalised integers were associated with objects (sacks or balloons) and at the same time with their consequences: going up or down is attached to the dirigible and the opposite is taken away from it. We are not still sure whether it is useful to introduce the substraction via a model. However, we believe that to get positive results with any such model the substraction should be introduced much earlier in the teaching. We think also that pupils should work within the model (using the symbolism of the model with the accompanying pictures and motion) for a longer time. This is the hypothesis we intend to check in a research project starting soon.

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TABLE 1

LA REUSSITE EN ARITHMETIQUE (AR.) ET DANS LE MODELE (MO.) (PERCENTAGE OF SUCCESS)

| | | Forme ** | Groupe A Ar. | Groupe A Mo. | Groupe A Mo. - Ar. | Groupe B Ar. | Groupe B Mo. | Groupe B Mo. - Ar. |
|---|---|---|---|---|---|---|---|---|
| A | 1 | + + > | 94,7 | 98,7 | 4,0 | 87,6 | 95,2 | 7,6 |
| D | 4 | - - > | 87,5 * | 97,3 | 9,8 | 67,6 | 93,7 | 26,1 |
| D | 5 | - - = | 79,6 * | 96,3 | 16,7 | 60,4 | 90,8 | 30,4 |
| I | 6 | - - < | 88,2 * | 94,7 | 6,5 | 61,9 | 91,3 | 29,4 |
| T | 7 | + - > | 88,8 | 92,0 x | 3,2 | 80,0 | 77,4 | - 2,6 |
| I | 8 | + - = | 90,8 * | 92,0 | 1,2 | 77,1 | 82,7 | 5,6 |
| O | 9 | + - < | 82,9 * | 87,3 x | 4,4 | 66,2 | 72,6 | 6,4 |
| N | 10 | - + > | 92,1 * | 93,3 x | 1,2 | 79,0 | 73,5 | - 5,5 |
| | 12 | - + < | 94,7 * | 91,3 x | - 3,4 | 69,5 | 73,1 | 3,6 |
| S | 17 | + + > | 89,5 | 89,3 | - 0,2 | 87,6 | 81,7 | - 5,9 |
| O | 18 | + + = | 89,5 | 90,7 | 1,2 | 84,8 | 94,2 | 9,4 |
| U | 20 | - - > | 77,2 | 91,3 | 14,1 | 85,7 | 90,4 | 4,7 |
| S | 21 | - - = | 85,5 | 89,3 | 3,8 | 87,6 | 91,3 | 3,7 |
| T | 22 | - - < | 78,9 | 70,7 | - 8,2 | 73,8 | 60,1 | -13,2 |
| R | 23 | + - > | 73,7 | 68,6 | - 5,1 | 73,3 | 71,1 | - 2,2 |
| A | 24 | + - = | 72,4 | 66,7 | - 5,7 | 68,6 | 65,4 | - 3,2 |
| C | 25 | + - < | 70,4 | 72,0 | 1,6 | 71,6 | 65,1 | - 6,5 |
| T | 26 | - + > | 72,4 * | 68,4 | - 4,0 | 60,0 | 67,0 | 7,0 |
| I | 27 | - + = | 63,1 | 66,6 | 3,5 | 51,4 | 60,5 | 9,1 |
| O | 32 | 0 - | 56,6 | 54,7 | - 1,9 | 70,9 | 67,7 | - 3,2 |
| N | | | | | | | | |

* Item in arithmetics with significant difference at 5% between group A and group B.

x Item in the model with significant difference at 5% between group A and group B.

** + - < for instance means that in a + b = ? a > 0 b < 0 and |a| < |b|.

# THE EQUATION-SOLVING ERRORS OF NOVICE AND INTERMEDIATE ALGEBRA STUDENTS

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*The equation-solving errors committed by six novice algebra students during a three-month teaching experiment are compared with the errors of nine intermediate subjects who had completed at least one year of algebra. The error analysis indicates that there was not much difference between the intermediate subjects and older more-experienced solvers, both in equation-solving methods and the mistakes they make. However, the novices' errors and solving methods were not the same as those of the intermediates. Two distinct patterns among novices suggest that we need two different characterizations of the skill of algebraic equation-solving in its early stages of learning.*

The errors which students make while solving algebraic equations can suggest how one might approach characterizing the skill of learning equation-solving. Lewis (1981) has pointed out that there are very few differences between skilled mathematicians and struggling college students in the area of equation-solving. In his study, the experts not only used similar strategies but also made the same mistakes in multiplying out, collecting, and moving terms to the opposite side of an equation. They stated that there was no real need in most situations to be more accurate. This suggests that a ceiling of "adequate expertness" with respect to accuracy and strategy choice occurs at some point in the learning of algebra, and that we ought to be looking at the solving methods and errors of more inexperienced algebra students to examine how the skill of equation-solving develops. Whereas we have acquired a rather large body of information on how college-level students and expert mathematicians attempt to solve equations and on the errors they make (e.g., Carry, Lewis, & Bernard, 1980), much less literature exists on the early learning of algebraic equation-solving. In order to be able to characterize the skill of equation-solving in its early period of learning, it seems useful to examine the errors which beginning students make, how these errors change over a certain period of time, and to compare these errors with those of more-experienced algebra students. This was one of the aims of a study which was carried out with novice and intermediate algebra students.

## The Study

The first phase of the three-phase equation-solving study involved interviews with ten seventh graders (12½ to 13½ years) who had not studied algebra before; this phase was

designed to uncover some of their pre-algebraic notions, in particular those on equations and equation-solving. A subset of this pre-algebra group (six children) was retained for the second phase of the study: a three-month teaching experiment on equation-solving. The solving method emphasized was that of performing the same operation on both sides of the equation. The teaching experiment with these novices included a pretest interview and two posttest interviews, one in June and the second one in September after the summer break. The third phase involved interviews with nine intermediate algebra students who had all taken at least one year of algebra. They were from grades 8 to 11 (6 from grade 8, one each from grades 9 to 11). This last phase was included so as to be able to compare the equation-solving procedures and errors of these students with those of the novices. At previous PME meetings, I have discussed some of the findings of this study: the pre-algebraic notions of the novices as seen in Phase 1; the equation-solving procedures of both the novices and intermediate subjects; and their methods for determining the equivalence of equations. This paper will focus on an analysis of the equation-solving errors of the two groups of subjects.

The data for this error analysis were drawn from the following equation-solving situations. The novices were presented with several sets of equations which they were asked to solve throughout Phase 2: in the pretest, at each session of the teaching experiment, in the two posttests. Each of the 14 equation-sets of the teaching experiment contained 12 equations of the following types (only the numbers were changed in each set): $6\underline{b} = 24$, $2\underline{x} - 6 = 4$, $\underline{y} + 596 = 1282$, $16\underline{x} - 215 = 265$, $\underline{n} + 6 = 18$, $13\underline{x} + 196 = 391$, $4\underline{c} + 3 = 11$, $32\underline{a} = 928$, $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$, $3\underline{a} + 5 + 4\underline{a} = 19$, $2 \times \underline{c} + 5 = 1 \times \underline{c} + 8$, $4\underline{x} + 9 = 7\underline{x}$. The pretest and posttests included equations of the same types, plus some extra ones (pretest: $37 - \underline{b} = 18$, $30 = \underline{x} + 7$; posttests: $\underline{x}/4 + 22 = 182$, $25\underline{y} + 13\underline{y} + 76 = 380$, $12 + 15\underline{a} - 7 + 6\underline{a} = 4\underline{a} + 107$). The intermediate subjects were asked to solve one set of equations during their Phase 3 interview. This was the same set as was presented to the novices in their Phase 2 pretest.

## Results

The equation-solving errors were divided into three categories:

1. The errors which the novices made which the intermediates also made;
2. The errors which the intermediates made which the novices did not make;
3. The errors which the novices made which the intermediates did not make.

Before examining these categories of errors, a few general results are mentioned. First, the three subjects in grades 9 to 11 did not make any errors in solving the equations which were presented to them. Thus, only the errors of the six students from grade 8 are included in the intermediate group. Second, the intermediates committed 17 errors on the 14 equations presented to them; the novices, 47 errors (in the pretest). However, the equations which provoked the errors among the two groups of subjects were not always the same. For example, the intermediates made no mistakes in solving $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$, while the novices committed eleven errors. On the other hand, the intermediates had two errors with $30 = \underline{x} + 7$ while the novices had none. Finally, the number of errors committed by the novices decreased throughout the study. By the time of the June posttest, on the twelve equations which were the same as in the pretest, the number of errors dropped from 44 to 16. But they were not the same as the errors committed by the intermediates. The error rate rose again in the September posttest to 33. This time, some of the novices' errors and the equations in which they committed the errors corresponded more closely to the errors committed in the same equations by the intermediates.

<u>Errors Common to Both Groups</u>

This category includes errors committed by the intermediates in their single interview session and by the novices in their several sessions:

-- Inversing a subtraction with a subtraction or failure to do so when necessary, e.g., solving $16\underline{x} - 215 = 265$ by subtracting 215 from 265 or solving $37 - \underline{b} = 18$ by adding 37 and 18.
-- Giving up when attempting to solve using the substitution procedure.
-- Inversing an addition with an addition, e.g., solving $30 = \underline{x} + 7$ by adding 7 to 30.
-- Computing a coefficient with a non-coefficient, e.g., solving $2 \times \underline{c} + 5 = 1 \times \underline{c} + 8$ by adding 2 with 5 on the left side.
-- Forgetting that concatenation means multiplication, e.g., considering $6\underline{b} = 24$ as $6 + \underline{b} = 24$.

These errors accounted for 10 of the 17 errors committed by the intermediate subjects. The first three of these error-types were committed by the novices during the pretest and also throughout the study; the fourth and fifth were seen from about midway through the teaching experiment. However, only one of these error-types occurred among the novices in the June posttest -- that of giving up before finding the correct solution while using the substitution procedure. All of the above errors recurred in the September posttest.

Errors Committed Only by the Intermediates

-- Leaving the unknown with a negative sign in front of it, e.g., $-\underline{x} = -17$.

-- Changing an addition to a subtraction when transposing, but then commuting the subtraction, e.g., $30 = \underline{x} + 7 \rightarrow 7 - 30 = \underline{x}$.

-- Transposing only the literal part of the term and leaving the coefficient behind, e.g., solving $7 \times \underline{c} = \underline{c} + 8$ by writing $7 - 8 = \underline{c} \div \underline{c}$.

-- Dividing larger by smaller rather than respecting the order for inversing, e.g., $11\underline{x} = 9 \rightarrow \underline{x} = 11/9$.

-- Computational error involving positive and negative numbers.

-- Inversing a one-operation addition equation twice by inversing the addition and then dividing the unknown by the result of the subtraction, e.g., solving $\underline{n} + 6 = 18$ by subtracting 6 from 18 and then attempting to divide $\underline{n}$ by 12.

Two of these errors involved negative numbers, a topic which the novices had not yet learned. They give us an indication of the kinds of errors which the novices had not yet committed but which could develop later. The remaining errors in this category include two transposing errors and two commutativity errors.

Errors Committed Only by the Novices

The most frequently committed errors in this category were:

-- Not using the order of operations convention (26).

-- Not knowing how to start solving a given equation-type (17).

-- Inversing a multi-operation equation before collecting together the multiplicative terms (13).

-- Not using the convention that two occurrences of the same unknown are the same number (9).

-- Giving precedence to an addition when it is preceded by a subtraction (7).

-- Inversing a two-operation equation only once and then using the result of that operation as the solution (7).

The numbers in brackets are the total number of times that these errors were committed throughout the study. Other common errors committed by the novices, which are not mentioned in this category because they were also committed by the intermediate subjects, but to a much lesser extent by the intermediates, were:

-- Giving up while substituting (58).

-- Inversing a subtraction with a subtraction (16).

-- Totaling a coefficient with a non-coefficient (12).

-- Inversing an addition with an addition (9).

## Discussion

Dividing the errors into these categories allowed us to make comparisons with the errors committed by more-experienced students as reported in other studies. The intermediates of this study were not unlike the older more-experienced subjects of Carry, Lewis, and Bernard (1980). For the most part, they all relied on the transposing method of equation-solving, i.e., bringing a term over to the other side and changing sign. Their errors tended to be related to the use of this algebraic solving method. In fact, they seemed to have permanently discarded the use of more arithmetical solving methods, such as, substitution, a method which would have allowed them to avoid several of the errors which they committed.

Though the novices committed some of the same errors as the intermediates, their overall performance particularly towards the beginning of the study was not like that of the intermediates. The novices at the outset of the study tended to fall into two groups. Half of them had a preference for arithmetic methods of equation-solving, such as, using known number facts or substitution; the other half preferred the algebraic method of transposing (Kieran, 1983). The errors which each novice made were clearly related to the type of equation-solving methods s/he preferred to use.

The changes in the novices' error patterns throughout the study were based on the way that their solving preferences were evolving. All subjects became more proficient with transposing one- and two-operation equations, even though this was not the algebraic method being taught in the teaching experiment. With the complex multi-operation equations, those who preferred arithmetic methods such as substitution continued to make the mistake of not being able to find the correct trial value. Those who preferred the algebraic method of transposing continued to make errors such as inversing before collecting similar terms.

Clearly, it was the errors of the novices which provided the most information toward characterizing the skill of equation-solving in its early stages. However, it was not enough to classify the errors into the three categories specified in this paper,

nor was it sufficient to group the errors around equation-types. It was necessary to analyze the errors of each subject in conjunction with the evolution of his/her approach to equation-solving.

In conclusion, acquiring the skill of equation-solving for beginning algebra students requires either adapting their already-existing algebraic method of transposing or replacing a strong preference for arithmetic solving methods with algebraic ones. The existence of these two different orientations among beginning algebra students implies the use of different techniques for making sense of equation-solving, i.e., for learning equation-solving. Thus, there is no one single way to characterize the learning of the skill of equation-solving. It depends on the background preferences of the learner. Thus, the main finding to emerge from the error analysis of this study is that there must be at least two different characterizations of the skill of algebraic equation-solving in its early stages of learning.

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# CHILDREN'S INFORMAL CONCEPTIONS OF INTEGER ARITHMETIC

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The results of tests and personal interviews indicate that some young students are able to perform correctly a number of cases of integer arithmetic prior to formal instruction, using a small number of very simple strategies successfully.

## 1. INTRODUCTION

As part of an ongoing research project on children's understanding of integer arithmetic, written tests were taken by two groups of students in the eighth and ninth grades of 34 schools in the Cape Town metropolitan area. The 993 eighth-grade students received only a brief, 20-minute introduction to the concept of negative number and the subtraction of a larger natural number from a smaller natural number, eg. 3 - 8. The 1331 ninth-grade students, who were from the same schools, had already received formal instruction in integer arithmetic.

Some cases of computation were performed correctly by a fair number of students prior to formal instruction. The success rates for the cases ${}^{-}7 + {}^{-}5$ and ${}^{-}12 - {}^{-}3$ were respectively 61% and 57%. Furthermore, formal instruction seems to have had little effect on the success rate for some of

| Computation | Success Rate(%) Eighth grade (993) | Success Rate(%) Ninth grade (1331) |
|---|---|---|
| ${}^{-}3 \times {}^{-}4$ | 33 | 85 |
| ${}^{-}7 \times 5$ | 45 | 84 |
| $6 \times {}^{-}4$ | 54 | 84 |
| ${}^{-}8 + 3$ | 52 | 78 |
| ${}^{-}4 + 7$ | 48 | 77 |
| $10 + {}^{-}3$ | 51 | 75 |
| ${}^{-}7 + {}^{-}5$ | 61 | 74 |
| $3 - 8$ | 48 | 69 |
| ${}^{-}12 - {}^{-}3$ | 57 | 63 |
| ${}^{-}5 - {}^{-}12$ | 34 | 55 |
| ${}^{-}7 - 4$ | 21 | 50 |
| $8 - {}^{-}3$ | 17 | 46 |

Table 1

High school students' responses to certain computations involving directed numbers, before and after instruction.

the cases. The ninth-graders' success rate for $^-12$ - $^-3$ was only 63% (a gain of 6%), although for other cases, notably for $^-5 \times {}^-4$, the gain was very large. (See Table 1.)

These results showed the need for more information on children's intuitions and informal knowledge of negative numbers and for an explanation why they are able to perform correctly, prior to formal instruction, certain computations involving directed numbers.

## 2. PERSONAL INTERVIEWS WITH YOUNGER STUDENTS

As a first step towards obtaining this information, personal interviews were conducted with 52 nine- to thirteen-year-old students who had been given only an introduction to the concept of negative numbers, illustrated by temperatures below zero on a thermometer.

The results of these interviews support the view that the concept of negative number is probably not alien to the experience of many young students, and that some of these students are able to construct simple and effective strategies (algorithms) for coping with at least some computational cases involving directed numbers. Furthermore, the meanings that students attach spontaneously to unfamiliar symbol sequences like $3 + {}^-5$ and $^-12 - {}^-3$, were also explored with encouraging results.

### The concept of a negative number

There was no evidence that younger students found the concept more difficult than older students. The very small number of students who declared that "you cannot subtract a larger number from a smaller number" were evenly spread over all the age groups. When these were questioned on the meaning they attached to a number like $^-5$, the following explanations were typical of the majority:

Anchen(11:3): "$^-5$ means that you have to get 5 before you have zero."

Steph(11:6): "$^-5$ means it is 5 below zero."

Huibri(10:7): "I've seen it on thermometers; it's less than zero."

### Reactions to the unfamiliar notation

Since the meanings of symbol sequences such as $3 + {}^-5$ and $5 - {}^-8$ were deliberately not explained in order to discover how intuitively acceptable these are to students, some students interpreted a case like 10 + -7 as

10 + 7 - 7, or transformed 5 - ⁻8 into 5 - 8, saying "the extra minus makes no difference". This last transformation may account for the persistent difficulties high school students have with the cases ⁻12 - ⁻3, ⁻5 - ⁻12 and 8 - ⁻3 (ninth grade success rates respectively 63%, 55% and 46%).

In general, however, the notation used, which was unfamiliar to them, raised no questions and was correctly interpreted by the majority of students.

## Students' computation strategies

It must be kept in mind that these students had received no instruction of any form concerning computations with negative numbers, and as they progressed through the different computational cases, they were not told whether their previous answers were correct or not. They were, therefore, operating in surroundings where their only criteria were their own intuitions concerning the mathematical and logical "rightness" of the thought processes they used to obtain an answer. One or two students made remarks like "It looks funny, but it must be so", about some of their answers, implying that they had enough confidence in their reasoning processes to accept an unexpected result.

The strategies employed were characterized by simplicity and by a willingness to reason, extrapolating from known facts about positive number arithmetic. Only four students stuck to the thermometer per se as embodiment; most used a vertical number line where appropriate, and none used a horizontal number line, although a horizontal number line is used at school from the fourth grade onwards. Students were willing to change their strategies to accommodate the different cases, e.g. starting off with a vertical number line to deal with 5 - 8, but solving cases like ⁻5 - ⁻2 or ⁻4 x 5 by extrapolating from known number facts.

The following strategies were employed:

1. A vertical number line.

   Jan(11:9): "⁻4 + ⁻4 + ⁻4 = ⁻12 they are minus therefore you go down.
   7 + ⁻7 = 0 because it is ⁻7 and then you go up 7."

2. Correspondences (analogies) with operations on whole numbers, which included regarding negative numbers as "discrete objects".

   Hennie(10:10): "⁻5 - ⁻2 = ⁻3 because 5 - 2 = 3 and these are minuses."
   Marcel(11:10): "⁻7 - ⁻7 = 0 because it is minus 7 take away minus 7 and that is nothing."
   Manus(11:6): "5 x ⁻4 = ⁻20. Minus 4 is 4 below zero, then 5 times further down. ⁻4 x 5 is the same."

The commutativity of multiplication (as used by Manus) was used intuitively by all the students who coped successfully with a case like $^-4 \times 5$ (80% of the pre-instruction group, which compares excellently with the success rate of 84% achieved by the ninth-graders after formal instruction).

3. Differences between the positive and negative integer operations, by employing logic.

| | |
|---|---|
| Hannes(12:6): | "$^-5 - {}^-8 = 3$ because if $5 - 8 = {}^-3$, $^-5 - {}^-8$ must be $^+3$." |
| Anchen(11:3): | "minus minus is the same as not not: if I am not not ill, I am ill." |
| Ron(12:1): | "$5 + {}^-3$ was subtraction, therefore $5 - {}^-3$ must be addition." |

<u>Misconceptions</u>

It has been mentioned that students apply commutativity to compute a case like $^-4 \times 5$, since $5 \times {}^-4$ is easily dealt with by transforming it to $^-4 + {}^-4 + {}^-4 + {}^-4 + {}^-4$ or to 5 times "four minuses". What has also emerged from the interviews is that students overgeneralise commutativity, in particular in the case of 2 - 5. The explanation goes: "2 - 5 is the same as 5 - 2. Since 5 - 2 = 3, therefore 2 - 5 is also 3". This may be part of the reason for what was regarded as a very puzzling item of information: the ninth-grade success rate for the case 3 - 8 is only 69%. Working at the time on the hypothesis that a person who could not perform 3 - 8 correctly, had no understanding of negative numbers, we could not explain why students who could not do 3 - 8 correctly, dealt successfully with $^-7 + {}^-5$, $^-4 + 7$, etc. The information since obtained suggests that a student may have some kind of understanding of negative numbers, but still fails on 3 - 8 because of a mistaken overgeneralisation of commutativity.

## 3. ERROR PATTERNS OF HIGH SCHOOL STUDENTS

As a check on the reproducibility of the response patterns, written tests were also taken by 846 ninth and tenth graders in four high schools of a large country town. Apart from showing the same pattern of success rates as was found among the earlier group of 1331 ninth graders, these results facilitate an analysis of student errors, because the original answers of the students are available. Although work on this has barely commenced, the following patterns were immediately obvious:

1. Although the success rates of the ninth graders are consistently lower than those of the tenth graders for all cases (in these schools mathe-

matics is optional only from the tenth grade onwards), the ninth and tenth grade distribution patterns for errors are similar.

2. For the four cases that have, in this research project, always proved the most difficult, the incorrect responses show that there is a significantly most dominant error response for each, and that this response has, in some cases, an even higher frequency than the correct answer.

| Computation | Correct response Ninth grade (514) | Correct response Tenth grade (332) | Error | Dominant error Ninth grade (514) | Dominant error Tenth grade (332) |
|---|---|---|---|---|---|
| $^{-}9-{}^{-}4$ | 58% | 63% | -13 | 26% | 30% |
| $^{-}5-{}^{-}9$ | 40% | 48% | -14 | 34% | 34% |
| $^{-}7-3$ | 36% | 47% | -4 | 52% | 45% |
| $7-{}^{-}5$ | 27% | 38% | 2 | 57% | 52% |

Table 2

High school students' responses to computations involving directed numbers, giving the percentages of the correct and dominant incorrect responses.

Different strategies may, of course, result in the same answer. For example, $7 - {}^{-}5$ yields 2 if the "extra minus" is ignored (as explained by the younger students), but it can also be the work of the "compulsive subtracter" who subtracts whenever he sees a "minus" anywhere. Incidentally, compulsive subtraction cannot be considered to be too common among these students, because the percentages of answers obtained by subtraction, or by subtracting and adding a negative sign, are high only for the cases $^{-}7 - 3$ and $7 - {}^{-}5$.

Interviews conducted with a group of 30 above-average ninth graders from a single school reveal that, for this particular group, incorrect responses were mainly due to misapplication of rules supplied by the teacher, and that students experienced difficulty, apart from parroting the rule, in explaining why a certain response was correct. Here are three examples:

Roux: "$5 + {}^{-}2 = {}^{-}3$ because a plus and a minus remain a minus."

Rikus: "$^{-}5 - {}^{-}9 = 14$ because you can change the signs. You cannot subtract $^{-}9$ from $^{-}5$, so the signs change."

Jan: "$^{-}7 - 3 = {}^{-}4$ because the value of the positive number is greater than that of the negative number, but smaller than the number. The answer is therefore a negative number, but less in value."

It was, however, not the main object of these interviews to obtain clarity on specific points, so that the whole problem of the strategies that cause the particularly high frequencies of the dominant incorrect response for a

certain case will have to be resolved by further interviews. If we could succeed in identifying one or two incorrect strategies that give rise to the very high error rate for these extremely difficult cases, suitable preventive and/or remedial teaching might have an enormous impact on student performances in this very troublesome area.

## 4. DISCUSSION

1. The low success rates of high school students for some of the cases, coupled with the extremely high frequency of a particular incorrect response for a particular case, point to deeply-rooted and widely-held misconceptions, and cannot be the result of carelessness. Since not all eighth-grade teachers favour exactly the same teaching strategies, this particular phenomenon can only be accounted for by a common denominator in the existing teaching strategies to which these 846 students had been exposed.

2. The ease with which the young students accepted negative numbers might indicate that the extension of the whole numbers to include the negative integers may need no more, and perhaps less, mental effort than to progress from the whole numbers to rational numbers. An earlier, but very careful introduction to negative numbers, possibly extending only to the addition of signed integers with no rules and/or algorithms mentioned, might therefore have good effects.

3. The amount of teaching that was necessary to establish a basic understanding of the concept of negative numbers was minimal. Although it is in no way advocated that teachers should spend as little time on their introductions where large classes are involved, it is possible that we explain too much and use too great a variety of concrete embodiments, or force an analogy with a concrete embodiment too far to achieve real understanding.

   The spontaneous strategies employed by the younger students suggest that they have a far greater affinity for numbers, number patterns, and simple logic than most of the teaching strategies employed at present seem to give them credit for. A teaching strategy that develops and reinforces students' existing understanding of this topic, allowing them time to think about and discuss carefully selected problems set by their teachers, might be more in tune with the information that has become available.

The teaching approach outlined above is being tried out on all the fourth to seventh-grade students (about 130) in a small country town school. It is, of course, not yet possible to tell how lasting the effects of of this teaching strategy will be, but the immediate results are excellent.

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obtained from this, the student population was stratified and a stratified sample was chosen for these clinical interviews (for a brief description, see Filloy / Rojano [ 4 ]). In this paper, we report observations performed through clinical interviews with children aged 12-13, who had had no previous instruction in algebra, but had showed a high proficiency in pre-algebra.

The school treatment preceeding the study included material which is designed to teach the student solutions to the simplest equations ($x \pm A = B$ , $Ax = B$ , $A \times (x \pm B) = C$), as well as problems arising therefrom (again, see Filloy / Rojano [ 4 ]).

**CLINICAL INTERVIEW.**- It includes five different series of items: 1) E equations series, for diagnosis verification; 2) C cancellation series, to determine the level of "conceptualization" of the equation as an equivalence; 3) Operating the unknown I series, in which we find items such as $Ax \pm B = Cx$ and $Ax \pm B = Cx \pm D$, to obtain insight into students's difficulties when facing this new type of equation . These series largely include an instructional section given by the interviewer, but referring to a concrete geometrical context. The results described in this paper are based on this series 4) A number finding series, to analyze the solving procedures in the different strata 5) Problem inventing P series, in which the students are supplied with equations of the forms presented in A, and are requested to figure out statements of problems that can be solved by applying them.

General Description of I series.

Series I : Operating the unknown. "Non-arithmetical" equations.

| | | | |
|---|---|---|---|
| $x + 2 = 2x$ | $7x + 15 = 8x$ | $3 + 2x = 5x$ | $13x + 20 = x + 164$ |
| $2x + 4 = 4x$ | $38x + 72 = 56x$ | $5x = 2x + 3$ | $10x - 18 = 4x$ |
| $3x + 8 = 7x$ | $37x + 852 = 250x$ | $5x = 3 + 2x$ | $10x - 18 = 4x + 6$ |
| $3x + 8 = 6x$ | $2x + 3 = 5x$ | $7x + 2 = 3x + 6$ | $8x - 10 = 6x - 4$ |

General Description of the Clinical Interview (just I series part).

a) Presentation of the first items.
b) At the moment the student can not approach them by any means, an instruction phase is developed using the following geometric model:
   - Proposed equation: $Ax + B = Cx$ ; A, B, C positive integers, $C > A$.
   - This equation is translated into a situation of the following sort:

Geometric Model

Statement: A man has a plot of land as A in the drawing (see figure 1). Next, he buys an adjacent plot with an area of B square metres.

A second man proposes to exchange this plot for another on the same street having the same area but a better shape (figure 1).

2. The modification of the arithmetical idea of equation which comes about:
   a. By facing different modalities of linear equations, in structures that do not necessarily conform to the "concrete" model provided to operate the unknown.
   b. Through the need not only to give meaning to the literal terms of the algebraic expressions, but to give sense to these new expressions and to the operations which are necessary for their utilization. By way of example, one way of giving them meaning occurs in the verification process, when a new significance is afforded to the algebraic expressions in which equality appears, as those where it is possible to perform a series of operations in such a way that a value for the unknown is obtained and when this is substituted in the left hand side, and the indicated operations are effected and the equivalent is done on the other side, the results coincide .

   Another way of giving sense to the whole of this part of the process comes from the recognition of the strategy of arriving at a new but simpler equation which was easy to solve before the instruction phase was developed.

3. The use of personal graphs (codes) to indicate the already completed actions and those to be realized on the elements of the equation in the solution process. This suggests the existence of a stage previous to the operational algebraic stage. In this previous one, there also occur obstacles which the afore-mentioned graphs impose when the equations become more complex, generating what afterwards, in a later study of algebra, are considered "usual syntactic errors": i.e. inadequate use of the equal signs, absence of these, loss of certain terms, etc.

   Graphs and their codes can be classified according to the tendency to reproduce situations in a more concrete language level or if they try to indicate the operations involved in the process. All of this is ignored by the usual teaching strategies of the subject.

4. The 'rooting' to the model, even in cases of complex representation, which lies beneath an apparent algebraic operativity on the elements of the equations. This is detected as the development of a capacity to adapt the use of the model to more complex equation modes ($Ax - B = Cx + D$, for instance). This happens to such an extent that the success attained by such process leads the student to retain in the context of the model, his or her interpretations and actions on the elements of the equation. The abstraction process of such actions, however, shows no preference towards the domain of algebra.

5. The detachment from the model, transferring the operativity on the coefficients to the operativity of the terms including unknowns, leading to the "usual error" of adding monomials of one degree to those of zero degree (the constants, the coefficients) i.e , the defective operation of the unknown.

   This was found in both the case of uses of the codes to represent situations and also in the presence of children trying to represent the operations. This seems to be one of the weakness of all the leading strategies which intend to use a more concrete model language in order to introduce more abstract objects and operations within them.

6. The presence of impediments, characteristics to each model, but which also preserve general behaviour patterns and that lead to awareness of the difficulties of the subject arising not only from the teaching method. Here it can be mentioned the need of a process of differentiation of each of the different situations that different equations propose, and the generalization needed to identify two situations that were found different before.

They need to find a new level of abstraction required for the last identification and the assimilation of the old objects and operations to the new abstract situations and operations, and also they need to construct new graphs and codes to represent all of it.

7. The recognition, through the model, of the diversity of first degree equations types, whose solution requires finally the same operations which nevertheless do not transfer from the simplest to the most complex. That is to say, once the use of the model for a certain mode of equation has been operationalized, transference to other, more complex modes, is not effected at the same operative level. In the best of cases, an attempt is made to use the model, by retaking –at an analytic level — the problem situation arising from the new equation type. This return to the analytic level is what marks recognition by the student of new situations ; and he or she is able to develop, in the courses of the interview, a capacity to discriminate among the various forms of the first, "non-arithmetical" equations of the first degree ($Ax + B = Cx$ ; $Ax + B = Cx + D$ ; $Ax + B = Cx - D$, etc.).

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# Visualizing Rectangular Solids Made of Small Cubes: Analyzing and Effecting Students' Performance

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ABSTRACT. This study presents evidence which suggests that students in grades 5 through 8 have difficulty relating isometric type drawings to the rectangular solids they represent. Items of the sort "How many cubes does it take to build a given (pictorially presented) rectangular solid" were used. The errors made by students were analyzed and the effect of instruction in spatial visualization activities on the performance of middle school students was assessed. The findings were examined relative to their psychological aspects and practical teaching implications.

BACKGROUND. Although we live in a three-dimensional world, most of the visual mathematical materials presented to children are two-dimensional. One of the first areas of mathematics in which children are forced to "read" and visualize information about objects from pictures is in the study of measurement, particularly, in the study of volume. As early as grade 3, drawings of rectangular solids, with lines suggesting unit cubes within the drawing, are used in textbooks to introduce children to the concept of volume. These drawings are sometimes isometric and sometimes turned almost horizontally as in Figure 1.

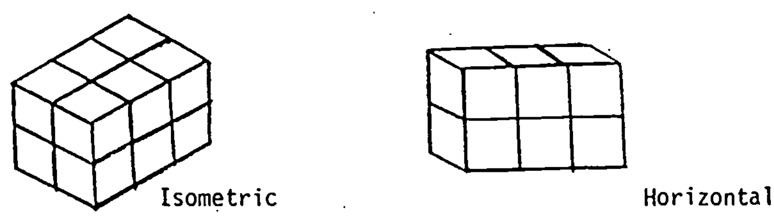



Figure 1: Types of figures used in textbooks to represent rectangular solids.

In addition to such representation of rectangular solids being used in textbooks, they are also included in many large-scale "educational assessment studies which purport to measure students' knowledge of the concept of volume (NAEP, 1979; MEAP, 1983; Hart, 1981). This is usually done by presenting the stimulus as in Figure 1 and the student is asked, "How many unit cubes does it take to make this solid?". The results from these assessment studies reveal that there are grade level as well as sex differences in the performance on such items. They also indicate that less than fifty percent of the students in the middle grades are able to correctly answer this type of item. The study reported in this paper will provide evidence to suggest that children have difficulty relating such pictures to solid objects and that in addition to directly teaching students to "read" such representations,

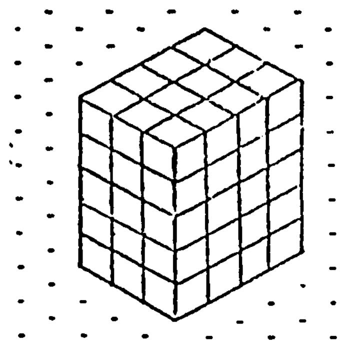
instruction may need to be complemented by introducing concrete experiences with solid objects even in the middle school.

RELATED STUDIES. The skill to "read" two-dimensional representation of solid objects is a part of the spatial visualization ability. Among the studies reported in the literature, the question of whether students can benefit from training is raised. In addition, the widespread assumption that male performance on spatial visualization superior to female performance (Maccoby & Jacklin, 1974; Sherman, 1980; Harris, 1981) has raised questions regarding the interaction of sex and instruction in affecting spatial ability (Eliot & Fralley, 1976; Maccoby & Jacklin, 1974; Harris, 1981). After a review of related studies on training in spatial visualization for students in the different levels of school and college, the conclusion is that the results are inconclusive and the field is still open for further research. In support, Bruner (1973) concludes "I don't think we have begun to scratch the surface of training in visualization".

PURPOSE OF THE STUDY. During a pilot study for developing spatial visualization activities appropriate for middle school students, difficulties students have in visualizing hidden parts of pictorially presented objects were encountered. In order to evaluate the effectiveness of the instructional activities, a spatial visualization test was created. Items of the sort "How many cubes does it take to build a given (pictorially presented) rectangular solid?" were included in the instrument. There were two interrelated purposes for including these items. The first was to determine whether students' performance would be affected by instruction in spatial visualization activities, and if so, whether the effect would vary by grade and by sex. The second was to study strategies used by students in attempting to answer this type of item.

CONSTRUCTION OF THE ITEM FORMAT. In order to develop a multiple choice format, the items were introduced in open-ended form. the students' responses were analyzed to produce categories of errors. Following the analysis of errors the students were interviewed to check the validity of the analysis. It was discovered that most students who missed the items were using one of the following four counting strategies: 1. Counting the actual number of faces showing. 2. Counting the actual number of faces showing and doubling that number. 3. Counting the actual number of cubes showing, and 4. Counting the actual number of cubes showing and doubling that number. These four strategies were chosen to construct the distractors for the two items included in the 32-item spatial visualization test. The two items were numbered 10 and 12. The actual text of item 12 is shown in Figure 2. Item 10 was similar with dimensions 2x3x4 and the options given were: 18, 24, 26, 36 and 52.

How many cubes are needed to build this rectangular solid?



| A | B | C | D | E |
|---|---|---|---|---|
| 36 | 47 | 60 | 72 | 94 |

Figure 2: The text of Item 12.

THE METHOD OF THE STUDY. In order to include students in grades 5 through 8 from a broad range of socio-economic status (SES), three sites, in and around a major midwestern city in the United States, were selected for the study. Site I can be described as urban, ranked the lowest on the SES, site II as rural and site III as suburban (ranked the highest on the SES). About 90% of the student body from the three sites, approximately 1000 students, participated in the study. The students were tested before and after three weeks of instruction in spatial visualization activities. The unit of instruction engages students in creating "buildings" from small cubes and drawing representations of these "buildings" in two ways, flat front or side views and isometric corner views. None of the activities, posed to the students during the instruction, specifically involved counting or evaluating the number of cubes needed to build rectangular solids.

RESULTS. Table I presents the percentages of students who correctly responded to each item and who got both items 10 & 12 correct by site, by grade and by time (pretest, posttest). Table II presents the percentages of boys and girls who got both items 10 & 12 correct by time, by site and by grade. Table III presents the percentages of students selecting the various options for item 12 on the pretest and posttest for each site and grade. Similar patterns were observed for item 10.

DISCUSSION AND CONCLUSIONS. Evidence obtained from responses to the items prior to the instruction indicate grade level differences (increasing with age), sex differences (favoring boys) and site differences (increasing with SES). The level of performance ranged from 25% to about 50%. After the instructional intervention the results show that middle school students, grades 5 through 8, regardless of site and grade level, significantly improved their performance. The improvement is judged by the percent of increase in the number of students who got both items correct. Those students could be said, in some sense to have mastered this type of items. In addition, in spite of the sex differences, boys and girls did not respond

Table I: Percentages of students correctly responded to each and both items 10 & 12 by time by site by grade

| Site | Grade | N | Item #10 Pretest | Item #10 Posttest | Item #12 Pretest | Item #12 Posttest | Items 10&12 Pretest | Items 10&12 Posttest |
|---|---|---|---|---|---|---|---|---|
| I | 5 | 102 | 29.4 | 41.2 | 26.5 | 31.4 | 20.6 | 26.3 |
| | 6 | 95 | 33.7 | 42.1 | 25.3 | 35.8 | 21.7 | 31.9* |
| II | 6 | 108 | 43.5 | 48.1 | 38.9 | 39.8 | 35.2 | 36.1 |
| | 7 | 159 | 51.6 | 69.8 | 44.0 | 62.9 | 40.5 | 60.1** |
| | 8 | 180 | 56.1 | 71.1 | 52.2 | 65.0 | 48.3 | 61.7** |
| III | 6 | 264 | 46.6 | 58.3 | 44.7 | 55.7 | 36.6 | 47.5** |
| | 7 | 70 | 77.1 | 80.0 | 68.6 | 84.3 | 67.1 | 78.6 |

*The significance level of the McNemar test was $p < .05$.

**The significance level of the McNemar test was $p < .001$.

Table II: Percentages of students correctly responded to both items 10 & 12 by sex by time by site by grade

| Site | Grade | Boys N | Boys Pretest | Boys Posttest | Girls N | Girls Pretest | Girls Posttest |
|---|---|---|---|---|---|---|---|
| I | 5 | 54 | 22.4 | 28.8 | 48 | 18.8 | 23.4 |
| | 6 | 46 | 27.3 | 34.8 | 49 | 16.7 | 29.2 |
| II | 6 | 54 | 46.3 | 40.7 | 54 | 24.1 | 31.5 |
| | 7 | 84 | 40.5 | 66.7 | 75 | 40.5 | 52.7 |
| | 8 | 79 | 57.0 | 75.9 | 101 | 41.6 | 50.5 |
| III | 6 | 145 | 43.4 | 50.0 | 119 | 28.6 | 44.5 |
| | 7 | 39 | 66.7 | 82.1 | 31 | 67.7 | 74.2 |

Table III: Percentages of students selecting the options to item 12 on the pretest and posttest by site by grade

| | | Site I Grade | | Site II Grade | | | Site III Grade | |
|---|---|---|---|---|---|---|---|---|
| | Options | 5 | 6 | 6 | 7 | 8 | 6 | 7 |
| Pretest item #12 | A | 13.7 | 11.6 | 7.4 | 8.8 | 4.4 | 10.2 | 2.9 |
| | B | 28.4 | 23.2 | 14.8 | 15.7 | 13.9 | 22.0 | 10.0 |
| | C* | 26.5 | 25.3 | 38.9 | 44.0 | 52.2 | 44.7 | 68.6 |
| | D | 4.9 | 6.3 | 13.9 | 6.3 | 7.2 | 6.1 | 10.0 |
| | E | 21.6 | 31.6 | 25.0 | 24.5 | 22.2 | 16.7 | 8.6 |
| Posttest item #12 | A | 25.5 | 14.7 | 14.8 | 14.5 | 9.4 | 15.9 | 2.9 |
| | B | 19.6 | 24.2 | 13.9 | 10.1 | 11.7 | 14.8 | 7.1 |
| | C* | 31.4 | 35.8 | 39.8 | 62.9 | 65.0 | 55.7 | 84.3 |
| | D | 6.9 | 5.3 | 11.1 | 3.8 | 3.3 | 3.4 | 0.0 |
| | E | 13.7 | 20.0 | 20.4 | 8.2 | 10.6 | 10.2 | 5.7 |

* Correct Answer

differentially to the instruction. This lends support to the conjecture that part of the differences in performance between boys and girls is the result of different environmental influences such as toys. Given an opportunity to make concrete representations with cubes, girls (and boys) do improve in their ability to accurately predict the number of cubes in a rectangular solid pictured isometrically. This strongly suggests that such concrete experience should be provided for children in the middle grades.

The data also clearly indicate that students who missed the items were employing incorrect counting strategies, counting faces or counting visible small cubes. Counting the faces is dealing with the picture strictly as a two-dimensional picture, while counting the visible small cubes is indicating an awareness of the three-dimensionality suggested by the picture. Another contrast is between those students who count faces or cubes and those who count and double the number of faces or the number of cubes. Those who do not double seem not to visualize the hidden portion of the figure. While on the pretest the most frequently chosen distractors were those related to strategies 1 and 2 (dealing with faces), on the posttest there was a trend toward strategy 3 (dealing with visible small cubes) and away from strategies 1 and 2. These patterns are generally observable for each grade within a site.

SUMMARY AND IMPLICATIONS. The evidence from the present study strongly suggests that students in grades 5 through 8 have difficulty relating isometric type drawings to the rectangular solids they represent. However, after three weeks of instruction in spatial visualization activities, students' performance improved significantly on the two items at all grade levels in all sites. This suggests that concrete experiences with cubes -- building, representing in 2-dimensional drawings, and reading such drawings -- are helpful in improving students' performance. The item format, based on the analysis of errors, can also be used to construct items to evaluate performance and diagnose errors in classroom situations or other assessment programs.

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# THE TEACHING OF REFLECTION IN FRANCE AND IN JAPAN

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In the world, France and Japan comprise two cultural poles of considerable impact. Their respective concepts of space and communication are treated in very different ways : these concepts contribute to the organization of the individuals'system of perception. The teaching of geometry constitutes a priviledged area for the observation of mental structures through the cognitive mode of understanding space, together with the linguistic uses and the carrying out of symbolic representations. The idea of reflection is one which is taught as well as being the product of the socio-cultural environment : we have chosen it as the matter of our study. Observations of various construction tasks have been carried out in parallel with French and Japanese students. They show that the errors made by both groups are basically the same but it appears that the elaboration of the notion is faster for Japanese students : cultural variables undoubtedly account for some of this.

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In the world, France and Japan comprise two cultural poles whose influence has had considerable impact over the centuries. Two kinds of societies have evolved : one accents autonomy, personal responsability, the other emphasizes confidence and cooperation; one is a "société de création" (creative society) and the other a "société de communication" (communicative society). (Gaudin, 1984)

The concept of space, the architecture, but also the language and the signs are handled in very different ways.

A quite extensive study of the concept of space and the architecture in France and in Japan carried out by Augustin Berque shows that French architecture prefers geometrical orders and perspectives whereas Japanese architecture shuns them, most notably symmetry. It appears that the tendance toward non-symmetry has evolved parallel to the development of the Japanese civilization. Western geometric urbanism is the extension of an "espace conçu" (conceived space) which has not developed naturally in Japan where "l'espace est vécu" (space is lived) : the rambling lay out is not without order, rather it promotes a progressive discovery and not a planned design.

In French, all stages in an argument are laid out and a single alphabet is used. In Japanese, the extent to which one expresses explicitly is greatly reduced, but the wealth of its signs used in writing is without comparison; two symbol-alphabets are used, each with 52 signs (coming from the simplification of Chinese characters) : Hiragana and Katakana (for foreign words) as well as about 7 000 Chinese characters (Kanji)* without forgetting the Latin alphabet (which is used in the context of mathematics, for example).

* 1 945 signs are taught during elementary and secondary school. About 3 000 are needed to read a newspaper.

Disciplines such as educational anthropology and crosscultural psychology have been developed in the last twenty years, especially in English speaking countries (Mauviel, 1984); they are beginning to take root in France, which already has a considerable background in this field.

Concerning learning, as Maurice Mauviel emphasizes, the way in which culture is acquired demands the greatest attention in the study of intercultural phenomena. The individual either semi-consciously or unconsciously draws on the symbolic h·ritage of his group (language, values system,bodily gestures...) which he transforms, and it is this process which contributes to the organization of the individuals' system of perception.

How do these different cultural contents act in school, in particular in the learning of mathematics and geometry ?

The teaching of geometry constitutes a priviledged area for the observation of mental structures through the cognitive mode of understanding space, together with the linguistic uses and the carrying out of symbolic representations.

The idea of reflection is one which is taught as well as being the product of the socio-cultural environment. We have chosen it as the matter of our study.

This idea appears at different stages in the French and Japanese programs.

French middle school* programs afford much flexibility to the teacher in the organization of his course; the third year of middle school emphasizes the learning of mathematical proofs. In plane geometry, translation, point symmetry, reflection and the projection on a line are taught as point transformations; the approachs vary greatly according to the textbook used.

Japanese middle school programs offer more manipulative tasks, construction exercises and measurements of plane and spatial figures before teaching the properties of these figures; few proofs are done. The textbooks are relatively uniform and the teachers follow them very closely. Axial symmetry is taught in the last year of elementary school starting from the idea of an axis of symmetry of a figure, and point symmetry is taught soon after.

Studies of the notion of transformation and in particular of reflection have already been carried out in several countries, in England by Kathleen Hart, in France by Régis Gras and Denise Grenier, in Germany by Veit Georg Schmidt; in these three countries, the students'conceptions seem to be quite similar.

We based our first comparison between French and Japanese students on the work of Denise Grenier and Michel Guillerault (Equipe de didactique des mathématiques et de l'informatique de Grenoble). They have attempted to determine what ideas about reflection a pupil of 11-15 years has before and after being taught in class, by observing the student in various construction tasks.

* In France, elementary school pupils are aged 6-11; in Japan they are 6-12. In France, middle school pupils are 11-15; in Japan 12-15.

For a start, they had groups of two students trace the reflected figure of given rectilinear figures relative to a straight line (cf. Denise Grenier). This activity has been proposed to pupils of 13-15 years in the third and fourth years of middle school in Grenoble before and after teaching.

The results have pointed out a certain number of procedures; it was interesting to test this activity in the Japanese school in Paris which follows Japanese programs. The tests were carried out with 13 year old pupils in the second year of middle school (who have already learned axial symmetry two years before). They showed that the Japanese pupils had already overcome the problems that the French students encountered : the agreement of the two students was immediate and the answer correct. In particular, the Japanese naturally exhibited behavior which was absent in the French pupils : when the axis is "oblique", the Japanese student rotates the paper so as to obtain a "vertical" axis.

These differences appeared important enough to motivate three series of observations carried out in parallel with the team in Grenoble, two in the Japanese school of Paris and the third in Japan* (analysis underway).

The first series of observations took place in the Japanese school in Paris in three classes, before the teaching of axial symmetry for one of them, after for the other two. The task proposed was a construction task similar to the previous one, but done individually. The results (Denys, 1984) were compared to those of French students aged 11-15. At the same time, etymological comparisons concerning the characters and terms used for teaching reflection in both cases have been begun.

The performance of 12 year old students of the Japanese school in Paris is comparable to that of 14 year old French middle school pupils. But one must note that the given tasks resemble ordinary Japanese classwork more than that of French schools. The same kind of mistakes, generally speaking, have been noticed on both sides at different ages. However, the Japanese student tends to want to complete the task by giving it an axis of symmetry. This is likely due to previous teaching (see above) and does not occur with French students. Two observations on the work of Japanese pupils who have not been previously exposed to teaching of axial symmetry :

- As early as 11 years old, the best results are obtained when the figures contains a "vertical" axis and a segment not intersecting the axis.
- When the axis is "oblique" and the segment crosses the axis, the rate of success is half of the previous.

The second series of observations took place in July 1984 in the Japanese school in Paris following further experimentation in Grenoble. Denise Grenier and Michel Guillerault gave a task to be performed individually to students aged 11-15 : the construction of the reflection of a segment relative to a line while varying the

* The first two experiments were performed at the Japanese school in Paris which is not necessarily representative of schools implanted in Japan.

orientation of the axis on the page, the position of the segment relative to the axis (above, below, right, left), the angle between the segment and the axis, the existence or absence of intersection between both, the kind of paper (blank or graph).

Among these figures, we have chosen twelve (adjoining page) to compare, taking into account the first observations, the role of these variables in the work of French and Japanese students. All the axes were drawn "oblique at 45°" and descending from left to right in six cases, from right to left in the other six figures. Half of the figures were on blank paper, the other half on graph paper.

A booklet containing all twelve figures was distributed to 11-14 year old pupils in the Japanese school in Paris (last year of elementary school and first two years of middle school). Three preliminary observations can be reported :

- On the whole, Japanese students have significantly more success than French pupils.
- The success rate improves consistently from year to year in the Japanese school where success is almost total by the second year of middle school (14 years old); on the other hand, the French students show no significant improvement between the first and last years of middle school (11 and 15 years respectively).
- On the whole, the success rate on graph paper is clearly below that on blank paper for all students alike (an exception for the French is figure 8). However, there has been no comparison of the same figure on blank and on graph paper.

Figures 1,4 and 5 had high success rates for both French and Japanese students. Apparently, the students have few problems when one of the angles between the axis and the segment is less than 45°, but not 0°.

For figures 2,3,7 and 10, the Japanese have a 10-20 % lead over French pupils; for figures 2,7 and 10, the mistakes are the same (the most frequent error is to draw a line parallel to the given line); on the contrary, for figure 3, nearly all the Japanese mistakes consist in extending the given line whereas the French mistakes are sometimes extensions and sometimes incorrect slopes. Apparently, the Japanese students, in extending, are trying to obtain a figure containing an axis of symmetry (see above the remarks about programs).

Figures 6 and 8 show the biggest differences in favor of the Japanese; for figure 8, which has a horizontal segment, the incorrect slope is much more frequent among the French than among the Japanese, and the mistake of tracing the given segment as its own image is sometimes seen among the French pupils but never among the Japanese. In figure 6, the most problematic for the French, the mistakes are the same on both sides but the difference between respective global scores is largely attributed to the nearly complete success of Japanese students in second year of middle school. For these two figures, the segment crosses the axis and none of the angles between the axis and the segment is less than 45° : it seems that the conjunction of these two factors throw the student off.

Figure 9 is the only one with a slightly higher success rate among the French than among the Japanese (with average success); here, the error of tracing the given line as its own reflection is less for the French student (see the above comments on figure 8); the Japanese pupil seeks once again an axis of symmetry as in figure 3, but not reduced to the axis.

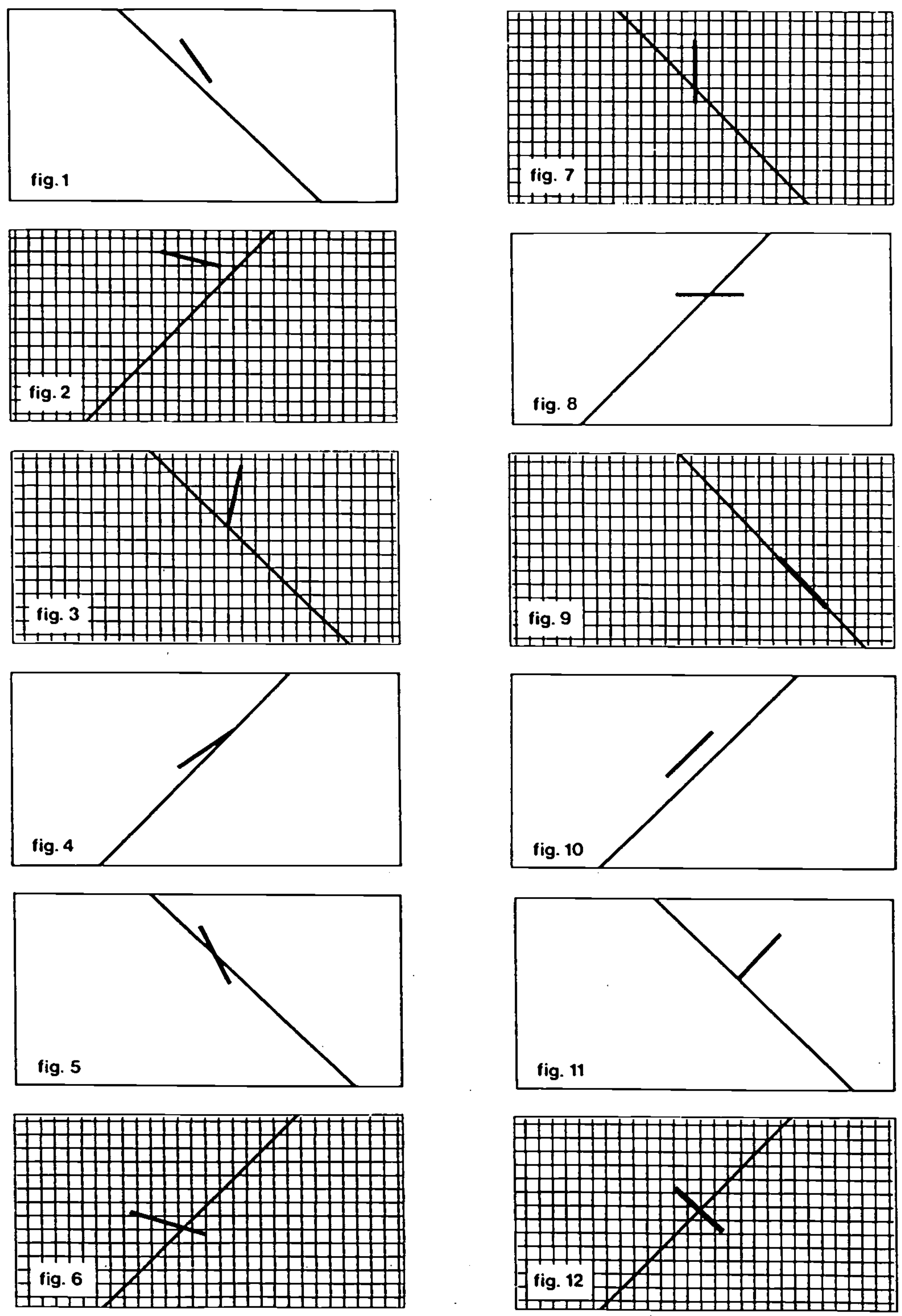

Figure 12 produced mediocre results on both sides; Japanese students undoubtedly had the same problems as in figure 9; as for the French pupils, the interpretation of their growing difficulties is not easy.

As for figure 11, it is hard to say if the good rate of success on both sides is real or due to an error of extension.

We have thus been led on to a third series of observations in which were systematically changed the following parameters : the position of the segment relative to the axis (above, below, right, left), one of the angles of the segment with the axis (30°, 45°, 60°), the existence or absence of a point of intersection between the axis and the segment, the right-to-left or left-to-right slope of the 45° axis; all the figures were done on blank and graph paper alike.

This third experimentation has been performed in Japan thanks to the active participation of Izumi NISHITANI, Hirokazu OKAMORI, Noboru OTAKE, Mitsuya YAMAUCHI and Hajime YOSHIDA.

The analysis underway will allow the clarification of the preceding observations. The results will be presented during the oral report.

These observations permit to extract a certain number of variables from the task and to test their influence on the performances and the kinds of errors made by Japanese and French students.

The nature of the errors is the same for both sides but it appears that the genesis of the notion of symmetry is more rapid with Japanese students and that they are less affected by variations of parameters. The kind of teaching in itself is probably one of the reasons for this phenomenon, but one may surmise that cultural variables also play a role; activities such as origami (paper folding) and calligraphy possibly have a non-negligible function : these frontiers remain to be explored.

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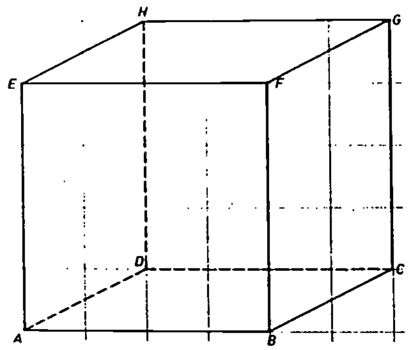
*Space Geometry Doesn't Fit in the Book* (A.Goddijn and M.Kindt)

We can recognize the section on space geometry in many textbooks by stereotyped figures such as in illustration 1.

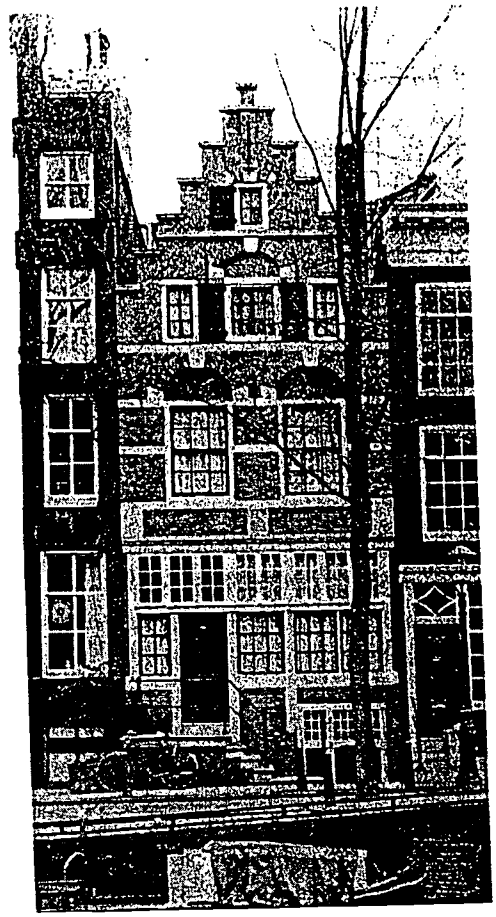



ill.1.

The most important object would appear to be the cube, and the most significant manner of reproduction the parallel projection, with the projecting lines invariably running alongside the readers right ear. If anything at all is done with central projection or perspective, it is mostly a question of showing how to draw a cube with the aid of two vanishing-points.

In illustration 1 as well as in the vanishing-point method, nothing is done off of the paper surface; everything is reduced to a drawing algorithm that limits itself to two dimensions. In the student material developed by the former IOWO and the present OW & OC (Research-group on math education of the University of Utrecht, the Netherlands) an attempt is made to give space geometry the place it deserves: in the three-demensional world around us, and not just in the flat world of the textbook.

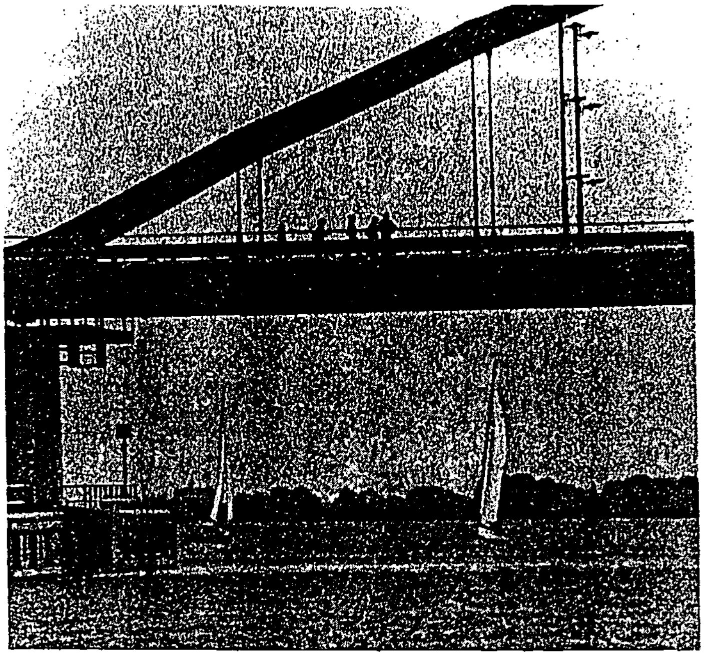
A number of aspects of this style of doing geometry will be presented here by means of a series of assignments. We start with an example from "Klein en Groot"(Large and Small), a textbook for 13 year olds. Important subjects are proportion and scale comprehension.



ill.2.

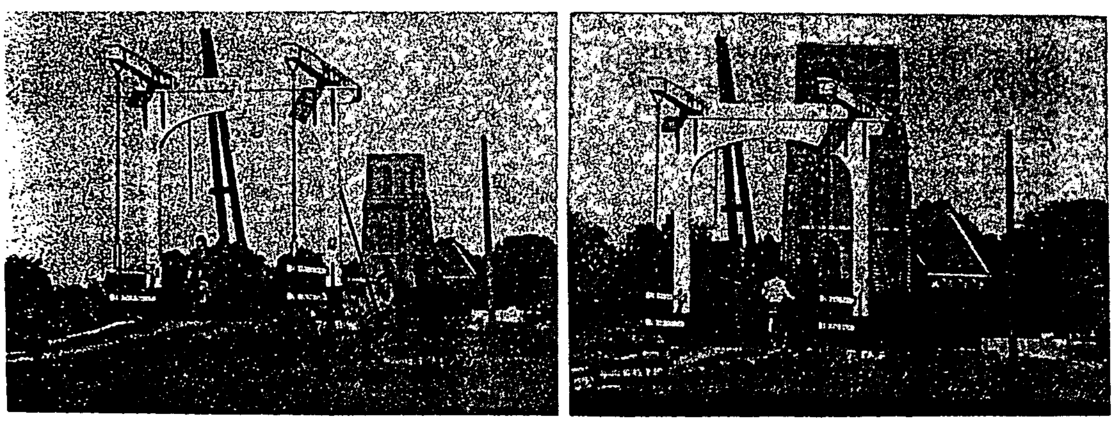
Various things must be measured against the canal house seen in the photo in illustration 2. Students use the bicycle as a unit and, by measuring it against the front of the building, arrive without too much effort at a height of approximately 17 meters for the entire house.

An important aspect of this example is that no rules are offered for calculating the proportion. The students find support for their way of finding the result in elements offered by the context. This is continually the case in the following examples. Critical reflection on the method followed is elicited by a question concerning measuring the clearance for ships under the Schellingwouder bridge in Amsterdam.



ill.3.

This is calculated by using the people seen on the bridge.
The question is whether the sailboat can pass under the bridge. The correct answer at this level is: you can't tell. This discussions with the students generally lead to this conclusion, although the situation makes them insecure: They are used to questions requiring one safe answer.
A much more extreme example of the mathematical problem concerned here is offered to some classes in another form.
Two photos are shown and the question is what is exactly the relation between the tower and the drawbridge. The students are asked te explain their answer with the aid of a drawing.



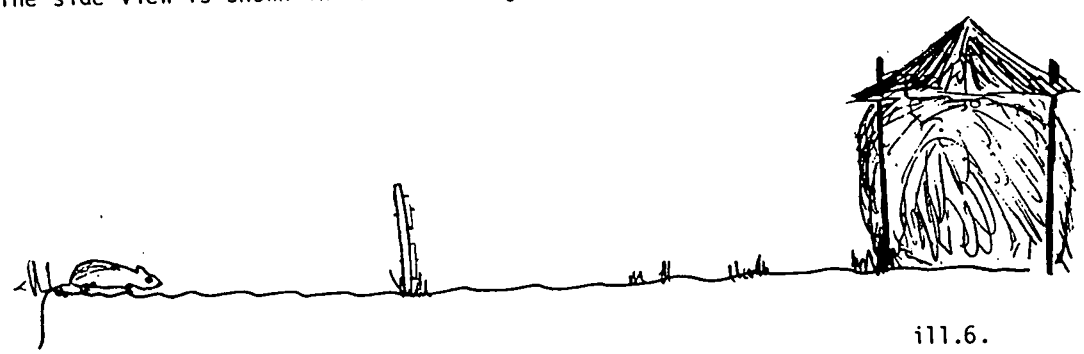
ill.4.

In illustration 5 we see some students involved with an explanatory drawing. In fact, this reproduces the photos; they didn't come up with a side-view.

ill.5.

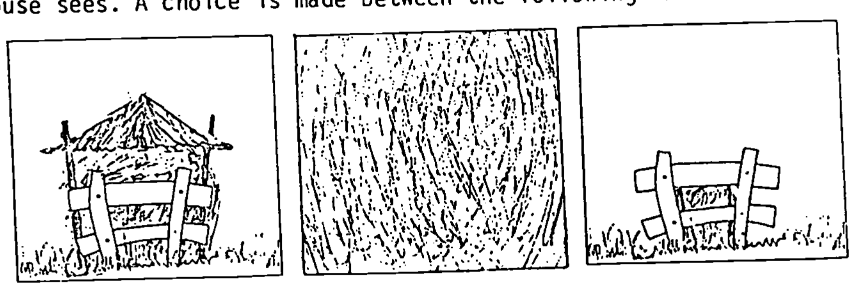
A number of attempts were made to give insight into this situation.

The side-view is shown in the following example.



ill.6.

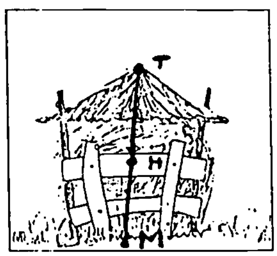
Through a hole in the fence the mouse can just see the top of the haystack. The students are asked to point out exactly where that hole is. Then they must show *what* the mouse sees. A choice is made between the following three sketches.



I II III

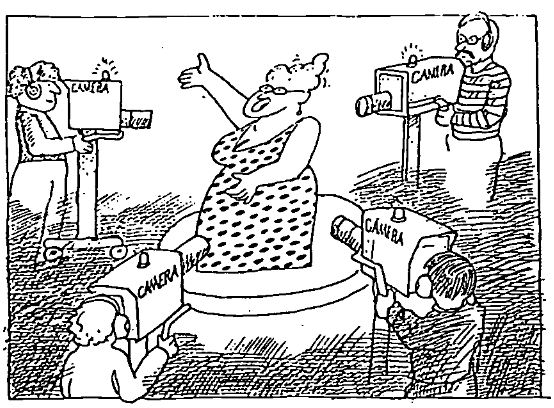
ill.9.

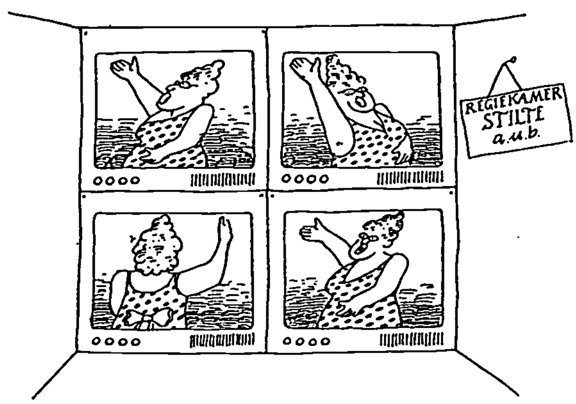
In the adjacent photo we see Eric (14) drawing the sight-line from the mouse's eye to the top of the haystack. Now it is certain to go well! But Eric chooses picture no.I. Why? 'Well, the mouse is sitting ther (M), looking through the hole (H) at the top (T).'




ill.10.

Placing the observer (the mouse) on the image being regarded is often done by the students, in photos as well!
A completely different assignment illustrates which 'mental activities are significant in the above situations.

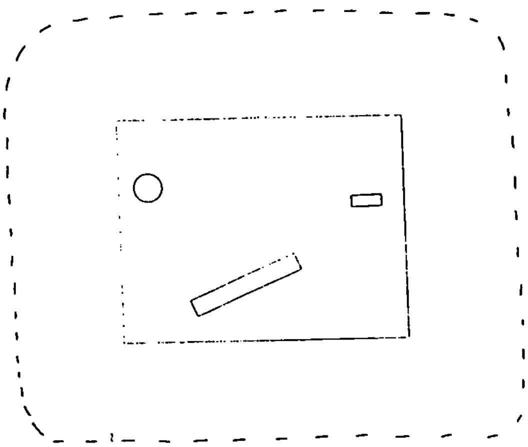


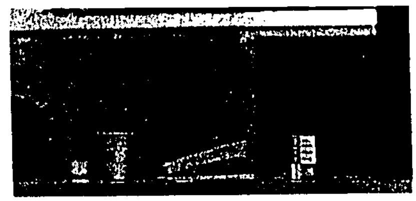
You must move yourself in your mind around the object being observed (in this case the singer). The four cameras invite this mental movement. Without error the correct monitor image is now paired with each camera.

Another example from the workbook 'Zie je wel' (So You See).

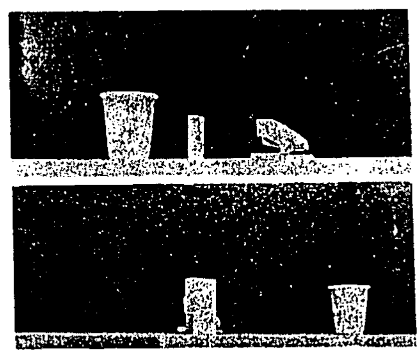
Seen from above.



Walk (in your mind) around the table in the drawing.



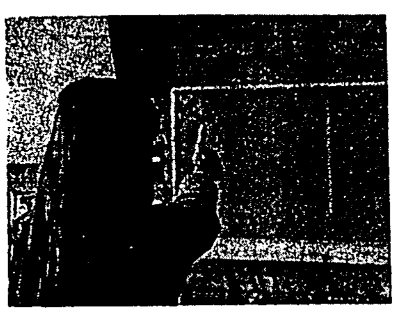
Where are you standing and in which direction are you looking when you see what the photo shows? Indicate that with an arrow in the adjacent drawing. Do this with the following photos as well.



ill.11.

For many students and adults as well it is, however, necessary to actually walk around the table with the three objects.

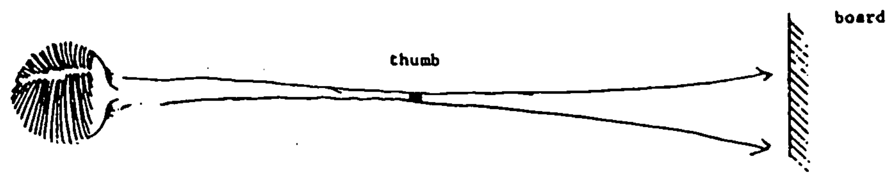
An experience familiar to everyone is investigated in the following assignment, also from 'Zie je wel'.



Assignment 2 (for 3 or 4 students)
Look with one eye at your tumb and stand so that you see your stumb against the line on the board. Hold your hand still. Now look with your other eye. What do you notice? Can you explain it?
Put lines on the board where everyone sees the thumb with the other eye. Each student stands at the same spot (not too close to the board). Why is it that the lines are in different places? Try to explain this with a drawing.

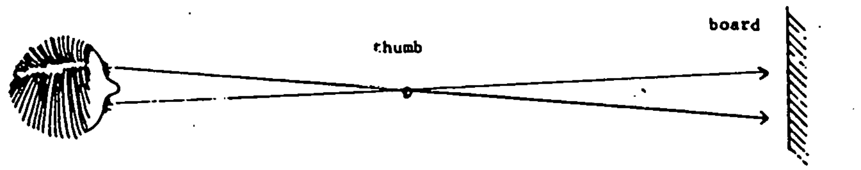
ill.12.

In a class discussion a top-view is used to explain the phenomenon.
In this way you can step out of the situation and observe everything, as it were, from the viewpoint of a fly on the ceiling. Another difficulty arises in reproducing the sight-lines from the eye to the board via the thumb. It is not clear without further preface that the lines have to do with the *rectilinear light path*. Some students appear to have this in mind:



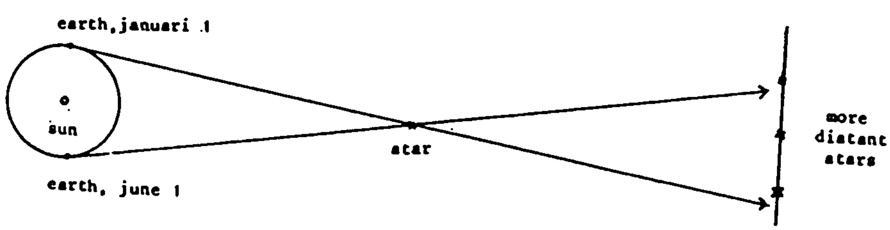

ill.13.

In order to elucidate the fact that insight into situation with the aid of the figure




ill.14.

is certainly not of a trivial nature, here is an application of the same figure for explaining the apparent parallax of stars, which makes it possible to determine the distance of stars from the earth.




ill.15.

Working with straight lines for the light path is dealt with explicitly in 'Shadow and Depth'.
The rectilinear path of the light is first investigated, then drawn on paper:

CATCHING SHADOWS

Now you can try things out yourself with light and shadow. It would be nice if the sun came out, but if it won't then you can make do with a bright lamp which casts nice sharp shadows.

▸ 17. Tape a small paper circle onto a window pane.
About so big:

If the sun isn't shining, then someone should hold the paper circle a couple of meters from the lamp.

Catch the shadow of the circle in the palm of your hand, close to the window. Lay the shadow very carefully on the floor; be careful that the shadow doesn't roll off your hand!
What kind of figure did your hand make as it moved through the air?
▸ 18. Make a drawing here which shows how your hand moved.
Draw the sun (or lamp) and the paper circle as well.

ill.16.

At the end of the section on shadow, the phases of the moon are observed. Among other things, a ball is illuminated from the side and moved in a circle around the students head. See photo.

ill.17.

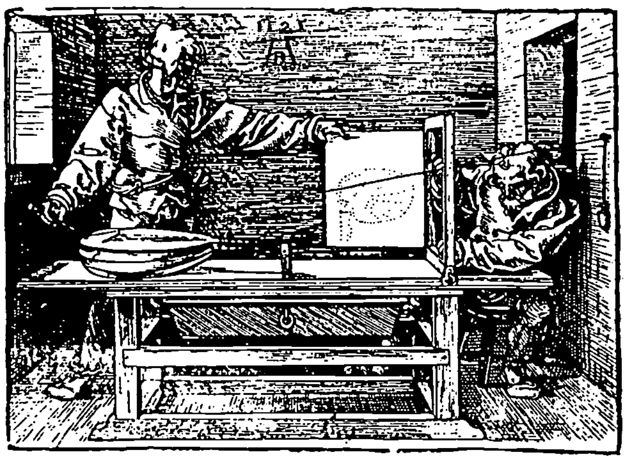
During the class discussion a girl at the back of the classroom is asked: 'What does the student with the ball see now?'
Moving in your mind in and out of the earth-moon system is a necessary mental activity for gaining insight into the phenomenon.
When doing perspective-drawing the student material does not aim towards efficient algorithmic methods, but rather illuminates continually the actual core of the cental projection.

Portraits are first drawn in this way with the aid of a plate of glass between drawer and model.

ill.18.

In a later phase, the bundle of sight-lines from the eye to all points of the object drawn are materialised with a string. We use Düreris famons woodcut as a guide-line.

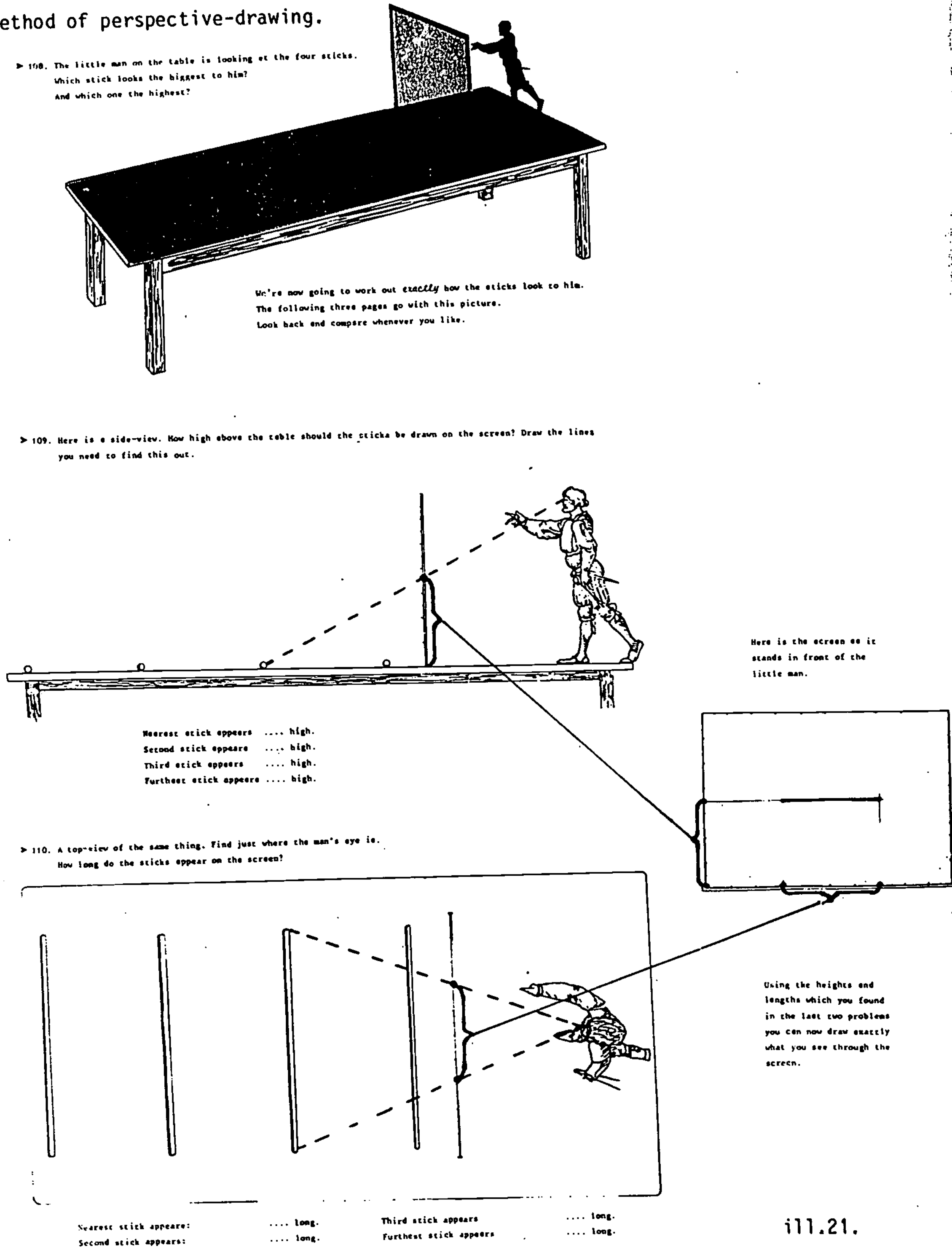


ill.19.

In practie, students and teachers show suprise that this method works so well when trying it thenselues with a folding blackboard or simply with the door te classroom.

ill.20.

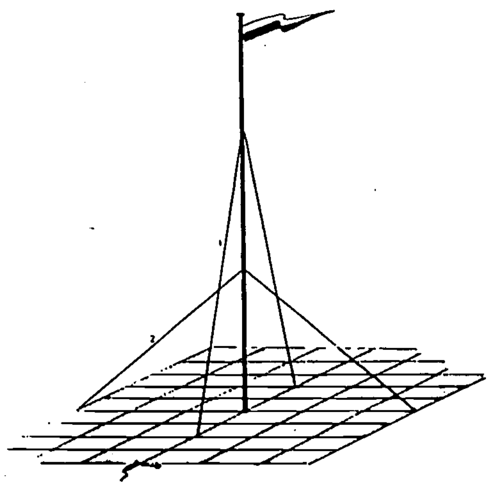
The weight at the right of Dürers print can be done without.
The person at the left can stretch the string herself. After this, a top-view and a side-view are again used. The following four illustrations summarize briefly this method of perspective-drawing.




ill.21.

By this means we let the students reach back to the original techniques of the 15th century Italian painters.

In the workbook 'Lessen in Ruimtemeetkunde' too (Lessons in Space Geometry, intened for 16 and 17 year olds in the upper grades of secondary education), this geometry of top- and side-view and sight-lines is a source of inspiration. There is more, however, after the perspective-drawing, for example a totally now manner of doing the classic assignment of constructing the line through a given point which intersects two skew lines.




> 70. Diagram 8.1.
>
> a. The ropes 1 and 2 seem to intersect.
> How do you know this is not the case?
>
> b. Where should a rope with same length as 1 be fixed on the floor?
> (The tiles on the floor are squares)
>
> c. A mouse sits on the floor (see diagram) and is looking at the flag-pole.
> Where are 1 and 2 'intersecting' according to its view?
>
> d. The mouse starts to move into the direction of the arrow. How does the apparent point of intersection at the ropes move along?
> In what position does the mouse see this point at eye level?

ill.22.

We find an unexpected application of the sight-line when we do draw cubes in parallel projection. By interpreting the apparent point of intersection of two ribs as a projection of one given projection-line (or sight-line), the direction of projection with respect to the cube can be determined.

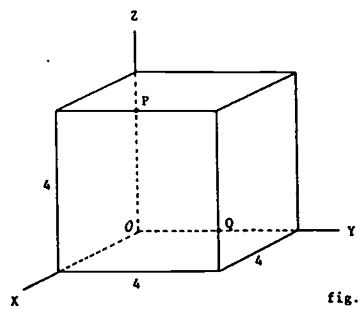



fig. 13.1

> 138. Parallel projection of a cube with coördinate system.

a. Bob: 'P has coördinates (4,2,4)'
Bill: 'I think the coördinates are (0,0,3)
Who is right?
b. What's the problem when looking for the coördinates of Q?
c. Bob indicates (4,2,1) in the diagram.
Do the same.
d. And where is (2,1,½)? And (-4,-2,-1)?
And (-40,-20,-10)?
e. What is the vector which represents the projection rays?

ill.23.

The last example shows clearly that the theme 'looking' does not only play a role at an elementary level, but is functionel as well in introducing more abstract matters such as vectors.
Reactions from the Dutch School system to the geometry as descibed here vary considerably, from extremely positive: 'finally real geometry' to extrevely negative 'The won't learn anything from this'.
A particular problem is the difficulty in testing the material. In the Hewet-program, the new mathematics program for the upper grades of secondary school, room has been made for a section on space geometry. But, although the suggested program is directed towards activities and less so towards acquiring factual knowledge, test questions still appear on the final exams which appear to be inspired by the traditionel program.
The fight for space outside of the book still goes on!

*Literature*

| | | |
|---|---|---|
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| | | Holland |

# MIDDLE SCHOOL PUPILS 'CONCEPTIONS ABOUT REFLECTIONS ACCORDING TO A TASK OF CONSTRUCTION

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Abstract
We give here the results of a research which was developed in French middle schools, related to pupils'conceptions about reflections. The two first steps of this research allow us to determine variables which have an effect upon the pupils'procedures and answers in a freehand construction task. The third step, which is the main matter of this paper, consists to examine more closely the influence of some of these variables and the range of the induced procedures.

## Introduction

This work was undertaken within a research group on geometric transformations, and we are interested by the notion of reflection for two main reasons :

-reflections have been the object of important research in Didactics which we could rely on ( especially K.Hart in England and R.Gras in France );

-reflections appear to be important geometric tranformations in mathematics, and they underly all paper folding activities that children usually do even when very young ( origami ).

The aim of our research is to construct a teaching sequence about reflections in a third form classroom of a French grammar school ( 13-14 year olds ) . For that we had to extend our knowledge of the pupils'conceptions about reflections, before teaching ( 11-12 year old pupils ) and after teaching ( 13-14 year olds ).

## Description of the two first experiments

In all this research, the task which we chose consisted in *freehand drawing the image of a geometric figure with respect to a given line* : this task corresponds to one of those proposed by K.Hart, and consequently we found analogous experimental conditions again. Moreover, we think that instrumental drawing raises specific difficulties for 11-14 year old pupils which may hide the nature of the problems raised by the notion of symmetry ; on the other hand, freehand construction gives us a better insight into pupils'global perception about reflections. We give all the

pupils the following definition : "imagine you fold the paper along the line , the figure you have drawn exactly covers the given figure" . Thus , they all have the same simple definition and , on the other hand , *folding may induce difficulties when the given figure is not in a half-plane according to the symmetry axis*. This task is not customary in French schools , where teaching about reflections emphasizes formal aspects of this notion : it follows that for pupils who have been taught in such a way ( 13-14 year old pupils ) , two conceptions may be in conflict : one analytic and one global .

The first step in our experiments was the observation of six dyads working on freehand construction tasks on symmetry , in an interactive situation . We have chosen some of the items proposed by K.Hart in (K.M.Hart 1981 ) and some others , and we asked the pupils to draw the image of the given picture with respect to the given symmetry axis.This interactive situation allowed us to observe pupils'solving procedures , since they had to agree before drawing a common solution . The interactive situation compelled these pupils to share their notions and perceptions .

This experiment raised some hypotheses which allowed us to choose some new geometric figures . These figures were gathered into a booklet which was dealt out in the classes at four different levels in local schools , with one hundred and fifteen 11-14 year old pupils . In this second experiment , we used the same task and a larger number of geometric figures , and each pupil had to solve by himself the booklet items within the usual classroom schedule . We especially wanted to establish a comparison between the difficulty in constructing reflections of figures made of dots and reflections of figures made of segments .

The results of these two experiments can be summed up as follows :

1-it appears that *the ability to construct the image of a single dot does not enable pupils to construct the image of a composite figure* . Pupils'answers are right when the components of the given figure are only dots , even if these dots are on the both parts of the symmetry axis . For exemple , 90 pupils worked successively on the two following figures :

fig. 1a

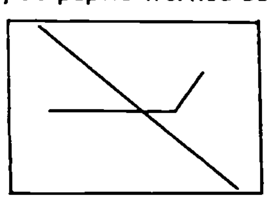
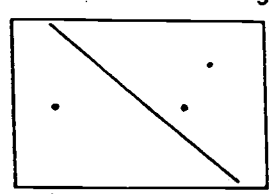


fig. 1b



Only 34 pupils out of the 90 gave a correct answer for fig.1a , as opposed to 72 pupils for fig.1b .

2-we were able to list and classify the pupils'answers into four principal types which we call as follows :

- *orthogonal procedure* : the image is obtained by drawing the

orthogonal symmetric of a few points ;

*-covering or prolonging procedures* : that is procedures in which some parts of the given figure are covered or prolonged by the image (fig.2a, 2b) ;

fig.2a

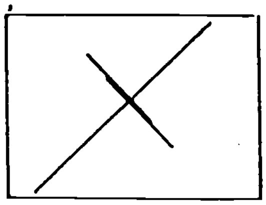
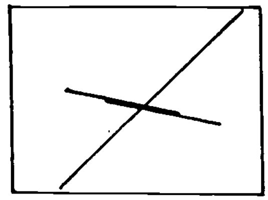


fig.2b



*-parallelism procedure* : the image is obtained by a global translation of the given figure (fig.3a , 3b) ;

fig.3a

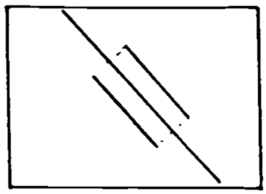
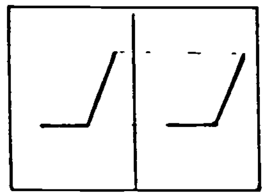


fig.3b



*-horizontal point sliding or vertical point sliding procedures* : vertices of the figure are moved along a horizontal or vertical line to unequal distancies related to their positions with the symmetry axis (fig.4a , 4b) .

fig.4a

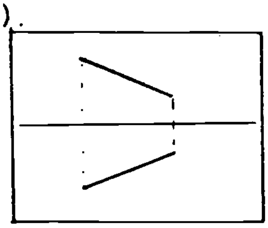
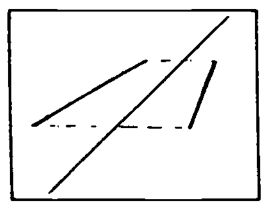


fig.4b



The orthogonal procedure gives a correct answer on condition that the distance to the symmetry axis is well determined .The other procedures may lead to correct answers in special cases (fig.2a , 3a , 4a), but may turn out to give uncorrect answers when the symmetry axis is neither vertical nor horizontal (fig.2b , 4b) or when the given figure has no symmetry property (fig.3b) .

3-Some of the proposed items were drawn on squared paper and others on plain , and it seemed to us that *the solving procedures differed according to the type of paper* : squared paper obviously lays emphasis on horizontal and vertical lines .

4-Other didactic variables ( i.ethe variations of these variables inducing changes in pupils'procedures and answers ) seemed to be important : e.g. the symmetry axis orientation and the positions of the different parts of the geometric figure . But these two experiments do not allow us to draw any conclusions on their influence .

These results and hypotheses were the basis of the third experiment which will now be described.

## Description of the third experiment

In the third step of our experiment, we wished to examine more closely the influence of some didactic variables. For this reason, we had to fix other variables, especially keeping unchanged the geometric figure to be reflected: we chose to keep only *the segment* because it is a simple figure which cannot be reduced to dots in this task.

We let the following variables vary more systematically:

- *the respective positions of the segment and the symmetry axis* :

a-axis non-intersecting the segment, or intersecting it at one of its ends or at another arbitrary point;

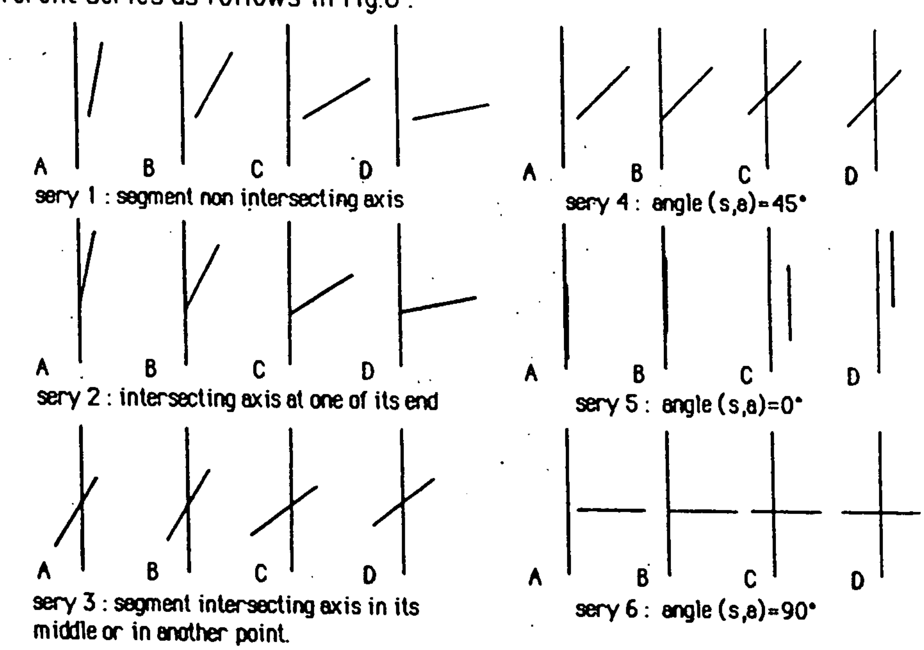
b-angle segment-axis (we always mean the acute angle and we shall call it *angle(s,a)* ); we have chosen four generic values for this angle : 11°, 29°, 61°, 79°(chosen in order to have the ends of a segment set on the intersections printed on the squared paper), and special ones : 0°,45°, 90°.

- *the axis orientation with respect to the edges of the sheet of paper* ( vertical , horizontal ,"right-oblique" (-135° clockwise ) and "left-oblique"(-45° clockwise ) ).

- *the type of paper* chosen as a support for the figure : either squared or plain.

The different positions of the global figure in the sheet were such that all procedures described above "could be used" by pupils, in the sense that they gave an image within the limits of the sheet of paper.

These criteria allowed us to choose 24 basic figures grouped in six different series as follows in fig.6 :



These 24 figures were set out on the sheet of paper in the four axis orientations explained above and each of the 96 resulting figures was drawn on the two types of paper. We obtained in this way 16 kinds of booklets of 12 geometric figures with horizontal and vertical symmetry axis ( 6 on squared paper and 6 on plain ), and 16 others booklets with "right oblique" and "left oblique" axis. Each pupil received two succesive booklets, the first one grouping only items with horizontal and vertical axes, and the second one grouping the same items rotated obliquely at -45°.

The booklets were submitted to 650 pupils in four forms (11-14 year olds) in different French schools .Since the notion of reflection is taught in the third form according to the school syllabus, pupils from the first and second forms had no notion of this teaching, whereas the pupils from the third and fourth forms had had lessons on reflections.

Description and analysis of the results :

-In items where the symmetry axis orientation is vertical and horizontal and do not intersect the segment or intersect it at one of its ends, the pupils'answers are usually right for all the angle(s,a) values and for both kinds of paper (series 1, 2 and fig.4A, 4B). The errors appear when the symmetry axis is on the slant and their number increases in the same order as the angle(s,a) value. The "prolonging or parallelism" answers can be seen on both kinds of paper, but they are more numerous as the angle(s,a) widens. The "horizontal or vertical point sliding" answers are very numerous especially on squared paper : they corroborate our hypothesis that the drawing of horizontal and vertical lines induce solving procedures which are different from those used on plain paper.

-Items where the symmetry axis intersect the segment lead to great difficulties (sery 3 and fig.4C,4D) : pupils consider spontaneously that reflection is a geometric tranformation from one half-plane to the other. A new type of answer appears in these items which we shall call "half-symmetry" procedure and which induces two kinds of mistakes :

-the image is a half-segment founded in one of the two half-planes limited by the symmetry axis : pupils dare not cross the axis (fig.7a, 7b).

-the image is a bent line on both sides of the axis : pupils draw both parts of the reflected segment with different procedures (fig.8).

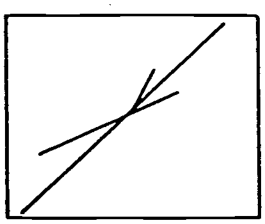


fig.7a

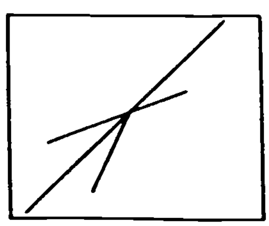


fig.7b

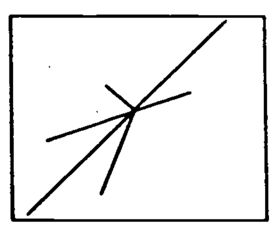


fig.8

These "half-symmetry" answers are more numerous on squared paper , which seems to induce counting procedures which prevent the pupils from global perception .

-Items where the symmetry axis is orthogonal or parallel to the segment are generally successful for any axis orientation on the sheet of paper of both types (series 5 , 6 ) : in this case , prolonging , parallelism or covering procedures lead to correct answers . The only difficulty is drawing the image when the segment is bended into the axis (fig .5A , 5B ) .

## Conclusions

In these experiment,we examined the different pupils'solving procedures and answers in a task of freehand construction and it appears that the variables which we studied have a great influence on these answers .The results of our works show how it is necessary to know the range where the pupils'conceptions are stable and allow to lead to correct answers. Whereover, these variables have a great influence also for the pupils who received lessons on reflections : we may think that the analytic procedures given in French teaching about this notion do not allow them to solve constructions problems . We did not see a real progression from 11 year old pupils to 14 year olds :the reason of this fact is to be investigated by means of studies of schoolbooks and teaching in classrooms.

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# GEOMETRIC CONCEPTS OF PORTUGUESE PRESERVICE PRIMARY TEACHERS: A VAN HIELE BASED RESEARCH

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In 1957 two Dutch researchers, D. van Hiele-Geldof and P. van Hiele, proposed that geometric learning occurs through a sequence of five levels. In Level 1 (Recognition) the student knows some vocabulary and recognizes shapes. At Level 2 (Analysis) properties of the figures are analysed, but it is only in Level 3 (Ordering) that a logical order of these properties is recognized and that the role of definitions is made clear. On Level 4 (Deduction) the student understands an hierarchical organization of geometry and the role of axioms, theorems, and deduction in general. At Level 5 (Rigor) geometry becomes a part of formal logic. To help students' learning, the van Hieles prescribed five phases: inquiry, directed orientation, expliciting, free orientation, and integration. Recent research has developed this model, that stands as an insightful global explanation of students' geometric reasoning.

Soviet researchers used van Hieles' ideas to analyse the geometry curricula. In consequence changes were design that led to a better organization of the curricula (Hoffer, 1983). van Hiele theory inspired several studies in the USA. Mayberry (1983) investigated the consistency of the theory. Usiskin (1982) directed a national program of tests to analyse correlations between van Hiele levels and the achievement of secondary students enrolled on a geometry course. A related study (Senk, 1982) focused on the ability to perform proofs. Two other studies (Hoffer, 1983; Fuys & Geddes, 1984), conducted interview based research looking for anedoctal descriptions of students' geometric perceptions. All these researches have enriched the theory.

There is a recent interest on the process of primary (Grades 1 through 4) teacher training in Portugal, the whole program being under revision. Geometry offers an interesting field: although a primary teacher must be able to teach geometry to children, this topic is seldom covered in EMP Schools (primary teacher training institutions). One investigation focused on EMP School students'

conceptions about area (Costa, 1984), and found that only one-third of first-year students, and a smaller part of third-year students, attained the basic concept of area. Students can enter EMP Schools with 9th, 11th, or 12th grade mathematics, and some educators sustain that it is not crucial for a preservice primary teacher to have different mathematical background. On the contrary, mathematics faculty at EMP Schools maintain that the entrance should be limited to students with at least 11th grade of Mathematics. Therefore it is of special interest to compare the geometric reasoning levels of these students.

## The Study

The purpose of this cross-sectional van Hiele based study was to investigate geometric concepts of EMP students, to detect the evolution of their geometrical ability, and to look for differences among entering students with different mathematical background.

The sample was composed of 143 first-year students, 148 second-year students, and 106 third-year students from three Portuguese EMP Schools (Beja, Castelo Branco, and Faro) that were present the day the test was administered. It represents 96% of the population of those schools. Subsamples with students from each of the three schools were tested for homogeneity on the variables sex, age, and mathematical background before entering EMP Schools. Only sex distribution revealed non-homogeneity in global comparisons, although subsamples from each year were found to be similar. Subsamples were merged for analysis.

Table 1
Mathematical background of students

| Year | Highest Mathematics grade level 9 | 11 | 12 |
|---|---|---|---|
| First year | 69 | 45 | 29 |
| Second year | 63 | 56 | 29 |
| Third year | 26 | 51 | 29 |

## The Test

A test developed by the CDASSG Project (Usiskin, 1982) using direct quotations from the van Hieles was employed. The task for each level was compounded of 5 items, and there were 5 tasks involved, corresponding at each level. For example Item 3 showed two rectangles and another trapezoid, and asked students to identify those that were rectangles. Item 13 prompted one square

and two other rectangles and asked the student to identify the rectangles. On Item 21 F-geometry was described as having exactly four points and six lines. Every line contained exactly two points.

. P

Q .

R . . S

Students were then asked to identify the correct relations of parallelism and intersection between lines. A student is at Level n (0<n<6) if he/she succeeded in each task from Level 1 to Level n and did not succeed in any task above Level n. Level 0 includes students that did not succeed in any task. Students that are not classifiable under this procedure constitute the Nofit Group. Two criteria were used to determine if a student succeded at a given cluster of items: (a) there is success at task n if at least 3 of its items are correct (3-of-5 criterion), or (b) if at least 4 of its items are correct (4-of-5 criterion).

## Results

One goal of this study was to describe the distribution of EMP students across the van Hiele levels. Table 2 presents students' performance under the two scoring criteria.

Table 2
Students' Performance by Levels

| Year | Task satisfaction criterion | | | |
|---|---|---|---|---|
| | 3-of-5 | | 4-of-5 | |
| | n | % | n | % |
| First-year | | | | |
| Level 0 | 3 | (2) | 33 | (23) |
| Level 1 | 32 | (22) | 45 | (31) |
| Level 2 | 60 | (42) | 41 | (29) |
| Level 3 | 21 | (15) | 5 | (3) |
| Level 4 | 5 | (3) | 0 | |
| Level 5 | 0 | | 0 | |
| Nofit | 22 | (15) | 19 | (13) |
| Second-year | | | | |
| Level 0 | 11 | (7) | 30 | (20) |
| Level 1 | 24 | (16) | 44 | (30) |
| Level 2 | 57 | (39) | 52 | (35) |
| Level 3 | 26 | (18) | 6 | (4) |
| Level 4 | 5 | (3) | 1 | (1) |
| Level 5 | 0 | | 0 | |
| Nofit | 25 | (17) | 15 | (10) |
| Third-year | | | | |
| Level 0 | 2 | (2) | 15 | (14) |
| Level 1 | 5 | (5) | 17 | (16) |
| Level 2 | 40 | (38) | 45 | (42) |
| Level 3 | 24 | (23) | 4 | (4) |
| Level 4 | 6 | (6) | 0 | |
| Level 5 | 2 | (2) | 0 | |
| Nofit | 27 | (25) | 25 | (24) |

Under the 3-of-5 criterion Level 2 contains the largest percentages of students for all years. Under the 4-of-5 criterion, Level 1 contains the largest percentage of first-year students, but for second-year and third-year students, Level 2 stands higher. Comparing cumulative percentages of students above Level 2 with those at and below Level 2 we observe that the majority of classifiable students of any of the three years is not reaching Level 3 under either criterion used. It is only in the third-year that, according to the most favourable criterion, a significant portion of students is above that level. The passage from Level 2 to Level 3 marks the passage from the analysis of several components of figures into properties to the local (non-global) organization of these properties. Level 3 is not yet a global hierarchical organization of knowledge, which appears only at Level 4, but presents an organization of the geometrical facts on islands of interrelated properties. Senk (1982) identified Level 2 as a gateway to success in a geometry course that uses proof. Portuguese primary curriculum does not require children to be at a van Hiele level higher than Level 2. But teachers at Level 2 are barely prepared to teach the present curriculum by the book, by root procedures and are not prepared to handle new situations inside the classroom that might require curricular innovation.

Comparisons were performed between the levels attained by students in each year (Table 3). Level distributions of first-year

Table 3
Qui-square analysis of performance of students of first, second and third year on the van Hiele levels

| Years under comparison | Qui-square statistics (df=4) | |
|---|---|---|
| | 3-of-5 criterion | 4-of-5 criterion |
| First year vs. second year | 0.71 | 2.17 |
| First year vs. third-year | 18.99** | 15.35* |
| Second year vs. third-year | 15.99* | 14.22* |

*p<.01. **p<.001.

and second-year students were not significatively different. But when the third-year distribution was compared with other distributions, differences became significant at the .01 level under both task satisfaction criteria. It appears that it is during the second year that students develop their geometric abilities and progress to Level 2. It is remarkable that this progression occurs as a "hidden curriculum" of EMP Schools, since geometry usually is not explicitly covered in the coursework. The interactions occuring

in EMP classrooms that facilitate this development ought to be studied but were not detected on this investigation.

Analysis was undertaken to look for differences among groups of entering students with different mathematical background. Conparisons between the 9th grade group and the 11+12th grade group were performed. Under the 3-of-5 criterion, $\chi^2$=10.86 ($p<.02$, df=3), and under the 4-of-5 criterion, $\chi^2$ =9.15 ($p<.05$, df=3). The most interesting fact is that there are very few topics of the 10th-12th grade curricula that are related to geometry (vectorial and analytic geometry). At least two reasons can explain this difference: a) the gap of several years on their mathematical instruction; b) a general lower ability for mathematics.

Although data showed that there is a progression of students geometric levels (Table 3), it was not clear if this progression was similar with groups of students having different mathematical background. So, further research was undertaken looking for differences among the levels of students with 9th grade mathematics, those with 11th grade, and those with 12th grade mathematics. Table 4 shows that only those students with 9th grade mathematics have a significant change of the levels.

Table 4
Comparison between groups of students with different years of Mathematics

| Groups of students | Chi-square statistics: 3-of-5 criterion (df=3) | Chi-square statistics: 4-of-5 criterion |
|---|---|---|
| Students with 9th grade mathematics | | |
| First year vs. second year | 4.67 | 1.57[b] |
| First year vs. third year | 11.41*** | 11.64***[a] |
| Second year vs. third year | 10.25** | 6.42*[a] |
| Students with 11th grade mathematics | | |
| First year vs. second year | 2.09 | 5.35[a] |
| First year vs. third year | 4.93 | 1.90[a] |
| Second year vs. third year | 2.72 | 2.02[a] |
| Students with 12th grade mathematics | | |
| First year vs. second year | 2.39 | 2.25[b] |
| First year vs. third year | 6.64 | 3.04[a] |
| Second year vs. third year | 6.37 | 5.18[a] |

[a]Note. df=2. [b]Note. df=3.
* $p<.05$. ** $p<.02$. *** $p<.01$.

## Some Misconceptions

This study obtained some clues about students' misconceptions, when analysis of responses to specific test items was performed.

Level 2 bias. Although 90% of the students correctly identified rectangles when squares were not present (Item 3), two-thirds of the students did not recognize a square as a

rectangle on Item 13. These two answers make perfectly good sense as Level 1 or 2 reasoning. This misconception is precisely one of the characteristics of the difference between Level 2 and Level 3. A confirmation of this was found on another item (14) that asked for class inclusion relations between squares, rectangles, and parallelograms that obtained only 6% correct answers.

Physical intuition bias. Although good descriptions of what was meant by "interception" and "parallel" were given in Item 21, phusical intuition led 62% of the students to answer that lines (P,R) and (Q,S) intersect.

## Conclusions

Schools were deliberately chosen for this study, restricting the generalizability of the results. It was also assumed that the test used is a good operational definition of the levels. With this constrains in mind we may conclude that:

1 - Students progress gradually from Level 1 to Level 2 in an "average" one-level increase. This progression is clearer in the passage from the second to the third year.

2 - This progress is significative only with students having 9th grade mathematics before entering EMP Schools. It appears as instruction in EMP produces effects only on students with less mathematical knowledge.

3 - EMP entering students with more mathematical background are those that present better geometrical knowledge.

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# A PRELIMINARY INVESTIGATION OF THE TRANSITION BETWEEN TWO LEVELS OF INTELLECTUAL FUNCTIONING

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This study follows on from studies reported at earlier PME conferences (Collis and Romberg, 1981; Collis, 1982; Jurdak, 1982; Romberg, 1982; Collis, 1983) which involved the use of the SOLO Taxonomy for evaluating students' problem solving in mathematics. These studies showed that it was possible to categorize children's mathematical thinking according to criteria derived from theories of intellectual skills development (Fischer, 1980; Case, 1980; Biggs and Collis, 1982). Moreover, they described a method for establishing the level of intellectual skills displayed by means of analysing an individual's response to a carefully constructed mathematical stimulus.

It will be recalled that the following criteria were used to set up the five levels of response. These were then used to devise appropriate problem-solving superitems for school mathematics;

| Level | Criteria |
|---|---|
| Prestructural | Irrelevant or 'avoiding the question' response. |
| Unistructural | Use of *one obvious* piece of information coming directly from the stem. |
| Multistructural | Use of two or more discrete closures directly related to *separate pieces* of information contained in the stem. |
| Relational | Use of two or more closures directly related to an *integrated* understanding of the information in the stem. |
| Extended Abstract | Use of an *abstract general principle* or hypothesis which is derived from or suggested by the information in the stem. |

Given the above, the study reported here set out to obtain some preliminary information regarding an individual's move from the multistructural level to the relational level. In terms of mathematics education this is a crucial step. It takes the student from the point where a mathematical or problem statement is regarded as a sequence of instructions to be followed to the point where the statement or problem is regarded as a balanced interrelated whole of which he/she has an overview. In the area of developing intellectual skills *generally* it is no less significant. The study to be reported consists of two parts.

In the first study a sample of children at the end of their elementary schooling who were likely to include individuals who were ready to move from multistructural responding to relational responding were tested on three items, one in each of the areas, number, space and volume. For the second study three boys, whose pattern of results in the first study showed the possibility of examining the move from multistructural to relational responding were selected and interviewed individually. The interview set out to investigate, (a) what strategies they were using to try to solve (or to actually solve) the relational question and (b) when appropriate, whether they could achieve the higher level with a little help.

## The First Study

Sample: A sample of 51 children (Mean age: 10 years 11 months; 29 boys and 22 girls) consisting of two classes in the final year of their elementary schooling at a local school in Sandy Bay, Tasmania were selected to do a group test of three items. The socio-economic status of the area in which they lived would be categorized as upper-middle class.

Items: (The items are summarized below as space prevents their presentation in full.)

<u>Number:</u> This consisted of a drawing of a 'number machine' which the children were told 'adds the number you put in three times and then adds 2 more. So, if you put in 4, it puts out 14' followed by the following three questions:

A. If 14 is put out, what number was put in?

B. If we put in a 5, what number will the machine put out?

C. If we got out 41, what number was put in?

<u>Space:</u> A diagram of a rectangle divided into regions by straight lines was followed by a definition of neighbouring regions. The three questions followed the same pattern as indicated above. The first asked about a neighbouring region which was dealt with in the stem of the item; the second asked <u>S</u> to find a specific neighbouring region given the definition; the third required <u>S</u> to make a deduction about a region from a set of concrete statements about neighbouring regions.

<u>Volume:</u> Diagrams demonstrated how buildings in block form could be made out of cubes of the same size. The first two questions simply asked how many cubes were required to make building blocks in diagrams given; the third required <u>S</u> to deduce the height of a building block given the base dimensions (in cubes) and the total number of cubes to be used.

The first two items had been used in previous studies and had shown high reproducibility coefficients (Guttman, 1941); the third was *based* on an item used earlier which had a suitably high coefficient. Thus the author was reasonably sure that the items would be valid in terms of the internal hierarchical structure of the responses required in each item. Subsequent analysis, using the data gathered from the first part of the study (described immediately below), showed the reproducibility coefficients to be .9B2, .945 and .889 respectively (the usual level of acceptability on this measure is $r > .85$).

Method: The three items were put together in a booklet and the students tested in their classes using standard group testing techniques. They marked their responses and did their working on the booklet provided.

The author carried out the testing in each of the two classes involved. Testing time available was 45 minutes but students were not given a time limit. They were all able to complete the items to their satisfaction well within the time available.

Analysis and Results: The booklets were marked and the results on each question for each student entered on a score sheet to show the achievement of each student individually. Although the main purpose of this part of the project was to select suitable Ss for the second part of the project it is appropriate at this point to summarize and comment on the results of the group as a whole. One aspect of these results is summarized in table 1 below.

Table 1

Percentage of Ss achieving Levels of Item

| Item | Unistructural | Multistructural | Relational |
|---|---|---|---|
| 1. Number | 89 | 80 | 61 |
| 2. Space | 96 | 78 | 30 |
| 3. Volume | 68 | 42 | 44 |

In addition it was observed that 61% of Ss had the Unistructural response correct in all three items, 5% had all three items completely correct - if one incorrect response is allowed out of the nine this percentage becomes, 28%. This last figure seems a more realistic estimate of Ss who could cope with all three items.

Some conclusions of interest can be drawn from the summary in Table 1. For example:

(a) number and the operations thereon are more readily achieved by this group at a relational level than problems involving space (in this case decisions about neighbours and regions) and volume;

(b) relational functioning with space and volume concepts is difficult for this group;

(c) multistructural functioning is not readily achieved with volume concepts, as tested in this study, and only the most elementary level of functioning in the volume problem was demonstrated by two-thirds of this group;

(d) the pattern of results in the relational column suggests that relational thinking is not a unitary trait as some developmental theories would imply.

## The Second Study

Sample: Three boys with differing patterns of achievement were selected to be interviewed. The first had achieved correct responses to the multistructural questions in number and space but could not answer correctly any of the volume questions; the second achieved the relational level in the number question but only the multistructural in the other two; the third achieved the relational level in the first two questions but failed to answer any of the volume questions correctly. The investigator felt that this variety of achievement would give the best chance of exploring the thinking involved in solving the problems in this pilot study.

Method: The investigator sat beside each child at a small table in the school library to carry out the interview. Each interview took about 20 minutes and was tape recorded.

The interviews were semi-structured and were conducted along the following lines:

(a) introductory, settling down remarks - including a statement that the investigator was interested in the way in which S had worked out his answers to the questions in the group test;

(b) S was given an unused copy of the group test and asked to do again the questions which he had done correctly in the earlier test and to say aloud for the investigator and the tape recorder what he was doing and how he decided to proceed in a particular way - he was not told that these were the correct items from his first test.

(c)(i) in the case of the first two Ss their attention was drawn to relational questions which they had answered incorrectly in the first test (again they were not told this) and asked to work out loud how they answered them; at this stage they were told that their multistructural answer was correct. At this point the interview became less structured as the investigator intervened when S indicated that he was getting 'lost' or about to give up. The intervention consisted of all or some of the following:

(1) questions to find out (a) whether S had grasped what the question was and (b) what S was trying to do at that particular time;

(2) suggesting a principle that might be involved, using as an example the data from the previous correctly answered question.

(3) suggesting a strategy - 'would it help if you...?'

(c)(ii) in the case of the third S the techniques described above in (c)(i) were adapted to the particular case involved in which S had already shown the ability to respond at the relational level in the two previous items;

(d) the interview was closed off by congratulating S on his efforts, giving him enough help to get the correct answers and letting him hear himself working on the tape if he so wished.

<u>Analysis and Results:</u> Analysis of the taped interviews and the notes made at the time both by <u>S</u> and the interviewer show some patterns which deserve closer investigation with a larger sample. The following results seemed clear in that each was exhibited by at least two out of the three <u>Ss</u> in the interview:

(a) <u>S</u> could do again, with little effort, the questions previously answered correctly;

(b) the technique of referring back to the question done correctly at the previous level and pointing out the required general principle was not usually helpful;

(c) when <u>S</u> lost 'control' of the problem he tended to close on irrelevant cues or begin guessing;

(d) when trying to follow strategy suggestions made by the investigator <u>S</u> tended to lose track of the variables in the problem.

It is interesting to note that all the results indicated above seem to be related to cognitive processing problems. The third student showed this up well - his major difficulty with the volume item was that he counted surfaces, instead of columes (cubes). When he realized this he still had trouble because he kept 'forgetting' and mixing the two as he tried to work on the particular question.

<u>Conclusion:</u> The results of these two studies show that a more thorough-going investigation along the lines described above would be viable and potentially very productive. The use of a larger and more representative sample for the first part of the study would not only produce practical information on levels and varieties of children's mathematical functioning in the school system but would also enable a larger number of individuals with a range of different, but typical, patterns of achievement, to be investigated.

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# THE DEVELOPMENT OF ABSTRACT REASONING - RESULTS FROM A LARGE SCALE MATHEMATICS STUDY IN AUSTRALIA AND NEW ZEALAND

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A large scale mathematics study was conducted in Australia and New Zealand in 1983 with 10,500 pupils aged between 11 and 18 years. The test administered was designed to cover a wide range of abstract reasoning, the same test being given to all pupils. These pupils came from diverse backgrounds, and their abilities, as reflected by their total score, ranged from remedial to gifted.

It is often erroneously assumed that, once a child begins to reason in abstract terms, s/he is capable of everything implied by Piaget's formal operations stage. This study sets out to examine the development that occurs over this period.

Mathematics problems were chosen to investigate a child's ability to reason in abstract terms as it is comparatively simple to increase the degree of abstraction by modifying the problem. The study spans an age range over which the development of abstract reasoning is most apparent - from about 11 or 12 years (Year 8 at school - the first year of Secondary schooling in South Australia and the second year of Intermediate schooling in New Zealand) to about 17 or 18 years (the final year of Secondary schooling - Year 12 in South Australia and Year 13 in New Zealand). We were therefore interested in whole school populations in Australia, extending down into Intermediate schools (Years 7 and 8) in New Zealand.

Eight secondary schools in the Adelaide metropolitan area in South Australia and 11 post-primary schools in the greater Wellington area (New Zealand) were chosen from regions of different socio-economic background to give a balanced sample.

## ANALYSIS OF THE RESULTS:

When large scale tests are carried out, the results are frequently presented in general terms such as class or Year means. Figures 1 and 2 give mean scores for Year 9 for Australian and New Zealand schools respectively. A wide range of means is evident, from 17.9 (standard deviation 8.7) to 32.4 (st. dev. 9.6) in the Australian sample schools, and 19.7 (st. dev. 8.1) to 33.6 (st. dev. 9.8) in the New Zealand sample schools. Although of interest, results such as these reflect nothing of the pattern of an individual's performance.

The aim in this research was to adopt an analysis technique that could be used to give information about the performance of individual pupils (other than score totals). Rasch analysis was therefore applied (Rasch, 1960; Wright and Stone, 1979).

Figure 1: Mean scores for Year 9 from schools in Australian sample

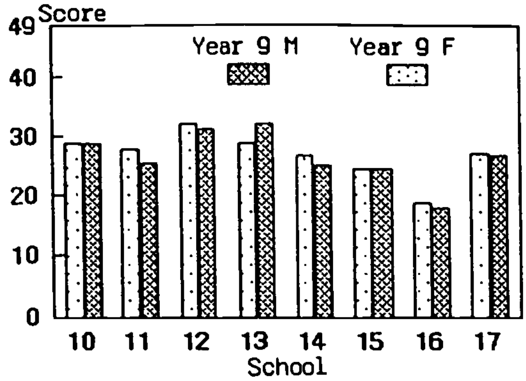
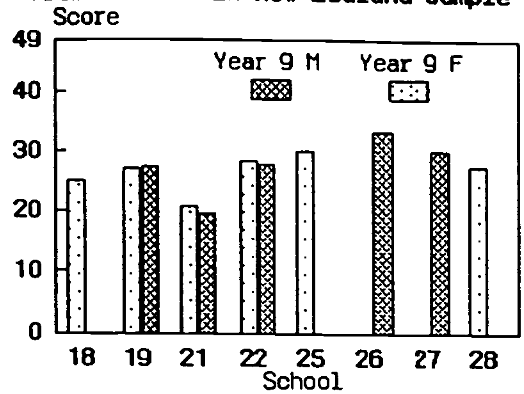



Figure 2: Mean Scores for Year 9 from schools in New Zealand sample




The validity of the test was ascertained by plotting approximate item characteristic curves (Izard et al, 1982). Each year level of each school was analysed separately. Pupils were divided into six equally sized groups of increasing ability on the basis of total test scores. The proportion of each group which answered each question correctly was plotted and a series of graphs such as that shown in Figure 3 was found. The slopes of the curves show that a higher proportion of correct responses is given by the higher scoring pupils. Similar slopes imply that different questions are discriminating between higher and lower scoring groups in similar ways.

Figure 3: Analysis of Test Questions

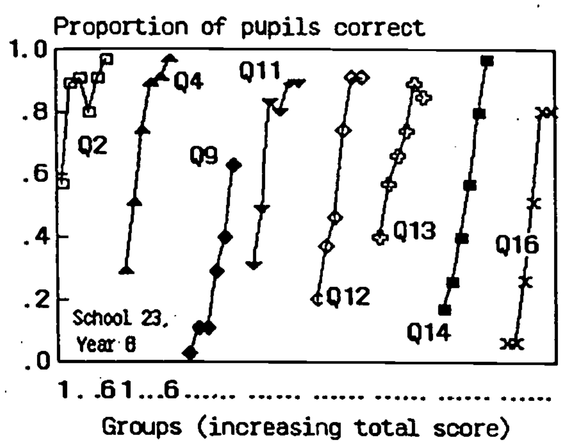
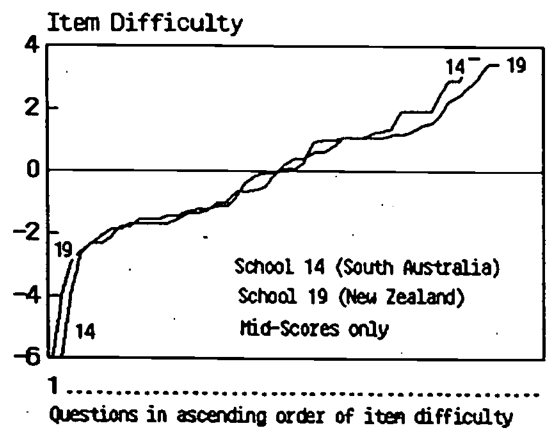



Figure 4: Item Difficulty of Questions




The item difficulty of each question was calculated by Rasch analysis. Sub-groups of pupils of equal (average) abilities were chosen from the large sample by analysing the results of pupils from whole school populations who achieved mid-scores in the test. The item difficulties found for forty two pupils from School 14 (Australia) and sixty pupils from School 19 (New Zealand) are plotted in Figure 4.

The curves are strikingly similar, in spite of the fact that the schools are located in different countries with different teachers, curricula and textbooks. The items cover the entire range of item difficulty from very easy for most pupils (about -6 Rasch units) to

very difficult for most students (about +6 units). An even distribution of item difficulty is important if a test is to differentiate between pupils over a wide ability range. (Izard et al, 1982).

What answering patterns are shown by individual pupils? Did pupils of similar reasoning ability show similar answering patterns? Kendall correlations were used to detect similar answering patterns. The Kendall correlations found for a group of 100 pupils from one New Zealand school who gained close to fifty percent for their test score were used to place the test questions into 8 groups. The correlations indicate that, if one or two questions in a given group are answered correctly (or incorrectly) then it is likely that the other questions in that group will also be answered correctly (or incorrectly).

Figure 5: Grouping of Questions

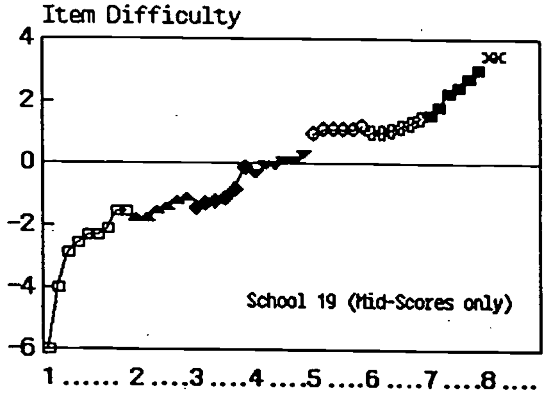
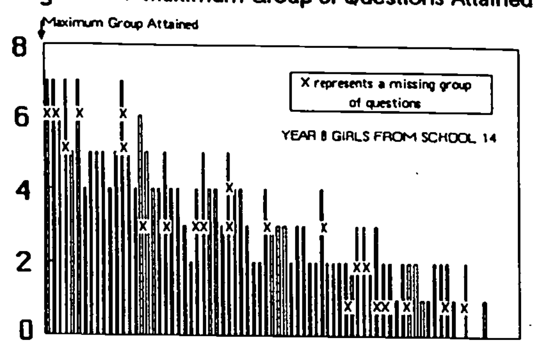



Figure 6: Maximum Group of Questions Attained




In Figure 5, a plot has been made showing the item difficulty of each item in its respective grouping. Figures 4 and 5 can be almost exactly superimposed, although Figure 4 shows simply the *questions* arranged in increasing order of item difficulty, while Figure 5 reflects the *grouping of questions* found with Kendall correlations, the questions in each group being arranged in increasing order of item difficulty.Thus the groups chosen with Kendall correlations are consistent with the Rasch analysis of item difficulty.

The implications extend beyond this simple observation, though. The item difficulties found for a particular set of questions for a given group of pupils reflect the cognitive functioning of that group of students. Thus patterns observed in the grouping of problems of similar item difficulty reflect patterns in the cognitive functioning of pupils. One question that arises is whether the same question groupings are found for all pupils. If pupils develop abstract reasoning capabilities in similar ways (but at different times), as shown by their ability to cope with and/or process successively more abstract information, then it can be postulated that the grouping of questions found for the subgroup will be applicable to the whole sample,regardless of ability or environment.

The hypothesis was tested by finding whether pupils achieved success in the first group of questions before achieving success in the second group, and so on. Success was taken as achieving two-thirds or more correct responses in each group. It is important to emphasize that the grouping of questions was chosen with one subgroup of pupils only, and that the same grouping of questions was applied to all age groups at all Year levels in the study. Evidence against the hypothesis would be shown as a less-than-two-thirds success rate for a significant number of question groupings, whether this is for a few students or spread over many students.

The hypothesis was confirmed for all pupils tested in both Australia and New Zealand, regardless of the pupils' ability, sex, teacher, school, background or country. The results for Year 8 girls at School 14 are shown in Figure 6.The diagram illustrates the maximum group of questions attained and any gaps (groups in which success was not achieved). When all missed groups are totalled and compared with the total groups in the sequences in which success was achieved, the gaps are not statistically significant and can be attributed to chance. These results confirm that a universal grouping of the questions in the test can be applied across the entire range of item difficulties.

The test questions were designed to include a number of similar types of problems, but at successively higher levels of abstraction. An example of one such sequence is given in the following six questions, the grouping for each question being indicated in parentheses. $18\times3=9\times\square$ (Group 1); $14\div7=28\div\square$ (Group 4); $6-1=\square-6$ (Group 5); scale of drawing problem (Group 6); $10\div2=\square\div10$ (Group 7); $1\div4=\square\times2$ (Group 8).

Development of capacity to process information and the idea of repetition of processes or stages in development or of cycles within each stage are included in recent theories of cognitive development. Pascual-Leone (1970) defined developmental levels in terms of capacity and discussed a central computing space which he denoted M-space; Case (1974, 1977) quantified M-space in a similar way to Pascual-Leone; Fischer (1980) explained cognitive development as a series of hierarchical skills; Ellerton (1980) introduced a cyclical approach to development; Biggs and Collis (1982) described five levels of structure of response within each Piagetian mode of functioning; and Halford (1982) defined three symbolic levels which represent increasingly complex environment systems.

The original proposal by Ellerton (1980) suggested that cognitive development proceeds via a series of levels, Level I incorporating Piaget's sensorimotor, preoperational and concrete stages, and Level II, the first of the formal levels, repeating the sensorimotor, preoperational and concrete stages but at a first level of abstraction. Thus, for example, just as young children can only cope with one object/event at a time in the sensorimotor stage, so in the Level II sensorimotor phase pupils can cope with only one idea from that

level of abstraction at a time. Similar parallels can be drawn for the preoperational and concrete stages at each Level. Development is pictured as continuous movement along a spiral.Although Fischer (1980), in his model, suggested a recurrence of the first four levels once individuals reached level 7, the four levels were only recycled once. Biggs and Collis (1982) suggest a repetition of formal operations at successively higher levels, with each of these repeating only the learning cycle that was postulated for each of the lower stages (unistructural, multistructural and relational), not a repetition of the lower stages themselves. In the spiral model above, repetition of three levels is envisaged for each change in the level of abstraction. In the data reported for the current study, eight groupings have been distinguished, and these include nothing beyond comparatively simple algebra. Thus the spiral model places greater emphasis on development through different levels of formal reasoning than do most other theories.

Horizontal decalage may be responsible for the gaps observed in the pattern of question groups illustrated in Figure 6. The model permits movement along the spiral even though some tasks at that level have not met with success.

The spiral model copes with the intuitive leaps made by some pupils by describing this in terms of jumping from one level to the next (Level II concrete to Level III concrete, for example). Vertical movement along the axis of the spiral is thus allowed as well as continuous movement around it, as illustrated in the sketch in Figure 7.The model is discussed more fully in Ellerton (1985).

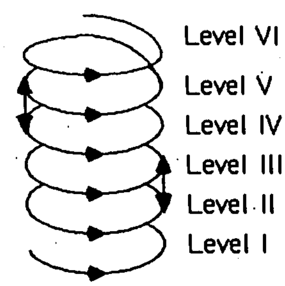



Figure 7: Spiral model for cognitive development.

Keats (1980) suggested that at least two parameters $c_i$ and $d_i$ are needed to represent the cognitive development of individual i, where $c_i$ is an individual differences parameter and $d_i$ is associated with the rate of development. Keats (1980) proposed that the two parameter model for the ability $A_{ik}$ of subject i at time j could be written as: $A_{ij} = t_j / (c_i t_j + d_i)$ or $1/A_{ij} = c_i + d_i/t_j$ . This is, in fact, equivalent to the developmental curve proposed by Halford and Keats (1978) and written as: $A_{ij} = M_i t_j/(t_j + k_i)$ or $1/A_{ij} = 1/M_i + k_i/M_i t_j$, where $M_i$ is the individual's ultimate ability and $k_i$ is a growth parameter, if $c_i = 1/M_i$ and $d_i = k_i/M_i$ . According to Keats (1978) and Keats (1980), both models can be interpreted in terms of Rasch analysis, and reference is made to the lack of longitudinal testing of the model. However, in the current study, when test data from a group of 35 students who had been retested 7 months after the initial test was applied to the models of Keats (1980) and Halford and Keats (1978), the resulting parameters could not be used to predict ability growth curves. The main problem appeared to be the prediction of negative $A_{ij}$ values (at t=200

months, for example), when the $A_{ij}$ values for t about 160 months (time of testing) was positive and increasing. The reason for the failure of the Halford and Keats (1978) and Keats (1980) models when tested with Rasch analysis data appears to arise from an inherent assumption in the models that either $A_{ij}>0$ or $A_{ij}<0$ but not both for the same individual. In Rasch analysis, $A_{ij}$ can be >0 or <0. Thus, if $A_{ij} < 0$, then either $M_i < 0$ or $k_i < -t$ since $t_i$ is always positive. But as development occurs, $A_{ij}$ *for the same individual* becomes positive. When this happens, $M_i t_j(t_j + k_i) > 0$. The same values for the parameters $M_i$ and $k_i$ cannot be used for both $A_{ij}>0$ and $A_{ij}<0$ and the models fail.

Attempts have been made to find a suitable model in which $A_{ij}$ can be less than zero. The simple model $A_{ij} = M_i - s_i/t_j$ where $M_i$ is the maximum ability of an individual i and $s_i$ is a growth parameter, allows $A_{ij}$ to be either positive or negative, but is invalid as $t_j$ approaches zero. The modification $A_{ij} = M_i - s_i M_i/(t_j+s_i)$ is, in fact, identical with that proposed by Halford and Keats (1978) and is not valid for Rasch analysis. The search for a suitable mathematical model for describing cognitive growth is continuing.

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# PUPIL COGNITIVE ABILITY LEVELS COMPARED WITH CURRICULAR DEMANDS WHEN SIX- TO EIGHT-YEAR-OLDS ARE TAUGHT ARITHMETIC OPERATIONS

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ABSTRACT

*This paper reports findings from cognitive assessments of the responses of 180 six- to eight-year-olds in the context of topics in arithmetic operations and from comparisons with curricular and textbook demands in the same topics.*

"Assessing Cognitive Levels in Classrooms" (ACLIC), a two-year cognitive assessment project in elementary school mathematics, culminated in June, 1985, with a report of its findings regarding student cognition levels and instructional cognitive demands in urban and rural centres throughout the province of Alberta, Canada (Marchand, Harrison, Bye, & Schroeder, 1985).

In the ACLIC research project, mathematics cognitive assessment data were collected from 360 six- to eight-year-old children by means of fourteen individual clinical interviews conducted by twenty-three "teacher-interviewers." The interview tasks were selected and adapted from the Piagetian and neo-Piagetian research literature (e.g., Lovell, K., "Piaget's Structure," personal communication, 1974; Biggs & Collis, 1982; Cornish & Wines, 1978) to assess student cognitive development levels in key school mathematics topics. Interview record sheets and audio recordings were used to rate student responses as Pre-operational, Early Concrete Operational, Late Concrete Operational, Early Formal Operational, or Formal Operational in eighteen different concept areas, including: Addition, Numerical Equivalence, Multiplication, and Subtraction/Place Value.

Cognitive assessment data were also collected by administering to eight-year-olds specially designed paper-and-pencil tests covering Operations, Number, Measurement and Geometry. The Operations paper-and-pencil test items were drawn (with permission) from the Operations Test developed for the Mathematics Profile Series of the Australian Council for Educational Research (Cornish & Wines, 1978). The comparability of the paper-and-pencil tests and the clinical interviews for assessing student cognitive response levels was appraised and confirmed by having sub-samples of eight-year-olds complete both paper-and-pencil tests and interviews.

The present paper reports ACLIC findings regarding the "Operations" topic strand as taught to six- to eight-year-olds. The PME 9 presentation will also include results from the nine- to eleven-year-olds.

## COGNITIVE ABILITIES AND DEMANDS: OPERATIONS

In the following paragraphs is included an abbreviated version of the Operations for six- to eight-year-olds "cognitive demand criteria" derived from Piagetian

overviews and from the cognitive ability criteria imbedded in the ACLIC student assessments.

ACLIC Cognitive Demand Criteria

Operations, Six- to Eight-Year-Olds

Pre-operational (PO)
Intuitive, perceptual handling of commutativity and multiplication by 1
Counting 1 to 10
Ability to match objects one-one but inability to conserve number because of perceptual distractions or inability to use transitive logic with numbers

Early Concrete Operational Thinking (EC)
Resisting premature closure (as in 213 + 342 = 342 + [])
Basic Facts (+, -, x, ÷)
Multidigit +, -, x, ÷ without regrouping
Three addends without regrouping
Demonstrating understanding of qualitative compensation with numbers but quantitative accuracy inconsistent
Ability to generalize one-step patterns from the concrete (e.g., commutative and associative properties of addition and subtraction)

Late Concrete Operational (LC)
Two or more steps or operations (e.g., x distributed over +)
Procedures generalized from concrete
Multiplication by 0
Multidigit +, -, x, ÷ with regrouping (multidigit by one-digit division)
Multiple addends with regrouping (i.e., sums > 18)
Accurate quantitative compensation with numbers
Successful systematic relating of two or more facts
Interpreting a story problem

Early Formal Operational (EF)
Multidigit Subtraction with borrowing across zero
Explanation and successful use of multidigit computational algorithms
Multidigit multipliers containing zeroes
Division with quotients containing zeroes
Multiplication by 10, 100,...
Use of "primitive placeholders" (e.g., "x as a raincheck")
Abstract, propositional thinking from concrete elements

Formal Operational (F)
Thinking characterized by formal structure where the elements are abstract (e.g., f - e = [] - f). Operations on relatively large numbers where successive steps require systematic persistence, even if arithmetic closure is feasible, e.g., (900 ÷ 30) ÷ 10 = [] ÷ (30 ÷ 10). (Cornish & Wines, 1978)

## RESEARCH QUESTIONS

The Final Report of the ACLIC Project addressed the following questions:

1) What levels of cognitive ability are demonstrated in mathematics topic contexts by Alberta students in each of Grades One through Six (ages six through eleven)?

2) What are the levels of cognitive demand made on students at each grade level by:

i) the curriculum objectives identified by the Elementary Mathematics Curriculum Guide, Alberta Education, 1982.

ii) The prescribed textual resources,
iii) teacher presentations, and
iv) teacher-made tests?

3) How well do the distributions of curricular demands (made by Curriculum Objectives, texts, teacher presentations, and tests) fit the distributions of student cognitive ability at each grade level in particular methematics topic strands?

## OPERATIONS INTERVIEW TASKS

There were five ACLIC Operations interviews. One of these explored early multiplication concepts and another assessed concepts of place value and multidigit subtraction, as indicated in the following capsule summaries:

Multiplication (Six-, seven-year-olds). Asked to distribute (10) flowers, one-for-one, into (10) vases and then seeing the flowers made into a "spread out" bouquet, the child was asked to distribute another (10) flowers into the vases. These flowers were then placed in a "tightly packed" bouquet and the child was asked: "Are there as many flowers here (spread out) as there (tightly packed). . . or are there more in one of the bouquets?" The child's grasp of the multiplication concepts contained in the task was then assessed by the question: "If we put all of the flowers in the vases with the same number in each vase, how many flowers will there be in each vase?" After a third set of (10) flowers was introduced, the question was repeated.

A response indicating that a child considered one of the bouquets to have more than the other was rated as Pre-operational. Accurate use of one-one correspondence but inability to predict the number of flowers per vase was rated as Early Concrete Operational. Successful prediction of the number of flowers per vase was rated as being at least at the Late Concrete Operational level. (Copeland, 1974, 137-139; Copeland, 1974b, 26-27; Piaget, 1952, 203)

Subtraction/Place Value (Six- to eight-year-olds). This interview began with a game using base ten blocks to ensure that the child had some familiarity with structured units, tens, hundreds, . . . material. Eight-year-olds were then asked: 1) to show 365 with the blocks, 2) to read 699, 3) to write the successor of 699, and 4) to write the number 2 less than 300 [Tasks 3) and 4) were suggested by items in a Chelsea Diagnostic Mathematics Test (Brown, Hart, & Kucheman, 1984, 6)]. Then, the following tasks were posed: 5) 527 - 332, 6) 702 - 25, and 7) 4 002 subtract 25 [6) and 7) were suggested by Davis & McKnight, 1980)] with the child being encouraged to describe what was done, and why, and to use the blocks, if needed.

Correct responses to one or more of 1) to 3), only, were rated as Early Concrete Operational. Correct responses to 4) and/or 5), in addition to the preceding, were rated Late Concrete Operational. Correct responses to 6) or 7) (with some explanation of the procedure, with or without blocks) were rated Early Formal Operational.

## OPERATIONS PAPER-AND-PENCIL TEST

The Operations Test chosen by the ACLIC Team was the 20-item version (for eight- and nine-year-olds) included in the Australian Council for Educational Research (ACER) "Mathematics Profile Series." The items are based on research by Collis

(1972, 1975) and they include student cognitions of commutative associative, and identity properties with elements that are either small or multidigit numbers. The only modification to the ACER Operations test items for the ACLIC study was the elimination of signed numbers (not introduced to Canadian six- to eight-year-olds). The ACER Operations test items are similar in format to: 2 + 5 = 5 + []; 8 x 1 = []; and (4 + 7) + 3 = [] + (7 + 3). (Cornish & Wines, 1978)

PROCEDURES AND FINDINGS: OPERATIONS, SIX- TO EIGHT-YEAR-OLDS

The responses from samples of pupils from typical rural and urban areas in Alberta were assessed as to cognitive level either by means of interviews or a paper-and-pencil test constructed from ACER items, or both.

The provincial curriculum objectives in elementary school mathematics prescribe certain performance expectations for the pupils for whom the curriculum is designed. Interpretations of the curriculum are supposed to reflect similar expectations. If some proportion of the Operations curriculum objectives intended for a given age level makes demands at an Early Concrete Level, for example, it seems reasonable that a comparable proportion of the students should be able to respond at least at that level. One interpretation of Research Question 3) [which subsumes Questions 1) and 2)] could take the form: What is the "goodness of fit" between the (Expected) Curricular Cognitive Demand frequency distributions and the (Observed) Pupil Response frequency distributions within the same topic strand. The results of the chi-square "goodness of fit" analyses comparing distributions of cognitive demands and pupil responses in the context of Operations are summarized in Table 1.

As the distributions of Curriculum and Textbook cognitive demands were taken as the "Expected" frequency distributions for the chi-square "goodness of fit" tests reported in Table 1, the Pre-operational and Early Concrete Operational categories had to be combined to remove Expected frequencies of zero. Consequently, in five of the seven Demand/Response contrasts in which the Null Hypothesis would not be rejected statistically, a significant percentage (10% to 45%) of student responses were at the Pre-operational level while there were no curricular demands at that level. The two exceptions were in the contrasts between the Curriculum and Textbook Demand distributions and the Subtraction/Place Value Pupil Response distributions of the eight-year-olds. In these cases there apears to have been a reasonable match between the curricular demands and the pupil response levels. In the remaining sixteen Demand/Response comparisons there is good reason to claim a mismatch between the Curricular Expectations and the Pupil Response Levels. Lack of provision for students at the Pre-operational level has already been noted. Additionally, the textbook demands in Operations topics for six-year-olds were found to be 91% at the Early Concrete Operational Level, compared with only 19% to 52% of the pupil responses being rated at that level (For seven-year-olds, the situation is similar.). On the other hand, 22% of the eight-year-olds' responses were at the Early Formal or Formal level, as compared with 4% and 8% of the Curriculum and Textbook components making

TABLE 1

CURRICULAR COGNITIVE DEMANDS COMPARED WITH PUPIL COGNITIVE RESPONSES
OPERATIONS, AGES SIX TO EIGHT

| Age 6 (Grade 1) | N | PO | EC | LC | EF | Chi-square Goodness of Fit Test* | |
|---|---|---|---|---|---|---|---|
| Demand Levels: | | | | | | | |
| Curriculum | 6 aims | 0 | 4 (67%) | 2 (33%) | 0 | | |
| Textbooks (2) | 3075 items | 0 | 2788 (91%) | 287 (9%) | 0 | with Curriculum | with Textbook |
| Response Levels: | | | | | | df = 1 | df = 1 |
| Addition | 60 pupils | 27 (45%) | 30 (50%) | 3 (5%) | 0 | 21.68* p<0.001 | 1.33* p>0.2 |
| Subtraction/PV | 60 pupils | 13 (22%) | 31 (52%) | 16 (27%) | 0 | 1.20* p>0.2 | 21.30* p<0.001 |
| Multiplication | 62 pupils | 15 (24%) | 12 (19%) | 35 (56%) | 0 | 14.91* p<0.001 | 162.66* p<0.001 |
| **Age 7 (Grade 2)** | **N** | **PO** | **EC** | **LC** | **EF** | **Chi-square Goodness of Fit Test*** | |
| Demand Levels: | | | | | | | |
| Curriculum | 9 aims | 0 | 7 (78%) | 2 (22%) | 0 | | |
| Textbook | 2248 items | 22 (1%) | 2017 (90%) | 209 (9%) | 0 | with Curriculum | with Textbook |
| Response Levels: | | | | | | df = 1 | df = 2 |
| Addition | 60 pupils | 12 (20%) | 17 (28%) | 31 (52%) | 0 | 30.10* p<0.001 | 362.88* p<0.001 |
| Subtraction/PV | 59 pupils | 0 | 8 (14%) | 50 (85%) | 1 (2%) | 140.78* p<0.001 | 416.38* p<0.001 |
| Multiplication | 59 pupils | 11 (19%) | 11 (19%) | 37 (63%) | 0 | 55.96* p<0.001 | 402.42* p<0.001 |
| **Age 8 (Grade 3)** | **N** | **PO** | **EC** | **LC** | **EF+F** | **Chi-square Goodness of Fit Test*** | |
| Demand Levels: | | | | | | | |
| Curriculum | 25 aims | 0 | 11 (44%) | 13 (52%) | 1 (4%) | | |
| Textbook | 2922 items | 0 | 1700 (558%) | 997 (34%) | 223+2 (8%) | with Curriculum | with Textbook |
| Response Levels: | | | | | | df = 2 | df = 3 |
| Addition | 60 pupils | 6 (10%) | 28 (47%) | 26 (43%) | 0 | 5.45* p>0.1 | 6.14* p>0.1 |
| Subtraction/PV | 60 pupils | 0 | 27 (45%) | 28 (47%) | 5 (8%) | 3.16* p>0.2 | 4.64* p>0.2 |
| ACER Operations | 94 pupils | 2 (2%) | 36 (38%) | 35 (37%)- | 15+6 (22%) | 83.26* p<0.001 | 561.49* p<0.001 |

PO, Pre-operational; EC, Early Concrete; LC, Late Concrete; EF, Early Formal; F, Formal

*Chi-square Goodness of Fit Tests: Each Cognitive Demand frequency distribution (Expected) was compared with each Pupil Response frequency distribution (Observed), after the Expected frequency distributions had been scaled to watch N Observed. (Siegel, 1956, 42-47, 249)

demands at those levels.

## CONCLUSIONS

In general, the findings indicate marked mismatches between curricular demands and pupil demonstrated abilities, with no Pre-operational demands to correspond to up to 45% of the pupil responses, with a shortage of Early Formal and Formal demands for the eight-year-olds, and with an overabundance of Early Concrete demands (which include "Basic Facts of + and -") for the six-and seven-year-olds.

Where such discrepancies are found between cognitive demand and pupil ability, a teacher might choose supplementary instructional resources that begin with additional lower level developmental experiences, such as counting, and then encourage the very able to engage in higher level activities like mathematical investigations that foster the kind of abstract and creative thinking characteristic of early formal or formal operational thinkers.

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# The Design of a National Pre-Vocational and Vocational Numeracy Scheme: The relationship of Psychology to Curriculum Development.

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Summary

The aim of the paper will be to describe a research and curriculum development initiative concerned with the design of a numeracy scheme for people aged 16+. The discussion will concentrate on the scheme

* philosophy and development
* framework and content
* assessment procedures.

The central theme of the paper is the provision of a framework for the teaching/learning process.

## 1. Philosophy and Development

### 1.1 The Philosophy of the Scheme

The Numeracy scheme is based on

i) a hierarchical model of concepts and skills. The non-contextual syllabus provides a structured framework for teachers and learners by giving a series of levels of progression through each topic.

ii) the principle of a Spiral Curriculum. In such a curriculum model an idea is introduced and then study at a later stage develops it further. As the idea is revisited consolidation occurs. The idea can then be developed and revised level by level.

The levels in the scheme provide the skeletal framework; the context in which the numeracy is used provides the flesh.

## 1.2 Background to the Development of the Scheme

The scheme has its roots in two complementary areas:

i) concern of the late 1960s about mathematical performance and attitudes towards mathematics

ii) research studies into the difficulties students experience in learning mathematics.

### 1.2.1 Concern about mathematical performance and attitudes

The scheme under discussion is being developed for the City and Guilds of London Institute (CGLI): Britain's largest technical testing and qualifying body. An independent organisation operating under a Royal Charter, it enjoys wide support from industry and works closely with the education service and Government departments.

The City and Guilds Numeracy scheme stems from the concern of the late 1960s about the apparently poor mathematical performance of craft and technician students and their negative attitudes towards mathematics. In the early 1970s the City and Guilds were approached by schools for schemes for young people, 15/16 year olds in their final years at school. Discussions with representatives from schools and industry resulted in the development of the CGLI Foundation Courses. One of the main aims of these courses was improvement in the basic skills of literacy and numeracy.

### 1.2.2 Reseach studies on learning mathematics at 16+

Rees (1973, 1974) highlighted some of the mathematical concepts and skills which were found to cause particular difficulty. Further, she revealed that lecturers were not aware of the nature and extent of these difficulties. Barry (1974), in a parallel study, observed that a disproportionate amount of time was spent on those elements of the mathematical curriculum which craft students did relatively well.

Furneaux and Rees (1976, 1978) and Rees (1978, 1981) in studies relating to the structure of mathematical ability have suggested that mathematical tasks can broadly be grouped into two categories. Some tasks people will be able to do, depending on their level of general ability, while other

mathematical tasks, which probe understanding, seem to require a more inferential ability.

Rees and Barr (1984) have summarised the findings of over a decade of diagnostic studies involving students age 10+ to 57 years of age and discussed how diagnostic information may be used to help teachers become aware of their students' difficulties.

### 1.3 The Development of the Scheme

The scheme has been developed on the results of the research outlined above and has also been influenced by the Cockcroft report (1982) and the work of the Assessment of Performance Unit (APU), Chelsea, Nottingham, et al. It is designed as a spiral curriculum, emphasising the careful development of conceptual structures based on a hierarchical system of levels; making use of the 'bottom up' strategy for determining the mathematical curriculum, which is promoted by Her Majesty's Inspectorate (HMI), see HMI (1985). This framework can be seen as a diagnostic structure with each level demanding more inference as the conceptual structures are allowed to develop.

The scheme not only aims to provide a foundation of mathematical concepts and skills, but also aims to assist students to gain confidence in their own ability to learn and manipulate mathematical ideas.

The design and aims of the scheme draw on classical ideas from the psychology of learning. The design draws on the ideas of learning hierarchies, cumulative learning theory, etc; while the aims draw on the ideas of positive reinforcement, and so on.

## 2. Framework and Content

### 2.1 A General Description of the Levels

The framework is basically concerned with "levels of content experience" (see table 1).

| Level | Description |
|---|---|
| 0 | **Doing**<br>eg. taking a waist measurement<br>**Knowing**<br>eg. 100 cm = 1 metre |
| 1 | **Simple formalisation**<br>eg. knowing that the area of a rectangle can be found by multiplying its length and width (ie. A = lb) |
| 2 | **Manipulation**<br>eg. given the area of a square find its length |
| 3 | **Further Abstraction**<br>leading to vocational applications |

table 1

## 2.2 A Specific Example of the Scheme Content

A standard syllabus may say:

> Students should be given the opportunity of developing an ability to ... perform the four operations on ... decimal fractions.

This is how this topic could be developed using the model discussed above.

| | level |
|---|---|
| Use the place value concepts for<br>Operations on decimal fractions | |
| Addition and Subtraction involving numbers including | |
| tenths and hundredths (in context only) | 0 |
| tenths, hundredths and thousandths | 1 |
| generalise from here | 2 |
| Multiplication involving numbers including | |
| tenths and hundredths by tens and hundreds (in context only) | 0 |
| tenths, hundredths, thousandths by whole numbers | 1 |
| tenths, hundredths and thousandths | 2 |
| generalise from here | 3 |
| Division involving numbers including | |
| tenths and hundredths by a whole number | 1 |
| tenths and hundrdths by a number including tenths and hundredths | 2 |
| generalise from here | 3 |

## 2.3 Applications of the framework in pre-vocational and vocational contexts

The applications for this numeracy framework will be taken from pre-vocational or vocational contexts. These can vary, for example, from 'life skill' activities to complex engineering/business problem solving situations. In most kinds of problem solving several levels of numeracy may be required.

# 3. Assessment Procedures

## 3.1 Background

There is at present a great deal of concern about the British examination system used at 16+. In Scotland the O grades are being replaced by the Standard grade as a consequence of the Dunning report (1977), while in England and Wales the GCE and CSE examinations will soon be replaced by the GCSE. Current expectations of assessment procedures tend to require a combination of internal assessment and external examination.

There is also work being carried out on the idea of a 'record of achievement' and in particular on graded assessment in mathematics. This work is being carried out in a number of reseach projects; for example, at the Oxford Delegacy (OCEA), National Foundation for Educational Research (NFER), etc.

## 3.2 Assessment Procedures: A New Pattern?

Working from the diagnostic results outlined above (1.2.1) and influenced by the APU, et al. the following assessment procedures are being developed and tested at a specific level: the Foundation level. These procedures incorporate as many as possible of the characteristics of good practice and are as follows.

The pattern is a mixture of internal and external assessment. The external assessment is based on tasks, assignments and a multiple choice examination. The tasks are practical, eg. measuring the width of the door; or formal questions, eg. how many centimetres are there in 3 metres? An assignment may consist of a mixture of practical and non-practical activities in a collection of tasks: testing a combination of competences

given a 'situation'. The tasks and assignments are administered by the teacher when a student has covered a given set of objectives. Teachers are encouraged to devise their own tasks that relate to the topics at the levels being taught. The teacher-devised tasks can be used in conjunction with the externally devised tasks. The teacher records on a Student Record Sheet successful performance on tasks and assignments. This internal assessment is externally moderated. A Profile which highlights the student's achievement can be awarded at the end of the course.

This pattern meets the demands of:

i) using local information about the student obtained during the course (in the form of the student record sheet)

ii) giving a national measure of achievement (in the form of the external examination).

## 4. Final Comment

The feedback we are currently receiving from teachers and students in this feasibility study is that the scheme

* helps students achieve
* gives students and teachers confidence
* makes monitoring progress easier
* has a realistic progression.

Could the psychological theory put into practice in this curriculum development initiative provide the model which will spiral into other subjects?!

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# EXPERIMENTAL STUDY OF PUPILS' TREATMENT OF REFUTATIONS IN A GEOMETRICAL CONTEXT

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**Abstract** : The aim of the present research has been to study on an experimental basis the decisions of pupils confronted, as a result of a problem-solving activity, with a *counterexample to a conjecture* that they have produced. The methode used is that of *situations of social interaction* which consists in two learners confronted with the joint task of finding a solution to a problem. The main result is that the overcoming of a contradiction does not necessarily constitute a progress with regards to scientific or school knowledge, but that it may be legitimate in terms of the *specific economy of the pupil activity*, within the given problem situation.

## I. Introduction

One of the basic ideas behind a certain number of studies in didactics of mathematics is to set up problem-situations by means of which pupils may become aware of a conflict between what they have produced (as a result of their current state of knowledge) and what is actually required. The underlying hypothesis of these studies is that awareness of the existence of such a contradiction will confront the pupil with the necessity of overcoming it. That would constitute the starting point for a progress of his knowledge.

A theoretical analysis of contradiction supplies evidence on two points :

(i) the large range of its possible consequences,

(ii) the determination of a choice among them that is non purely cognitive (Balacheff 1984).

In the context of mathematical activity, and within the theoretical framework of the analysis of Lakatos, the set of possible effects of a counterexample to a given conjecture includes the modification of the conjecture, or of its proof, the modification of the background knowledge of the problem-solver or of its rational basement, or even the rejection of the counterexample itself.

The aim of our research has been to study on an experimental basis the decisions of pupils confronted, as a result of a problem-solving activity, with a counterexample to a conjecture that they have produced.

## II. The experimental approach

### II. 1. The problem

The methode used in order to approach these questions was that of situations which encouraged confrontation of different view-points as to the solution of the problem and hence, a verbal exchange making explicit the refutations, the treatement of the refutations and thus the proof process associated with the solution.

Pairs of 13-14 year-old pupils were required to solve the following problem :

> *give a way of calculating the number of diagonals of a polygon once the number of its vertices is known*

The answer to the question was to be expressed in a message addressed to, and to be used by other 13-14 year-old pupils.

### II. 2. The situation

The communication was only invoked (that is to say : communication is merely projected, the other participants are absent. Mugny,1985, Chap.10). Nethertheless, this situation which we call *situation for communication*, structures the pupils'activity, and more particularly it solicits a verbal formulation of the counting procedure. This is a thing that the pupils do not normally do straight away even if they are technically capable of it. At the same time, the desire to supply a reliable tool to the other group means that the pupil pair pays more attention to their formulation.

Lastly, the two pupils have access to as much paper as they want, but, on the other hand to only one pencil. This condition reinforces the cooperative nature of the situation, while at the same time giving us more direct access to the dynamics of the two confronting knowledge systems; especially in cases of decision making.

The observer intervenes only after the pupils have claimed that they have produced a final solution. At this stage he abandons his stance of neutrality and asks the pupils to deal with counterexamples that he advances. Thus we have access to two phases during the observation, one quasi-independent of the observer and the other with strong observer-pupil interaction.

## II. 3. The field of knowledge

The problem that as been selected calls upon pupil knowledge that is more cultural than scholastic, in so far as that, while polygons may have been studied in the primary school in activities of geometrical classification, they are no longer, for these pupils, a part of the taught curriculum. Sometimes they are mentioned before passing rapidly on to the study of triangles and quadrilaterals. Diagonal, however, is more present in the curriculum, playing an essential role in the study of parallelogram.

Not unlike the case for solids in the 18th century, pupils'conceptions related to the mathematical content concerned are not theorized. In order to solve the problem pupils may have to specify the objects to which it is related. In this experimental study we find ourselves in a context which is favourable to the emergence of processes like those described by Lakatos, and thus to the examination their relevance for an analysis of the behaviour of pupils confronted with counterexamples to a conjecture that they have produced.

# III. Four case studies

## III. 1. Introduction

Observations, which began in 1981, were carried out mainly during the first semester of 1982. 14 pairs were observed a 80-120 minutes period. A research report of the result analysis is currently being written, in collaboration with C. Dupuis (Université de Strasbourg).

The observed problem-solving procedures are closely linked to the meaning given by the pupils to the "objects" *polygon* and *diagonal*, and particularly the interpretation of *regular polygon* for polygon, *diameter* for diagonal leads the pupils to the conjecture $\int(n)=n/2$ (from now on $\int$ will be used to indicate a relation between the number of vertices end the number of diagonals of a polygon). This interpretation is in most cases reinforced by (school) knowledge of the properties of the parallelogram (which is in most case the rectangle or the square). The contradiction profferred by regular polygon with an *odd* number of vertices is overcome in various different ways. We do not intend to make a detail analysis of this here, but it is worth pointing out that these objects are either rejected as polygons or brushed aside as exceptions, or on the other hand, lead to modification of the conjecture by suggesting two different calculations depending on whether the polygons are odd or even.

The three types of solution which "effectively" provided the required number

of diagonals were : $\int(n)=\int(n-1)+(n-2)$ , $\int(n)=(n-3)+\Sigma(n-p)$ , and the classical formula $\int(n)=n(n-3)/2$. The rational basis for these conjectures are most often empirical, but the nature of these proof processes can vary considerably from one solution to another (Balacheff 1985).

We will now go on to examine four cases particularly significant (from now on $P_n$ will be used to refer to a polygon with n vertices)

## III. 2. Georges & Olivier

These pupils, after studying three examples ($P_4, P_5, P_6$) and after making certain counting errors in the empirical drawing of the diagonals conclude that $\int(n)=2\int(n-1)$. Attempts at verification on $P_7$ fail because of its complexity. The pupils then try to organize the drawing of the diagonals, they identify the following property : there are n-3 diagonals at each vertex, from which they deduce $\int(n)= n(n-3)$. On the other hand one of the pupils notice that certain diagonals are obtained twice : " *we can get them like that, but they are there already* " (Geo. 204). The procedure which has been used up to then is then rejected for a procedure where the diagonals are only counted once. They then reach $\int(n)=(n-3)+\Sigma(n-p)$ which describes the procedure for drawing diagonals which they have been systematically using. This last conjecture is validated by a "crucial experiment" run on $P_{14}$.

The observer then advances the triangle which is brushed aside by the pupils as an exception, because its first term is zero the pupils procedure does not enable them to deal with it. When confronted with other counterexamples (non-convex polygons) the first reaction of the pupils is to reject them, but " *that's making a lot of exceptions* " (Oli. 625) they then, unsuccessfully, go on to search for a formula for this class of polygons.

## III. 4. Blandine & Isabelle

After examining two even polygons ($P_4, P_6$) they induce $\int(n)=n/2$. This conjecture is, however refuted by $P_9$. Then the pupils abandon their initial diameter oriented conception in favour of the traditional one. They notice from the study of $P_7$ that there are three diagonals from each vertex and that " *x-3 multiplied by x equals the number of diagonals* " (Isa. 46). But the contradiction between the result of the multiplication (4x7=28) and that of the enumeration (14) on the figure leads the pupils to reconcider their conception of a diagonal. They revert to their previous diameter conception. Their analysis of the contradiction revealed by the odd polygons is that one of the vertices is not matched, they therefore conclude $\int(n)=n/2$ for even polygons and $\int(n)=(n-1)/2$ for odd polygons.

The triangle advanced by the observer is treated as an exception, but non-convex polygons result in the incorporation of a condition in the conjecture.

## III. 4. Hamdi & Fabrice

These pupils use a combinatorics approach to the problem. Without making a distinction between sides and diagonals, they conclude (in the case of $\mathbf{P}_7$) that there will be 7x7=49 diagonals. They do not draw a figure and they affirm that $\mathbf{P}_4$ must have 16 diagonals. The triangle however, blocks them. They reconsider the definition of *diagonal* and with reference to the rectangle they firstly, make the distinction between "side" and "diagonal", and secondly, they insist on the property of intersection of diagonals. However this property is stretched by accepting that diagonals intersect outside the polygon. Three competing propositions appear at this stage : (i) the number of diagonal is "*the number of vertices squared minus [something]*" (Fab.152) which is an adaptation of the initial conjecture, (ii) "*each [vertex] has the number of vertices minus 3*" (Fab.160) which explains that $f(n)=n^2$ gives too high a number (however this property is not otherwise exploited, (iii) "*according to the number of sides of the polygon each vertex has one or several diagonals*" (Ham.159).

Finally, they notice that the diagonals "*go in both directions*" (Fab.184), but advance the conjecture that "*as soon as you know the number of vertices you take your $x^2$ which is the number of diagonals*". In fact it is likaly to be a case of coding confusion of 2x and $x^2$ which is frequent at this school level.

It is important to notice that although the necessary properties for the establishment of the classical formula have appeared they are not used in this way. This formula will be the result of the adaptation of the pupils to successive counterexamples provided by the observer. By means of $\mathbf{P}_4$ he refutes $f(n)=n^2$ which is definitaly rejected. The following formulation advanced by the pupils is " *according to the number of sides of the polygon each vertex has one or several diagonals* " , it is rejected as it does not answer the problem. Pupils therefore advance that *this number* is of n-3 at each vertex, and finally, $f(n)=n(n-3)$; except for triangles which are not considered as polygons, and for quadrilaterals.

To this last conjecture the observer opposes $\mathbf{P}_6$ and this leads to the adaptation $f(n)=n(n-3)/2$ , triangle and quadrilaterals still being excluded. In the end the latter are incorporated into the domain of validity of the conjecture, however triangles "*as they haven't got any diagonals are of no interest to the pupils*" (Fab.553). In order to prove their conjecture the pupils propose " *we'll make a big one [polygon]* " (Ham.462). They begin to construct a

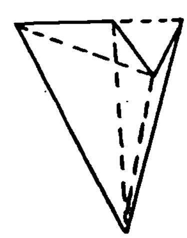
$P_{16}$ which remains unfinished.

The observer advances coupled polygons as counterexamples, the pupils reject these from the domain of validity of their conjecture incorporating the condition : "*it doesn't work when there is a vertex which links several vertices at the same time*" (Fab.692).

### III. 5. Christophe & Bertrand

The diagonal are interpreted according to the traditional meaning. Examination of $P_5$ makes it possible to establish that "*each vertex has two neighbours*" (Ber.123) and in the counting of the diagonals "*these neighbours and the vertex itself*" (Ber.268) are excluded. This means that there are n-3 diagonals for each vertex. This is verified on larger polygons ($P_{10}$, and then $P_7$). The constant number of diagonals at each vertex leads to : $f(n)=n(n-3)$. Verification on $P_7$ refutes this conjecture. For Cristophe this is enough to motivate his rejection of it and its replacement by $f(n)=2n$ which satisfies $P_7$. Bertrand however, taking as a starting point his remark that each diagonal is counted twice, seeks to adapt the conjecture. This leads him to suggest first $f(n)=n(n-3)-n$, but then $f(n)=n(n-3)/2$. Christophe resists this new conjecture strongly and defends $f(n)=n/2$ based only on $P_7$. $P_8$, taken as the decisive case, refutes $f(n)=n/2$ and upholds Bertrand's conjecture. Bertrand justifies this using $P_7$ as a "generic example".

The observer advances the following counterexample which pupils reject showing that a diagonal is in fact hidden by the drawing. However, through lack of time, they were unable to get the end of their treatment of a crossed quadrilateral which they interpreted as a pentagon and in which, again, they sought for hidden diagonals.



## IV. Conclusion

In the present paper we ignore the type of validation employed by the pupils. We merely mention that while frequently bearing the stamp of empiricism, these validations reveal profound differences (cf. Balacheff 1985). We will focus in the present paper on the pupil treatment of contradictions, main aspects of which have been illustrated by the preceding cases studies.

Firstly, and this is essentially confirmation of a widely held view-point, the evolution of the solution procedures that were observed was in close relation to the evolution of the meaning assigned to the concepts by the pupils. An objection that is often brought up is that the result would have been different if the definition of *polygon* and *diagonal* had been given to the pupils. Experiments using this mode are currently being carried out and we are already in a position to say that, fundamentally, the results are not changed : it is the meanings assigned to the definitions which evolve during the resolution of the problem.

Secondly, pupils'behaviour demonstrates the basic category shown by Lakatos as a result of his epistemological analysis. Overcoming a contradiction is a complex process, which, doubtless, is determined by the nature of the conceptions in play, but also by a global conception of what mathematics is (for example, the acceptability or not of the existence of exceptions) or again, by an interpretation of the experimental situation.

The overcoming of a contradiction can be done in different ways, *it does not necesserily constitute a progress* whith regards to scientific or school knowledge, but each may be legitimate in terms of the specific *economy* of the pupil activity, within the given problem *situation*.

Futhermore, the meaning of a solution is in no way independent of the conditions of its production. For example, the step from n(n-3) to n(n-3)/2 can stem from an *ad hoc* adaptation to a contradiction (the opposition of 14 to 28 for $\mathbf{P}_7$) or alternatively from the identification of the reasons underlying the contradiction (the diagonals are counted twice).

The design of a teaching situation based on learner errors must take this complexity into account. Especially to reveal the constraints on the task and on the situation which are liable to orient the treatment of contradictions in a direction compatible with the epistemological aims that are being pursued : in other words, it must be able to orient in such a way as to guarrantee the *meaning* of knowledge constructed in teaching situations.

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# Problem Solving Processes of Gifted Students

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Abstract

The purpose of this study was to analyze the processes used by gifted students in solving word problems. The subjects were the nineteen students in a class for gifted sixth-graders. The procedure was to videotape each student doing word problems. The students were asked to "think aloud" as they worked each problem. Seven problems were selected from seventh and eighth grade textbooks.

Findings of the study indicate that with reading and problem interpretation difficulties grouped together, the percentages of errors falling in each category were almost identical to those found by Ballew and Cunningham in 1982. Students also tended to label the numbers they used in solving problems.

## Statement of the Problem

Purpose. The purpose of this study was to analyze the processes used by gifted students in solving word problems. The study was based on the assumption that these processes could be described and categorized, and that certain patterns of behavior would emerge when gifted students are confronted with mathematical word problems.

Significance of the study. Word problem solving appears difficult for most students to learn and difficult for most teachers to teach. Perhaps the difficulty continues to exist because little is known about how to teach people problem solving skills.

It seemed likely that gifted students might use some thinking, some processes, and some techniques that could be taught to others. It might be that gifted students, in solving problems, do some things so quickly and so automatically that neither they nor their teachers are aware of what processes they go through. It might also be that the types of errors gifted students make could be a source of knowledge leading to their own improvement and to the improvement of others. Less gifted students, by observing the errors of others, could perhaps see some ways to avoid similar errors, and could also gain some psychological sustenance by seeing that they are not alone in having difficulty with word problems.

Questions addressed by the study. This report is on two of the questions addressed by the study:

1. Can analysis of problem solving processes of gifted students yield error patterns that might be useful in helping others learn to avoid errors?
2. Can analysis of problem solving processes of gifted students yield patterns of successful strategies and techniques that might be useful in helping others improve their problem solving skills?

## Background

Several researchers (e.g., Ballew and Cunningham, 1982; Clement, 1982; Knifong and Holtan, 1976, 1977) have studied the errors that students make while solving problems. The work of Ballew and Cunningham (1982) focuses on four types of errors -- computation, problem interpretation, reading, and integration -- that can be diagnosed as a student's main source of difficulty in solving word problems. Clement's study (1982) implies that students who are taught a standard approach to solving problems often retain an intuitive, nonstandard approach which contradicts what they have been taught and can "take over" when students are solving problems. Beginning with the work of Kilpatrick (1967), the method of investigation known as protocol analysis (or think aloud techniques) emerged as a way of studying thought processes in word problem solving. Using this procedure, the investigator asks students to verbalize their thoughts as they work through selected problems. The researcher records these sessions on audio- or videotape and later produces a written transcription of the verbal protocols which are then coded and analyzed. Kantowski (1977) and Mandell (1980) each used protocol analysis to identify successful problem-solving strategies of their subjects. For example, Kantowski (1977) found that certain goal-oriented heuristic strategies were more likely to be used as students developed their problem-solving ability. Webb (1979) used the think aloud method in a study that demonstrated an interaction between conceptual knowledge and problem-solving processes.

## Design

Subjects. The study was carried out in a school system that had already created a special class for the most gifted sixth grade students in the system. All of the nineteen sixth grade students in this special class participated in the study.

Problems. The investigators selected seven word problems from seventh and eighth grade textbooks. Criteria for selection were:

1. Problems were chosen to represent different situations such as one-step problems, multiple-step problems, problems with extraneous information, and problems with insufficient information.
2. Problems were judged to be solvable by most of the students in the study.
3. Problems were judged to be sufficiently difficult to cause some thinking so that students' processes might be observed and analyzed.

A pilot study was done on seventh grade students who had been in the gifted class the previous year, and the problems were judged to meet the criteria. A complete set of the problems used is available from the investigators.

Procedures. The method used was to videotape each individual student doing the problems. The students were instructed to do each problem just as they would any problem, with one exception. The students were asked to "think aloud" as they worked on each problem. The pilot study showed the investigators that some practice might be helpful in getting the students accustomed to the thinking aloud process. The procedure in the actual study, then, was to have each student do two practice problems before the taping began. The practice was followed by a discussion with the students of their thinking aloud. All questions by the students were answered before they began the seven-problem sequence of the study.

The seven problems were given in random order to each student. Instructions were read to each student so that each student would be certain to get exactly the same information. A set of these instructions is available from the investigators.

This procedure yielded approximately nine hours of video tape. The team of investigators, consisting of the teacher of the gifted class, two university professors, and one research assistant, then coded the data and analyzed it into behavior patterns across students within problems and across problems within students. The investigators were looking for patterns of successful strategies and types of errors.

## Findings

Error analysis. The first question addressed by this study was concerned with types of errors made in problems that were missed. Four broad categories of word problem solving difficulties were postulated by Ballew and Cunningham (1982). The names given to these categories, together with their definitions, are:

> Computation: The ability to add, subtract, multiply, and divide whole numbers, common fractions, and decimal fractions.
>
> Problem Interpretation: The ability to set word problems up for solution using the correct operations, and the correct order of operations.
>
> Reading: The ability to read a word problem without help, and to set it up correctly.
>
> Integration: The ability to combine computation, problem interpretation, and reading into the complete and correct solution of word problems.

The existence of these four categories was verified in a field-test of 250 sixth grade students (Ballew and Cunningham, 1982). In that study the percentages of

students having each of these four categories as their main source of difficulty were: Computation, 26%; Problem Interpretation, 19%; Reading, 29%; Integration, 26%.

The nature of the present study made it not feasible to distinguish between the categories of problem interpretation and reading difficulties, so these two categories were combined. Two of the investigators analyzed all of the video tapes, the written transcripts, and the original written work of the students. All problems worked incorrectly were identified and the main source of difficulty was classified as computation, reading/problem interpretation, or integration by the two investigators working independently. Finally, the two investigators compared results and resolved all disagreements in interpretations. The results of this procedure are shown in Tables 1 and 2.

Table 1. Main Source of Difficulty, by Type, for Each Problem

| Type of Difficulty | Problem Number 1 | 2 | 3 | 4 | 5 | 6 | 7 | Totals | % of Errors* |
|---|---|---|---|---|---|---|---|---|---|
| Computation | 0 | 5 | 3 | 0 | 1 | 0 | 1 | 10 | 26 |
| Reading/ Prob.Int. | 1 | 0 | 5 | 3 | 3 | 0 | 6 | 18 | 47 |
| Integration | 1 | 1 | 4 | 4 | 0 | 0 | 0 | 10 | 26 |
| Totals | 2 | 6 | 12 | 7 | 4 | 0 | 7 | 38 | |

* Total is not 100% because of rounding error.

Table 2. Frequency of Incorrect Problems

| Number of Problems Missed | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency | 3 | 3 | 7 | 3 | 3 |

The two investigators, working independently, agreed on the classification of 34 of the 38 errors, for an inter-rater reliability rate of .89. The agreement was 1.00 on the computation errors, .94 on the reading/problem interpretation errors, and .70 on the integration errors. Part of the difference resulted from overlooking errors rather than from disagreement on the classification.

Successful strategies. The second question addressed by this study was concerned with finding successful patterns used by students in solving problems. It was hypothesized that using labels with numbers as students worked word problems would be a helpful strategy.

Two of the investigators analyzed subjects' use of numbers and their labels in all verbal protocols. It was determined that there were two kinds of numbers that appeared in student verbal protocols: 1) problem related numbers, i.e., numbers given in the problem with appropriate labels (oranges, miles, cents, etc.) as well as student-provided answer(s) to the problem, and 2) computational numbers, i.e., numbers from the problem being considered by the students as they translate those numbers into computational procedures. Computational numbers are evidenced by such terms as "added to," "take away," "times," etc. The answers to computational procedures are always counted as problem related numbers. Numbers produced as steps in an algorithm are not counted at all in this analysis.

Both problem related and computational numbers are stated in (or are implied by) the written problem as presented to the students. The difference between them is that problem related numbers are used in rehearsing the information given in the problem, while computational numbers are used to describe the arithmetic procedures being used to solve that problem. The inter-rater reliability for number of numbers per problem per subject was .91 for problem related numbers and .89 for computational numbers. The inter-rater reliability for classifying each number as labeled or not was .99. All discrepancies were resolved by consultation.

This analysis yielded four variables: 1) total number of problem related numbers, 883, 2) proportion of problem related numbers labeled in subjects' verbal protocols, 78.37 percent, 3) total number of computational numbers, 542, and 4) proportion of computational numbers labeled in subjects' protocols, 24.91 percent.

Clearly, gifted word problem solvers have a strong tendency to retain the labels of problem related numbers when rehearsing them in protocols but not when using these numbers in computational procedures used to solve the problems. In addition, gifted word problem solvers rehearse the numbers from a problem more as problem related numbers than as computational numbers.

These same four variables were also determined for each of the seven problems and correlated with the difficulty of each problem as determined by the numbers of subjects missing each problem. These analyses resulted in four correlations. The correlation between problem difficulty and problem related numbers was .881—the harder the problem, the more problem related numbers in the protocols. The correlation between difficulty and proportion of problem related numbers labeled was -.066; in essence, there was no relationship between problem difficulty and tendency to have problem related numbers labeled. The correlation between computational numbers and difficulty was also high, .709, but the correlation between proportion of computational numbers

labeled and difficulty was -.367, suggesting that subjects only labeled computational problems when those problems were relatively easy to solve.

## Conclusions

Error patterns by gifted students in solving word problems were found to be similar to those found earlier in a study of ungrouped sixth grade students. It is believed that students, viewing videotapes of themselves working word problems, can correct many of their own errors.

The technique of labeling numbers during rehearsal (i.e., problem-related numbers) is quite prevalent among gifted students across all problems regardless of difficulty level. Labeling of computational numbers occurs most frequently in the easier problems. Thus, labels seem to help students understand what the problem requires but they may interfere with pure computation. In addition, the high correlations found between 1) problem difficulty and 2) the numbers of problem related and computational numbers in the students' verbal protocols (.881 and .709, respectively) lend support to the validity of protocol analysis as a means of investigating word problem solving.

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# ABSTRACTION AND ACTIVITY

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## 1. *Preliminaries*

The theoretical approach put forward in this paper has two roots. One is the article [4] by von Glasersfeld and the other is an analysis of the relationship of mathematical operations to material actions cf. [3]. The latter one shows that mathematical operations (like addition, differentiation) relate (numerical and/or geometric characteristics of) certain points or states which are passed through in the course of the material actions. For this relating to be possible the respective points or states have to be accentuated which can be viewed as focussing one's attention to them. This relation - established by the mathematical operation - is abstract in the sense that it can not be perceived by our senses. But nevertheless we can become conscious and aware of this relation, we can talk about it and its properties and it obtains a concrete and objective quality in one's consciousness which is reflected in the usual way of defining mathematical concepts. So the natural question arises how to conceive a cognitive representation of those relations and the origin and genesis of it which could explain the phenomenological facts and observations. There the already mentioned acts of focussing one's attention come in. Glasersfeld [4] proposes a view in which units and numbers are cognitively represented by certain complexes of acts of attention. Here a non-formalized modification of this basic idea is used to find an answer to the above asked question.

## 2. *Outline of the theory*

The proposed theory is intended to offer a model for the fact that abstract relations mostly are experienced like objects and are given names which emphasize this object-like character. In short, conceptions of abstract relations as developed by the individual are considered to be cognitive representations of actions (or activities) of focussing or concentrating one's attention. These representations can be conceived like frames in the sense used for instance by R.B. Davis [1]. In such a frame the schema of the actions together with the knowledge of the individual about their applicability is stored. A well developed frame acts rather automatically and in a highly curtailed way such that the action of the frame (i.e. the process) remains hidden for the individual and only the result of the action becomes conscious. In our case this result is the psychological (subjective) existence of the abstract relations whereas the process of focussing the attention may be an unconscious and automatic one triggered by certain signals which call upon the respective frame for action. This does not mean that the result of the action is any kind of product in

our mind but the accomplished action itself is the psychic result which creates the impression of the objective character of the relations. "Attention" in this model is used as a basic term and the ability of man for focussing one's attention is postulated . Thereby the attention can be directed to events in the sensual experience but also to mental objects, imaginated objects and to one's own cognitive processes and their effects. Further - following Glasersfeld - attention is not conceived to be continuous but as a sequence of seperated acts of attention. A kind of demystification of "abstract" is intended by basing abstractness on human activity (cf. [5]), by equating it with cognitive representations of/for this activity and by conceiving existence of abstract objects as a cognitively constructed schema for certain (mental) actions. Though each individual constructs his frames through his activity this activity is essentially a social one and mediated by societal tools. This is the condition for the congruency of frames of different individuals. We shall not dwell upon this social character of activity and mentioning it is only intended to avoid the misunderstanding that an individual just by himself could construct the relevant frames. Further, no ontological questions are discussed concerning for instance the problem of an objective existence of the relations outside of human activity. The general position will be exemplified and expanded in the following examples.

### 3. *Example: Formation of sets*

Objects are given for sensual perception or mental imagination as seperated, single entities. Combining objects into a set establishes a relation and connection between the originally separated objects. The resulting set is an abstract object (though it may consist of material objects) and as such not given to our senses. The cognitive reality (which we feel) of a certain set is now explained as being just the activity of focussing one's attention simultaneously onto the objects of the set. The objects of the set are excised out of the flow of perception (or imagination) by acts of attention (cf. Glasersfeld) and combined to an attentional complex. This complex is formed by directing one's attention ("opening parenthesis") and terminating attention ("closing parenthesis"). Glasersfeld for his enclosing of attention acts has proposed void or blank acts of attention. Like every activity also this one has a motivation and a goal which regulate the activity. This can be for instance: Similarity of the objects (with respect to appearance or to usage), hints by other people (teachers!), enactive, iconic or symbolic hints (circle around the objects on the paper with a pencil or with one's hands, making a list, putting written objects in parenthesis). In all cases the constructed set is given for the consciousness as the respective activity of guiding and focussing one's attention. The abstract set for the individual is just this activity or rather the frame which controls it and has stored the schema of this activity. Such, the abstract set is real since the constructive activity is real.

The focussing and limiting of attention by several acts in an early stage of the development of the corresponding frame will proceed in an unfolded and conscious form. Later on it will get more and more automated, curtailed and unconscious resulting in the mental construction of the respective set: One feels as if the set were "perceived" instantaneously. The curtailment and automation make it possible to recognize, to become conscious of the whole process of the actitivy (like standing outside of it, viewing it from a distance) which then causes the impression of an object - like character of the set. The development of the frame leads from the temporal extension of a process (focussing and guiding attention) to the timeless, momentary impression of recognizing an object. If the objects of the set are material ones (including written symbols) the mental construction of the set is accompanied by a process of perceiving the objects by scanning them with one's eyes. This is the external sign for focussing one's attention in the above described way. Similarly imaginated objects are "scanned" mentally during forming the set. Being confronted with hitherto unfamiliar objects these processes will be unfolded ones and the construction of the set is performed consciously.

So far we have referred to the formation of a single set as an abstract relation. The construction of various sets will develop into a more general frame which then represents the schema of forming sets: Unite several objects by seperating them from the surrounding context (the parenthesis!). There will be a transition to a verbalization of the role of the frame, which regulates focussing attention according to the formation of sets. The meaning of the verbalization stems from the frame and the activity regulated by it. Due to the far reaching schematization and to the verbalization the frame can be applied to arbitrary objects and to infinitely many (ideal) objects. The verbalization apparently can be used (for communication and thinking) almost completely detached from the corresponding frame. Introspection shows that when thinking of the set of real numbers in a way one scans the number line or when thinking of the set of natural numbers one steps through the sequence 1,2,3,... Every formation of a set is an activation of the general frame.

Similar cognitive models can be applied wherever a kind of union, of combining of objects of perception or of thought occurs. For instance, a matrix can be viewed as a certain form of organizing one's attention in a sequence of acts of attention which again as a whole are separated from the context (this corresponds to the parentheses in the denotation of matrices). The acts of attention can be organized by columns or by rows; curtailment gives rise to conceptions like column vector or row vector. The acquisition of the concept of a matrix therefore will be supported if the teacher by gesture, visual and verbal hints tries to guide the attention of the learner. But the corresponding activity has to be carried out by the latter himself!

### 4. *Example: Correspondences and function*

A correspondence (relation) between two or more objects is as abstract as a set. In our model the cognitive representation of a correspondence of A to B can be conceived as follows: There are two acts of attention, the first being focussed at A, the second at B and both separated from the context by void acts of attention. The objects again can be material or mental ones (for instance sets as results of the activity of other frames). A frame "correspondence" will then represent the schema of this complex of attentional activity and the curtailed application of this frame will generate the psychic impression of the existence of the correspondence and of its recognition which in fact is a construction carried out by the frame. External guidance can be given in many ways: Pointing to the objects (or symbols) in the respective order, writing or drawing paper resspresentations in the same order; using arrows or the denotation as an ordered pair; executing actions or operations (cf. the Preliminaries) leading from A to B, for instance transforming A into B: formulas, calculations, movements and the like where A corresponds to the starting point and B to the final point. In the last examples the action will be the means for focussing one's attention first to A and then to B and such it establishes the correspondence of A to B. Probably this is the most common and effective way for constructing correspondences (and also more complicated abstract relations). The action leading in a certain way from A to B (for example calculating B from A) should not be confused with the acts of attention which give rise to the constructive recognition of the correspondence: Carrying out the action does not guarantee the cognitive construction of the correspondence, the guidance of attention may fail. By a set-formation process single correspondences (A to B) can be united to what in mathematics is a correspondence or relation and under additional conditions one obtains a cognitive model for functions. In this set formation the objects are already attentional complexes (represented by frames) onto which now the attention has to be focussed. It is conceivable that this - if iterated - leads to a complex hierarchy of attentional complexes.

### 5. *Example: Ratio of magnitudes*

Let A,B be two rods representing different values of length. Already simple forms of the ratio of the length of the rods like "longer than" are based on certain actions (putting A,B alongside for comparison, cutting to the same length) which direct the attention to the relative positions of the endpoints of the rods. A frame representing "longer than" would comprise schematizations of those actions and of the attentional complex generated by them which on the other hand excises certain states or points in the process of the actions. Focussing one's attention to (the activation of) this frame will then constitute the cognitive reality of "longer than". For the construction of the quantitative, numerical ratio additional actions are

necessary: Dividing B until A=m (1/n B) or copying A and B until nA=mB. The ratio A:B=m:n is then an attentional complex focussing at the configuration obtained by these actions. Again: Ratio as an abstract relationship between A and B is established (for the consciousness of the individual) by certain actions (or mental operations) which guide the acts of attention in the corresponding way.

## 6. *Concluding remarks*

We have put forward the view that cognitive representations of (abstract) relations (conceptions) are frames of/for certain attentional complexes and (systems of) actions. This does not imply that abstract concepts like mathematical ones are taught in a way which permits or fosters the construction of the adequate frames by the learner. The teaching of concepts more often than not is restricted to offering a verbal definition (corresponding roughly to the above mentioned verbalization). This definition usually presents the abstract concept completely detached from its basic activities as a verbal object (in the form of a noun): A set is the union ..., two magnitudes have the ratio ... and so on. It appears to be questionable if beyond this verbal expression at the learner an understanding for and a meaning of the concept will be created. It may well be that the phrasing of the definition activates inadequate frames (attributing some meaning to the nouns) which then results in a biased or incorrect subjective conception. Many empirical results from interviews point into this direction.

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# TOWARD UNDERSTANDING MATHEMATICAL THINKING

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Following Halmos and others it is shown that professional mathematicians characterize mathematical thinking by the insight it provides and the elgance of its reasoning. Using appropriate problems, a diverse selection of college students were found to almost completely lack such mathematical thinking.

Halmos [1968] describes mathematics as "the logical dovetailing of a carefully selected sparse set of assumptions with their surprising conclusions via a conceptually elegant proof" (p.360). To exemplify this characterization, he gives the Tennis Player Problem:

> 463 tennis players are enrolled in a single elimination tournament. They are paired at random for each round. If in any round the number of players is odd, one player receives a bye. How many matches must be played, in all rounds of the tournament together, to determine the winner?

While this problem is rather easily solved, it exhibits a number of features which are relevant for the investigation of mathematical thinking. A problem which is similar in this respect, but very difficult to solve, has recently been mentioned in a colloquium talk by Erdős:

> Given n points in the plane, not all collinear, show that there are (at least) n straight lines, each containing two or more of the n points.

The relevant characteristics these problems have in common are the following: They are easily stated and easily understood. There is more than one "natural" approach to their solution (see Pederson and Polya [1984] for more such problems and how to use them in teaching). One of these solutions is considered by most mathematicians to be particularly elegant. This elegant solution achieves a rather powerful

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result with amazingly elementary methods and, in so doing, provides some deep insight into the subject matter. Problems with these characteristics can be used to illustrate the power and elegance of mathematical thinking. A prime example, representing mathematical thinking at its zenith, is Euclid's proof that there are infinitely many prime numbers. This proof starts by assuming that there are only finitely many prime numbers; it then goes on to show that if P is their product, then P+1 cannot be a composite number; indeed, since all primes divide P, no prime can divide P+1; a contradiction is thus obtained and the assumption must be wrong. This proof exhibits all of the characteristics mentioned above.

It appears to be intuitively clear to mathematicians what constitutes good mathematical thinking. In spite of this, most scholars would be hard pressed to define mathematical thinking in an inclusive and exclusive manner. Loosely stated, mathematical thinking is concerned with the thought processes associated with proving and problem solving; and the assessment of the level of mathematical thinking involves metrics of insight and elegance which are placed upon the process of the problem solution as well as its final product. In spite of the lack of precision in this characterization, several questions immediately arise. For example, is mathematical thinking independent of subject matter? Burton [1984] based her dynamic model of mathematical thinking on four underlying processes which are independent of their contextual setting: specializing, conjecturing, generalizing and convincing. Together with these processes, the dynamics of mathematical thinking become context and subject independent in her helical model. This limits the model to such a high level of generality that it subsumes, together with mathematical thinking as exhibited above, a large class of problem solving activities which are valuable as such but have little to do with mathematical insight or elegance of thought. In other words, the model becomes difficult to verify (or falsify) and therefore is of limited usefulness in practice.

To illustrate this, consider the problem of finding the roots to the equation $x^2 + 4x = 0$; suppose person A approached its solution with the quadratic formula, person B with the factorization $x^2 + 4x = x(x+4) = 0$ and person C proceeded by trial and error. What is their level of

mathematical thinking? Which solutions show mathematical understanding and insight? And what about their measure of elegance? Would your evaluation change if you knew that B was a college math major and C a third grader? How would Burton's model distinguish the three solutions? In particular, how should one characterize B's thinking pattern when restructuring the quadratic expression by factoring? From the point of view of mathematical thinking B's line of action is intimately linked to the context and elicited by the mathematical subject matter at hand. Burton's model seems to deal with generalized problem solving rather than with the mathematical insight and elegance of reasoning which professional mathematicians seem to consider indicative of a high level of mathematical thinking.

Halmos [Albers, 1982] as well Davis and Hersh [1981] make this last point very clear when they stress the poetry and beauty of mathematical reasoning and contrast it with the proof of the Four-Color-Map Problem by Appel and Haken [1977]. This proof certainly constitutes a major mathematical achievement. It is also an example for good problem solving. In fact, it exhibits all aspects of Burton's model. It is, however, lacking of insight and elegance to such an extent that Halmos expressed his desire and expectation to see a four page proof, whereas Davis and Hersh were led to state that "... it wasn't a good problem after all." (p.384)

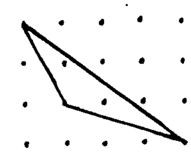
If we take for granted that one of the goals of mathematics instruction is to engender mathematical thinking in the above sense, the question arises whether, in problems of the Tennis Player type, students can see the elegant solution, and which background variables affect their ability to do so. The study reported here aimed to answer that question.

A set of problems were selected according to the characteristics mentioned earlier: easy to understand, and having more than one solution path, at least one of them providing insight in an elegant way. Their solution did not require more than high school level mathematics, often less. Sample questions are presented below:

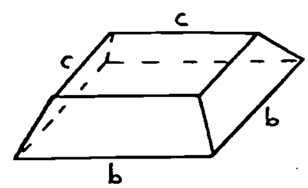
1. Suppose you decided to write down all whole numbers from 1 to 9999. How many times would you have to write the digit 1?

2. Find the remainder when $10^{10}$ is divided by 98.

3. Find the area of the triangle if the grid points are separated by one unit of length.



4. The body in the figure has a square base of length b and a square top of length c. Its height is h. Find its volume.




5. The zeros of a function $f(x)$ are $x = -2$ and $x = 7$. For what values of t is $f(3t) = 0$ ?

6. Find the minimum of the function $f(x) = 2x^2 + 1$.

Several problems were given in written or interview form to each of 32 college level students in the United States and in Israel. The students were classified as weak, intermediate or strong according to their mathematics background, as measured by courses they had taken and grades they had obtained. Their solution processes were analyzed to determine: (i) paths and patterns considered by the student in trying to construct a solution, and (ii) whether the elegant solution path was considered and if so, which background information might have led the student on this path.

It was found that the background information on the students could be ignored. Indeed, regardless of their background, the students (1) rushed toward an answer, often bypassing a rational analysis of the givens, (2) used known procedures uncritically, (3) seldom thought whether alternative routes were available before attacking a problem, (4) did not recapitulate nor check whether their answers were reasonable, (5) did not generalize unless they were explicitly required to do so. As a consequence of (1), (2), and (3), the elegant solutions were almost entirely missing. Moreover, each problem was approached as a separate

entity; similarities in process and intent, such as in problems 3 and 4, were not perceived. The students' background had some influence on their ability to understand the problems (e.g., problem 6) but not on the level of their mathematical thinking.

These observations are discouraging. At the very least, we expected many correct and some elegant solutions. These expectations were not fulfilled. The solution processes we found were procedural rather than conceptual. Attempts to see the "Gestalt" of a problem were lacking, even among the best of the students involved in the study. Elegance of mathematical thought was foreign to them.

If we accept, with Halmos [1985], that "... if you don't know the pure-thought solution of the problem, you are missing a great deal of insight and beauty" (p.16), then the results of this study show that something is terribly amiss in the curriculum. Students' understanding, on the whole, appears to be procedural rather than conceptual. Silver [in press] has argued that in order to diminish procedural errors the conceptual bases need to be improved. How can this be done? We turn, once more, to Halmos [1980] who encourages us to concentrate "... on [students'] ability to get to the heart of the matter quickly, and on their intelligently searching questions..." (p.523). He points out that, at least at the college level, this might be achieved at the expense of covering everything that we want our students to learn. We hope that by refining the notion of mathematical thinking, this study has paved the way for progress toward a type of teaching which promotes insightful, elegant mathematical thinking.

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# THE DESCRIPTION OF ERRORS BY MODELS OF COGNITIVE SCIENCE

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Summary: In clinical interviews, grade 7 students of average abilities worked out some problems on the topic of fractions. The problems were given to the children in different forms of representation. From examples it is illustrated that, even if children are dealing with similiar exercises and even if they had been taught in the same way, their mathematical thinking might be quite different. The differences in the children's behavior seem to come from differences in the internal representation of their knowledge of the concepts and procedures.

To explain why the children handled as they did, a theoretical framework is needed. In this paper, Davis & McKnight's "hypothetical mechanisms" are used to explain children's understanding of fractions. A revised version of Minsky's and Davis' frame-model seems to be especially useful to explain the relationship between the teacher's notions, the children's "concept images" and the real-world aspects of mathematics.

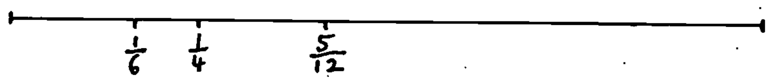
To give an idea of the problems under investigation, some examples will be presented:

Yvonne (aged 13):

1. Here you see a straight line:

At first, mark $\frac{1}{4}$ of the line; then mark $\frac{1}{6}$ of the line. Put these two new lines together. Now what fraction of the whole line have you got?

Yvonne had no problems with this item: By use of the measure on her ruler, she marked the fractions correctly, and she gave the correct answer: $\frac{5}{12}$.




In the same way, she correctly solved two further items (2. a) and b)), presenting the same straight line, but asking for "$\frac{2}{3}$ and $\frac{1}{6}$" and "$\frac{3}{4}$ and $\frac{2}{3}$" of the line, resp.

Let us now look at Yvonne's computations (students' results are indicated by handwriting):

3. Calculate: a) $\frac{1}{3} + \frac{1}{3} = \frac{2}{6}$ b) $\frac{2}{3} + \frac{2}{3} = \frac{4}{6}$ c) $\frac{3}{7} + \frac{4}{7} = \frac{7}{14}$

4. Calculate: a) $\frac{1}{4} + \frac{1}{8} = \left( \frac{2}{8} + \frac{1}{8} = \frac{3}{} \right) \frac{2}{12}$

(Yvonne put her first attempt to solve item 4. a) in parenthesis, and said: "Now I've made a mistake."

Interviewer: "Why?"

Yvonne: "There is no need to change them, one can simply add here.")

b) $\frac{2}{5} + \frac{2}{15} = \frac{4}{20}$ c) $\frac{2}{5} + \frac{5}{7} = \frac{7}{12}$ d) $\frac{5}{10} + \frac{25}{100} + \frac{125}{1000} = \frac{155}{1110}$

5. Calculate: $\frac{1}{4} + \frac{1}{6} = \frac{2}{10}$ $\frac{2}{2} + \frac{1}{6} = \frac{3}{9}$ $\frac{3}{4} + \frac{2}{3} = \frac{5}{7}$

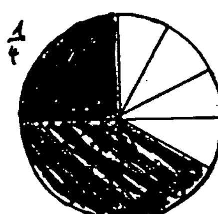
Although from items 1. und 2. it becomes clear that Yvonne had a correct idea of fractions, in all computational questions she consciously used the rule "numerator plus numerator, denominator plus denominator". She used this rule even in item 5., where the same fractions were given as in items 1. und 2. When comparing the answers of these items she believed that "both answers" - these on 1. und 2., <u>and</u> those on 5. - were correct, and there was no "cognitive conflict" at all in the girl's mind.

<u>Andree</u> (aged 13):

Andree solved all computational questions in items 3., 4. and 5. correctly, he had no problems at all with adding fractions by using a rule. The questions in his items 1. and 2. were a little different from those Yvonne was asked:

1. Shade in first $\frac{1}{4}$ of the circle and then $\frac{1}{6}$ of the circle as well. What fraction of the circle have you now shaded in altogether?

$(\frac{1}{4})$ $\frac{1}{4}$

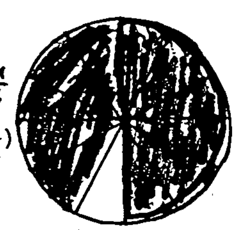


$2\frac{1}{2}$ $(2\frac{1}{2})$ $\frac{1}{6}$ $(\frac{1}{6})$

2. a) The same question with $\frac{1}{6}$ and $\frac{1}{8}$.
   b) The same question with $\frac{3}{4}$ and $\frac{1}{6}$.

a)

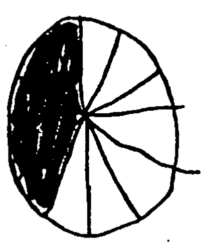
$3\frac{1}{2}$ $(3\frac{1}{2})$ $\frac{1}{6}$ $(\frac{1}{6})$



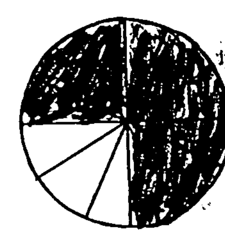
$\frac{1}{8}$ $(\frac{1}{8})$

b)

$\frac{1}{6}$ $(\frac{1}{6})$



nicht lösbar (not solvable)



$\frac{3}{4}$ $(\frac{3}{4})$

Neither in the computational questions nor in the diagrams is it really a problem to realize <u>how</u> the children came to their solutions, i.e. it is no problem to reconstruct their strategies. However, it would be much more interesting to know <u>why</u> the children answered in that way. To

explain children's mathematical behavior, of course the mathematical concepts or procedures which are usually difficult for children must be known precisely, and also the teaching methods which were used to teach them these concepts and procedures must be known. But knowing these facts is not enough: Yvonne and Andree were dealing with equal or similiar exercises, and they had been taught in the same way. Nevertheless, their dealing with the items is completely different. As there had been no formal instructions on fractions since at least half a year, the differences in the children's mathematical behavior cannot be explained as a result of actual instructions; instead we may assume that these differences come from differences in the internal representation of their knowledge of the concepts and procedures in question. Therefore, a theoretical framework is needed to explain their behavior.

What should such framework look like?

Models or theories should

- begin with children's individual ideas, conceptions and misconceptions, and not with the teacher's notions or those in the textbooks; because from the interviews it becomes clear that children's "concept images" are often quite different from the "concept definitions" the teacher has in mind (cf. Vinner and Hershkowitz 1983);
- distinguish between a child's knowledge about some mathematical content and his or her ability to use this knowledge (cf. Skemp 1979a, who uses the terms "knowing that", "knowing how" and "being able");
- enable us to speak about children's individual ideas, i.e. they should provide a language or a set of metaphors such that we are able to make assumptions about the kind of representation of knowledge in the children's minds.

To be brief: Even if a child's behavior seems to be non-sensical or contradictory, the models should enable us to describe this behavior in such a way that - from the child's point of view - it becomes consistent.

Of course, there are no models or theories satisfying all these conditions. However, there are models that might be regarded as steps in this direction: for example the van Hiele levels (1976), some models of understanding from Skemp (1979b) or Herscovics and Bergeron (1983), Vergnaud's "cognitive theory" (1981, 1982), Davis and McKnight's "hypothetical mechanisms" (1979), and the concepts of "microwords" or "do-

mains of subjective experience" (Lawler 1981, Bauersfeld 1983; see also Kilpatrick 1985).

In my opinion, Davis and McKnight's model of "hypothetical mechanisms in mathematical thought" seems to be a most promising approach, especially concerning the three mechanisms "procedure", "visually-moderated sequence" and "frame". Even if the frame-notion seems to be a little vague in Davis' papers, I think that it is possible to characterize it more precisely (cf. Hayes 1980, Hasemann 1983, 1984):

The actual concepts built up in a person's mind are known to be individual and personal interpretations that might be quite different from the corresponding "objective" concepts and notions often cited in textbooks. Frames are regarded as the representations of these individual concepts in a certain language. Frames might differ from person to person in two ways: not only the actual concepts in the persons' minds might be different, but also the kinds of representation of these personal concepts. Regarding this second aspect, for example when reading item 1. (in the circle-version mentioned above) one child might "see" a quarter of a circle as a holistic image and, to solve the problem, might shade in a part of the given circle such that the fraction on the paper and the fraction image in mind become equal; another child might remember how to construct any fraction by dividing the whole circle in as many parts as the denominator names and taking as many of these parts as the numerator names; a third child might realize that the circle is already divided into 12 parts and take 12 : 4 = 3 twelfths of the circle for $\frac{1}{4}$. In these cases, an appropriate concept image of $\frac{1}{4}$ is actually represented in the children's minds, but in different ways.

In fact, this revised version of the frame model enables us to describe Yvonne's and Andree's conceptions of fractions in such a way that their dealing with the items which was mentioned above becomes consistent:

Both children had an idea of adding fractions by calculation. When reading the "+"-sign, a special program was retrieved by the children that told them what to do. They worked out the procedures belonging to this program without any further consideration (except Yvonne who considered the "new" type of fractions in item 4. for a moment, but then she realized that - relating to her program - there was no difference between items 3. and 4.). So there was on the one hand a frame (the "calculation-frame") in the children's minds related to the addition of fractions by calculation. On the other hand, in each child there was an-

other frame related to the diagrammatical problems: in items 1. and 2., Yvonne marked all fractions on the straight line correctly, and she gave correct answers. But she did not realize that the different answers in items: 1., 2. and 5., resp., were contradictory. In the interview, she pointed out that "both answers" were correct, and she was right from her point of view: the answers she had got by drawing and those she had got by calculating came from procedures which belonged to different frames, and as these two frames were not connected in her mind there has no reason at all to be unduly concerned by different answers.

With Andree, it is the same case: he also retrieved different frames related to the calculations on the one hand and the diagrams on the other. But his frame related to the diagrams is worth noting: regarding simple fractions like $\frac{1}{4}$ or $\frac{3}{4}$, he retrieved holistic images, but regarding more "difficult" fractions like $\frac{1}{6}$ or $\frac{1}{8}$, he refered to some concept of whole numbers such that for example the fraction $\frac{1}{6}$ equaled "$\frac{1}{4}$ plus 2 sub-units of the circle" (whereby the sub-units were the twelfths the circle was divided into). There were a lot of children who identified "$\frac{1}{n}$ of the circle" with "n parts of the circle", and - correspondingly - "$\frac{1}{n}$ of the line" with "a length of n cm"; i.e. this "sub-unit frame" was very common in the sample of children we interviewed.

Obviously, these children had not yet built up an appropriate concept of fractions in their minds, or, to be precise: they had built up different concepts related to the different aspects of the concept of fractions, and these concepts were not yet connected in their minds. So when they obtained different solutions from their different frames, there was no reason for a cognitive conflict. However, later on in the interview, Yvonne was deeply affected by a strong conflict when she realized that her two different answers to a problem taken from everyday life could not be accepted, and this realization led to a violent fit of coughing which forced us to stop the interview.

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# GRAPHIC TOOL FOR CALCULUS

## Drawing and using by students in physics problem-solving situation

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## I. INTRODUCTION

Graphic representation in mathematics, specially good interpretation of cartesian graphs by students, is an important aim in mathematics education. But in applying mathematics to physics students often have considerable difficulties. Our goal is to analyse drawing and using graphic tools for calculus by students who are solving problem in the context of a subject matter they have studied (here electrostatics).

Graphic tool for calculus is understood as a graphic representation, that is a medium of intellectual operations in problem-solving processes. The graphic tools have perceptive characteristics - related to the vision and the drawing of the graphic tool - and operational features related to methods of calculation used to obtain the required results. Perceptive or operational characteristics can help or impede the student's demonstration.

We present here two studies.

A. Students have to draw and use a graphic tool for calculus after reading a problem statement including no diagram. We analyse the produced graphic tools to find the electric field and potential of a charged disk and the dipole ones. It is possible to exhibit a standard graphic tool of calculus from a corpus of 17 anglo-saxon and french text-books, frequently used by students. The elements represented on the standard diagram have been grouped in two categories : Basic Elements : those used frequently - more than 60 % - by experts, and Additional Elements.

Each element can be characterized in terms of orientation element, calculation element, element belonging to required relations, mentioned element in the problem statement, etc... or by its vectorial or scalar nature.

B. A diagram is given with the problem statement and students have to complete it. Here it is more judicious to examine the groups of Added and/or Omitted Elements with respect to the given diagram.

In this type of study we assume that the student difficulties with graphic representation cannot be simply attributed to an inadequate preparation in mathematics : often students lack the ability to apply the skills they have learned in their mathematics course to a new scientific domain. Also this is a descriptive study of difficulties students encounter in performing graphic representation and using it in a problem-solving situation. For this we analyse the exam papers of 347 1st year university students.

## II. GRAPHIC SYMBOLIZATION FOR THE CHARGED DISK PROBLEM

For this problem, the experts use two standard graphic tools for calculus which depend on if the electric field is firstly requested (Fig. 1a) or secondly after a potential calculation (Fig. 1b).

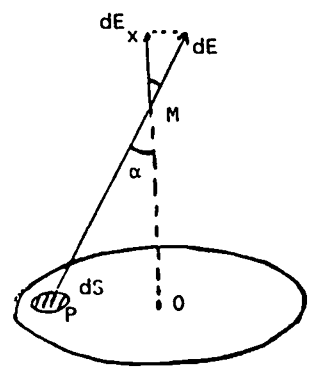


Fig. 1a

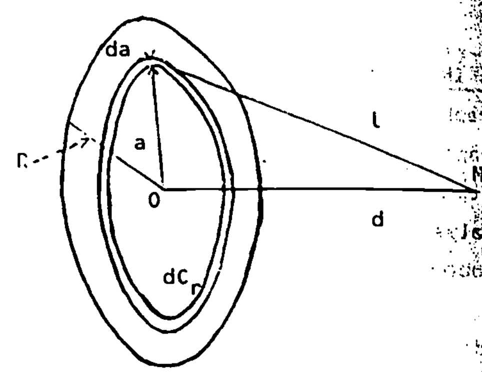


Fig. 1b

We examine only Figure 1. From the corpus and the problem statement, the Basic Elements can be categorized in :

- mentioned basic element {M,O} ;
- basic element of current position {P}
- calculation basic element $\{dS, \alpha, d\vec{E}, d\vec{E}_x, l\}$.

In the same way Additional Elements of the standard graphic tool are categorized in mentioned additional elements $(E, \sigma)$, additional elements belonging to required relations $\{\theta, d, R\}$ and additional elements of calculation $\{d\Omega\}$.

The papers are divided in two classes : the papers of 65 successful students with marks above the average and the 61 unsuccessful student'ones. The data from students papers are shown in Table I, where the frequency of an element present on the diagram is given for each student class. To compare students with experts we use this frequency and sometimes calculate the standard deviation σ centred on the average frequency for all students.

| Table I | dS | α | P | M | $d\vec{E}$ | $d\vec{E}_x$ | O | l | Other Elements | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Basic Elements | | | | | | | | | |
| "Successful" class | 71 | 75 | 17 | 78 | 91 | 57 | 78 | 63 | 60R,35θ,(30a,3da), 35d, | 9σ, 3E |
| "Unsuccessful" class | 34 | 52 | 11 | 74 | 62 | 34 | 57 | 44 | 46R,33θ,(36a,11da),46d,18$dC_r$, | 10σ, 5E |
| "All students" class | 53 | 64 | 14 | 76 | 77 | 46 | 68 | 54 | 53R,34θ,(35a,7da), 40d,9$dC_r$, | 10σ, 4E |
| Experts | 100 | 86 | 100 | 86 | 100 | 57 | 100 | 71 | 29R,29θ, ,29d, 29dΩ, | 43σ,43E |

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This table can be completed by some remarks, specially on perceptive features of the graphic tool : for example 106 students reproduce the expert configuration associated to the first calculation of the electric field (Fig. 1).

The analysis of the Table I leads to the following conclusions :

All students forget the element of current position (P) and figure frequently the elements mentioned in the problem statement. Unsuccessful students fail to represent the calculation elements necessary to perform them.

Successful students and experts have practically the same system of basic elements. However their additional element systems are different. For example students use the inadequate calculation element {a} whereas this is never used by experts.

The unsucessful students use a system of Basic Elements only centred on elements mentioned in the statement, {M,O} ; their system of Additional Elements is composed of calculation elements ($a$,$da$,$dC_r$,$dE$) adequate or not, and elements belonging to required relation ($\theta$,$R$,$d$).

In conclusion, students and experts do not use exactly the same elements in a graphic tool of calculus. 25 (among the 65) successful students and 33 (among the 61) unsuccessful students produce a wrong graphic tool. For the two kinds of students the rate of successful (or unsuccessful) solving is related to a correct and complete (or incorrect and incomplete) diagram.

For the unsuccessful students, the primary source of wrong graphic tool of calculus comes from perceptive features (frontality and incorrect perspective). Operational features as source of wrong graphic tool have the same frequency in the two classes.

## III. GRAPHIC SYMBOLIZATION FOR THE DIPOLE PROBLEM

The standard graphic tool of calculus is shown on Fig. 2. The Basic Elements are grouped in

- mentioned basic elements {±q,ℓ} ;
- basic elements of position {O,P,M,N} ;
- basic elements of calculation {$r_1$,$r_2$,H} ;
- basic elements belonging to required relations {$\theta$,r}.

In the same way we have mentioned additional element {$\vec{E}$}, additional element of orientation {x}, additional element belonging to required relations {$\vec{E}_r$,$\vec{E}_\theta$}.

Among the students who draw a diagram, 60 students success and 32 fail. The table II gives the corresponding data.

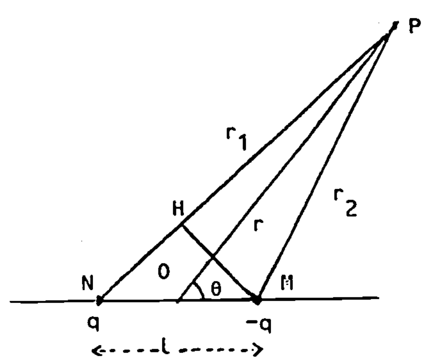


Fig. 2

| Table II | Basic Elements | | | | | | | | | Other Elements |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\theta$ | r | $r_{12}$ | $\pm q$ | H | l | O | P | MN | |
| "Successful" class | 92 | 93 | 87 | 98 | 12 | 77 | 70 | 78 | 67 | $52(\vec{E}_r,\vec{E}_0)$,33E,$32\vec{u}_r$,$17\vec{u}_\theta$,13x |
| "Unsuccessful" class | 75 | 88 | 88 | 81 | 3 | 69 | 59 | 88 | 44 | $13(\vec{E}_r,\vec{E}_\theta)$, ,$13\vec{u}_r$, $9\vec{u}_\theta$, ,$28(\vec{E}_q,\vec{E}_{-q})$ |
| "All students" class | 86 | 91 | 87 | 92 | 9 | 74 | 66 | 82 | 59 | $38(\vec{E}_r,\vec{E}_\theta)$,24E,$25\vec{u}_r$,$14\vec{u}_\theta$,12x,$16(\vec{E}_q,\vec{E}_{-q})$ |
| Experts | 100 | 91 | 91 | 100 | 82 | 64 | 100 | 100 | 91 | $36(\vec{E}_r,\vec{E}_\theta)$,36E, ,45x |

Some remarks on perceptive features of the graphic tool complete the table : all students use the disposition (segment MN horizontal) of the standard graphic tool and the majority associate it to the procedure of an expert's calculation.

The presence of calculation and orientation elements has a low frequency. The mentioned elements and the ones belonging to required relations are highly represented on the graphic tool. The "successful" and "unsuccessful" classes represent differently the angular and orientation elements. The two classes are similar in representing the elements belonging to required relations and elements mentioned in the problem statement.

The system of Basic Elements is not the same for experts and students. The comparison with the expert's one shows that successful students add elements belonging to required relations $(\vec{E}_r,\vec{E}_\theta)$ and unsuccessful students neglect the position elements (M,N).

Students have a system of Additional Elements different from the expert one : they add other orientation elements $(\vec{u}_r,\vec{u}_\theta)$. Moreover the unsuccessful students add inadequate elements of calculation $(\vec{E}_q,\vec{E}_{-q})$.

## IV. GRAPHIC SYMBOLIZATION ON A GIVEN DIAGRAM

The problem was to find the electric field and potential at the point A for a given distribution of charges. In the statement a diagram (Fig. 3) is given. The problem statement and student copies examinations enable us to give the lists of Added Elements and Omitted Elements (Table III).

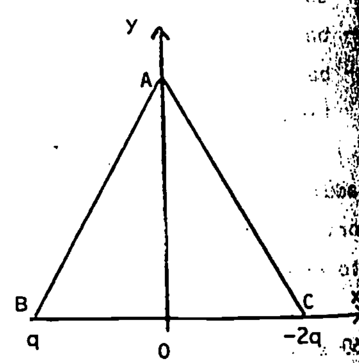


Fig. 3

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| Table III | Added Elements | | | | | | | | | | | | Omitted Elements | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\vec{E}_C$ | | $\vec{E}_B$ | | $\vec{E}_A$ | | $E_{Ax}$ | $E_{Ay}$ | AX | α | a | θ | A | B | C | O | q | −2q | $O\vec{x}$ | $O\vec{y}$ |
| | → | ‖ ‖ | → | ‖ ‖ | → | ‖ ‖ | | | | | | | | | | | | | | |
| "Successful" class | 95 | 90 | 95 | 90 | 88 | 88 | 7 | 7 | 73 | 49 | 49 | 80 | 10 | 7 | 7 | 29 | 7 | 7 | 22 | 17 |
| "Unsuccessful" class | 92 | 83 | 94 | 85 | 81 | 79 | 13 | 13 | 52 | 31 | 48 | 38 | 4 | 6 | 6 | 37 | 8 | 8 | 25 | 25 |
| "All students" class | 94 | 86 | 95 | 87 | 84 | 83 | 11 | 11 | 61 | 38 | 48 | 57 | 6 | 6 | 6 | 33 | 8 | 8 | 24 | 22 |
| σ | 1.5 | 3.5 | 0.5 | 2.5 | 3.5 | 4.5 | 2 | 2 | 10.5 | 9 | 0.5 | 21 | 3 | 0.5 | 0.5 | 4 | 0.5 | 0.5 | 1.5 | 4 |

The analysis of this table leads to the following conclusions. The shape of the given diagram is faithfully reproduced. The best symbolized elements are the vectorial entities, the mentioned elements in the problem statement and elements of required relations. The less frequent symbolized elements are the scalar entities, the orientation elements and those which are not given in the statement. There are important differences on the way students figure orientation and angular elements. Unsuccessful students better represent lenghts than angles.

## V. COMPLEMENTARY STUDY

In this study students had to produce a graphic tool simple, but not necessary taught in standard physics course. It was for the calculation of field and potential of a uniformly charged spheric corona. The analysis of 33 copies suggests the following conclusions.

The elements of shape, which limit the domain of a physical system (here the two spheres) are figured with high frequency. Vectorial elements (the electric field $\vec{E}(r)$) are highly figured. However, students with low abilities are unable to draw vectorial entities at a current point : they represent them only on limiting surfaces. The space separation by a surface which limits a physical system introduces in the student mind that all quantities including the electric potential are non-continuous through the surface.

## VI. CONCLUSIONS

In drawing graphic tools for calculus and its use, students reproduce faithfully, in a large extent, the profile disposition (direction of symmetry axis, nature of integration element, dS or $dC_r$) of the graphic tool of experts when it exists ; and they associate the calculation method of experts. But the source of unsuccess comes from a mesure of calculation elements and failing students used principally mentioned elements in statement problem or elements belonging to formulae ; and then after, more or less adequate calculation elements. Perceptive features are primary source of student wrong graphic tools for calculus : the failing students use incorrect perspectives like frontality

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# AFFECTIVE INFLUENCES ON MATHEMATICAL PROBLEM SOLVING

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Recent research has made substantial progress in characterizing the cognitive processes that are important to success in mathematical problem solving. However, the influence of affective factors on these cognitive processes has yet to be studied in detail. The purpose of this paper is to propose a theoretical framework for investigating the affective factors that help or hinder performance in mathematical problem solving.

When students are given a non-routine mathematical problem to solve, their reactions express a lot of emotion. If they work on the problem over an extended period, the emotional responses frequently become quite intense. Most students will begin work on the problem with some enthusiasm, treating it like a puzzle or game. After some time, the reactions become more negative. The students who have a plan to solve the problem may get stuck trying to carry out the plan. Usually they become quite tense, and often try to implement the same plan repeatedly, getting more frustrated with each unsuccessful attempt. If the students obtain a solution to the problem, they express feelings of satisfaction. If they do not reach a solution, they may angrily insist on help so that they can reduce their frustration.

Despite these rather intense reactions by students, curriculum developers have generally neglected affective issues in problem solving. Most instructional material on problem solving does not deal with affect in any substantive way.

Given the intensity of the emotional responses, it is a bit surprising that research on mathematical problem solving has not looked into affective issues more seriously. Occasionally researchers do provide anecdotal information on affective issues (Schoenfeld, 1983), and some studies have gathered data on affective factors like confidence in problem solving (Charles & Lester, 1984). However, most research on problem solving ignores affect, and no one has attempted to develop a theoretical framework for research on affect. The role of affect remains a major underrepresented theme in research on problem solving (Silver, 1985).

Although most research and development efforts pay little attention to affect, an exception to this rule is the book Thinking Mathematically, by Mason, Burton, and Stacey (1982). They treat affect in a systematic way, and suggest techniques for dealing with the frustrations of being "stuck". They also emphasize the satisfactions of problem solving, such as the joy of making up your own conjectures.

## AFFECTIVE ISSUES AND COGNITIVE PSYCHOLOGY

Research on problem solving needs a theoretical framework for affective issues that is compatible with the current emphasis on cognitive processes in problem solving. Mandler's work on "mind and emotion" (1975, 1984) provides a sound foundation for such a framework.

Mandler is a well-known cognitive theorist who also works on affective issues. He emphasizes the role of interruptions in an individual's plans or planned behavior as a major source of emotion. In psychological terms, these plans result from the activation of a schema that produces an action sequence, and this sequence has "a tendency to completion" (Mandler, 1984, p. 173). When the sequence is interrupted, the normal pattern of completion cannot occur. The result of the blockage is the physiological arousal of the individual. This arousal might be observed as muscle tension or rapid heartbeat, for example. Along with the arousal, the individual evaluates the meaning of the interruption, and interprets it as surprise, frustration, joy, or some other emotion. Mandler's analysis of emotion seems particularly appropriate for mathematical problem solving. A problem is typically defined as a task where the goal is not immediately attainable, and where there is no obvious algorithm for the student to use. In other words, the problem solver is blocked. These blockages are a central part of mathematical problem solving, and constitute a major source of affect in problem solving.

## DIMENSIONS OF A THEORY

Building on Mandler's work, the following major dimensions of affective factors in problem solving are proposed:

1. Magnitude and Direction. Affective influences vary in their magnitude (or intensity) and direction (positive or negative). The emotions observed in problem solving (frustration, joy) are generally more intense than the attitudes that research on the affective domain has usually addressed. Although we tend to emphasize the negative aspects of affect (e.g., the frustration of getting stuck on a problem), more positive emotions (such as the satisfaction of finding a solution, or the joy of making a conjecture) are also powerful influences on problem solving.

2. Awareness and Control. Problem solvers are frequently unaware of the affective factors that are influencing them (Kilpatrick, 1984). By bringing the emotion (e.g., frustration) to consciousness, students can become aware of its influence and bring it under control. The role of unconscious, as well as conscious, processing of information is important. The tendency of mental processes to become automatic (and hence unconscious) is well known, and seems to apply to the affective domain as well as the cognitive. It seems clear, for example, that some problem solvers automatically decide to quit whenever their first attempt at a solution is unsuccessful.

3. Types of Cognitive Processes. Affective reactions may have different influences on different cognitive processes. Metacognitive and managerial processes seem particularly susceptible to the influence of the emotions. Also, the influence of affect is different at the entry or attack phase of problem solving than it is during the review phase. During the entry phase, the most typical emotion is frustration, where an inexperienced problem solver repeatedly uses the same plan and repeatedly finds that the path to a solution is blocked. During the review phase, the problem solver typically feels satisfaction with finding an answer--any answer--and then neglects to check the answer for reasonableness or to look for more elegant solutions. Thus we see that the emotions (negative in one phase, positive in the other) can keep students from performing at their best in problem solving.

4. Types of Instructional Environments. The influence of affect appears to be quite different in small-group, rather than individual, instruction. It will also vary according to the amount of guidance provided by the teacher, the availability of computers or other laboratory equipment, the kinds of assessment that students expect, and many other factors.

## SUMMARY AND IMPLICATIONS

In summary, we need to investigate affective influences on problem solving, including their magnitude and direction, students' awareness and control of them, and their influence on cognitive processes in various instructional settings. These investigations should provide a theoretical structure with a sound basis in cognitive psychology that can provide direction to research and development in the teaching of mathematical problem solving.

The development of such a theory has important implications for mathematics education. It should help to improve instruction in mathematical problem solving for all students, especially for those who find problem solving particularly stressful. Moreover, research on affective issues should help explain the persistent and troubling differences between boys and girls on problem solving tasks. Previous efforts to explain these differences have concentrated on factors like spatial ability, course enrollment patterns, and biological issues. None of these approaches has been very helpful in explaining sex differences in problem solving achievement. More detailed investigations of affective issues appear to be a more promising approach to determining the sources of these differences in performance (Reyes, 1984).

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# IDENTIFYING AND DISSEMINATING THE TRAITS OF "NATURAL" PROBLEM SOLVERS

by

Donald M. Peck and Stanley M. Jencks

In the course of researching children's conceptual understandings of mathematics, a small number of "natural" problem solvers have been identified. These students possess concrete referents for the processes and symbols of mathematics and are able to determine for themselves, on the basis of these referents, whether or not their reasoning is sensible. Helping children acquire a concrete system of referents in an altered psychological setting appears to increase the number of students who can solve problems and think mathematically.

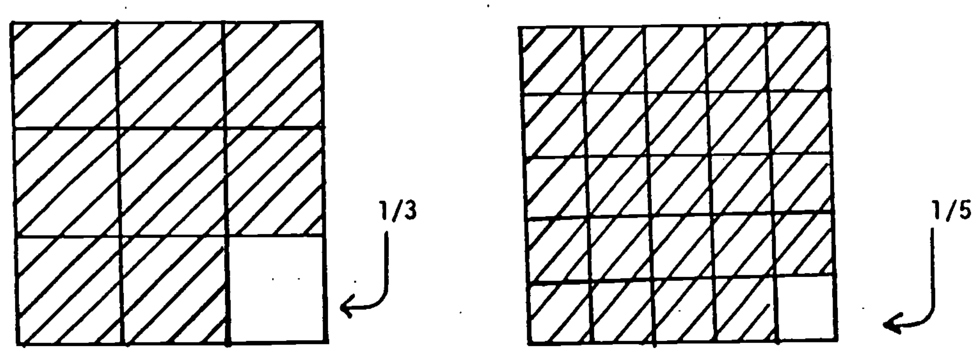
The authors have spent considerable time investigating children's conceptual understandings of mathematical ideas. Occasionally they come across students who view mathematics and mentally organize it in ways that are conducive to mathematical thinking and problem solving (4). The thought processes of these "natural" problem solvers are markedly different from the majority of students, even though the other students may be considered equally capable in most other respects (1,4). The ability of these few problem solvers to attack and solve problems seems to have been developed independently since most other students do not possess the same insights and characteristics of strategy. Recent work with third, fifth and sixth grade children suggests that other students can acquire some of the same characteristics possessed by "natural" problem solvers if the approaches and psychological settings of their mathematical instruction are altered. The purpose of this paper is to outline the research approach which identified "natural" problem solvers, to discuss several of their desirable characteristics, and to suggest some conditions necessary to promote the same traits in other students.

The method of research utilized interviews with students in an attempt to determine the depth of students' understanding of mathematical topics. It was not sufficient for a child to show he had a pattern or a set of correct rules for performing a required task. The researcher inferred understanding only if a child could respond to the required task without using rules or patterns. This is because many children use rules or patterns they have superficially learned without being able to connect them to their experience in a meaningful way (3,4). Hence, if a child used a rule or pattern, the interviewer tried to

determine if it was a shortcut of convenience or one of necessity, and what mental constructs it rested upon.

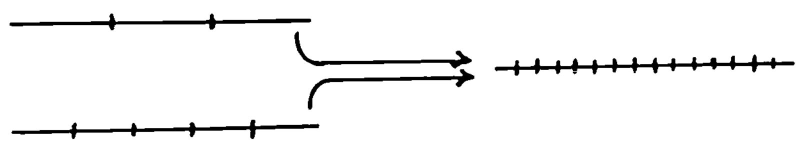
As the investigators explored the conceptual understandings of students, they found that "natural" problem solvers have concrete referents or models with which they gauge the correctness of their reasoning (4). Thus, these students have a way to assume ownership over the issue of correctness of their work. From common sense specifics, the "natural" problem solvers construct broad generalizations and mental structures which they use to reason about fundamental mathematical ideas. On the other hand, the majority of students seem unable to relate the symbols and operations of mathematics to personal experience; they are left to struggle with abstract rules without a foundation to make the rules sensible (3).

Consider some excerpts from interviews with bright ninth graders, all rated in the upper third of their class by standardized examinations, teacher judgement, and past performance. A boy in his second year of algebra correctly solved the problem 2/5 + 1/3 = . The interviewer tried to uncover the mental constructs this student had for fractions. When asked to show what 1/3 and 1/5 would look like, the student drew the figures below:




This student viewed mathematics in terms of rules and processes. Although classified as an advanced student, his conceptual understandings were not conducive to attacking problems in a logical way. He was dependent upon memorized rules. He had difficulty solving problems.

Another ninth grade boy, who was able to solve wide varieties of mathematical problems and seemed to be a "natural" problem solver, handled the situation of 2/5 + 1/3 by sketching two "sticks" and cutting one into thirds and the other into fifths in the following manner:



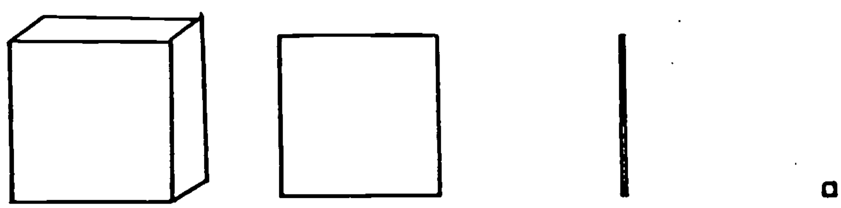
He argued: If I divide each third into five pieces and each fifth into three pieces, I get two sticks of the same size with fifteen pieces each. Two fifths is six pieces and 1/3 is five pieces. Altogether there are eleven pieces, so 11/15 is correct.

This student had concrete referents for fractions and demonstrated an ability to attach the symbols to reality in a variety of ways and then reason about them. A difference between the student and the "natural" problem solver is that the "natural" problem solver does this easily and extends this way of thinking to geniune problem situations. On the other hand, the nonproblem solver concerns himself with the mechanics of mathematics rather than with ideas underlying the processes and symbols.

An attempt was made with a heterogeneous class of fifth graders to see if they could come to view the symbols and processes of mathematics in terms of concrete referents similar to "natural" problem solvers and whether this would help them more adequately address problem situations. Some conditions seemed necessary (5). First, in order to build sound conceptual referents, objects and reasoning from them became the central instructional concern. No rules were given or allowed in arguments except as they evolved in the minds of the students and resulted from their own thinking. Second, arguments as to what was going on in terms of concrete objects replaced correct answers as the goal. The children were expected to explain their conclusions to the teacher using concrete objects. Third, an effort was made to help children internalize the objects and form a "mental image" of the work performed. They were asked to imagine the objects and explain their solutions to problems without the objects present. Finally, the

teacher never declared an effort correct nor accepted a casual argument. The children made the decisions as to whether their arguments were sensible.

The curriculum for the fifth grade required that two concepts be well grounded: place value and rational number. The operations were developed by student investigations of these two fundamental ideas. The materials were presented in formats that encouraged development of strategies for organizing to count. The general structure of the class moved from experiences with concrete materials to sketches or other representational schemes, to the formation of a "mental image" which the students used to make predictions and justify results. Within the framework of exercises, problems were posed which required students to organize and modify their ways of viewing the concepts of fraction and place value. For example, 1/11, 3/11, and 4/11 were nestled in among a set of problems asking children to find ways to give decimal equivalents for fractions. The children discovered that these resulted in repeating decimals. One student argued from some base ten blocks as follows:



1 0.1 0.01 0.001

STUDENT: If I take this as one (using the large block sketched above), break it up into tenths and share it with eleven (divide it by the denominator of 1/11) no one will get any of the tenths. So I will break it up into hundredths. Now there are a hundred of them so each share is nine, so I write .09. But there is one of the sticks left over, so I'll break it up into little blocks. Each of them is a thousandth, but there is only ten of them so I can't share them with eleven, so I'll get out my laser and cut each of these little cubes into ten teeny flats which are ten thousandths. Now there are a hundred of them, so each gets nine of them with one left over, so I write another 09 (0.0909). Each time it goes the same and there is one left over so it goes again. It never stops.

Some of the children investigated other fractions and found ways to predict 11ths, 9ths, 7ths, etc. The instructor posed the following problem: "Do you suppose there is a fraction that would result in the repeating decimal 0.121212 . . .?" Two days later, the children said they could find a fraction for any repeating decimal. The instructor expressed disbelief and asked them to explain for 0.121212 . . .. They argued:

> We that knew if we got a number like 0.010101 . . ., or 0.020202 . . ., or 0.030303 . . ., or 0.040404 . . ., or 0.060606 . . ., we would have it since 0.121212 . . . is just something times any one of these. At first we tried a bunch of things like 25ths or so, but we couldn't find anything exact. Then we thought that we would need a number (denominator) big enough that there wouldn't be any tenths when the cube (the unit cube) was divided up. If we could get just one hundreth for every one and have one left over, we could get 0.010101 . . .. So if we use a fraction with 99 in the bottom, when we divide up the cube, there would be no tenths. Then when we break it (each tenth) into ten each, we get hundredths. There is enough for one hundredth each with one left over. You break that into thousandths. No one gets any, so we cut it up into ten little flats (ten thousandths) and everyone gets one with one left over. That happens every time so the number goes over and over. That gives us 0.010101 . . . and 12 of them gives us 0.121212 . . ., so 12/99 is the answer.

The students went on to explain how to get the fractional equivalents for decimals having blocks of three, four or more numerals in the repeating pattern. What would have been a formidable problem for many became accessible because of the referent systems these students had for place value and rational number.

Of the thirty-two students in the group, only two appeared to be "natural" problem solvers at the beginning of the school year. At the end of the school year, all but six of these students had built sound referent systems for rational and whole numbers. The referent systems helped them to solve problems more effectively than many of their counterparts in other fifth grade classes. More importantly, they possessed a foremost trait of the problem solver: they knew for themselves when they were being sensible.

5

There is promise of producing more problem solvers if the students can build and internalize concrete referents for the symbols and processes of mathematics. Simultaneously, we must be willing to abandon rule-memory approaches to mathematics and the assumption that rules and facts must be in memory before problem solving can be addressed.

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# Different Modes of Thought in Stochastic Thinking

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Abstract: This paper introduces the dichotomy between the intuitive and analytic mode of thought in order to conceptualize strategies in stochastic thinking, especially probability judgments. The proposed modes of thought are introduced by two lists of nine features. Two experiments were run, one with a sample of 118 students, the other with a sample of 10 university professors, in order to validate the proposed definition of the modes of thought. The experimental paradigm used was various base rate problems. The rating of the strategies was based on written protocols and provided a satisfactory interrater-reliability.

Introduction: Mathematics education and teacher training are not the only areas that have voiced complaint about unsatisfactory performance in stochastic thinking. Expert studies, natural field studies, and many psychological laboratory experiments have repeatedly demonstrated difficulties with stochastic thinking (evidence and references for these three areas are provided in the corresponding chapters of KAHNEMAN, SLOVIC, and TVERSKY, 1982; NISBETT and ROSS, 1980; SCHOLZ, 1981, 1983). According to current psychological research, intuitive judgmental heuristics, like the so called representativeness heuristic (cf. KAHNEMAN & TVERSKY, 1972), the availability heuristic (cf. TVERSKY & KAHNEMAN, 1973), the causal schema (cf. TVERSKY & KAHNEMAN, 1979), or the relevance construct (BAR-HILLEL, 1980) are predominantly regarded as being responsible for the insufficient performance and the "biasedness" in individual response behavior.

A prototypical example of such biased performance which has been investigated in some depth is the so called base rate fallacy. Generally, the term base rate fallacy denotes the phenomenon, that when people are exposed to additional or diagnostic information about the probability of an event, they often show insufficient consideration of prior actuarial knowledge, hypotheses, or base rates. The subsequent Hit Parade problem is a variant of the so called Taxi Cab problem which was introduced by TVERSKY and KAHNEMAN (1979).

## Hit Parade Problem

In a South German radio station's hit parade, a studio guest chosen from listeners' fan mail is asked to tell whether a newly presented song will become a hit, i.e. whether a newly presented title will be among the 10 songs most named by the listeners.

One of this program's fans now established on extensive statistics. During the last five years, 10 % of the newly presented songs were accepted into the hit parade; 90 % of these songs were not.

He (the fan who recorded the data) also found out that the studio guests had made 80 % correct predictions, both for the titles that became hits, and the titles that were rejected. In other words, the studio guests made a correct hit-prediction for 80 % of the titles that actually became hits, and a correct no-hit-prediction for 80 % of the titles that did not become hits.

Now the question: What is the probability that a song chosen at random by the studio guest as a prospective hit will be named as a hit by the listeners?

In variants of the Cab problem like the Hit Parade story, two kinds of information are given. First there is the base-rate information, which may be called a priori information or statistical information as it is often - but not always - known in advance or has been produced by statistical data. In the Hit Parade problem the base-rate information provides the a priori odds $p(H)/p(\bar{H})$, which is .1/.9 = 0.11. The studio guest´s accuracy provides the additional or diagnostic information and thus the likelihood ratio $p(D/H)/p(D/\bar{H})$. In the case of the Hit Parade problem $p(D/H)$ denotes the probability that the studio guest said "Hit" given a title that actually became a Hit. The probability $p(D/H)$ will also be called diagnosticity. The requested probability $p(H/D)$ (which is .31 in the Hit Parade problem) may be derived from the simplest form of Bayes theorem that is:

$$\frac{p(H/D)}{p(\bar{H}/D)} = \frac{p(D/H)}{p(D/(\bar{H})} \times \frac{p(H)}{p(\bar{H})} .$$

Traditional psychological research on the base-rate fallacy has been characterized by two issues: a) the ´normative solution´ has been taken as the yardstick for judging subjects behavior, and b) inferences on the underlying cognitive process resulting in a probability judgment are mostly based solely on an analysis of the product of the decision process. Both assumptions may be critizized. First: mapping between words and mathematics is very delicate. As SCHOLZ and BENTRUP (1984) have demonstrated, one can also easily derive other resonable solutions than p (H/D), e.g. by introducing so called information weights, or by reevaluating the story parameter by referring to different variants of the probability concept. The label ´normative solution´ which will be used for p (H/D) must therefore be placed in semiquotes. Second: the above judgmental heuristics, which have been developed as (post hoc) models of the judgmental process may only describe 20 to 40 % of the subjects response behavior as they only may explain the diagnosticity responses.

Based on an analysis of thinking aloud protocols, of more than one hundred written protocols, and of interviews into the subjects' proceeding in the course of probability judgments, the author hypothezised that different modes of thinking guide the judgmental process. We propose that if we want to understand the cognitive process of probability judgment in problems like the Hit Parade story, it is essential to distinguish between intuitive and analytic thinking. The subsequent list of attributes and features has been developed with reference to various works on cognitive psychology (e.g. BASTICK, 1982; HAMMOND, HAMM, GRASSIA & PEARSON, 1983), and mathematics education (e.g. FISCHBEIN, 1980, WACHSMUTH, 1981). This is introduced as a definition of the prototypical intuitive and analytic modes of thought in stochastic thinking (c.f. Table 1). A description of the attributes and features is given in SCHOLZ (1985).

Table 1: List of attributes and features of intuitive and analytic thought

| | Intuitive thought | Analytic thought |
|---|---|---|
| A | preconscious<br>- information acquisition<br>- processing of information | conscious<br>- information acquisition and selection<br>- processing of information |
| B | understanding by feeling and instinct of empathy | pure intellect or logical reasoning, independent of temporary moods and physiology |
| C | sudden, synthetical, parallel processing of a global field of knowledge | sequential, linear, step-by-step, ordered cognitive activity |
| D | Treating the problem structure as a whole ("Gestalt erkennend") | separating details of information |
| E | dependent on personal experience | independent on personal experience |
| F | pictorial metaphors | conceptual or numerical patterns |
| G | low cognitive control | high cognitive control |
| H | emotional involvement | cold, emotion-free activity |
| I | feeling of certainty towards the product of thinking | uncertainty towards the product of thinking |

A series of experiments were performed in order to validate the proposed definition, especially the interrater reliability, and to investigate the effect of the activated mode of thought. We will give a brief account of two of those experiments.

Experiment 1

Subjects: The subjects were 88 students volunteers from a German Gesamtschule ranging from grade 7 to 13, and 30 postgraduate students who had received their Vordiplom (a pre-diploma, received after two years of university studies).

Procedure: Subjects were asked to work with concentration on a questionnaire that included three variants of the Cab Problem (e.g. the above Hit Parade problem). On the last pages of the questionnaire, the first problem was repeated, and the subjects were asked to write down an extensive justification of their judgment.

Dependent variables and Rating procedure: The subjects provided probability judgments and written justifications. The latter were the object of a rating procedure. Four raters independently classified each written protocol according to a coder instruction on the nine features of intuitive and analytic thought. The rating intuitive, analytic or intermediate/non-classified could be coded for each dimension. According to rater´s coding, a strategy is labeled analytic, if at least two more features are rated as analytic than intuitive. The same procedure in reverse was used for an intuitive rating.

Results: Various measures were applied to check the interrater reliability of the intuitiveness - analyticity rating. We will limit ourselves to reporting the results of the analysis using the well approved Z-value, which is $Z_{i,j} = 2\,M / (C_i + C_j)$ (where M is the total number of matchings of classified cases and $C_k$ the total number of rater k´s analytic - intuitive ratings, k = i,j) and the $Chi^2$-values for the 3 x 3 classification of pairwise ratings (cf. Table 2). According to a rule of thumb, Z-values above .8 may be considered as providing a good interrater-reliability. As can be inferred from Table 2, the average Z-values are .76 and thus a little below the critical value of .8. This is obviously due to rater 4´s relatively low matchings with the other raters who (if rater 4 is excluded) result in a .82 average Z-value. The $Chi^2$-values are all well above 13.8, which corresponds to a .001 level of significance with 3 degrees of freedom.

Table 2: Z-values and $Chi^2$-values (dF = 4) of interrater-reliability of strategy classification for all pairs of raters

| | Pairs of Raters | | | | | |
|---|---|---|---|---|---|---|
| | 1 x 2 | 1 x 3 | 1 x 4 | 2 x 3 | 2 x 4 | 3 x 4 |
| Z-value | .87 | .78 | .70 | .81 | .72 | .68 |
| $Chi^2$-value | 74.8 | 50.0 | 33.4 | 59.4 | 39.7 | 34.7 |

In our sample, 29 % of the subjects responded with the diagnosticity information, only 3 % gave a "normative" repsonse, and more than 20 % produced unsophisticated responses outside the interval (p (H), p (D/H)). According to all four raters' codings, most of the diagnosticity responses were given in the analytic mode (cf. Table 3).

Table 3: Frequency of diagnosticity responses split by the different types of strategies according to rater classification. In brackets: total number of strategies according to the rater

| | Mode of thought | | |
|---|---|---|---|
| | analytic | intermediate/ non-classified | intuitive |
| Rater 1 | 30 (69) | 2 (12) | 3 (37) |
| Rater 2 | 31 (75) | 3 (7) | 1 (36) |
| Rater 3 | 33 (85) | 1 (13) | 1 (20) |
| Rater 4 | 27 (56) | 5 (13) | 3 (49) |

Experiment 2:

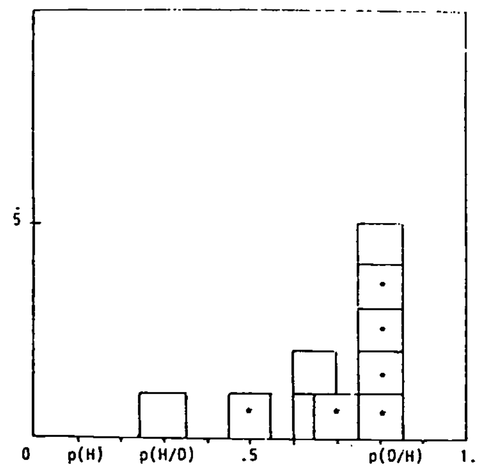
Subjects: The subjects were ten university professors working in statistics and related fields. The sample was made up of four mathematicians, three theoretical physicists, and three social scientists.

Procedure: The professors were hired for an expert study on stochastic thinking. The first part of the experiment (reported here) consisted of an investigation of experts' handling of base rate problems. In a first trial, they were asked to produce an intuitive response (and not to look for a formal solution, although one presumably existed), and in a second trial, to use formal statistics to deal with the problem.

Results: The experts' justifications in the "intuitive condition" were submitted to an analytic vs. intuitive rating by the author. Six subjects were judged to process intuitively, three analytically, and one subject could not be classified. One of the three analytical subjects

refused to respond intuitively, as he knew that such an answer would be "cogently" incorrect and hence inappropriate. The experts' response distribution in the intuitive mode is presented in Figure 1. In the analytic condition, two experts did not finish the task, presumably due to experimental stress. Five subjects provided the "normative solution" whereas 3 yielded a diagnosticity response. Finally, we want to note that in a postexperimental interview, all but one expert agreed on the "standard normative solution" as "the" adequate answer.

Figure 1: Expert's response distribution in the "intuitive" condition . x-axis: probability judgments, y-axis: frequency. An inserted * denotes that the subject was judged to process actually intuitively.




Discussion of the Experiment's results:

In order to conceptualize individuals' cognitive strategies when assessing probability judgments, we defined the dichotonomy of intuitive and analytic mode of thought by lists of features. Two experiments using various base rate problems were run with samples who show a wide range in age and educational background. The samples produced typical, rather unsatisfactory response distributions, as has been repeatedly observed (cf. BAR-HILLEL, 1980 etc.) by base rate problems. With reference to the introduced modes of thought, the results permit three conclusions:

a) The rating procedure of the applied modes of thought, based on written justifications, provided a satisfactory interrater-reliability. (A closer analysis of the reliability of the single features, e.g. also the theoretically and also empirically somewhat dubious confidence feature 1 may be found in SCHOLZ, 1985).

b) The necessity of distinguishing between the introduced modes of thought has been demonstrated by an analysis of the response distributions produced by the different modes. Surprisingly most of the diagnosticity responses were produced by students who were judged

to process in an analytic mode. This finding is in contradiction to the widely supposed intuitive nature of the base rate fallacy.

c) Unlike to the students, a (small sample) of university professors produced a lot of diagnosticity responses when processing intuitively. We suppose that these differences between students´ and experts´ behavior might point to different knowledge and heuristic structures guiding the two samples´ intuitive probability judgments and their stochastic thinking.

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# Disregarding the hierarchy in a behavioural program as an error type in pupils' computation tasks

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In written subtraction tasks pupils occasionally display a certain kind of erroneous solution, the underlying strategy coming out from the following (assumed) examples:

$$\begin{array}{r} 675 \\ -\ 238 \\ \hline 443 \end{array} \qquad \begin{array}{r} 675 \\ -\ 298 \\ \hline 423 \end{array}$$

The results are obtained by constantly forming the difference between the bigger and the smaller digit number irrespective of whether these belong to the minuend or to the subtrahend, such as

$8 - 5 = 3$ , $7 - 3 = 4$ , $9 - 7 = 2$ , and so on, in the examples.

Many researchers have reported and analyzed this error type. Among the approaches dealing with it, we have to take into consideration expecially those explanatory models of cognitive processes in which this strategy assumes the function of a paradigm: H. Bauersfeld (1983), developing the concept of "subjektive Erfahrungsbereiche" (domains of subjective experiences), and Brown and Burton (1978), Young and O'Shea (1981), being based on production systems, both approaches presenting the above mentioned error strategy as a local strategy.

It is the intention of this paper to contrast these local explaining patterns with a rather global one, comprising the error type in subtracting natural numbers and comparable error strategies occuring in other subject areas. Expecially in computation tasks with rational numbers the erroneous strategy can be observed. In order to describe the common traits of this strategy in different subject areas, the correct subtraction algorithm with rational numbers for a certain task type shall at first be represented by a flow-chart (figure 1) and set over against the erroneous strategy (figure 2). In this context the error strategy can be described as confusing parts of the main program and of the sub-program. It consists in disregarding the hierarchy within a program representing the behaviour in carrying out the computation.

Little modifications of the flow-chart for the erroneous strategy are suitable as representations of this strategy used in other number domains (such as written subtraction with natural numbers, as shown above), and even with other operations (examples being reported later).

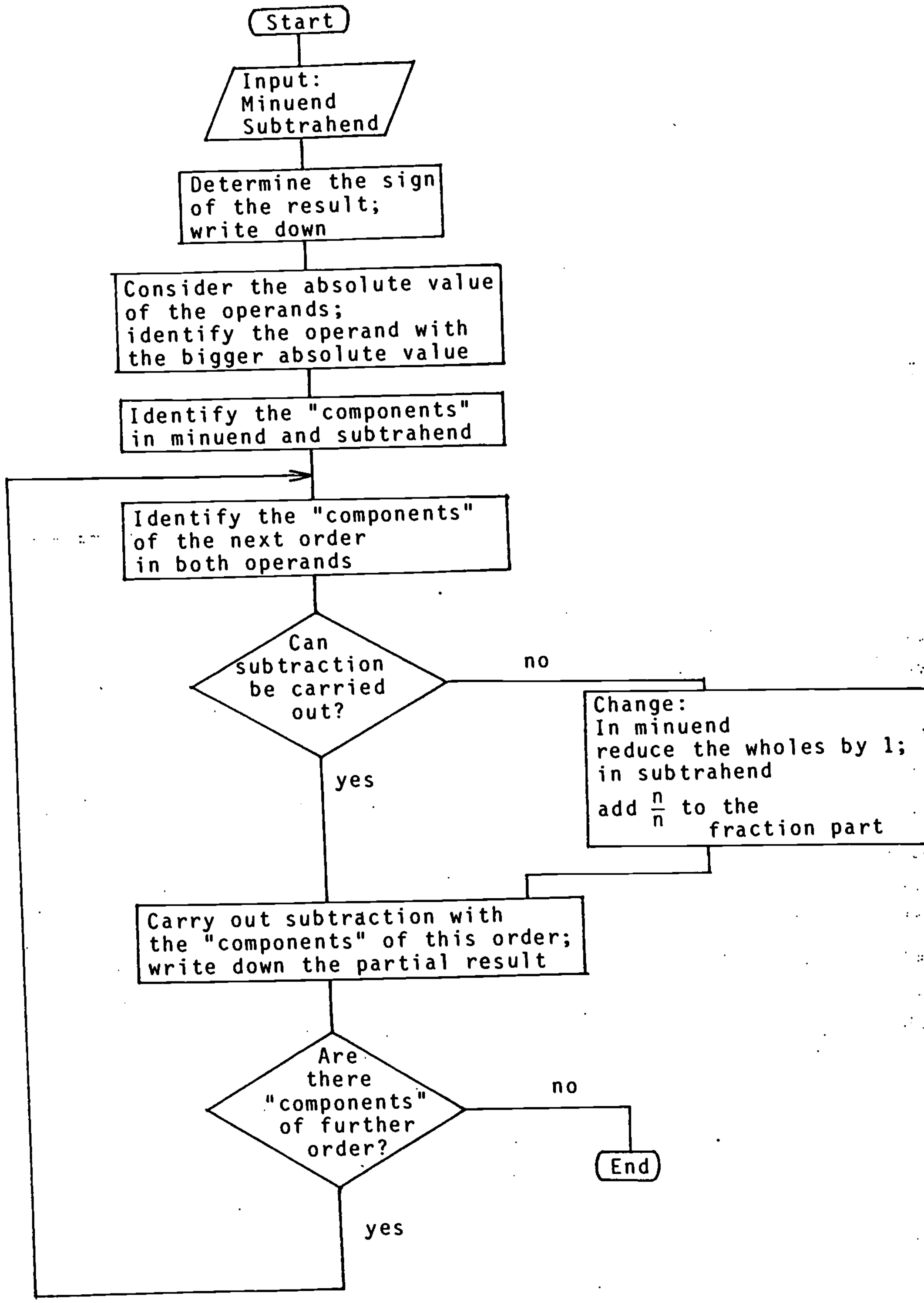


Figure 1: Flow-chart: Subtraction, correct strategy

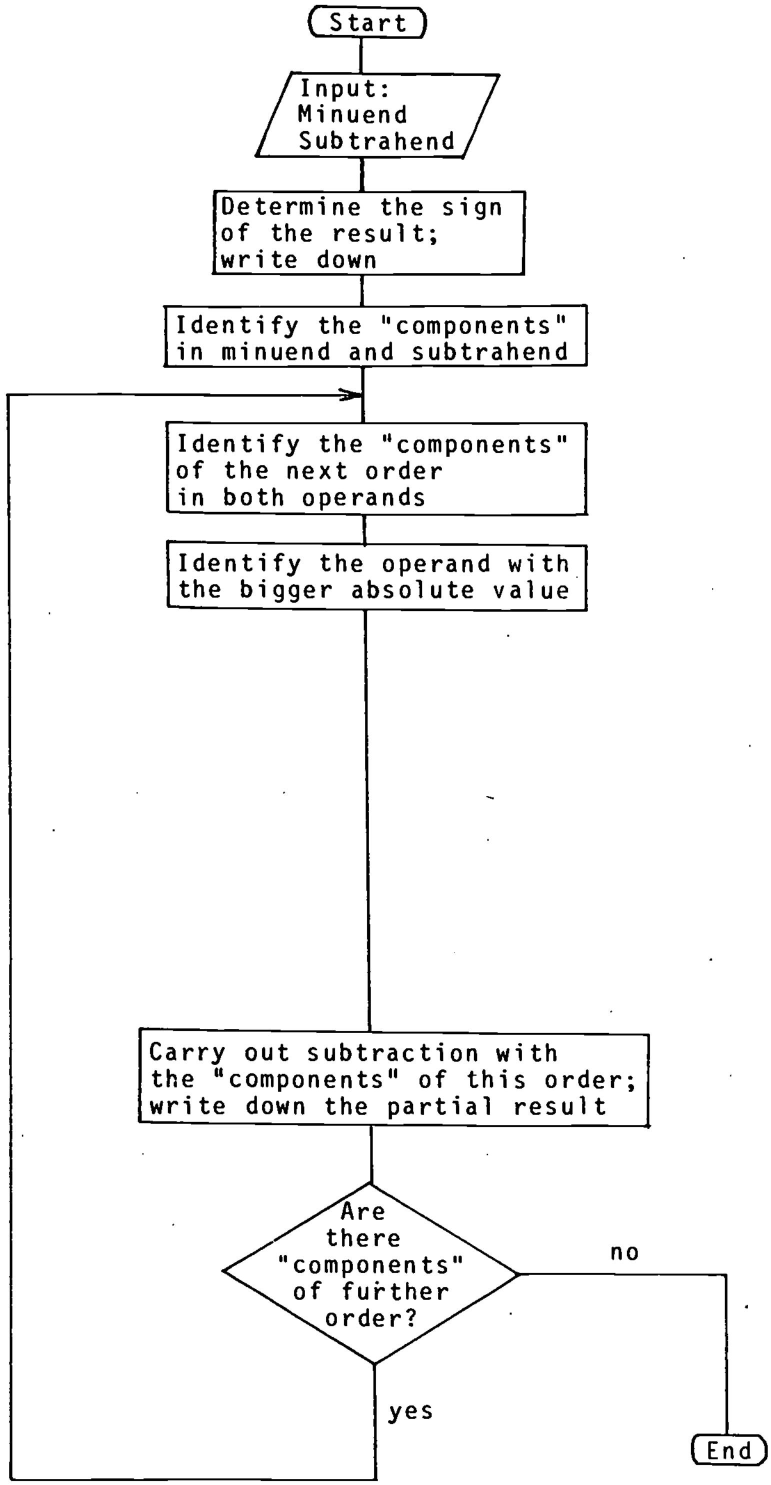


Figure 2: Flow-chart: Subtraction, erroneous strategy

Thus, the strategy in question is not as specific as assumed in the local explanatory models. For instance, 7-th and 8-th graders did as follows:

(1) $17 - (12\frac{1}{5} + 13\frac{5}{6}) = \ldots = 17 - 26\frac{1}{30} = 16\frac{30}{30} - 26\frac{1}{30} = -10\frac{29}{30}$

(2) $\ldots = 17 - 26\frac{1}{30} = -8\frac{29}{30}$

(3) $(-17\frac{3}{14} + 16\frac{5}{6})\cdot(-5\frac{11}{16}) - \frac{2}{15} = (-17\frac{9}{42} + 16\frac{35}{42}) \ldots = -1\frac{26}{42} \ldots$

These examples refer to the case that the absolute value of the minuend is bigger than that of the subtrahend number, whereas the fraction part of the former is smaller than that of the latter. The flow-chart (figure 1) representing the correct strategy for this case contains, among others, the following steps:

- identifying the first operand and the second operand among two given rational numbers; this may be imagined as becoming aware that the two numbers are different, and selecting the bigger one;
- identifying the "components" of both operands, i.e. the wholes and the fraction parts. These "components" are imaginated as bearing an order; in case of rational numbers the latter corresponds to the succession from wholes to fraction parts; referring to other examples, such as written subtraction with natural numbers, we may think of the exponent of the matching power of 10, in ascending or descending order;
- carrying out an operation, in the present topic subtraction with partial items, namely the "components" of the order in question.

The flow-chart is not at all formalized, nor are the partial steps formulated in detail. The different steps "Input" and "Identifying 1. and 2. operand" respectively shall express the fact that optical or acoustical perception or even writing down the numbers is an act separate from conscious noticing both numbers and retrieving subtraction algorithm.

The flow-chart contains a loop, describing the sub-program of carrying out subtraction with the "components" iteratively. Within the sub-program we find a branching, referring to whether or not the operation with the components can be carried out immediately; if not, the components have to be changed to make the operation workable, f.i. by writing $6\frac{1}{3} - 1\frac{2}{3}$ as $5\frac{4}{3} - 1\frac{2}{3}$ .

The erroneous strategy is represented by the flow-chart in figure 2. Identifying the bigger number has become a step in the sub-program,

and deciding whether the operation is workable is no longer necessary, and so is the fairly complex step of changeing, because in the sub-program the bigger of both components is determined separately for each order.

The modification of the strategy leads to a considerable reduction of complexity and consists in a confusion of program levels: comparison and subtraction is performed in a sub-program instead of the main program; hierarchy within the behavioural program steering the operation is disrespected. In case that both components of the minuend number are bigger than those of the subtrahend, the modified program would yield the same correct result as the original program. So far, the modified strategy is an over-generalization of a correct strategy, too.

We can observe carrying out an operation component by component even in multiplying rational numbers. Examples, taken from classroom tests, are as follows:

(4) $(-\ 1\frac{21}{30})\cdot(-\ 9\frac{4}{9}) = 9\frac{84}{270}$

(5) $\ldots = (2\frac{11}{12})\cdot(-\ 6\frac{2}{7}) = (2\frac{11}{6})\cdot(-\ 6\frac{1}{7}) = -\ 12\frac{11}{42}$

(6) $(11\frac{5}{6} - 12\frac{2}{15})\cdot(-\ 9\frac{4}{9}) - \frac{5}{8} = (11\frac{25}{30} - 12\frac{4}{30})\cdot\ \ldots$

$$= (-\ 1\frac{21}{30})\cdot(-\ 9\frac{4}{9}) - \ldots = +\ 9\ \frac{84}{270} - \ldots$$

example (5) showing the erroneous strategy together with reducing the fraction parts (which might have drawn the pupil's attention to algorithms for computations with fractions, which, however, did not prevent the error), (6) displaying the same strategy in subtraction as well as in multiplication.

Thus, the strategy of separating "components" and processing them in isolated sub-programs is not a specific one. It was found also in occasional tasks to be done by heart, such as 13·16, for which the result 100+3·6, thus 118, was given. It is not surprising to observe that even those pupils who explicitely reduce the fraction parts additionally in both factors, do not realize any contradiction between the result due to their strategy and a rough estimation of the result (provided they actually did estimate roughly); for, we know that different symbol systems with their specific rules coexist in pupil's mind without being connected.

Analyzing these errors in detail leads to three issues:

1. the longevity of acquired strategies
2. their instability and the dependency of their reactivation on situative conditions, expecially on the number data and their complexity
3. their domain-independency (at least to a certain extent).

The erroneous strategy actually originates from formerly acquired correct strategies through over-generalization. However, we may raise the question, why they are activated in a special situation. In general there is the explanation that limited capacity of working-memory restricts selection of strategies to pregnant and less complex procedures, which do not require such a high amount of attention as the correct strategy would do. However, further explanations have to include the dynamics of the ovserved person's individual motives and his or her impetus structure.

There are many questions for further investigation arising from the phenomenon reported here:
Do errors according to the described strategy occur in other types of tasks than those which have turned out to be susceptible to the error strategy in the present instructional context? (F.i. in tasks with both components of bigger absolute value in the subtrahend)
May the negative sign be the releaser for the erroneous strategy?
Are there other possible releasers?
Can we find topics in quite other subject areas (such as geometry) in which this strategy or a comparable one is demonstrable?

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# THE USE OF MODELS IN THE CONSTRUCTION OF NUMERATION: THE EVOLUTION OF REPRESENTATIONS TOWARDS AN EFFICIENT SYMBOLISM

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In our research, different models were used in order to bring children to the construction of the concept of numeration. We shall see how these models allowed us to promote a learning with is consistent with our conception. We shall examine more closely one of these models and see how it was used inside an evolution process from descriptive written representations to a meaningful and efficient symbolism.

In a previous research project on numeration, we described how it is presently taught at the primary level and have identified the false conceptions developped by those teaching practices. A careful study of these difficulties and an epistemological analysis of the concept has led us to develop conceptions of numeration, and of what its teaching should be, which are different from those considered in current mathematics teaching at the primary level.

In a longitudinal study (1980-1983), the researchers took charge of the teaching of mathematics with a cohort of pupils during three years, from first grade to third grade. We shall examine the teaching strategy we then developped, in view of bringing children to construct progressively a meaningful and efficient symbolic system for numbers.

To reach our goal, different models of the concept were used. Three factors determine our selection of models:

a) our conceptions of numeration and its learning;
b) the children readiness to use the model;
c) the nature of the links between the model and reality.

We shall first make explicit those three points in representing one of our model. Afterwards we shall examine how we incite children to construct and develop their representations within the model.

## The flower model: a realisation of our conception of numeration

A teaching of numeration which is in accordance with our conception of the concept has as major preoccupation to instigate work and reflection on groupings. Indeed for us the learning of numeration is not limited to the study of the decimal system of numeration and its syntactic rules but is rather a process of representation of numbers, which will lead to this conventional writing.

The models we use are collections with rules governing the making of groupings and in which operations are defined. Those are essential because in addition to giving intentions for carrying out transformations on groupings, they inject meaning for those transformations (to make groupings, to "unmake" them...). Moreover by performing operations on collections, one is constantly obliged to establish relations between groupings and by doing so the written representations he uses (drawings, schema or symbolic writing) are more and more meaningful for him. Another and not the least reason in favor of considering operation is the need to become more effective in operating on collections and in giving information on those, that forces one to have recourse to written representations and to refine them.

The fundamental idea behind our model is the construction by children of paper flowers. The flowers are made from 10 super imposed two dimension flowers which are cut from thin paper. The basic elements of the model are: two-dimension flowers, three-dimension flowers (made as describe above), bunches of flowers (made from 10 two-dimension flowers tied altogether), and garlands (made of flowers from 10 bunches).

We recognize here a model of numeration since we have a collection of "feuilles" with grouping modalities and operations (addition, subtraction, multiplication, division...) can be defined on this collection.

## The flower model: its accessibility to children

We have three criteria to determine the accessibility of a model for children:

1) groupings presented are clearly identifiable;
2) the relation between presented groupings is explicit;
3) the actions to apply to groupings are direct and do not require substitution.

These criteria are used to differenciate our models and decide in which sequence we shall use them for teaching. For example, we wouldn't with young children start with a model such as the abacus in which groupings are not identifiable but given by a convention. With the abacus, relations between groupings are not explicit but conventional rules and the actions to apply on groupings require substitution.

In the flower model, groupings are clearly identifiable, the relations between presented groupings are not explicit. However, the children have access to them, which is not the case for the abacus. As for the third criteria, the children can operate directly on groupings with no contraints of substitution. Indeed, they can easily "undo" a flower, a bunch and a garland.

Prior to using the flower model with the children, another model should be used, as we did, in which the relation between presented groupings is really explicit. Children would then never need to interrupt a task to find or verify a relation not yet mastered or memorized.

Having apprehended this model of numeration, the children can reinvest and develop more over the basic skills which are called into play in numeration that is: making groupings, "unmaking" them, making groupings of groupings, exchanging one grouping of a given order for units or groupings of an inferior order, or inversely...

In our first research on numeration, we made obvious the difficulties encountered by children relatively to those skills and more particularly the skill to coordinate two different groupings simultaneously.

In all of our models, there are at least two groupings. One can verify this in the flower model where three groupings are presented.

We choose the models to manage so that the children can use and develop these skills at a level accessible to them right from the begining of their learning of numeration.

<u>The "flower model": links with reality</u>

We want our models to be usable in contexts in which the operations and relations have an a priori meaning or at least easily constructable. We then can rely on knowledges already aquired further more since the elements with their rules and contraints are embedded in the child's environment the number of instructions dictate to the child and that he will have to memorize is

minimized but most importantly artificial instructions are no longer likely to be used.

In the flower model, there is no necessary need of contexts to give meanings to the groupings and modalities of grouping. Yet the contexts (the store, workshop of flower makers...) allow us to give meaning to the operations on collections, to the comparisons between collections and to the need to use written representations.

## CONSTRUCTION AND EVOLUTION FOR CHILDREN'S REPRESENTATIONS WITHIN THE MODEL

We take the most of a context by inventing diverse variations on the same theme. For the context of a workshop of flower makers, some of the possible variations are: to compare the daily production of workers, to calculate a worker's production on two weeks, to fill an order from a store... These situations urge children to operate, compare, communicate over all to construct written representations and make them become more and more efficient.

From the beginning of the learning in which we use a new model, children operate on real collections they make concrete actions on groupings and observes their effect. In order to communicate information on collections as well as on transformations made on those, children make use of different means: oral langage, written langage, drawings, mimes, schematization, symbolic writings. The first written representations of numbers attach to collections aim at reproducing the collections as the first representations of actions made on collections try to describe all what was done and observed. To make these descriptive written representations evolve, the teaching strategy is to work with situation in which the need to communicate and operate becomes more and more hard to fill. Children are then forced to question their representations and ajust them so that they become more formal and efficient.

Therefore with the use of several contexts and using a single model, the child can go through the entire process of numeration, from the handling of collections to the efficient use of symbolism.

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# The Decimal System: Understanding and Notation

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The importance of knowledge of place value for our educational practices goes beyond its use as a conventional notation in mathematics since this particular notation is also the basis of the computation algorithms we teach in school. Column multiplication and addition and subtraction with carrying and borrowing, for example, can hardly be performed on Roman numerals. There is no proper way of interpreting the instruction "carry the one" when one is calculating the sum of XXIII and LVIII. Three aspects may be distinguished when we discuss the question of knowledge of our number system: (a) the ability to generate number names according to a system (in which a virtually infinite set of numbers can be named using a small set of linguistic labels); (b) the ability to understand the number meanings represented by the number names; and (c) the ability to use place value notation system (in the present case with Hindu-Arabic numerals). Curiously, the literature on the decimal system does not systematically distinguish between these aspects. Very enlightening studies are reported by Resnick (1983) and Brown (1981) with children who have already received systematic instruction on place value at school. It is difficult under these circumstances to distinguish between children's understanding of the decimal system and their knowledge of written conventions. Ginsburg (1977) documented young children's knowledge of the system underlying number words to generate number names but his findings have been difficult to reconcile with the problems teachers face when instructing children on the number system. Some apparent contradictions in the literature may perhaps be resolved by keeping in mind that the three differents aspects mentioned above may develop and interact interact with one another in different fashions.

The present study attempted to investigate whether children display an understanding of the meaning of number words compatible with the notion of a decimal system before they receive systematic school instruction on place value notation. It also investigated this question in illiterate adults, who have, by virtue of using money in their daily lives at least, a large experience with numbers.

## Study 1

The present study investigated children with no previous school instruction on the number system with regard to: (a) their understanding of relative value; (b) their ability to sequentially sum token coins of the same value; and (c) their ability to combine token coins of varied values to obtain desired amounts. These abilities were tested independently of children´s knowledge of written numbers.

All children were given a Relative Values Task, a Money Counting Task, and a Conservation of Number Task. Order of taks followed the Latin Square. Children were tested individually in their schools in the presence of an observer. Interviews were tape recorded and transcribed verbatim.

### Subjects

Subjects were 72 Brazilian pre-school children in the age range 5 to 7 years who attended either a private or a state supported kindergarten. In Recife, Brazil, where the study was conducted, the distinction between these two types of schol corresponds closely to different socio-economic levels, with private schools being a priviledge of children from middle- and upper-income homes.

While some of the children could recognize digits, none knew how to use the Hindu-Arabic numerals to systematically represent numbers greater than 9.

### Procedure

Children were initially pre-tested on their counting ability. Their performance in this initial task determined how the subsequent part of the interview was to be carried out.

1. The Relative Values Task.

Children were asked to play a money game (adapted from Carraher and Schliemann, 1983) in which different coloured tokens represented different values. For children who could count above ten and generate number names after 20 (even if with the help of the examiner for the choice of labels for tens) the tokens were introduced with values of one and ten. Nine tokens of each denomination were availabel in this case. For children who did not meet these conditions in the counting task, the different tokens were introduced with values of one and five. Four tokens of each denomination were then availabel.

To investigate the children´s understanding of relative value, they were asked to judge which of two arrays of tokens had more money. The arrays were constructed in such a way that responses based on the number of tokens in the arrays were

distinct from those based on the value (e.g., five tokens worth ten were compared to five tokens worth one "cruzeiro").

Children's performance on this task was classified into one of the following three categories: (1) does not take into account relative value when answering; (2) takes into account the relative value of the tokens but is unable to justify the response by counting the totals; (3) takes into account the relative value of the tokens and justifies the answer by counting the totals in each array. The first item, when failed by the child, was considered as an example. In that case, the experimenter counted with the child the amount of money in each array and repeated the question. Performance in this case was evaluated considering the next two items.

2. Counting money.

In this task children were invited to play a store game in which the experimenter was teh sales clerk and determined the prices of the toys. The child purchased toys with the same play money availabel in the previous task.

Four trials were used for purchases using a singlle denomination, two for with units and two for higher value tokens where amounts could not be obtained with units (e.g., 10 and 20 were values which could not be obtained with units because there were only nine tokens with value one). A passing score for each level was given to children who succeeded on both trials.

Children's ability to combine different values was tested in four trials. For children working with numbers up to ten, these trials involved paying six, nine, eight and seven "cruzeiros" to the experimenter. (N.B. The "cruzeiro" is the standard Brazilian monetary unit.) For children working with numbers above ten, the values were 13, 26, 35 and 42. If the child attempted the first item without success, the experimenter inverted the roles of buyer and seller and payed to the child the desired amount, asking the child to count the money. If the child showed difficulty in counting, the experimenter counted the money with the child. The roles were then reversed again and the next three trials were presented. A passing score on this task was given to children who succeeded in three out of four items.

3. Conservation of number.

The conservation task was carried out with six real coins all of the same value and following the procedure outlined by Piaget and Szeminska (1964). The conservation question in this task, as in the Relative Values Task, is posed as "Is one person richer than the other?" or "Can one of the people buy more candy than the other one?".

Results

Of the 72 children, 47.2% could not satisfy the counting

conditions and were tested accordingly with tokens with values one and five. The remaining 62.8% were tested with tokens of values one and ten.

In the Relative Values Task, 40.3% of the children did not display an understanding of relative value, taking into account only the number of tokens in the array. For example, two arrays with the same number of tokens might be regarded as equal even though the child was able to recall the value of the tokens. An array with a token worth five or ten was judged as worth less than one with four or nine tokens, respectively, even though the set of fewer coins had a higher value per coin. Fourteen (48%) of these children insisted upon their answer even after naming the values of both arrays correctly. For example, Simone, 6 years(comparing an array with four units with an array of one token worth fave "cruzeiros") asserted that the experimenter, who had the money in the array with four one" cruzeiro" coins, was richer than herself. When asked to justify her answer, she proceeded to count the tokens in the array. The experimenter then asked her how much money there was in the first array. She answered "Four". When asked how much money there was inthe second array, she correctly answered "Five". The questions "Who is richer?" and "Who can buy more candy?" were posed again but Simone reasserted that the experimenter, who had the array with four "cruzeiros" was richer and could buy more candy with her money.

In contrast to these children's behavior, 15.3% of all children were able to respond correctly in this task even though they were unable to count the total value of the arrays under comparison. When asked to justify their answer, they would simply mention the value of the tokens in the array (e.g., "Because mine are worth ten and yours are worth one"). When asked how much money each one had, they either made errors in the array with higher value (units were always counted correctly) or gave up counting the values and responded, for example, "five tens".

Finally, 44.4% of the 72 children, including some five year olds, displayed good understanding of relative value and were able to justify their answers by referring to the total values of the arrays.

Performance on the Money Counting Task can be summarized as follows: (a) only two children were unable to solve both problems with singles or any of the other problems, failing in all three parts of the task; (b) two children showed results inconsistent with a scalogram for the three parts in the task, passing both higher value items and failing one of the items with singles; (c) 27.8% of all children passed both items with units and failed at least one of the two higher value items (i.e., tokens worth five or ten "cruzeiros"); (d) 27.8% passed all items with same-value tokens but failed the items with different-value

tokens; (e) 38.9% of the children passed all same-value items and at least three of the four items with different values combined.

Errors in the items in which different values were combined includedseveral interesting error types. Some children were unable to shift scales in the middle of their counting, treating units as tens after starting a string of tens or tens as units after starting a string of units. Sometimes children started counting with a ten "cruzeiro" token and had difficulty in finding the proper label, reading off the amount as "ten and one" ("dez e um") instead of "eleven" ("onze") or "ten and two" instead of "twelve". The latter type of error was only observed in the initial trial, when the experimenter and the child reversed their roles and the child was asked to tell how much money the experimenter had given for the item purchased.

Of the 28 children who were able to correctly combine tokens of different values, 22 had been tested with units and tens; 6 had been tested with units and fives because they had not met the criterion in the pre-test.

Performance in the Conservation of Number Task can be summarized as follows: (a) 7.9% of the children were unable to establish the initial equivalence between the arrays; (b) 63.5% were able to establish the initial equivalence of arrays but did not conserve this equivalence when the perceptual organization of the arrays was transformed; and (c) 28.6% were able to establish the equivalence between the arrays and conserve it even after the perceptual transformation.

The correlations between performance in the three different tasks were low. Tables 1, 2, and 3 present the contingency tables obtained for these intercorrelations.

## Discussion

It is remarkable that a large proportion (38.9%) of children showed the ability to coordinate two scales with different intervals to obtain one integrated system even though they had no previous formal instruction on the number system. A question that one would like to answer is, thus, how did these children acquire this knowledge of number systems? Did they discover these aspects of number meanings from having learned the linguistic decimal system? Did they figure out that "twenty two" means "twenty and two", "twenty three" means "twenty and three" and so forth, thereby building the idea of two coordinated scales? It is possible. However, if this is the case, knowledge of the linguistic system is neither a sufficient nor a necessary condition for children to construct the idea of combining scales of different values. Only some (51.2%) of the children who were able to generate number nouns in a systematic way were also able to succeed in counting money with different denominations. It will also be recalled that this knowledge cannot be seen as a necessary condition for the emergence of the ability to combine

the different values into one single scale since six (21.4%) of the children who displayed this ability had not met the criterion established in the counting pre-test.

Other sources of experience for the construction of this ability must be availabel in children's daily lives. A possible such source is a solid understanding of addition, which would allow these children to "count on" from any number in any known scale without mixing up the scales. The error types mentioned previously when children attempted to count tokens of different values in fact suggest a dificulty with "counting on", which would lead to responses as "ten and one" instead of eleven. The children who worked with tokens worth one and five had to be able to add five and five in order to correctly solve one of the items with same-value tokens. Since children this young do not receive training in counting in fives in Brazil, they did not readily know the answer to the item in which they were asked to pay the experimenter ten "cruzeiros". They had to count their one "cruzeiro" tokens, find out that they didn't have enough of them to get up to ten, count their singles adding them to one five-cruzeiro token, find out that this was still not enough, and finally figure out that what they actually needed was two fives. This was a complex response which heavily depended upon their ability to add.

Study 2

The study of illiterate adults can offer interesting information regarding the relationship between the understanding of a number system and the ability to use a particular type of notation for this same system. Adults who have no systematic instruction on number systems must construct their understanding on the basis of their cultural experiences with number. If unschooled adults display an understanding of the number system and little knowledge of our notation system, then these two aspects must be distinguished. If only adults who understand the number system discover how our notation system works in the absence of school instruction, this finding would be suggestive of a need to understand the system in order to understand its notation. However, if there are those who write numbers well but fail to display a basic understanding of the mathematical characteristics of the number system, one could be led to the conclusion that the linguistic system may provide the necessary information for learning how to write numbers even in the absence of comprehension. As unlikely as this may seem to a psychologist, one cannot deny that the number label "twenty two" may be enough for the adult to realize that the number is written with a 2 for twenty and another 2 for the two; similarly, thirty nine would take a 3 for thirty and a 9 for the nine. Important checks in this case would be numbers with empty classes where zero is used as a place holder. These are not represented in the

language but are represented on paper. What would happen in this case? On the other hand, "twenty two" may be seen as the representation of "twenty and two", in which case the number would be written as 202, an error compatible with a good understanding of the number system in the absence of knowledge of the notation system.

The information available on the number system in the culture

The following description examines what types of information an uninstructed adult may find on number system in daily life simply from being a functioning adult in Recife, Brazil.

The first source of information to consider is the language. In Portuguese, number names reveal the following number information. Numbers zero to fifteen are labled by different words which do not reveal the existence of a system. Even though there are similarities between the labels for 3 and 13 ("tres" e "treze"), 4 and 14 ("quatro" e "quatorze"), 5 and 15 ("cinco" e "quinze"), these similarities do not reveal anything about the number system. Starting with 16, the number system can be extracted from the words by the speaker, even though some transformation in the pronunciation of the words is necessary for the identification of the system. The word for 16 ("dezesseis") is composed of three words in principle, "dez" (10), "e" (and), and "seis" (6), but the pronunciation of "dez" is slightly changed with respect to the vowel in this process of composition. There is normally no immediate awareness of the morpheme "dez" in its pronunciation when forming the composite words for numbers 16 to 19. From number 20 on, the language structure is quite revealing of the number system. "Vinte e um" (21) keeps its elements apart and unites them by a conjunction, just as do all other numbers with tens and units. The denominations for the tens show at best a faint relationship to the number of tens. "Vinte" (20) shows no relationship to "dois" (2) and "trinta" (30) shows at best a faint similarity to "tres" (3) just as "quarenta" (40) and "quatro" etc. While for instructed adults these can be treated as obvious relationships between the words, they may not seem so obvious to uninstructed adults. With respect to the hundreds, one hundred ("cem") is a completely "new" word in the system (and, incidentally, it is pronounced exactly as the word for "without"). A second word, which is used to refer to "one hundred things" ("cento"), is the one which enters the composition of the labels for the hundreds. It suffers a slight transformation when 200, 300, and 500 are composed but remains perfectly recognizable when united with the remanining hundreds.

Again, to the ears of instructed adults, the relationships between "duzentos" (200) and "dois centos" (two hundred-things) may appear obvious but it may not seem so to unschooled adults.

The second source of information on number systems readily available to unschooled adults is the monetary system, based on the Brazilian "cruzeiro". The following denominations, in coins and bills, were available at the time of this study: 1, 5, 10, 20, 50, 100, 200, 500, 1000, and 5000 "cruzeiros". These denominations do not describe the decimal system since values such as 5 and 20 or 50 are intermediary values, in a way unnecessary for the composition on any particular value. However, the existence of coins and bills with these denominations is useful information about the number system. The label "vinte e um" coupled with the use of a coin worth 20 plus a coin worth 1 informs the user of this language and monetary system that "vinte e um" is a composite word referring to 20 and 1. This represents a very important piece of information about the number system because it reveals its additive properties: tens and units are added, creating a single scale which coordinates both types of interval. "Duzentos cruzeiros" (200) can be obtained by the use of one note worth 200 or two notes worth 100 each. It is conceivable that the implicit multiplication (2 x 100) in this composition could be understood by the users of the system although it is more likely that they will conceive of this situation as 100 + 100 than as 2 x 100.

The monetary system presently used in Brazil consists of a devaluation of another system with the same name, "cruzeiros", obtained by the division of the value of each previous note by 1000. Certain values were eliminated in this process and all values were recoded. At the time this study was conducted, the inflationary process had resulted in the elimination of "centavos", worth one hundredth "cruzeiro", but in fact worthless for all practical purposes. However, linguistic habits had not yet totally adapted to the previous official devaluation of 1968. Many people still referred to 1 "cruzeiro" as "mil" (thousand) and to 1000 as "milhao" (million), a usage which did lead to occasional misunderstandings. This practice can be observed even in young children from lower socio-economic status who were too young to have lived in the time of the "old cruzeiro", as it is popularly called, but who may have heard these expressions at home and much more often than the referrences to the "new cruzeiro". The adjetives "old" and "new", which were current right after the devaluation and helped users avoid confusion, are no longer used. It appears that now when people use the "thousands" of the old system confusion is avoided simply on the basis of the estimated valued of what is being talked about. For example, when someone refers to the price of a lemon as 250 000, the listener understands this to refer to 250 in view of what one may reasonably pay for a lemon today.

Subjects

Subjects in this study were six illiterate adults who had never been in school. Five were women, two of which work in the cleaning staff of the university while the remaining three work as domestic helpers in private homes. The man in the sample is a gifted blue-collar worker with many specialities, including setting window-panes, an occupation which he worked in for nine years and which afforded him with a lot of measurement experience. All were fully functioning adults with family and job responsabilities. They all carried out their monetary transactions regularly in daily life.

Procedure

All subjects were individually tested by one of two interviewers well known to them. Interviews were tape recorded and transcribed verbatim.

The following items were presented to each subject always in the same order: (1) one item on relative value (Who has more money, you with five ten "cruzeiros" notes or I with five one hundred "cruzeiros" notes?); (2) three items on the decomposition of amounts (Using only notes of 100, 10, and 1, and always using the largest notes you can, how would you pay me 365, 843, and 208 "cruzeiros"?); (3)three additions with carrying and one subtraction with borrowing (999 + 1; 353 + 8; 134 + 67; 1000 - 750); (4) writing down two two-digit numbers, two three-digit numbers one of which had a place-holding zero, and one three-digit number.

Interviews allowed for slight adaptations when necessary such as questioning the subject to clarify responses.

Results

All six subjects correctly solved the question on relative value. Morevoer, four solved all the questions involving the combinations of different values to obtain previously specified amounts. However, these were not easy questions. Three subjects solved them by slowly composing the amounts by counting outloud the different strings of hundreds while keeping track on their fingers of the notes needed. One example is transcribed below.

D.D., female, 49 years [attempting to pay 365 "cruzeiros by using only bills of 100, 10, and 1 and using the largest values possible]: "Three hundred and sixty five?" (E: Yes) "At home I do all my calculations, but I don´t know, I learned so little, I didn´t go to school..." (E: But you do it at home, don´t you?) "Yes. One note of one hundred, one hundred. With

one hundred, two hundred (while keeping track of the number of notes with the help of her fingers], three hundred. Three hundred and sixty five. Sixty five more, right?" (E: Yes) "Ten, twenty, thirty, with one more ten, fourty, fifty, sixty [while keeping track of the number of 10 "cruzeiro" notes], and five. Three notes of one hundred, six tens and five notes of one."

This process was clearly improved on over the three trials and all three subjects were able to respond almost promptly to the last question. The improvement suggests that they were quite capable of coping with decomposition of amounts in the decimal system but may have had little or no practice in solving money questions under the specific form they had in this study.

The errors presented by the two subjects who were not able to solve the items on combination of different values into a single scale are indicative of difficulties in addition and in recalling the number names appropriately. One such error was the counting of five tens as the number of notes needed to get 60 "cruzeiros" which resulted from skipping over 40 in the process of counting. A second type of error which was observed suggested a confusion between the "new" and the "old cruzeiro". This type of error may not have appeared if the task had included reference to the price of an object, which could then work as a means of keeping the value in mind. For example, one subject, after correctly counting eight as the number of hundreds needed to get 800 "cruzeiros", went on to suggest that a 50 cents coin should be used for paying the remaing 43 "cruzeiros" mentioned in the task.

Of the four subjects who succeeded in the previous task, three solved all the additions and two solved all the subtractions presented.

Neither of the two subjects who failed the items on decomposition of amounts solved any of the four computations.

Some errors in addition seemed to reflect difficulties in counting on when using a system which shortens the number names. For example, when adding 353 and 8, one subject, appropriately reversing the units in the augend and the addend, counted: "358. 9, 10, 11. 311." Another subject, working out 134 + 67, counted "134, 35, 36, 37, 38, 39, 40, 50", and then proceeded to add 60 to this result.

The relationship between decomposing amounts in the decimal system and writing numbers showed the following results.

(1) The subject with large experience in measurement initially refused the task but on a later occasion was observed taking notes of measurements without difficulty.

(2) None of the other three subjects who succeeded in the problems of decomposition of amounts was able to write numbers. Some of their errors reflected a representation of the separate classes in the numbers: 10053 for 153, 909 for 99 etc. Other errors suggeested a specific influence of the practice of reading

prices. "Centavos" (cents) had just been elimitaed from the monetary system when this study was conducted but had influenced the way prices were written for a long time up to then through the systematic addition of two zeros to the right of the price tags. One subject used two zeros to the right of all numbers dictated. All of these errors are suggestive of the important role which money may play in helping unschooled adults come to grips with our number system.

(3) Both subjects who did not succeed in the decomposition of amounts items showed a comparatively better performance when writting numbers. One of them succeeded in writing all five numbers dictated and the other one only failed the three digit number with a place-holding zero (writing 34 for 304). Two interpretations of these results seem plausible. The first would be that these subjects were poorly tested on decomposition of amounts and that their performance did not adequately reflect their competence. The second is that these subjects were perhaps able to derive the writing of numbers from number names.

Discussion

If we consider the results of these two studies in conjunction, it is possible to come to some interesting conclusions. First, the psychological processes involved in understanding the number system and using place value notation must be distinguished. Knowledge of one does not imply knowledge of the other. Second, it is likely that the understanding of the number system is based on a good knowledge of addition. The errors observed in children when trying to coordinate units and tens into one single scale suggested difficulties in counting on. It is interesting to recall at this point the studies by Saxe (1982) and Saxe and Posner (1983), which show a connection between the need to perform computations with money and the introduction of a base into a non-base number system. It cannot be seen as a mere accident that the monetary system introduced in the culture was the "old pound" and that the base adopted in this number system was 20. Third, the ability to perform addittions and subtractions with regrouping is independent of the knowledge of place value. Unschooled adults who showed some ability in adding and subtracting also showed ability in decomposing amounts according to our number system but three of them were unable to use place value notation appropriately. This represents a very important observation for mathematical educators. While it seems that these adults would not be able to understand the "carrying" and "borrowing" procedure, which are based upon the notation system itself, they were capable of understanding the notions required for addition and subtraction with regrouping when performing oral computation. Fourth, errors such as writing "20011" for "211"

may in fact reflect a sound comprehension of the number system in the absence of knowledge of place value. They may constitute an attempt by the individual to represent all the component values in the total amount. If these errors reflect the understanding of the number system, they should be treated as such and attempts to teach place value to those who exhibit them should take their knowledge into account. Fifth, learning how to write numbers from the numbers labels seems to be possible. However, this type of learning does not seem to contribute to the development of computational abilities in the same way that the understanding of the number system appears to.

A final comment about our educational practices seems in order. If we consider them carefully, a strong emphasis on written exercises cannot be denied. Computational procedures are intimately tied to the notational system and perhaps are introduced in ways that work against children's understanding of the meaning of what they are doing when adding. A 6 year old child, who happened to have learned how to perform additions with regrouping before he was taught how to do them in school, demonstrated to me the innapropriateness of our ways of teaching by developing his own way of writing the "ones that we carry". When adding 18 + 24, instead of writing first the digit 2 and then carrying the one, he wrote down the one on top of the other tens and then the 2, thereby preserving the meaning of the sum, 12, and using the written procedure as a representation of what he had in fact understood. It seems that our educational practices may profit from treating written mathematics initially as a notational system for meanings which must be understood and only at a later point transform the written system into an instrument of thinking.

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Table 1

Frequency per level of performance in the Relative Values and the Counting Money Tasks.

| Relative * Values Task | Counting Money: Counts Units | Counts Tens | Counts with Mixed Values |
|---|---|---|---|
| 1 | 14 | 11 | 6 |
| 2 | 4 | 7 | 1 |
| 3 | 3 | 3 | 21 |

N = 70 **

Chi square = 25.58; df = 4; $p \ll .001$

* 1 = Responses are based upon number of tokens.

2 = Correct responses in the absence of counting totals.

3 = Counts totals and responds correctly.

** = Loss of subjects was due to inconsistency in the scale in the Counting Money Task.

Table 2

Frequency per level of performance in the Conservation and Counting Money Task.

Counting Money

| Conservation * | Counts Units | Counts Tens | Counts with Mixed Values | |
|---|---|---|---|---|
| NC | 4 | 1 | 0 | 5 |
| I | 13 | 17 | 14 | 44 |
| C | 3 | 3 | 12 | 18 |
| | 20 | 21 | 26 | |

N = 67 **

chi square = 12.36; df = 4; $p < .02$

*NC: Fails to build equivalent rows.

I: Builds equivalence but does not conserve it.

C: Builds and conserves equivalence.

** Loss of subjects was due to inconsistency in the scale in Counting Money and to disagreement between the two judges for the Conservation Task.

Table 3

Frequency per level of performance in the Conservation and Relative Values Tasks.

| Conservation * | Relative Values Task * 1 | 2 | 3 | |
|---|---|---|---|---|
| NC | 3 | 2 | 0 | 5 |
| I | 24 | 9 | 10 | 43 |
| C | 2 | 1 | 16 | 19 |
| | 29 | 12 | 26 | 67 ** |

chi square = 24.73; df = 4; $p \ll .001$

* See coding of levels in previous tables.

** Loss of subjects was due to disagreement between the two judges in the Conservation Task.

# WORKING WITH SIMPLE WORD PROBLEMS IN EARLY MATHEMATICS INSTRUCTION

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*Abstract*

*The main characteristics of an experimental program for the teaching of addition and subtraction word problems in the first grade of primary school are presented. This program is based on research in our center concerning children's solution processes of elementary arithmetic word problems. The design and results of an exploratory teaching experiment, in which this program was implemented and evaluated in a first-grade class of a primary school are also discussed.*

## INTRODUCTION

Since 1977 we are working on a research project in which we try to contribute to a better understanding of the development of young children's arithmetic problem-solving skills and processes and of the influence of formal instruction on that development. In one longitudinal investigation we collected empirical data on the development of children's representations and solution processes with respect to elementary arithmetic word problems (De Corte & Verschaffel, 1984, 1985, in press). In another recent study we analysed a representative sample of six instructional programs that are frequently used in first grade mathematics education in Flemish elementary schools. The analysis focused on teaching word problems in the first grade (De Corte, Verschaffel, Janssens & Joillet, 1984). Based on the findings of both studies, we criticized current educational practice with respect to elementary arithmetic word problems and formulated suggestions for its improvement (De Corte et al., 1984). Starting from these research findings, and the derived criticisms and suggestions, Zwerts (1984) constructed an experimental teaching program for the teaching of word problems in the first grade. In the next section we summarize the main characteristics of this program. Thereafter we discuss an exploratory teaching experiment in which this program is implemented and evaluated.

## MAIN CHARACTERISTICS OF THE EXPERIMENTAL PROGRAM

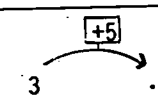
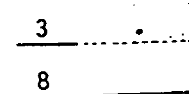
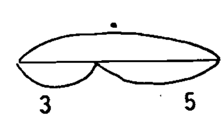
In current programs for mathematics instruction in the first grade, word problems are introduced after children have learned the formal operations of addition and subtraction, and are able to solve simple number sentences successfully. In other words, word problems are assigned an application function : it is expected that, through solving them, the pupils will learn to apply the formal arithmetic operations to cope intelligently with different kinds of (real) problem situations. It is questionable whether it is proper to assign to word problems only this application function in elementary mathematics. Indeed, recent work on addition and subtraction word problems has produced strong evidence that young children who have not yet had instruction in formal arithmetic, can nevertheless already solve those problems successfully using a wide range of informal strategies that model closely the semantic structure and meaning of the distinct problem types (Carpenter & Moser, 1982; De Corte & Verschaffel, 1984). This finding suggests that word problems could, more than has been the case hitherto, be mobilized in the first grade to promote a thorough understanding of the formal arithmetic concepts and operations. As Carpenter & Moser (1982, p. 9) have stated, verbal problems "could represent a viable alternative for developing addition and subtraction concepts in school". In accordance with this view, Zwerts' (1984) program does not postpone the teaching of word problems until children have learned the formal operations of addition and subtraction; on the contrary, word problems are presented before introducing these arithmetic operations and the related numerical sentences.

As already suggested in the preceding paragraph recent investigations have revealed a strong relationship between children's performances, solution strategies, and errors on simple addition and subtraction word problems and the semantic structure of these problems. A widely accepted categorization of simple addition and subtraction problems nowadays, distinguishes three semantically different types of problems : change, combine, and compare. Change problems refer to dynamic situations in which some event changes the value of a quantity. In combine problems two amounts are involved, which are considered either separately or in combination. Compare problems involve two amounts that are compared and the differences between them. Each of these three classes is further subdivided in distinct problems types depending on the identity of the unknown quantity; for the change and the compare problems further distinctions are made depending on the direction of the event (increase or decrease) or the relationship (more or less). Combining these three characteristics fourteen problem types are distinguished (De Corte & Verschaffel, 1984, in press; see also Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983). In our analysis of six instructional program currently used in first grade mathematics

education in Flanders, the verbal problems from these programs were classified according to the fourteen different types of simple addition and subtraction problems. This revealed a remarkable restricted, one-sided and stereotyped support of verbal problems in current instructional practice : in several programs there was a substantial preponderance of certain problem types, while other problem types were very scarce or even totally absent. For example, in half of the programs no single compare problem was found (De Corte et al., 1984). Taking into account the recent evidence concerning the importance of the semantic structure on children's solutions of word problems, in Zwerts' (1984) program the formal arithmetic operations of addition and subtraction are introduced and exercised using the whole range of problem types distinguished in the above-mentioned classification schema.

In almost all the instructional programs that we analysed, children are taught to solve verbal problems using one type of visual representation of the relations between the quantities involved in the problem, either an arrow diagram or a venn diagram; after making this graphic representation they must then choose (and write down) the appropriate arithmetic operation (De Corte et al., 1984). This raises the following question : is it appropriate and justified to teach children only one form of graphic representation, for example the arrow diagram, to solve all kinds of word problems ? In our opinion, the arrow diagram is very appropriate for representing the dynamic nature of change problems, but much less suitable for addition and subtraction problems with a combine and compare structure. Other kinds of graphic representation are probably more appropriate for representing the main relations between the quantities in these categories of verbal problems, namely the part-whole and the matching schema respectively.

Table 1. Diagrams for representing the three main categories of addition and subtraction word problems

| Type | Example | Diagram |
|---|---|---|
| Change | Pete had 3 apples; Ann gave him 5 more apples; howe many apples does Pete have now ? |  |
| Compare | Pete has 3 apples; Ann has 8 apples; how many apples does Ann have more than Pete ? |  |
| Combine | Pete has 3 apples; Ann has 5 apples; how many apples do they have altogether ? |  |

When children have one generalized and uniform schema imposed on them, they are forced to reinterpret the verbal text of many of the problems in terms of this schema which is totally unnecessary for finding the solution and can bear a negative influence on the development of their understanding of the related arithmetic operations. In Zwerts' (1984) program, children are taught different models for representing different types of simple word problems (see Table 1).

Finally, Zwerts' (1984) program also contains a few tasks (such as unsolvable word problems) intended to develop the so-called word problem schema in children. This schema reflects the subject's knowledge about the typical structure and function of arithmetic word problems in general. In our research we have shown that beginning first graders sometimes fail on elementary arithmetic word problems just because of their unfamiliarity with this new and peculiar kind of texts (De Corte & Verschaffel, in press).

## THE TEACHING EXPERIMENT

During the school year 1983-84, Zwerts (1984) implemented this program in one first-grade class of a primary school. The experimental group was administered a pretest and a posttest consisting of eight addition and subtraction word problems, representing eight different categories of the above-mentioned classification schema. The results were compared with those of a control group taught according to the regular arithmetic program; the main characteristics of which are in accordance with the current instructional programs as described in the previous section.

Table 2 presents the results of the quantitative analysis of the performances of both groups on the pre- and the posttest. It shows that the experimental group made nearly twice as much progress from the pretest to the posttest as the control group.

Table 2. Results (in %) of the experimental and the control groups (n=18) on the pretest and the posttest

| | Pretest | Posttest |
|---|---|---|
| Experimental group | 47% | 87% |
| Control group | 43% | 63% |

A qualitative analysis of the solution processes of the children who solved a problem correctly reveals that at the time of the posttest both groups attacked the problems differently. Most control children used only one kind of diagram to represent all the problem types, namely a venn diagram. On the contrary, the children of the experimental group spontaneously applied different diagrams to represent the distinct problem types.

Finally, a qualitative analysis of the wrong answers of the control children on the posttest suggests that many of these errors result directly from teaching them only one type of diagram. Indeed, many children erroneously answered a problem with the result of the opposite operation (an addition with the two given numbers instead of a subtraction, or vice versa) after filling in the numbers inappropriately in their uniform graphic representation, in this case the venn diagram. In the experimental group such "wrong operation" errors resulting from filling in the known and unknown numbers incorrectly in the selected schema, were found very rarely.

## CONCLUSIONS

It is a well-documented finding that elementary school children have a lot of difficulties in solving arithmetic word problems. Zwerts' (1984) exploratory teaching experiment suggests that an instructional program presenting word problems to children before they master the corresponding numerical sentences, and stimulating them to use distinct diagrams for representing different problem types, has a positive effect on their problem solving capacities. However, the limited scope of this study and several methodological weaknesses (some of which relate to our concern for external validity), warn against hasty and definitive conclusions. Taking into account similar findings obtained by other researchers (Lindvall, Tamburino & Robinson, 1982; Wolters, 1983), it is nevertheless our conviction that the present study justifies continuation of this line of work. But this future research will have to correct the main weaknesses and deficiencies of Zwerts' (1984) teaching experiment. We mention two examples. First, in the present version of the program, too little attention is paid to the transition from children's initial informal counting strategies to the formal arithmetic operations of addition and subtraction. A second critic relates to the relatively "poor" content and stereotyped nature of the word problems. To accomplish more effectively their function in the acquisition of the concept of addition and subtraction it is probably necessary to introduce a varied set of more realistic word problems than was the case in the present investigation. In this respect Treffers & Goffree (1982) distinguish between traditional school word problems, and so-called context problems. This latter type of problems have the following features : (1) their format is less

stereotyped; (2) the situations described are more attractive and challenging to children; (3) in solving context problems the child's knowledge of the real world is mostly useful, often even necessary, while this knowledge may often be hindering in solving traditional school word problems.

Note

L. Verschaffel is a Senior Research Assistant of the National Fund for Scientific Research, Belgium.

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# THE DEVELOPMENT OF NUMBER CONCEPTS IN LOW ATTAINERS IN MATHEMATICS AGED SEVEN TO NINE YEARS

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## Introduction

The strategies which low attaining pupils used to answer questions about number were observed in a series of interviews and used to infer pupils' understanding of counting, addition and subtraction and place value. Hypotheses about the orders of acquisition of number concepts were tested and a descriptive framework was found which included 'hierarchical strands', that is some skills were found to be 'pre-requisite' for others. A diagnostic assessment instrument was developed alongside the descriptive framework and this was continually modified until it gave valid and reliable results. Subsequently pupils' performances in the diagnostic assessment were examined in relation to the hierarchical framework and used as the basis for a remedial teaching programme.

## The descriptive framework and the diagnostic assessment interview

The descriptive framework (Figure 1a) is based on responses given by three small samples of pupils shown in Table I.

TABLE I

Samples used in developing the descriptive framework

| Study | n | Age Range | No. of Interviews |
|---|---|---|---|
| Pilot study | 5 | 8-2 to 9-1 | 6 |
| Main assessment study | | 7-5 to 9-3 | 6 |
| Longitudinal study | 7 | 7-5 to 11-2 | 4 |
| Diagnostic assessment trials | 41 | 7-5 to 9-5 | 1 |

Different types of skills were assessed including, for example, 'rote' learning as in the sequence of counting words; experiential learning such as familiarity with different ways of representing numbers; and application as in the solution of certain categories of word problems. The forty seven assessed skills are listed in Table II.

TABLE II

Skills assessed in D.A.I. in order of difficulty

3. Mentally carried out two digit 'take away' with regrouping
6. Uses multiplication facts to solve a 'sharing' word problem
47. Perceives 'compare difference unknown' word problem as subtraction

TABLE II (continued)

4. Models two digit 'take away' with regrouping using apparatus
45. Fully appreciates concept of class inclusion
7. Mentally carries out two digit take away without regrouping
5. Uses multiplication facts to solve a 'lots of' word problem
2. Mentally carries out two digit addition with regrouping
20. Uses counting back/up/down strategy for 'take away'
33. Groups objects to enumerate a partly grouped collection
15. Uses repeated addition/subtraction for a 'sharing' word problem
46. Partly appreciates concept of class inclusion
12. Uses derived facts for addition
8. Mentally carries out two digit addition without regrouping
24. Can count in 10s from a non-decade two digit number
25. Can count backwards in 10s from a non-decade two digit number
34. Quantitatively compares differently grouped collections
9. 'Knows answer' when taking ten away from a two digit number
10. 'Knows answer' when adding ten to a two digit number
1. Models two digit addition with regrouping using apparatus
17. Knows number bonds (not just the 'doubles')
26. Interpolates between decade numbers on a number line
13. Models two digit 'take away' (no regrouping) using apparatus
16. Uses repeated addition for a 'lots of' word problem
19. Solves 'compare (more) difference unknown' word problem
21. Counts in 2s and 1s for a collection grouped in 2s
11. 'Knows answer' when adding units to a decade number
14. Models two digit addition (no regrouping) using apparatus
22. Counts in 5s and 1s for a collection grouped in 5s
18. Solves 'compare (more) compared set unknown' word problem
23. Counts in 10s and 1s for a collection grouped in 10s
40. Knows numbers backwards from 20
27. Orders a selection of non-sequential two digit numerals
35. Appreciates structure of grouped collections
38. Solves sharing problem by direct physical modelling
39. Solves 'lots of' problem by direct physical modelling
44. Appreciates conservation of number
28. Appreciates commutativity of addition for '1 + n'
29. Uses a counting-on strategy for addition
30. Reads a selection of non-sequential two digit numerals

TABLE II (continued)

32. Knows number sequence for counting in 2s, 5s and 10s
37. Uses counting on strategy when provoked
31. Repeats numbers in correct sequence to 99
41. Knows numbers backwards from 10
36. Compares collections and states whether equal
42. Knows number sequence to 20, can add and take away with cubes
43. Makes 1:1 correspondence

For the pupils interviewed, some skills appeared to be pre-requisite for others. In order for skill A to be deemed pre-requisite for skill B, three criteria had to be satisfied.

(i) All, or nearly all the pupils who 'passed' on skill A must also have 'passed' on skill B. In practice values of H > 0.80 (Loevinger, 1947) were accepted

(ii) In the longitudinal study pupils should acquire skill A before skill B

(iii) There should be, in the researcher's opinion, a logical reason why skill A should be mastered before skill B.

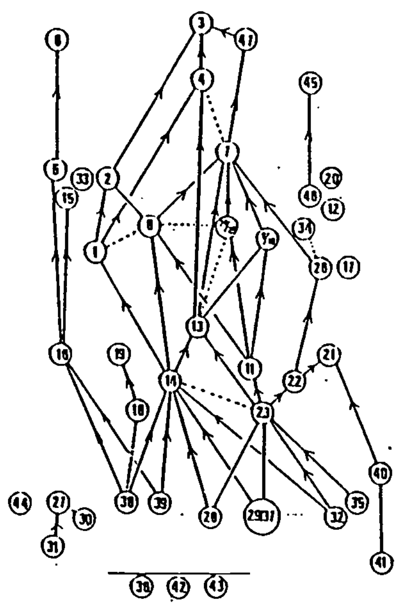


Figure 1a

Descriptive framework

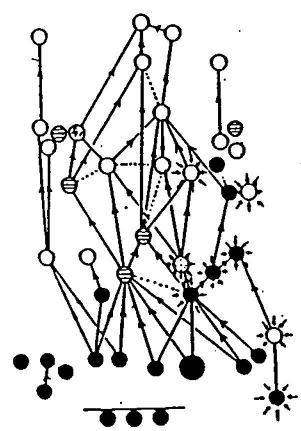


Figure 1b - Change in performance during pilot teaching study for Jy

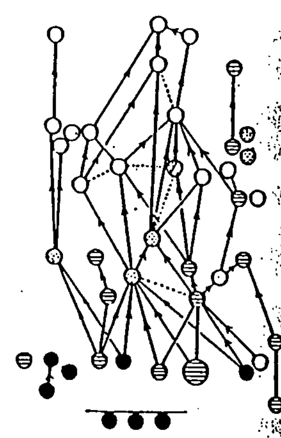


Figure 1c

Change in performance during main teaching study for Nan.

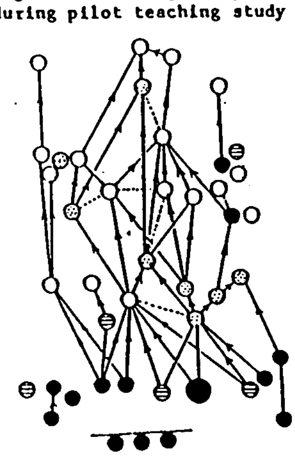
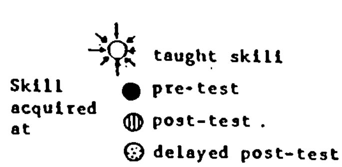


Figure 1d

Change in performance during main study for Jos.



The teaching studies

Pupils' performances in the diagnostic assessment, examined in relation to the descriptive framework provide a basis for designing and evaluating a teaching programme. In a pilot teaching study seven children (the main assessment study sample) were taught individually and in the main teaching study, twelve children (a sub-sample of the Diagnostic Assessment trial sample) were taught in Group A (n = 5) or Group B (n = 7).

The model of learning adopted might be termed an 'Ausubelian, skill-integrationist' approach. Following Piaget (1952) individuals are seen to learn as a result of their physical actions on the environment and their mental actions relating to these experiences. They are thought to integrate skills which they have already acquired when these are simultaneously called in to play in their mental actions, (Schaeffer et al, 1974). Consequently a suitable starting point for the design of activities for a particular learner is 'what the learner already knows' (Ausubel et al, 1978). The view is also taken that perceiving the 'equivalence of different intellectual strategies' (Bryant, 1982) is likely to lead to an association between mental skills which could result in integration of these skills, a notion that is supported by Lawler's (1981) work.

This model of learning has several important implications for teaching:

1. Children must interact with the physical world and also reflect on these interactions.
2. Ideas and materials presented must be related to what children already know.
3. In order to achieve integration of skills there may need to be repetition of the mental process involved in appropriate tasks.
4. In order to acquire mathematical concepts, children will need, as Dienes (1960) suggested, a variety of examples of those concepts in different mathematical forms, different contexts and, possibly, in different modes.
5. It is likely that as Bryant (1982) and Lawler (1981) suggest, children learn when different intellectual strategies turn out to produce the same result, especially if, in Lawler's word 'none was anticipated'.

In the pilot study each child was taught individually because

(i) The children possessed different skills so the selected teaching points were different for each child

(ii) The speed at which a new idea or skill was grasped would vary from one child to another

(iii) It was intended to maintain a stance of accepting and respecting each child's view during the teaching programme, in order to allow room for reflection, but in a group, the children's shared perceptions of 'right' and 'wrong' would make this difficult to achieve.

In the Main Teaching Programme there were two major changes:

1. In order to promote discussion and to limit children's self conciousness, pupils were taught in a group instead of individually. This had the advantage of being more readily transferrable to normal classroom practice.
2. Children were given general number activities which embodied concepts which they had already grasped as well as concepts which they might acquire. Where appropriate, new skills or concepts would be suggested but there would be a less specific intention to teach particular points.

Results and conclusions

The changes in performance from pre- to post- to delayed post-test for Jy (pilot teaching study), Nan (main teaching study) and Jos (main teaching study) are shown in relation to the descriptive framework in figures 1b, 1c and 1d respectively. Figure 1b also shows which skills Jy was taught.

The main conclusions of the teaching studies were:

1. The hierarchical framework for describing children's acquisition of number concepts can be used in designing a remedial teaching programme for teaching number to 7-9 year old low attainers.
2. The hierarchy is most useful for
   (a) describing each child's knowledge of number which can then be used as a basis for extending the child's understanding
   (b) establishing which cognitive skills the child is likely to learn.

   It is less useful as a predictor of precisely which, or how many, skills a child will learn.

3. The cognitive skills which children acquire fit the hierarchical framework tolerably well although child's performance at any stage of learning is unlikely to fit it exactly.
4. Children made substantial progress when they were taught using the hierarchical framework as a basis. This happened whether the children were taught individually or as a group who were at a roughly similar level of performance, although in fact, the overall progress of children taught in a group was greater than that of those taught individually.
5. Progress made during the teaching programme, was in nearly every case not merely maintained, but increased in the following months. This suggests that the children had made real gains in conceptual understanding which allowed them to integrate some new concepts into their framework of understanding.
6. The children who made most progress were those who appeared to involve themselves most in the practical tasks and the discussion.

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# Teaching Multi-Digit Addition and Subtraction

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First- and second-grade children were successfully taught to add and subtract multi-digit numbers by first adding and subtracting with pieces of wood differing in size. Most children successfully generalized the addition and subtraction procedures to ten-digit symbolic problems.

First-grade children (age 6 at school entry) of above-average mathematics ability and second-grade children of all ability levels were taught to add and to subtract multi-digit numbers by first adding and subtracting with a physical representation of base ten numbers in which the pieces for each place differed in size by a multiple of ten. The ones (unit) pieces were white Cuisenaire rods (wooden cubic centimeters), the tens pieces were orange Cuisenaire rods (1 cm x 1 cm x 10 cm), the hundreds pieces were orange Cuisenaire flats (1 cm x 10 cm x 10 cm) and the thousands pieces were orange Cuisenaire large wooden cubes (10 cm x 10 cm x 10 cm) or cardboard versions of these. Calculations were done on a cardboard calculation sheet divided into three horizontal rows (for the two addends and the sum in addition or for the minuend, subtrahend, and difference in subtraction) and into four vertical columns labeled by place name (ones, tens, hundreds, thousands) and by a picture of the wood piece for that column. To add, the wooden pieces representing an addend were placed across the top row (e.g., for 2765, 2 large cubes, 7 flats, 6 longs, and 5 little cubes were used). Wooden pieces for the other addend were placed across the second row. Addition was done one column at a time by moving the pieces in a column down to the bottom row. A somewhat simplified subtraction algorithm was used. Children did all of the trades (borrowing, regrouping) with the wood first (until every top number was larger than the bottom), and then did all of the subtracting. We hoped that this would be easier than alternating between trading and subtracting and that it would help to decrease one of the most common subtraction errors: subtracting the top number from the bottom number.

Two measures were taken to help children relate the wood to the base ten symbols and relate the operations with the wood to the operations with the base-ten symbols. First, in earlier work with the Dienes multi-base blocks with teachers and with children (the resulting teacher-training materials are in Bell, Fuson, & Lesh, 1976), I found that the wood functions only as an answer-getting device, as a wooden calculator, if users do not link their actions with the wood very closely to their actions with the symbols. Therefore, in that work and in the present study recording was done in the symbolic problem <u>as each move</u> (trading, adding, or subtracting) was done with the wood. Second, the wood and symbols were related by frequent teacher and child verbalizations of three sets of terms: wood names (e.g., two big cubes, seven flats, six longs, five little cubes), our number words (e.g., two thousand

seven hundred sixty five), and base ten words (e.g., two seven six five). Two functions of these verbalizations were to give meanings to our number words and number symbols and to provide a vocabulary for later discussion of operations on the wood when the wood was not physically present.

All of the children in the study had learned earlier in the year to add and to subtract single-digit numbers using one-handed finger patterns to keep track of the number counted on (for addition) or counted up (for subtraction) (see Fuson, in press; Fuson & Secada, 1985; Hall, Fuson, & Willis, 1985). Therefore they did not have trouble finding the sum or the difference in any given column. Seven classes ranging from above-average first graders through all ability levels of second graders participated in the study. Project staff did all of the teaching in the first class taught and did the introduction in the next two classes taught. Teacher lesson plans were prepared to accompany the student worksheets for the teachers of the other four classes; these teachers did the teaching for their classes. Each teacher had an initial orientation to the teaching units and some classroom visits from the project staff. The multi-digit instruction was begun at different times in the year. To date, the addition teaching has been completed in six classes and the subtraction teaching, in three classes. Retention tests were given at monthly intervals after instruction, followed by a short review and another test. One final round of testing is scheduled for May and is not presented in this paper. The children in the study represented a range of socioeconomic and racial/cultural groups.

The performance data in Tables 1, 2, and 3 indicate that the addition and subtraction instructional units were successful at each of the ability levels and grades taught. Mean scores for every classroom improved significantly and considerably from the pretest to the posttest on both timed and untimed tests.

For addition, four of the classes were near ceiling on the test of 2- and 3-digit sums, and the other two classes had a mean above 75% correct. The means on the more difficult 4- and 6-digit problems ranged from 45% to 85% correct performance. Scores dropped on all addition tests after the teaching of multi-digit subtraction. Some children seemed to experience interference between the two multi-digit operations. However, scores generally rose to posttest levels or above on the test given in the next month. In those classes in which time permitted following for at least two months after instruction, children got considerably faster at multi-digit addition; scores on the timed test rose between 25% and 66%. The high initial scores on the easy (2- and 3-digit) untimed test left little room for improvement. On the more difficult (4- and 6-digit) test only the class with five months of test data showed a significant improvement. Children were fairly accurate on the timed test (between 67% and 81% correct problems of those problems tried).

To test generalization of the multi-digit addition procedure, a single ten-digit plus ten-digit addition problem was given. Three non-adjacent columns of the problem did not require trading (carrying, regrouping) in order to test nonrote application

of the trading procedure. One point was given for each correct digit in the answer. Children loved doing such large problems. The posttest means on this problem ranged from 8.4 to 9.4 out of the possible 10, indicating highly accurate trading and single-digit calculation when the children were highly motivated.

In subtraction, all 3 classes were near ceiling on the test of 2- and 3-digit differences and had means between 50% and 80% on the test of 4- and 6-digit differences. In the two classes for which results at least 2 months after instruction are available, children got considerably (about 50%) faster at subtracting multi-digit numbers. All classes showed competence at dealing with problems having zeros in the minuend, with the class having the most months of practice reaching ceiling on this task. The posttest means on the ten-digit subtraction problem were 8.5 for all classes, again indicating highly accurate trading (borrowing, regrouping) and single-digit calculation.

Interviews were held with children at each ability level whose scores indicated low performance on one or more tests. Preliminary analyses of these interviews indicate that the primary difficulty was interference between addition and subtraction, manifesting itself chiefly in an inability (temporary, in most cases) to get started adding or subtracting. In most cases general comments by the interviewer which required the child to remember the procedure done with the wood (e.g., "What did you do with the wood?") were sufficient for the child to reconstruct the correct symbolic procedure and continue to use it on subsequent problems. Some children had to go back to the wood temporarily, but all could do the procedure with the wood and almost all could then complete the symbolic procedure without the wood.

These results indicate that multi-digit addition and subtraction can be successfully taught much earlier than it is now taught in American schools and that the almost universal practice of doling out one more place in each successive grade (2-digits in second grade, 3-digits in third grade, etc.) is unnecessary and may even be counterproductive.

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Table 1

Mean Number Correct and Mean Percent Correct Timed Addition Test of Two-digit to Six-digit Sums

| Grade Ability School | n | Pre | Post | | 1 Month BR[a] | 1 Month AR[b] | | 2 Month BR | 2 Month AR | 3 Month BR | 3 Month AR | 4 month BR | 4 month AR | 5 Month BR | 5 Month AR |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 High A | 23 | 0.4 | < 6.5 | s | | | | | | | | | | | |
| 2 Low A | 14 | 1.8 | < 5.7 | s | 4.9 | 6.1 | s | | | | | | | | |
| 2 Low B | 17 | 1.9 | < 5.2 | s | | | | | | | | | | | |
| 2 Average A | 25 | 1.3 | < 6.3 | | $^{+}8.3$ | < $9.1^{+}$ | s | 6.1 | 6.3 | $^{+}7.8$ | < $9.1^{+}$ | | | | |
| 2 Average B | 21 | 2.7 | < 5.9 | s | 5.2 | 4.9 | | 6.2 | < $7.5^{+}$ | | | | | | |
| 2 High A | 24 | X | 5.9 | s | 5.0 | 5.6 | | 6.6 | $6.9^{+}$ | $^{+}8.4$ | $8.5^{+}$ | $^{+}8.0$ | < $9.5^{+}$ | $^{+}8.7$ | < $9.8^{+}$ |
| | | | | | | | | | | | | | | | |
| 1 High A | 23 | 15% | < 74% | s | | | | | | | | | | | |
| 2 Low A | 14 | 27% | < 74% | s | 68% | 77% | s | | | | | | | | |
| 2 Low B | 17 | 19% | < 67% | s | | | | | | | | | | | |
| 2 Average A | 25 | 15% | < 80% | | 87% | 87% | s | 75% | > 65% | 90% | 86% | | | | |
| 2 Average B | 21 | 52% | < 81% | s | 73% | 66% | | 75% | 81% | | | | | | |
| 2 High A | 24 | X | 79% | s | 75% | 78% | | 84% | 89% | 88% | 84% | 76% | 86% | 85% | 89% |

Note. Each score in the top 6 rows is the mean number of correct answers on an 11-item 3-minute test of 2, 1, 4, 1, 3 addition problems of 2, 3, 4, 5, and 6-digit numbers, respectively. One 2-digit problem had no carries, and 2, 3, 3, 1, and 1 problems had 1, 2, 3, 5, and 6 carries respectively. Each score in the bottom 6 rows is the percent of problems correct out of the problems tried within the 3-minute time limit. The ability groups are math ability (assigned before any of our instruction) based on tests and teacher judgment.

< indicates a t-test significant at the .05 level.
\- indicates a score significantly lower than the posttest ($p < .05$).
\+ indicates a score significantly higher than the posttest ($p < .05$).
s denotes the timing of the subtraction teaching.
X denotes no test given (the first pretest given was not difficult enough).

[a]Before Review [b]After Review

Table 2

Mean Number Correct on Untimed Addition Tests of 2- and 3-Digits and of 4- and 6-Digits

| Grade Ability School | Pre | Post | | 1 Month BR[a] | 1 Month AR[b] | | 2 Month Br | 2 Month AR | 3 Month BR | 3 Month AR | 4 Month BR | 4 Month AR | 5 Month BR | 5 Month AR |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 High A | 0.1 < | 3.8 | s | | | | | | | | | | | |
| 2 Low A | 1.0 < | 2.6 | s | 2.8 | 3.1 | s | | | | | | | | |
| 2 Low B | 0.9 < | 3.1 | s | | | | | | | | | | | |
| 2 Average A | 0.4 < | 3.5 | | 3.8 | 3.5 | s | -2.4 | 2.6- | 3.4 | 3.9 | | | | |
| 2 Average B | 2.3 < | 3.7 | s | -2.4 | 2.4- | | 3.4 | 3.6 | | | | | | |
| 2 High A | X | 3.5 | s | 3.2 | 3.4 | | X | X | 3.7 | 3.7 | 3.6 | 3.8 | 3.6 | 3.7 |
| | | | | | | | | | | | | | | |
| 1 High A | 0.0 < | 3.4 | s | | | | | | | | | | | |
| 2 Low A | 0.9 < | 2.6 | s | 2.5 | 2.5 | s | | | | | | | | |
| 2 Low B | 0.2 < | 1.8 | s | | | | | | | | | | | |
| 2 Average A | 0.2 < | 3.2 | | 3.3 | 3.0 | s | 2.1 | 1.9- | 2.8 | 3.2 | | | | |
| 2 Average B | 1.0 < | 2.6 | s | 1.9 | 2.4 | | 2.6 | 2.8 | | | | | | |
| 2 High A | X | 2.8 | s | 2.0 | 2.1 | | X | X | 3.2 | 3.0 | +3.3 | 3.3+ | +3.1 | 3.2+ |

Note. Each score in the top 6 rows is the mean number of correct answers on a 4-item untimed test of 2 2-digit and 1 3-digit addition problems with one carry each and 1 3-digit problem with 2 adjacent carries. Each score in the bottom 6 rows is the mean number of correct answers on a 4-item untimed test of 2 4-digit and 2 6-digit addition problems with 2 non-adjacent, 2 adjacent, 5 adjacent, and 6 adjacent carries, respectively.

< indicates a t-test significant at the .05 level.
\- indicates a score significantly lower than the posttest ($p < .05$).
\+ indicates a score significantly higher than the posttest ($p < .05$).
s denotes the timing of the subtraction teaching.
X denotes no test given.

[a]Before Review [b]After Review

Before Review After Review

Table 3

Mean Number Correct and Mean Percent Correct on Timed Subtraction Test and

Mean Number Correct on Untimed Tests of 2- and 3-Digits, of 4- and 6-Digits, and of Minuends with Zeros

| Grade Ability School | Pre Post | 1 Month | | 2 Month | | 3 Month | | 4 Month | | Possible Score |
|---|---|---|---|---|---|---|---|---|---|---|
| | | BR[a] | AR[b] | BR | AR | BR | AR | BR | AR | |
| 2 Average A | 1.0 < 7.5 | -6.4 | 7.0 | | | | | | | (11) |
| 2 Average B | 1.0 < 5.1 | 5.7 | 6.3+ | +6.7 | 7.4+ | | | | | |
| 2 High A | 1.0 < 6.4 | 6.7 | X | +7.6 | 7.8+ | +8.3 < | 9.3+ | +9.0 | 9.6+ | |
| 2 Average A | 7% < 76% | 67% | 72% | | | | | | | (100%) |
| 2 Average B | 11% < 71% | 68% | 71% | 69% | 76% | | | | | |
| 2 High A | 9% < 80% | 82% | 80% | 84% | 84% | 82% | 86% | 83% | 81% | |
| 2 Average A | 0.1 < 3.2 | 3.1 | 3.4 | | | | | | | (4) |
| 2 Average B | 0.2 < 3.1 | 3.0 | 3.1 | 3.3 | 3.6 | | | | | |
| 2 High A | 0.3 < 3.5 | X | 3.2 | X | X | 3.6 | 3.5 | 3.6 | 3.4 | |
| 2 Average A | 0.1 < 2.6 | 2.3 | 2.2 | | | | | | | (4) |
| 2 Average B | 0.0 < 2.0 | 2.2 | 2.3 | 2.5 | 2.6 | | | | | |
| 2 High A | 0.1 < 2.6 | X | 2.9 | X | X | 3.0 | 3.2 | +3.2 | 3.1+ | |
| 2 Average A | .0 < 4.7 | 4.5 | 4.2 | | | | | | | (6) |
| 2 Average B | .0 < 3.6 | 3.9 | 4.2 | 3.8 | 4.6 | | | | | |
| 2 High A | .0 < 3.9 | X | 5.0+ | X | X | +5.6 | 5.1 | +5.2 | 5.5+ | |

Note. The addition tests in Tables 1 and 2 were used to construct the subtraction tests for the top 12 rows. Each subtraction item is an inverse problem of the corresponding addition item. Each score in the bottom 3 rows is the mean number of correct answers on a 6-item untimed test of 2, 2, and 2 subtraction problems of 2-, 3-, and 4-digits, respectively, with 3 problems with one 0 in the minuend, 2 problems with 2 0's, and 1 problem with 3 0's in the minuend. Six minutes were given for the timed subtraction test, whose scores are reported in the top 6 rows.

< indicates a t-test significant at the .05 level.
- indicates a score significantly lower than the posttest ($p < .05$).
+ indicates a score significantly higher than the posttest ($p < .05$).
X denotes no test given.

[a]Before Review [b]After Review

Teaching Counting On for Addition and Counting Up for Subtraction

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The present study extends earlier attempts to develop and test teaching units to move children from primitive object solution procedures to the use of the more efficient counting-on procedure for addition and counting-up procedure for subtraction. Evidence for the effectiveness and feasibility of these units is described, and issues concerning spaced review and interference effects are considered.

Children in many countries follow a developmental progression in the counting solutions which they spontaneously invent to solve addition and subtraction problems (see Carpenter & Moser, 1984, and DeCorte & Verschaffel, 1985, for recent reports and Carpenter & Moser, 1983, and Riley, Greeno, & Heller, 1983, for recent reviews). In earlier work (Fuson, in press; Fuson & Hall, 1984; Fuson & Secada, 1985), teaching units were developed to move children from primitive object solution procedures (counting all the objects for addition and taking-away some objects for subtraction) to more advanced and efficient verbal sequence solution procedures: counting on for addition (for 8 + 5, "8, 9, 10, 11, 12, 13") and counting up for subtraction (for 13 - 8, "8, 9, 10, 11, 12, 13; five more so the answer is 5"). Five regular classroom teachers successfully used the teaching units to teach children to count on both with and without concrete objects and to count up for subtraction. The units contained problems in which both addends (or the subtrahend and difference) were single-digit numbers and the sum (or minuend) was between 11 and 18.

Details regarding those teaching units were described in the reports cited above and will not be given here except to note the implementation of an unusual procedure by which children were taught to keep track of their progress in counting. Many children use their fingers to keep track of the words counted as they count up or count on, and both hands often are used for this purpose when the counting exceeds five. Children often put down their pencils to do a problem, either to use their writing-hand fingers or the fingers on both hands; this putting down is slow and distracting. We developed and successfully implemented a finger-pattern technique by which only the non-writing hand is used to keep track (see Figure 1).

The present study extended that earlier work in several ways: More classrooms (10) and more lower ability children were involved, teachers chose where to fit in the units, and progress was monitored and reviews were conducted over a longer period of time. The teaching units consisted of minor modifications of those used earlier. Our objectives were to explore more fully, and with a wider variety of children and teachers, the effectiveness and feasibility of our teaching units, and of a spaced review procedure following initial teaching.

## Method

Six classes of Grade 1 (ages 6 and 7) and 4 classes of Grade 2 children (ages 7 and 8) participated. The classes were grouped by ability in mathematics and ranged from low-ability children in Grade 1 to average-ability children in Grade 2(see Tables 1 and 2). Of the 10 teachers involved, 5 were familiar with our units from previous work. Some of the second-grade children had had the units in the previous year.

The timing of implementation of the teaching units and the amount of related instruction that had occurred previously varied considerably among the classes, mainly as a function of the ability level of the class. Eight of the classes required a period between 3 and 5 weeks to complete the addition units. The remaining two classes (the low-ability Grade 1 classes) preceded these units with other units in which lower sums (between 8 and 12) were involved, so that the total time for addition in those classes was about 8 weeks. The subtraction units required less than 3 weeks for all classes. Tests were given just before and just after completion of the addition and of the subtraction phases. In each of the following months the children were tested, given a brief (about 5-min.) review, then immediately tested again. Final testing in May is scheduled.

## Results and Discussion

The performance data in the left-hand columns in Tables 1 and 2 indicate that both the addition and the subtraction teaching were successful at every ability level in both grades. Children in every classroom improved significantly on the counting-on test, the addition test, and the subtraction test. Subtraction posttest scores did not differ significantly from the addition posttest scores. As in the earlier study, teaching subtraction as counting up was quite easy after children knew how to count on. This seems to be true partly because counting up is easier to carry out than is counting on because the feed-back loop for stopping the counting in counting up is visual and does not depend upon overlearning of the finger patterns.

The decline in some addition scores from the initial posttest to the first 1-month test may be due to the teaching of subtraction during that interval. However, there were several classes in which that decline did not occur even though subtraction was taught in that interval. Perhaps initial learning was not as strong in the former cases or perhaps the teachers in the latter classes took specific steps to reduce such interference. Note that in 2 of the classes in which addition performance declined, the short review was sufficient to restore performance to the posttest level. It would seem worthwhile to study such interference effects and means by which they might be reduced (Hall, 1984).

Another way to assess the success of the present units is to examine the achievement of the children on subsequent topics for which high levels of skill in subtraction and addition of single-digit numbers (i.e., addends through 9) are prerequisite. Successful learning of multi-digit addition by all six of our most

advanced classes, and of multi-digit subtraction by the two most advanced classes (teaching is in progress in the other four classes), seems encouraging in that respect (Fuson, 1985).

Turning now to the monthly test-review sessions, it appears that some combination of the initial test and brief review were beneficial, in that performance often improved from the first to the second test on those occasions. It is not possible here to determine the separate contributions of the initial tests and the reviews within these sessions. Note also that in general there seems to have been improvement in performance over the several-month period following the posttest. However, in some cases it was not possible to continuously track improvement through the year because of ceiling effects which necessitated shifting to more difficult tests. One purpose of the monthly sessions was to begin working toward a program of spaced brief review, a procedure by which forgetting may be retarded and a level of relatively effortless performance may be obtained. Although the advantages of spaced review have been well known for many years, schools in the United States typically do not implement such procedures in any systematic way.

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Table 1

Counting On and Addition Test Means by Class

| Grade Ability School | Counting On Pre | | Counting On Post | Addition Pre | | Addition Post | | 1 Month BR[a] | | 1 Month AR[b] | | 2 Month BR | 2 Month AR | 3 Month BR | | 3 Month AR | 4 Month BR | | 4 Month AR | 5 Month BR | | 5 Month AR |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 Low A | 3 | < | 11 | 1 | < | 9 | | 9 | | 11 | s | 9 | 11 | | | | | | | | | |
| 1 Low B | 3 | < | 11 | 2 | < | 10 | s | $^{-}6$ | < | 10 | | 9 | 11 | | | | | | | | | |
| 1 Average A | 3 | < | 14 | 2 | < | 10 | | $^{+}12$ | | $12^{+}$ | | | | | | | | | | | | |
| 1 Average B | 3 | < | 12 | 2 | < | 11 | s | $^{-}5$ | | $6^{-}$ | | X | X | $^{-}8$ | < | 13 | 10 | < | $13^{+}$ | | | |
| 1 High A | 5 | < | 12 | 3 | < | 13 | s | 13 | < | $16^{+}$ | | X | X | 14 | | $16^{+}$ | 48 | < | 51 | 55 | < | 59 |
| 1 High B | 6 | < | 13 | 7 | < | 13 | s | $^{-}11$ | < | 15 | | X | X | $^{+}16$ | < | 19 | 32 | < | 38 | 52 | < | 66 |
| 2 Low A | 10 | < | 14 | 11 | < | 14 | s | 14 | < | $17^{+}$ | | X | X | 16 | < | $17^{+}$ | 47 | | 52 | 61 | | 59 |
| 2 Low B | 8 | < | 14 | 7 | < | 14 | s | $^{-}11$ | | $11^{-}$ | | X | X | 15 | < | $17^{+}$ | 45 | | 44 | 47 | < | 51 |
| 2 Average A | 10 | < | 15 | 11 | < | 17 | s | 18 | | 18 | | X | X | 18 | < | $18^{+}$ | 65 | | 65 | 66 | > | 60 |
| 2 Average B | 12 | < | 15 | 13 | < | 17 | s | 16 | | 17 | | X | X | $^{-}16$ | | $15^{-}$ | 65 | | 66 | 60 | < | 68 |

Note. The Counting On scores are the mean number of items correct on a 2-minute 20-item test of addition problems with dots and numerals given for each addend; scores rise considerably when children change from counting all the dots to just counting on the dots for the second addend. Addition scores are the mean number of items correct on a 2-minute 20-item test of the more difficult single-digit sums--those with sums between 11 and 18, excluding doubles (a + a). The addition scores for 4M and 5M of the bottom 6 rows in the table are the mean number of items correct on a 3½ minute test of all 100 addition facts. The ability groups are math ability (assigned before any of our instruction) based on tests and teacher judgment.

< indicates a t-test significant at the .05 level.
\- indicates a score significantly lower than the posttest ($p < .05$).
\+ indicates a score significantly higher than the posttest ($p < .05$); the scores on the 100 item test cannot be tested with the posttest.
s denotes the timing of the subtraction teaching.
X denotes no test given.

[a]Before Review

[b]After Review

Table 2

Subtraction Test Means by Class

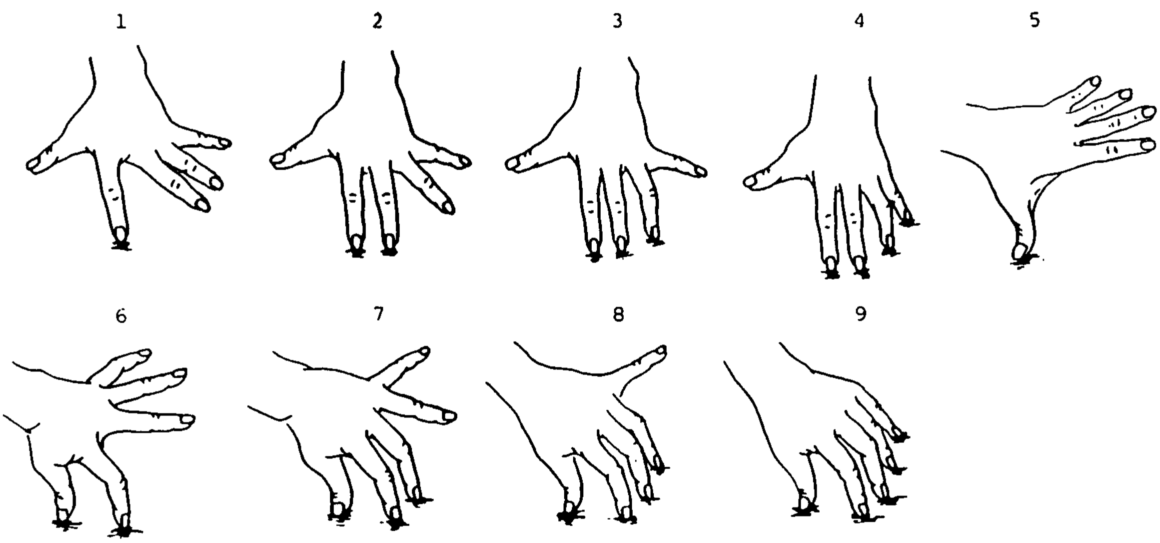
| Grade Ability School | Subtraction Pre Post | 1 Month BR[a] | 1 Month AR[b] | 2 Month BR | 2 Month AR | 3 Month BR | 3 Month AR | 4 Month BR | 4 Month AR | 5 Month BR | 5 Month AR |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 Low A | 1 < 11 | | | | | | | | | | |
| 1 Low B | 2 < 8 | 9 | 9 | 8 | 11 | | | | | | |
| 1 Average A | | | | | | | | | | | |
| 1 Average B | 0 < 11 | 9 | 9 | 10 < | 13 | | | | | | |
| 1 High A | 1 < 16 | 16 < | 17 | 16 < | 18+ | 56 < | 60 | 52 | 55 | | |
| 1 High B | 2 < 14 | +16 < | 17+ | +18 | 18+ | 25 < | 37 | X | 59 | 61 | 65 |
| 2 Low A | 3 < 13 | 12 < | 16 | 12 < | 15+ | 43 < | 55 | 45 | 49 | | |
| 2 Low B | 2 < 16 | 15 | 16 | -13 < | 17 | 39 | 45 | 41 | 44 | | |
| 2 Average A | 10 < 17 | +18 | 18+ | +18 | 18+ | 27 < | 30 | 72 < | 77 | | |
| 2 Average B | 7 < 15 | 15 | 16 | +18 | 18+ | 66 < | 75 | 64 < | 72 | | |

Note. Subtraction test scores are the mean number of items correct on a 2-minute 20-item test of the more difficult single-digit differences--those with sums between 11 and 18, excluding doubles (a + a). The scores for 3M, 4M, 5M of the bottom 6 rows in the table are the mean number of items correct on a 6 minute test of all 100 subtraction facts. The ability groups are math ability (assigned before any of our instruction) based on tests and teacher judgment.

< indicates a t-test significant at the .05 level.
- indicates a score significantly lower than the posttest ($p < .05$)
+ indicates a score significantly higher than the posttest ($p < .05$); the scores on the 100-item test cannot be tested with the posttest.

[a]Before Review [b]After Review

The finger patterns for 1 through 9 are made by touching certain fingers and/or the thumb to some surface such as a table. Thus there is kinaesthetic as well as visual feedback for the finger patterns. The finger patterns use a subbase of 5. The thumb is 5, and 6 is the thumb plus the 1 finger (6 = 5 + 1), 7 is the thumb plus the two fingers (7 = 5 + 2), etc. The motion from 4 to 5 is a very strong and definite motion--the fingers all go up and the thumb goes down, all in one sharp motion with the wrist twisting.

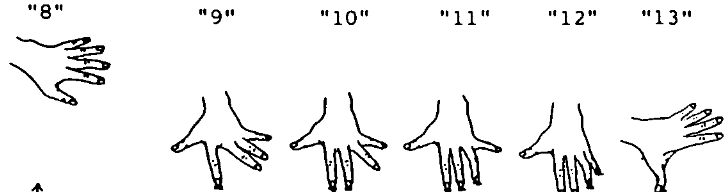



The finger patterns above are the patterns used in Chisenbop. We use the finger patterns differently from the way in which they are used in Chisenbop. We use them in the way that children spontaneously use fingers on both hands to keep track--the fingers just match the counting words which are counting on through the second addend (or counting up to the sum in subtraction).

| Addition | Subtraction |
|---|---|
| 8 + 5 = ? | 13 - 8 = ? |
| The counting-on procedure: | The counting-up procedure: |
| 1. Count on 5 more words from 8. | 1. Count up from 8 to 13. |
| 2. Stop when finger pattern for 5 is made. | 2. Stop when say 13. |
| 3. Answer is last <u>word</u> said (13). | 3. Answer is what the <u>hand</u> says--the finger pattern for 5. |

Words said: "8" "9" "10" "11" "12" "13"

Finger patterns:



Hand up in air ready to begin the finger pattern for 1.

Figure 1: Counting On and Counting Up To with Finger Patterns

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# THE DEVELOPMENT OF SPATIAL IMAGINATION ABILITIES AND CONTEXTUALISATION STRATEGIES: MODELS BASED ON THE TEACHING OF FRACTIONS

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The present work is a report on results attained during the first stage of a research project. The purpose of this stage is to further clarify the relationship between the acquisition of the fraction concept and the development of those abilities required to interpret and use the geometric language involved in the diagrams that frequently appear in teaching vehicles (especially textbooks) for the contextualisation of the fraction concept.

**INTRODUCTION.-** The use of diagrams and designs in the teaching of mathematics is more and more frequent. In many situations, especially at the elementary and secondary levels, perception and visualisation are called upon for the contextualisation of mathematical notions and concepts. Most of such diagrams, however, involve the use of visual and geometric conventions, and more often than not these turn out to hamper the interpretation or the reading of the diagrams. (Some authors have already pointed out a few of the obstacles in question: see, for instance[2 ]).

Reports are found throughout the literature on this field of research, that would show evidence that graphic materials, as presented in textbooks and other visual supports, lead to the acquisition of erroneous conceptions. Good results have been obtained, however, by means of learning strategies prepared with adequate visual supports (see [15 ]). Some papers also report that a sizable percentage of children prefer diagrams to words (see [3] ), and in others there is evidence that diagrams generally helped children towards the solution of problems (see [ 7 ]).

In the teaching process, the introduction of the fraction concept at the basic level finds a fundamental support in contextualisation through diagrams; frequently, these represent plane geometric forms, and emphasis is placed on the concept of fraction of the unit.

The main purpose of our present research activities is to determine the relationship between the acquisition of the fraction concept, and the development of those abilities that are required for the interpretation and utilization of the geometric symbol-language involved in the diagrams. This paper is a report on the results achieved during the first stage of our research. A complete report on the results of diagnosis and on clinical interviews can be found in [5] .

**BACKGROUND.-** In the course of two experimental studies ( [ 11] , [ 12 ]) that were conducted with children 9-16 years old in Mexico City, in 1982,[1] a number of obstacles related to the fraction concept came to light. In [11] , an analysis was made on the evolution of those diffi-

(1) Research work with children was carried out, essentially, at the Centro Escolar Hermanos Revueltas.

culties, through the results obtained by students of various ages, with questionnaires that concentrated several aspects of fractions. And in [12], work was done with a group of students in the first year of secondary school, among which difficulties had been detected at the beginning of the school-term by means of a diagnosis evaluation; in the course of the year, activities were implemented with these children, aimed at helping them overcome the formerly detected hindrances. Most of the conclusions and observations from these two research works are in agreement with those mentioned by other investigators whose results appear in the literature on this field (see, for instance, [1], [7], [8], [13], [14]).

**First Stage**

A. Preparation of a diagnosis questionnaire. Research was initiated by preparing a questionnaire that would allow us to clarify the level of obstacles involved in the understanding and application of the fraction concept.

In order to perform this task, we analyzed the questionnaires which were used in the two experimental studies mentioned above, and also other questionnaires that have appeared in the literature (e.g., [7]). This analysis lead us to a classification of those actions that the student must perform to solve an specific task. In this classification we found several levels of complexity; therefore, we decided to include in our questionnaires some items concerned with the use of more primitive notions, but which were closely linked to the fraction concept. Examples of these are: the partition of a whole in equal parts, part-whole relationships, the exhaustive division of the whole, the equivalence of parts, etc.

On the other hand, an analysis of the learning strategies for this topic at the elementary level[2], will allow us to identify obstacles that arise from the structure of the learning process itself [6]. It was also important in this analysis of elementary level learning strategies, to examine the types of diagrams employed for contextualisation of the fraction concept, as well as other diagrams which are used for different purposes, but which also involve the primitive notions already mentioned, as for example, for the area or volume concepts, in the measuring process, etc.

Based on the above, we designed a number of items that would conform the diagnosis questionnaire.

All items required both diagram interpretation, and graphic representation of operations and notions related to the fraction concept (the partition of a whole in equal parts, part-whole relationships, identifying the denominator and numerator of a fraction, and so forth). Diagrams of plane figures, graphic representations of tri-dimensional figures, and designs of collections of objects, all appear as items, in such a way that these cover the meaning of a

(2) In Mexico, teaching at the elementary level is based on a common syllabus for the whole country. Curricular principles, learning strategies, and activities aimed at developing these strategies appear in the cost-free text-books [9] -- which are the ones used by children —, as well as the teachers' manuals [10].

fraction when considered as a part of a whole, the meaning of a fraction as a subset of a set, and more complex relations, as the comparison between relations of different wholes and parts of them. Also, in the wording of questions both usual language and symbolic language were included.

The questionnaire is structured so as to contemplate different levels of complexity with respect to the actions the student must perform to find a solution to the questions asked; therefore, we grouped items that we considered having the same degree of difficulty with respect to that criterion.

B. **Exploratory Interview**

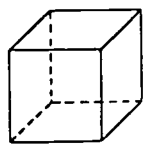
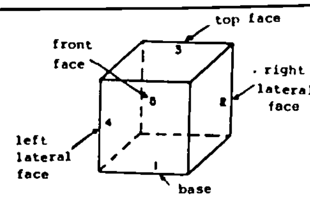
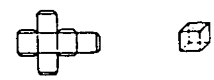
Due to the fact that our questionnaire contained items ranging from elementary notions to the application of the fraction of the unit concept to complex situations, we decided to perform some exploratory interviews.

These interviews would help us to clarify what difficulties were involved in the items, would allow us to observe the children's behaviour when faced with problems that are not usual in teaching of fractions (as, for instance, identifying a fraction of a solid body), and would give us further elements to choose those questions that would conform a diagnosis evaluation, which we would apply to a group of the first year of secondary, in order to proceed with the second stage of our research. Children 11-13 were chosen to carry out these interviews

From an analysis of the interviews, the following are some of the observations that seem to be more relevant:

- In the solving process of problems with diagrams representing a divided whole, the numerator of the fraction took preference; nevertheless, when justifying the result, the denominator was primarily used. These roles of the numerator and denominator kept alternating throughout the solution of problems where the division of the whole was required.
- The equalness of the parts in a divided whole was never questioned at all. When dividing a whole in equal parts equalness was taken into account, but the focus was on the number of parts in which the whole must be divided.
- The interpretation of diagrams relating to blocks made up of cubes, caused some troubles even when in this case the problem statement gave a guideline for the identification of the parts of the figure. (In [ 4 ], with reference to the volume concept, we reported on the problem detected as to the reading of this class of graphical representation).
- It was necessary to contextualise tri-dimensional figures represented by diagrams, to enrich the latter perceptually by evoking concrete objects, and to have the children undergo long instructions processes, if they were to interpret and understand the geometric symbol-language that such diagrams imply; this had to be done in spite of the fact that similar designs —or the same ones — appear in the learning strategies of the elementary level.

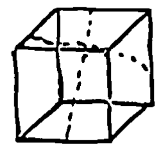
— Frequently, the first reading of tri-dimensional representations of objects referred to a figure conformed exclusively by the visible faces. (This observation will be illustrated by transcribing here a part of one of the interviews).

| Problem | How the interview developped | Remarks |
|---|---|---|
| Represent ¼ of the following figure.<br> | Laura: "The base is one of them (points to face 1, as indicated in the sketch under Remarks); this face is another one (2); this is another face (3); the one behind is another one (4); and this one, up front, is still another one (5)".<br>Teacher: "The faces".<br>Laura: "Yes".<br>Teacher: "How many faces are there on the cube?"<br>Laura: "One of them could be like this (with her marker, she marks the edges of the base); then two (she marks face 2)".<br>Teacher: "That one, which face is it?"<br>Laura: "The one on the side".<br>Teacher: "Which one is the face below?"<br>Laura: "This one" (indicates 1).<br>Teacher: "That one is the one below".<br>Laura: "Yes. This one (points now to face 3) is the one on top. This one, for instance, this way (indicates the edge of face 4). There would only be left the front one, but it seems to be covered ( she indicates face 5, without marking it). But there would only be this one (points to faces 3 and 2 at the same time), this (the base), and this (face 4)". | <br>From the way she describes the cube, she indicates her conception of it as a sum of faces. It is to be noted furthermore, that she is omitting one of these faces (the back one) and that she mentions a back face when she refers to the left lateral face. She is still using the concept of flatness, associating it perhaps to a developed cube. The graphic representation of a cube is frequently found as follows, in the textbooks:<br><br>This can probably produce obstructions for the conception of the cube as a solid body. From her wording a certain degree of confusion is observed when it comes to recognize the spatial location of the faces. It should be noted that se identifies the front face as a "covered" one. |

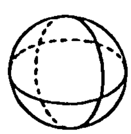
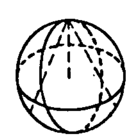
— In spite of the fact that the fraction concept has been mastered, to the extent of using it at an operative level, different strategies are used to approach problems with diagramas belonging to familiar figures (the usual ones appearing in textbooks) that are called upon to attack non familiar figures; transference in both cases, however, was correctly carried out. Difficulties arose only when trying to apply that knowledge to graphic representations of tri-dimensional figures.

With respect to this same problem, which is illustrated by the above transcription, we will show the result attained by Laura on her first try to solve the problem:

| Problem | How the interview developped | Remarks |
|---|---|---|
| | Method:<br>1. She draws a dotted straight line, joining a back edge of face 3 with the lower edge of face 5. | With this first cut she would indeed obtain a half of the cube; however, she cannot represent it graphically, although her theory |

| Problem | How the interview developped | Remarks |
|---|---|---|
| | 2. In a similar way, she joins the left lateral edge of face 4 with the back edge of face 2.<br><br>Result :<br> | about it is correct. (A badly represented dichotomy).<br><br>The second cut is not very fortunated (see the marks on the edges); it confirms the preceding assumption: a badly represented dichotomy.<br><br>It could be, though, that she is dividing a flat surface<br><br>and that she is only considering the perimeter, a process which has been successful in the case of a plane figure. She cannot be equally precise, however, in the case of the other cut. |

— There is an essential difference between understanding those partitions that are indicated in a diagram by means of conventions (as for instance, dotted lines for non-visible edges), by evoking actions on concrete objects (as when slicing an onion or a plasticine figure), and doing the graphic representation of the results of such actions. (This observation is also illustrated by the transcription of a part of an interview).

| Problem | How the interview developped | Remarks |
|---|---|---|
| No. 9.<br><br>Represent $\frac{1}{3}$ of the following figure.<br> | Teacher: "You are going to cut . . . ".<br>Laura: "Like this .. (with her hand, she follows the cuts of her fist), from the half towards here".<br><br>Teacher: "From the half towards there".<br>Laura: "I cut from here to there (she repeats her outlining on her fist). Then like this, and then like this (see these approximate cuts under Remarks)."<br><br>Teacher: "And, doesn't that leave you with four parts?"<br>Laura: "No, because I cut one here, and one more here (she repeats the same motions, indicating the cuts on her fist); that gives us three parts" (she indicates them with counting).<br>Teacher: "There, now; could you shade the parts you are referring to?" | <br><br>By the use of her fist she is able to imagine temporarily the body she has been proposed; besides, she can indicate the cuts in a more precise way.<br><br>The fist and the indications on it, show the understanding and the certainty of a correct execution. She proposes three cuts that produce three parts of the sphere.<br><br> Uses a straight dotted line. |

| Problem | How the interview developped | Remarks |
|---|---|---|
| | Laura: "No, I cannot shade it, unless I go back there...". She starts drawing a straight dotted line between the previous two, (she had already - drawn). See the drawing of this last line, under Remarks. | Even so, she does not succeed in stating what was inferred in the graphic repre - sentation of the previous demonstration . She still cannot use the equator and median lines as a reference to indicate the - cuts she has so clearly signalled on her - - fist. |

— A further observation was that a certain kind of strategies which were successful in the solution of previous problems, were then employed in a repetitive manner.

— In the course of the interview, a number of learning sequences were gradually introduced, with strategies focused on the processes of figure contextualisation, of evocation of object images and the cuts that can be performed on such objects, the location of the represented elements (e.g., the ones that are placed forward or behind), and the definition of those planes that leave such location perfectly established. By means of these strategies, significant advances were achieved, both in diagram interpretation and in making it possible for the students to use and resort to diagrammatic representations in order to reach new results.

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# CHANGES IN DECIMAL REASONING

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Abstract
This paper describes the transition from the state of a novice to that of an expert in the domain of decimals. The research has two parts to it. The first study identifies the systems of rules, while the second study captures a developmental trend in their use. The findings will be supported by evidence from parallel studies carried out in France, the USA and England.

## The Tasks

The main task consists of comparisons of pairs of numbers (i.e. the child is asked to tell which number is bigger). An analysis of the possible combinations of length and size of the numbers together with the possible ways of answering (correctly/incorrectly) yields two interesting and informative cases. In one case, the longer number is actually the smaller, yet the child answers incorrectly that "this (the longer) number is bigger"; In the other case the shorter number is smaller, yet the child answers incorrectly that "this (short) number is bigger". A zero in the tenth place (as in 4.08 vs. 4.7) usually calls for special considerations. This kind of task was found to be of interest in the pilot studies (and in the study of Leonard et al, 1981). This analysis together with findings of pilot and parallel studies yields the following list of hypothesized rules. The rules correspond to those found by Leonard et al (1981), but are extended to include a rationale for each one.

Rule 1: "The decimal which has a bigger number to the right of the decimal point is bigger". Sometimes it can be stated as "Longer numbers are bigger". This rule relies on knowledge of whole numbers. In this case children claim that 4.63 is bigger than 4.8, "because 63 is bigger than 8".

Rule 2: "Shorter numbers are bigger". This rule relies on knowledge of rational numbers. In this case children claim that 3.2 is bigger than 3.47, "because the shorter number has tenths and the longer number has hundredths and thenths are bigger than hundredths".

Rule 3: In this case children who usually employ Rule 1, will change their rule and say in the case of zero in the tenths' place (e.g. 4.08 and 4.7), "that 0 in the tenths makes the number always smaller".

The First Study

A pre-structured interview (about 45 min.) was individually administered to 21 children of the sixth grade, who had just finished studying the chapter about decimals. The interviewer was free to stop the child at any point and ask him to explain his correct or incorrect answers. The structured interview, which consisted of about 60 questions, included two-numberthe comparisons using pairs of numbers chosen so as to elicit the use of hypothesized rules. It also included tasks relating to additional knowledge to give an idea of what the child knows and help in determining the source of his difficulties (for example: questions about place-value, the density of decimals and integration of different number systems).

Table 1 presents part of the data of the first study. Each number in the table stands for the rule a child is using in a given problem. This number was not determined by that problem alone, but by the child's overall performance. A child that answered correctly, for example, that 4.7 is bigger than 4.08, could be theoretically using Rule 3 or Rule 2. If he used Rule 1 in problems without a zero in the tenths, a "3" would be written in the table for this answer. If he used Rule 2 in other problems, a "2" would be written there.

Table 1 - The rules used to produce the answers

| Items | | R1 Ss. | R3 Ss. | R2 Ss. | not clear | Experts |
|---|---|---|---|---|---|---|
| detecting R1 | | | | | | |
| 4.8 | 4.63 | 1 1 1 1 | 1 1 1 | 2 2 x 2 2 2 2 | 2 2 2 | c c c c |
| 0.36 | 0.5 | 1 1 1 c | c c 1 | 2 2 x 2 2 2 2 | 2 2 2 | c c c c |
| 2.305 | 2.35 | 1 1 1 1 | 1 1 1 | 2 2 2 2 2 2 2 | 1 1 2 | c c c c |
| 13/100 | 0.125 | 1 1 1 1 | 1 c 1 | 2 x 2 2 2 2 x | 2 2 2 | c c c c |
| 7/10 | 0.678 | c 1 1 c | c c 1 | 2 2 2 2 2 2 2 | 1 2 2 | c c c c |
| 0 on left | | | | | | |
| 4.7 | 4.08 | 1 1 1 1 | 3 3 3 | 2 2 x 2 2 2 2 | 2 2 2 | c c c c |
| 2.621 | 2.0687986 | 1 1 1 1 | 3 3 3 | 2 2 2 2 2 2 2 | 2 2 2 | c c c c |
| 2.035 | 2.35 | 1 1 1 1 | x 3 3 | 2 2 2 2 2 2 2 | 2 2 2 | c c c c |
| 4/100 | 0.038 | c 1 c c | x 3 x | 2 x 2 2 2 2 x | 1 1 1 | c c c c |
| 1.067 | 1.4 | 1 c 1 1 | 3 3 3 | 2 2 2 2 2 2 2 | 2 1 1 | c c c c |
| 0 on right | | | | | | |
| 0.25 | 0.100 | 1 1 1 1 | 1 c 1 | 2 2 2 2 x 2 2 | 1 2 2 | c c c c |
| 21/100 | 0.130 | - 1 1 c | 1 c 1 | 2 x 2 2 2 2 2 | 2 2 2 | c c c c |
| equivalent | | | | | | |
| 2.350 | 2.35 | c c c 1 | c 1 1 | c c c c c 2 c | c c c | c c c c |
| 23/100 | 0.230 | - c c x | c 2 1 | c c c c c 2 c | c c c | c c c c |
| 20/100 | 2/10 | c c c 2 | c 2 2 | c c c c c 2 c | c c c | c c c c |
| hidden digits | | | | | | |
| 0.- | 0.--- | c 2 x c | 1 c c | 2 2 2 2 2 2 2 | c c c | c c c c |
| -.--- | -.- | c c c c | 1 c 1 | 2 2 2 2 c 2 2 | c c c | c c c c |
| detecting R2 | | | | | | |
| 3.2 | 3.47 | - - - 1 | - 1 1 | - - - - 2 2 2 | - - 2 | - - c c |
| 4.45 | 4.4502 | 1 1 1 1 | 1 x 1 | 2 2 2 c 2 2 2 | 2 1 2 | c c c c |
| 4/10 | 0.457 | - 1 1 - | 1 x 1 | 2 2 2 2 2 2 2 | - 1 1 | c c c c |

The children's explanation during the interview were an important source of information for filling up this table. For example: one child marked that 0.1<0.25 explaining that one can add zeroes and actually compare 0.100 and 0.25. This explanation makes his answer consistant with his further use of Rule 1, which would not be so without it. Results relating to additional knowledge are shown in table 2.

Table 2 - Performance on writing decimals and 'place-value'

| | Holding Rule 1 | | Holding Rule 2 | Experts |
|---|---|---|---|---|
| Writing decimals: | | | | |
| 6 tenths & 2 hths | - - + - | - - - | - + + + + + - | + - + + |
| 3 wholes & 6 hths | - - + - | - + - | + + + + + + + | + + + + |
| Value of columns: | | | | |
| 5 in 1.54 & 2.45 | - - + - | - - - | + + - + + - + | + + + + |

The Second Study

A written questionnaire was given to 240 children in grades 7,8 & 9. The questionnaire included most of tasks of the first study and had 6 subsets each of which had 5 items in it. The subsets were designed to identify users of Rule 1 and 2 and to find out the intervention of 0 for those using Rules 1 and 2, separately.

The questionnaire enabled us to validate the tasks that verified our hypotheses about the system of rules. A factor analysis performed for the 30 items yielded 3 factors:

Factor 1 consisted of all the items aimed at discerning children who operate according to Rule 2, except for one item, which will be discussed in discussing factor 3.

Factor 2 consisted of all the items intended to discern Rule 1.

Factor 3 included two items which consisted of equivalent numbers.

It seems that the task of comparing different numbers is not the same as that of comparing equivalent ones. This was already found out in the first study (See the equivalent items in Table 1).

The loading on each factor was very high and similar for all the items in a given subset. These three factor explained 87% of the explained variance. Table 3 will present the six subsets of the test; the rules they intend to test; which factor they appear on; and the percentage of success in a given subset.

Table 3 - Main factor for each subset

| The subset | Discerning Rule | Factor # | % errors |
|---|---|---|---|
| A | Rule 1 | 2 | 17.7 |
| B | Rule 3 (0 on left) | 2 | 11.8 |
| C | Rule 1 (0 on right) | 2 | 18.2 |
| D | Rule 2 | 1 | 31.4 |
| E | Rule 2 (0 on left) | 1 | 26.2 |
| F | Rule 2 (0 on right) | 1 | 28.9 |
| G | Equivalent numbers | 3 | 14.3 |

In addition to the validation of the basic instrument that we were using to discern the hypothesized system of rules, the second study also demonstrated the changes in the distribution of rules in the various grades and ability groups. The results are summarized in table 4. The numbers in the table are percentages of children using a certain rule (indicated by the column's name).

Table 4 - The distribution of the rules in different grades

| | Rule 1 | Rule 2 | Correct | Unclear |
|---|---|---|---|---|
| High ability | | | | |
| grade 7 | 0.07 | 0.19 | 0.56 | 0.16 |
| grade 8 | 0.04 | 0.21 | 0.62 | 0.14 |
| grade 9 | 0.03 | 0.12 | 0.82 | 0.03 |
| low ability | | | | |
| grade 7 | 0.27 | 0.27 | 0.14 | 0.32 |
| grade 8 | 0.22 | 0.40 | 0.11 | 0.27 |
| grade 9 | 0.07 | 0.37 | 0.11 | 0.44 |

Among the low ability group the percentage of children using rule 1 or rule 3 decreases as grade gets higher, while the percentage of users of rule 2 increases. Exactly the same pattern of changes appears in the higher ability group. Here the percentage of users of rule 1 or rule 3 is already very small in 7th grade and it continues to diminish, while the percentage of rule 2 users in grade 7 is relatively high (which fits with the claim that this is a more advanced state) and decreases in grade 9.

Additional data that will support these results later, in the discussion, is drawn from studies by Swan (1983) and by Resnick and Nesher (1983). Swan found that 50% of the 12 year olds he tested were using Rule 1 or Rule 3 and 28% used Rule 2. Resnick and Nesher summarized interviews of children in Pittsburgh at age 10-11 and found 7 children using rule 1 or 3 (41%) and only 3 (18%) using rule 2. Kidron and Vinner (1983) found 33% of 10 th graders (n=91) and 13% of 11 th

graders (n=85) used rule 2 (Though their items were a little different using pairs such as: 0.333' o.33333, where rule 2 users made the error of claiming that 0.333'<0.33333)).

Discussion

The almost complete correspondence between the pattern of each child's set of answers and the answers expected of him, when one finds he is using a certain rule, was made evident by the uniformity of the columns in Table 1, and confirmed the assumptions about the formulation of these rules. This, together with the children's explanations showed us that children rely on the knowledge of one of two sets of numbers: children who used Rule 1, relied on knowledge of whole numbers. Their rationale is based on comparing whole numbers. Children who used Rule 2, relied on knowledge of rational numbers. Their rationale is based on comparisons of parts (or fractions).

This information in table 2 suggests that those using Rule 1 have a weaker understanding of previous knowledge of the number system. They have probably compared whole numbers by using a more primitive rule that says: "Longer numbers are bigger". This rule served them perfectly in the domain of whole numbers; In moving to decimals, though, they continued to see a decimal number as 'two whole numbers combined by a decimal point' and compared the decimal parts of the numbers as if they were whole numbers (i.e. "The longer is bigger"). Some of the users of Rule 1 changed their mind only if 0 was involved, thus becoming users of Rule 3 which is an improvement over Rule 1.

Children who operated under Rule 2, are those who understood that the part right to the decimal point is a fraction, but they failed to make the coordination between the size of the part, the fraction (e.g. tenths, hundredths etc) and the number of such parts in question. Thus

in the transition from rational numbers, written as a ratio between the numerator and denominator to decimal numbers in which the size of the parts (denominator) is given by the column location and not mentioned explicitly, they failed to make the full integration. In perceiving the explicitly mentioned number (right to the decimal point), some of these children would give the numbers the role of a denominator, and some of them, the role of a numerator, or would mix them all up, failing to make the needed comparison. Evidence for such a confusion, can be found in the following item: when asked to write the rational number 3/4 in decimal form some wrote: 3.4 ; 0.3 or 0.34.

The second study yielded two important outcomes; one relating to the choice of items and the diagnosis of the answers; the other relating to the process of transition from a novice's state to the expert's state. The results of this study supported the assumption that children are basically using two rules: rule 1 and rule 2. They also supported the mapping from items to rules, as it showed that all the items that were chosen to single out rule 1 users, for example, turned out to have a high loading on the same factor.

From the results, together with the data from studies by Swan (1983) and by Resnick & Nesher (1983) detailed earlier, there emerged a clear picture of the trend of change in the distribution of rules. A high percentage of children using rule 1 or 3 in the lower grades was found in the study by Resnick and Nesher, in Swan's study and in our first study. The study by Kidron & Vinner together with our second study show that more children use rule 2 in higher grades, while less are using rule 1 and with the years there is then the beginning of a decrease of users of rule 2 so that the total number of mal-rules users decreases.

## Summary

Some important points have emerged from the studies. It was found that children are consistent in their use of a system of rules to operate with when they deal with tasks concerning decimals. These systems of rules seem to emerge from previous knowledge. One from the domain of whole numbers; and the other from the domain of rational numbers. The child that relies on an importation from the domain of whole numbers, does not have in fact a full grasp of the notion of "place value". This fits with the developmental trend which was found in the use of these systems of rules. At an early stage in learning decimals one can find more children employing "Whole-numbers"' considerations (Rule 1) than those who employ "Rational-numbers"' considerations (Rule 2). At a later stage only a few will be using Rule 1, but more will be using Rule 2. They hold on to Rule 2 quite stubbornly, probably failing to make the coordination between the size of a fractional part and the number of parts. Thus it seems that the transition of a child from a novice state to an expert state in decimals, depends on fully acquiring knowledge in basic concepts. This long transition to an expert's state, could be characterized by intermediate, transitional systems of rules, each relies on a previous partly learned knowledge, that can be so ordered as to form a learning path for this domain.

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# Selected Results From The Rational Number Project

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This paper is intended to serve as an updated compendium of Rational Number Project activities. Several major project strands are described. Each description is followed by several references to published materials dealing with that particuliar strand.

The descriptions provided are of necessity very brief. Interested persons should consult the appropriate references for more detailed information.

The Rational Number Project (RNP) was a four-year (1979-83) U.S.-based research project funded by the National Science Foundation (NSF). The project involved three universities (Northern Illinois, Minnesota and Northwestern) and untilized well-defined theory-based instructional and evaluation components as well as an overall plan for validating project generated hypotheses. The project's intent was to describe rational number development from its beginnings to its formal operational level in well-defined instructional settings. The major goal is the identification of psychological and mathematical variables which impede and/or promote the learning of rational number conepts.

The project has recently been re-funded by the NSF (1984-86) and is at present focusing on the role of rational number concepts in the development of proportional reasoning skills.

We are indebted to the following people who assisted in this research: Nadine Bezuk, Kathleen Cramer, Mary Pat Roberts, Robert Rycek, Constance Sherman and Juanita Squire. Special thanks also to Mary Welty for her typing and organizational skills.

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## - CONTEXTUAL CONCERNS -

The task of assessing children's ability to utilize rational number knowledge in applicational situations is difficult. Children often are unable to transfer ideas to contexts they have not enountered before. Rational Number Project, addressed the issue by providing a rich foundation of rational number concepts utilizing a broad range of perceptual variables in a manner consistent with the ideas of Dienes.

Concurrent with instruction, interviews which stressed children's functional rational number knowledge were conducted. An evaluation of these interviews suggests the following: Only subjects exhibiting consistent success in a variety of applied situations can be assumed to have developed a generalized understanding of rational number. Children who do not have a workable concept of rational number size cannot be expected to exhibit satisfactory performace across a set of tasks which varies the context in which the number concept of fraction is involved.

In one study, 5th grade children were required to select digits from a provided list to form two fractions whose sum was as close to 1 as possible. In a second study, the same children were to suggest target rational numbers on a number line. These were to be hit by an electronic "dart" flying across a video screen. In a third study, these children were to interpret a given set of fraction symbols as rations for black-ink/water mixtures and to associate them with a scale of increasingly darker gray levels.

Findings suggest that a coordinated use of order and equivalence knowledge, combined with skill in estimating the size of rational numbers, enabled some children to be successful across all three tasks. An ability to perceive the ordered pair in a fraction symbol as a conceptual unit (rather than 2 individual numbers) was also found to be an indicator for successful performance.

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## PERCEPTUAL DISTRACTORS

In the work of the Rational Number Project, it has been observed that certain components of a manipulative aid or pictorial display that are essential to illustrate one basic concept frequently impair the child's ability to use the aid for another concept. In particular, various types of perceptual cues can negatively influence children's thinking. In some cases, these perceptual cues act as distractors and overwhelm children's logical thought processes.

In the instructional component of the Rational Number Project, we found that children tend to assume that physical conditions within which problems are presented are relevant to and consistent with the task. This tendency is probably an artifact of their learning from a textbook- or worksheet-dominated instructional program that places little emphasis on manipulative materials. Within such a program, problem conditions are necessarily static in nature, providing little opportunity for children to manipulate problem conditions. Students expect that mathematical problems conditions (context) conform to the intended task and, therefore, are not in need of restructuring or rethinking. Children learn that one simply takes what is given, and proceeds directly to the solution.

Perceptual distractors represent one class of instructional conditions that make some types of problems more difficult for children to solve. Knowledge of their impact will be helpful in the design of more effective instructional sequences for children. It seems reasonable to suggest that initial examples might be given wherein the potential impact of perceptual distractors is minimized, but that later examples should deliberately provoke children to resolve conflicts that arise in association with perceptual distractors.

Although performance with rational numbers is affected by the presence of distractors, children can be taught to overcome their influence. Furthermore, the strategies generated by children to overcome these distractors lead to more stable rational number concepts.

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ORDER AND EQUIVALENCE

Understanding the order and equivalence of fractions required an understanding of the compensatory relation between the size and number of equal parts in a partitioned unit. A small percentage of children are able to exhibit an understanding of this relation after only brief instruction. Other children grasp it after additional lessons. For still others, the relation remains elusive even after they have had ample opportunities to learn and practice.

Instruction aimed at developing an understanding of the compensatory relation will require more instructional time than has been given in most curricula, in addition to a careful spiraling of the concept through several grade levels. We recommend that fractions be introduced in the third grade. The introduction should be limited to establishing elementary meanings for fractions, with a heavy emphasis on unit fractions. As the compensatory relation is being learned, its application to the problem of ordering unit fractions can begin. Such experience would provide a good foundation for establishing a quantitative concept of rational number. At the end of the third grade or at the beginning of the fourth, instruction would incorporate the concept of nonunit fractions, which would be developed through the iteration of unit fractions. The concept of order would be extended to fractions with the same numerators and then to fractions with different numerators and denominators.

Our observations suggest that children whose rational number concepts are insecure tend to have a continuing interference from their knowledge of whole numbers. This interference needs careful consideration by researchers, curriculum developers, and teachers. It would clearly be inadequate simply to inform children when the schemata they have developed for dealing with whole numbers are appropriate and when they are not; children need to learn how to make such determinations on their own.

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STRATEGIES

Many children develop or invent strategies for dealing with fraction order and equivalence tasks which likely have origins in whole number arithmetic or even more elementary experiences. It was observed that the idea of strategy is fairly fluid among many children. It appears, however, that children's strategies are frequently local strategies. That is, the strategy employed is often a function of the specific task, and does not necessarily persist through or transfer to different situations. This was found between children and within particular individuals. Variation in the numerical characteristics of a problem frequently will generate different solution strategies even within a single individual. It was illustrated above that the same child within a short period of time might employ two vastly different algorithms for similar questions, one algorithm referring to the physical aspect of a number of pieces, the other being based upon number relationships and thus dealing at a higher level of abstraction. This supports a hypothesis of interaction between solution strategies and the numerical characteristics of the task situation.

It is interesting to note that children employ strategies which have not been taught directly. The residual and transitive strategies are examples of this. Residual: $5/6 < 7/8$ because they both have one piece left over (to make a whole) and since $1/6$ is greater than $1/8$, $5/6$ $(1-1/6)$ must be less. Transitive: $4/9 < 5/8$ because $4/9 < 1/2$ and $5/8 > 1/2$). Such strategies seem to be natural extensions of those previously used.

It was also observed that a self-generated strategy was less likely in tasks with fractions less than 1. It may be hypothesized that this is because a proper fraction such as 3/5 can be dealt with more easily by imagining its "bigness" as part of a physical unit, than would be the case for 12/5.

Children often invent strategies (many of which are incorrect) when they are asked to compare two (not equivalent) fractions which neither have like numerators nor like denominators. For example, 3/5 and 5/8 are difficult to order by any means other than an abstract approach such as converting to a decimal, using a common denominator, or using a cross multiplication.

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## CONCEPT OF UNIT

Three different percent problems are represented in the statement x is P% of y. Problems of the type to find x given y are more difficult for junior high school children than ones of the type to find y given x. The former are analogous to problems which are emphasized in elementary school fraction instruction: Find 3/4 of . The latter is analogous to a problem type which gets little or no attention in elementay school instruction: If this is 3/4, find the whole. The differential difficulty between the two percent problem types for junior high school children may result from the differential emphasis which we give in elementary school to the two analogous fraction problem types.

If instruction did emphasize both of these fraction problem types in the elementary school, children might acquire a better concept of fraction than is currently the case. These two fraction problem types exemplify Piaget's concept of reversibility; the operations of finding a fractional part of a unit and of finding the unit of which a given fraction is part are inverse operations. Ability to do one of the operations but not the other suggests an incomplete understanding of the concept of fraction.

We gave problems of the type "If x is m/n of y, find y" in various forms:

(a) x was either a continuous region or a discrete set,
(b) x contained or did not contain a perceptual distractor,
(c) x was either a unit fraction, a non unit fraction less than one, or a fraction greater than one.
(d) y was equal to one (the unit) or greater than 1.

The data from grade 5 children indicate use of 4 different strategies for solution. Two strategies which usually lead to a correct solution were similar; the child first partitioned the given fractional part into n equi-sized pieces and then referred to each piece as (a) one n-th or (b) one part. After this the child found the whole by iterating this piece while counting and saying, (a) 1/n-th, 2n-ths, ..., n/n-ths or (b) 1 part, 2 parts, ..., n parts.

Most unsuccessful solution attempts involved one of two strategies:

(a) The child treated the <u>given</u> fractional part as the unit fraction 1/n and iterated this n times or (b) The child treated the fractional part as the unit and showed m/n-th of it.

Reference: Behr, M., Post, T., Lesh, R., and Waschsmuth, I. Understanding Rational Numbers: The Unit Concept. Paper in preparation.

ESTIMATION

Whether or not a child understands the concept of size of a fraction is an indicator of the depth of the child's understanding of the fraction concept. Many children do not have this understanding; an indicator of this are the achievement results from an NAEP item which asked 13- and 17-year olds to choose from among 1,2, 19, and 21 an estimate for 12/13 + 7/8. Frequent choices of 19 and 21 suggest that many children lack understanding for the size of fractions.

We take the position that estimation skill is closely related to the concept of number size. The understanding of the size of numbers--whole numbers, fractions, decimals--is essential to the ability to make estimates. We also believe that instruction in and practice with estimation will help children develop an understanding of number size.

We investigated children's ability to estimate rational numbers in the context of a "construct-a-sum" task. Children were asked to choose whole numbers from among 1, 3, 4, 5, 6, 7, to form two fractions whose sums would be as close to, but not equal to, 1 as possible.

Grade 5 children exhibited essentially 5 strategies in dealing with this task. One strategy involved the use of a reference number such as 1/2, 3/4, or 1. In another strategy children did a mental manipulation of a correct addition algorithm, including mental computation of equivalent fractions. Each of these two successful strategies involved good understanding of fraction equivalence. Two unsuccessful strategies represented difficulty with the use of a reference point or inaccurate mental manipulation of a correct algorithm or mental manipulation of an incorrect algorithm. A third successful strategy was based on very inaccurate estimates of fraction size. Unsuccessful performance reflected poor understanding of fraction equivalence.

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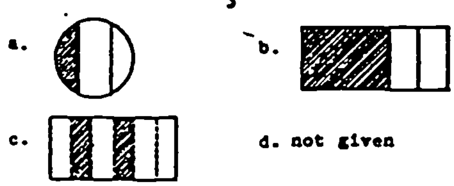
REPRESENTATIONS, TRANSLATIONS, & PROBLEM SOLVING

Several articles from the Rational Number and Proportional reasoning Projects have described roles that representations, and translations among representations, play in mathematical learning and problem solving. In the book, The Acquistion of Mathematics Concepts and Processes, Behr, Lesh, Post, & Silver (1983) focus on the following five distinct representation systems: (1) experience-based "scripts" - in which knowledge is organized around "real world" events which serve as general contexts for interpreting and solving problems, (2) manipulative models - like Cuisenaire rods, arithmetic blocks, fraction bars, number lines, etc. in which the "elements" in the system have little meaning per se, but the "built in" relationships and operations fit many everyday situations, (3) pictures or diagrams - static figural models which, like manipulatable models, can be internalized as "images," (4) spoken languages - including specialized sub-langusges related to domains like logic, etc., (5) written symbols which can involve specialized sentences and phrases ($X + 3 = 7$, $a(b + c) = ab + ac$ as well as normal English sentences and phrases.

The item below taken from a proportional reasoning examiniation (Lesh, Behr, & Post, 1985) illustrates a "written symbol to picture" translation.

31. What picture shows $\frac{1}{3}$ shaded?




d. not given    e. I don't know

In our chapter in a book about Representations in Mathematics Learning Problem Solving, edited by Janvier (1985), we discuss the fact that part of what educators mean when they say that a student "understands" an idea like "1/3" is that: (a) s/he can recognize the idea embedded in a variety of qualitatively different representational systems, (b) s/he can flexibly manipulate the idea within given representational systems, and (c) s/he can translate the idea accurately from one system to another. We also discuss ways that these translation abilities are reflected in problem solving capabilities. For example, consider item 29 (below), adapted for our research from a recent "National Assessment" examination.

29. The ratio of boys to girls in a class is 3 to 8. How many girls were in the class if there were 9 boys?

a. 17    b. 14    c. 24    d. not given    e. I don't know

Educators familiar with results from recent "National Assessments" may not be surprised that U.S. students' success rates for problem 29 were only: 11% for 4th graders, 13% for 5th graders, 30% for 6th graders, 29% for 7th graders, 51% for 8th graders. Success rates on the seemingly simpler problem 31, however, even lower: 4% for 4th graders, 8% for 5th graders, 19% for 6th graders, 21% for 7th graders, 24% graders for 8th graders. On the translation item 31, only 1 in 4 students answered correctly: 43% selected answer choice (a); 4% selected (b); 15% selected (c); 34% selected (d); 3% selected (e); and 2% did not give a response.

One major conclusion from this research is apparent from the preceding examples; not only do most 4th-8th graders have seriously deficient understandings in the context of "word problems" and "pencil and paper computations," many have equally deficient understandings about the models and language(s) needed to represent (describe and illustrate) and manipulate these ideas. To remediate these deficiencies, our research has focused heavily on the role that translations and transformations play in the acquisition and use of elementary mathematical ideas (Lesh, 1985).

The RN & PR projects conducted in conjunction with Lesh's Applied Mathematical Problem Solving (AMPS) project, have shown that students' solutions to problems like #29 (above) typically involve the use of spoken language (together with accompanying translations anad transformations), in addition to pure written symbol manipulations (i.e.. transformations). On the other hand these studies also how that repeated drill on problems like #29 does not necessarily provide the type of instruction related to developing an understanding of the underlying translations.

Lesh, Landau, & Hamiltion (1984), suggested that purportedly realistic word problems often differ significantly from their real world counter-parts in difficulty level, the processes most often used in solutions, and in the types of errors that occur. Real problems often occur in a form that inherently involves more than one representational system. Furthermore, during solution processes, student's frequently changed the representation of an aspect of their situation form one form to another; or at any given stage, two or more representational systems, were used, each illuminating some aspects of the situation while deemphazing or distorting others.

Other links between problem solving capabilities and conceptual understandings are discussed in Using Mathematics in Everyday Situations (Lesh, 1985). For example, one chapter deals with a proportional reasoning problem in which the phases that students typically passed through during 40 minute solution attempts exactly paralleled stages that the RN & PR projects observed over periods of several years in the development of the underlying concepts required to do the problem. The "local conceptual development" character of AMPS problem solving sessions means that we are able to apply

to AMPS -style applied problem solving many of theoretical perspectives developed by the RN & PR projects, and vice versa.

Finally, relationships between problem solving, conceptual understandings, and representation system capabilities are being explored in some of the instructional materials currently under development at the World Institute for Computer Assisted Teaching (WICAT). A modified and enhanced version of the "symbol manipulator/equation solver" (SAM) that was developed for WICAT's IBM Algebra and Calculus courses SAM is being enhanced with the ability to produce "dynamic models or pictures" illustrating a range of typical "proportional reasoning and/or units arithmetic" problem types, and with the ability to operate on measurement levels in addition to numbers and variables. Using such utilities, students can focus on graphic representations of the processes they use to arrive at solutions.

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# COGNITIVE PROCESSES AND PRODUCTS IN PROPORTIONAL REASONING

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Numerous studies have shown that children and many adults have a great deal of difficulty with basic concepts of fraction, ratio, and proportion, and especially with problems involving these concepts. These studies have discovered correct and incorrect thinking strategies children employ in problems involving these concepts. Little research seems to have been addressed to the question of how these concepts develop in children. Some studies (Noelting, 1980; Karplus et al., 1983) indicate that many children use faulty qualitative reasoning, both incorrect and inappropriate. Some children use additive comparisons where multiplicative comparisons are required (Hart, 1981). Apparently many children do not see cause and effect relationships between components of equations such as $a/b = c/d$ or $a/b = k$. However, observation of important cause and effect relationships are possible through qualitative reasoning alone.

In a multisite project about rational numbers and proportional reasoning, children are being studied while learning these concepts (clinical interviews), individual differences between students are analyzed (by large scale testing) and an instructional program is being developed (computer implementation of a part-whole-world). The schema construct will be the center of interest. We will distinguish between two kinds of schemata, the Part-Whole-Schema and the Equalized-Wholes-Schema, to describe the cognitive processes involved in proportional reasoning tasks.

We hypothesize two kinds of schemata to be involved in proportional reasoning that deal with two kinds of ratios. The first concerns the ratio of two extensive values given with respect to a common unit of measure, whereas the second concerns two dissimilar extensive quantities A and B, that is, two extensive quantities whose measures cannot be given in a common unit (distance and time for example).

Involved in the first type of proportional reasoning schema is the cognition of an A and a B as conceptual entities, each of which are part of a common whole. If A and B represent two "mixable" quantities whose measures, extensive values, are respectively $\underline{a}$ and $\underline{b}$, where $\underline{a}$ and $\underline{b}$ are given w.r.t. the same unit of measure, then one can conceive of a part-whole system with components A, B, and A B. Measures associated with these parts are then $\underline{a}$, $\underline{b}$, and $\underline{a} + \underline{b}$. Of interest in this system is how the values of $\underline{a}$, $\underline{b}$, and $\underline{a} + \underline{b}$ interact with and affect the ratios $\underline{a} : \underline{b}$ and the fraction $\underline{a}/(\underline{a} + \underline{b})$, an intensive quantity which is a measure of the concentration of A in $A \cup B$. Of particular initial importance, at this point, is the realization

that <u>any</u> proportion problem of this type can be conceptualized as a part-whole situation. To describe the cognitive processes involved we talk of a part-whole-schema.

The second kind of schema refers to the ratio of two non-mixable quantities A and B; that is, two extensive quantities whose measures cannot be given in a common unit. Involved in the cognition of this type of ratio are three important cognitive structures: (a) The conceptualization of subparts of A into a conceptual entity--A, (b) The conceptualization of subparts of B into a conceptual entity--B, (c) A conceptual <u>equalization</u> of the two wholes A and B.

Because of this last characteristic we are referring to it as the Equalized-Wholes-Schema. From these initial cognitive structures for this ratio situation, together with conservation of proportionality under transformation <u>within</u> and <u>on</u> the two conceptual entities A and B, arise the important notions: (a) That ratio means the rate of change in A with respect to change in B, (b) The concept of unit ratio, i.e., the extent of A in one unit of B.

Many situations in which two extensive quantities are expressed in different measure spaces the multiplicative relationships between them can be expressed in terms of an Equalized-Wholes-Schema (EWS). An EWS is useful to represent the ratio (i.e., rate) between two extensive quantities which are not mixable into a common whole because of the measure spaces from which they are derived. For situations where an EWS is applicable the relationship of interest is the intensive multiplicative relationship between the two quantities.

So what are the most relevant characteristics to distinguish between the Part-Whole-Schema and the Equalized-Wholes-Schema?

1) Part-Whole-Schema

Important in children's ability to judge the equality or inequality of two fractions, and in doing proportional reasoning is (a) to know when additive comparisons are appropriate or inappropriate, and the same for multiplicative comparisons, and (b) to know the cause and effect relationship between and within additive and multiplicative comparisons due to changes in components of a part-whole structure. Moreover, it is important from our theory base that a child be able to reason qualitatively about the cause and effect relationship which exists between and within additive and multiplicative comparisons.

The comparison of the multiplicative relationship between the $P_1 : W$ ratio in two part-whole structures could be considered to be the initial and final states of a transformation of a part-whole structure (PWS). Under certain part-whole transformations the value of a part-to-whole ratio (as well as others) is invariant,

but not under all transformations. Under certain part-whole transformations, whether or not the value of the part-to-whole ratio is invariant or whether the change is an increase or decrease, can be determined qualitatively; under other transformations this will be ambiguous and will require quantitative (i.e., computation based) reasoning. Underlying these questions are PWS transformation principles. The extension of a child's part-whole schema to include at least implicit knowledge of these principles we hypothesize to be essential for children's ability to meaningfully perform on proportional reasoning tasks.

2) Equalized-Wholes-Schema

Important in children's ability to judge the equality or inequality of two rates, and in applying concepts of rates to problem solving situations is to (a) know the effect that an additive transformation of the components of an EWS has on the multiplicative relationship between these components and (b) know the effect of multiplicative transformations of the components of an EWS on the multiplicative relationship between these components. Moreover, it is important from our theory base, that a child be able to reason qualitatively about the cause and effect relationship between the additive and multiplicative transformations and the ratio between the components of the EWS.

Proportional thinking is frequently called for in problem situations in which three of four data are given and the problem solver is to find the missing one. The solution to such a problem essentially requires application of a restatement of the EWS transformations. The fact that two ratios form a proportion means that the multiplicative relationship between $W_1$ and $W_2$, the two components of an EWS, remains equal under an EWS transformation. A missing data proportional problem then begins with the two conditions: that (a) the multiplicative relationship (i.e., the ratio) is constant and (b) a given transformation on one of the two components of the EWS is given. The problem solver must then determine (a) the transformation to perform on the other component of the EWS and (b) the result of this transformation. That is, the problem solver must determine the required action under the given conditions.

On that basis we have constructed a proportional reasoning test which takes into consideration (1) four different ratios: speed, mixture, scaling, and density; and (2) different settings within each type. The different settings for each ratio are (a) speed: running laps and driving cars, (b) mixture: orange juice, paint, (c) scaling: making a classroom map or a city map, (d) density: standing in a movie line, hammering nails into a board. (3) the sequencing of quantitative and qualitative questions (missing value questions with numerical comparisons first vs.

questions without any numerical comparisons first), (4) the sequencing of rational number and proportional reasoning questions (rational number first vs. proportions first).

This test gives a 4 x 2 x 2 x 2 design so that the different variables can be analyzed in terms of their influence on the procedure. About 1100 students have been tested and these results will be presented soon (Post, 1985). As this approach concentrates on the outcome of the test and not on the process of the task solution, we are going to observe a limited number of students more intensively in the context of teaching experiments.

For that purpose a part-whole-world has been developed for personal computers which serves as a microworld in order to explore the concept of proportion by means of visualizations and active modifications of part-whole-diagrams. Children learn to transform real life situations into part-whole-diagrams, to compare these situations by comparing the two diagrams and to construct new situations after transformations of the part-whole-diagram. In that sense the microcomputer can be used to develop a concept of part and wholes, of fractions, of ratios, and later of proportions. The reactions of the students while working on problems on the other hand can be used to determine where they have difficulties. This in turn will be used to modify the presentation of the problems.

Abbreviations

$P_1$: Part 1

$P_2$: Part 2

W: Whole

$W_1$: Whole 1

$W_2$: Whole 2

PWS: Part-Whole-Schema

EWS: Equalized-Wholes-Schema

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# The Concept of Repeating and non-Repeating Decimals at the Senior High Level

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§ *1. Introduction* In a previous study (Kidron and Vinner, 1982) we examined various aspects of the rational numbers in high school students. The decimal representation of the rational numbers played there an important role. In this paper, we will consider the whole domain of real numbers and here the decimal representation will have even a crucial role. Infinite decimals are generally included in the senior high curriculum. They are mentioned in most textbooks, sometimes briefly and sometimes in a more detailed manner. The impression is that, in general, at least in classes, the whole topic is introduced in a superficial and unsatisfactory way. Moreover, after it has been introduced, it is not mentioned any more during the senior high level.

Nevertheless, on the basis of this limited exposure some beliefs or intuitions about the real numbers are formed in the students. We are trying to expose some of these beliefs or intuitions here. We do it by means of various non-routine questions.

§ *2. Fractions and Repeating Decimals*

The following question was posed to 91 tenth graders and 97 eleventh graders learning at an academic high school in Jerusalem.

Question 1: *A teacher asked his students what is the 50th digit in the decimal equal to 1/7. One student claimed that it is too much work to carry out 50 division steps. Another student claimed that it is possible to find the answer in less than 50 steps. Who is right? Please, explain!*

In their answers all the students took for granted that there exists a decimal equal to 1/7. It is hard to tell whether it is so because the question suggested it or because they really believe that every fraction has a decimal representation. Our impression is that the second alternative is true but we did not have a special question to examine it in our questionnaire. (It is absolutely possible that there are students at the same grade levels in different populations who do not share the same belief.) The method of finding out the decimal equal to a given fraction is, of course, to divide the numerator by the denominator. Our aim in the above question was to find out whether the students know that the decimal representing 1/7 is repeating. We were not interested in its 50th digit. Hence our analysis will not relate to this point. Also, the distributions of the answers to question 1 of the 10th graders and the 11th graders were almost identical. Therefore, we do not distinguish between them in our analysis.

Table I

Distribution of Answers to Question 1

| Categories | Percentages (N = 188) |
|---|---|
| I. 1/7 is not repeating | 10 |
| II. It is impossible to tell ahead of time whether 1/7 is repeating or not | 12 |
| III. 1/7 is repeating | 74 |
| IV. No relevant answer or no answer | 4 |

Here are some illustrations to categories I-III above.

Category I: * *In this division* (1:7) *we do not get a repeating decimal. Therefore, 50 division steps are required to get the answer.* (This answer was typical of students who started to divide 1 by 7 and decided to stop before the decimal started to repeat.)

Category II: * *If 1/7 is repeating then it is possible to find the 50th digit in less than 50 steps but if it is not repeating then one must perform 50 division steps.*

* *The only way to find out whether 1/7 is repeating or not is experimentation.*

Category III: * *The chance that we will get a repeating decimal is very good.*

* *Out of 50 possibilities it is absolutely certain that we will reach a digit which will repeat itself and then it is absolutely certain that the decimal will repeat itself.*

* *The second student is right because in this particular case 1/7 is a repeating decimal. It is* $.\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$. *This I have discovered by dividing 1 by 7. After getting this number* (.142857) *I saw that again I should divide 1 by 7. I can assume that there will be no change in the results when I start the division again and so on till infinity.*

* *I think that the second student is right because all the time we divide by 7 and the remainder is also divided by 7. The number of the possible remainders is 6 and I think that already after the 7th division we will get a repeating decimal.*

After examining Table I we may ask ourselves what can be assumed about the percentage of students who really know that every given fraction is equal to a repeating or finite decimal (not only 1/7). In order to answer this question we should look at the different explanations students gave to establish that 1/7 is repeating. General explanations, like the last one in category III above (which are *not* based on dividing 1 by 7) can be considered as indicating that the student knows in general the relation between fractions and repeating decimals. On the other hand, explanations based on dividing 1 by

7 (like the one before the last in category III above) can be considered as indicating that this general relation between fractions and repeating decimals is unknown (at least it is tacit). The required information is given in Table II.

Table II

Distribution of the Explanations to the Answers in Category III, Table I

| Categories | Percentages (N = 188) |
|---|---|
| I. It is claimed in one way or another that $1/7 = .\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$ | 49 |
| II. No explanation | 21 |
| III. A general argument which explains why 1/7 must be a repeating decimal | 3 |

If we combine categories II and III in Table II we see that only 24% did not use division in order to claim that 1/7 is repeating. Thus (giving perhaps too much credit to the students in category II), it is possible to assume that perhaps only 24% know that every fraction is equal to a repeating or finite decimal. This estimate is based on the assumption that students in category I, Table II are unaware of the above relation between fractions and repeating decimals. We believe that it is a reasonable assumption but we admit it should be examined elsewhere. Considering Table I again one should note that the difference between category II and category III there is not necessarily essential. A student who believes that "the only way to find out whether 1/7 is repeating is experimentation" and carries out the "required experiment" will finally find himself in category III.

§ 3. *How is an Infinite Decimal Formed?*

The following question was posed to the same sample as in § 2:

Question 2: *A teacher asked his students to give him an example of an infinite decimal.*

*Dan: I'll look for two whole numbers such that when I divide them I won't get a finite decimal; for instance: 1 and 3.*

*Ron: I'll write down in a sequence digits that occur to me arbitrarily, for instance: 1.236418 . . .*

*Dan: Such a number does not exist because what you write down is not a result of a division of 2 whole numbers.*

*Ron: Who told you that what you write down must be the result of a division of 2 whole numbers?*

*Who is right? Please explain!*

Question 2 is supposed to examine whether the students are mathematically matured for the idea of the irrational numbers as infinite non-repeating decimals. (We asked the students in the sample whether they studied the concept of irrational numbers in the past. 77% of the 10th graders claimed that they studied it, 7% claimed that they do not remember whether they studied it or not and the rest claimed that they had not studied it at all. In the 11th graders, 78% claimed that they studied it, 12% claimed that they do not remember whether they studied it or not and the rest claimed that they had not studied it at all. It was impossible to verify it since they came from various junior high schools.) The distribution of the answers to question 2 is given in Table III.

Table III

Distribution of Answers to Question 2

| Categories | Percentages of 10th graders (N = 91) | Percentages of 11th graders (N = 97) |
|---|---|---|
| I. Any decimal must be a result of a division of 2 whole numbers | 56% | 54% |
| II. An infinite decimal can be obtained not only as a result of a division of 2 whole numbers | 23% | 43% |
| III. No answer | 21% | 3% |

Here are some answers to illustrate categories I-II in Table III.

Category I: ** I think Dan is right because when you write a number and after that you write numbers at the right side of the decimal point then it is the result of a division of 2 whole numbers.*

** Dan is right because we asserted that an infinite number, namely, an infinite decimal is a certain kind of a rational number and in order to obtain a rational number we should divide two whole numbers.*

** If the number is not the result of a division of 2 whole numbers then it is impossible to define it or to express it.*

** I think Dan is right contrary to Ron who creates something out of nothing, a meaning-*

*less number.*

Category II: ** Finally, such a number given by Ron must exist.*

** The number exists although the thought process of Dan is safer.*

** Ron is right because it is clear that the number that he wrote exists and not every decimal must be the result of a division of 2 whole numbers.*

** One can obtain a number merely by writing down its digits.*

From Table III we learn that 55% of the students identify the set of all decimals (finite or infinite) with the set of all rational numbers. Since around 80% claimed that they had learned about irrational numbers, it follows that *at least* 35% (80 + 55 - 100) do not know that an irrational number can be expressed as an infinite decimal. In fact, only 4% of the 10th graders and 33% of the 11th graders showed awareness of the existence of non-repeating infinite decimals.

It is well known that the same stimulus (question 2 in our case) can evoke totally different reactions; but when it happens, an explanation should be suggested. From reading the answers in category II (Table III) it is quite clear that the students who belong to this category have a mental ability which does not exist in the students who belong to category I. It is the ability to imagine an infinite procedure of writing digits, in an arbitrary way, to the right of the decimal point. This is an example of what we called (Vinner and Tall, 1982) an imagination act. A failure of performing the above imagination act is probably the reason of rejecting Ron's suggestion.

The difference between the 10th graders and the 11th graders in Table III is statistically significant ($\chi^2 = 18.489$, $p < 0.001$). Note that 21% of the 10th graders could not even answer the question. The difference can be explained either by maturation or by mortality (the weak students of the 10th grade do not study M<thematics in the group of the 11th graders of our sample). Of course, as always, the possibility that the result is random also exists.

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# INCONSISTENT STUDENT BEHAVIOR IN APPLICATIONAL SITUATIONS OF MATHEMATICS

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## APPLICATIONAL SITUATIONS OF MATHEMATICS

Mathematics is a field that frequently requires a coordination of diverse knowledge structures which were acquired at different occasions, in order to successfully deal with a task situation. Vergnaud (1983) has pointed out the importance of obtaining insights into children's use of mathematical knowledge in applicational situations, since the knowledge to be learned has to be related to situations for which this knowledge is functional. It is not sufficient that children can demonstrate the use of certain mathematical knowledge in pure settings, if one wants them to be able to utilize this knowledge in situations that are not specifically designed for, or do not explicitly call for, application of a given rule. Even when all necessary pieces are known, lack of coordination of relevant knowledge can be a reason of failure.

## INCONSISTENCY OF STUDENT BEHAVIOR

Frequently one finds that learners master the standard problems in a certain field but stumble in contexts they have not encountered before, even though the knowledge necessary to master the situation should be available. It may happen that alternative knowledge frameworks in the mind of a learner are evoked by the situation and override knowledge more adequate for the task. One effect is that the answers to the same mathematical question posed in different contexts can differ. In this sense, the behavior of a student can be inconsistent across a variety of situations.

One attempt to shed light on this phenomenon is the work of Thomas Seiler (1973). Seiler has conducted a series of experiments showing that juveniles already thinking formally (in the sense of Piaget) are not always able to use formal thinking operations in all problem solving tasks (not even in all tasks used by Piaget). Therefore Seiler considers it necessary to introduce a "situation and range-specific factor" in the developing cognitive structure which inhibits its generalization. Seiler's statements include that while traditional cognitive theories can well explain the big leaps of generalization that are certainly observable, they tend to ignore the common fact that, without further guidance, an individual never can apply a given rule in all novel situations. Formal thinking structures arise from the individual's experience with specific problems in specific situations and barely reach an unrestricted, universal generality. Further, it is very probable that, within one individual and with respect to one subject domain, different thinking structures can

coexist which can become activated alternatively, depending on the symbol system primarily triggered or cued by a situation (in particular cf. Seiler, 1973, p. 268).

As a consequence, we would not expect it to be a question of "yes" or "no" whether a student has acquired a certain concept or rule but expect that s/he might be unable to apply that concept or rule in all circumstances. To discover inconsistent behavior, then, does not mean to find that a student has not gotten a concept or rule so far, but rather, to find evidence for a restriction of the range in which the student's concept or rule applies. To identify the conditions and laws of inconsistent behavior means to prepare the grounds for remediation towards further maturization of the student's concept.

In this paper we shall discuss a student's inconsistent behavior in an applicational situation in the domain of rational number learning and attempt to clarify the conditions of its origin.

## CONTEXT

The notion of rational number comprises a conceptual field (Vergnaud, 1983) that involves a large number of subconcepts and subaspects. Thus it constitutes a rich domain to study children's grasp and use of mathematical ideas. In a series of studies conducted by the Rational Number Project (see acknowledgment), situations were constructed that did not expressively call for, but required a coordinated application of, several subconcepts of rational number in order to succeed. One of these studies is the "Gray Levels Study" which was also the background for an earlier discussion of structures and mechanisms of children's mathematical knowledge (Wachsmuth, 1984). After 30 weeks of experimental instruction, each child in a group of sixteen 5th-graders was presented with a complex problem solving task, embedded in a video-taped clinical interview.

The task involved a set of 12 fractions, written as symbols a/b on little cards, which were said to represent ink mixtures with a parts black ink in b parts solution. The fractions were to be ordered by size and to be associated with stages on a scale of 11 distinct gray levels arranged by increasing "grayness" from 0% (white) to 100% (black) in stages of 10%. Presented were the fractions 0/20, 1/5, 2/7, 6/20, 2/5, 4/10, 6/15, 2/4, 4/8, 4/6, 6/9, and 12/15. Although the visual information could be used as a guidance, it was necessary to use the numerical information and apply various pieces of rational number knowledge in order to perform well on the task. Thus the gray levels task was expected to elicit how children bring their rational number knowledge to function in a complex applicational situation.

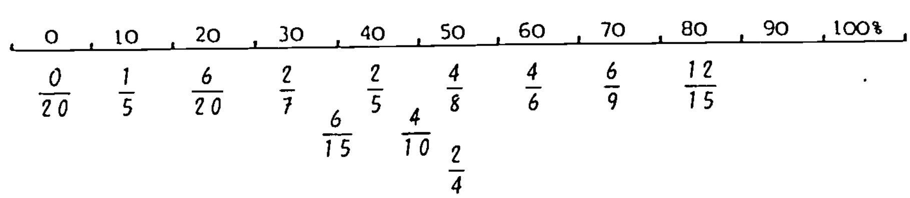
Three of the sixteen children were so successful that the average deviation in their card displacement was less than half a stage off the correct location. Other children were much less successful. The differences are assumed to be due to different ability in activating relevant domains of fraction knowledge and coordinating it on the task. A low performer, Terri, was discussed earlier (Wachsmuth, 1984). Here, a much better but still not perfect performer, Bert (not his real name), is the subject to generate further hypotheses about factors that impede optimal functioning of a developing cognitive structure.

A CASE OF INCONSISTENT BEHAVIOR

In his performance on the gray levels task, Bert, in general a relatively high-achieving subject, exhibits inconsistent behavior in the following respect. In the beginning Bert recognizes the equivalence of the fractions 2/4 and 4/8, and also of 4/6 and 6/9. At this time, he is able to infer that equivalent fractions should be associated with the same gray value. However, in the course of working the problem, Bert associates the fractions 4/6 and 6/9 which he had previously regarded as equal with different but adjacent gray levels at about the correct location. That is, independently of his knowledge about fraction equivalence, Bert exhibits a good perception of the size of these fractions. When he is reminded at his previous statement in the follow-up interview, he realizes his mistake and corrects it. The interview begins with Bert's initial observations on some of the fractions.

0. BERT: (Early-on, sorts the cards and puts 2/4 and 4/8 together on table.)
1. INTERVIEWER: You put two-fourths and four-eighths together?
2. BERT: (picks them up) They're equal.
3. INTERVIEWER: I see... you would put them on the same card (i.e., gray level)?
4. BERT: Yeah... (now puts 6/9 together with 4/6) These two are equal...

In his final placement of all cards at the scale, Bert has put 4/6 at the 60% level and 6/9 at the 70% level. That is, with respect to placement on the scale, Bert rates these fractions as very close but has lost sight of their equivalence. Similarly, he has put 2/5 at 40%, 4/10 at 45%, and 6/15 at 35% (see figure below; the percent marks were not present on the gray level scale).



5. INTERVIEWER: (after the whole task has been completed) You put four-tenths left of four-eighths, why did you do that?

6. BERT: Because four-tenths is less,... less than half a unit.

7. INTERVIEWER: And you put six-ninths right of four-sixths, why did you do that?

8. BERT: Because four-ninths-and-a-half would be half a unit...

9. INTERVIEWER: ... Before, you mentioned that they are equal ... four-sixths and six-ninths ...

10. BERT: Oh yeah, they are! (picks up 6/9 and 4/6) I think they'd be right there (puts both cards on 60%).

11. INTERVIEWER: Why did you put twelve-fifteenths over there? (points to 80%)

12. BERT: Because that's only three-fifteenths away from a whole.

13. INTERVIEWER: Why did you put six-fifteenths over here? (points to 35%)

14. BERT: Because that's... that's less than half a unit... (thinks)... umm, seven and a-half a unit would be... seven and a-half would be...

15. INTERVIEWER: What did you think about when you put six-twentieths? (points at 20%)

16. BERT: Because six-twentieths is greater than one-fifth; one-fifth equals four-twentieths.

17. INTERVIEWER: You put one-fifth right here (points to 10%) and two-fifths here (points to 40%)...

18. BERT: (points to 20% and 30%) Well see, there could be fractions between there.

19. INTERVIEWER: You put two-fifths there (40%) and four-tenths there (45%). What was your thinking?

20. BERT: Well, four-tenths would probably be ... well ... they're equal! (laughs, puts 4/10 over 2/5 on 40%) I didn't notice this.

## DISCUSSION

With respect to Piaget's stages of cognitive development, Bert (age 10;11;24) could be considered transitional from the concrete to the formal-operational stage. In an earlier interview assessing children's ability to compare pairs of fractions and pairs of ratios presented in a symbolical form (cf. Wachsmuth, Behr, & Post, 1983), Bert had mastered each of 18 (2 x 9) tasks of varying difficulty. Thus, the above document seems suited to illuminate some critical aspects about Bert's developing cognitive structure with respect to the range specificity of his rational number knowledge. In the following we attempt an explanation for the inconsistent behavior exhibited by Bert.

1. Bert has a repertory of rules that he can use when the task is to make a judgment about equivalence or non-equivalence of fractions. One of his rules might be

formulated as follows: "Two fractions are equal, if one can be transformed into the other by multiplying its numerator and denominator by a common factor." For example, in the same interview session Bert found 4/6 and 20/30 (presented as symbols) to be equal, explaining "six times five is thirty and four times five is twenty." He also has a rule to determine the equivalence of two fractions which could match for 4/6 and 6/9 (i.e. where the corresponding terms are not multiples of one another), like: "Two fractions are equal, if a lower-terms fraction of one of them can be found from which the other one can be generated as a higher-terms fraction." For example, on a different item in the same interview, Bert stated that 8/10 and 20/25 are equal, explaining that "four-fifths is lower-terms, so four times five is twenty, and five times five is twenty-five."

To recognize the equivalence of fractions, however, requires that this rule is "activated", i.e. is attempted to be used in the situation. This might explain why Bert has stated that 4/6 and 6/9 are equal right in the beginning, but has "lost sight" of this fact when placing the fractions. We conclude that both of the following are true: (1) The fact "4/6 = 6/9" is no longer present in Bert's short-term memory when he places these fractions at the scale, and (2) the rule that could establish the equivalence of the two fractions is no longer "active". But when the interviewer reminds Bert at his earlier statement (line 9 of the transcript), Bert immediately adjusts his solution.

2. Bert has a repertory of rules that he can use to determine the sequence (in magnitude) of non-equivalent fractions. In several instances, Bert's explanations indicate a successful attempt to estimate the size of a fraction by using 1/2, 1, or some other fraction as a point of reference. Bert used 1/2 in his placement of 4/10 (line 6 of the interview transcript). He used 1 as a point of reference to place 12/15 (line 12), and 1/5 as reference point in placing 6/20 (line 16). Using this strategy may include the generation (not: recognition!) of equivalents of some fraction (e.g., 1/5 is transformed into 4/20 in line 16). A characteristic in his placement originating from this process of estimation is the little "slack" in placing the fractions.

Even without making use of all equivalences, Bert's placement of equivalent fractions was considerably close to correct. His explanations suggest that he has again employed his estimation strategy which can generate an approximate size judgment (estimate) independently of recognition of equivalences. For example, he apparently has used 1/2 (transformed to 4 1/2 ninths) in his placement of 6/9 (line 8), and (as 7 1/2 fifteenths) in placing 6/15 (line 14), and possibly also in placing 2/5

and 4/10. What results again is a little "slack" in placing fractions that are actually equal in size such as 4/6 and 6/9. (Had he noticed that 4/6 and 6/9 are equal amounts away from 1/2, namely by 1/6 and 1 1/2 ninths, respectively, application of this strategy would have been absolutely successful.)

This analysis leads us to the following conclusion: To find that two rationals are equal, there are two totally different ways that involve different repertories of rules: (a) find the fraction representatives to be equivalent, thus representing the same number; (b) make a judgment on the size of each fraction separately and find them to be equal in size. While the former is based on a procedure of algorithmic nature, it is natural that the latter, as it pertains to the field of estimation, depends on less precise arguments. It is heavily supported by Bert's explanations that in the course of working the problem he has switched to the second rule repertory and that this is the reason for his close-to-correct placement of 4/6 and 6/9, and of 2/5, 4/10, and 6/15. The rule that could have established equivalence of 4/6 and 6/9 as representatives of the same number was no more active at this point. But when the interviewer calls Bert's attention to 2/5 and 4/10 (line 19), he does recognize their equivalence.

3. From the above discussion results that at least two different sets of rules are part of Bert's cognitive structure. The fact that they are not always coordinated presumably has given rise to the inconsistent behavior observed. Two points that absolutely need discussion here are the following:

- Is that what happened a sole event, or does it indicate an important restriction in Bert's developing cognitive structure?
- Is that what happened unique to Bert, or could it be an interindividual phenomenon that is also relevant to other children?

Concerning the first point the following is notable. In a task-based interview conducted about one-half hour later under a different format (ratio symbols were used in place of fractions, e.g., 2:3 in place of 2/5, etc.), Bert displayed similar behavior: There he put 2:3, 4:6, and 6:9 at different but adjacent gray levels. Concerning the second point it is noted that four other subjects placed 4/6 and 6/9 at different but adjacent gray levels close to the correct position. So the observation apparently does not concern a factor totally unique to Bert.

In conclusion: At least, these observations suggest to further explore the idea that a disparity of the rule repertories a learner possesses can restrict the range in which the rules can be applied. The effect is a lack in the coordination of rules

such that in some cases relevant rules are not applied because one set of rules has "taken over" control of the learner's doing. For a "good conception" of rational number size it would certainly be necessary that a learner obtains an awareness of the different sets of rules that are available to him/her to make judgments and that s/he is able to coordinate these. Exactly here is the point where remedial work can be invested to help the learner progress towards more stable performance also in complex applicational situations. For example, if Bert could be made to always check for equivalences before making a size judgment for one in a set of rationals, his performance should become very close to optimal.

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# MISCONCEPTIONS IN GRAPHING *

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Abstract: This paper attempts to organize some recent findings in the literature concerning the errors students make on tasks involving graphs and proposes some initial cognitive explanations for patterns in these errors. Some basic characteristics of a competence model are proposed for knowledge structures used in comprehending and generating graphs. Two types of common misconceptions, treating the graph as a picture and slope-height confusions, are discussed in terms of this model.

## Competence Model

The ability to interpret graphs is important for mathematical literacy and for understanding the concepts of function and variable, as well as for developing basic concepts in calculus. I will first propose a partial model of four types of knowledge structures needed to comprehend graphs, shown in Figs. 1a and 1b. This should not be viewed as a Platonic "given", but as a plausible competence model which provides an initial theoretical framework for interpreting students' errors. I will consider some of the conceptions needed to solve the problem of drawing the qualitative shape of the graph of speed vs. time for a bicycle rider coming down from the top of a hill. In particular, I want to incorporate (a) the connection to a real world context and (b) the idea of variation.

Static model. Conception (1) in Fig. 1a is a naive practical representation incorporating everyday knowledge about the problem situation based on one's concrete experience with watching and riding bicycles, including the sensation of speeding up as one rides down a hill. Here I will simply refer to each knowledge structure as a conception, but they can also be thought of as occurring at different levels of representation. Conception (2) represents the idea that at a particular time, the bicycle is at a particular speed. Thus, the subject must have adequately developed concepts for speed and time and must be able to isolate these variables in the problem situation. In conception (3), I show the subject forming a spatial distance

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metaphor for these variables. The notation here is related to but not equivalent to that of Driver (1973). Time and speed are represented as lengths of line segments. The segments in 3 can then be mapped onto distances in the graph in conception (4). Here the length representing time must be mapped onto the distance of a point from the y axis, and the length representing speed must be mapped onto distance from the x axis.

Thus in this view a graph is essentially a model of a relationship between variables based on a metaphorical representation of variable values as lengths on line segments. The model is analogical in the sense that the relationships between line segments in the model are isomorphic to the relationships between variables in the problem context.

The concept of variation in a dynamic model. One limitation of the model in Fig. 1a is that it only shows correspondences between individual static values of the variables. The notion of variation seems to be missing. What is needed is a representation of correspondences between changes in variables. An attempt to model some aspects of the notion of variation is illustrated in Fig. 1b. Here conception (1) is the same as in Fig. 1a, but in conception (2) we find the variables of change in time and change in speed instead of simply time and speed.

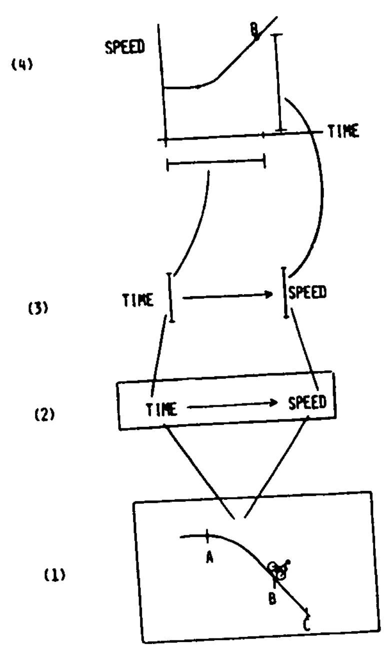


FIG 1A

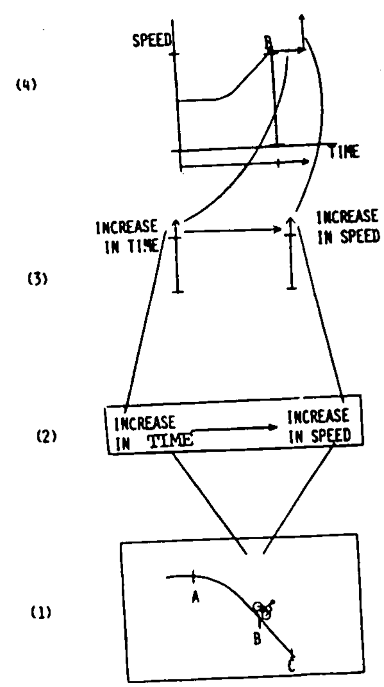


FIG 1B

Conception (2) in Fig. 1b embodies the idea that as the time elapsed increases the speed of the bicycle increases. It is this idea that would seem to carry the basic notion that "the speed is increasing with time" in which the variables are truly thought of as varying. In conception (3) the correspondence of direction of change in time with direction of change in speed is represented in terms of the spatial metaphor of line segments. An increasing value of speed is represented by an increasing length of line segment. Thus whereas the metaphor for the value of a variable was length, the metaphor for an increase in value is an increase in length, and the metaphor for continuous increase in value is a continuously growing line segment. In conception (4) the increases in the two variables are mapped onto changes in the x and y coordinates of the graph as the graph point moves. Thus motion of a point on the graph in two dimensions is used to represent changes in two variables at once. These more dynamic concepts in Fig. 1b are thought of as existing in conjunction with the more static concepts in Fig. 1a.

In this case we are still dealing at a very basic conceptual level, since we are only representing the qualitative concept of the direction of change in two variables. In summary, this competence model proposes some of the basic knowledge structures that are needed to comprehend qualitative graphs, and indicates that even at this low level, the knowledge required can be somewhat complex.

Graphing Misconceptions

Slope-height confusion. In a study of science-oriented college students who anticipated difficulty in taking science courses, McDermott, et. al. (1983) asked the following question referring to the graph in Fig. 2: "At the instant t = 2 seconds, is the speed of object A greater than, less than, or equal to the speed of object B?" As many as half the students answered incorrectly. One interpretation of this error is that students are mistakenly using the graphical feature of height instead of slope to represent speed. The model proposed here for the cognitive source of this error is a misplaced link between a successfully isolated variable and an incorrect feature of the graph, as illustrated in Fig. 2, solution 1. However, as McDermott points out, it is difficult to assign a single cause to this error since students have been observed to confound the physical concepts of speed and relative position at a conceptual level in other tasks which do not involve graphs. This second interpretation is illustrated in Fig. 2, solution 2.

The height-for-slope error has also been reported by Janvier (1978). In one of the problems in his extensive thesis on graphing, he asked students to draw graphs of height vs. time for the water level in different jars being filled with water. In one task he showed them the graph (A) in Fig. 3 for a wide jar and asked them to

draw the graph for a narrower jar being filled from the same water source. Some of the students drew the parallel line of dots (B) in Fig. 3 instead of a line (C) with an increased slope. Other errors not discussed here, such as height-for-difference, slope-for-height, and slope-for-curvature substitutions also fall into this general category.

Treating the Graph as a Picture.

In another type of error the student appears to treat the graph as a literal picture of the problem situation. This error can occur, for example, when students are asked to draw a graph of speed vs. time for a bicycle traveling over a hill. In classroom observations of a college science course we noted that many students would simply draw a picture of a hill. This can happen even when the student first demonstrates the ability to describe the changes in speed verbally.

This type of error has been discussed by Monk, (1975), Kerslake (1977), Janvier (1978), and McDermott, et.al. (1983). For example, Janvier interviewed students solving a problem about a graph of speed vs. distance travelled for a race car going around a track with a number of curves in it. Many students erred when asked if they could tell how many curves were in the race track by looking at the oscillating graph. They tended to confuse this with the number of curves in the graph itself.

In making the bicycle problem error of drawing the shape of the hill, the student appears to be making a figurative correspondence between the shape of the graph and some visual characteristics of the problem scene, as shown in Fig. 4. This simplistic process contrasts to the relatively complicated process in the competence model shown in Fig. 1. Notice that in Fig. 1 the student must differentiate between and coordinate at least two separate images: (a) the problem

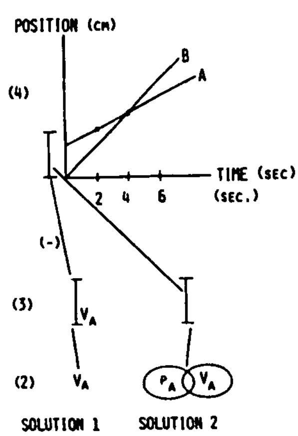


FIG. 2

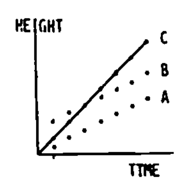


FIG 3

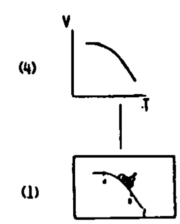


FIG 4

situation and (b) the graph. Students making the error apparently have difficulty in maintaining this differentiation.

<u>Two types of graph-as-picture errors</u>. In classroom observations with college remedial mathematics students, we found that many students generating a graph such as the one shown in Fig. 5 would say incorrectly that the two cars represented in the graph were passing each other at the point where the graphs cross. Here they seemed not to be drawing a whole picture, but to be mapping a local visual feature of the problem scene (same location) onto a similar feature of the graph. However this response could alternatively be interpreted as confounding the physical concepts of same speed and same location, as in the problem studied by McDermott et. al. mentioned earlier.

As part of an effort to develop problems elucidating specific misconceptions more clearly, David Brown of our group and I developed the following Intersection problem referring to the graph in Fig. 6.:

> Two cars (A & B) are driving toward an intersection. One car is coming from the north on Main Street, the other car is coming from the east on Green Street. Below is a graph showing each car's distance from the intersection versus time. At what time or times are the cars at the same location?

This problem does not involve speed as a variable and so avoids the alternative interpretation referred to above. When we gave it to 16 students studying remedial level mathematics in college, 13 said that 2 and 6 were times when the cars would be in the same location. Here the students seem to be focussing on the local feature of the graphs being in the same location and treating this as representing the cars being in the same location. These point by point visual correspondences can occur even if the student is not treating the graph as a complete picture of the paths the cars are driving on.

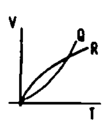


FIG. 5

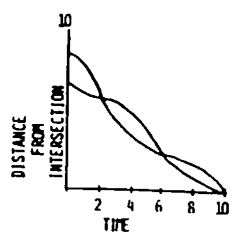


FIG. 6

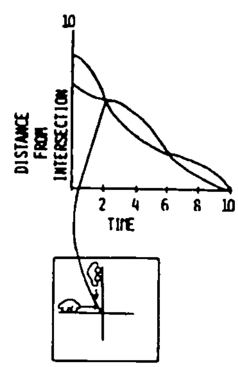


FIG 7

We can now distinguish between two types of graph-as-picture errors. We refer to the bicycle error as a global correspondence error. Here the shape of an entire problem scene is matched to the shape of the entire graph in a global manner, as shown in Fig. 4. This contrasts with the intersection problem error, which we call a feature correspondence error. Here a specific visual feature of the problem scene (path intersection) is matched to a specific feature of the graph, as shown in Fig. 7.

## Conclusion

Our partial taxonomy of misconceptions in graphing now looks like:

1) Link to Incorrect Graph Feature
   a) Height for Slope        b) Slope for Height
   c) Height for Difference   d) Slope for Curvature

2) Graph as Picture
   a) Global Correspondence   b) Feature Correspondence

More generally, we can refer to all of the above as non-standard symbolization strategies. Non-standard symbolization strategies also occur in the area of algebra word problems (Clement, 1982). In addition students can confound physical context variables, which can occur in problems with or without graphs.

Space has allowed a discussion of only two of the many observable types of graphing errors. A competence model portraying both static and dynamic aspects of graphing concepts has allowed us to describe different possible sources for the errors. This model is still limited in its present form however, and its extension, refinement, and empirical validation are important tasks in our current research program.

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# THE MATHEMATICS OF PSYCHOLOGY EDUCATION

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Problems of teaching statistics and other mathematical applications to university psychology students are discussed. Results from tests on about 400 students are reported, showing substantial weakness in calculation, estimation and conceptual understanding.

"The teaching of statistics to psychology students is probably the biggest area of failure and frustration in most psychology departments" (Evans, 1976). These problems can be attributed to the conceptual difficulty of the subject, compounded by the fallibility of probabilistic intuition (see eg Kahneman, Slovic and Tversky, 1982). The situation is further complicated by theoretical controversies. In particular, there is an identifiable crisis at present in that criticisms (which have been stirring for some time) of the traditional null-hypothesis-testing methods of inferential statistics are becoming hard to ignore, yet such methods remain standard in published research, in most undergraduate texts, and in commercial software. Massive inertia in the system resists innovation; witness the very limited impact of Bayesian methods over more than twenty years despite some powerful advocates. The EDA (Exploratory Data Analysis) approach, spearheaded by Tukey, has achieved rather more impact in a relatively short time (even SPSS now includes a few EDA techniques).

Apart from statistics, there are many other applications or mathematics in psychology: Psychophysics, Information Theory, Decision Theory, Game theory, etc. Introductory and elementary treatments of such topics are appropriate at the undergraduate level and suitable texts are available. In my opinion, however, there is a lack of a book giving general coverage of Mathematical Psychology pitched at the right level. Most such books are written by experts for experts; the most suitable one I know of is Miller's "Mathematics and Psychology" published back in 1964.

There are also books, such as Bishir and Drewes (1970), which systematically cover the mathematics - but in too formal a manner. Take the concept of function. In Bishir and Drewes, a function is defined in Chapter 4 (after chapters on sets, symbolic logic and relations) as a special kind of binary relation. Such a formal approach is off-putting and does not make contact with the students intuitions. Why not start with the concept of variable (with associated ideas of number systems and measurement) and define a function as a rule which determines the value of one variable when the value of the other is known? Links with general concepts of causality and prediction can then be made. Key functions (linear, exponential etc.) should be discussed as models of situations with certain general characteristics which give rise to these functions.

Faced with these problems, it would seem appropriate for psychologists to study their students, yet, as far as I know, very little research has been done. Here I will report some results from exploratory and unsystematic studies of three samples of psychology students: (a) first-year students at Queen's University Belfast studying psychology as one of their subjects (N = 287), tested near the beginning of the academic year (b) first-year students selected for Honours Psychology at Nottingham (N = 37) tested at the end of the year, and (c) second-year students at Edinburgh (N = 93), tested at the beginning of the year. For simplicity, results will be reported here as percentages for the combined group (N = 417).

Some of the questions tested computation:

| | | | |
|---|---|---|---|
| (1) | $\sqrt{0.09}$ | Correct : 60% | 0.3 : 27% |
| (2) | .02 X .12 | Correct : 72% | 0.24 or 0.024 : 23% |
| (3) | If a = 3, b = -2 and c = 7 what is the value of $3b^2 - abc$? | | Correct : 63% |

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With the availability of calculators and computers, estimation becomes important. Judging by this example, the students are not very good at it:

(4) Estimate (do not attempt to calculate)

$$\frac{5.6832 \times 0.623}{0.07689}$$

Correct : 41%

(the answer is about 46 - estimates between 20 and 100 accepted)

Most of the questions, however, were designed to test conceptual understanding. This item, taken from Krutetskii (1976) is based on the fundamental notion of estimating a statistic for a population from that for a sample:

(5) Forty fish were caught from a pond with a net; each one was marked and thrown back into the pond. On another day 60 fish were netted from the pond, and there were 4 marked fish among them. Estimate how many fish are in the pond.

Correct : 42% 96 or 100 : 27%

The frequently observed incorrect answers 96 and 100 look reminiscent of the additive strategies used by many children in Hart's work on ratio. Indeed, on one of Hart's items, 18% of the students tested on a ratio question (4 is to 6 as 7 is to ? in the context of geometrically similar figures) were "adders" (gave the answer 9).

A question taken from Tversky and Kahneman (1974) tests awareness of the effects of sample size on variability:

(6) A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50 per cent of all babies are boys. The exact percentage of baby boys, however, varies from day to day. Sometimes it may be higher than

50 per cent, sometimes lower. For a period of one year, each hospital recorded the days on which more than 60 per cent of the babies born were boys. Which hospital do you think recorded more such days?

Larger hospital . . . Smaller hospital . . . About the same . . .

Correct : 22% About the same : 67% Larger hospital : 10%

Another question testing the same concept in the context of playing a game of squash to 9 or 15 points (see Kahneman and Tversky, 1982) gave very similar results.

One question relating directly to statistics was used (this was only given to the Nottingham students):

(7) In an experiment, the temperatures at which subjects began to complain of discomfort were measured (in Centigrade) for male and female subjects. Some of the statistics calculated were as follows:

Temperature reached (male subjects) - mean: 20, standard deviation: 2

Correlation between temperature reached and extraversion: 0.15

t-test for difference between males and females: 1.25

If the temperatures had been measured in Fahrenheit instead, what would these figures have been? The formula for converting Centigrade to Fahrenheit is F = 32 + (1.8 X C)

| | | | | |
|---|---|---|---|---|
| Mean | Correct : 84% | | | |
| SD | Correct : 11% | 35.6 : 35% | 2 : 27% | 6.8 : 8% |
| Correlation | Correct : 62% | no answer : 30% | 31.73 : 8% | |
| t-test | Correct : 54% | no answer : 32% | 34.25 : 14% | |

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Few of the students, therefore, were able to think through the correct transformation for the standard deviation; the errors made indicate lack of understanding of its basic nature as a measure of spread independent of location. The third and fourth parts of the question could be answered through consideration of the invariance properties of the formulae for the two statistics, but the students were not expected to know that. Rather, they could have arrived at the answer by realizing that neither statistic could be of much use if it could be changed by such a change of scale.

For items (5) and (6) the results for the Nottingham and Edinburgh students may be regarded as more significant, in that these students had been given instruction on the concepts underlying the questions. This instruction had relatively little impact, however. This reflects a familiar observation namely that students who "know" about, for example, the estimation of a population statistic from a sample statistic, or the effects of sample size on variability in a binomial distribution may not recognize situations in which that knowledge is applicable.

Most people who teach statistics have anecdotes illustrating students' inability to think logically once they have labelled a problem as statistical; here is one from my experience. A student, who was of above-average ability generally, did a thesis project in which mice were tested for aggression, the measure of aggression being the time in seconds between initial contact of two mice, and the onset of fighting. Some mice did not fight at all. Thier scores were entered as zero, included with the other scores, and these figures were analyzed by analysis of variance. (By the way, the member of staff supervising this thesis did not notice anything amiss).

Of course, as well as conceptual problems, there are many to do with past experience with mathematics, sex-role stereotyping, "learned helplessness" etc. Besides more systematic used of pencil-and-paper tests, therefore, students should be interviewed to probe some of these aspects as well as probing further the misconceptions revealed by the tests.

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THE ACQUISITION OF BASIC MULTIPLICATIONAL SKILLS

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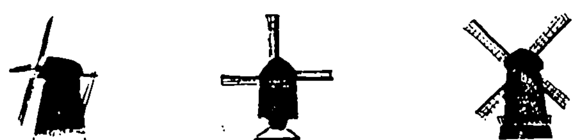
The intent of this contribution is to give an impression of education directed towards teaching children basic multiplications along new lines. This is, to all appearances, a generally recognised and accepted part of arithmetic, but those who are familiar with actual arithmetic instruction for 7 to 9 year olds will note numerous differences in outlook upon reading this paper.
Regarded superficially, the aim of the old and new methods is the same. One goal of the old form of instruction was the learning of the basic multiplications by heart. This took place in a very special manner, by learning the multiplication tables. We can see in a Dutch method from the 1960s how, for example, the four times-table was introduced. On the very same page the children are encouraged to learn the table by heart. There are, however, no directions as to how to go about this.

1



Deze molen heeft .. wieken.

Deze molen heeft natuurlijk ook .. wieken.

Hoeveel wieken heeft deze molen?

Hoeveel wieken hebben deze molens met elkaar?
Je kunt het zo opschrijven 4 + 4 + 4 = ..  Het kan ook zo 3 × 4 = ..

2 **Nu gaan we de tafel van 4 leren**

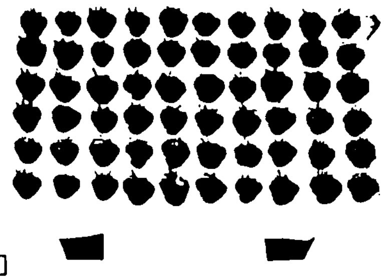
| | |
|---|---|
| 4 | 1 × 4 = 4 |
| 4 + 4 = .. | 2 × 4 = 8 |
| 4 + 4 + 4 = .. | 3 × 4 = 12 |
| 4 + 4 + 4 + 4 = .. | 4 × 4 = 16 |
| 4 + 4 + 4 + 4 + 4 = .. | 5 × 4 = 20 |
| 4 + 4 + 4 + 4 + 4 + 4 = .. | 6 × 4 = 24 |
| 4 + 4 + 4 + 4 + 4 + 4 + 4 = .. | 7 × 4 = 28 |
| 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = .. | 8 × 4 = 32 |
| 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = .. | 9 × 4 = 36 |
| 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = .. | 10 × 4 = 40 |

3 Leer de tafel van 4 uit je hoofd.

figure 1 from: "Naar Zelfstandig Rekenen"

But we also recognize the formal approach at many points in methods which advocate a modern view of arithmetic instruction.

Dit is de tafel van zes.



$6 = 1 \times 6 = 6$

$6 + 6 = 2 \times 6 = 12$

$6 + 6 + 6 = 3 \times 6 = \square$

$6 + 6 + 6 + \square = 4 \times 6 = \square$

$6 + \square + \square + \square + \square = \square \times 6 = \square$

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figure 2 from: "Taltaal"

An essential part of current arithmetic instruction would seem to be the recitation of the multiplication tables. This often degenerates into rattling them off. The idea behind this was that, by pompous reptition, the facts would then stick in the student's memory and would remain by association: if someone should say "6 x 7", then the child would respond - sometimes uncousciously - with "42". An association would have to be created between two sounds that always appear together. Memorisation would then be furthered by continuous recitation of the multiplication tables.

Much criticism can be made of this procedure from an educational standpoint. Seen pedagogically, it is a weak manner of working: it can demotivate children and turn arithmetic into an unattractive subject. But psychologists, too, express their doubts. Van Parreren writes, for example:

"You can take a row of nonsense material, but also a row of words, and even a meaningful text and recite it "without thinking", rattle it off, babble it, or however one wishes to call it ... In our context the significance is that this

procedure is hardly manoeuvrable at all. Test persons are only able to perform in this way by reciting the row from beginning to end, that is, in the normal order and, moreover, in a rapid tempo".

The essential question is how the basic multiplications can be imprinted without causing damage to the applicability. Ready knowledge which is inflexible is indeed not of much use to children. Knowledge of basic multiplications is of great significance for further development in arithmetic, for learning column arithmetic as well as for solving problems, estimating and mental arithmetic. The heart of the matter is therefore to develop a method of instruction which can lead to manoeuvrable, flexible ready knowledge.

How could we establish such a method? We went to the children themselves for advice. We examined how they worked out multiplications. The results of this research have been or shall be published elsewhere, among other places in an article in Educational Studies in Mathematics (1985). We ascertained that children tend to use a number of arithmetic strategies aside from the usual lines of learning (multiplication tables), viz:

- the commutative property (9 x 2 = 2 x 9)
- the factor 10 (10 x 6 = 6 with a zero added)
- doubling (2 x 6 = 12; 4 x 6 = 12 doubled)
- halving (10 x 6 = 60; 5 x 6 = half of 60)
- one more (5 x 7 = 35; 6 x 7 = 35 + 7 = 42)
- one less (10 x 7 = 70; 9 x 7 = 70 - 7 = 63)

We ascertained as well that children acquire supports which can be used in order to learn to calculate other multiplications. Supports and arithmetic strategies, was our supposition, should be at the base of the fund of knowledge of basic multiplications. The new line of instruction was arranged in four phases, from the introduction to the application of ready knowledge.

The phases are:

1. Introduction.

   The students are confronted with a number of situations of a multiplicative character. At that moment the children have only addition and subtraction skills at their disposal. With the means available to them they must try to describe the multiplication situations. It is clear that finding solutions for the problems can take place at various levels. These solution levels are used in order to raise the entire class up to a higher plane. For this purpose the levels are discussed and commented on with the class.

2. Development of arithmetic strategies.

   In the first phase the students learned to solve multiplications whereby, more or less by chance, they learned separate basic multiplications by heart. These basic multiplications acquire the function of supports, enabling the terrain to be explored further. In this phase, attention is paid explicitly to the strategies which children can use for calculating multiplications, as well as for expansion of their ready knowledge of basic multiplications. The children are on their way to a formal approach to the terrain.

3. Imprinting.

   We assume that the children are fairly skilled in calculating basic multiplications with the aid of arithmetic strategies and that they already know a large number of supports as facts. In this phase, that overlaps the previous one to a great extent, attention is paid explicitly to imprinting the basic multiplications. If it turns out that children do not know a certain product by heart, this need not be a problem for them. They can still always calculate the answer (with a certain amount of

skill). It is therefore not a question of all or nothing.

The "surroundings" play a large part in the imprinting of more or less separate multiplication facts. A fact such as, for instance 5 x 7 = 35 can be memorised by keeping 10 x 7 = 70 in the back of the mind. Certain supports remain invisible but available by writing them on the back of the board. If a student forgets a certain fact, the board can be opened up, revealing the fact. With increasing frequency the children will "see" the multiplication facts without them being visible.

4. Application.

   About a year has passed since the introduction. Most of the children know many multiplication facts by heart, and they can work out the facts which they don't know by other means. Both aspects - knowing and being able to work out - must be maintained in application situations. In order to further the manoeuvrability of the factual knowledge, many situations and contexts are chosen which have a multiplicative character. Aside from this flexibility, maintenance and sharpening of the knowledge can be mentioned as important goals of the education in this phase.

The four phases described above cannot be sharply divided, but rather overlap each other.

Moreover, in each phase regular attention is paid to division. Once children have acquired a certain ability in multiplication then they can divide as well. Calculating 42 : 6 = comes to the same thing as calculating the multiplication ... x 6 = 42 or 6 x ... = 42. In context problems, too, we see that division and multiplication can be interchanged.

The following two points are characteristic for this program:

- the program frequently makes use of situations which the children can solve through multiplication

- one of the program's points of departure is the ideas which children hold. Children's idea and solutions are also used in order to stimulate progress in development.

The central question is whether the desired flexibility in the basic knowledge of multiplication and division is furthered by this program. Instructor's opinions have been positive on this point. It can be seen in the work of older students that a flexible use of knowledge also takes place in other areas of arithmetic. It is therefore suggested that these children have a somewhat different attitude in this respect than do children who have been in contact with a more traditional program.
A certain amount of caution should be exercised here. Other causes can play a rôle, for instance, the instructors' increasing enthusiasm for the new program. This enthusiasm can have a positive effect upon the students.

An important question is how teachers can deal with the program. In other words, what sort of teacher must you be (or become!) in order to work with this program?
Two significant matters are touched upon here.
In the first place, instructors must be prepared to alter their standpoint as to the manner in which students gain factual knowledge in arithmetic instruction. There does seem to be traditionally a general consensus about the question of how this memorisation takes place.
A very convincing point in favour of the new approach would seem to be the changes which instructors observe in their students. This matter also concerns the value which instructors place on situations and contexts. Practicing the multiplications tabels is an old established, rigid learning process, disconnected from any context.
A second matter deals with the way the teacher views the students. Does the

teacher value the students' contributions? Does she make room for them? Or does she stick closely to the textbook instructions?

In other words, it is of utmost importance whether the instructor can grow in her profession, whether she is prepared to learn from the students and whether she maintains an investigative attitude. It is possible that instructors are steadily acquiring a different attitude with regard to the renovation of their arithmetic education. These matters could be dealt with in future research.

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# ESTIMATION: PRACTICE AND PROCESS

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Everybody seems to be agreed that the development of estimation skills is important. (See, for example, Cockcroft, 1982; Reys and Bestgen, 1981). In the mathematics curriculum in Israel, estimation as a subject appears in a very limited way in the elementary school, and not at all in the junior high school or high school curricula.

In order to get an idea, if and how students deal with estimation tasks, and what processes they use, we administered a questionnaire to sixth grade students. The results obtained suggested the necessity for treatment (Markovits et al., PME 1984), and formed the basis for a unit on estimation, developed for seventh grade students. The unit includes items on the following topics: meaningful accuracy, estimation in "every day life", in computation, in measurement, algorithmic estimation of quantities and checking the reasonableness of results. For the evaluation of the unit we used experimental and control groups with pre and posttests. The students were asked to explain their "estimates" thus enabling us also to study the processes used, and their misconceptions. The experimental group (about 350 students from 15 classes) was given an ability test in mathematics, the pretest, the unit and the posttest. The control group (about 200 students from 7 classes) was given the ability test in mathematics and the posttest.

So far we have analysed the pretest results, and we report here these results on sample pretest problems. In our presentation we will compare these results to those obtained on the posttest.

In general, we found that students have little ability to estimate. For example, when we asked them to estimate the height of a 4 floor building, about half the students ( n = 300) gave an aswer between 10 and 20 meters (the range we considered to include any reasonable answer). To the question "how many litres of water in a bucket?", only about 20% wrote 8-15 litres (which we allowed as the reasonable range). About 70% underestimated. The situation was about the same when we asked for an estimate of the speed of a plane which flies from Tel Aviv to New York, and here, about a quarter of the students said that the speed would be between 0-100 km/hour.

To illustrate the processes students used, we bring two examples.

Example 1

Algorithmic estimation of quantities

(The percentage of responses for each answer are given in brackets: n = 333).

*How many words in a 200 page book?*

*Indicate the answer you consider to be reasonable.*

*(4%)* *a) About 1000 words.*

*(17%)* *b) About 10,000 words.*

*(33%)* *c) About 100,000 words. (Reasonable answer)*

*(46%)* *d) About 1,000,000 words.*

*Explain your choice.*

The explanations are categorized in the following table.

| Category of explanation | % students in category | % students giving reasonable answer |
|---|---|---|
| No explanation | 26% | 10% |
| "Irrational" explanation<br>e.g. "This is what I think"<br>"This is the most reasonable answer" | 21% | 8% |
| Rational explanation<br>Level 1 - Naive explanation<br>e.g. "There are many pages, so there will be many words"<br>"The book is very thick, there must be many words" | 19% | 3% |
| Level 2 - Algorithm based on the number of words per page x number of pages.<br>e.g. "There will be about 100,000 words, because there are about 400 words on a page, and 400 x 200 = 80,000" | 26% | 10% |
| Level 3 - Algorithm based on a reasonable number of words per line x number of lines per page x number of pages.<br>e.g. "There are about 15 words on a line and about 30 lines on a page. | 2% | 2% |
| Something else | 6% | |
| Total (n = 333) | 100% | 33% |

All students who used the three-step (level 3) algorithm (words per line, lines per page, pages) gave the correct response, whereas less than half (10%) of the students who used a two-step (level 2) algorithm (words per page, number of pages) obtained the correct answer. An estimation of the responses shows that the other 16%, used an unreasonable estimate for the number of words per page. Clearly, it is easier to estimate reasonably the (smaller) number of words per line, than the (larger) number of words per page. But this approach would seem to need to be learnt explicitly.

Example 2

Checking the reasonableness of the results

(The percentage of responses for each option are given in brackets: n = 376)

*Painter A paints a wall in 3 hours.*
*Painter B paints the same wall in 7 hours.*
*The two painters worked together.*
*In about how many hours will they paint the wall?*
*Indicate the answer you consider to be reasonable.*

| | |
|---|---|
| *(2%)* | *a) 21 hours.* |
| *(26%)* | *b) 10 hours.* |
| *(12%)* | *c) less than 10 hours but more than 5.* |
| *(32%)* | *d) 5 hours.* |
| *(11%)* | *e) 4 hours.* |
| *(3%)* | *f) 3 hours.* |
| *(13%)* | *g) About 2 hours. (Reasonable answer).* |
| *(1%)* | *h) About half an hour.* |

*Explain your choice.*

The majority did not make the simple logical deduction that if one painter paints the wall in 3 hours, two painters should finish the work in less than 3 hours. Instead, most of the students manipulated the given numbers using an arbitrary familiar algorithm (see table below).

If the students were aware of the necessity of checking the reasonableness of their results, maybe some of them would come to the conclusion that something is wrong with the algorithm they used. But, the majority of the students evidently do not pay attention to the reasonableness of their answer.

| Category of explanation | % of students in categories |
|---|---|
| No explanation | 9% |
| Average $(\frac{7 + 3}{2})$ | 31% |
| Addition (7 + 3) | 22% |
| Subtraction (7 - 3) | 4% |
| Something near the correct explanation | 5% |
| Correct explanation | 10% |
| Something else | 19% |

We can use problems like this, firstly to convince the students of the desirability of checking their answers, which is no less important than solving the problem itself, and secondly to show that not every problem need be (or even can be) solved by a standard algorithm. In many cases (logical) reflection by itself can lead to the reasonable answer.

Observing these and other reactions of students to the questions included in the pretest, there is no doubt that development of the ability to estimate needs deliberate fostering.

The comparison with the posttest will show whether the proposed treatment is effective in promoting estimation abilities.

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# A TRANSFERTEST FOR MATHEMATICS, CONTAINING ITEMS WITH CUMULATIVE HINTS *)

Paper to be presented at the 9th conference of the International Group for the Psychology of Mathematics Education (IGPME) dd. july .. at Noordwijkerhout, Netherlands

## ABSTRACT

The purpose of this research is to construct a test, that can measure transfer capacity in the mathematical domain in a differentiated fashion. Classical transfertests merely result in right/wrong scores for each item and do not consider the existence of learning potential on the items, that were 'wrong'.

By offering optional help with transfer items, it is possible to determine the learning potential (i.e. the developmental possibilities) of a student. Accounting for the amount of help needed for solution of the item renders a measure for learning potential and possibly a more refined measure for transfer capacity.

Investigation of correlations of transfertest scores with other measures for achievement will contribute to the construct validation of learning potential.

Finally it is expected that this test can be used to measure adequately differences in amount and type of transfer, resulting from the use of different teaching methods in the classroom.

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## 1. Introduction and theoretical background

It is often noted in education that students have considerable difficulty in applying knowledge and procedures, that they were taught in some specific context, in other domains. The ability to apply acquired skills and knowledge in other domains is usually termed transfer capacity. Attempts to measure the transfer capacity of individual students by objective tests always result in dichotomous scores for each item, because subjects will either give a right or a wrong answer. Especially if such a test is very homogeneous, an all or none total test score may be the result.

Assessment procedures such as clinical interviews render more refined results. This is because partial credit can be given for items that subjects can only answer correctly with some help from the interviewer in the individual testing situation. Krutetskii (1976) gives many examples of this procedure in his study into the nature of mathematical abilities. He offers help to students while they are trying to solve a number of math problems, that are increasingly difficult (a so called "stair case" of problems). The level of difficulty of the last problem solved by the student is used as a messure of the ability to transfer.

Because of lack of standardization in the testing situation, where the amount and kind of help given are not controlled, such a test is not reliable, let alone valid. Trismen (1981,1982) constructed mathematics items with hints in paper and pencil form, but no attempt was made to investigate reliability and validity of a series of these items.

The purpose of this research project then is to construct a transfertest for mathematics, containing items with hints, that fulfils the demands of reliability and validity. Because partial credit will be given for items that can only be solved with some help, we expect to be able to use this test for measuring transfer capacities of individual students more accurately.

The concept of learning potential is based on the difference between independent achievement and achievement in situations where help is offered. This discrepancy was studied in Russian psychology first and termed "zone of proximal development" (Vygotsky, 1964; Van Parreren & Carpay, 1980). Vygotsky's theory on the devslopment of higher mental processes stresses the idea that children gradually develop higher mental functions by a process of social interaction with adults and peers. He claims that the tasks they can only perform with help presently, they will be able to do independently in the nearby future. This view implies that the amount of help needed to solve a problem that is above the actual level of mastery of a subject should be a reasonably good predictor of future performance level.

That is, by studying and assessing the learning potential of students, or as Ginsburg and Allardice (1983) put it: "their knowledge and skills which are still in embryonic form", one should get an impression of the future mastery level. Therefore we propose to use the transfer test for mathematics for predicting future performance in schoolmathematics in order to investigate the validity of the concept of learning potential.

Another application of the transfertest is the study of differences (qualitative and quantitative) in tranafer capacitiss that may

result from the use of different teaching methods for schoolmathematics. In order to formulate hypotheses about these differences, textbooks of three widely used teaching methods for schoolmathemstics in the Netherlands are investigated.

Present views on the didactics of mathematics education stress the necessity of embedding msthematical applications in a wide variety of realistic contexts. Teaching methods, based on this principle will probably enhance students' abilities to recognize the underlying mathematical structure of transfer problems , embedded in non mathematical contexts, i.e. realistic situations. This type of transfer is usually labeled "horizontal" (cf. de Leeuw, 1983), because the level of difficulty of the mathematical knowledge and procedures that need to be applied in order to solve the problem, is not different from that of the subject matter that was taught. Once the problem, that is stated in natural language, is translated into a formal mathematical representation, it can be solved by a known method.

Mathematical problems with an essentially different, unknown underlying structure cannot be solved by merely applying known solving methods. The ability to solve such problems usually requires integration of mathematical knowledge and skills, or discovery of new knowledge components, and is often labeled "vertical transfer" because these problems are more difficult than the subject matter that was taught. It is questionable whether teaching methods, that are based on the "realiatic" didactical approach, will enhance thia type of transfer. More likely, teaching methods that stress the formal structure and abstract nature of mathematical concepts will do better in this case.

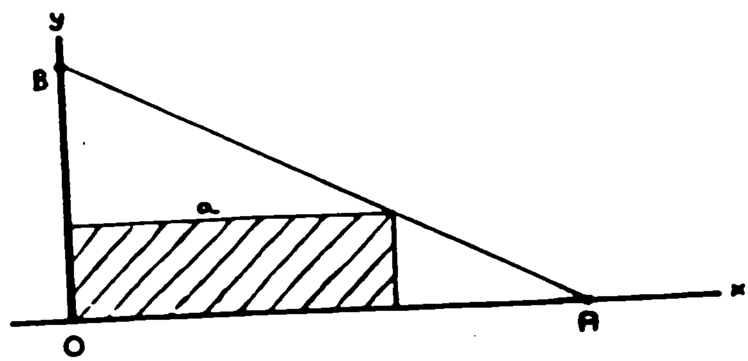
## 2. Mathematical ability

Although it is generally assumed that differences in mathematical ability can be ascribed to different amounts of talent or giftedness (apart from the results of practice), it is very difficult to define the factors, comprising mathematical talent. Definitions of components of mathematical insight, understanding or ability are usually task related. Kolmogorov, a russian mathematician, stated that "the ability to work accurately with mathematical symbols" was a feature of the mathematically gifted. Van Hiele (1973) gave a definition of mathematical ability that is not based on a description of task related skills, which reads: "being able to perform intentionally and adequately in new situations". However, this definition does not convey the specific characteristics of mathematical talent to us, since it merely describes intelligent behaviour in general.

Krutetskii developed separate tests to measure six different components of mathematical ability and confirmed his hypotheses concerning the composition of mathematical giftedness by means of factor analysis. According to Krutetskii, the mathematically gifted can be discriminated from the untalented by their formalized mathematical perception, their ability to think logically, their ability to generalize mathematical subject matter and solution methods and their ability to remember these generalizations. Furthermore they are characterized by flexibility and automation of thought processes.

The following example concerns the topic of generalized solution methods. The first version of the transfertest was presented to seventy two students in the fourth grade of secondary education. All students were given only three items, because there was relatively

little time available. One of the items of the first version of the transfertest is shown in figure 1.




Line AB is described by the formula: $y=-1/2x + 8$
For which value of a is the surface of the rectangle maximal ?

Figure 1
An item in the first version of the test

Only three subjects succeeded in solving this problem. All of these used an essentially heuristic method: systematically trying out several values for "a", calculating f(a) and multiplying both to find the surface. Next, they inspected the values found for the surface with different values of "a" and chose the largest.

Although this solving method is legitimate, it does not <u>prove</u> that the value found for "a" renders the largest possible surface of the rectangle. Expressing the surface as a quadratic function of "a" and determining the maximum value of it would be the appropriate analytic solving method. Krutetskii would probably consider our subjects quite untalented, and in fact, it may be concluded that, although these students found a correct answer, they did not show any transfer. Making a table of values of a function and comparing these to find the

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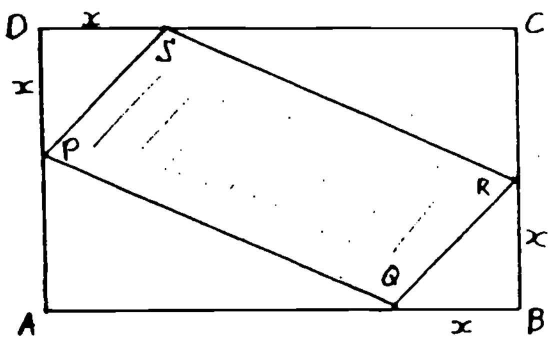
largest is a very common method, which they probably applied many times before.

Krutetskii observed that some of his very bright subjects would find the solution for such "optimizing" problems and immediately after they stated the generalized solution method that can be applied to a whole class of similar problems, such as the one in figure 2.




Rectangle ABCD has length 8 and width 6. In this rectangle, a parallelogram PQRS is inscribed as can be seen in the sketch.
For which value of x is the surface of parallelogram PQRS maximal ?

Figure 2
Another item in the first version of the test
(Item by courtesy of A. van Streun, State University of Groningen)

Solving this problem can proceed by essentially the same method; it is only more difficult to express the surface of PQRS as a quadratic function of x. In fact, none of our subjects solved this problem, even if they had consulted all available hints. We did not ask subjects afterwards to remember the problems they were confronted with or the solutions they found. Krutetskii would have predicted that better students will not remember the exact numerical givens, but rather the

generalized problem class and solving method, while the less able students will only remember parts of givens of problems quite accurately, but will never state any general principle.

Remembering general types of domain specific problems and appropriate solving methods is often mentioned as a feature of expertise in particular domains of knowledge (Chi, Feltovich & Glaser, 1981). The superficial structures of the math problems Krutetskii used to measure flexibility of thought, suggest inappropriate solving methods because of their similarity with problem types with a different underlying structure. In many investigations about differences between experts and novices it has been observed that experts identify adequate solving methods by attending to features of the underlying structure of problems, while novices tend to make mistakes because they concentrate on superficial features of the problem statement. This fallacy can be demonstrated by posing another question about the problem in Figure 1. Suppose the question is not to find the value of "a" for which the surface of the rectangle is maximal, but to find the value of "a" for which the surface is minimal. It can of course readily be aeen that this is so for a=0 or for a=16 because in both cases the rectangle will reduce to a line.

Presupposing that the novice ia familiar with the solving method mentioned earlier (i.e. expressing the surface of the inscribed figure as a function of x and determining the maximum of this function), he or she may tend to apply it on this problem as well. This would lead to the difficulty of finding the minimum of the function: $f(x) \dashrightarrow -1/2x^2 + 8x$. Thia function only has a maximum value, because the sign of the first parameter is negative.

A sketch of the parabola is shown in figure 3.

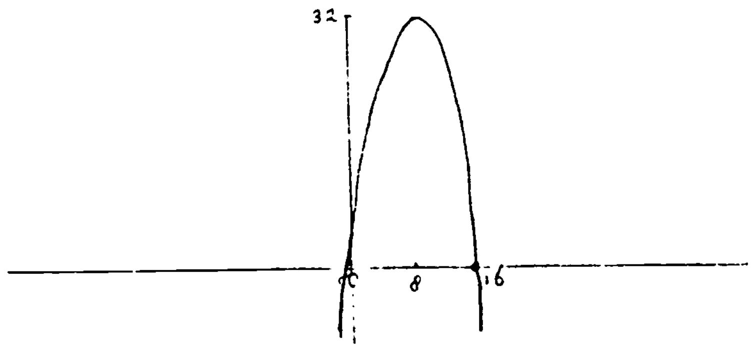



Figure 3
The graph of the function f: x--> $-1/2x^2+8x$

Realizing that the part of the graph under the ordinate axis is not meaningful in relation to this problem, because negative surface does not exist, one could interpret the intersection points of the function and the ordinate axis as minima, because they are the smallest function values in the relevant domain of the function, and thus still find the correct answer. The standard method taught, however, would be to first calculate the intersection points with the ordinate axis and subsequently find an extreme value by averaging these points. If the function value resulting from this turns out to be a maximum value, this could clearly cause an impasse. Obviously, without a proper prior analysis, using a known solution method on the basis of a superficial analogy between a known and a new problem can lead to considerable difficulty.

Considering the similarity between features, that are claimed to comprise mathematical abilities and features of intelligence and expertise, we must conclude that, given the present state of theoretical views, it is impossible to discriminate mathematical ability from intelligence or expertise. Becsuse it seems likely that

mathematically gifted persons will acquire expertise in this domain more rapidly than the untalented, the question is only of theoretical importance. Discussion of the subject can hardly be evaded because we are in need of a theory about the development of mathematical abilities.

## 3. Design, method and relevant variables

Corresponding to the threefold aim of this research project, construction of a test for the prediction of future performance in mathematics education, validation of the concept "learning potential", and comparison of the effects of teaching methods, we need to distinguish adequate designs for each of these purposes.

### 3.1. Predicting future math performance

The transfertest for mathematics will be presented to third grade students in secondary education. The subject matter of the test items is focused on mathematical functions. Future performance in school-mathematics is operationalized as the score on a criterion test, in which no help will be offered, that will be presented in the fourth grade. This score is the dependent variable.

Independent variables are predictors such as transfer capacity, that is operationalized as the total transfertest score. Every item in this test is scored polytomous, dependent on the amount of help (i.e. the number of hints) needed by the student to solve it. Another predictor is the learning potential score, which is calculated as the sum of scores on all items where hints were actually used.

Because future performance may also be partly determined by the

actual level of mastery of mathematics subject matter in the third grade, this will also be measured. This actual mastery test will consist of items, that every student should be able to solve on the basis of what has been taught in the math curriculum.

Since personality factors may influence learning results and transfer capacity some personality traits will be measured as well. Need for achievement and fear of failure are particularly relevant, because these may influence the tendency to ask for help. An important cognitive style is field(in)dependence, assuming that field independent students will be more capable to distinguish relevant and irrelevant features of the problem statements. Finally, the attitude towards (school)mathematics may influence the development of performance.

The relative importance of these predictors will be tested by means of multiple regression analysis. Variables on other accumulation levels, such as socio economic status, the quality of education, where the didactical qualities of the teacher are particurlarly important, must be accounted and controlled for.

### 3.2. Validating the concept of learning potential

Although there has been an uprise in interest for the concept of learning potential, its exact meaning has only been studied in an objective, empirical way in research on intelligence testing (cf. Babad & Budoff, 1974). The claim is that a measure for learning potential reflects the developmental possibilities of a student better than a static measure for ability to generalise or transfer capacity. This would imply that the learning potential score will predict future per-

formance better than the sum score on items, where no help was used. Furthermore, we should be able to make a clear cut distinction between the actual level of mastery of a student and his/her learning potential. While the first should reflect the tasks that can be performed presently without help from other persons, the second should indicate developmental possibilities by means of a more dynamic assessment of performance.

Therefore, we would expect mastery level scores to correlate high with a concurrent criterion, such as present school grades, while they should correlate lower with a future criterion, such as future school grades or the criterion test already mentioned. The learning potential scores, on the other hand, should show high predictive validity in comparison to their concurrent validity.

However, since actual mastery level and future performance level almost invariably show a positive correlation, and because the strength of this relationship may also depend on the teaching method used in the classroom, these differences might be very small and difficult to interpret.

### 3.3. Comparison of teaching methods

In order to investigate the effect of different teaching methods, i.e. book series for schoolmathematics, mean transfer test and learning potential scores for groups of students, using different methoda on their schools, will be compared by means of analysis of variance. Use of certain schoolbooks is considered as a "quasi experimental" treatment here with the aforementioned acores and a measure for attitude towards mathematics as dependent variables.

Possible interactions between personality traits of students and teaching methods will be investigated as well. It is plausible that students, who are characterized by a high level of fear to fail, will profit more from strongly structured learning situations. Therefore, it may be better to teach such students algorithmic approaches to problem solving, whilst heuristic teaching methods may be more suited for students with minimal needs for structuring subject matter in advance.

Because there is neither random assigment of groups to the experimental variable nor is there any control on the entry level of these groups, it is necessary to control for possibly influencing factors, such as the way the book is put into educational practice by the teacher (cf. Freeman c.s., 1983a, 1983b), mean socio economic status of students and the orientation towards achievement on the school.

## 4. Construction of items and hints

Items have to be "fair" towards different teaching methods, i.e. they have to be independent of a certain method. Students should not be (dis)advantaged for certain items because the subject matter taught by their schoolbook fits the subject of the item better or worse. Therefore, book series were analyzed concerning content, terminology, instruction of problem solving methods and use of symbols.

Transfer items were constructed by putting known mathematical subject matters into non mathematical contexts, introducing unknown combinations of known subject matters or by introducing subject matter that is usually only taught later on in the curriculum.

The main problem with constructing hints is that the type of

assistance given to every student is standardized in the test. Because different solution methods may be applied by students, there is really no guarantee that the hints given are always compatible with the solution process that is already in progress.

It was attempted to minimize this problem by giving students some control over the usage of hints, and by choosing transfer problems, that can in principle be solved by only one particular method. With every item a total of six hints could be consulted. These were divided into three main and three follow up hints. Main hints usually were general indications, while the follow up was a specification or clarification of the main hint and can be conceived of as a more concrete clue. This so called open end structure of hints was used by Trismen (1981,1982) earlier. Follow ups could be skipped if the main hint was considered unnecessary, superfluous or incompatible with the particular solution method used by the subject. This was done in order to prevent the subtraction of extra points for the consultance of hints that contained information, that appeared to be irrelevant for a particular subject.

The following is an example of the hints that were available with the problem in Figure 1.

Hint 1 The surface of the rectangle is a function of ....
Express the surface of the rectangle by means of a

Hint 1 (follow up) The width of the rectangle is ......
Is the height of the rectangle dependent on the value of a as well ?

Hint 2 The height of the rectangle is the distance between the ordinate axis and ....

Hint 2 (follow up) The height of the rectangle is exactly the function value belonging with x= ......

Hint 3 The surface of the rectangle is a function of a. This function can be described by a parabola with a minimum/maximum value. (Cross out the wrong alternative)

Hint 3 (follow up) Which value of a belongs with the maximum of the function ?

## 5. Preliminary results

The try out of the first version of the transfertest took place in the fourth grade at three schools, each using a different teaching method for schoolmathematics. In general, the items turned out to be too difficult; four items were not answered correctly by any of the students.

Omitting these items, Table 1 outlines the results of the try out.

| | without hints | with hints | |
|---|---|---|---|
| right answers | 36 - (21%) | 33 - (20%) | 69 (41%) |
| wrong answers | 18 - (11%) | 80 - (48%) | 98 (59%) |
| | 54 - (32%) | 113- (68%) | 167 |

Table 1
Results of the try out of the transfertest

There is a rather large percentage of items, that were not solved in spite of the usage of hints. This relative ineffectivity of the hints can partially be ascribed to the mean difficulty of the items. The "distance" between the initial situation of the subject and the final solution is too large to be crossed, even if assistance is given.

No overall effect of the number of hints consulted on finding a solution could be demonstrated; students giving a correct answer after consulting hints used just as many hints as students who still failed

after using help. This may be caused by the mean difficulty of the items as well. There was a significant relationship between mean number of hints used by students and difficulty of an item. This suggests that the lack of an average effect of using more hints on the chance of finding a solution, can be explained by students that went on consulting hints while solution of the problem was well out of reach of their capacity, even if they used all available hints.

In stead of accumulating the number of hints used by every subject, it is also possible to accumulate the number of subjects over every sequence of hints used, and relate this to the chance of finding a solution. Then we find that subjects using 1 to 4 hints have a mean chance of 6.5% of finding a correct answer, while subjects using 4 or more hints have a mean chance of 28% of finding the solution. This suggests that persisting to consult hints <u>does</u> increase the chance of finding the right solution, at least for some students.

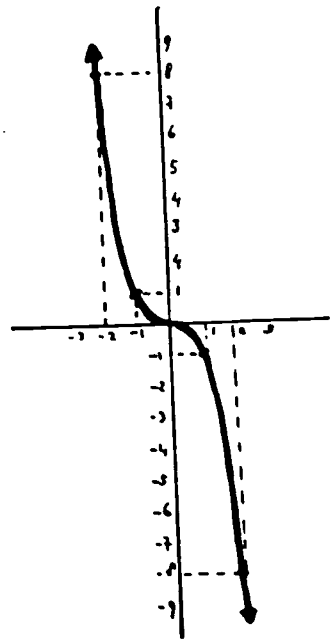
We also asked students to judge the content of the hints. They were asked to label them "useful", "superfluous", "difficult" or "confusing". Surprisingly, the number of times hints were considered superfluous correlated positively with their effectivity. Effectivity was defined as the proportion of correct answers among students that used hints. Twenty percent of the hints, belonging to items, that could not be solved by certain subjects, were judged to be difficult. This was only three percent for items that were answered correctly. Thus it appears that (lack of) understanding of hints is dependent on the difficulty of an item.

## 6. Conclusions

Because the items were too difficult (even for fourth graders) it is obvious that either easier items have to be constructed or the help given in the form of hints must be made more effective. We decided to try both. Some problem statements were reformulated in order to make the problem easier to solve. Also, new items were added that are relatively easier.

Because it is suspected that the ineffectivity of the hint sequences can, at least partially, be explained by lack of understanding separate hints, the concept of open end hints was abandoned. Instead, a hint structure whereby it is possible to give feedback is presently being investigated. With every item, two hints are provided in the form of multiple choice questions. There are four alternatives: the correct answer, two plausible mistakes and the category: "I don't know", which was added in order to avoid guessing. In the first case, it is assumed that the subject has understood the hint and needs no further explication, so she is only informed that her answer was correct and urged to try to find a solution for the problem again. In the other three cases, feedback consists of revealing the correct answer and a clarification of the reason that it is right. For consulting a hint question one point is subtracted from the maximum item score. Getting feedback costs another point. Feedback is withheld if the answer to the question is correct, because the main hint was apparently well understood. Consequently, no extra point needs to be subtracted in this case.

The following is an example of an item with multiple choice hints:



This is the graph of a function f. Try to find the formula that describes f. (The graph continues where there are arrows, but there is no space left on the paper)

Hint 1: Looking at the form of the graph, you may conclude:

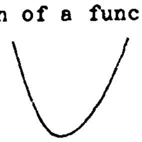
*A f is a constant function or a function of the first degree

*B f is a function of the second degree

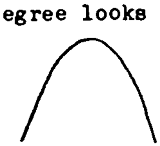
*C f is another kind of function

*D I don't know

Answer 1: C is the correct answer. A constant function and a function of the first degree can be described by a straight line. The graph of a function of the second degree looks like this:



or this:



Now try to solve the problem again.

Hint 2: For some values of x it is possible to find the function value exactly by looking at the graph: $f(+2)=-8$, $f(+1)=-1$, $f(0)=0$, $f(-1)=1$ and $f(-2)=8$.
Why is $g: x \rightarrow x^3$ <u>not</u> the function you are looking for ?

*A Sometimes f(x) is positive and sometimes it is negative, while $x^3$ can never be negative

*B Sometimes f(x) is positive and sometimes it is negative, while $x^3$ can never be positive

*C The sign of f(x) is always the opposite of the sign of x, while $x^3$ always has the same sign as x

*D I don't know

Answer 2: C is the correct answer.
$(-2)^3 = -2.-2.-2 = -8$; $-2$ and $(-2)^3$ share the same sign
$(+2)^3 = +2.+2.+2 = +8$; $+2$ and $(+2)^3$ share the same sign
For other values of x the same principle holds, so $x^3$ always

has the same sign as x; therefore g: x--> $x^3$ cannot be the function you are looking for.
Now try to solve the problem again.

## 7. Perspective

Efficacy of the multiple choice form of the hints is currently being investigated by presenting the test to individual students during thinking aloud sessions. Protocol analysis is used to support the assessment of efficacy of this type of assistence. Analysis of the statements of individual students will convey common misconceptions and errors, so that these can be anticipated. This is important for the adaptation of the content of the hints. We also hope to be able to study the influence of hints on the solving processes of students more accurately in these individual thinking aloud sessions than could be done on the basis of the collective try out of the first version of the test.

The development of tests with cumulative assistence has some interesting educational implications. The assignment of students to various types of education as well as the choice of subjects within the school curriculum may be favourably influenced if a differentiated measure for learning potential predicts future performance better than conventional tests. The application of this method (whereby assistence is offered in the form of questions and if necessary, feedback is given as well) for instructional purposes is particularly promising for computer supported instruction. Offering standardized assistence can be conceived of as a compromise between drill and practice programs, which have many didactical disadvantages, and highly advanced tutorial course ware, which is very difficult to implement.

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# GEOMETRICAL AND NUMERICAL STRATEGIES IN STUDENTS' FUNCTIONAL REASONING

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This paper reports some results of a research project on the reasoning processes used by students in handling numerical functions and interpreting Cartesian graphs. Further information is provided in Ponte (1984).

Functions are a central topic of the high school and college undergraduate mathematics curriculum. Many properties of numerical functions are naturally described by their graphs, thus making particularly relevant the dialetic between numerical and geometrical aspects. Following the terminology of Karplus (1979), in this paper functional reasoning refers to the mental processes associated with the classical notion of functional dependence--that is, with the notion that in some circumstances the value assumed by a certain variable depends in a well-defined way on the value taken by another variable.

## Theoretical Background

Two kinds of thinking based on different systems of symbolic representation were distinguished by Skemp (1971). Discursive-verbal thinking focuses attention on only one part of a squeme at a time and is especially suited for analytical tasks. Global-visual thinking is more holistic. It concerns the ways in which parts relate to each other and to the whole and is especially suited for synthetic enterprises. Skemp suggested a complementary role for these two forms of language systems. But he indicated that the socialized character of our knowledge and our difficulty in externalizing images may have caused an imbalance in the development of our ability to use these two modes of thinking.

In their study of the role of visual imagery, Piaget and Inhelder (1971) indicated that images constituted a valuable auxiliary means--in many instances even a necessary support--for the functioning of operations. Images can play the role of springboards for deduction, enabling one "to outline in rough what the operations

extend and bring to conclusion" (p. 379). Other writers have also discussed the role of imagery as a system of symbolic representation distinct from the discursive-verbal and have identified imagery as especially relevant in problem-solving processes and in creative mathematical thinking (Ponte, 1984).

The relationship between numerical and geometrical reasoning was addressed by Janvier (1978). He reported that in some problems involving graph interpretation a control group following a table method outperformed an experimental group using innovative graphical techniques. Janvier concluded that for many students the use of tables proved to be an useful tool to study processes of variation.

## The Study

The sample included 262 U.S. students divided into four groups: algebra 11th graders, nonalgebra 11th graders, preservice elementary school teachers, and preservice secondary school teachers. The algebra students were taking or had already taken Algebra II, whereas nonalgebra students never had enrolled on that course. The preservice teachers were just finishing their academic requirements to begin teaching. The preservice secondary school teachers were mathematics education majors whereas the preservice elementary school teachers were not.

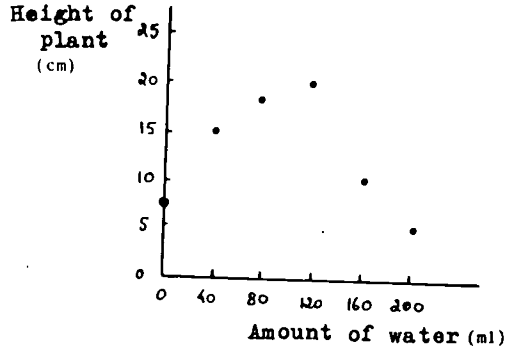
The students were given a 36 item test on graph reading and interpretation. Among the students who took this test, 26 were interviewed. The first interview focused on the responses to the written test. In the following interviews was used a set of open-ended tasks involving the construction or the interpretation of graphs.

## Interpolation

In the tasks used in this research most of the information was conveyed graphically. One test item concerned an interpolation question in a discrete context (Figure 1). The facility level of this item is shown in Table 1. Some students used numerical strategies in the interviews when they explained how they had solved this question. They read the ordinates of the given points and averaged them. As one student explained:

Rose owns a flower shop. She gave different amounts of water to six similar plants each day and measured the height of each plant after three weeks.

Use the information in the graph below to answer questions 1 and 2.




1.- If one plant was given 140 ml of water daily for three weeks, what would be its expected height?

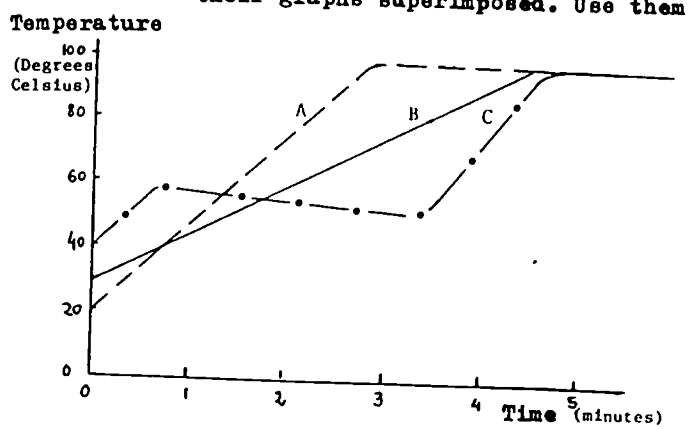
A. 15 cm
B. 19 cm
C. 23 cm
D. 25 cm

2.- How tall would you expect a plant to be after three weeks if given 220 ml of water each day?

A. Less than 5 cm
B. About 8 cm
C. About 10 cm
D. More than 13 cm

During an experiment on boiling water in their science class, three groups of students plotted a graph showing the temperatures at each instant.

We have their graphs superimposed. Use them to answer questions 3, 4, and 5.




3.- If Group C had not had the problem with the flame at about what time would they have had their water boiling?

A. 1.8 minutes
B. 2.2 minutes
C. 2.8 minutes
D. 5.1 minutes

4.- In bringing water from 70 degrees Celsius to 90 degrees Celsius,

A. Group A was fastest and Group C slowest
B. Group B was faster than Group C
C. All the groups took about the same time
D. Groups A and C took about the same time

5.- The temparature of water increases faster with a stronger flame. Is seems to be the case that

A. Group B seemed to have the weakest flame being A and C about the same
B. The flame of Group A was hotter than the flames of B and C
C. The flame of Group C was hotter than the flame of Group A during the first minute of the experiment
D. All the flames were about the same intensity

Table 1

Item Facilities

| Item | Eleventh Graders Nonalgebra n=76 | Eleventh Graders Algebra n=103 | Preservice Teachers Elementary n=52 | Preservice Teachers Secondary n=31 |
|---|---|---|---|---|
| Interpolation | | | | |
| 1 | .46 | .73 | .81 | .97 |
| Extrapolation | | | | |
| 2 | .75 | .96 | .98 | 1.00 |
| 3 | .45 | .58 | .52 | .48 |
| Rates of Change | | | | |
| 4 | .20 | .56 | .38 | .68 |
| 5 | .21 | .30 | .19 | .26 |

"I would just have... Get... Like we get for 120 ml and the 160. To find out where their points are, 120 is 20, 160 is 10. And I get an average, I mean the number in between the two, which would be 15."

Other students used geometric strategies, often with a strong flavor of linearity. Trying to determine the location of the point corresponding to 140 ml, they usually disregarded all the points in the graph except those corresponding to 120 ml and 160 ml.

## Extrapolation

Extrapolation appeared to be an easy task in simple situations were some linear pattern could be followed but was quite demanding on situations involving complex patterns of variation. Two test items concerned extrapolation, one in a discrete context and another in a continuous context (Figures 1 and 2). Item 2 proved to be easy, but in the interviews some students semed to have reasoned without much concern for the encurving pattern when the amount of water increases from 120 ml to 200 ml. As one student said: "If you give more than that [200 ml] it just doesn't grow. It would die. You can tell by the way it goes down."

Item 3 assessed the comprehension of patterns in a process of variation. In the interviews most students showed a general understanding of this point but were not very attentive in considering all its aspects. Many responses seemed to involve just a form of rough qualitative thinking:

"Just from looking at how A, B, and C did start off, C seems to have an head start... So obviously it would have had started boiling before B and A... And I thought it was at 3 minutes that A had the water boiling... So my answer was 2.8..."

A sophisticated numerical strategy was used in Item 3 by a preservice secondary school teacher. She decomposed the graph of group C into several parts disregarding the "inactive period" and the associated loss in temperature, summed up the times taken to heat the water from 40 to 60 and from 60 to 100 degrees, and found the correct answer to the question!

## Rates of Change

Some students failed in Items 4 and 5, involving rates of change, because they had difficulty in focusing in the specific questions asked. Others, apperared to have confused amount and rate. In the interviews, some students used numerical strategies reading the coordinates of points and performing arithmetical operations. For example, a student compared this way the rates of change of the linear sections of the Boiling Water Experiment:

"In the first minute C goes up 20 [degrees], but so does A. And over here... A goes from 60 to 100 in about 2 minutes and C goes from 60 to 100... In about 2 minutes.... C may be a little bit faster... It goes from under 60 to 100 in less time than A goes from 60 to 100."

But other students were clearly thinking in geometrical terms, speaking of "angles," "slant," "being paralel." The high school students never expressed themselves in terms of "slope," but the preservice secondary school teachers were familiar with this notion. However, some were somewhat confused about the relation between it and the inclination of a line.

Most students did not use a method based on the notion of rate of change to compare increases or decreases in different intervals but used a numerical strategy comparing the values of the function at the extremes of the interval. They could deal easily with interval comparisons, but they had little idea of how the method of comparing values at the extremes of intervals can be extended through a limiting process to define rate of change in a single point.

## Conclusion

The research reported in this paper suggests that many students do not feel at ease processing geometrical information and have trouble making the connection between graphical and numerical data. Most students made extensive use of numerical strategies even when geometrical approaches would have been straightforward. One of them explicitly acknowledged this preference for dealing with numbers instead of graphs saying that "numbers are easier for me to understand." Students' difficulty in relating numerical and geometrical information should be a concern in the teaching of mathematics. This difficulty has been identified in earlier work with the number line, notably in dealing with scales (Sullivan, 1982). Many students appear to understand deficiently the properties of integer, rational, and real numbers and their geometrical representations.

We still have little knowledge about students' conceptions and difficulties. The in-depth study of students' thought processes should be systematically undertaken. We need to know how their natural strategies can be developed and broaded. Teachers need this information to design appropriate educational experiences to help students to overcome their difficulties.

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# Development of Audio-Visual Testing for the Improvement of Teaching-Learning Process in Mathematics Education

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It is necessary for students to combine visual information from the text, OHP and the blackboard with the acoustic image corresponding to the explanation by the teacher in order to understand Teaching-Learning Process. If some students fail to combine a visual information with its acoustic image correctly, they will not be able to understand the conception given in Teaching-Learning Process and as the result their learning attitude will sometimes become worse for it. On the other hand,when the students recognize that the combinaton is easy, their learning attitude will become too relax. However this process was not observable in ordinary paper-pencil type test. We have developed a new Audio-Visual Test(AV-Test) aiming at the correct combination of visual information and acoustic image and we have investigated the design of the new Teaching-Learning Process in mathematics education. In this paper we show a design of math class based on the new AV-Test.

## 1. Audio-Test

Our Audio-Visual Test is composed of two components,Audio-Test and Visual-Test. First,we show the construction of the Audio-Test. Let us consider the case where we convey a conception to the students by telephone or radio. If the students succeed to construct their images corresponding to the voice of the teacher, they will understand the conceptions. In Audio-Test students write the problems on a sheet of paper listening to the teacher's voice and solve the problems following the teacher's instructions. Our Audio-Test is not a usual type. The scenario of Audio-Test itself reflects problem-solving process or Thinking process in Teaching-Learning Process of the student.

## 2. Visual-Test

Let $(W_n)$ be the word sequence corresponding to the time sequence $(t_n)$ in Audio-Test. By combining some suitable words, in their mind, the students are able to construct a fundamental conception. Therefore, the students are able to obtain a fundamental conception sequence $(C_n)$ which corresponds to $(W_n)$. Of course each $W_n$ is composed of words. The picture sequence $(P_n^i)$ on the television screen is given by the visual-

ization of $C_n$ in the following way.

(2.1) We represent the fundamental conception $C_n$ only by use of a picture $P_n$.

(2.2) From $P_n$, we make a picture sequence $P_n^1, P_n^2, \ldots, P_n^m$ ,usually m=3 , which satisfy the following conditions.

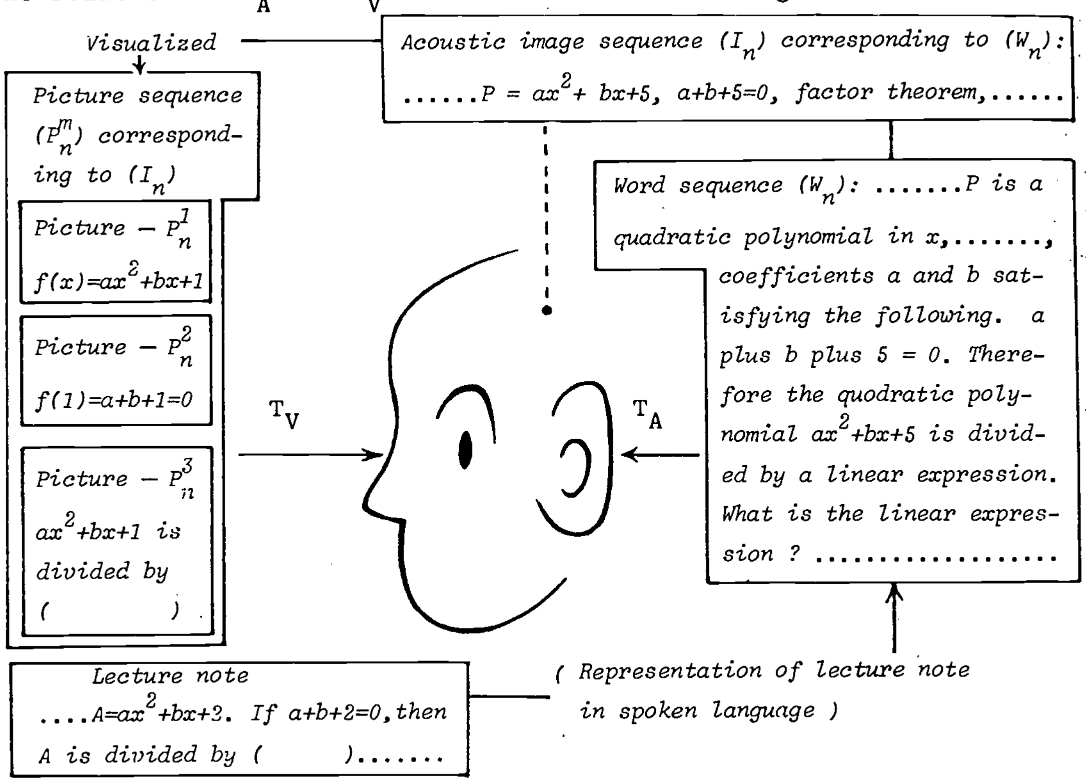
(For simplification, we also denote the content of $P_n, P_n^1, P_n^2, \ldots, P_n^m$ with the same notation.)

(2.2-1) $P_n = P_n^1 \cup P_n^2 \cup \cdots\cdots \cup P_n^m$ .

(2.2-2) $P_n^i \neq P_n^j \cap P_n^i$ and $P_n^i \cap P_n^j \neq P_n^j$ , if $i \neq j$ .

For smplification, we denote our Audio-Test and Visual-Test as well as their scores by $T_A$ and $T_V$ respectively.

The relation of $T_A$ and $T_V$ is shown in the following.




( Fig — 1 ) The Relation of $T_A$ and $T_V$

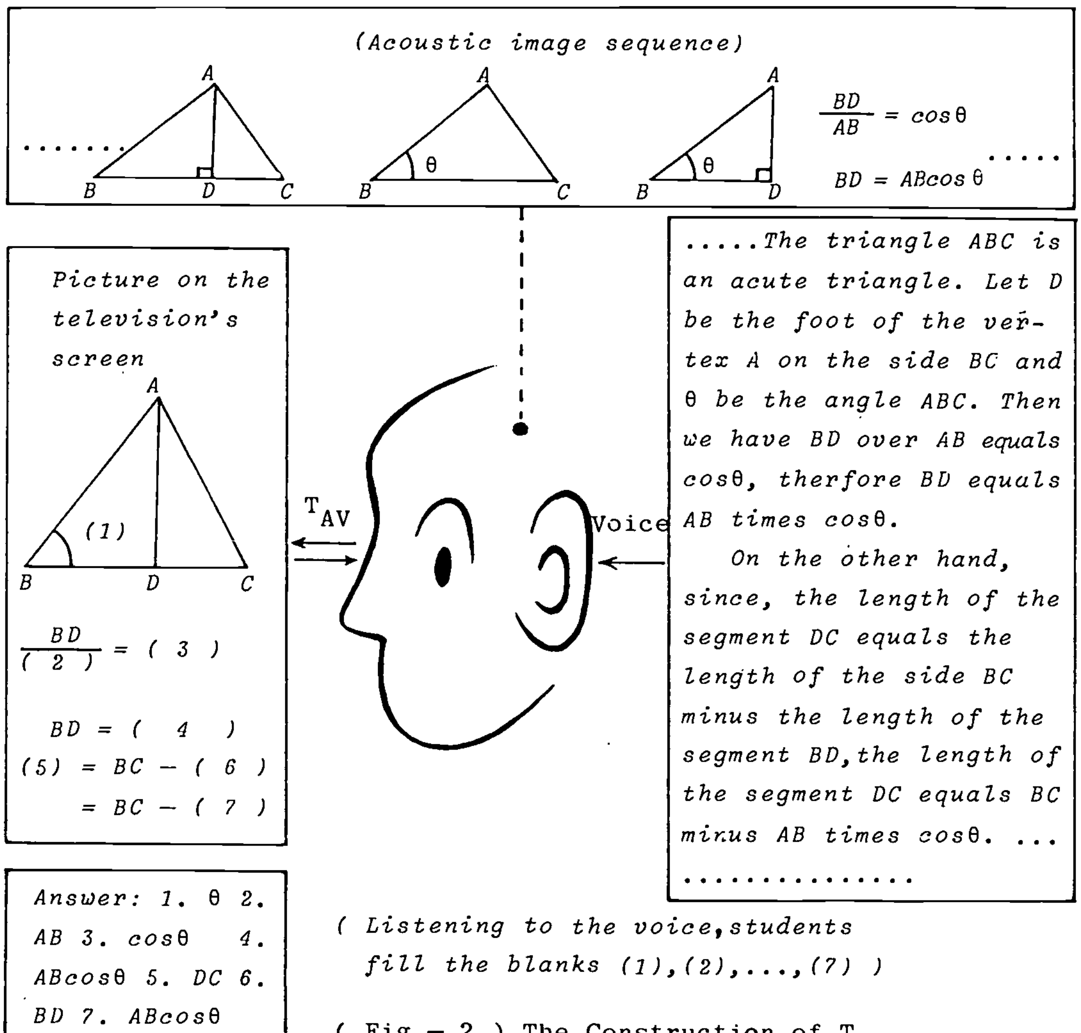
(Remark) (1) The same conceptions are treated in $T_A$ and $T_V$, but some of the notations are changed. (2) In $T_V$ we give the notations for the answer by the voice. (3) Each $P_n^i$ in $T_V$ is shown for about 5 seconds.

We showed that $T_V$ represents the slow learner's achievement unable to see by ordinary paper-pencil type tests.( see 2 ) For example,

it is not rare that $X_P^k = \inf(X_P^i)$ and $X_V^k >$ (mean of ( $X_V^i$ ) ), where $X_P^i$ is the score of student k on the Paper-Test corresponding to the scenario of $T_A$ and $X_V^i$ is the score of student i on $T_V$.

3. Audio-Visual Test $T_{AV}$ and Paper Test $T_{AV}^P$ corresponding to $T_{AV}$

It is necessary for students to combine visual information from the text and blackboard with the acoustic image corresponding to the explanation by the teacher in order to understand Teaching-Learning Process. Therefore we see if the students are capable of cmbining the visual information with the acoustic image by the use of our Audio-Visual Test $T_{AV}$ which requires the correct combination of visual information with the acoustic image. The construction of $T_{AV}$ is shown in Fig-2.

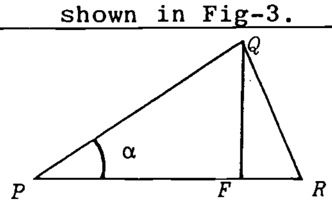



( Fig — 2 ) The Construction of $T_{AV}$

But we can not judge if the students were able to understand the coception $C_n$ only by $T_{AV}$ . Because,in $T_{AV}$ , there are some blanks which the students are able to fill with the right answer without complete understanding of the conception $C_n$. But if the students are able to understand our Teaching-Learning Process well, they can change the notations of memories which correspond to our Teaching-Learning Process. Therefore, if they can't change the notations of memories, it means that they don't understand the conception yet. Now, we may diagnose whether the students are able to understand the conceptions in Teaching-Learning Process or not in the following way.

(3.1) At first, we carry out $T_{AV}$.

(3.2) On a answer sheet, we present the picture corresponding to $T_{AV}$ as shown in Fig-3. Remembering the Teaching-Learning Process in $T_{AV}$, the students fill the blanks 1,2,..,7, to make the picture correspond to the conception $C_n$.




$\angle QPR = (\ 1\ )$

$\dfrac{PF}{(\ 2\ )} = (\ 3\ )\quad PF = (\ 4\ )$

$(\ 5\ ) = PR - (\ 6\ )$
$\quad = PR - (\ 7\ )$

(Fig-3) Paper-Test $T_{AV}^{p'}$

We denote our Paper-Test by $T_{AV}^{p'}$.

Next, we consider a case where the student could understand only some parts of $C_n$. Then there exists a blank k ($1 \leqq k \leqq 7$) in Fig-3 which satisfies the following condition.

Condition: They can't fill the blank k,but, if some other suitable blanks are filled with right answers, they will succeed to fill it. Therefore, we have tried to allow the students to see some right answers by removing the covering on the blanks, if they want to see them by all means. Then we have

a) Some students, who fill the blanks with careful cosideration,tend to fill the blanks without seeing the right answer.

b) Some of the students show a tendency to remove the covering without much consideration.

Hence we have decided that each student should be given the right answers corresponding to his errors in $T_{AV}$.

We denote the kind of this Paper-Test by $T_{AV}^{p}$. $T_{AV}^{p}$ has the characteristics as follows.

(3.3) The students, who were able to understand only some parts of the conception, succeed to fill the blanks of $T_{AV}^{p}$ corresponding to this parts.

(3.4) In $T_{AV}^{p}$, every student's attitude is very serious,indepent of com-

prehension degree of the conceptions.

## 4. By Using of $T_{AV}$ and $T^{P}_{AV}$, a Design for Teaching-Learning Process and its Evaluation

It is very important in our Teaching-Learning Process to satisfy the following conditions.

(4.1) Every student always keeps serious attitude in our class.

(4.2) The suitable audio-visual informations are given to him.

(4.3) Every student is able to consider the whole coception at his own pace.

Using $T_{AV}$ and $T^{P}_{AV}$, we design our Teaching-Learning Process and evaluate them as follows.

( Preliminaries )

(4.4) We pick out the fundamental conception sequence $(C_n)$ and draw up the scenarios corresponding to each $C_n$ in the same way as in $T_A$. (see 2.1) Of course , generally, the conceptions of $(C_n)$ are independent to one another.

(4.5) For each $C_n$, by use of $T_{AV}$, we see if the students are able to combine the visual information with the acoustic images corresponding to the sub-sequence of $(W_n)$.

(4.6) By use of $T^{P}_{AV}$, we clarify whether the students are able to understand the conception $C_n$ at his own pace or not.

( Remark ) Since $C_n$ in ( 4.5, 4.6 ) are arranged at random, it is impossible for the students to consider what $T_{AV}$ means as a whole.

( Design for Teaching-Learning Process and its evaluation )

(4.7) We draw up the scenario of Teaching-Learning Process by combining the conceptions of $(C_n)$ in (4.4) and add the complementary explanations to it if necessary, where notations of $C_n$ in (4.7) are different from $C_n$ in (4.4).

(4.8) We make $T_{AV}$ and $T^{P}_{AV}$ correspond to each $C_n$ and carry out the Teaching-Learning Process with $T_{AV}$.

(4.9) When the Teaching-Learning Process is over, we evaluate it by use of $T^{P}_{AV}$in (4.8).

## 5. Some Considerations on Teaching-Learning Process by $T_{AV}$ and $T^{P}_{AV}$

We denote the scores of Test in (4.5,....,4.9) as well as these

tests by $T_{AV}$ and $T^{P}_{AV}$ respectively. To distinguish our Audio-Visual Test and corresponding Paper-Test in (4.7,...,4.9) from those in (4.4,.....,4.6) , we denote those tests in (4.7,....,4.9) by $T^{*}_{AV}$ and $T^{P^*}_{AV}$ respectively. Then we have the following.

(5.1) If $T^{P}_{AV}(C_n) < T^{P^*}_{AV}(C_n)$ $(T^{P}_{AV}(C_n) > T^{P^*}_{AV}(C_n))$, then the Teaching-Learning Process gave positive( negative ) effect to the comprehension of $C_n$.

(5.2) If a student's scores satisfy that $T^{*}_{AV}(C_n) < T^{P^*}_{AV}(C_n)$, then his learning attitude on $C_n$ was not good.

(5.3) If the student's scores satisfy $T^{*}_{AV}(C_n) > T^{P^*}_{AV}(C_n)$ , then it is concluded that he has not understood $C_n$ well yet, though he can follow the explanations of the teacher.

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# ERRORS, EXPECTATION AND THE TEACHER

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A model relating teacher expectation and student errors has been developed and applied in a series of studies by the authors since 1978 (Zehavi and Bruckheimer, 1981, 1983a, 1983b). The main findings were:

1. Accuracy of estimation depends on the topic (e.g. traditional vs. modern), and on the cognitive level of the item.
2. The differences in estimation between teachers relate strongly to their experience and education.
3. Although the actual success on a test varies from item to item, the predicted success has a very flat pattern.

Schwarzenberger (1984) and other claim that we should learn from student mistakes how to improve teaching, and involve teachers in the testing procedures. In 1983 we started to use the model of errors, expectation and the teacher as an inservice tool. Items for which previous application indicated mismatch, were redeveloped in tests and applied to high ability and medium ability classes. In inservice workshops the teachers first discussed the difficulties they expected students to have and constructed a catalogue of errors. Then they estimated expected percentage success. Their average expectation was compared with actual results obtained in the sample. This is illustrated in Figure 1 for a ten-item questionnaire devoted to common algebraic techniques (see below). There is evident overestimation for the medium ability (see below) classes on all items, and for the high ability classes for the last four items.

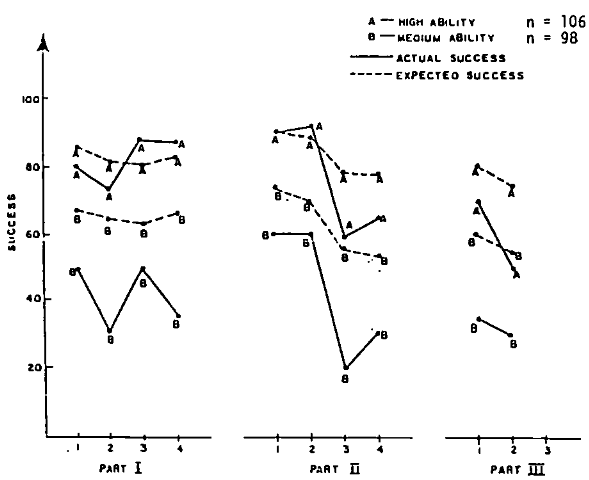



Figure 1 : Student Performance and Teacher Expectation
n = 23

Figure 1 "photographes" the state, but we can use it as motivation to investigate the learning/teaching process and to influence it. This is the spirit of Wheeler's suggestion (1983), that one of the benefits that teachers could derive from research is that it could be used to raise consciousness, particularly when it surprises and attracts attention to items of experience that have been overlooked.

## INVESTIGATION OF MISMATCH

We start with the text of the test.

*Part I : Remove parenthesis and simplify*

1. (a-3)(a-2)      2. (6a+1)(2a-5)
3. a(a+b) - b(a-b)      4. (a+2a)(2b-b)

*Part II: Find the solution set*

1. 8-2x+2(1-x)=0      2. 5(x-1) - 4(x-3) = 7
3. 2(x-4) - 3(x-5) < 12      4. 4(2x-2) + 5-4x > 3(4-x)

*Part III: Find the solution set*

1. 2(x-y) + 3y = 5; y = 3x - 10      2. 2(x+2) + 3(y+1) = 9; 3(x+3) - 2(y-1) = 14

Table 1 presents actual (A) and expected (E) success and errors for Part I.

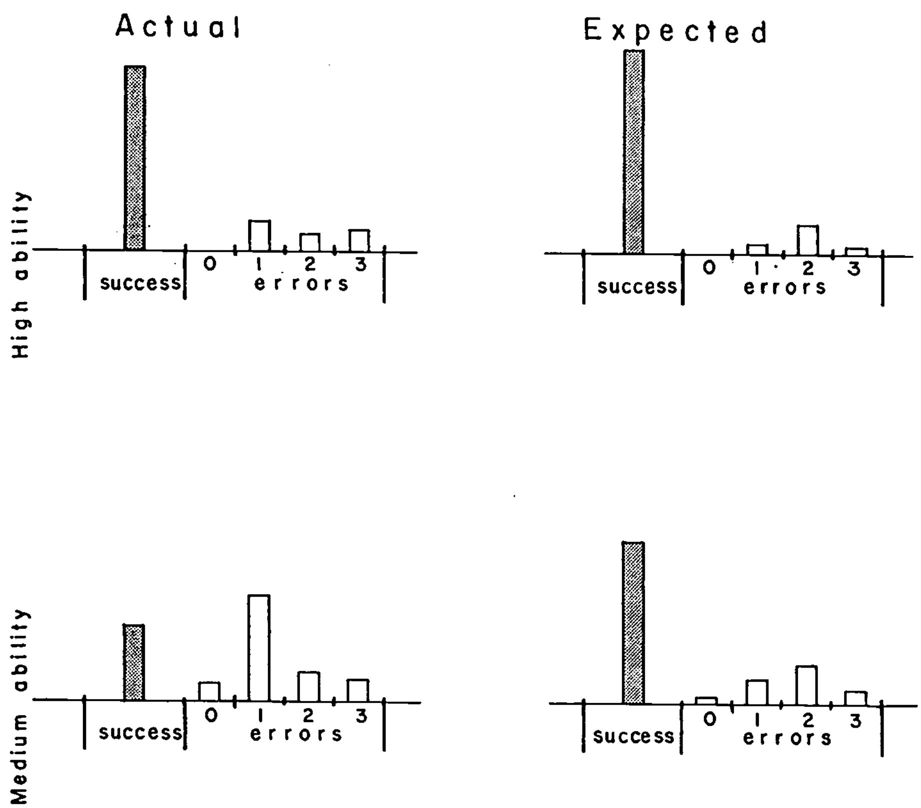
| | | | Success | 0 unable to get started | Errors in: 1 monomial multiplication | 2 distribution | 3 concept of like terms |
|---|---|---|---|---|---|---|---|
| High ability | 1 | A | 78 | - | - | 16 | 6 |
| | | E | 86 | - | 5 | 5 | 4 |
| | 2 | A | 73 | - | 12 | 7 | 8 |
| | | E | 81 | - | 4 | 12 | 3 |
| | 3 | A | 88 | - | 3 | 3 | 6 |
| | | E | 82 | - | 2 | 9 | 7 |
| | 4 | A | 87 | - | - | 2 | 11 |
| | | E | 85 | - | 1 | | 6 |
| Medium ability | 1 | A | 50 | 1 | 13 | 24 | 13 |
| | | E | 68 | - | 11 | 16 | 5 |
| | 2 | A | 30 | 7 | 42 | 12 | 9 |
| | | E | 65 | 3 | 10 | 16 | 6 |
| | 3 | A | 50 | 9 | 10 | 7 | 24 |
| | | E | 64 | - | 2 | 23 | 11 |
| | 4 | A | 35 | 10 | 18 | 12 | 25 |
| | | E | 67 | - | 2 | 15 | 16 |

Table 1:

Actual (A) and Expected (E) success and errors (in percentage) for Part I

A lively analysis of student performance took place during the workshop. The teachers were very much involved in the process, trying to "justify" their estimation. For example consider exercise $I_2$.

This exercise was correctly answered by 73% of the high ability students and by only 30% of the medium ability students. The dominant error occured in the monomial multiplication (6a*2a=12a). About 12% of the former and 42% of the latter made this error. The teachers were very surprised; they expected the dominant errors would occur in the binomial distribution. The gap between their estimate and the actual result can be explained by this observation alone (see Figure 2).



Remark:

Table 1 does not give the whole story. There were wrong responses that belongeg to more than one category, e.g. (6a+1)(2a-5)=12a-5. When the responses were checked, the first error which occurred was coded for the classification. This is also how the teachers were asked to organize their expected distribution of errors.

The discussion led to the suggestion that students who also erred and wrote a*a=a in item I-1, need altogether different treatment from students who only erred when there were coefficients (6a*2a).

An illustrative mismatch occurred in Part II. The teachers expected the two inequalities to be a little harder than the two equations. In reality, student achievement on the inequalities was much lower than on the equations (see Figure 1). Further, the teachers expected about the same success on the two inequalities. They explained that in No. 3 there would be difficulties in dividing by a negative number, and thus changing the direction of the inequality. In No. 4 there is more transposing work. Their guess was that these difficulties would compensate. In reality, the difficulty in proceeding from $-x < 5$ was the greater: 35% of the high ability students and 52% of the medium ability failed.

This finding fits our previous experience when we discussed cases of mismatch with teachers, and heard the excuse, "when I was in school we did not do inequalities" (Zehavi and Bruckheimer, 1981). In fact, the situation is even more serious. In a study which evaluated the effect of some inservice courses, Fresko and Ben-Chaim (1984) examined teachers' knowledge and confidence in the topics covered in the course. They found that teachers had serious difficulty in solving inequalities even after the course. One must admit that there is a problem with the teaching of the topic. This leads to the next phase of our studies.

## IMPLICATIONS FOR TEACHING

In 1983, studies like the one described above, were carried out in one-day workshops. At the end of the day, we could discuss with the teachers what action might be taken in class, only very briefly. A follow-up inservice application of the model in 1984-5 took us one step further. A group of teachers (n=25) on sabbatical, attended a one-year course. The syllabus contains mathematical enhancement, use of new approaches and teaching strategies, testing and evaluation, using technology in teaching and improvement of teaching in general.

In the meeting devoted to testing and evaluation, we presented to the teachers the series of studies performed within the model which relates teacher expectation and student difficulties. Discussions of the findings and their impact were held. Then the teachers

became investigators themselves and conducted projects which were essentially repetitions of parts of our studies. Teams of 2-3 teachers prepared tests, applied them in some classes and compared actual results with the class-teacher's prediction. Each team asked all other participants to express their expectations. The group's expectations were averaged and also compared with actual success. In the following meetings, each team presented its findings to the group, and together we discussed the implications for teaching. The projects and the discussions were documented by the teachers and submitted.

In order to be able to evaluate the effect of this treatment on the participants, a pre-post assessment took place*. At the beginning of the course the teachers were asked to assign their expectations to the test quoted above. The picture was very similar to the one described earlier. Then the teachers were asked to write essays about the implications of the findings for classroom teaching. The essays were not annotated nor discussed at that stage. Toward the end of the course, the teachers were again asked to write essays on the same topic regarding the same findings.

A comparison of the pre-essays with the post-essays indicated some major contributions to the teachers' awareness of mismatch and its implications. In the pre- treatment stage, the teacher suggested in a very general way: decrease expectation, do more practice, devote more time to inequalities, give up some topics for less able students, etc. - one can sense a sort of pessimism.

The post-treatment essays were optimistic and indicated practical action that might be taken in the classroom. We bring some examples related to the items which were discussed above.

For Part I, a diagnostic graded worksheet was suggested. For example,

| | | | |
|---|---|---|---|
| i) | (a-3)(a-2) | ii) | (3-a)(a-2) |
| iii) | (2a-3)(a-2) | iv) | (a-3)(2a-2) |
| v) | (5a-2)(2a-3) | vi) | (2-5a)(2a-3) |

For Part II, it was recommended that the teaching of equalities and inequalities be closely linked, together with related open phrases. For example,

True or false?

| | | | |
|---|---|---|---|
| i) | $2(-1-4)-3(-1-5)=12$ | ii) | $2(7-4)-3(7-5) < 12$ |
| iii) | $2(-5-4)-3(-5-5)=12$ | iv) | $2(-8-4)-3(-8-4) < 12$ |

* Acknowledgment to Professor Adrian P. Van Mondfrans from Brigham Young University, Provo UTAH, for a fruitful discussion concerned this study.

Find the solution set

i) $2(x-4)-3(x-5)=12$ ii) $2(x-4)-3(x-5) < 12$

## CONCLUSION

The model which relates errors with expectation has been used iteratively to identify needs and treatment strategies. It is hoped that, after a series of such activities, teachers will improve their subjective in-situ evaluation and teaching strategies. The next iteration will probably develop from the projects submitted by the teachers.

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# THE RELATIVE IMPORTANCE GIVEN BY THE TEACHER TO THE STUDENT'S ANSWER AND REASONING

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*In the teaching of mathematics, teachers often attach more importance to the mathematical products than to the mathematical processes, often reducing understanding to the acquisitions of abilities. This is reflected by a tendency to focus more on the student's answer than on his reasoning. To counteract this tendency, we developed an approach to teacher training based on their initiation to conceptual analysis using a model of understanding. We used this approach with a group of teachers enrolled in a 45-hour in-service course. To evaluate the relative importance given by these teachers to the students' answers and to their reasoning, we developed and tested a new tool called "The Correction Test". In the following, we describe this tool and the results we obtained: these results indicate that the teachers initiated to conceptual analysis attached more importance to the reasoning than those in a control group.*

In the teaching of mathematics, teachers often reveal an instrumental and formalist perception of mathematics as well as a rather behaviorist vision of the learning process. In such a perspective, teachers often attach more importance to the mathematical products than to the mathematical processes, often reducing understanding to the acquisition of abilities. This is reflected by a tendency to focus more on the student's answer than on his reasoning.

In an attempt to change their perception, we developed an approach to teacher training based on their initiation to conceptual analysis using a model of understanding. This approach, described by Bergeron et al. (1981) and by Herscovics et al. (1981), links the psychological, epistemological and pedagogical aspects of the teaching of mathematics to the mathematical content itself. Mathematical concepts are examined in the context of the model of understanding; the focus on the cognitive aspects of teaching leads the teacher to reflect on the mental processes involved in the elaboration of a particular notion. This, in turn, may also lead to the teacher restructuring his teaching strategies to conform more closely with what he has understood about the learning of the notion.

Along with the approach, we have developed a tool, called "The Correction Test" to evaluate the relative importance the teachers place on the students' answers and on their reasoning.

## The Correction Test: description and experimentation

The test is composed of two elementary problems, one in geometry (what is the area of a rectangular garden whose sides are 3 m and 6 m respectively?) and the other one in

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arithmetic (Peter has four dozen eggs. Jack has only two dozen. How many more eggs does Peter have?). Each problem was given to teachers along with four possible student solutions. The solutions were of the following types:

1) a right answer stemming from a correct reasoning;
2) a right answer by itself, without any trace of reasoning;
3) a right answer but originating from a faulty reasoning;
4) a wrong answer in spite of a good reasoning.

Teachers were asked to grade (on 10) each solution and then to justify their grading. We thought that a teacher who assigned more importance to the answer than to the reasoning would give low marks when the answer is wrong, even if the reasoning is correct (type 4) and high marks when the answer is right, despite a faulty reasoning (type 3) or no evidence of reasoning (type 2). On the contrary, a teacher more interested in reasoning would give high marks when the reasoning is correct even if the answer is wrong (type 4) and lower marks when the reasoning is incorrect (type 3) or missing (type 2). The justifications were required in order to validate our interpretation of the marking.

We used this test with an experimental group (n=18) of teachers enrolled in an in-service program at the "Faculté d'Education Permanente" at the Université de Montréal. They took a 45-hour mathematics education course in which our approach was used. We administered the test immediately before and after the course to see if there were any changes in the way they weighted the students's answers versus their reasoning. The same test was used in similar conditions with a control group (n=16) consisting of teachers enrolled in the same program but taking a course in a field other than mathematics or mathematics education.

## Results

### Type 1 solutions

Type 1 solutions are perfect: they show right answers stemming from clear correct reasoning. For this solution, our aim was not only to detect any change in the marks given before and after the courses, such a change reflecting inconsistency on the part of the teachers being tested, but to make sure that these teachers understood the problems and were able to grasp the soundness of the reasoning and the accuracy of answers.

On the geometry problem, 15 out of the 16 teachers in the control group gave 10/10 in the pre and the post-tests (see table next page). In arithmetic, 15 teachers awarded 10/10 in the pre-test and all 16 gave that perfect score in the post-test. This was a good first indication of consistency on the part of the teachers. teacher no. 4 was the only exception: this teacher's justifications revealed a perfectionnist nature which led her to deduct marks for minor formal details while recognizing the overall quality of the solution.

| Marks given<br>Type 1 - solution<br>Right answer, good reasoning | | | |
|---|---|---|---|
| Geometry | | Arithmetic | |
| Control | Experim. | Control | Experim. |
| Pre Post | Pre Post | Pre Post | Pre Post |
| 10- 10 | 10- 8 | 10- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| 10- 10 | 9- 10 | 10- 10 | 10- 10 |
| 8- 7 | 10- 10 | 9- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| 10- 10 | 9- 10 | 10- 10 | 10- 10 |
| 10- 10 | 8- 10 | 10- 10 | 10- 10 |
| 10- 10 | 10- 9 | 10- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| 10- 10 | 5- 10 | 10- 10 | 10- 10 |
| 10- 10 | 8- 8 | 10- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| 10- 10 | 10- 10 | 10- 10 | 10- 10 |
| | 10- 10 | | 10- 10 |
| | 10- 10 | | 10- 10 |

In the experimental group, perfect scores were awarded by all teachers for the arithmetic problem on both the pre and post-tests while in geometry, only 11 of 18 teachers gave perfect marks in the two cases. Though the latter was at first alarming, the teachers' comments showed that 17 out of the 18 teachers acknowledged the correctness of the reasoning and the answer in the pre-test, and that all 18 acknowledged this in the post-test. The slight variations observed in the grades reflect a more stringent attention to details in the presentation of the solution. The only real discrepancy was teacher no. 13 in the pre-test, whose reaction can be explained by the fact that she looked at the type 3 solution (right answer but faulty reasoning) immediatly prior to evaluating type 1 and was thus biased.

The preceding demonstrates that:

- all the teachers faced the task of grading in a responsible and competent mannner;
- both groups seemed consistent in their grading though teachers in the experimental group proved to be more demanding with respect to form. Fortunately, this tendency can easily be monitored by their own justifications;
- the two groups seemed similar enough to be used in further comparaisons.

### Type 2 solutions

Type 2 solutions are the shortest; in the arithmetic problem, all we find on the page is the answer, 24 eggs; the geometry question is much the same with the addition of a simple drawing to the answer, 18 $m^2$. There is absolutely no trace of the reasoning involved present on either answer sheet. We predicted that a teacher for whom the answer is the most important would grade these answers generously while, rather, a teacher who is more interested in the reasoning would give low marks unless he was to give the student the benefit of doubt. Thus, a shift of focus from the answer to the reasoning would be reflected by a drop in the marks awarded.

This, however, is not what was observed. Both groups behaved similarly in the pre and post-tests, the marks given follow no distinct trend:

| Marks given<br>Type 2 - solution<br>Right answer, no reasoning | | | |
|---|---|---|---|
| Geometry | | Arithmetic | |
| Control | Experim. | Control | Experim. |
| Pre Post | Pre Post | Pre Post | Pre Post |
| 10- 10 | 6- 10 | 10- 10 | 6- ? |
| 5- 0 | 3- 2 | 1- 0 | 3- 2 |
| 5- 5 | 5- 7 | 5- 5 | 3- 6 |
| 5- 2 | 0- 3 | 0- 0 | 0- 0 |
| 5- 10 | 0- 0 | 5- 10 | 0- 0 |
| 10- 8 | 8- 3 | 10- 4 | 0- 1 |
| 7- 3 | 10- 10 | 1- 1 | 10- 10 |
| 3- ? | 4- 0 | 3- 3 | 4- 0 |
| 5- 0 | 7- 10 | 0- 0 | 5- 2 |
| 5- 8 | 4- 4 | 2- 2 | 2- 2 |
| 0- 2 | 10- 9 | 0- 0 | 10- 8 |
| 4- ? | 2- 0 | 4- ? | 2- 0 |
| 5- 2,5 | 5- 7 | 0- 2,5 | 5- 6 |
| 2- 2 | 5- 8 | 2- 2 | 5- 5 |
| 5- 5 | 5- 1 | 5- ? | 0- 1 |
| 8- 5 | 5- 5 | 5- 3 | 5- 5 |
| | 5- 8 | | 5- 5 |
| | 10- 9 | | 0- 10 |

- in the control group, among the 14 teachers who did complete the test for this type of solution, 4 awarded the same mark before and after in geometry while 3 of them increased the grade and 7 lowered it. In arithmetic, there were 9 the same, 2 increases and 3 decreases;
- in the experimental group, out of 18 teachers in geometry, 4 stayed the same, 7 decreased and 7 increased their marks while in arithmetic, 7 went unchanged, 5 increased and 5 decreased, 1 of the original 18 teachers not completing the post-test.

No definite conclusions can be drawn from the type 2 solutions since the variations seemed inconsistent. The only possible observation is that a large majority of teachers in both groups commented on the absence of detailed answers and stated that they would not accept this kind of solution from their own students.

Type 3 solutions

Type 3 solutions are the most disturbing: correct answers are presented, stemming from a faulty reasoning (the perimeter is calculated instead of the area in geometry) or from a fragmented reasoning (only the number of eggs that Jack has is calculated in arithmetic). Once again, we expected lower grades from the teachers who are more concerned with reasoning and higher grades from the ones concerned with the answer, and a shift of focus from the answer to the reasoning being reflected by a drop in marks awarded.

But once again, our observations differed from our expectations:

- in the control group, 8 of the 15 marks awarded in geometry remained the same in the pre and post-test, 5 increased and 2

| Marks given<br>Type 3 - solution<br>Right answer, faulty reasoning | | | |
|---|---|---|---|
| Geometry | | Arithmetic | |
| Control | Experim. | Control | Experim. |
| Pre Post | Pre Post | Pre Post | Pre Post |
| 0- 0 | 3- 10 | 0- 0 | 5- 10 |
| 0- 0 | 3- 5 | 5- 6 | 3- 3 |
| 2- 3 | 10- 0 | 5- 5 | 4- 4 |
| 2- 0 | 0- 0 | 3- 3 | 5- 4 |
| 0- 0 | 0- 0 | 0- 0 | 0- 0 |
| 2- 4 | 10- 0 | 3- 8 | 0- 0 |
| 5- 0 | 0- 10 | 3- 3 | 10- 10 |
| 0- 0 | 0- 0 | 8- 5 | 5- 5 |
| 10- 10 | 9- 10 | 3- 3 | 0- 2 |
| 0- 0 | 4- 4 | 3- 5 | 2,5-10 |
| 0- 2 | 10- 8 | 4- 5 | 0- 9 |
| 0- 0 | 2- 1 | 8- 3 | 5- 5 |
| 0- 10 | 5- 5 | 0- 0 | 7- 7 |
| 2- 2 | 1- 7 | 6- 6 | 6- 7 |
| 0- 10 | 2- 0 | 3- 10 | 3- 3 |
| ?- 0 | 0- 5 | 5- 5 | 7- 8 |
| | 0- 8 | | 5- 3 |
| | 10- 0 | | 0- 0 |

decreased; in arithmetic, 9 marks out of 16 remained unchanged in the post-test while 5 increased and 2 decreased;

- in the **experimental** group, 5 marks out of 18 were unchanged for the geometry problem, 7 marks increased and 6 decreased; in arithmetic, 10 marks stayed the same, 6 increased and 2 decreased.

There is no clear trend leading to a defenite conclusion in these variations since both in the experimental and control groups, almost equal numbers of marks increased in the arithmetic problem as well as in the geometry problem. Exactly the same number of marks decreased in the arithmetic problem in both groups; the only exception was the geometry problem where there were 2 decreases in the control group contrasting with 6 decreases in the experimental group. Moreover, about one third of the teachers in both groups did not detect the mistakes in the reasoning in the pre or in the post-test or in both cases, although every teacher commented on these reasonings in his justifications: some of the teachers found them correct, some said that they were faulty, the others, disturbed by the exactness of the answers, did not know how to deal with these faulty reasonings and seemed to think that "something" ought to be good "somewhere"....

Type 4 **solutions**

Type 4 solutions are almost perfect: they show correct reasoning but wrong answers because of poor calculations (6x3=16 in the geometry problem) or mistaken transcription of number (42 instead of 48 in the arithmetic problem). We predicted that a teacher more concerned with the answer would give low marks for these solutions and a teacher more interested in the reasoning would award higher marks, a shift of focus from the answer to the reasoning being reflected by an increase of the marks awarded.

| Marks given<br>Type 4 - solution<br>Wrong answer, good reasoning | | | |
|---|---|---|---|
| Geometry | | Arithmetic | |
| Control<br>Pre Post | Experim.<br>Pre Post | Control<br>Pre Post | Experim<br>Pre Pos |
| 5- 8 | 7- 8 | 8- 8 | 8- 8 |
| 8- 8 | 7- 8 | 8- 8 | 7- 9 |
| 8- 8 | 7- 9 | 7- 8 | 8- 9 |
| 7- 6 | 7- 8 | 6- 6 | 7- 8 |
| 8- 9 | 8- 8 | 9- 9 | 8- 8 |
| 8- 9 | 8- 8 | 9- 9 | 8- 6 |
| 7- 7 | 0- 0 | 9- 9 | 0- 0 |
| 8- 8 | 6- 8 | 8- 8 | 8- 8 |
| 8- 8 | 4- 8 | 8- 9 | 7- 8 |
| 8- 8 | 5- 7 | 7- 8 | 8- 8 |
| 8- 8 | 8- 8 | 8- 8 | 8- 7 |
| 7- 7 | 8- 8 | 7- 7 | 8- 8 |
| 0- 7,5 | 5- 8 | 5- 10 | 7- 8 |
| 8- 8 | 7- 7 | 8- 8 | 5- 6 |
| 0- 0 | 8- 9 | 6- 6 | 8- 9 |
| 5- 5 | 7- 8 | 5- 5 | 7- 8 |
| | 5- 8 | | 5- 8 |
| | 5- 8 | | 8- 8 |

This time, we observed a marked difference between the experimental group and the control group:

- in the **control group**, 11 teachers out of 16 gave exactly the same marks in the pre and post-tests for the geometry problem and 12 of them did the same for the arithmetic problem. Moreover, 4 of the 5 remaining marks changed for the geometry problem and the 4 remaining marks for the

arithmetic problem increased. This could be interpreted as a "natural" trend to increase the marks;

- in the experimental group, 12 teachers out of 18 increased their marks in the post-test for the geometry problem while 6 other marks remained unchanged. For the arithmetic problem, 9 marks were increased, 2 were decreased, the remaining 7 staying the same in the pre and post-test.

This seems to confirm our hypothesis that the course in conceptual analysis would bring about a change in the way teachers weighted the reasoning and the answer. Our results indicate that, while 4 out of 16, 25%, of the teachers in the control group "naturally" increased their grading from the pre to the post-test for the geometry problem, this percentage climbs to a striking 67% (12 out of 18) in the experimental group. This trend is strongly confirmed by what is observed for the arithmetic problem: the same 25% upward shift is present in the control group, while 50% (9 out of 18) of the teachers in the experimental group awarded a higher mark in the post-test than they did in the pre-test for this solution.

## Conclusion

The focus the teachers put on the student's answer or on his reasoning remains hard to mesure because it is a tendency and it is not as clearcut as one may think. No teacher would consider only one of the two elements, answer or reasoning, and completely exclude the other from his preoccupations: this is what type 2 solutions (none of the teachers would accept an answer alone) and type 3 solutions (where a good answer proved to be disturbing for a large number of teachers) showed. Thus, the variations observed in the marks given were often thinner than one may expect.

However, type 4 solutions clearly indicate that a change occured in our experimental group: teachers initiated to conceptual analysis proved to be more concerned with the reasoning of the child in the post-test than they were in the pre-test. A change in the same direction was also observed in the control group but was, by far, less important.

Let us add that this "Correction Test" was used along with a "questionnaire" and with long interviews (6 in each group). These allowed us to lead twelve detailed case studies in which we collected a large number of converging indications on the teachers' perceptions of mathematics and mathematics education, confirming the reality of the evolution described here.

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# CONFIDENCE, COMPETENCE AND INSERVICE EDUCATION FOR MATH TEACHERS

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## SUMMARY

Investigation was made of six inservice courses for strengthening teachers' subject matter competencies while introducing them to a mathematics curricula for Grades 7-9. Participants were administered a knowledge test and confidence measures. Results were consistent across courses: greater self-confidence in knowledge of curricular materials, greater self-confidence in ability to teach the curriculum, and a corresponding increase in mathematics skills were manifested following inservice.

## 1. INTRODUCTION

Inservice education has become increasingly important as the main channel through which teachers are kept abreast of instructional innovations, curricular change, technological developments, and new discoveries in their specific fields. Not only can inservice provide information and opportunities for teachers to develop on a cognitive level but it possesses socio-psychological benefits as well. In this sense inservice can function to: 1) reduce anxiety by building up teachers' confidence concerning their subject matter competency and teaching ability, 2) integrate them more fully into the educational system by placing them in contact with other teachers and by exposing them to new ideas and developments at the frontiers of their field, and 3) reinforce and maintain in them positive attitudes towards both subject matter and teaching.

In the past, efforts to evaluate the impact of inservice activities on teachers have been few in number and generally have focussed upon intellectual growth (Friederwirtzer & Berman, 1983; Welch & Walberg, 1968), or attitudinal change (Henderson, 1976; Rothman, 1968; Rothman, Walberg & Welch, 1968; Spooner, Szabo & Simpson, 1982; Zurhellen & Johnson, 1972). Little attention, however, has been paid to its anxiety - reducing potential. The purpose of the present paper is to explore the effect of inservice courses for mathematics teachers on the confidence levels and cognitive skills of participants.

Six courses were evaluated, two of which were held in July 1983 and four of which took place in July 1984, at the Department of Science Teaching at the Weizmann Institute of Science in Israel. All courses were intended for mathematics teachers whose formal mathematics background was weak, with the explicit purpose of introducing them to the content and instructional approaches of the Rehovot Math Curriculum for Grades 7, 8 and 9.

Table 1 summarizes the basic characteristics of each inservice course evaluated.

## 2. PARTICIPANTS

Teachers who attended the inservice courses were varied with respect to general teaching experience, some had little experience while others had more than 20 years experience in the field. Participants were similar to one another, however, with respect to two important characteristics: 1) only about 20% - 25% had had any formal math education, and 2) very few had experience teaching math above the Grade 6 level.

Table 1 : Characteristics of inservice courses

| Course | | Grade Level | Duration (# hours) | Year evaluated | # of Participants |
|---|---|---|---|---|---|
| A1 | Integers, number bases and algebraic expressions | 7 | 60 | 83 | 48 |
| A2 | (same as above) | 7 | 60 | 84 | 73 |
| B1 | Set algebra, inequalities and equations | 8 | 60 | 83 | 49 |
| B2 | (same as above) | 8 | 60 | 84 | 50 |
| C | Linear and quadratic functions | 9 | 30 | 84 | 34 |
| D | Euclidean geometry | 8, 9 | 30 | 84 | 34 |

## 3. METHOD

In a pre-test post-test quasi-experimental design, participating teachers were administered measures which tested teachers' self-confidence as to subject matter skills, their confidence about their ability to teach curricular topics and their actual knowledge of mathematical content. The measures included mathematical problems which required only a pupil's level of knowledge in order to solve. Respondents were first asked to indicate confidence in their ability to solve each and then indicate their confidence to teach each topic. In both instances, a 4-point scale was provided on which respondents could indicate that they were very confident (4), somewhat confident (3), not very confident (2), or not at all confident (1). They were then asked to solve similar problems. In the scoring of the latter, partial credit was given to answers not fully correct and total scores were computed in percentages.

The instrument for Course A contained 9 mathematical exercises, for Course B there were

8 exercises, for Course C 6 exercises and for Course D 4 exercises.

## 4. FINDINGS

The evaluations carried out on Courses A1 and B1 during the summer of 1983 were intended as pilot studies to try out the research instruments and general methodological approach. Since results from this pilot were replicated in 1984, only the data from the second year are displayed below to simplify presentation. (For full details of the 1983 evaluations, see Fresko & Ben-Chaim, 1984).

In Table 2, mean scores and standard deviations are presented per measure per course for pre-test and post-test. Statistically significant upward shifts were noted on all measures, meaning that mathematical knowledge, confidence in solving and confidence in teaching were enhanced for inservice participants in all four courses.

As can be seen, participants in each course began with an average score of about 53% and finished with an average of about 84%. This relatively low starting level on material requiring only a pupil's knowledge attests to a very real need of these teachers for inservice instruction in order to prepare them for teaching the curricular material.

Regarding confidence in their ability to solve the exercises, teachers began at about the "somewhat confident" level and completed the course with scores which approached the "very confident" level. Confidence in teaching was consistently lower than confidence in solving across courses, although differences were of smaller magnitude at the conclusion of inservice than at the start.

Correlations between confidence in solving ability and confidence in teaching ability were relatively strong on both pre- and post-tests, ranging from .62 to .92. In other words, teachers who lacked confidence in their ability to teach the content of the various exercises were those who also doubted their mathematical competence.

Correlations between solving confidence and actual ability to solve exercises, i.e. knowledge, were calculated in order to determine to what extent a teacher's confidence was based upon reality. Results on the pre-test measures were not consistent: in two (Courses A2 and C) correlations were strong, in one (Course B2) of medium strength, and in one (Course D) quite weak. Correlations on the post-test were considerably weaker for Courses A2, B2 and C and remained about the same for Course D. Thus, it seems that one outcome of inservice participation was to undermine the relationship between confidence and actual skill. An inspection of the data indicated that this was a consequence of many participants improving their knowledge more than their self-confidence.

Table 2: Means and standard deviations per course on knowledge, solving confidence and teaching confidence measures*

| Course | | Knowledge Pre | Knowledge Post | Solving Confidence Pre | Solving Confidence Post | Teaching Confidence Pre | Teaching Confidence Post |
|---|---|---|---|---|---|---|---|
| A2 | $\overline{X}$ | 53.0 | 83.5 | 2.84 | 3.73 | 2.54 | 3.55 |
| (N=50) | S | 26.0 | 14.5 | .76 | .30 | .76 | .38 |
| B2 | $\overline{X}$ | 46.5 | 85.5 | 3.01 | 3.76 | 2.86 | 3.71 |
| (N=23) | S | 29.0 | 19.5 | .78 | .33 | .86 | .35 |
| C | $\overline{X}$ | 54.0 | 83.5 | 2.81 | 3.68 | 2.64 | 3.51 |
| (N=24) | S | 31.0 | 14.0 | 1.02 | .46 | 1.00 | .52 |
| D | $\overline{X}$ | 58.0 | 84.5 | 3.04 | 3.60 | 2.81 | 3.55 |
| (N=25) | S | 18.5 | 20.5 | .77 | .38 | .88 | .42 |

* All twelve pre-post comparisons are statistically significant using t-test, at $p < .001$.

## 5. DISCUSSION

Several conclusions emanate from the data which have implications for the planning of inservice activities for math teachers. First, content-oriented inservice instruction is required, that is, many math teachers need to learn mathematics better. Secondly, teachers who lack subject matter competence tend to lack self-confidence. It appears that competence must initially be treated, followed by efforts to reinforce self-confidence both with respect to competency and with respect to teaching capability. Inservice activities can be planned which are structured along this hierarchical principle, offering the teacher with the appropriate reinforcement at each stage.

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# ACTIVITIES WITH TEACHERS BASED ON COGNITIVE RESEARCH

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In previous studies (Hershkowitz and Vinner, 1984) we found that concept images of students (age 10-14) and elementary school teachers, in basic geometry, have similar patterns.

The first impression is that students conceptions are just a "mirror reflection" of their teachers' conceptions.

A deeper look at the results leads to another conclusion:- There are processes in perception and concept formation which are inherent to the concepts themselves and which have the same inhibiting effects both on students and teachers.

This includes perceptual factors (like the orientation factor), the effect of critical and non-critical attributes of examples and non-examples of a given concept, and the effect of teaching strategies, like the use or misuse of definitions or concept rational sets, (the set of examples and relevant non-examples used to introduce a given concept), etc.

It seems that without systematic intervention, directed to deal with those factors, we cannot expect significant improvements in that geometrical knowledge, from adolescence to maturity. The purpose of our presentation is to describe such an intervention - a set of in-service activities with teachers, based on our previous research. It was clear for us that teaching the teacher basic geometrical concepts, in the same ways that he is used to teach them, would be inneffective, boring and insulting too.

Here we will describe one example from this set of activities including some teachers reactions.

## The *trianquad* activity (see figure 1)

Population - Twenty mathematics senior teachers in elementary schools - without any academic background in mathematics.

All of them took a questionnaire which served as the basic research tool in our previous studies.

Goals - 1) Improving teachers' concept images of some geometrical concepts.
2) Increasing teachers' awareness to their students difficulties and misconceptions.
3) Increasing teachers' ability to analyse concepts; - critical and non-critical attributes, - examples and non-examples.
4) Improving teachers understanding of the role of definition.
5) Gaining experience in teaching strategies which take into account 2), 3) and 4) above.

## The activity

Stage 1) An unstructured discussion on the ways of teaching basic geometrical concepts from the early childhood, with special focus on the role of definition, examples and non-examples of the concept.

Stage 2) The *trianquad* exercise is given to the participants (see figure 1), as a new teaching strategy (Herron et al 1976).

This task presents concept formation of an artificial concept, through successive trials in a set of examples and non-examples (see part of it in figure 1). The participant makes a guess successively, in each frame - of the exercise, and receives feedback on his previous guessing. He gradually discovers what attributes a *trianquad* has and does not have. At the end he is asked to give the definition of *trianquad*.

The teachers worked individually on this exercise. They were amused and stimulated to discuss it. Seventeen teachers out of twenty, gave a definition which includes all the three critical attributes of the *trianquad*, (it is a geometrical figure consisting of a quadrilateral and a triangle which have a common vertex). The remaining 3 teachers had mistakes in one of the attributes.

Four teachers, out of the seventeen added unnecessary or incorrect information, based on non critical attributes, like:

- *The figures could be contained in one another overlap, or be separated from one another. (unnecessary information)*
- *The triangle is "smaller" than the quadrilateral. (incorrect information).*
- *Trianquad is a seven sided polygon. (incorrect).*

It is worthwhile to note that the teachers' ability to verbalize was quite good (much more than the ability of students of grades 6 and 8, who took a similar activity).

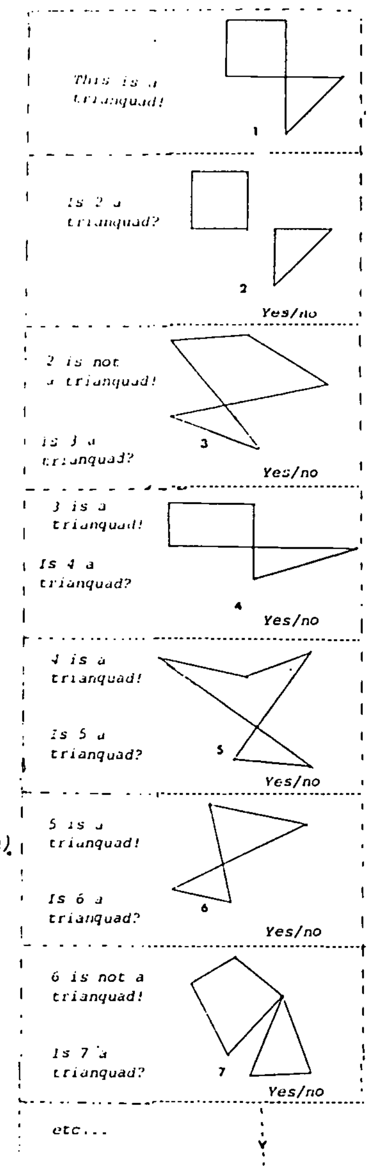



*Figure 1: Part of the trianquad exercise*

Stage 3) Analysis of the *trianquad* exercise. This was done by guided discussion. The following are parts from it. (R - denotes the researcher and T - denotes a teacher who reacts to the researcher or to another teacher).

*R : - How is the concept of trianquad conveyed through this way of teaching?*

*T : - ... Through examples*

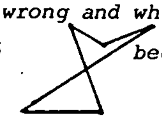
*T : - Part of them were mistakes - the examples were not true.*

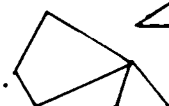
*T : - "They" pointed our errors.*

*T : - There was a feedback too ...*

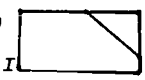
After that the teachers together with the researcher concluded that this process of learning a concept is consisted of examples, non-examples, guessing and feedback.

*R : In what items was your guess wrong and why?*

*T : I made a wrong guess in No. 5*  *because this is not a "regular" quadrilateral.*

*T : I was wrong in No. 7*  *because I had the impression that the sides must lie on one line.*

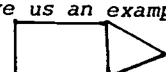
*T : The examples there teach us, but on the other side we should be careful.*

*T : Number 10*  *was not clear, I was not wrong there, but it was hard for me since I saw it as a pentagon and a triangle with a common side.*

*T : What about the common vertex, does it have to be only one?*

*R : What is your opinion?*

*Several teachers : Yes. No. Only one. No at least one ...*

*T : "They" did not give us an example of trianquad which has more than one common vertex, like this*  *, so it is only one.*

*T : Yes, but on the other side "they" did not give us an example with two common vertices and said that it is not a quadrilateral, so we don't know.*

While this discussion went on, teachers understood the meaning of examples, non examples, critical attributes and non-critical attributes. They noticed non-critical attributes which can have a misleading effect, and factors which inhibit our ability to recognize some examples of the concept (perceptual factors). They discussed the characteristics of concept-definition, (that it must be minimal on one side but unambiguous on the other side), and the characteristics of valuable non-examples.

Stage 4) The use of the *trianquad exercise* as a model for other concepts acquisition. Here we choose concepts whose images were very poor in both populations' students (grades 5 to 8) and teachers (student teachers and teachers).

For instance: The altitude to the side a in a given triangle.

In our group only 8 teachers (out of 20), knew how to construct an altitude to a given

side in all 4 kinds of triangles: isosceles, acute, obtuse, and right angled triangles, (see Figure 2).

Their wrong constructions were very similar to those of the students. At the beginning "we" (the researcher and the teachers) analysed together the concept of altitude in a given triangle, and then the researcher presented a few typical student mistakes, on the overhead projector, and discussed them with the teachers.

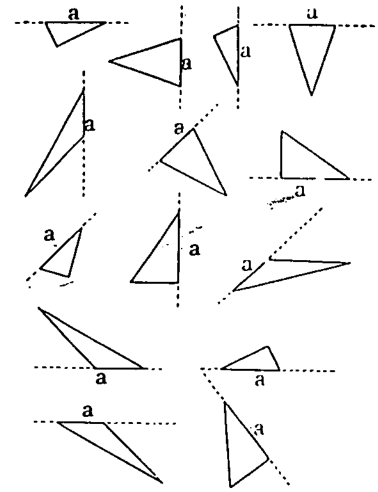


*Figure 2: The given triangles for the altitude task*

For example: (See Figure 3)

*R : What about this student, what did he draw?*

*T : He succeeded in the isosceles triangle and in the second one ...*

*T : In the obtuse - angle triangle he decided that the vertex is at the bottom.*

*T : .... This was easier for him.*

*T : Here the altitude is to the continuation of side a. This business of an outside altitude is not clear to him ...*

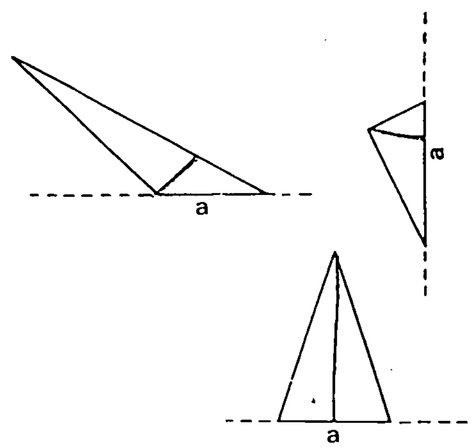


*Figure 3: One student constraction of an altitude*

Note : Eleven teachers, out of the twenty in this group did not know how to draw an altitude to side a in the obtuse triangle. Five of them had the above mistake.

After examining and discussing together a few more students' mistakes, the teachers were asked to work in a group of two or three, on a collection of students' mistakes on the altitude task. They had to analyse first the mistakes that each student made, to explain

them and then to make a hierarchy of student answers on this task,(from the worst errors to the correct answer). Most of the above working groups chose the path from answers that did not have even one critical attribute of the altitude, to those which had at least one and so on until the correct drawing.

As a homework teachers were asked to create by themselves an exercise, on forming the concept of the altitude, using the *trianquad exercise* model and the student mistakes as the basis to the relevant non-examples.

## Conclusion

One month after having this activity, teachers were asked to construct again altitudes in the same collection of triangles (see figure 2). Because of administrative reasons only 14 teachers (out of the twenty) took this "postest". *All of them* got the right answers. Before the activity, only 7 out of these 14 got it right.

We can say that as a result of our activity the altitude image in this group was improved. Our impression is that also the other goals of our activity were at least partially obtained although it is not easy to measure it scientifically.

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P M P (Primary Mathematics Project for the Intelligent Teaching of Mathematics):

# A PROGRESS REPORT

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Summary

This paper is about an eight-year project for bridging the gap between theory and the classroom. As Dewey (1) has written, "Theory is in the end ... the most practical of all things." The process of developing a theory is a lengthy one; but producing a practical embodiment is at least as lengthy; and we still have to get people to use it, and use it sensibly. This paper will give a pinhead introduction to the theory; a brief outline of the process by which a practical embodiment has been developed; and a few examples from the 358 practical activities which we have devised and field tested for classroom use.

* * * * *

In my country and others, for more than twenty years, much time and money have been spent in efforts to improve the teaching of mathematics in our nations' schools. Nevertheless, the problem is still with us. And there is no reason to suppose that future efforts will be any more successful than those of the past, unless we can identify and remedy at least some of the causes of this large-scale failure. In the present paper, I shall suggest two of these, and outline the remedies which are being developed in the present project.

First Cause: Projects based on mathematical rather than psychological and epistomelogical models. (To be expanded verbally).

Second Cause: Lack of success of in-service education. (The reasons for this require a paper to themselves).

The present project is based on the belief that mathematics is one of the most powerful and adaptable mental tools which the intelligence of man has constructed for his own use cooperatively over the centuries. Hence its importance in today's world of rapidly advancing science, technology, manufacture, and commerce. From this, it is predictable that unless we can find out how to teach mathematics in ways which bring children's intelligence into use, rather than habit learning, there is little likelihood of

progress. On these assumptions, necessary conditions for putting things right include the following:-

1) A model of intelligence which helps us to understand its function, and its relation to learning, rather than one which is centred on "I.Q." and its measurement.
2) Finding out how to apply this model to the learning and teaching of mathematics.
3) Developing and field testing methods and activities based on (2), sufficient for a feasibility study.
4) Finding out how to help teachers to use these methods with understanding: a model for in-service and pre-service education.
5) Expansion of (3) into a full curriculum.
4) Dissemination.

I will take these one at a time.

1) <u>Pinhead introduction to the new model of intelligence</u>.

A full exposition takes nearly 200 pages (2). A longer one than the one which follows can be found in the proceedings of ICME IV (3). Here, I will mention just two if its most important features.

The first of these is the contrast between habit learning and intelligent learning. To learn someone's telephone number is a matter of rote memorising, which is a form of habit learning. When we have rote-memorised something, say a telephone number, this enables us to do just one job - dial a particular subscriber. It is not adaptable to other uses, nor does it help us to learn other telephone numbers. If our telephone number is changed, the old habit gets in the way of the new one. Moreover, this kind of learning is for most people tedious and laborious. If passing an examination for qualification as a secretary involved memorising fifty telephone numbers, few would pass and the work would be regarded as boring and difficult.

In contrast, a person asked to learn the following sequence of numbers

3 6 9 12 15 18 ... (a hundred of these in all)

would experience little difficulty. He would not try to memorise the sequence of numbers itself. He would learn a pattern, a <u>generating structure</u> from which the whole sequence can be derived.

In so doing, he would be using intelligent learning.

Alao in contrast to habit learning, knowledge of this second kind is highly adapatable. This exemplifies another essential feature. If one were asked what would be the hundredth number, or the ninety-ninth, this question would present no difficulty. Moreover, having learnt to recognise a pattern of this kind, the knowledge would be helpful in recognising other patterns such as

9 22 35 48 61 ...

in which the regularity is of the same kind but less obvious. In the process of so doing, the learner would be building up a more general concept, that of all sequences of numbers exhibiting this pattern. This would be useful to him in finding the pattern of the following series

1 2 4 7 11 16 ...

Here, there is a pattern within a pattern. The increase by which each new term is obtained is now not the same each time; but it shows a regular pattern which someone who has already learnt the earlier patterns has a good chance of recognising. Without this knowledge, to perceive this regularity would be more difficult. Our existing knowledge structures (schemas) act as mental tools for the acquisition of new knowledge. Though almost all conceptual knowledge is pre-requisite for later learning and understanding, mathematical knowledge exhibits this hierarchic nature particularly strongly.

Whereas rote-learning allows the production of results in only a limited class of cases, those to which the rules apply, the learning of conceptual structures is not only less taxing on the memory, but more adaptable. In general, the essence of intelligence is adaptability. The adaptability of mathematical knowledge in particular is discussed at greater length in reference (3).

Another aspect of the new theory emphasises the ways in which these knowledge structures (schemas) are constructed. The term "construct" is used here to mean a combination of building and testing. This will be expanded in the verbal presentation. (See also reference 4).

In conventional mathematics teaching, the main emphasis is on communication by the teacher of a method to be followed by the pupil, followed by practice in using this method. Thus, only one (at most)

of the six available means for intelligent learning is made available to the child. In the present approach to the intelligent learning of mathematics, all six are used. A wide variety of physical experiences embodying mathematical ideas have been devised. These enable the learners to build mathematical models which yield testable predictions. The latter is one of the major uses of mathematics in science, technology, and commerce, but so far it is almost entirely lacking in school mathematics. Many of these activities take the form of games, providing shared experiences which give rise to mathematical discussion among the children. This peer group interaction is an important method of learning, and children correct each others' mistakes in a way which is much less threatening than being told they are wrong by a teacher. Learning in this way, the natural creativity of the child often comes into action spontaneously. This may show itself either in devising new ways of arriving at solutions, which puts them on the road to successful problem solving: or by extrapolating a mathematical pattern which they have learnt to new situations.

2), 3). These occupied the first three years of the project. It took the form of a close collaboration with a selected group of primary teachers. Our purpose was to find out how to apply the new model of intelligence to bring about intelligent teaching and learning of mathematics in the classroom. In this, we were together breaking new ground, both in theory and application.

A close relationship was developed in which they learned from me about human intelligence, and improved their knowledge of mathematics; while I learned from them about the practicalities of the primary school classroom.

4) The next stage of the project addressed itself to the second of the two problems described above, that of finding an effective approach to the in-service education of teachers. It was based on the assumptions that this needed to be (i) based in the schools themselves, rather than outside; (ii) to involve at least half the teachers in the school; and to be low-intensity, over a period, rather than high-intensity and short-term. This last was related to the great amount which teachers need to learn, and the importance of giving them time to assimilate it at their own pace.

As originally envisaged, this was a consultancy approach. It

involved regular weekly visits to the school by myself and my research associate. But clearly, only a very limited number of schools could be helped in this way.

A major step forward became possible when it was found that the newly developed materials themselves provide a basis for an in-service education for teachers, independently of a consultant. Since each of the activities embodies the new theory of intelligence, in its practical application, we now find that by the use of these activities teachers can learn about the theory by seeing it in action. This relates their learning directly to their own classroom needs. Teachers' learning is further helped by the fact that the P M P activities externalise the children's thinking, and thereby enable teachers to use their own observations as a way of increasing their understanding of how children learn and think. We thus get 'two for the price of one', time-wise. The children learn mathematics better, and the teachers learn about the intelligent learning of mathematics by observing their own children in the process.

A new, self-help, model for school based in-service education thus emerges.

(i) The teachers first need to try out these activities with each other, as a preliminary to using them in the classroom. This also improves their own mathematical knowledge.

(ii) They then learn about children's learning processes, by observing their own children using these activities. Experienced teachers have already said that they learn more from these observations than from all the books they read at college.

(iii) These experiences are then shared in weekly seminars with other teachers, and the ensuing discussion consolidates and expands teachers individual experiences. There also emerges, by consensus, a unified cooperative approach to the teaching of mathematics in the school concerned.

The self-help model has already shown itself successful in several schools which I am only able to visit once or twice a year.

5) This is where we are now (1985, seventh year). We currently have about 350 of these activities, covering approximately the first seven years of school. So there are less than two new activities

a week for teachers to learn, based on a 36 week teaching year.

(Here will follow a demonstration of a few of the 358 activities).

6) I hope that the materials will be ready for dissemination in about eighteen months. This coincides rather conveniently with the beginning of my retirement from my present appointment at Warwick University. So I hope to spend some of my time travelling, visiting my academic friends, and offering summer schools and short introductory courses in the use of the new materials.

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# The evaluation of two types of knowledge at different points in time after study

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The fact that knowledge can be of different kinds and be put to different uses is of relevance to anyone interested in the psychology of school education. What kinds of knowledge students should have depends partly on the properties of the various knowledge types.

In the present study, two different types of knowledge, declarative and procedural knowledge are studied with respect to their relation to metacognitive judgements.

Declarative and procedural knowledge are assumed to differ with respect to their function and their introspectability. Declarative knowledge can be characterized as mental representations of the world. The size of the units of declarative knowledge is currently a matter of controversey,(see e.g., Anderson, 1983, Chapter I.) But the contents of these units are consciously introspectable and the contents, as such, are not directly related to any particular behavior. Thus, choice of behavior can be influenced but is not directly determined by the content of the specific declarative knowledge attended to.

In contrast, procedural knowledge can be seen as representations of abilities. That is, activating procedural knowledge means carrying out the behavior (covert or overt) specified in the procedure. Furthermore, procedural knowledge is usually not considered to be consciously introspectable (e.g., Neves & Anderson, 1981).

Since, in practice, there is no generally acceptable way to diagnose whether a particular knowledge content is stored in a declarative or procedural form (Johnson-Laird, 1983; Newell, 1972; Winograd, 1975), we can only aim at reasonable operationalizations of knowledge formats.

In the present study, problem solving is seen as an operationalization of procedural knowledge and the ability to answer knowledge questions as an operationalization of declarative knowledge. The metacognitive task studied is to give evaluations with respect to correctness of performance, either when solving problems or when answering knowledge questions. In this connection, the effect of different

amounts of each type of knowledge is investigated.

It is not immediately obvious that subjects will give more valid confidence judgements for either type of knowledge. For both types of knowledge evaluations will either be based on recognition of the item to be evaluated or by consideration of the implications of associations to other knowledge elements. (Often recognition judgements may also utilize such associations.) Thus, for both knowledge types, the associations will presumably primarily be between the item to be evaluated and some declaratively stored knowledge. Furthermore, subjects' own conceptions about the respective types of knowledge may affect the validity of their judgements.

With respect to the effect of the amount of knowledge on the performance evaluation task, two previous studies (Lichstenstein & Fischhoff, 1977; Chi, 1978) suggest that the validity of subjects' confidence ratings will decrease with lesser amounts of knowledge. Lichtenstein and Fischhoff (1977) found that, up to a point, (about 80% correct of the test items) more knowledgeable subjects made more valid confidence judgements of the correctness of their performance compared to less knowledgeable subjects. Subjects who had less than 80% correct of the test items tended to be overconfident and subjects who had more than 80% correct tended to be underconfident. Chi (1978) studying a different metacognitive task, found that people (both children and adults) with better knowledge about chess, tended to perform better when asked to predict how many trials they would need to reproduce the positions of all the pieces in an organized arrangement on a chessboard. Thus, again less knowledge was associated with poorer performance on a metacognitive task.

## Methods

The subjects were 54 students studying first year statistics at the University of Göteborg, Sweden.

Three different groups of subjects with 17, 19 and 18 subjects, respectively, were run in the study. The first group participated directly after an examination (relevant to the problems and questions used in the study) which all subjects took, and passed. The second and third group were run after one and two months, respectively.

The subjects first attempted to solve two statistical problems.

The first problem was a time-series problem where the subject, given sales values for each quarter over a period of three years, was asked to analyze the development of the sales by using an appropriate model. The task included calculating seasonal indices and sales values adjusted for seasonal influence. The second problem was a regression problem, wherein the subject was asked to calculate a regression line of the size of a harvest on the amount of fertilizer used, and to calculate the linear correlation between the two variables. Furthermore, the subject was asked to calculate the mean increase in harvest for each unit of fertilizer used, as well as the expected size of harvest on an area when a specified amount of fertilizer was used. (The problem instructions and the solutions can be found in Allwood, 1984, or can be obtained from the author). After the experimenter had broken down the subjects' written solutions to the two problems into six and seven substeps, respectively, the subjects were asked to make confidence ratings of the correctness of each substep of their solutions. For this purpose, subjects were provided with five-point scales with the two end points defined as: "I am completely certain that this part of the problem was solved <u>completely correctly</u>" and "I am completely certain that <u>some error was made</u> in this part of the problem solution".

Next, subjects were asked to answer in writing, 28 written questions aimed at assessing subjects' knowledge and understanding of concepts and solution methods related to the two problems given. (These questions can also be obtained from the author).

Finally, subjects made confidence ratings on the correctness of their answers to each of the knowledge questions. Subjects made these ratings on five-point scales, similar to those used in phase two of the experiment. No time limit was set for subjects in any phase of the experiment.

## Results and discussion

Due to space limitations, it is not possible to give a detailed presentation of the results. Therefore, only a brief outline of the main results will be given. The effects mentioned below are regarded as significant or reliable when $p < .05$. A two-way analysis of variance (ANOVA) with time since study and type of knowledge (as evidenced in problem solving performance and answers to knowledge

questions) as factors showed that there was a reliable decrease in both types of knowledge over the time span studied. This analysis also showed that there was no significant difference between the two types of knowledge in decrease of retention. However, subjects (overall) scored significantly higher (in relation to the maximum score possible) on the problem solving task compared to the knowledge question task. The reason for this difference is not clear since it is difficult to compare the level of difficulty between the problems given and the questions asked.

We deal next with subjects' confidence ratings. A two-way ANOVA showed that subjects' confidence ratings decreased significantly over time since study. Furthermore, the level of the confidence ratings differed significantly between the two tasks. This is expected since subjects solved (correctly) a larger proportion of the problems given, compared to the proportion of knowledge questions answered correctly. The same pattern of results held when the correct problem solutions and question answers were analyzed separately. However, when erroneous problem solutions and question answers were analyzed, the decrease in level for subjects' confidence ratings did not quite reach significance for either time since study or type of task.

Finally, we deal with the validity of subjects' confidence ratings, i.e., we ask if there were any differences over time or between knowledge types in subjects' ability to discriminate between erroneous and correct question answers or problem solutions. This analysis focused on subjects "hits" and "false alarms".

By subjects' "hits" is meant his/her correct responses rated with the highest rating category 5, ("I'm completely certain that this [item] was solved completely correctly"). By subjects' "false alarms" is meant his/her erroneous responses rated with the highest category 5.

Table 1 shows subjects' hits and false alarms for the different times since study and the different types of knowledge. Two-way ANOVAS showed a significant effect for type of knowledge both with respect to subjects' hits and with respect to their false alarms. Inspection of the data shows that problem solving knowledge had a higher score for both hits and false alarms. The analyses also show that there was no significant effect of time since study either for hits or for false alarms and, furthermore, that there was no sigificant interaction

between types of knowledge and time since study.

Table 1. Subjects "hits" and "false alarms" for different types of knowledge and times since study.

| | Time since study | | | | | |
|---|---|---|---|---|---|---|
| | A few days | | One month | | Two months | |
| | problem solving | knowledge questions | problem solving | knowledge questions | problem solving | knowledge questions |
| "Hits" | .74 | .53 | .59 | .48 | .57 | .40 |
| "False alarms | .58 | .29 | .45 | .19 | .28 | .24 |

Seen in signal-detection terms, the pattern in the data on subjects' hits and false alarms suggests that there may be systematic differences between the two knowledge types with respect to the criterion subjects used as a cut-off point when deciding whether to rate their performance with the highest confidence rating 5, or to give a lower rating. (Lindsay and Norman, Appendix B, give a brief introduction to signal detection theory.) This possibility was tested by creating a measure of the criterion used by the subjects. A measure of the criterion was constructed for each type of knowledge and point in time since study by taking the average of the negative z-scores of the proportions of hits and false alarms for that type of knowledge and point in time since study. A two-way ANOVA with the criterion as the dependent variable showed a significant difference in the criterion both for type of knowledge and for time since study. The data also show that subjects use a more liberal criterion for problem solving knowledge as compared to the criterion used for question answering knowledge. It may be that subjects' higher level of retention of procedural knowledge explains why subjects used a more liberal criterion when evaluating such knowledge. Other aspects, such as properties of the two knowledge types or subjects' conceptions of the two knowledge types, may also have contributed. More research is needed here.

The change in criterion over time since study means that subjects over time became more conservative in their confidence judgements, i.e. they had progressively fewer of both hits and false alarms. It may be that subjects in the conditions with longer time spans since study were aware that a long time had elapsed since they studied the material and accordingly adjusted their confidence judgements on the basis of this knowledge.

In contrast to the studies on meta-cognitive performance by Lichtenstein and Fischhoff (1977) and Chi (1978) reviewed above, subjects with less knowledge, in the present study, did not perform worse than subjects with more knowledge in the meta-cognitive task investigated (to give confidence judgements of their own performance). This difference between the present study and the two others may mean that "having less knowledge" is not a unitary phenomena. Results in the present study suggest that subjects' awareness that their knowledge has deteriorated might influence meta-cognitive performance in domains in which the subjects are less knowledgeable. Thus, the degree of such awareness may be an important distinguishing factor between individuals "having less knowledge".

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# A CONSTRUCTIVIST VS A FORMALIST APPROACH IN THE TEACHING OF THE EVEN-ODD NUMBER CONCEPT AT THE ELEMENTARY LEVEL

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*The first part of our paper concerns the general topic of a constructivist approach in the teaching of mathematics. We examine the possible contributions of a prior mathematical analysis of the concept to be taught, the need for a psychological analysis of the cognitive obstacles involved, and the necessity of a contextual analysis. These analyses are necessary elements but do not in themselves constitute a model suggesting the construction of conceptual schemes. The model we use is based on the constructivist psychological learning theories of Piaget and Bruner. The second part of this paper is devoted to the application of these theoretical considerations to the even-odd number concept.*
*number concept.*

In our most recent communication at the Sydney meeting of PME (Herscovics & Bergeron, 1984), we identified two different ways used in the teaching of mathematics by trying to characterize a constructivist approach and a formalist approach. A formalist approach emphasizes rigor, precision and form in expressing mathematical ideas. Teachers who have achieved a high level of competency perceive the mathematics they teach with such clarity that they may simply try to transmit this knowledge to the pupil in its finished form, forgetting the long and arduous journey that brought them there. In such a case, the mathematical form is often mistaken for the mathematical content. Of course, we do not mean to imply that the formal aspect of mathematics is not important. On the contrary, we think of it as essential in the mathematical development of the student. However, the question raised is "When should it be introduced?". A formalist approach introduces prematurely the formal aspect of a mathematical concept and this often proves to be pedagogically counterproductive. On the other hand, by deferring it towards the end of the student's construction of the given concept, the formal aspect then becomes a way of formalizing mathematical ideas he has already grasped.

While a formalist approach is essentially concerned with the transmission of mathematical knowledge, a constructivist approach focuses primarily on the learner's construction of this knowledge. If in the first instance the pupil can be treated as a passive recipient, in the second one he must be considered as an active agent of the learning process. As pointed out by Hermine Sinclair (1985), a constructivist

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approach requires a mathematical analysis of the concept to be taught, a psychological study of the learning processes involved, and an analysis of the various situations which might provide an appropriate context.

The objective of the mathematical analysis of a concept is to delineate the meaning mathematicians attach to it. The importance of a given concept must also be considered in relation to the mathematical system or structure in which it is embedded and within this system or structure, its mathematical pre-requisites need to be identified. Of course, it would be a mistake to believe that definitions, importance and pre-requisites are set in stone. In fact, mathematicians do not necessarily agree on a definition, and quite often, a brief historical study reveals major changes in their perception. For instance, a computer scientist may view a variable as a placeholder, an algebraist may perceive it as a set representative, and an analyst may define it in terms of functional dependence. Historically, the very notion of function has changed from its dynamic definition as primarily being a rule relating an independent and dependent variable, to a static definition viewing a function as a special set of ordered pairs.

A psychological analysis of the learning processes involved must bring out the cognitive obstacles which may be encountered in the learner's construction of a given concept. Of course, such obstacles will vary according to the pupils, their age and their mathematical background. Even if one has identified the mathematical pre-requisites, these do not necessarily correspond to the psychological ones. At the elementary level, the perception of zero as a number is a very good example. Mathematically it can be defined as the cardinality of the empty set, for which the pre-requisites are the notions of quantity, number, and empty set, or it can be defined as the origin of the natural numbers on the number line, in which case the pre-requisites are the notion of length, number and measure of length. But neither of these presentations will bring about the perception of zero as a number. A child may very well write "0" as the answer to 3 - 3 = ? but use the numeral or the word "zero" just as another word for "nothing" since this is the result of "three, take away three". In conceptualizing zero as a number, the child is handicaped by the very notion of number which signifies the measure of a quantity. However, since there are no objects in an empty set, there is no quantity to measure, and thus zero cannot be viewed initially as a number. This cognitive obstacle can be overcome by the gradual detachment from an enactive mode of representation to a symbolic one. Only in the latter case does "operating" with zero become possible, as in 5 + 0 = ?; 0 + 6 = ?; 7 - 0 = ?. It is in the arithmetic context that zero takes on its meaning as number, for at the enactive level it makes little sense to add or take away "nothing".

This last example shows how even a change in the mode of representation can provide a new context which may help resolve a cognitive problem uncovered through prior psychological analysis. Both the concrete and symbolic representations provide the learner with a specific inference system in which different classes of problems become significant. This illustrates but one aspect of contextual analysis. An equally important one deals with the need to create situations in which the learner is confronted with a cognitive disequilibrium. If students are constantly faced with problems they can solve within their existing inference system, the essential stimulation necessary for further learning is missing. For instance, a young child who can solve addition problems by counting all the objects starting from one, has little incentive to use the more efficient procedure of counting on from one of the terms unless he has to handle sums such as $16 + 3$ in which case counting all is rather inefficient. But even then, many children will persist in using their more primitive procedure. For them, it is only when they are unable to count one of the sets, which may be hidden or contained in a sealed jar, that a cognitive obstacle is created. While they still understand the problem at hand, their existing method of solution is no longer adequate. This new class of problems provides a context favoring the acquisition of the more evolved procedure.

While the above example illustrates how a cognitive disequilibrium can be induced by a perceived lack of efficiency or simply a lack of knowledge, a third source of disequilibrium can result from conflicting cognitive schemes. The classical Piagetian conservation of number tasks reveal how the counting scheme can come into conflict with a judgment based on configuration (Greco & Morf, 1962): based on his perception, a child may infer that the longer row contains more objects, while equality can be inferred from counting. In fact, it is the eventual resolution of this cognitive conflict that results in the child's conservation of number.

We have tried to justify the need for three types of analyses, a mathematical one, a cognitive one, and a contextual one, for each notion to be taught. However, although these analyses are necessary elements, they do not in themselves constitute a model describing the construction of conceptual schemes. Such a model can only emerge from constructivist psychological learning theories. Bruner and Piaget are two psychologists who provide us with such theories. Bruner (1966) identifies three modes of representation, enactive, iconic, symbolic, and suggests that this sequence corresponds to the child's intellectual development. Ginsburg and Opper (1979) identify three levels in Piaget's theory of intellectual development: the level of <u>action</u> (knowledge is preserved in the form of schemes which allow the actions to be repeated in identical situations and generalized to new ones), the level of <u>conceptualization</u> (actions that were previously performed directly on objects are recons-

tructed internally by the child), the level of abstraction (the child can reflect on his own thought).

Of course, these psychological theories are quite general and do not take into account the particular nature of each discipline. And when it comes to mathematics, Piaget (1966,1973) explicitly recognizes that its construction involves the transition from an initial "intuitive" stage to an eventual level of "formalization". Thus, when it comes to mathematical conceptual schemes, we recognize four stages in their construction, an intuitive stage, a procedural one, a third stage involving abstraction, and a final one, that of formalization. We consider these four stages as major cognitive steps and have called them levels of understanding. Hence we use the terms "model of construction" and "model of understanding" interchangeably. Since a description of these four levels has been presented in our Sydney paper (Herscovics & Bergeron,1984), we will restrict ourselves to the application of our model to the even-odd number concept.

## THE CONSTRUCTION OF THE EVEN-ODD NUMBER CONCEPT

This notion proves to be particularly interesting since it can be introduced by using two mathematically equivalent definitions ("a number is even if it is divisible by 2", or "a number is even if its last digit is 0, 2, 4, 6 or 8") in the sense that each one can be derived from the other. However, they are far from being pedagogically equivalent. Indeed, while the first definition is independent of the numeration system used, the second one is not. For instance, representing 14 in Roman numerals, XIV, shows that the second definition does not hold. Furthermore, the first definition leads to a transition from division to divisibility by associating the latter with the remainder and this can be extended to the study of divisibility by any other number leading to an intuitive perception of the division algorithm (given any two natural numbers a,b, $0 < b \leqslant a$, there exist unique integers q, r such that $a = qb + r$ and $0 \leqslant r < b$).

Neither can the two definitions be considered equivalent from a psychological perspective. Indeed, the first definition can easily be linked to an action-scheme within the reach of even a very young child. The concept of even-odd number can be associated with the idea of splitting a set into two equal parts. Such a task can make him aware that for certain numbers the corresponding sets can be equally divided while others leave a remainder of 1. Even as early as 4 or5 years, children can partition a given set into two equivalent subsets by splitting pairs of objects or by "sharing" the set with a partner one element at a time (one for me, one for you). However, as pointed by Hermine Sinclair (1985), this does not imply that

they necessarily perceive the equivalence of the two subsets they have constructed. This awareness comes somewhat later when a 1:1 correspondence can be used more globally to compare two sets, not only for partitioning one. Of course, even a child in kindergarten can handle this action-scheme. But does it imply an understanding of the even-odd number concept? If the task we have described is not within an arithmetic framework, it merely involves the partitioning of a set. However, a set is never even or odd, only numbers are. At this enactive level, only the child's informal mathematical knowledge is required. This is why we consider this action-scheme as an intuitive understanding of the even-odd number concept.

The partition scheme described above can reach fairly high levels of efficiency since sets corresponding to given numbers can be rearranged into two rows of equally spaced objects which, at a glance, tell us if the two rows are of the same length or not. However, this procedure becomes quite cumbersome when larger numbers are involved (e.g. 784). What is then required is an arithmetic procedure which eliminates the need to rely on objects. Since we use division by 2 as our criterion, the division algorithm is the arithmetic procedure which is needed here. The quotient provides us with the number of possible pairs, while the remainder can only be 0 or 1 "since otherwise more pairs would be available". When a student uses the division algorithm to decide the parity of a given number, he thereby demonstrates a procedural understanding of this concept. Of course, we don't expect this to be achieved before the middle grades of elementary school.

A student using the division algorithm to verify if a number is even or odd has achieved a certain abstraction in the psychological sense for he has freed himself from the need to rely on physical objects. But this does not mean that he has reached a level of understanding which we can qualify as mathematical abstraction. In the mathematical sense, abstraction here involves a process of generalization extending the even-odd property from individual numbers to classes of numbers. It is only when the parity concept is perceived as the property of a class of numbers that it can be used to partition the natural numbers into corresponding disjoint classes. The pupil might then be led to discover that by adding 1 to any number belonging to one class he obtains a number from the other class.

One cannot reasonably expect the symbolic representations of these classes to be accessible to elementary school children. Notation such as $[2k]$, $[2k + 1]$ is far too formal to be meaningful at this early age. In fact, the notation 2k or 2 x k conveys more the sense of multiple than the sense of division, the latter being far closer to an action-scheme than the former. Nevertheless, we still can conceive of a fourth level of understanding in grades 4-6, that of formalization. And this is

where the alternate definition has its place. To identify an even number as one whose last digit is 0, 2, 4, 6 or 8 is a procedure which enables one to find the parity of a number even more efficiently than through division. But it is only when it can be related to the initial definition that it can be considered as this fourth level of understanding. The argument that every even number must terminate by one of these digits is fairly easy to grasp: since all numbers end by 0, 1, 2, 3, ..., 9, multiplying them by 2 will yield numbers ending by 0, 2, 4, 6 or 8. The converse is harder to convey. Of course one should not attempt a formal proof at the primary level but rather rely on logical explanations. This is not too difficult for any number can be decomposed into tens and units (493 = 490 + 3 = 49 tens + 3 units). Since tens are always divisible by 2, the parity of the number will depend on the parity of the units.

BY WAY OF CONCLUSION. Using the even-odd number concept, we have provided both a mathematical and a psychological analysis and have used our model of understanding to identify four stages in a possible construction of this notion. For each stage, we have designed a work sheet aimed at achieving the corresponding cognitive objectives. In doing so, we have in some sense inverted the usual approach to contextual analysis which consists in determining the cognitive scope of a given activity. Rather than search for available activities which might in some way be useful to us, we preferred to design our own in order to tailor them more closely to our cognitive objectives. Our first companion paper will present the four work sheets and will report on how primary school teachers ordered these into a teaching sequence without any prior discussion of this concept.

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# THE SEQUENCING OF ACTIVITIES FOR THE TEACHING OF THE EVEN-ODD NUMBER NOTION BY PRIMARY SCHOOLTEACHERS WITHOUT THEIR PRIOR ANALYSIS OF THIS CONCEPT

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*An analysis of the even-odd number concept has identified four stages in a possible construction of this notion. Four worksheets reflecting these four stages were prepared. The ordering of these worksheets into a teaching sequence by an experimental group of prospective elementary school teachers was compared with that of a control group. The experimental group was trained over a period of two months in the analysis of arithmetical concepts. While the post-test indicates that this training had a positive overall effect, no evidence of any significant transfer to the even-odd number concept was obtained.*

Following the excessive formalism of the new math, the tendency has been to develop a more constructivist approach to the teaching of mathematics. For quite some time, psychologists have been recommending it. Piaget's influence has been manifest in most of our teachers' recepe "go from concrete, to semi-concrete, to abstract" and Bruner's influence by his model of representation "move from enactive, to iconic, to symbolic". Much as these paradigms are valuable sources of insight, they prove to be too general to provide us with an operational model for the design and ordering of instructional material. While there is often a concensus on what is meant by "concrete" and "enactive", opinions differ widely on what is abstract and symbolic, especially between psychologists and mathematicians. Most mathematical concepts are far too complex to be analysed within the confines of these paradigms. What is needed is an analysis from the perspective of a "pedagogical" epistemology, that is an epistemology which does not limit itself to spontaneous phenomena, but takes into account pedagogical intervention.

Such an analysis has been achieved through the use of our model of understanding. Its four levels of understanding can be identified with four stages in a possible construction of a mathematical concept. In our previous paper we have described four stages in a construction of the even-odd number concept. Based on this analysis we have prepared four corresponding work sheets.

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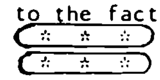
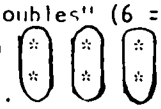
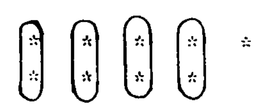
Having prepared these work sheets, a question of importance to us was whether teachers might order them in a teaching sequence similar to ours and if exposure to our model of understanding would affect their ordering. On the other hand, if their order differed, what was their underlying justification ? To find answers to these questions we selected a class of future teachers (n=46 reduced to n=40) enrolled in an undergraduate elementary education program at the Université de Montréal. During the Spring of 1984 these students attended 15 three-hour sessions (twice a week) in a mathematics education class in which our model of understanding was used to train them in the analysis of fundamental arithmetical notions (number, the four operations, place value notation, addition and subtraction algorithms). However, they did not analyze the even-odd number concept. They were asked to order the four work sheets in a teaching sequence both at the beginning and, seven weeks later, at the end of the course. We were hoping to detect some influence of their training in conceptual analysis on their ordering. To determine the effect of the test itself, a control group was sought. Since we needed teachers without prior exposure to our model but who also, at the time, would not be influenced by other mathematics education courses, we selected a group of students enrolled in a Bachelor's degree in Orthopédagogie (General Remediation) (n = 48 reduced to n = 32). They were asked twice, at five weeks interval, to order the four worksheets. A summary of these follows (copies of the originals will be available at the presentation).

Worksheet 1

This worksheet points out that using chips, some numbers (6) can be represented by two equal rows while others (9) cannot. Attention is first directed to the fact that those represented by two equal rows are "doubles" (6 = 2 x 3)  and then that they can also be viewed as "pairs"  . By contrast with other numbers like 9, one chip will be left over. This is how even and odd numbers are defined.



Worksheet 2

This worksheet raises the questions: Is 694 an even or odd number ? Is there an arithmetical operation which might enable us to decide ? In answering these questions, we introduce the operation of division ( $2\overline{)694}$ ) and justify it by relating the quotient to the possible number of pairs while focusing on the significance of the remainder (0 or 1).

Worksheet 3

This worksheet raises two important questions: Are there as many even numbers as

odd ones ? Are there natural numbers which are neither even nor odd ? Students are asked to separate into two columns the even and odd numbers between 1 and 30. The fact that odd and even numbers alternate in the natural number sequence is invoked to answer the first question. The fact that upon division by 2 the remainder can only be 0 or 1 answers the second question.

Worksheet 4

The student is asked to examine two columns of consecutive even and odd numbers respectively and to note a pattern in their last digit. He is then asked to use this last digit criterion to decide the parity of 463, 357 and 882, to verify it by dividing by 2 and examining the remainder. Questions are then raised regarding the generality and validity of this criterion: Will it be true for any number? If yes, how can we explain it? The explanation given for numbers ending by 0 is that they constitute tens which can always be divided into two equal parts. For the even numbers 2, 4, 6 or 8, by adding to them any number of tens, the result is always even. A similar reasoning can be used for numbers ending by 1, 3, 5, 7 or 9.

After assigning letters D, A, C, B to worksheets 1, 2, 3 and 4 respectively, and distributing them in random sequences, teachers were given 45 minutes to indicate in which order they would use them if they were to teach this concept to 4th grade pupils with a knowledge of division involving numbers with several digits. They were also asked to justify their ordering. The following table provides the number of subjects having chosen a particular order.

| Ordering of the four worksheets | Experimental group (n=40) | | Control group (n=32) | |
|---|---|---|---|---|
| | Pretest | Postest | Pretest | Postest |
| D C A B | 15 (.38) | 14 (.35) | 10 (.31) | 4 (.13) |
| D C B A | 6 (.15) | 4 (.10) | 6 (.19) | 13 (.41) |
| D B A C | 5 (.13) | 2 (.05) | 5 (.16) | 2 (.06) |
| D B C A | 5 (.13) | 3 (.08) | 1 (.03) | 5 (.16) |
| D A B C | 4 (.10) | 8 (.20) | 3 (.09) | 1 (.03) |
| D A C B | 4 (.10) | 5 (.13) | 4 (.13) | 4 (.13) |
| A C D B | 1 (.03) | - | - | - |
| A D C B | - | - | 2 | - |
| C D B A | - | 1 (.03) | 1 (.03) | - |
| C D A B | - | 1 (.03) | - | 1 (.03) |
| C B D A | - | 1 (.03) | - | - |
| C B A D | - | 1 (.03) | - | - |
| B A D C | - | - | - | 1 (.03) |
| A D B C | - | - | - | 1 (.03) |

Let us first analyze the results of our control group. We first notice that 29 out of 32 subjects (0.91) chose D as the initial worksheet in their ordering, thus indicating that they would start their teaching sequence from the intuitive level. The strength of this conviction is evidenced by the fact that the same high percentage (0.91) appears in the post-test. Reasons invoked by them are the "concreteness" of the situation, the possibility of manipulating objects, the visual nature of the definition. We also notice that hardly any of them, 4 out of 32, chose the same ordering as ours (D A C B).

An important issue which must be raised is the stability of these orderings. If we compare the pre-test to the post-test, we note that only 7 out of 32 subjects (0.22) in our control group repeated their initial order. If we limit ourselves to those who have chosen intuition (D) as their starting point and regroup them into three distinct classes according to their first two choices DC, DB, DA, it would appear that these classes are quite stable since the number of subjects choosing DC changes from 16 (10 + 6) to 17 (4 + 13), those opting for DB change from 6 (5 + 1) to 7 (2 + 5), and those selecting DA change from 7 (3 + 4) to 5 (1 + 4). However, this stability is only superficial since an examination of those subjects who remained in the same class from pre-test to post-test reveals that while 11 out of 16 (0.69) have stayed in DC, only 2 out of 6 (0.33) remained in DB and 2 out of 7 (0.29) in DA.

Thus a total of 15 subjects out of 32 (0.47) have remained within the same class between the pre-test and the post-test. Perhaps the instability of the other 17 subjects could be generally attributed to their lack of knowledge both in mathematics and mathematics education but more specifically to their lack of clear criteria which might have enabled them to be more consistent in their ordering. This lack of criteria becomes evident when we examine the justifications provided by the teachers for their sequence. Most of them were unable to formulate the reasons behind their ordering, except for their choice of D as their starting point. Only among those choosing DCAB or DCBA do we find a greater incidence of justifications as well as clearer criteria. Their choice of C as the second activity was explained in terms of numerical pattern recognition as well as the size of numbers (1 to 30). Since these were relatively small, activity C was perceived as a follow up of activity D. C was viewed as easier than either A or B since activity A involved larger numbers and the division algorithm, while B required reasoning based on a prior understanding of place value notation.

Analysis of the experimental group. We first notice that by choosing D as the initial activity (.98 in the pre-test and .90 in the post-test), the experimental group is comparable to the control group (.91). But the similarity also extends to

the distribution of their initial orderings. Continuing with the classes obtained using the first two choices, the experimental group splits up on the pre-test into 53% selecting DC, 26% opting for DB with 20% choosing DA, which does not differ greatly from the percentages found in the control group (50%, 19% and 22% respectively).

But before presenting a detailed analysis of our data, we should point out the results that might be expected following the two months of training of our experimental group: (1) we should expect the number of subjects choosing DC as their first two activities to be rather large and stable since, based on our results from our control group, the criteria for this ordering are quite clear in terms of "concreteness" and size of numbers; (2) a second hypothesis concerns the choice of DA as the beginning of the sequence. We thought that experience with our model of understanding would bring them to view arithmetic division by 2 as procedural understanding, hence justifying A in second position. This rationale should translate on the one hand, into an increased number of subjects choosing DA in the post-test, and on the other hand, a greater stability among the subjects having chosen DA in the pre-test.

(3) A third result that could be anticipated concerns those choosing DB as their initial ordering. Since activity B involves an informal mathematical proof, the training in conceptual analysis should have induced the experimental group to perceive this as formalization, the fourth level of understanding, thus reducing the number of subjects selecting DB in the post-test; (4) a fourth expectation deals with the relative order of activities A and B. A closer examination of B (worksheet 4) shows that students are asked to use the last digit criterion in deciding the parity of 463, 357 and 882 and to check this by dividing by 2 and looking at the remainder. Hence, activity B presumes that division by 2 has been related to the question of parity beforehand, making A (worksheet 2) a pre-requisite of B. Thus, finding B prior to A in any ordering (DCBA, DBAC, DBCA) seems somewhat contradictory and we find it difficult to explain this phenomenon or the fact that in our control group this incidence actually increased from the pre-test (6 + 5 + 1 = 12) to the post-test (13 + 2 + 5 = 20). Perhaps it could be due to a superficial reading of the worksheets leading these subjects to overlook the need for pre-requisites. Should this be the case, the experimental group can be expected to perform much better since pre-requisites have been a constant concern in their analyses of concepts.

Regarding our first hypothesis, the data bear out our expectation. The number of subjects in the experimental group choosing DC in the post-test has dropped from 21 to 18 but remains large making this class the dominant one (0.45). It is also relatively stable since 12 out of the 21 subjects (0.57) remain in the same class from pre-test to post-test and that 9 out of these 12 even provide identical sequences in both cases.

Regarding our second hypothesis, whereas the control group had shrunk from 7 (0.22) to 5 (0.16) the number of subjects in the experimental group choosing DA as their first two worksheets increased from 8 (0.20) to 13 (0.33). Those in this class at the post-test originated from within the class (4), from DC (5) and from DB (4) at the pre-test. Regarding the stability of this class, 4 out of the initial 8 (0.50) remained , while of the 4 subjects choosing another order in the post-test, 3 adopted a sequence starting with DC, another acceptable ordering.

The third result that we anticipated was also achieved. Indeed, while the number of control subjects choosing DB in the post-test (7) as their first two worksheets remained similar to the number of subjects in the pre-test (6), the tendency is definitely downward in the experimental group where the number of subjects drops from 10 in the pre-test to 5 in the post-test.

Our fourth expectation is also borne out by the results. While in the control group the number of subjects starting with D,but placing B before A in their ordering, increased from 12 to 20, the opposite trend is evidenced in the experimental group where these numbers have decreased from 16 (6 + 5 + 5) to 9 (4 + 2 + 3).

Conclusions. While the results we obtained with our experimental group were encouraging, they also proved to be somewhat disappointing. The two months spent in training our teachers in the analysis of basic arithmetic concepts had the intended effect as evidenced by the tendencies gathered from our data. However, too many of them still chose DC as the first two activities in the post-test, thereby indicating that in their judgment an enactive procedure such as making rectangular arrays (worksheet D) was sufficient to enlarge the even-odd property of specific numbers to classes of numbers. But this is asking the child to generalize inductively on the basis of the first 30 numbers, which would not be very convincing for larger numbers. Following D by A would provide the division algorithm with an application to larger numbers and assure a stronger inductive basis. Another cause of disappointment was the relatively large number of our experimental subjects who, after having rightly chosen D as their first activity, still put B before A in their ordering, thus overlooking the fact that the procedure in A was needed for the formalization in B.

This first experiment has made it clearer than ever before that different orderings (DCAB, DABC, DACB) could all be justified from a constructivist perspective. While for these three classes the control group dropped from 17 to 9, the trend was reversed with the experimental group increasing from 23 to 27. This, however, is but a minor increase which indicates that the expected transfer from the analysis of other mathematical concepts to the even-odd number notion was not achieved to any significant extent. This motivated us to investigate the possible effect of prior analysis of the concept in question. Results are reported in a companion paper by Nantais et al.

# THE SEQUENCING OF ACTIVITIES FOR THE TEACHING OF THE EVEN-ODD NUMBER NOTION BY PRIMARY SCHOOLTEACHERS FOLLOWING THEIR PRIOR ANALYSIS OF THIS CONCEPT

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*An analysis of the even-odd number concept has identified four stages in a possible construction of this notion. Four worksheets reflecting these four stages were prepared. The ordering of these worksheets into a teaching sequence by a group of prospective elementary shcool teachers and a group of practicing "orthopédagogues" is examined following their training in the analysis of arithmetical concepts including the notion of even-odd number. The results obtained are then compared with those of a first experiment reported in a prior companion paper.*

In a first companion paper we have described how our model of understanding enabled us to identify four levels of comprehension of the even-odd number concept, intuitive understanding, procedural understanding, abstraction and formalization. The second companion paper provided a summary of four worksheets, each one covering activities aimed at achieving a corresponding level of understanding. Their ordering in the above sequence would constitute a possible constructivist approach for the teaching of this notion. A question of importance to us was whether teachers trained in using our model of understanding to analyze arithmetical concepts might transfer this knowledge to the even-odd number concept. Such a transfer might be reflected in their ordering of the four worksheets.

In our second paper, we reported on the outcome of a first experiment with a class of prospective elementary schoolteachers and compared them with a control group. While the results we obtained with our experimental group were promising, in the sense that we managed to reverse some of the trends observed in the control group, nevertheless, we could not claim that any significant transfer had been achieved. The two months spent in training our subjects in the analysis of fundamental arithmetical concepts did not seem to be reflected in their ordering of the four activities. Such transfer proving to be too difficult, our investigation moved on to study the degree of acceptability of our analysis. Would prior exposure to our analysis of the even-odd number concept (an expanded version of the one in our first companion paper) affect the teachers' ordering of the four worksheets?

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To answer this question we selected two groups, one of prospective elementary teachers (n=43), the other,a group of practicing "orthopédagogues" assigned to handle remedial work at the primary level (n = 17). The training of the two groups was quite different. In the case of the prospective teachers, the training was identical to the one given to the subjects in our first experiment. Like them, they were enrolled in a mathematics education course in which, through general discussions, they were guided in their analysis of number, the four arithmetic operations, place value notation, the addition and subtraction algorithms. This second group of prospective teachers enjoyed a more leisurely paced program spread out over a 15-week period in the Fall of 1984, at the rate of 3 hours per week. On the other hand, our group of "orthopédagogues" consisted of participants in a 5-day in-service course organized at the request of a school board. Each day involved six hours of work,and all the concepts analyzed by the other group, excepting multiplication and division of small numbers, were dealt with in the first three days.

As a pre-test the four worksheets were distributed to both groups in random sequences at the beginning of the course. Moreover, the even-odd number concept was discussed in class and followed by the distribution of a 5-page analysis of this notion. Six weeks later, the four worksheets were again given to both groups and, as in the pre-test, teachers were asked to order them and justify their sequence. Worksheets D, A, C, B reflect the four levels of understanding, intuitive, procedural, abstraction and formalization, and these are described in a companion paper by Bergeron & Herscovics.

| Ordering of the four worksheets | Prospective teachers (n=43) | | "orthopédagogues" (n=17) | |
|---|---|---|---|---|
| | Pretest | Postest | Pretest | Postest |
| D C A B | 10 (.23) | 9 (.21) | 2 (.12) | 2 (.12) |
| D C B A | 10 (.23) | 7 (.16) | 7 (.41) | 5 (.29) |
| D B A C | 3 (.07) | - | 2 (.12) | 1 (.06) |
| D B C A | 1 (.02) | 2 (.05) | 1 (.06) | 2 (.12) |
| D A B C | 10 (.23) | 3 (.07) | 3 (.18) | 1 (.06) |
| D A C B | 7 (.16) | 21 (.49) | 1 (.06) | 4 (.24) |
| C D A B | 1 (.02) | - | - | 1 (.06) |
| C D B A | - | 1 (.02) | - | - |
| B D C A | 1 (.02) | - | - | - |
| B C A D | - | - | 1 (.06) | - |
| B C D A | - | - | - | 1 (.06) |

Analysis of the prospective teachers group. A comparison of these subjects with those of our first experiment indicates that in all cases the choice of worksheet D (intuition) as a first activity is dominant (0.95 at the pretest and 0.98 at the post-test). Where it differs, however, is in the distribution of the orderings in the pretest: if we consider the three classes determined by the first two choices DC, DB and DA, we find that on the pretest the number of teachers choosing DC is comparable to the numbers obtained in our first experiment (0.46 vs 0.53 and 0.50 for the exeperimental and control groups respectively), but that those choosing DB are much fewer (0.09 vs 0.26 and 0.19) while those selecting DA are far more numerous (0.39 vs 0.20 and 0.22). In any conclusions we may reach, such initial differences must be taken into account.

A first observation concerns the attraction of the DC choice. That the proportion of prospective teachers choosing this option at the post-test remains high (.37) is quite surprising in view of their extensive training in conceptual analysis which included the even-odd number concept. As in our first experiment, a likely explanation would be the size of the numbers chosen for activity C which were relatively small. The strength of the DC ordering is evidenced by the fact that 9 out of the 20 teachers who had made this choice in the pretest remained with it in the post-test. Of the 16 subjects in this class at the post-test, 13 made no reference whatsoever to their course work in the justification they provided for their ordering. This is in sharp contrast with the subjects in the DA class. Out of these 24 teachers, 17 made explicit reference to levels of understanding.

Regarding the number of teachers who chose DA in the post-test, the proportion seems very impressive (.56). However, if we take into consideration the fact that even in the pretest 17 teachers had made this choice (.39), the increase of 17% is comparable to the increase of 13% obtained in our first experiment. Thus, it seems that a prior analysis of the even-odd number concept had no major effect on the choice of DA as the first two activities. However, while in our first experiment only 4 out of the initial 8 subjects having chosen DA in the pretest had remained in DA for the post-test, in our present group a strikingly high number of teachers had done so (14 out of 17). Another remarkable difference can be seen in the proportion of teachers having chosen DABC and DACB in the post-test. Whereas in the first experiment the percentages for these two classes were .20 and .13 respectively, in the present group they are .07 and .49 which is also a reversal of the percentages obtained in the pretest (.23 and .16). Since both these groups received the same training, it is difficult to assess the possible reason for these differences. Perhaps they are due to the more leisurely pace that was possible with the present group (3 hours weekly for 15 weeks vs 6 hours weekly for 7 weeks).

Finally, with regards to the teachers' perception of activity A (the division algorithm) being a pre-requisite for activity B (formalization), the number of subjects in the present group ignoring it in their ordering has dropped from 14 in the pretest (10 + 3 + 1) to 9 at the post-test (7 + 0 + 2) which is comparable to the results in our first experiment (16 reduced to 9).

Analysis of the "orthopédagogues" group. Even if their number is small, what is interesting is that on the basis of the pretest, this last group seems to be close to the two groups in our first experiment. Not only did they choose D overwhelmingly as the first activity (.94 in the pretest and .88 in the post-test) but on the pretest they distribute themselves similarly into the three classes: 53% selected DC, compared to 53% for the first experimental group and 50% for the control group; 18% chose DB as compared with 26% and 19% respectively, while 24% opted for DA, compared to 20% and 22% in the other two groups.

The strength of the DC option is again evidenced in the post-test. Not only does this class remain the largest one, but moreover, 4 out of the 9 teachers choosing this order in the pretest remain with it in the post-test, giving us a ratio of .44 which compares to the .45 obtained with the prospective teachers. Regarding the number of subjects in each of the other two classes DB and DA, their small size prevents us from making any valuable inference. Nevertheless, we seem to have had little effect on this group. Perhaps this is due to the very condensed training they received, that is 30 hours in 5 days.

## CONCLUSION

As pointed out earlier, our lack of success in inducing an expected transfer in our first experiment prompted us to investigate the degree of acceptability of our analysis of the even-odd number concept. Would teachers trained in the analysis of arithmetical conceptual schemes, including this notion of parity, order the worksheets in the sequence we had suggested ? If we judge this by the number of subjects choosing, on the post-test, DA as the first two activities, the results are ambivalent, a majority of the prospective teachers (56%) opting for this sequence while the number of "orthopédagogues" is merely 5 out of 17. However, this can hardly be attributed to our training, since 39% of the prospective teachers had already chosen this class in the pretest. Nevertheless, their training in conceptual analysis enabled them to provide clearer and more explicit justifications for their ordering. Another effect we can detect within this group is their perception of activity B as one of formalization which is the fourth level of understanding in our model. This is evidenced by the decrease of DABC (from 10 to 3) to the benefit of DACB (from 7 to 21).

A further observation related to the question of acceptability of our analysis is the re-occurring importance of the DC class. An examination of the four groups of teachers in this study reveals that in each one of them this particular class is the largest one at the pretest (.53, .50, .46, .53). In the post-test, this class increases in the control group (to .54), it remains the dominant class for both the first experimental group and the "orthopédagogues" group (.45 and .41 respectively), while remaining large and important with the prospective teachers (.37). Moreover, we have also shown that the DC class was quite stable in the sense of the same subjects being likely to select these first two activities in the pretest as well as in the post-test. On the basis of both numbers and stability, it seems that our training had little impact on this group which seemed to continue favoring its initial choice.

Whatever the logic of our suggested construction, the degree of acceptability by the teachers will vary depending on whether they have other criteria of their own on which to base their ordering and the strength of their conviction. For instance, the relative size of the numbers appearing on the worksheets played a definite part in our subjects' ordering. In each group, at least two or three teachers explicitly justified following up D by C on the basis of the relatively small numbers used in C. The size of numbers used was also invoked by several of these teachers in explaining why they had left A to be the third or fourth activity, since the numbers involved here were 3-digit numbers. But of course, this was not the only reason. Many teachers left A for the last position because they felt long hand division by 2 was the most symbolic form of activity and that it could be used to "prove" the parity of a number. This rationale is quite surprising since our instructions specified that they could assume that these activities were aimed at 4th grade pupils who already had a knowledge of the division algorithm and also, since worksheet B clearly implied that worksheet A was a pre-requisite.

An analysis of the different sequences selected by the teachers reveals that, in addition to the one we proposed, two others can also be considered as constructivist. Indeed, if we consider DCAB, we must agree that the action-scheme in D (using two rows of chips) is adequate to determine the parity of small numbers and that it can be used for activity C (discovering the existence of two disjoint classes on the basis of the first 30 numbers). In fact, no harm results from leaving A (the division algorithm) until later, as long as it precedes activity B. Such a sequence has the advantage of bringing the child to a certain level of abstraction of the even-odd number concept quite early, even before being introduced to the division algorithm needed for larger numbers. This ordering would thus reflect a spiral approach to the teaching of this notion. The other sequence, DABC, which is also constructivist, is fairly close to our own (DACB). Worksheet B still looks at an efficient way of determining if a number is even or odd and this can be handled prior to the explicit recognition of two disjoint classes.

On the other hand, various sequences cannot be considered as constructivist. For instance, any sequence not beginning with D avoids using the child's informal knowledge as a starting point. This is tantamount to requiring him to learn new material without the opportunity of relating it to his existing cognition and prior experience. From a constructivist viewpoint, both DBAC and DBCA must be rejected since B here is nothing but a "mathematical trick" which cannot be related to D at this point. Using the last digit to recognize the parity of a number is a way of using the "form" in which it is expressed in our decimal notational system and as such, worksheet B constitutes a piece of "formal" mathematics. It is its premature introduction that we consider as an example of a "formalist" approach to teaching.

As pointed out in an earlier paper (Herscovics & Bergeron,1984), since a formalist approach starts from a given mathematical notion which then needs to be related to the student's cognition, for the learner, there is inevitably at the outset a cognitive discontinuity between what he knows and what has to be learned. Thus, a premature introduction of worksheet B necessarily creates a cognitive gap for the pupil. By contrast, a student subjected to what we consider as constructivist sequences does not have to face such cognitive discontinuity for, by starting from his existing knowledge he can rely on it to climb the various steps in the construction of the concept. This illustrates how, for a student in such a learning situation, each step is an extension of his accrued knowledge, thereby endowing the learning process with continuity.

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# Towards a Framework for Constructivist Instruction

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Abstract: The article documents the difficulties students encounter in learning mathematics and classifies them into narrow and positivist conceptions of mathematics and unexamined assumptions about how mathematics is learned. She calls for a radical restructuring of mathematics classrooms using constructivist theory and proposes such a framework for it using the concepts of reflection, communication/ interpretation and the use of resources.

Introduction: The difficulties in learning mathematics are widely documented; successful practices which overcome those difficulties are rare. Since research findings seldom translate directly into actions, I suggest that in much of the research the lack of theory (which would allow the interpretation of the difficulties within a single framework) impedes the development of successful practices. In my opinion, constructivism is a candidate for such a theory. Its selection, however, requires the development of a radically different model of teaching and learning mathematics. In this paper, I will discuss constructivism and offer one framework for instruction which is compatible with it. In doing so, I will rely heavily on my experience developing and directing the SummerMath program at Mount Holyoke College, a program with explicitly stated constructivist underpinings. Thus, what will be presented herein is in part a philosophical argument and in part a discussion of the lessons we drew from our experiences in that program.

The problems in mathematics education can be described as follows:

1. *Students' mathematical knowledge is limited and rigid*. They focus on answers; they expect whole number solutions; they lack multiple representations; they rely on memorization and imitation of examples; and their powers of generalization, abstraction, curtailment and flexibility are weak. They believe mathematics is objective, external and absolute. (See Confrey, 1981; Confrey and Lanier, 1980.)

2. *Mathematics is classified as formal knowledge, and hence is isolated from common experience and common sense-making*. Studies of students' misconceptions in mathematics and science emphasize that students differentiate school learning from

everyday experience. When these two spheres conflict, we find that the student often sees no such tension and hence sees no need of resolution. The result is sometimes an isolation of the two spheres and sometimes a rather capricious interaction. Furthermore, we find that students experience serious difficulties in applying what they have learned in mathematics, and their beliefs in the objectivity of numbers allow them to ignore fundamental questions of value in quantitative issues. (See Ginsberg, 1980; Clement, 1982.)

3. *Students rely on external sources of authority to evaluate their competence in mathematics.* From student interviews and journals, Confrey (1984) described how students appeal to an external authority, a teacher or a book, and doubt their own ability to assess their performance; how students felt intimidated by the pace demands of the mathematics classroom; and how students had negative reactions to the public exposure of the evaluation system. In all of these instances, we see evidence of students relinquishing control of their own learning, and accepting a system in which such control is external.

4. *Students are alienated from mathematics and experience fear, avoidance, anger and apathy towards it.* The affective consequences of the alienation of students from mathematics are experienced widely. In the general population, even very bright and influential people openly discuss their negative feelings towards mathematics. In significantly large disadvantaged groups, minorities (except Pacific-Asian) and women, this alienation is heightened. Mathematics students hold very little conviction about their ideas, and there is a tendency to waiver in their answers when asked to explain their beliefs.

I believe that these descriptions of the difficulties in mathematics learning can be classified even further into two categories:

1. Student's conceptions of mathematics
2. Student's beliefs about how mathematics is learned.

Changing students' beliefs and experiences in these two areas requires, in my opinion, a radical restructuring of the mathematics classroom. Such a revision can be undertaken more easily with a guiding theory and in the next section, I will offer constructivism as such a theory. In the last section, I will discuss a framework for teaching which I believe is compatible with constructivism, and provide examples of how it might alter some of our traditional perspectives on mathematics education.

Constructivism: The theory of constructivism is a theory about the limits of knowledge. Knowledge consists of that which we can know. What is knowable is necessarily a product of our own mental acts or constructions. Thus, as I understand it, a fundamental tenet of constructivism can be stated as follows:

> A person's knowledge is necessarily the product of her/his own constructions, or mental acts. Thus, s/he can have no direct or unmediated knowledge of any objective reality. From sensary experiences to meditative reflections, the intellect, emotions, memories and expectations mediate experience of the world. If a person chooses to believe in an objective reality, that belief is an act of faith, another sort of construction. Knowledge consists of a mental action on the world; it is a way of seeing, of organizing experience.

The implications of such a theory may be made clearer by an example. In introducing my son to the idea of symmetry, I showed him a fern and folded it in half to show how the two sides "landed on top of each other". Doing so entailed an action; in this case, a physical action. (When I am alone, it is an imaginative action.) Then I began to speak about the axis of symmetry, pointing it out as though it were "in" the plant. This was perplexing to him, and rightly so. The axis of symmetry is a label for a mental construct involved in the action of folding the plant, physically or mentally. It is no more present in the plant than is the number three in a set of three oranges. Mathematical ideas are essentially mental constructs which we use to view the world, yet we often speak as if they are "out there".

Once we have gone through the construction enough times to "recognize" it with ease, we act and speak as if the construction has become a part or quality of the object. In answer to the question, "Where is the axis of symmetry?", we answer, "It's there, down the stem of the plant": the construct is described as being in the object. This process of positing the perceptions and cognitions we make in objects leads us quickly to a belief that we are not constructing, but describing the world as it is.

Not only do we speak as though we were engaged in description, but we further tend to talk as though the language encapsulates a precise meaning for the word, and thus fail to remember that it is also a representation; it exerts its own force on its own construct. Vinner (1983) reminded us of this when he wrote of the significance

examining the history of the discipline. Culture plays a significant role in its evolution (Wilder (1968); Toeplitz (1963). History provides the impetus for a first revision in mathematics education: *the portrayal of knowledge as an evolving and competing set of constructions by human beings.*

Criticisms of constructivism often center on its susceptibility to relativism. Critics claim that if there is no appeal to *certain* knowledge, then everyone's knowledge must be equally valid. Such criticism ignores a fundamental aspect of constructivism; the role of others in the construction process. Others influence all aspects of construction. A construction is a representation of an object; most often its purpose is in its potential to aid in communication with others. Thus, in our language, in our symbol systems, in our choice of problems, our level of rigor, our standards of proof, the community of others with whom we orient ourselves plays a fundamental role in construction. Just at the process of reflection serves to provide feedback to us on our individual constructions, the act of communicating and interpreting the acts of others is another source of feedback. Thus communication/interpretation is used as the second organizing concept of the framework. (See von Glaserfeld, 1983, 1984, a, b)

It must be stressed that as a constructivist, there can be no *certain* knowledge that another person's construct is identical to one's own. By speaking as if the constructs are given in the objects, we are lulled into believing that they must be objectively given and that therefore the duplication of the construct for different people is assured. A constructivist sees such assurance as problematic. S/He believes that much of our intellectual lives are spent creating evidence to judge if another's constructs correspond to our own. To do this in mathematics, we create definitions on which we think we can agree and which seem to capture essential qualities, we develop logical methods of argument ot express our seemingly common chains of reasoning, and we identify instances of our claims and see if our stated conclusions sound the same. This process of *negotiation of meaning* is also an essential part of the constructive process, and it provides us a second source of feedback.

If students' do not understand the constructive process, it is likely they will produce weak constructions. Many of the difficulties described at the outset can be interpreted as ample evidence of weak constructions. Thus, a second revision proposed by a constructivist is *to aid students in learning to create more powerful constructions*. Doing this requires the devotion of considerable time to the

of "concept images" in the meaning for "function". For the most part, students' experiences with mathematics lead them to see it as Platonists, as though they are learning the definitions and properties of objects which exist in some ideal way independent of human beings.

I don't want to suggest that the process of objectifying the mental act is insignificant. In fact, quite the contrary. I believe that in mathematics, we objectify our cognitions for the purpose of creating a bootstrap so to speak. Through that process of objectification, we use the construction we've just created as the next position from which we can look about, assessing its potential, and freeing up space for new constructions. This looping back on our knowledge is an essential process for providing us feedback in our intellectual system. This process of objectifying knowledge (previous constructions) and then assessing their potential is what reflection is all about. Relfection is the first organizing concept in the framework for teaching presented in the next section.

Presenting new knowledge to students in "objectified" form, exemplified by lecture or direct instruction, is fraught with perils according to constructivism. Firstly, students are denied the opportunity to construct the ideas in their first encounters. Even if the context is ripe for a construction, the demands of pace alone mediate against it. Secondly, because students are unlikely to be constructivists themselves after years in our system of schooling, it is unlikely that they would even consider constructing. Thus, hearing an "objective" presentation simply reinforces thier beliefs that mathematical knowledge is certain, everlasting, and inhuman. Mastery to them is a function of learning to perform the appropriate behaviors; their personal experiences are seen as irrelevant.

A constructivist, believing that *certain* knowledge is unattainable, is free to concentrate on how to account for the relative stability of some knowledge and instability of others. A glance at the history of the discipline quickly reveals its controversial and multifacted development. One sees competing concepts, multiple symbol systems and a geneology of problems (Toulmin, 1972) that provides the sort of richness to the discipline that would compel even the most reticient of students. Powerful constructions are those which unite a variety of ways of thinking; which yield to multiple representations, which can account for a variety of phenomena and which are convincing to the reigning group of scholars. (For a discussion of powerful constructions, see Confrey, 1985.) Another fundamental lesson is gained by

processes of construction, reflection and the negotiation of meaning. In order to accomplish this, it is suggested that students must learn to be more effective in the *use of resources*. The effective use of resources is the third and final concept in the framework for teaching.

A Framework for Constructivist Instruction: In summary, I have suggested two fundamental revisions: the portrayal of mathematical knowledge as an evolving and competing set of human constructions and the instruction of students in how to create more powerful constructions. I have developed three guiding concepts to use in these revisions: REFLECTION, INTERPRETATION/COMMUNICATION AND THE USE OF RESOURCES. That is, I believe that effective mathematics instruction will develop if the instructional system is designed to promote reflection by individual students on the mathematical concepts, to allow for interaction among students in pairs and small groups stressing the importance and complexity of communicating to others, and to deemphasize mindless performance of role algorithms by providing rich resources and teaching students their effective use.

Such revisions require a dramatic new configuration of the role of the teacher.
[1]See Confrey and Upchurch (1985) for an examination of the instructional practices of a constructivist instructor. No longer will s/he be the ultimate authority in the classroom, for personal autonomy is the backbone of the construction of meaning. The teaching role will require the gathering of evidence on individual students' understanding, the development of a rich environment for learning and the careful organization of the classroom into groups that work effectively. Experience in clinical interviewing, ease and flexibility with the content, the facilitation of group interactions and modelling of problem solving will provide the most effective educational background. The teacher will become more intellectually active as s/he seeks to allow for the variations in approaches which will result from freeing students from the dictates of prescribed mathematics.

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# REPRESENTATION AND DEDUCTION

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Abstract: *Re*-presentation, designating a *re-play* of conceptual structures, is contrasted with the popular idea of mental images of external things. The author claims that all deductive procedures require *re-play* of sensory-motor content as well as operational routines and demonstrates this in the classical syllogism and an example of simple addition. He concludes that the generation of deductive abilities in both logic and mathematics must be based on the practise of inductive inference.

> ...for the roses had the look
> of flowers that are looked at.
> *T.S.Eliot*[1]

The epigraph was chosen from a poet rather than from a mathematician or scientist, to suggest from the outset that my paper is intended as a contemplative stroll through theory and not as a report on empirical research. We are all concerned with teaching, and poets, I'm sure you will agree, have had some success in that area. They may have achieved it unintentionally, but their lack of didactic ambition does not seem to have impeded the learning of those who *wanted* to learn, nor has it diminished their readers *desire to learn*. I have no statistical evidence for these assertions, but I nevertheless intend to push further in the direction they indicate.

Poets know, perhaps better than others, that readers or listeners cannot be given ready-made thoughts, images, and ideas. They can only be given words. Being given words, however, they will inevitably bring forth thoughts and images of their own; and by presenting particular combinations of words, one can, at least to a modest extent, guide the conceptual construction of the *meaning* which, eventually, the readers or listeners will believe to have found in the text.

Poets also know (only too well) that it is no use trying to tell a listener or reader that his or her interpretation is "wrong". Paul Valéry said:

> Once published, a text is like an appliance of which anyone can make use the way he likes and according to his means; it is not sure that the builder could use it better than others. Besides, he knows well what he wanted to make, and that knowledge always interferes with his perception of what he has made.[2]

Since teachers of mathematics, as a rule, *know* well what they are explaining, that knowledge invariably interferes with their own perception of their explanation. Consciously or unconsciously they take for granted that certain things are "self-evident", and they forget that what seems evident in mathematics is always contingent upon the habit of performing specific mathematical operations.

As seasoned users of language, we all tend to develop an unwarranted faith in the efficacy of linguistic communication. We act as though it could be taken for granted that the words we utter will automatically call forth in the listener the particular concepts and relations we intend to "express". We tend to illude ourselves that speech "conveys" ideas or mental representations. But words, be they spoken or written, do not *convey* anything. They can only call forth what is already there. This should become clear every time we test the *representations* our words have called forth in a listener; but we don't *see* it because of our unwarranted presuppositions concerning the process of "communication".

One misapprehension stems from the general notion of "representation". As that term is used in psychology and cognitive development, it is ambiguous in more than one way. First, like many words ending in "-ion", "representation" can indicate either an activity or its result. This ambiguity rarely creates difficulties. Far more serious is the *epistemological* ambiguity to which the word gives rise. It creates an unwholesome conceptual confusion.

The distinction I want to make clear concerns two concepts which, for instance in German, are expressed by two words, *Darstellung* and *Vorstellung*; both are usually rendered in English by "representation". The first designates an item that corresponds in an iconic sense to another item, an "original" to which it *refers*. The second designates a conceptual construct that has no explicit reference to something else of which it could be considered a replica or picture. (In fact, *Vorstellung* would be better translated into English as "idea" or "conception".)

Thus, if one uses the word in the second sense, it would help to spell it "*re*-presentation". The hyphenated "re" could be taken to indicate repetition of something one has experienced before. This would lessen the illusion that mental *re*-presentations are replicas or images of objects in some "real" world. It would help to focus attention on the fact that what one *re*-presents to oneself is never an independent external entity but rather the *re-pl*ay of a conceptual item one has derived from experience by means of some sort of abstraction.[3]

The ability to *re*-present to oneself prior constructs is an essential part of all cognitive activities. It

comes into play when you ask yourself whether the soufflé you are eating now is really as exquisite as the one you had in Dijon twenty years ago. Whenever you compare sensory experiences, and one of them is not in your actual perceptual field, that experience must be *re*-presented. As one gets older, one realizes that the memory from which one *re-plays* such past sensory experiences cannot always be trusted. Soufflés (and other sinful experiences) from one's distant past tend to seem sweeter than present ones. But this nostalgic tendency is not what I want to discuss here.

The ability to *re*-present is just as crucial in the use of symbols. The so-called "semantic nexus" that ties a symbol to what it is supposed to stand for, ceases to function when the symbol user is not able to *re*-present the symbol's *meaning*. Memory, clearly, plays no less a part in the symbolic domain than in that of sensory experience.

Irrespective of the particular position you may have adopted concerning the foundations of mathematics, you will all agree that symbols such as "+", "-", "x", and ":" refer to operations and can, in fact, be interpreted as imperatives (*add!, subtract!, multiply!, divide!*). To obey any such imperative, one must not only "know" the operation it refers to, but also how to carry it out; one must know how to *re-pl*ay the symbolized operation with whatever material happens to be at hand. That is to say, if operator-signs are to function as *symbols*, the operations to which they refer must have been abstracted by the symbol user from the sensory-motor material with which they were implemented in that symbol user's own prior experience.

The semantic nexus between an operator-symbol and the abstracted operation it designates is no less indispensable in logic than it is in mathematics. Quine speaks of "the inseparability of the truths of logic from the meanings of the logical vocabulary".[4] Logical truth, of course, refers to the reliability of deductive inferences that can be derived from the chosen premises; it does not pertain to the experiential foundation of either premises or conclusions. If a syllogism were formed with the premises "All socialists are evil" and "Snoopy is a socialist" it would as *logically* lead to the conclusion that "Snoopy is evil" as the traditional syllogism leads to the conclusion that Socrates is mortal. The logical "truth" of a deduction is not impaired by the experiential falseness of the premises.

Although the logic of the syllogism is in no way tied to what seems likely or unlikely in the thinker's experiential world, *following the rules* and *carrying out the operations* that are called forth by the use of words be-

longing to the "logical vocabulary" is nevertheless an activity and, as such, requires an active, thinking agent. Hume saw this, and concluded that deduction, because it involved a psychological process, could not be as infallible as classical logicians like to believe.[5] If this introduction of doubt were legitimate, doubt would eventually infest also the realm of mathematical operations. To discuss it may therefore not be an idle exercise -- especially if, as I believe, Hume's notion can be tied to the theory of *re*-presentation.

As far as deductive logic is concerned, what the premises say should always be explicitly *posited* rather than taken as statements of fact. Their relation to the experiential world is irrelevant. What matters is that they be taken *as though* they were unquestionable, as hypotheses which one accepts for the time being, and that their hypothetical status should always be carried over to the conclusion. In other words, we should always explicitly say:

*If all* men are mortal,
and *If* Socrates is a man,
*Then* Socrates is mortal.

This emphasizes two things: first, that one is dealing with assumptions whose experiential validity one has decided not to question for the moment; and, second, that the logical certainty one attributes to the conclusion pertains to the operations that are called forth by the *logical terms* "if", "all", "and", and "then". These two aspects are the basis of our faith in the infallibility of deductive procedures.

John Stuart Mill, in an attempt to subvert faith in the syllogism, argued that, in order truthfully to formulate the premise that *all* men are mortal, one should have to examine *all* members of the class called "men" with respect to their mortality. If, having done this, no exception to the rule had been found, one would know that Socrates is mortal, because, being a man, he must have been tested for mortality. If, on the other hand, he had not been tested, this could only mean that either he was not considered a "man", or that the use of the term "all" in the premise is unwarranted. This is a neat argument, but it shows that Mill did not see the premises of the syllogism as deliberate assumptions but as statements of experiential fact. Once this is understood, the argument no longer goes against the syllogism but against the misconception that deductive inferences should automatically be "true" in the experiential domain.

There may, however, be other problems. If the premises of syllogisms are understood as deliberately hypothetical conceptual structures (which one agrees not to question), one may still want to examine the deductive procedure, a

procedure that involves several steps. Having constructed the premises, one must call up the logical operations designated by the tokens of the logical vocabulary and *re-play* these operations with the *re*-presentations of the premised conceptual structures. That is to say, in order to come to a conclusion, the conceptual construct created for the major premise must have been maintained unchanged, at least long enough to be available for *re*-presentation when one has created the conceptual construct for the minor premise and is ready to proceed with the logical operations that relate the two premises so as to produce the conclusion.

Whether or not one believes with Kant that the deductive operations called forth by *logical terms* are part of the inherent, a priori repertoire of the human mind, it seems plausible that, rather than being created each time anew, they are *re-played*, much like preprogrammed subroutines, when the associated symbol or sequence of symbols gains the agent's attention in an appropriate context.[6] If this is the case, some form of memory would be required for the performing of logical operations, and since memory would have to be considered a psychological phenomenon, one might be tempted to invoke Hume's doubt.

The question of memory arises even more clearly in connection with the hypothetical conceptual structures that are generated in response to the not specifically *logical* components of the premises, i.e., the hypothetical conceptual structures to which the logical operations must be applied. All deductive procedures require that we trust our ability to maintain, and *re*-present as they were, the conceptual structures and the operational routines we intend to use. If we doubt this ability, all logic goes by the board. We are not inclined that way. It would be as disruptive as doubting the reliability of memory and all the other electronic devices in a computer.

However, we may still question *how* we acquire logical operations. Professional philosophers usually dismiss any consideration of the developmental aspects of thought as "genetic fallacy" and pretend that logicians and other users of logical operations do not have to construct the required procedures but have them ready-made in their minds even if they do not always use them. Like Piaget, I find this an absurd contention. Instead, I would suggest that it is precisely the experiential success of inductively derived rules that provides both the occasions and the motivation for the abstraction of the specific logical operations that are then associated with symbols and used *without* reference to experience.

From that perspective, it seems clear that, in the construction of the syllogistic procedure, the components of the premises that are not the specifically *logical* terms

must be interpretable by the active agent in a way that makes sense in the context of that agent's experience. It seems likely that we come to make the necessary reflective abstractions when we apply rules that work, rather than rules that are countermanded by experience. If we have never formulated a tentative rule of the kind "all roses I have seen, smelled sweet", we would not be tempted to say: "this flower looks like a rose -- therefore it will smell sweet." In other words, if we have had no success with inductive inferences, we are unlikely to proceed to deductive ones.

To conclude, let me try to apply this line of thought to the basic understanding of numbers and how they interact. A child can no doubt learn by heart expressions such as "5+8 = 13". However, in order to *understand* them, she must be able to *re*-present the meanings of the involved symbols. As in the syllogism, the parts of such numerical expressions involve assumptions. "5" *means* that one assumes a plurality of countable items which, *if they were counted* (i.e. if number words were coordinated with them one-to-one), they would use up the number words from "one" to "five". The "+", then, signifies that a second plurality of items which, by itself, would use up the number words from "one" to "eight", is to be counted with the number words that follow upon "five".[7] Children may *re*-present these pluralities and the counting activity in many different ways. The sensory-motor material they use to implement the abstracted patterns is irrelevant. What matters is that they have abstracted these patterns and can *re-pl*ay them in whatever context they might be needed. For I would claim that only if they have acquired a solid facility in the generation of this kind of *re*-presentation can they possibly enter into the garden of mathematical delights.

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# Educational and psychological considerations in the development of synthetic geometry curriculum for Israeli low ability students

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Abstract

Synthetic geometry is taught in Israeli junior high schools to low ability students by a curriculum which is based on theory and research in cognitive psychology. The paper describes the curriculum development and the ways in which it accomodates the instruction to teachers' mathematical background, students' prior declarative and procedural knowledge, and the cognitive requirements of the tasks. Illustrations are provided for the major undertakings of the curriculum.

## I The background

The instruction of geometry in the Israeli curriculum consists of two cycles. In the first (grades K to 7)--informal (propodeutic) geometry is taught, which consists of perceptual analysis and description of geometrical concepts and figures, and in the second (grades 8 to 10)--synthetic (Euclidean) geometry is taught. Synthetic geometry was never removed from the curriculum in the new mathematics reform for two major reasons. Mathematically, synthetic geometry is the only deductive system that can be taught at the level of junior high school. Educationally, it was suspected here that Euclid was expelled from the new mathematics curriculum partly because teachers found it very difficult to teach and students--to learn. M. Maschler, a pioneer of the new mathematics in Israel, strongly insisted that the failure to teach synthetic geometry is a result of insufficient understanding of how learning occurs, which should not mislead mathematics educators to conclude that it is a mission impossible.

In the Jerusalem junior high school mathematics curriculum synthetic geometry is taught to students of all ability levels: A (high), B (middle) and C (low). At the outset it was thought that students in the higher groups (A and B) can learn from available textbooks, and a special textbook was written for the group C. After three years it was realized that (a) no problems have arisen with A group, (b) the regular textbooks proved too difficult for students in group B, and (c) the textbook for group C proved too difficult and not addressing most of the cognitive deficiencies characterizing students in group C.

To solve these problems it was decided that students in group B will learn the textbook which was originally written for group C, and a new textbook will be written to meet the special needs of the students in group C. I was appointed to this task and worked on it in collaboration with M. Maschler.

My first task was to identify the most serious cognitive deficiencies of the students in group C. To this end several activities were conducted. A biweekly seminar was organized, which was comprised of several school mathematics teachers, university mathematics teachers and cognitive educational psychologists. In this seminar, mathematical and cognitive analyses were made of the central geometrical concepts, based on our teaching experience and visits to classrooms. Then a series of tests designed to reveal the precise nature of the difficulties were administered to students in group C. Then, original, more appropriate, ideas were proposed the seminar to teach some central concepts and trying them out in several classrooms. These ideas were inspired by the works of Polya (1957, 1962), Dienes (1960, 1963), Skemp (1971), and Newell and Simon (1972).

This extensive activity yielded certain information about the students and their teachers which were lacking untill then. The student characteristics are as follows: They suffered continuous failure in all school subjects, had many behavioral problems, and had remedial classes experience and treatment by school psychologists or counsellor; they had unfavorable self image and poor self concept, self reliance, and

motivation towards learning and lacked work habits; they had low IQ (65-90), deficient organization of long and short term memory, and were able to remember (recall or recognize) surprisingly small amounts of information; they had very short (1-3 minutes) attention span, poor visual pattern recognition and reasoning (induction, deduction, analogy) skills, poor ability to recall and apply procedures in general and mathematical algorithms in particular, and poor imaginative ability. These students were not only reluctant to learn but also difficult to be taught.

In addition to these general features, these students have four characteristics that pertain to learning to prove theorems:

1. It is hard to convince them of the necessity to prove intuitively or percetually "obvious" facts and features.
2. Since they have encountered no formal proof in any school subject before, for them justifications are based on either self experience or authority.
3. They have had no experience with either axiomatic or deductive systems.
4. It is almost impossible for them to reason systematically and logically, and they are unable to think in divergent, convergent, deductive and inductive modes according to either few or simple principles.

Many of the teachers in group C lack not only the needed mathematical background, but also the knowledge thay are supposed to teach. This is the result of their being elementary teachers (who usually get little or no mathematical education) who were administrtively transferred to junior high school as part of the reform of Israeli schools. These teachers are unable to observe, diagnose and understand mathematics learning and to identify the cognitive difficulties of their students; they view their students as uninstructible, especially in mathematics. Many of them are rather reluctant to teach mathematics, and are satisfied to keep the students in the classroom rather on the streets.

## II The educational principles

The geometry curriculum in general and for group C in particular was guided by the following educational principles.

1. Ability grouping is desirable, since it is easier and more efficient to teach students in homogeneous groups.
2. Intra-group mobility should be possible. Thus, all ability groups must have a core curriculum, and all groups should progress at approximately the same rate, which is determined by the curriculum.
3. Process is of equal importance as product. Thus, procedural knowledge should be emphasized by explicating and exemplifying every minute detail of the procedures.
4. Individual differences persist even in homogeneous groups, and instruction should accomodate for these individual diferences. Hence, the textbook should be in the form of a workbook, the instruction should consist of a rather short direct teacher presentation of the lesson material, and followed by individual guided practice of the taught content which appears in the workbook.
5. To provide for the teachers' weak mathematical background, teacher guide should precede every lesson, explains the purposes and didactical rationale for the lesson, and includes solutions to all problems.
6. To emphasize the central issues in each subject and help the teachers to identify the most serious difficulties, sample test items should be included in the guide.

## III The curriculum

In grade 7 students learn basic geometrical concepts both general and specific (e.g., straight line, triangle, bisector, midpoint) (book 1). In grade 8 students learn (1) the concept of triangle congruence, congruence theorem s.a.s., isosceles triangle and rhombus (book 2); (2) congruence theorems a.s.a. and s.s.s., triangle construction, distances between two points and between a point and a line, the

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application of congruence theorems in complex situations, the concepts of direct and converse theorems (through application to perpendicular bisector), proof by recuctio ad absurdum, and the concept of parallel lines (book 3). In grade 9 students learn (1) theorems of parallel lines, sum of the angles of triangle and polygon, right triangle, external angle of a tringle, square, retangle, parallelogram, trapezoid and the mutual relationships among these quadrilaterals (book 4); (2) congruence theorem s.s.a., cirle and its features, inscribed and circumscribed circles in triangle and rectangle, loci (e.g., bisector of an angle, arc) (book 5); (3) proportion and triangle similarity (book 6).

This curriculum is taught in Israeli junior high schools since 1974.

## IV Teaching concepts

The finding that students in group C do not retain the correct meaning of many geometrical concepts probably reflects their improper learning strategies or inadequate information processing skills. Therefore, it was impossible to start teaching these students the usual curriculum, and about 25 hours are devoted to teaching them the basic concepts and the processing skills needed for their acquisition. Since skills cannot be developed in vacuum, they are developed simultaneously with, and through the slow and in-depth instruction of some very basic concepts, such as line, line segment, angle, adjacent angles, and triangle. The didactics for this curriculum rest on the combinations of the following student characteritics and the cognitive processes involved in learning these concepts.

1. Basic skill develoment at the ages of 12-14 is a very difficult task, which must proceed not only gradually but also slowly. Since long term retention is better for material that was learned in spread than in massed practice, the preparatory part of the curriculum is taught in grade 7 in 23 weekly lessons, each of which is devoted to the instruction of a single concept.

2. Concepts (abstract entities) differ from their names (concrete symbols), and the two are acquired by different processes (Ausubel et al., 1978). The combination of concept and concept name proved very confusing for students in group C, and therefore their acquisition is separated: the concept is taught first, and the name is assigned to it only after most crucial feaures of the concept were acquired. For instance, to teach the concept of isosceles triangle, we start with the construction of such a triangle, proceed to distinguish isosceles from other triangles, and identify its crucial features.

3. Concept acquisition involves both declarative knowledge, i.e., knowledge of the fearures of the concept, and procedural knowledge, i.e., being able to perform the procedures needed for either deciding whether a given figure is an instance of a certain concept or using the concept in problem solving. For instance, to teach the meaning of the point and arrowhead at both sides of the diagram of a ray, these meanings are repeated with every ray; they are exemplified by realizing that lengthening the ray is possible only in the arrowhead direction, when asked to decide whether a point is on the given ray, when it is located on both sides of the diagram of the ray.

4. Concept acquisition is facilitated, and even requires to present the learner with both examples and nonexamples of the concept (e.g., Skemp, 1971). Therefore, for the instruction of each concept, numerous examples and nonexamples are introduced and carefully examined from the very beginning. To teach procedural knowledge pertaining to concepts the following means are used: a) the precise processes involved in the former paragraph are specified and exemplified explicitly, and b) concepts are taught by inferring their critical features from the examples and nonexamples. This is illustrated by Example 1 in the Appendix.

5. Concept designation requires the use of several symbols, such as English characters, points, numerals and arrowheads. Students in group C have great difficulties not only in remembering the English alphabeth but also in retaining the

common meaning of other symbols. To minimize processing effort in perception (recognizing) and conception (recalling the meaning or designation), only few characters are used. The meaning of each symbol is established by two means: (a) Verbal explication of the meaning whenever it apears, (b) using different symbols to convey all aspects of the concept, and (c) construction of the meaning through the application of procedures implied by this meaning. For instance, instead of designating an angle with three English upper case letters, either a single numeral (1 or 2) or a single greek letter ($\alpha$, $\beta$) is used; The term "midpoint" is used interchangeably with "dividing into two equal segments" and "equidistant from both ends".

6. To reduce the memory load required for certain tasks and adapt it to the poor memory organization and skills of the students in group C two means are used:

(a) Concepts that are relatively easy to learn are introduced right when they are needed, rather than at the beginning of the chapter or subject. For instance, although angles are introduced in the third month, vertical angles are introduced only about one year after it, and an angle of a triangle is introduced at the end of the first year, when they are actually needed.

(b) For concepts that are relatively difficult to learn perceptual schemas are gradually constructed by introducing these concepts ahead in time, by using them repeatedly in contexts that are both simpler and similar to those in which they appear later on. This is illustrated by Example 2 in the Appendix.

7. To overcome memory deficiencies, special problems are designed for each lesson that practice only the single concept taught in this lesson.

8. To prohibit misconception formation, the problems in each lesson probe into a variety of concept features that are prone to produce micsonceptions if they remain unattended or inadequately or inappropriately processed. For instance, when the bisetor of the vertex angle of an isosceles traingle and two of its features were studied, some assignment problems required only the identification of this bisector and distinguishing it from other bisectors (e.g., which of the following line segments is the bisector of the vertex angle?).

9. Special lessons are devoted to provide the students in group C with general concepts and skills that students of higher ability already possess. For instance, to teach the concept of a common side to two tringles, adjacent triangles are presented in which either whole or part sides coincide; To prevent losing the whole figure for its parts, it is required to identify all line segments in diagrams containing a line segment with several points marked on it, a triangle with several line segment drawn from one of its vertices, or an angle with several rays emanating from its vertex.

## V Teaching to solve "proof problems"

The major objective of this part of the curriculum is to develop the cognitive skill of finding a proof to a given proposition (whether it is a theorem or a "problem to prove"), which is referred to as strategic knowledge (Greeno, 1978). To this end the curriculum uses the following means.

1. To provide for the deficient cognitive skills a single theorem is taught in every lesson, and difficult theorems even require several lessons.

2. The ability to find proofs is composed of the following skills: a) distinction between information that is given or suoght, b) interpretation of relationships and other kinds of geometrical information, c) application of concepts and theorems, plans, solution methods and principles to a given situation, and d) setting and achieving goals.

3. The axiomtic structure of the system is loosened, and some arguments can be justified ("proved") by concrete rather than formal reasoning. For instance, the straight line axiom is treated on an intuitive and perceptual level, which is based mainly on observation.

4. A distinction is made between finding a proof and writing it down, with the latter

being postponed until the complete proof is found.
5. Special tables are provided in which the various parts of the proof are filled in, and these tables are used throughout the curriculum.
6. Prescriptions (both algorithmic and heuristic) regarding how to find proofs are provided, which are based on and use the following principles and means:
(i) To reduce memory load, given and obtained information is displayed on the diagram, which serves as external memory, and enables the application of strong heuristics to check various arguments (Newell and Simon, 1972).
(ii) Strategic knowledge concerning the direction of proof and goal setting are taught explicitly. This is illustrated by Example 3 of the Appendix.
(iii) Complex procedures were broken down to their composite primitive procedures, which are taught explicity. If students are able to compose them--they are allowed and encouraged to do so; if they cannot--in grade 9 they are gradually taught how to achieve it.
(iv) Solution methods (plans, procedures) are dealt with explicitly, i.e., by mentioning, formulating, discussing and applying them with full awareness. For instance, to make the application of solution methods more flexible and to enrich the repretoire of solution mehtods, similar goals are attained by a variety of methods instead of using only correct methods; methods that attain specific types of goals (including very basic ones) are formulated explicitly. For instance, (a) to prove that two triangles have a common side, the following method is formulated: To identify a common side check whether the two triangles share two vertices; If yes--the side included between them is the common side. If no--the triangles do not have a common side, (b) to prove that two angles are equal, the following methods are proposed: 1) find suitable congruent triangles, each containing one of the angles, and prove their congruence, 2) find a third angle which is equal to both angles, and 3) if each angle is the sum (or difference) of two other angles, prove that the corresponding addends (terms) are equal.
(v) Both forward and backward searh processes are taught explicitly.
(vi) Proofs are explicitly characterized as being obtained by the application of either the method or result of previous proofs (solutions). This is illustrated by Example 4 in the Appendix.
(vii) To minimize the need to recall previously learned strategic knowledge, to enable keep the learned procedures separate and uncomposed for some time, the first problems in each lesson practice only or mainly the learned elements taught in this lesson.

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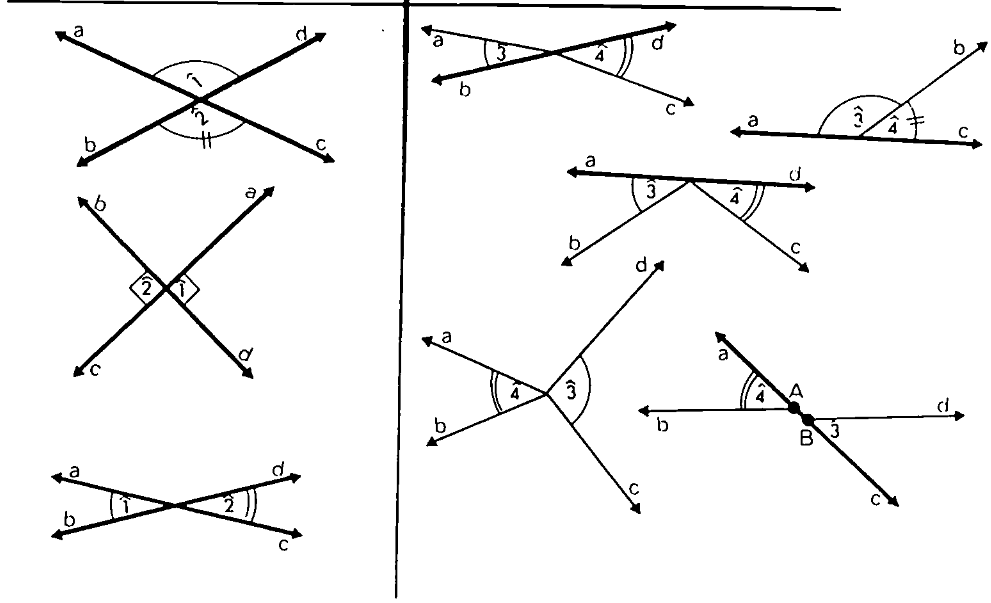
## VII Appendix

Example 1: Explain what vertical angles are from the following diagrams of pairs of angles that are and are not vertical angles.

| $\hat{1}$ and $\hat{2}$ are VERTICAL ANGLES | $\hat{3}$ and $\hat{4}$ are NON-VERTICAL ANGLES |
|---|---|

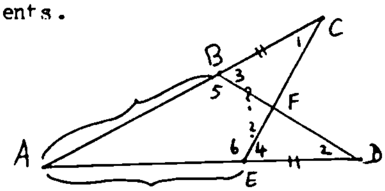



Example 2: A common angle to two triangles appear in diagrams like the following about one year before this concept is needed to solve complex congruence problems.

Example 3: Biderctional solution is taught in the following problem. The major steps are shown as presented to the students.

Given: AB=AE and BC=DE.

Prove: BF=EF




Proof: To prove BF=EF we want to prove $\Delta BCF \cong \Delta DEF$ (BACKWARD)

This cannot be achieved. So look if something useful can be obtained from the data.

We can prove that $\Delta ABD \cong \Delta ACE$.

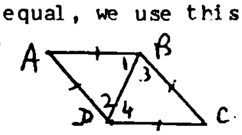
Therefore $\hat{5}=\hat{6}$ and $\hat{1}=\hat{2}$

Hence $\hat{3}=\hat{4}$, and we can now prove that $\Delta BCF \cong \Delta DEF$ (FORWARD)

Example 4: After using the method (theorem) of proving the equality of angles by showing that their opposite sides in a triangle are equal, we use this method by explicitly recapitulating it while solving another problem.

Given: ABCD is a rhombus

Prove: $\hat{B}=\hat{D}$




Proof: If we could show that angles B and D are sums of angles we could use the method of corresponding addends. They can be divided into two angles by the diagonal BD. Since $\hat{B}=\hat{1}+\hat{3}$, and $\hat{D}=\hat{2}+\hat{4}$, it is sufficient to show that $\hat{1}=\hat{2}$ and $\hat{3}=\hat{4}$. These can be obtained from the features of the isosceles triangles in which these are the base angles.

## PASSIVE VERSUS ACTIVE USE OF KNOWLEDGE

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**ABSTRACT:**

A unique problem involving the concept of group was given through one of three presentations to high-school math teachers. Although they possess a good knowledge of the concept, they tend not to make use of it at first; they use it the most for the geometrical presentation as was expected. Their performance on transfer from one problem to another is better than the performance of the non mathematicians.

The question to be answered is how to make cognitive organizations performant if the context and the contents of the problem to be solved are unusual.

## THE EXPERIMENT.

The experiment was deviced to show that the use people can make of a well known concept is different depending on the presentation of the problem to be solved. Under certain conditions experts (as we call them in opposition to novices) may make no use at all of their knowledge.

### 1. The Task

Given a set of three elements $E=\{x,y,z\}$ there are six bijective mappings on E, which are:

$$
\begin{array}{llllll}
 & x \to x & & x \to x & & x \to z \\
Id: & y \to y & Sx: & y \to z & Sy: & y \to y \\
 & z \to z & & z \to y & & z \to x \\
 & & & & & \\
 & x \to y & & x \to z & & x \to y \\
Sz: & y \to x & P1: & y \to x & P2: & y \to z \\
 & z \to z & & z \to y & & z \to x
\end{array}
$$

The set of these mappings is a group G, whose Pythagoras table is given on figure 1. This well known property is tought during the first year of university, and is used in high schools as an example of a group before that time.

The six lines

| | | | |
|---|---|---|---|
| (a) | x | y | z |
| (b) | y | z | x |
| (c) | x | z | y |
| (d) | z | y | x |
| (e) | z | x | y |
| (f) | y | x | z |

represent the six mappings.Line (b) is the image of line (a) by P2; line (c) is the image of line (a) by Sx, and so on. Now let us consider lines (a) and (c) on one hand and lines (e) and (f) on the other hand: line (c) is the image of line (a) by the mapping Sx, and line (f) is the image of line (e) by the same mapping, and so are lines (b) and (d); Sx being an involutive mapping, (a) is the image of (c) as well as (c) is the image of (a).

In other words, one obtains line (c) from line (a), in the same way as one obtains line (f) from line (e).

The subject was given in a random list the set of the 30 couples (a,b), (a,c),..., one can obtain with the six lines excluding those which have the same line as first and second element. His task was to sort them and write the five subsets corresponding to the five mappings different from the identity. For P1 for example the subset is:
(a,e) (b,a) (c,f) (d,c) (e,b) (f,d).
The subject was not told how many subsets were necessary to solve the problem properly, nor the number of couples to be writen is each subset.
There were three presentations of the problem as shown on figure 2:

-abstract : the set E being the set of three symbols $\{ \square , \times , \circ \}$

-geometrical : the set E being the set of the three summits of an equilateral triangle

-concrete : the set E being the set of the three hours :$\{4, 8, 12\}$ which placed on a clock correspond to the geometrical presentation.

One can solve the problem without using any of the properties of the group. It is a long and tedious work but it can be made. One can make total or partial use of the properties of the group concept: we shall give a few examples:
each mapping has an inverse: thus if (a,b) and (c,d) are together in a subset G1, (b,a) and (d,c) are together in a subset G2; as we noticed before some mappings are involutive, then if (a,c) and (c,a) are in the same subset and so is (e,f), then (f,e) must be there to; all subsets have the same number of couples, and so on.

## 2. The Population

The experimental group was a group of math teachers. The control group was a group of non mathematician teachers.

## 3. The Hypothesis

a) the use of the concept of group, or of certain properties of the group would be different depending on the presentation. We thought that they would be mostly used for the geometrical presentation, because at school geometrical mappings such as rotations and symetries are often used and combined. We made the hypothesis that the concrete presentation would induce some calculations leading to a cul de sac in the resolution of the problem.

b) the time for solving the problem would be different for novices and experts, as the experts could make use, even in a non verbalised way, of some properties of the group.

c) after he had solved the problem under one presentation the subject was given another presentation and asked to solve it. Our hypothesis was that in this transfer situation, the experts would justify the identity of the two problems better than the novices and in particular refer to the group structure explicitely.

## 4. The Results

a) for experts the time of resolution ranged from 10 minutes to 1 hour. For novices 1 hour was necessary.

b) no math teacher refered explicitely to the concept of group at first. While

solving the problem they would make some use of the properties of the group.

c) the novices would also make use of some properties of the group; this is not a surprise: in fact, as Piaget explained, the concept of group underlines the knowledge all individuals build up during the genesis of intelligence. There was a difference in the use the two populations would made of the properties : if an incorrect use would lead to a difficulty the experts would detect it easily and understand the mathematical nature of the error.

d) in the transfer situation experts would very easily justify the nature of the identity between the two problems, without starting to solve the second one except for a few subjects. The novices would not see any similarity between the two problems except in the question.

These results show that the knowledge is not immediately accessible if the situation is not a situation when the subject expects to have to make use of it. Once it has been activated in the process of resolution, even if no explicit reference is made to it, it can be used in different ways. At the beginning experts are not different from novices, they become different if the task induces the use of the specific knowledge they possess.

## A THEORETICAL MODEL

Mathematical knowledge is acquirred through learning and through problem solving activities. It is assumed that the problem solving activity has a most important role in learning mathematics and students perform many exercices.

We shall try to explicit briefly an analysis of the construction of such a knowledge using a concept that we would like to discuss with other researchers.

While learning, a subject builds up what can be called "cognitive organizations". These cognitive organizations are strongly related to the context and the contents which have been used at first to introduce them. Thus the understanding a subject can have is not very general, and its use can lead to errors. Cognitive organizations need practice to become altogether correct, general and powerful. This fact can be explained through the concept of "releasing components". A cognitive organization is accompagnied by different releasing components which recognize the task on which a cognitive organization is to be used, the time when it should be left for another and so on. It seems interesting to us to make the differences between cognitive organizations and releasing components as the former can be well defined through mathematical theory, and so an agrement can be obtained on what the knowledge of group is. Someone who solves problems on groups has the corresponding cognitive organization. If he encounters difficulties in identifying questions on which the concept is to be used, then his releasing components are nor performant.

The releasing components are built up through the practice of numerous and various problem solving situations using the cognitive organization. Thus we can understand that in our experiment the math skills were used the most in the geometrical situation as they are mostly practiced in high school in this field of mathemetics. Altogether it shows that intensive practice in one field impedes the practice in other fields and leads to a lack of generality. The building up of releasing components is dependant on the field of practice.

It is also dependant on the situation which leads to the problem solving task: a psychological experiment is not a situation when one has to make use of mathematical knowledge.

There is another point on which the concept of releasing component sheds light, that is the transfer situation: throuh the releasing components the task is identified as a work on the group concept, the cognitive organization is activated and will remain so as long as no other releasing component gives an end to its work.

Some questions remain nevertheless unanswered: what are the mathematical and the psychological natures of the releasing components and how are they built up? We have said, and it is well assumed that practice is necessary: math teachers are the ones who practice the most their cognitive organizations through exercices, years after years, and yet they may fail in identifying the nature of a problem.

## CONCLUSION

Our work was intended to point out that even experts in a specific field may not use their knowledge under certain conditions. These findings have some bearings both on pedagogy and on research.

As math teachers we must not expect our students to use their newly built up knowledge in a problem if their are not told to do so. We must accept at this moment that their knowledge remains fragmented. But we must also consider what teaching means. It seems clear to us that, even at a high level (high school or university), teaching mathematics is not only teaching mathematic theories. It is necessary to allow the releasing components to be settled together with the cognitive organizations, otherwise the latter will not become a real knowledge usable inside and outside school on different contents.

As researchers in mathematics education, we must try to answer the questions:

-what are the releasing components for that precise concept of mathematics,

-can we identify in the different uses of a concept some general releasing component which induces them,

-if we manage to know better what the releasing components are, how can one teach mathematics so that they get settled?

Figure 1:

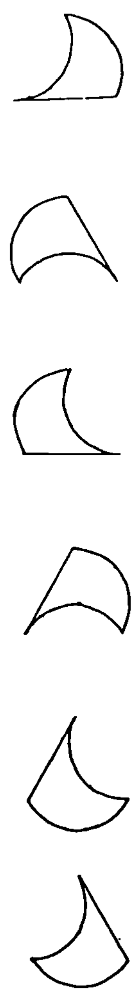
Pythagoras table for group G

| | Id | Sx | Sy | Sz | P1 | P2 |
|---|---|---|---|---|---|---|
| Id | Id | Sx | Sy | Sz | P1 | P2 |
| Sx | Sx | Id | P2 | P1 | Sz | Sy |
| Sy | Sy | P1 | Id | P2 | Sx | Sz |
| Sz | Sz | P2 | P1 | Id | Sy | Sx |
| P1 | P1 | Sy | Sz | Sx | P2 | Id |
| P2 | P2 | Sz | Sx | Sy | Id | P1 |

Figure 2:

The three presentations of the problem

a
b
c
d
e
f



| | | | |
|---|---|---|---|
| a | □ | * | ○ |
| b | * | ○ | □ |
| c | □ | ○ | * |
| d | ○ | * | □ |
| e | ○ | □ | * |
| f | * | □ | ○ |

| | | | |
|---|---|---|---|
| a | 4h | 8h | 12h |
| b | 8h | 12h | 4h |
| c | 4h | 12h | 8h |
| d | 8h | 4h | 12h |
| e | 12h | 4h | 8h |
| f | 12h | 8h | 4h |

# The Teaching of Infinity

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Infinity is undeniably one of the main concepts in philosophy, science and mathematics. Recent psycho-didactical studies have shown that students are facing great difficulties in acquiring this concept and particularly the notion of actual infinity. One of the reasons for these difficulties is the existence of biased primary intuitive attitudes towards this concept. (Fischbein et al, 1979; Tall, 1980; Duval, 1983). As a result, the intuitive solutions of problems dealing with the equivalence of infinite sets are often erroneous.

In order to overcome these difficulties, new approaches towards the concept of infinity should be developed.

Consequently, the main objectives of this research are:

1. To identify the inner conflicts in the intuitive understanding of the various aspects of the notion of actual infinity.
2. To try to improve the high school students' intuitive understanding of the notion related to actual infinity through systematic instruction.

## Method

The Teaching Program: Twenty lessons, suitable for tenth grade students, were developed. The program included: 1. Basic notions of set theory; 2. The concept of cardinal number; 3. Equivalence of infinite sets; 4. Enumerable sets; 5. Non-enumerable sets.

Various didactical procedures were devised for improving the intuitive attitudes and the analytical understanding of the concept of infinity. One of these was the conflict-teaching approach, which was intended to make the students aware of their own intra-intuitive conflicts concerning the notion of actual infinity. Awareness of these conflicts may cause a state of inner disequilibrium which is the optimum time for creating new modified concepts.

Subjects. Two hundred and eighty students, from eight tenth grade classes chosen from two academically selective high schools in Tel-Aviv, participated in this research. Four classes belonged to the experimental group (158 students) and four classes (122 students) belonged to the control group.

To assess the effect of teaching on the students' attitudes towards the equivalence of infinity sets, a questionnaire was administered to both experimental and control groups before and immediately after the instruction. In addition, the students were interviewed in person.

The questionnaire contained sixteen mathematical problems, which the students had never dealt with before. In each of the problems two infinite sets were given. The students had to decide whether these two sets were equivalent; In nine out of the sixteen problems two equivalent infinite non-enumerable sets were given. The cardinal number of both these sets was c. In the remaining seven problems, two non-equivalent infinite sets were given, one of these two sets had the cardinal number $\aleph_0$ whereas the other set had the cardinal number c. A brief description of the sixteen mathematical problems will be found in Tables 1 and 2.

## Results and Conclusions

Only the distribution of the experimental group answers to the mathematical problems are given in Table 1 and 2.

Table 1: Categories of answers to questions dealing with two equivalent infinite sets* (in %)

| The Sets Compared A | vs B | Pre-Test: Same Number of Elements | Pre-Test: Different Number of Elements | Pre-Test: No Answer | Post-Test: Same Number of Elements | Post-Test: Different Number of Elements |
|---|---|---|---|---|---|---|
| All points of a line | All points of a square | 70,3 | 24,4 | 4,8 | 90,0 | 10.0 |
| All points of a triangle | All points of an arc | 63,2 | 36.8 | | 84,2 | 15.8 |
| All points of a plane | All points of a line | 63,2 | 34,2 | 2,6 | 76.3 | 23.7 |
| All points of a circle | All points of the interior of a circle | 58.6 | 34.2 | 7.2 | 82.9 | 17.1 |
| All points of a square | All points of a line segment | 55,2 | 42,1 | 2.6 | 84.2 | 15.8 |
| All points of a square | All points of a triangle | 55.2 | 42.1 | 2.6 | 91.7 | 8.3 |
| All points of a plane | All points of the interior of a circle | 50.0 | 47.4 | 2.6 | 76.3 | 23.7 |
| All points of a square | All points of a larger square | 47.4 | 50.0 | 2.6 | 94.7 | 5.3 |
| All triangles in a plane | All equilateral triangles in a plane | 43.9 | 56.1 | - | 80.0 | 20.0 |

* All these sets have the infinite cardinal number c.

Table 2: Categories of answers to questions dealing with two Non-equivalent infinite sets* (in %)

| The Sets Compared: A | vs. B | Pre-test: Set A has more elements | Pre-test: Set B has more elements | Pre-test: It is not specified which has more elements | Pre-test: Same number of elements | Pre-test: No answer | Post-test: Set A has more elements | Post-test: Same number of elements | Post-test: No answer |
|---|---|---|---|---|---|---|---|---|---|
| All points of a plane | The Natural numbers | 47.5 | - | - | 47,4 | 5.1 | 81.6 | 18.4 | - |
| All points of a plane | The Integers | 42.7 | - | 3.5 | 47.5 | 6.3 | 75.0 | 25.0 | - |
| All points of a plane | The Rational Number | 22,5 | 2.5 | - | 67.5 | 7.5 | 82.9 | 12.2 | 4.9 |
| All points of a line | All the "Rational points" of a plane | 20.0 | - | 3.7 | 65.0 | 11,3 | 78.0 | 14.3 | 7.7 |
| All points of the interior of a circle | All the "Rational points" of a plane | 15,0 | 6,7 | 2,0 | 63,7 | 12.6 | 83.7 | 16,3 | - |
| All points of a line segment | All the "Rational points" of a plane | 6,3 | 34.5 | 1.7 | 50,0 | 7.5 | 62,9 | 28.8 | 8.3 |
| All the "Irrational points of a plane" | All the "Rational points" of a plane | 32.2 | 28.1 | - | 20.5 | 19.2 | 90.2 | 9.8 | - |

* In all these questions, Set A has the infinite cardinal number c whereas Set B has the infinite cardinal number $\aleph_0$.

Through the answers to the pre-test questions, the primary intuitive attitudes of the tenth grade students towards the concept of infinity were examined;

## Results Concerning the Answers to the Pre-Test Questions

1. The equivalence claims: The percentages of the "same number of elements" answers were between 43.9% and 73.2% (see Tables 1 and 2). In most of the questions, the percentages of the students, who claim that both sets have the same number of elements is higher than the percentages of those who did not claim so.

The main argument given by the students for the equivalence claims was that "only one kind of infinity exists, therefore all the infinite sets have the same number of elements". This idea of equivalence corresponds to the primary intuitive understanding of the infinite as an endless process. (Fischbein and al, 1979).

2. The non-equivalence claims: The percentages of the "different number of elements" answers were between 19,5% and 56,1% (see Table 1 and 2).

The students justified the "non-equivalence claims" by three main arguments:

a. "A set contains more elements than its proper subset." In other words: "The whole is bigger than its parts." (for instance questions 7-9, Table 1).

b. "A non-bounded set contains more elements than a bounded set" (for instance question 5 Table 2).

c. "A two dimensional set contains more elements than a linear set" (for instance questions 1-5, Table 1).

The students' justification for the "non-equivalence claims" is based on methods derived from their daily experience connected with finite sets. These methods are unsuitable for infinite sets. Furthermore, the students had no intuitive tendencies concerning the existence of different Cantorian's cardinal numbers.

3. Basic difficulties in the intuitive understanding of the actual infinity were revealed in the answers to the pre-test questions and during instruction. The following inner conflicts which may explain these difficulties were identified:

a. The conflict between two tendencies:
   1. The tendency to consider that all the infinite sets are equivalent.
   2. The tendency to compare the cardinal numbers of two sets according to the principle: "The whole is bigger than its parts".

b. The conflict between two statements:
   1. The intuitively acceptable statement: "A proper subset of a given set has a smaller cardinal number than the whole set".
   2. The formal statement: "Every infinite set has a proper subset which has the same cardinal number."

83.7% of the students were inconsistent in their answers. In part of the answers they claimed that all the infinite sets were equivalent whereas, in other answers they claimed that one infinite set had a smaller cardinal number than the other set. It must be stressed that only 5.7% of the students were aware of the deep contradiction between these two claims.

To sum up the findings of the pre-test: 1. All primary intuitive methods used by the students are unsuitable when determining whether two infinite sets are equivalent. 2. The students are not aware of the inner contradiction in their primary intuitive attitudes towards the notion of actual infinity.

Comparison of Pre-test and Post-test Answers

1. The percentages of the correct answers to the post-test questions were higher than those of the pre-test.;

The percentages of the correct answers to problems dealing with two equivalent infinite sets were between 76.3% and 94.7% (see Table 1), whereas, the percentages of the correct answers to the non-equivalent problems were between 62.9% and 90.2% (see Table 2).

2. In the post-test all the students who claimed, correctly, that two sets were non-equivalent, chose the proper set as having a higher cardinal, whereas in the pre-test they did not always do so.

3. The reasoning used by the students to justify their post-test answers were different from those given by them to justify their answers to the pre-test.

Four main arguments were used to justify the equivalence claims:

a. About 50% of the students made an attempt to find a one to one correspondence between the two infinite sets.
b. About 30% of the students proved the equivalence claims by using the mathematical theorems which they had been taught when learning the set theory unit.
c. About 10% of the students justified their equivalence claims by pointing out the analogy between the given equivalence problem and an equivalence problem learned in the set theory unit.
d. Only 10% still used biased primary intuitive attitudes towards the concept of infinity.

Three main arguments were used to justify the non-equivalence claims:

a. About 80% of the students made an attempt to prove that one of the sets was non-enumerable whereas the other one was enumerable.
b. About 10% of the students pointed out the analogy between the given non-equivalence problem and a non-equivalance problem learned in the set theory unit in order to justify the non-equivalence claims.
c. Only 10% of the students still used their former primary intuitive techniques to justify the non-equivalence claims.

These results enable us to draw the following conclusions:

1. About 70% of the students were made aware of their own intra-intuitive conflicts concerning the concept of actual infinity as a result of the systematic instruction. These students acquired the necessary set theory notions and used only the adequate procedures for establishing the equivalence of infinite sets.

2. After the systematic instruction, about 20% of the students were still in a state of inner disequilibrum with regards to the concept of infinity. They were able to solve correctly only some of the problems whereas in respect to others they failed and resorted again to non-adequate intuitive techniques.

3. About 10% of the students had not been able to free themselves from their primary intuitive constraints and used only nonadequate intuitive techniques in solving the mathematical problems.

The main conclusion of the set theory instruction is that by using suitable teaching methods, including an active didactical approach towards the intuitive tendencies of the students, it is possible to improve the high school students' intuitive understanding of very complex mathematical concepts.

An unanticipated achievement of the instruction was that the students' awareness of the inner conflicts in their intuitive ways of thinking produced in them a much deeper understanding of the need and the importance of formal mathematical proofs in contrast with their biased primary intuitive evaluations.

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