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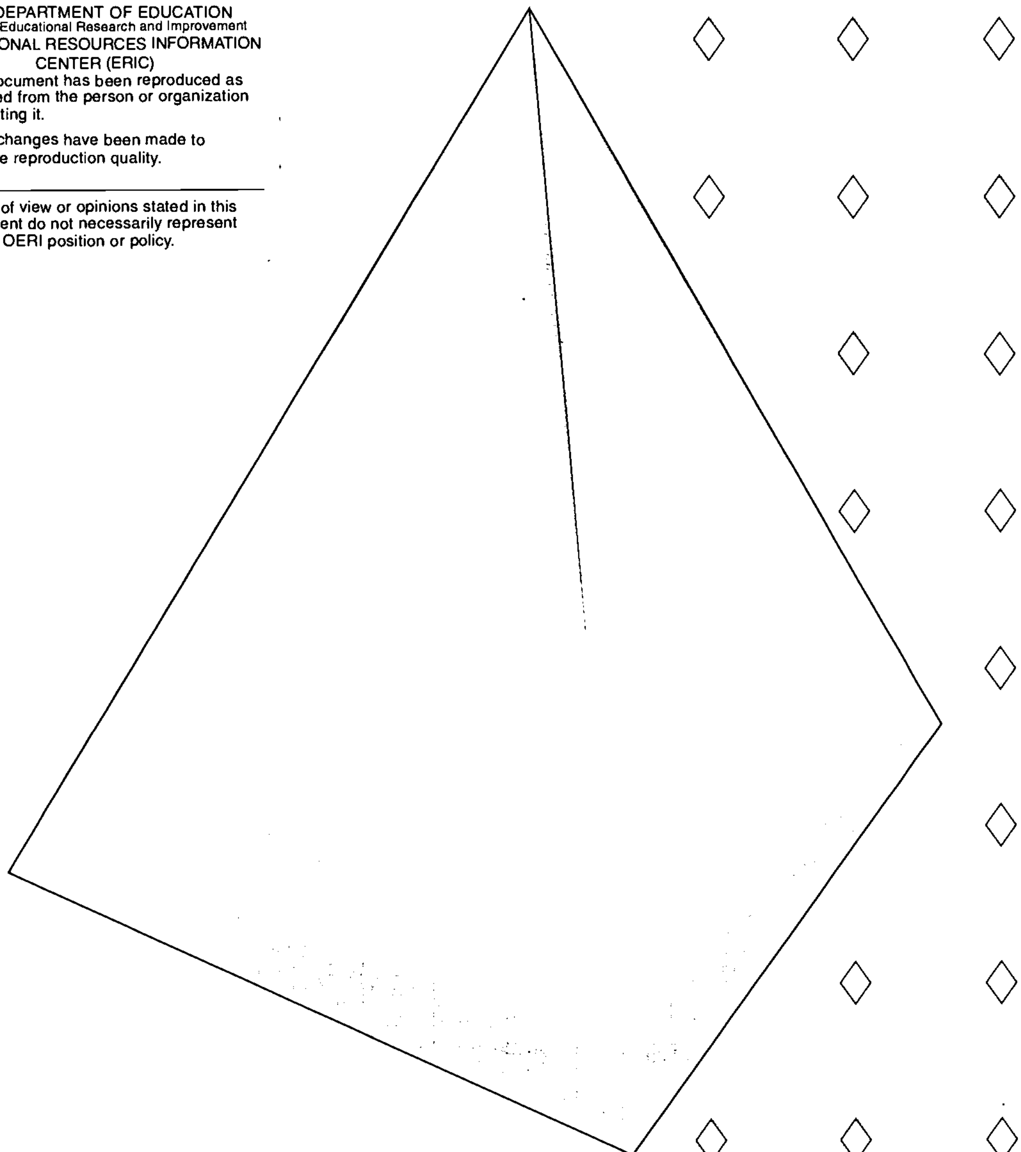
ABSTRACT

These excerpts from the National Council of Teachers of Mathematics (NCTM) Standards support the Third International Mathematics and Science Study (TIMSS) Resource Kit by offering some context for discussing the teaching of algebra and geometry seen in the videotapes that are also part of the kit. The Standards describe the mathematics content that all students should know and be able to do. This document is divided into grade levels K-4, 5-8, and 9-12 with each level having 12 to 14 standards. The curriculum standards for grades K-4 include 13 standards and excerpts on geometry/spatial sense and patterns/relationships are presented. The curriculum standards for grades 5-8 contain an overview of those 13 standards plus a list of features which should be included in the 5-8 curriculum. Summaries are presented for the standards on patterns and functions, algebra, and geometry. The section on standards for grades 9-12 discusses background, underlying assumptions, features of this content, and patterns of instruction. The excerpts for grades 9-12 address algebra, functions, geometry from a synthetic perspective, and geometry from an algebraic perspective. (AIM)

Fostering Algebraic and Geometric Thinking

Selections from the NCTM *Standards*



NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

FOSTERING ALGEBRAIC AND GEOMETRIC THINKING

SELECTIONS FROM THE NCTM *STANDARDS*

July 1997

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PREFACE

The National Council of Teachers of Mathematics is pleased to support *Achieving Excellence: A TIMSS Resource Kit* by supplying excerpts from the *Curriculum and Evaluation Standards for School Mathematics* (1989) and the *Professional Standards for Teaching Mathematics* (1991). The selected Curriculum Standards offer some context for discussing the teaching of algebra and geometry seen in the videotapes that are part of this kit. The selected Teaching Standards can be considered as the context for instruction in a variety of areas. Occasional reference is made to portions of the complete documents that are not included here. The complete set of Curriculum and Evaluation Standards can be found by accessing the NCTM Web site at www.nctm.org and selecting the heading "About NCTM." The complete books can be ordered from the National Council of Teachers of Mathematics at (800) 220-8483.

By way of additional background, the *Curriculum and Evaluation Standards for School Mathematics* describes the mathematics content that all students should know and be able to do. The document is divided into grade levels K–4, 5–8, and 9–12, with each level having twelve to fourteen standards.

These standards identify the basic skills and understandings that students should have in number and number theory, geometry, measurement, probability and statistics, patterns and functions, discrete mathematics, algebra, and beyond. While students are mastering such important basic skills, they must also—

◊ learn to value mathematics;

◊ become confident in their ability to do mathematics;

◊ become mathematical problem solvers;

◊ learn to communicate mathematically;

◊ learn to reason mathematically.

Given these content goals, the *Professional Teaching Standards for Mathematics* illustrates ways that teachers can structure classroom activities to encourage such learning. In four major areas, teachers strongly influence students' opportunities to make sense of mathematics by—

◊ choosing worthwhile mathematical tasks;

◊ establishing and promoting classroom discussion;

◊ creating an environment for learning; and

◊ analyzing one's own teaching, including the efficacy of assessing students' learning.

In addition, the Council has published *Assessment Standards for School Mathematics* (1995), which provides a set of principles for teachers and others to use in examining assessment practices. Educators must ensure that assessment reflects the mathematics that all students need to know and be able to do. Those responsible for mathematics education must be able to draw valid inferences about students' mathematics learning so that assessment results can be used to modify the teaching process as needed. Moreover, assessment must promote equity, so that all students have the opportunity to demonstrate their mathematical understanding and all

teachers work to help students when understanding is not yet complete. For all this to occur, assessment must be an open, coherent process.

We hope that *Fostering Algebraic and Geometric Thinking* along with the other items in this resource kit will prompt reflection on the goals of teaching mathematics as well as related classroom practice in an international context. We can learn a great deal from this setting that can lead to improving the mathematical growth of children and the professional development of teachers.

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EXCERPTS FROM THE

CURRICULUM AND EVALUATION STANDARDS FOR SCHOOL MATHEMATICS

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CURRICULUM STANDARDS FOR GRADES K–4

OVERVIEW

This section presents thirteen curriculum standards for grades K–4:

1. ***Mathematics as Problem Solving***
2. ***Mathematics as Communication***
3. ***Mathematics as Reasoning***
4. ***Mathematical Connections***
5. ***Estimation***
6. ***Number Sense and Numeration***
7. ***Concepts of Whole Number Operations***
8. ***Whole Number Computation***
9. ***Geometry and Spatial Sense***
10. ***Measurement***
11. ***Statistics and Probability***
12. ***Fractions and Decimals***
13. ***Patterns and Relationships***

The Need for Change

The need for curricular reform in K–4 mathematics is clear. Such reform must address both the content and emphasis of the curriculum as well as approaches to instruction. A long-standing preoccupation with computation and other traditional skills has dominated both *what* mathematics is taught and *the way* mathematics is taught at this level. As a result, the present K–4 curriculum is narrow in scope; fails to foster mathematical insight, reasoning, and problem solving; and emphasizes rote activities. Even more significant is that children begin to lose their belief that learning mathematics is a sense-making experience. They become passive receivers of rules and procedures rather than active participants in creating knowledge.

The Direction of Change

The Introduction describes a vision for school mathematics built around five overall curricular goals for students to achieve: learning to value mathematics, becoming confident in one's own ability, becoming a mathematical problem solver, learning to communicate mathematically, and learning to reason mathematically. This vision addresses what mathematics is, what it means to know and do mathematics, what teachers should do when they teach mathematics, and what children should do when they learn mathematics. The K-4 standards reflect the implications of this vision for the curriculum in the early grades and present a coherent viewpoint about mathematics, about children, and about the learning of mathematics by children.

Children and Mathematics: Implications for the K–4 Curriculum

An appropriate curriculum for young children that reflects the Standards' overall goals must do the following:

1. *Address the relationship between young children and mathematics.* Children enter kindergarten with considerable mathematical experience, a partial understanding of many concepts, and some important skills, including counting. Nonetheless, it takes careful planning to create a curriculum that capitalizes on children's intuitive insights and language in selecting and teaching mathematical ideas and skills. It is clear that children's intellectual, social, and emotional development should guide the kind of mathematical experiences they should have in light of the overall goals for learning mathematics. The notion of a *developmentally appropriate* curriculum is an important one.

A developmentally appropriate curriculum encourages the exploration of a wide variety of mathematical ideas in such a way that children retain their enjoyment of, and curiosity about, mathematics. It incorporates real-world contexts, children's experiences, and children's language in developing ideas. It recognizes that children need considerable time to construct sound understandings and develop the ability to reason and communicate mathematically. It looks beyond what children appear to know to determine how they think about ideas. It provides repeated contact with important ideas in varying contexts throughout the year and from year to year.

Programs that provide limited developmental work, that emphasize symbol manipulation and computational rules, and that rely heavily on paper-and-pencil worksheets do not fit the natural learning patterns of children and do not contribute to important aspects of children's mathematical development.

2. *Recognize the importance of the qualitative dimensions of children's learning.* The mathematical ideas that children acquire in grades K-4 form the basis for all further study of mathematics. Although quantitative considerations have frequently dominated discussions in recent years, qualitative considerations have greater significance. Thus, how well children come to understand mathematical ideas is far more important than how many skills they acquire. The success with which programs at later grade levels achieve their goals depends largely on the quality of the foundation that is established during the first five years of school.

3. *Build beliefs about what mathematics is, about what it means to know and do mathematics, and about children's view of themselves as mathematics learners.* The beliefs that young children form influence not only their thinking and performance during this time but also their attitude and decisions about studying mathematics in later years. Beliefs also become more resistant to change as children grow older. Thus, affective dimensions of learning play a significant role in, and must influence, curriculum and instruction.

ASSUMPTIONS

Several basic assumptions governed the selection and shaping of the K–4 standards.

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1. *The K–4 curriculum should be conceptually oriented.* The view that the K–4 curriculum should emphasize the development of mathematical understandings and relationships is reflected in the discussions about the content and emphasis of the curriculum. A conceptual approach enables children to acquire clear and stable concepts by constructing meanings in the context of physical situations and allows mathematical abstractions to emerge from empirical experience. A strong conceptual framework also provides anchoring for skill acquisition. Skills can be acquired in ways that make sense to children and in ways that result in more effective learning. A strong emphasis on mathematical concepts and understandings also supports the development of problem solving.

Emphasizing mathematical concepts and relationships means devoting substantial time to the development of understandings. It also means relating this knowledge to the learning of skills by establishing relationships between the conceptual and procedural aspects of tasks. The time required to build an adequate conceptual base should cause educators to rethink when children are expected to demonstrate a mastery of complex skills. A conceptually oriented curriculum is consistent with the overall curricular goals in this report and can result in programs that are better balanced, more dynamic, and more appropriate to the intellectual needs and abilities of children.

2. *The K–4 curriculum should actively involve children in doing mathematics.* Young children are active individuals who construct, modify, and integrate ideas by interacting with the physical world, materials, and other children. Given these facts, it is clear that the learning of mathematics must be an active process. Throughout the Standards, such verbs as *explore, justify, represent, solve, construct, discuss, use, investigate, describe, develop,* and *predict* are used to convey this active physical and mental involvement of children in learning the content of the curriculum.

The importance of active learning by children has many implications for mathematics education. Teachers need to create an environment that encourages children to explore, develop, test, discuss, and apply ideas. They need to listen carefully to children and to guide the development of their ideas. They need to make extensive and thoughtful use of physical materials to foster the learning of abstract ideas.

K–4 classrooms need to be equipped with a wide variety of physical materials and supplies. Classrooms should have ample quantities of such materials as counters; interlocking cubes; connecting links; base-ten, attribute, and pattern blocks; tiles; geometric models; rulers; spinners; colored rods; geoboards; balances; fraction pieces; and graph, grid, and dot paper. Simple household objects, such as buttons, dried beans, shells, egg cartons, and milk cartons, also can be used.

3. *The K–4 curriculum should emphasize the development of children's mathematical thinking and reasoning abilities.* An individual's future uses and needs for mathematics make the ability to think, reason, and solve problems a primary goal for the study of mathematics. Thus, the curriculum must take seriously the goal of instilling in students a sense of confidence in their ability to think and communicate mathematically, to solve problems, to demonstrate flexibility in working with mathematical ideas

and problems, to make appropriate decisions in selecting strategies and techniques, to recognize familiar mathematical structures in unfamiliar settings, to detect patterns, and to analyze data. The K–4 standards reflect the view that mathematics instruction should promote these abilities so that students understand that knowledge is empowering and that individual pieces of content are all related to this broader perspective.

Developing these characteristics in children requires that schools build appropriate reasoning and problem-solving experiences into the curriculum from the outset. Further, this goal needs to influence the way mathematics is taught and the way students encounter and apply mathematics throughout their education.

4. *The K–4 curriculum should emphasize the application of mathematics.* If children are to view mathematics as a practical, useful subject, they must understand that it can be applied to a wide variety of real-world problems and phenomena. Even though most mathematical ideas in the K–4 curriculum arise *from* the everyday world, they must be regularly applied to real-world situations. Children also need to understand that mathematics is an integral part of real-world situations and activities in other curricular areas. The mathematical aspects of that work should be highlighted.

Learning mathematics has a purpose. At the K–4 level, one major purpose is helping children understand and interpret their world and solve problems that occur in it. Children learn computation to solve problems; they learn to measure because measurement helps them answer questions about how much, how big, how long, and so on; and they learn to collect and organize data because doing so permits them to answer other questions. By applying mathematics, they learn to appreciate the power of mathematics.

5. *The K–4 curriculum should include a broad range of content.* To become mathematically literate, students must know more than arithmetic. They must possess a knowledge of such important branches of mathematics as measurement, geometry, statistics, probability, and algebra. These increasingly important and useful branches of mathematics have significant and growing applications in many disciplines and occupations.

The curriculum at all levels needs to place substantial emphasis on these branches of mathematics. Mathematical ideas grow and expand as children work with them throughout the curriculum. The informal approach at this level establishes the foundation for further study and permits children to acquire additional knowledge they will need. These topics are highly appropriate for young learners because they make important contributions to children's mathematical development and help them see the usefulness of mathematics. They also provide productive, intriguing activities and applications.

The inclusion of a broad range of content in the curriculum also allows children to see the interrelated nature of mathematical knowledge. When teachers take advantage of the opportunity to relate one mathematical idea to others and to other areas of the curriculum, as will be described in Standard 4, children acquire broader notions about the interconnected-

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ness of mathematics and its relationships to other fields. The curriculum should enable all children to do a substantial amount of work in each of these topics at each grade level.

6. *The K–4 curriculum should make appropriate and ongoing use of calculators and computers*. Calculators must be accepted at the K–4 level as valuable tools for learning mathematics. Calculators enable children to explore number ideas and patterns, to have valuable concept-development experiences, to focus on problem-solving processes, and to investigate realistic applications. The thoughtful use of calculators can increase the quality of the curriculum as well as the quality of children's learning.

Calculators do not replace the need to learn basic facts, to compute mentally, or to do reasonable paper-and-pencil computation. Classroom experience indicates that young children take a commonsense view about calculators and recognize the importance of not relying on them when it is more appropriate to compute in other ways. The availability of calculators means, however, that educators must develop a broader view of the various ways computation can be carried out and must place less emphasis on complex paper-and-pencil computation. Calculators also highlight the importance of teaching children to recognize whether computed results are reasonable.

The power of computers also needs to be used in contemporary mathematics programs. Computer languages that are geometric in nature help young children become familiar with important geometric ideas. Computer simulations of mathematical ideas, such as modeling the renaming of numbers, are an important aid in helping children identify the key features of the mathematics. Many software programs provide interesting problem-solving situations and applications.

The thoughtful and creative use of technology can greatly improve both the quality of the curriculum and the quality of children's learning. Integrating calculators and computers into school mathematics programs is critical in meeting the goals of a redefined curriculum.

STANDARD 9:
GEOMETRY AND SPATIAL SENSE

In grades K–4, the mathematics curriculum should include two- and three-dimensional geometry so that students can—

- ◊ ***describe, model, draw, and classify shapes;***
- ◊ ***investigate and predict the results of combining, subdividing, and changing shapes;***
- ◊ ***develop spatial sense;***
- ◊ ***relate geometric ideas to number and measurement ideas;***
- ◊ ***recognize and appreciate geometry in their world.***

Focus

Geometry is an important component of the K–4 mathematics curriculum because geometric knowledge, relationships, and insights are useful in everyday situations and are connected to other mathematical topics and school subjects. Geometry helps us represent and describe in an orderly manner the world in which we live. Children are naturally interested in geometry and find it intriguing and motivating; their spatial capabilities frequently exceed their numerical skills, and tapping these strengths can foster an interest in mathematics and improve number understandings and skills.

Spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world. Insights and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense. Children who develop a strong sense of spatial relationships and who master the concepts and language of geometry are better prepared to learn number and measurement ideas, as well as other advanced mathematical topics.

In learning geometry, children need to investigate, experiment, and explore with everyday objects and other physical materials. Exercises that ask children to visualize, draw, and compare shapes in various positions will help develop their spatial sense. Although a facility with the language of geometry is important, it should not be the focus of the geometry program but rather should grow naturally from exploration and experience. Explorations can range from simple activities to challenging problem-solving situations that develop useful mathematical thinking skills.

Evidence suggests that the development of geometric ideas progresses through a hierarchy of levels. Students first learn to recognize whole shapes and then to analyze the relevant properties of a shape. Later they can see relationships between shapes and make simple deductions. Curriculum development and instruction must consider this hierarchy because although learning can occur at several levels simultaneously, the learning of more complex concepts and strategies requires a firm foundation of basic skills.

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STANDARD 13: PATTERNS AND RELATIONSHIPS

In grades K–4, the mathematics curriculum should include the study of patterns and relationships so that students can—

- ◊ ***recognize, describe, extend, and create a wide variety of patterns;***
- ◊ ***represent and describe mathematical relationships;***
- ◊ ***explore the use of variables and open sentences to express relationships.***

Focus

Patterns are everywhere. Children who are encouraged to look for patterns and to express them mathematically begin to understand how mathematics applies to the world in which they live. Identifying and working with a wide variety of patterns help children to develop the ability to classify and organize information. Relating patterns in numbers, geometry, and measurement helps them understand connections among mathematical topics. Such connections foster the kind of mathematical thinking that serves as a foundation for the more abstract ideas studied in later grades.

From the earliest grades, the curriculum should give students opportunities to focus on regularities in events, shapes, designs, and sets of numbers. Children should begin to see that regularity is the essence of mathematics. The idea of a functional relationship can be intuitively developed through observations of regularity and work with generalizable patterns.

Physical materials and pictorial displays should be used to help children recognize and create patterns and relationships. Observing varied representations of the same pattern helps children identify its properties. The use of letters and other symbols in generalizing descriptions of these properties prepares children to use variables in the future. This experience builds readiness for a generalized view of mathematics and the later study of algebra.

OVERVIEW

This section presents thirteen curriculum standards for grades 5–8:

1. ***Mathematics as Problem Solving***
2. ***Mathematics as Communication***
3. ***Mathematics as Reasoning***
4. ***Mathematical Connections***
5. ***Number and Number Relationships***
6. ***Number Systems and Number Theory***
7. ***Computation and Estimation***
8. ***Patterns and Functions***
9. ***Algebra***
10. ***Statistics***
11. ***Probability***
12. ***Geometry***
13. ***Measurement***

The Need for Change

Mathematics is a useful, exciting, and creative area of study that can be appreciated and enjoyed by all students in grades 5–8. It helps them develop their ability to solve problems and reason logically. It offers to these curious, energetic students a way to explore and make sense of their world. However, many students view the current mathematics curriculum in grades 5–8 as irrelevant, dull, and routine. Instruction has emphasized computational facility at the expense of a broad, integrated view of mathematics and has reflected neither the vitality of the subject nor the characteristics of the students.

An ideal 5–8 mathematics curriculum would expand students' knowledge of numbers, computation, estimation, measurement, geometry, statistics, probability, patterns and functions, and the fundamental concepts of algebra. The need for this kind of broadened curriculum is acute. An examination of textbook series shows the repetition of topics, approach, and level of presentation in grade after grade. A comparison of the tables of contents shows little change over grades 5–8. It is even more disconcerting to realize that the very chapters that contain the most new material, such as probability, statistics, geometry, and prealgebra, are covered in the last half of the books—the sections most often skipped by teachers for lack of time. The result is an ineffective curriculum that rehashes material students already have seen. Such a curriculum promotes a negative image of mathematics and fails to give students an adequate background for secondary school mathematics.

These thirteen standards promote a broad curriculum for students in grades 5–8. Developing certain computational skills is important but con-

stitutes only a part of this curriculum. Nevertheless, the existing curriculum in some schools prohibits many students from studying a broader curriculum until they have "mastered" basic computational skills. Shifting the focus to a broader curriculum is important for the following reasons:

1. Basic skills today and in the future mean far more than computational proficiency. Moreover, the calculator renders obsolete much of the complex paper-and-pencil proficiency traditionally emphasized in mathematics courses. Topics such as geometry, probability, statistics, and algebra have become increasingly more important and accessible to students through technology.
2. If students have not been successful in "mastering" basic computational skills in previous years, why should they be successful now, especially if the same methods that failed in the past are merely repeated? In fact, considering the effect of failure on students' attitudes, we might argue that further efforts toward mastering computational skills are counterproductive.
3. Many of the mathematics topics that are omitted actually can help students recognize the need for arithmetic concepts and skills and provide fresh settings for their use. For example, in probability, students have many opportunities to add and multiply fractions.

The vision articulated in the 5–8 standards is of a broad, concept-driven curriculum, one that reflects the full breadth of relevant mathematics and its interrelationships with technology. This vision is built on five overall curricular goals for students: learning to value mathematics, becoming confident in their ability, becoming a mathematical problem solver, learning to communicate mathematically, and learning to reason mathematically. The teaching of this curriculum should be related to the characteristics of middle school students and their current and future needs.

Features of the Mathematics Curriculum

The 5–8 curriculum should include the following features:

- ◊ Problem situations that establish the need for new ideas and motivate students should serve as the context for mathematics in grades 5–8. Although a specific idea might be forgotten, the context in which it is learned can be remembered and the idea re-created. In developing the problem situations, teachers should emphasize the application of mathematics to real-world problems as well as to other settings relevant to middle school students.
- ◊ Communication with and about mathematics and mathematical reasoning should permeate the 5–8 curriculum.
- ◊ A broad range of topics should be taught, including number concepts, computation, estimation, functions, algebra, statistics, probability, geometry, and measurement. Although each of these areas is valid mathematics in its own right, they should be taught as an integrated whole, not as isolated topics; the connections among them should be a prominent feature of the curriculum.
- ◊ Technology, including calculators, computers, and videos, should be used when appropriate. These devices and formats free students from tedious computations and allow them to concentrate on problem solving

and other important content. They also give them new means to explore content. As paper-and-pencil computation becomes less important, the skills and understanding required to make proficient use of calculators and computers become more important.

Instruction

The standards are not intended to each constitute a chapter in a text or a particular unit of instruction; rather, learning activities should incorporate topics and ideas across standards. For example, an instructional activity might involve problem solving and use geometry, measurement, and computation. All mathematics should be studied in contexts that give the ideas and concepts meaning. Problems should arise from situations that are not always well formed. Students should have opportunities to formulate problems and questions that stem from their own interests.

Learning should engage students both intellectually and physically. They must become active learners, challenged to apply their prior knowledge and experience in new and increasingly more difficult situations. Instructional approaches should engage students in the process of learning rather than transmit information for them to receive. Middle grade students are especially responsive to hands-on activities in tactile, auditory, and visual instructional modes.

Classroom activities should provide students the opportunity to work both individually and in small- and large-group arrangements. The arrangement should be determined by the instructional goals as well as the nature of the activity. Individual work can help students develop confidence in their own ability to solve problems but should constitute only a portion of the middle school experience. Working in small groups provides students with opportunities to talk about ideas and listen to their peers, enables teachers to interact more closely with students, takes positive advantage of the social characteristics of the middle school student, and provides opportunities for students to exchange ideas and hence develops their ability to communicate and reason. Small-group work can involve collaborative or cooperative as well as independent work. Projects and small-group work can empower students to become more independent in their own learning. Whole-class discussions require students to synthesize, critique, and summarize strategies, ideas, or conjectures that are the products of individual and group work. These mathematical ideas can be expanded to, and integrated with, other subjects.

Materials

The 5–8 standards make the following assumptions about classroom materials:

- Every classroom will be equipped with ample sets of manipulative materials and supplies (e.g., spinners, cubes, tiles, geoboards, pattern blocks, scales, compasses, scissors, rulers, protractors, graph paper, grid-and-dot paper).
- Teachers and students will have access to appropriate resource materials from which to develop problems and ideas for explorations.
- All students will have a calculator with functions consistent with the tasks envisioned in this curriculum. Calculators should include the follow-

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ing features: algebraic logic including order of operations; computation in decimal and common fraction form; constant function for addition, subtraction, multiplication, and division; and memory, percent, square root, exponent, reciprocal, and +/− keys.

◊ Every classroom will have at least one computer available at all times for demonstrations and student use. Additional computers should be available for individual, small-group, and whole-class use.

STANDARD 8: PATTERNS AND FUNCTIONS

In grades 5–8, the mathematics curriculum should include explorations of patterns and functions so that students can—

- ***describe, extend, analyze, and create a wide variety of patterns;***
- ***describe and represent relationships with tables, graphs, and rules;***
- ***analyze functional relationships to explain how a change in one quantity results in a change in another;***
- ***use patterns and functions to represent and solve problems.***

Focus

One of the central themes of mathematics is the study of patterns and functions. This study requires students to recognize, describe, and generalize patterns and build mathematical models to predict the behavior of real-world phenomena that exhibit the observed pattern. The widespread occurrence of regular and chaotic pattern behavior makes the study of patterns and functions important. Exploring patterns helps students develop mathematical power and instills in them an appreciation for the beauty of mathematics.

The study of patterns in grades 5–8 builds on students' experiences in K–4 but shifts emphasis to an exploration of functions. However, work with patterns continues to be informal and relatively unburdened by symbolism. Students have opportunities to generalize and describe patterns and functions in many ways and to explore the relationships among them. When students make graphs, data tables, expressions, equations, or verbal descriptions to represent a single relationship, they discover that different representations yield different interpretations of a situation. In informal ways, students develop an understanding that functions are composed of variables that have a dynamic relationship: Changes in one variable result in change in another. The identification of the special characteristics of a relationship, such as minimum or maximum values or points at which the value of one of the variables is 0 (x- and y-intercepts), lays the foundation for a more formal study of functions in grades 9–12.

The theme of patterns and functions is woven throughout the 5–8 standards. It begins in K–4, is extended and made more central in 5–8, and reaches maturity with a natural extension to symbolic representation and supporting concepts, such as domain and range, in grades 9–12. Examples appropriate for grades 5–8 are incorporated into other standards for this age group.

STANDARD 9: ALGEBRA

In grades 5–8, the mathematics curriculum should include explorations of algebraic concepts and processes so that students can—

- ◊ ***understand the concepts of variable, expression, and equation;***
- ◊ ***represent situations and number patterns with tables, graphs, verbal rules, and equations and explore the interrelationships of these representations;***
- ◊ ***analyze tables and graphs to identify properties and relationships;***
- ◊ ***develop confidence in solving linear equations using concrete, informal, and formal methods;***
- ◊ ***investigate inequalities and nonlinear equations informally;***
- ◊ ***apply algebraic methods to solve a variety of real-world and mathematical problems.***

Focus

The middle school mathematics curriculum is, in many ways, a bridge between the concrete elementary school curriculum and the more formal mathematics curriculum of the high school. One critical transition is that between arithmetic and algebra. It is thus essential that in grades 5–8, students explore algebraic concepts in an informal way to build a foundation for the subsequent formal study of algebra. Such informal explorations should emphasize physical models, data, graphs, and other mathematical representations rather than facility with formal algebraic manipulation. Students should be taught to generalize number patterns to model, represent, or describe observed physical patterns, regularities, and problems. These informal explorations of algebraic concepts should help students to gain confidence in their ability to abstract relationships from contextual information and use a variety of representations to describe those relationships.

Activities in grades 5–8 should build on students' K–4 experiences with patterns. They should continue to emphasize concrete situations that allow students to investigate patterns in number sequences, make predictions, and formulate verbal rules to describe patterns. Learning to recognize patterns and regularities in mathematics and make generalizations about them requires practice and experience. Expanding the amount of time that students have to make this transition to more abstract ways of thinking increases their chances of success. By integrating informal algebraic experiences throughout the K–8 curriculum, students will develop confidence in using algebra to represent and solve problems. In addition, by the end of the eighth grade, students should be able to solve linear equations by formal methods and some nonlinear equations by informal means.

STANDARD 12: GEOMETRY

In grades 5–8, the mathematics curriculum should include the study of the geometry of one, two, and three dimensions in a variety of situations so that students can—

- ***identify, describe, compare, and classify geometric figures;***
- ***visualize and represent geometric figures with special attention to developing spatial sense;***
- ***explore transformations of geometric figures;***
- ***represent and solve problems using geometric models;***
- ***understand and apply geometric properties and relationships;***
- ***develop an appreciation of geometry as a means of describing the physical world.***

Focus

> Geometry is grasping space ... that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it. (Freudenthal 1973, p. 403).

The study of geometry helps students represent and make sense of the world. Geometric models provide a perspective from which students can analyze and solve problems, and geometric interpretations can help make an abstract (symbolic) representation more easily understood. Many ideas about number and measurement arise from attempts to quantify real-world objects that can be viewed geometrically. For example, the use of area models provides an interpretation for much of the arithmetic of decimals, fractions, ratios, proportions, and percents.

Students discover relationships and develop spatial sense by constructing, drawing, measuring, visualizing, comparing, transforming, and classifying geometric figures. Discussing ideas, conjecturing, and testing hypotheses precede the development of more formal summary statements. In the process, definitions become meaningful, relationships among figures are understood, and students are prepared to use these ideas to develop informal arguments. The informal exploration of geometry can be exciting and mathematically productive for middle school students. At this level, geometry should focus on investigating and using geometric ideas and relationships rather than on memorizing definitions and formulas.

The study of geometry in grades 5–8 links the informal explorations begun in grades K–4 to the more formalized processes studied in grades 9–12. The expanding logical capabilities of students in grades 5–8 allow them to draw inferences and make logical deductions from geometric problem situations. This does not imply that the study of geometry in grades 5–8 should be a formalized endeavor; rather, it should simply provide increased opportunities for students to engage in more systematic explorations.

CURRICULUM STANDARDS FOR GRADES 9–12

OVERVIEW

This section presents fourteen curriculum standards for grades 9–12:

1. ***Mathematics as Problem Solving***
2. ***Mathematics as Communication***
3. ***Mathematics as Reasoning***
4. ***Mathematical Connections***
5. ***Algebra***
6. ***Functions***
7. ***Geometry from a Synthetic Perspective***
8. ***Geometry from an Algebraic Perspective***
9. ***Trigonometry***
10. ***Statistics***
11. ***Probability***
12. ***Discrete Mathematics***
13. ***Conceptual Underpinnings of Calculus***
14. ***Mathematical Structure***

Background

Historically, the purposes of secondary school mathematics have been to provide students with opportunities to acquire the mathematical knowledge, skills, and modes of thought needed for daily life and effective citizenship, to prepare students for occupations that do not require formal study after graduation, and to prepare students for postsecondary education, particularly college. The *Standards'* Introduction describes a vision of school mathematics in which these purposes are embedded in a context that is both broader and more consistent with accelerating changes in today's society. High school graduates during the remainder of this century can expect to have four or more career changes. To develop the requisite adaptability, high school mathematics instruction must adopt broader goals for *all* students. It must provide experiences that encourage and enable students to value mathematics, gain confidence in their own mathematical ability, become mathematical problem solvers, communicate mathematically, and reason mathematically. The fourteen standards for grades 9–12 establish a framework for a core curriculum that reflects the needs of all students, explicitly recognizing that they will spend their adult lives in a society increasingly dominated by technology and quantitative methods.

In view of existing disparities in educational opportunity in mathematics and the increasing necessity that all individuals have options for further education and alternative careers, each standard identifies the mathemati-

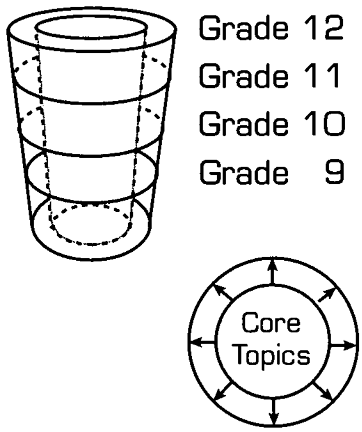

Fig. 1. A differentiated core curriculum

cal content or processes and the associated student activities that should be included in the curriculum for *all* students. As suggested by figure 1, the core curriculum is intended to provide a common body of mathematical ideas accessible to all students. We recognize that students entering high school differ in many ways, including mathematical achievement, but we believe these differences are best addressed by enrichment and extensions of the proposed content rather than by deletions. The mathematics curriculum must set high, but reasonable, expectations for *all* students.

The core curriculum can be extended in a variety of ways to meet the needs, interests, and performance levels of individual students or groups of students. To illustrate, many of the standards also specify topics that should be studied by college-intending students. We use the term *college-intending* not in an exclusionary sense, but only as a means by which to identify the additional mathematical topics that should be studied by students who plan to attend college. In fact, we believe that these additional curricular topics should be studied by all students who have demonstrated interest and achievement in mathematics.

A school curriculum in line with these standards should be organized so as to permit all students to progress as far into the mathematics proposed here as their achievement with the topic allows. In particular, students with exceptional mathematical talent who advance through the material more quickly than others may continue to college-level work in the mathematical sciences. However, we strongly recommend against acceleration that either omits content identified in these standards or advances students through it superficially.

Figure 1 also is intended to portray an expectation that mathematical ideas will grow and deepen as students progress through the curriculum and that the consolidation of learning is essential for all students during the senior year. Such a synthesis of mathematical knowledge will enhance students' prospects for securing employment and for both entering and successfully completing collegiate programs. It is, therefore, an underpinning of the proposed curriculum.

Underlying Assumptions

The standards for grades 9–12 are based on the following assumptions:

◊ Students entering grade 9 will have experienced mathematics in the context of the broad, rich curriculum outlined in the K–8 standards.

◊ The level of computational proficiency suggested in the K–8 standards will be expected of all students; however, no student will be denied access to the study of mathematics in grades 9–12 because of a lack of computational facility.

◊ Although arithmetic computation will not be a direct object of study in grades 9–12, number and operation sense, estimation skills, and the ability to judge the reasonableness of results will be strengthened in the context of applications and problem solving, including those situations dealing with issues of scientific computation.

◊ Scientific calculators with graphing capabilities will be available to all students at all times.

◊ A computer will be available at all times in every classroom for demon-

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stration purposes, and all students will have access to computers for individual and group work.

- ◊ At least three years of mathematical study will be required of all secondary school students.
- ◊ These three years of mathematical study will revolve around a core curriculum differentiated by the depth and breadth of the treatment of topics and by the nature of applications.
- ◊ Four years of mathematical study will be required of all college-intending students.
- ◊ These four years of mathematical study will revolve around a broadened curriculum that includes extensions of the core topics and for which calculus is no longer viewed as the capstone experience.
- ◊ All students will study appropriate mathematics during their senior year.

Features of the Mathematics Content

Initially, it may appear that an excessive amount of curriculum content is described in the 9–12 standards. When this content is evaluated, however, it should be remembered that the proposed 5–8 curriculum will enable students to enter high school with substantial gains in their conceptual and procedural understandings of algebra, in their knowledge of geometric concepts and relationships, and in their familiarity with informal, but conceptually based, methods for dealing with data and situations involving uncertainty. Moreover, additional instructional time can be gained by organizing the curriculum so that student learning is systematically maintained and review is embedded in the context of new topics or problem situations. With these conditions satisfied, it is our belief that it will be possible to address the recommended content within a three- or four-year sequence with the expectation of a reasonable level of student proficiency.

Traditional topics of algebra, geometry, trigonometry, and functions remain important components of the secondary school mathematics curriculum. However, the 9–12 standards call for a shift in emphasis from a curriculum dominated by memorization of isolated facts and procedures and by proficiency with paper-and-pencil skills to one that emphasizes conceptual understandings, multiple representations and connections, mathematical modeling, and mathematical problem solving. The integration of ideas from algebra and geometry is particularly strong, with graphical representation playing an important connecting role. Thus, frequent reference to graphing utilities will be found throughout these standards; by this we mean a computer with appropriate graphing software or a graphing calculator. In addition, topics from statistics, probability, and discrete mathematics are elevated to a more central position in the curriculum for all students. Specific topics that should be given either increased or reduced emphasis are summarized in the chart.

Patterns of Instruction

The broadened view of mathematics described in the Introduction to this document under the rubric *mathematical power*, together with the capabilities of available and emerging technology, suggests a need for changes in instructional patterns and in the roles of both teachers and students.

A variety of instructional methods should be used in classrooms in order to cultivate students' abilities to investigate, to make sense of, and to construct meanings from new situations; to make and provide arguments for conjectures; and to use a flexible set of strategies to solve problems from both within and outside mathematics. In addition to traditional teacher demonstrations and teacher-led discussions, greater opportunities should be provided for small-group work, individual explorations, peer instruction, and whole-class discussions in which the teacher serves as a moderator.

These alternative methods of instruction will require the teacher's role to shift from dispensing information to facilitating learning, from that of director to that of catalyst and coach. The introduction of new topics and most subsumed objectives should, whenever possible, be embedded in problem situations posed in an environment that encourages students to explore, formulate and test conjectures, prove generalizations, and discuss and apply the results of their investigations. Such an instructional setting enables students to approach the learning of mathematics both creatively and independently and thereby strengthen their confidence and skill in doing mathematics.

The role of students in the learning process in grades 9–12 should shift in preparation for their entrance into the work force or higher education. Experiences designed to foster continued intellectual curiosity and increasing independence should encourage students to become self-directed learners who routinely engage in constructing, symbolizing, applying, and generalizing mathematical ideas. Such experiences are essential in order for students to develop the capability for their own lifelong learning and to internalize the view that mathematics is a process, a body of knowledge, and a human creation.

The use of technology in instruction should further alter both the teaching and the learning of mathematics. Computer software can be used effectively for class demonstrations and independently by students to explore additional examples, perform independent investigations, generate and summarize data as part of a project, or complete assignments. Calculators and computers with appropriate software transform the mathematics classroom into a laboratory much like the environment in many science classes, where students use technology to investigate, conjecture, and verify their findings. In this setting, the teacher encourages experimentation and provides opportunities for students to summarize ideas and establish connections with previously studied topics.

The most fundamental consequence of changes in patterns of instruction in response to technology-rich classroom environments is the emergence of a new classroom dynamic in which teachers and students become natural partners in developing mathematical ideas and solving mathematical problems.

Assessment of student learning should be viewed as an integral part of instruction and should be aligned with key aspects of instruction, such as the use of technology.

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STANDARD 5: ALGEBRA

In grades 9–12, the mathematics curriculum should include the continued study of algebraic concepts and methods so that all students can—

- ◊ ***represent situations that involve variable quantities with expressions, equations, inequalities, and matrices;***
- ◊ ***use tables and graphs as tools to interpret expressions, equations, and inequalities;***
- ◊ ***operate on expressions and matrices, and solve equations and inequalities;***
- ◊ ***appreciate the power of mathematical abstraction and symbolism;***

and so that, in addition, college-intending students can—

- ◊ ***use matrices to solve linear systems;***
- ◊ ***demonstrate technical facility with algebraic transformations, including techniques based on the theory of equations.***

Focus

Algebra is the language through which most of mathematics is communicated. It also provides a means of operating with concepts at an abstract level and then applying them, a process that often fosters generalizations and insights beyond the original context.

Aspects of this standard represent extensions of algebraic concepts developed first in grades 5–8. Whereas this earlier work was developed as a generalization of arithmetic, algebra in grades 9–12 will focus on its own logical framework and consistency. As a result, for example, algebraic symbols may represent objects rather than numbers, as in "p + q" representing the sum of two polynomials. This more sophisticated understanding of algebraic representation is a prerequisite to further formal work in virtually all mathematical subjects, including statistics, linear algebra, discrete mathematics, and calculus. Moreover, the increasing use of quantitative methods, both in the natural sciences and in such disciplines as economics, psychology, and sociology, have made algebraic processing an important tool for applying mathematics.

The proposed algebra curriculum will move away from a tight focus on manipulative facility to include a greater emphasis on conceptual understanding, on algebra as a means of representation, and on algebraic methods as a problem-solving tool. For the core program, this represents a trade-off in instructional time as well as in emphasis. For college-intending students who can expect to use their algebraic skills more often, an appropriate level of proficiency remains a goal. Even for these students, however, available and projected technology forces a rethinking of the level of skill expectations.

STANDARD 6: FUNCTIONS

In grades 9–12, the mathematics curriculum should include the continued study of functions so that all students can—

- ***model real-world phenomena with a variety of functions;***
- ***represent and analyze relationships using tables, verbal rules, equations, and graphs;***
- ***translate among tabular, symbolic, and graphical representations of functions;***
- ***recognize that a variety of problem situations can be modeled by the same type of function;***
- ***analyze the effects of parameter changes on the graphs of functions;***

and so that, in addition, college-intending students can—

- ***understand operations on, and the general properties and behavior of, classes of functions.***

Focus

The concept of function is an important unifying idea in mathematics. Functions, which are special correspondences between the elements of two sets, are common throughout the curriculum. In arithmetic, functions appear as the usual operations on numbers, where a pair of numbers corresponds to a single number, such as the sum of the pair; in algebra, functions are relationships between variables that represent numbers; in geometry, functions relate sets of points to their images under motions such as flips, slides, and turns; and in probability, they relate events to their likelihoods. The function concept also is important because it is a mathematical representation of many input-output situations found in the real world, including those that recently have arisen as a result of technological advances. An obvious example is the [$\sqrt{x}$] key on a calculator.

STANDARD 7: GEOMETRY FROM A SYNTHETIC PERSPECTIVE

In grades 9–12, the mathematics curriculum should include the continued study of the geometry of two and three dimensions so that all students can—

- ***interpret and draw three-dimensional objects;***
- ***represent problem situations with geometric models and apply properties of figures;***
- ***classify figures in terms of congruence and similarity and apply these relationships;***
- ***deduce properties of, and relationships between, figures from given assumptions;***

and so that, in addition, college-intending students can—

- ***develop an understanding of an axiomatic system through investigating and comparing various geometries.***

Focus

This component of the 9–12 geometry strand should provide experiences that deepen students' understanding of shapes and their properties, with an emphasis on their wide applicability in human activity. The curriculum should be infused with examples of how geometry is used in recreations (as in billiards or sailing); in practical tasks (as in purchasing paint for a room); in the sciences (as in the description and analysis of mineral crystals); and in the arts (as in perspective drawing).

High school geometry should build on the strong conceptual foundation students develop in the new K–8 programs. Students should have opportunities to visualize and work with three-dimensional figures in order to develop spatial skills fundamental to everyday life and to many careers. Physical models and other real-world objects should be used to provide a strong base for the development of students' geometric intuition so that they can draw on these experiences in their work with abstract ideas.

STANDARD 8: GEOMETRY FROM AN ALGEBRAIC PERSPECTIVE

In grades 9–12, the mathematics curriculum should include the study of the geometry of two and three dimensions from an algebraic point of view so that all students can—

- ***translate between synthetic and coordinate representations;***
- ***deduce properties of figures using transformations and using coordinates;***
- ***identify congruent and similar figures using transformations;***
- ***analyze properties of Euclidean transformations and relate translations to vectors;***

and so that, in addition, college-intending students can—

- ***deduce properties of figures using vectors;***
- ***apply transformations, coordinates, and vectors in problem solving.***

Focus

One of the most important connections in all of mathematics is that between geometry and algebra. Historically, mathematics took a great stride forward in the seventeenth century when the geometric ideas of the ancients were expressed in the language of coordinate geometry, thus providing new tools for the solution of a wide range of problems.

More recently, the study of geometry through the use of transformations-the geometric counterpart of functions—has changed the subject from static to dynamic, providing in the process great additional power that can be used, for example, to describe and produce moving figures on a video screen. Viewed as an algebraic system, transformations also provide college-intending students with valuable experiences with properties of function composition and group structure.

The interplay between geometry and algebra strengthens students' ability to formulate and analyze problems from situations both within and outside mathematics. Although students will at times work separately in synthetic, coordinate, and transformation geometry, they should have as many opportunities as possible to compare, contrast, and translate among these systems. A fundamental idea students should come to understand is that specific problems are often more easily solved in one or another of these systems.

EXCERPTS FROM THE

PROFESSIONAL STANDARDS FOR TEACHING MATHEMATICS

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STANDARDS FOR TEACHING MATHEMATICS

The *Professional Standards for Teaching Mathematics* is designed, along with the *Curriculum and Evaluation Standards for School Mathematics*, to establish a broad framework to guide reform in school mathematics in the next decade. In particular, these standards present a vision of what teaching should entail to support the changes in curriculum set out in the *Curriculum and Evaluation Standards*. This document spells out what teachers need to know to teach toward new goals for mathematics education and how teaching should be evaluated for the purpose of improvement. We challenge all who have responsibility for any part of the support and development of mathematics teachers and teaching to use these standards as a basis for discussion and for making needed change so that we can reach our goal of a quality mathematics education for every child.

—From the preface of *Professional Standards for Teaching Mathematics*

OVERVIEW

This section presents six standards for the teaching of mathematics organized under four categories.

Tasks

Standard 1. Worthwhile Mathematical Tasks

Discourse

Standard 2. Teacher's Role in Discourse

Standard 3. Students' Role in Discourse

Standard 4. Tools for Enhancing Discourse

Environment

Standard 5. Learning Environment

Analysis

Standard 6. Analysis of Teaching and Learning

INTRODUCTION

The *Curriculum and Evaluation Standards for School Mathematics* represents NCTM's vision of what students should learn in mathematics classrooms. Congruent with the aims and rhetoric of the current reform movement in mathematics education (e.g., National Research Council 1989, 1990), the *Standards* is threaded with a commitment to developing the mathematical literacy and power of all students. Being mathematically literate includes having an appreciation of the value and beauty of mathematics as well as being able and inclined to appraise and use quantitative information. Mathematical power encompasses the ability to "explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems" and the self-confidence and disposition to do so (National Council of Teachers of Mathematics 1989, p. 5). It also includes being able to formulate and solve problems, to judge the role of mathematical reasoning in a real-life situation, and to communicate mathematically.

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ASSUMPTIONS

The standards for teaching are based on four assumptions about the practice of mathematics teaching:

1. *The goal of teaching mathematics is to help all students develop mathematical power.* The *Curriculum and Evaluation Standards for School Mathematics* furnishes the basis for a curriculum in which mathematical reasoning, communication, problem solving, and connections are central. Teachers must help every student develop conceptual and procedural understandings of number, operations, geometry, measurement, statistics, probability, functions, and algebra and the connections among ideas. They must engage all students in formulating and solving a wide variety of problems, making conjectures and constructing arguments, validating solutions, and evaluating the reasonableness of mathematical claims. Along with all this, teachers must foster in students the disposition to use and engage in mathematics, an appreciation of its beauty and utility, and a tolerance for getting stuck or sidetracked. Teachers must help students realize that mathematical thinking involves dead ends and detours and encourage them to persevere when confronted with a puzzling problem and to develop the self-confidence and interest to do so.

2. *WHAT students learn is fundamentally connected with how they learn it.* Students' opportunities to learn mathematics are a function of the setting and the kinds of tasks and discourse in which they participate. What students learn—about particular concepts and procedures as well as about thinking mathematically—depends on the ways in which they engage in mathematical activity in their classrooms. Their dispositions toward mathematics are also shaped by such experiences. Consequently, the goal of developing students' mathematical power requires careful attention to pedagogy as well as to curriculum.

3. *All students can learn to think mathematically.* The goals described in the *Curriculum and Evaluation Standards for School Mathematics* are goals for all students. Goals such as learning to make conjectures, to argue about mathematics using mathematical evidence, to formulate and solve problems—even perplexing ones—and to make sense of mathematical ideas are not just for some group thought to be "bright" or "mathematically able." Every student can—and should—learn to reason and solve problems, to make connections across a rich web of topics and experiences, and to communicate mathematical ideas. By "every student" we mean specifically—

◊ students who have been denied access in any way to educational opportunities as well as those who have not;

◊ students who are African American, Hispanic, American Indian, and other minorities as well as those who are considered to be part of the majority;

◊ students who are female as well as those who are male;

◊ students who have not been successful as well as those who have been successful in school and in mathematics.

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This assumption is supported by the vignettes, which were drawn from classrooms with students of diverse cultural, linguistic, and socioeconomic backgrounds.

4. *Teaching is a complex practice and hence not reducible to recipes or prescriptions*. First of all, teaching mathematics draws on knowledge from several domains: knowledge of mathematics, of diverse learners, of how students learn mathematics, of the contexts of classroom, school, and society. Such knowledge is general. However, teachers must also consider the particular, for teaching is context-specific. Theoretical knowledge about adolescent development, for instance, can only partly inform a decision about particular students learning a particular mathematical concept in a given context. Second, as teachers weave together knowledge from these different domains to decide how to respond to a student's question, how to represent a particular mathematical idea, how long to pursue the discussion of a problem, or what task to use to engage students in a new topic, they often find themselves having to balance multiple goals and considerations. Making such decisions depends on a variety of factors that cannot be determined in the abstract or governed by rules of thumb.

Because teaching mathematics well is a complex endeavor, it cannot be reduced to a recipe for helping students learn. Instead, good teaching depends on a host of considerations and understandings. Good teaching demands that teachers reason about pedagogy in professionally defensible ways within the particular contexts of their own work. The standards for teaching mathematics are designed to help guide the processes of such reasoning, highlighting issues that are crucial in creating the kind of teaching practice that supports the learning goals of the *Curriculum and Evaluation Standards for School Mathematics*. This section circumscribes themes and values but does not—indeed, it could not—prescribe "right" practice.

STANDARD 1: WORTHWHILE MATHEMATICAL TASKS

The teacher of mathematics should pose tasks that are based on—

- ***sound and significant mathematics;***
- ***knowledge of students' understandings, interests, and experiences;***
- ***knowledge of the range of ways that diverse students learn mathematics;***

and that

- ***engage students' intellect;***
- ***develop students' mathematical understandings and skills;***
- ***stimulate students to make connections and develop a coherent framework for mathematical ideas;***
- ***call for problem formulation, problem solving, and mathematical reasoning;***
- ***promote communication about mathematics;***
- ***represent mathematics as an ongoing human activity;***
- ***display sensitivity to, and draw on, students' diverse background experiences and dispositions;***
- ***promote the development of all students' dispositions to do mathematics.***

Elaboration

Teachers are responsible for the quality of the mathematical tasks in which students engage. A wide range of materials exists for teaching mathematics: problem booklets, computer software, practice sheets, puzzles, manipulative materials, calculators, textbooks, and so on. These materials contain tasks from which teachers can choose. Also, teachers often create their own tasks for students: projects, problems, worksheets, and the like. Some tasks grow out of students' conjectures or questions. Teachers should choose and develop tasks that are likely to promote the development of students' understandings of concepts and procedures in a way that also fosters their ability to solve problems and to reason and communicate mathematically. Good tasks are ones that do not separate mathematical thinking from mathematical concepts or skills, that capture students' curiosity, and that invite them to speculate and to pursue their hunches. Many such tasks can be approached in more than one interesting and legitimate way; some have more than one reasonable solution. These tasks, consequently, facilitate significant classroom discourse, for they require that students reason about different strategies and outcomes, weigh the pros and cons of alternatives, and pursue particular paths.

In selecting, adapting, or generating mathematical tasks, teachers must base their decisions on three areas of concern: the mathematical content, the students, and the ways in which students learn mathematics.

In considering the mathematical content of a task, teachers should consider how appropriately the task represents the concepts and procedures entailed. For example, if students are to gather, summarize, and interpret

data, are the statistics they are expected to generate appropriate? Does it make sense to calculate a mean? If there is an explanation of a procedure, such as calculating a mean, does that explanation focus on the underlying concepts or is it merely mechanical? Teachers must also use a curricular perspective, considering the potential of a task to help students progress in their cumulative understanding of a particular domain and to make connections among ideas they have studied in the past and those they will encounter in the future.

A second content consideration is to assess what the task conveys about what is entailed in doing mathematics. Some tasks, although they deal nicely with the concepts and procedures, involve students in simply producing right answers. Others require students to speculate, to pursue alternatives, to face decisions about whether or not their approaches are valid. For example, one task might require students to find means, medians, and modes for given sets of data. Another might require them to decide whether to calculate means, medians, or modes as the best measures of central tendency, given particular sets of data and particular claims they would like to make about the data, then to calculate those statistics, and finally to explain and defend their decisions. Like the first task, the second would offer students the opportunity to practice finding means, medians, and modes. Only the second, however, conveys the important point that summarizing data involves decisions related to the data and the purposes for which the analysis is being used. Tasks should foster students' sense that mathematics is a changing and evolving domain, one in which ideas grow and develop over time and to which many cultural groups have contributed. Drawing on the history of mathematics can help teachers to portray this idea: exploring alternative numeration systems or investigating non-Euclidean geometries, for example. Fractions evolved out of the Egyptians' attempts to divide quantities—four things shared among ten people. This fact could provide the explicit basis for a teacher's approach to introducing fractions.

A third content consideration centers on the development of appropriate skill and automaticity. Teachers must assess the extent to which skills play a role in the context of particular mathematical topics. A goal is to create contexts that foster skill development even as students engage in problem solving and reasoning. For example, elementary school students should develop rapid facility with addition and multiplication combinations. Rolling pairs of dice as part of an investigation of probability can simultaneously provide students with practice with addition. Trying to figure out how many ways 36 desks can be arranged in equal-sized groups—and whether there are more or fewer possible groupings with 36, 37, 38, 39, or 40 desks—presses students to produce each number's factors quickly. As they work on this problem, students have concurrent opportunities to practice multiplication facts and to develop a sense of what factors are. Further, the problem may provoke interesting questions: How many factors does a number have? Do larger numbers necessarily have more factors? Is there a number that has more factors than 36? Even as students pursue such questions, they practice and use multiplication facts, for skill plays a role in problem solving at all levels. Teachers of algebra and geometry must similarly consider which skills are essential and why and seek ways to develop essential skills in the contexts in which they matter. What do students need to memorize? How can that be facilitated?

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The content is unquestionably a crucial consideration in appraising the value of a particular task. Defensible reasoning about the mathematics of a task must be based on a thoughtful understanding of the topic at hand as well as of the goals and purposes of carrying out particular mathematical processes.

Teachers must also consider the students in deciding on the appropriateness of a given task. They must consider what they know about their particular students as well as what they know more generally about students from psychological, cultural, sociological, and political perspectives. For example, teachers should consider gender issues in selecting tasks, deliberating about ways in which the tasks may be an advantage either to boys or to girls—and a disadvantage to the others—in some systematic way.

In thinking about their particular students, teachers must weigh several factors. One centers on what their students already know and can do, what they need to work on, and how much they seem ready to stretch intellectually. Well-chosen tasks afford teachers opportunities to learn about their students' understandings even as the tasks also press the students forward. Another factor is their students' interests, dispositions, and experiences. Teachers should aim for tasks that are likely to engage their students' interests. Sometimes this means choosing familiar application contexts: for example, having students explore issues related to the finances of a school store or something in the students' community. Not always, however, should concern for "interest" limit the teacher to tasks that relate to the familiar everyday worlds of the students; theoretical or fanciful tasks that challenge students intellectually are also interesting: number theory problems, for instance. When teachers work with groups of students for whom the notion of "argument" is uncomfortable or at variance with community norms of interaction, teachers must consider carefully the ways in which they help students to engage in mathematical discourse. Defensible reasoning about students must be based on the assumption that all students can learn and do mathematics, that each one is worthy of being challenged intellectually. Sensitivity to the diversity of students' backgrounds and experiences is crucial in selecting worthwhile tasks.

Knowledge about ways in which students learn mathematics is a third basis for appraising tasks. The mode of activity, the kind of thinking required, and the way in which students are led to explore the particular content all contribute to the kind of learning opportunity afforded by the task. Knowing that students need opportunities to model concepts concretely and pictorially, for example, might lead a teacher to select a task that involves such representations. An awareness of common student confusions or misconceptions around a certain mathematical topic would help a teacher to select tasks that engage students in exploring critical ideas that often underlie those confusions. Understanding that writing about one's ideas helps to clarify and develop one's understandings would make a task that requires students to write explanations look attractive. Teachers' understandings about how students learn mathematics should be informed by research as well as their own experience. Just as teachers can learn more about students' understandings from the tasks they provide students, so, too, can they gain insights into how students learn mathematics. To capitalize on the opportunity, teachers should deliberately select tasks that provide them with windows on students' thinking.

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STANDARD 2: THE TEACHER'S ROLE IN DISCOURSE

The teacher of mathematics should orchestrate discourse by—

- ***posing questions and tasks that elicit, engage, and challenge each student's thinking;***
- ***listening carefully to students' ideas;***
- ***asking students to clarify and justify their ideas orally and in writing;***
- ***deciding what to pursue in depth from among the ideas that students bring up during a discussion;***
- ***deciding when and how to attach mathematical notation and language to students' ideas;***
- ***deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let a student struggle with a difficulty;***
- ***monitoring students' participation in discussions and deciding when and how to encourage each student to participate.***

Elaboration

Like a piece of music, the classroom discourse has themes that pull together to create a whole that has meaning. The teacher has a central role in orchestrating the oral and written discourse in ways that contribute to students' understanding of mathematics.

The kind of mathematical discourse described above does not occur spontaneously in most classrooms. It requires an environment in which everyone's thinking is respected and in which reasoning and arguing about mathematical meanings is the norm. Students, used to the teacher doing most of the talking while they remain passive, need guidance and encouragement in order to participate actively in the discourse of a collaborative community. Some students, particularly those who have been successful in more traditional mathematics classrooms, may be resistant to talking, writing, and reasoning together about mathematics.

One aspect of the teacher's role is to provoke students' reasoning about mathematics. Teachers must do this through the tasks they provide and the questions they ask. For example, teachers should regularly follow students' statements with, "Why?" or by asking them to explain. Doing this consistently, irrespective of the correctness of students' statements, is an important part of establishing a discourse centered on mathematical reasoning. Cultivating a tone of interest when asking a student to explain or elaborate on an idea helps to establish norms of civility and respect rather than criticism and doubt. Teachers also stimulate discourse by asking students to write explanations for their solutions and provide justifications for their ideas.

Emphasizing tasks that focus on thinking and reasoning serves to provide the teacher with ongoing assessment information. Well-posed questions

can simultaneously elicit and extend students' thinking. The teacher's skill at formulating questions to orchestrate the oral and written discourse in the direction of mathematical reasoning is crucial.

A second feature of the teacher's role is to be active in a different way from that in traditional classroom discourse. Instead of doing virtually all the talking, modeling, and explaining themselves, teachers must encourage and expect students to do so. Teachers must do more listening, students more reasoning. For the discourse to promote students' learning, teachers must orchestrate it carefully. Because many more ideas will come up than are fruitful to pursue at the moment, teachers must filter and direct the students' explorations by picking up on some points and by leaving others behind. Doing this prevents student activity and talk from becoming too diffuse and unfocused. Knowledge of mathematics, of the curriculum, and of students should guide the teacher's decisions about the path of the discourse. Other key decisions concern the teacher's role in contributing to the discourse. Beyond asking clarifying or provocative questions, teachers should also, at times, provide information and lead students. Decisions about when to let students struggle to make sense of an idea or a problem without direct teacher input, when to ask leading questions, and when to tell students something directly are crucial to orchestrating productive mathematical discourse in the classroom. Such decisions depend on teachers' understandings of mathematics and of their students—on judgments about the things that students can figure out on their own or collectively and those for which they will need input.

A third aspect of the teacher's role in orchestrating classroom discourse is to monitor and organize students' participation. Who is volunteering comments and who is not? How are students responding to one another? What are different students able to record or represent on paper about their thinking? What are they able to put into words, in what kinds of contexts? Teachers must be committed to engaging every student in contributing to the thinking of the class. Teachers must judge when students should work and talk in small groups and when the whole group is the most useful context. They must make sensitive decisions about how turns to speak are shared in the large group—for example, whom to call on when and whether to call on particular students who do not volunteer. Substantively, if the discourse is to focus on making sense of mathematics, on learning to reason mathematically, teachers must refrain from calling only on students who seem to have right answers or valid ideas to allow a broader spectrum of thinking to be explored in the discourse. By modeling respect for students' thinking and conveying the assumption that students make sense, teachers can encourage students to participate within a norm that expects group members to justify their ideas. Teachers must think broadly about a variety of ways for students to contribute to the class's thinking—using means that are written or pictorial, concrete or representational, as well as oral.

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STANDARD 3: STUDENTS' ROLE IN DISCOURSE

The teacher of mathematics should promote classroom discourse in which students—

- ◊ ***listen to, respond to, and question the teacher and one another;***
- ◊ ***use a variety of tools to reason, make connections, solve problems, and communicate;***
- ◊ ***initiate problems and questions;***
- ◊ ***make conjectures and present solutions;***
- ◊ ***explore examples and counterexamples to investigate a conjecture;***
- ◊ ***try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers;***
- ◊ ***rely on mathematical evidence and argument to determine validity.***

Elaboration

The nature of classroom discourse is a major influence on what students learn about mathematics. Students should engage in making conjectures, proposing approaches and solutions to problems, and arguing about the validity of particular claims. They should learn to verify, revise, and discard claims on the basis of mathematical evidence and use a variety of mathematical tools. Whether working in small or large groups, they should be the audience for one another's comments—that is, they should speak to one another, aiming to convince or to question their peers. Above all, the discourse should be focused on making sense of mathematical ideas, on using mathematical ideas sensibly in setting up and solving problems.

STANDARD 4: TOOLS FOR ENHANCING DISCOURSE

The teacher of mathematics, in order to enhance discourse, should encourage and accept the use of—

- ***computers, calculators, and other technology;***
- ***concrete materials used as models;***
- ***pictures, diagrams, tables, and graphs;***
- ***invented and conventional terms and symbols;***
- ***metaphors, analogies, and stories;***
- ***written hypotheses, explanations, and arguments;***
- ***oral presentations and dramatizations.***

Elaboration

In order to establish a discourse that is focused on exploring mathematical ideas, not just on reporting correct answers, the means of mathematical communication and approaches to mathematical reasoning must be broad and varied. Teachers must value and encourage the use of a variety of tools rather than placing excessive emphasis on conventional mathematical symbols. Various means for communicating about mathematics should be accepted, including drawings, diagrams, invented symbols, and analogies. The teacher should introduce conventional notation at points when doing so can further the work or the discourse at hand. Teachers should also help students learn to use calculators, computers, and other technological devices as tools for mathematical discourse. Given the range of mathematical tools available, teachers should often allow and encourage students to select the means they find most useful for working on or discussing a particular mathematical problem. At other times, in order to develop students' repertoire of mathematical tools, teachers may specify the means students are to use.

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STANDARD 5: LEARNING ENVIRONMENT

The teacher of mathematics should create a learning environment that fosters the development of each student's mathematical power by—

- ◊ ***providing and structuring the time necessary to explore sound mathematics and grapple with significant ideas and problems;***
- ◊ ***using the physical space and materials in ways that facilitate students' learning of mathematics;***
- ◊ ***providing a context that encourages the development of mathematical skill and proficiency;***
- ◊ ***respecting and valuing students' ideas, ways of thinking, and mathematical dispositions;***

and by consistently expecting and encouraging students to—

- ◊ ***work independently or collaboratively to make sense of mathematics;***
- ◊ ***take intellectual risks by raising questions and formulating conjectures;***
- ◊ ***display a sense of mathematical competence by validating and supporting ideas with mathematical argument.***

Elaboration

This standard focuses on key dimensions of a learning environment in which serious mathematical thinking can take place: a genuine respect for others' ideas, a valuing of reason and sense-making, pacing and timing that allow students to puzzle and to think, and the forging of a social and intellectual community. Such a learning environment should help all students believe in themselves as successful mathematical thinkers.

What teachers convey about the value and sense of students' ideas affects students' mathematical dispositions in the classroom. Students are more likely to take risks in proposing their conjectures, strategies, and solutions in an environment in which the teacher respects students' ideas, whether conventional or nonstandard, whether valid or invalid. Teachers convey this kind of respect by probing students' thinking, by showing interest in understanding students' approaches and ideas, and by refraining from ridiculing students. Furthermore, and equally important, teachers must teach students to respect and be interested in one another's ideas.

Demonstrating respect for students' ideas does not imply, however, that teachers or students accept all ideas as reasonable or valid. The purpose of valuing students' ideas and ways of thinking is not just to make students feel good but to foster the development of their understanding of, and power with, mathematics. Therefore, the central focus of the classroom environment must be on sense-making. Mathematical concepts and procedures—indeed, mathematical skills—are central to making sense of mathematics and to reasoning mathematically. Teachers should consistently expect students to explain their ideas, to justify their solutions, and to

persevere when they are stuck. Teachers must also help students learn to expect and ask for justifications and explanations from one another. Teachers' own explanations must similarly focus on underlying meanings; something a teacher says is not true simply because he or she "said so."

Emphasizing reasoning and justification implies that students should be encouraged and expected to question one another's ideas and to explain and support their own ideas in the face of others' challenges. Teachers must help students learn how to do this: Students need to learn how to question another's conjecture or solution with respect for that person's thinking and knowledge. They also need to learn how to justify their own claims without becoming hostile or defensive.

Serious mathematical thinking takes time as well as intellectual courage and skills. A learning environment that supports problem solving must allow time for students to puzzle, to be stuck, to try alternative approaches, and to confer with one another and with the teacher. Furthermore, for many worthwhile mathematical tasks, tasks that require reasoning and problem solving, the speed, pace, and quantity of students' work are inappropriate criteria for "doing well." Too often, students have developed the idea that if they cannot answer a mathematical question almost immediately, then they might as well give up. Teachers must encourage and expect students to persevere, to take the time to figure things out. In discussions, the teacher must allow time for students to respond to questions and must also expect students to give one another time to think, without bursting in, frantically waving hands, or showing impatience.

Students' learning of mathematics is enhanced in a learning environment that is built as a community of people collaborating to make sense of mathematical ideas. It is a key function of the teacher to develop and nurture students' abilities to learn with and from others—to clarify definitions and terms to one another, consider one another's ideas and solutions, and argue together about the validity of alternative approaches and answers. Classroom structures that can encourage and support this collaboration are varied: students may at times work independently, conferring with others as necessary; at other times students may work in pairs or in small groups. Whole-class discussions are yet another profitable format. No single arrangement will work at all times; teachers should use these arrangements flexibly to pursue their goals.

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STANDARD 6: ANALYSIS OF TEACHING AND LEARNING

The teacher of mathematics should engage in ongoing analysis of teaching and learning by—

- ***observing, listening to, and gathering other information about students to assess what they are learning;***
- ***examining effects of the tasks, discourse, and learning environment on students' mathematical knowledge, skills, and dispositions;***

in order to—

- ***ensure that every student is learning sound and significant mathematics and is developing a positive disposition toward mathematics;***
- ***challenge and extend students' ideas;***
- ***adapt or change activities while teaching;***
- ***make plans, both short- and long-range;***
- ***describe and comment on each student's learning to parents and administrators, as well as to the students themselves.***

Elaboration

Assessment of students and analysis of instruction are fundamentally interconnected. Mathematics teachers should monitor students' learning on an ongoing basis in order to assess and adjust their teaching. Observing and listening to students during class can help teachers, on the spot, tailor their questions or tasks to provoke and extend students' thinking and understanding. Teachers must also use information about what students are understanding to revise and adapt their short- and long-range plans: for the tasks they select and for the approaches they choose to orchestrate the classroom discourse. Similarly, students' understandings and dispositions should guide teachers in shaping and reshaping the learning environment of the classroom. Additionally, teachers have the responsibility of describing and commenting on students' learning to administrators, to parents, and to the students themselves.

Students' mathematical power depends on a varied set of understandings, skills, and dispositions. Teachers must attend to the broad array of dimensions that contribute to students' mathematical competence as outlined in the *Curriculum and Evaluation Standards for School Mathematics*. They should assess students' understandings of concepts and procedures, including the connections they make among various concepts and procedures. Teachers must also assess the development of students' ability to reason mathematically—to make conjectures, to justify and revise claims on the basis of mathematical evidence, and to analyze and solve problems. Students' dispositions toward mathematics—their confidence, interest, enjoyment, and perseverance—are yet another key dimension that teachers should monitor.

Paper-and-pencil tests, although one useful medium for judging some aspects of students' mathematical knowledge, cannot suffice to provide teachers with the insights they need about their students' understandings in order to make instruction as effectively responsive as possible. Teachers need information gathered in a variety of ways and using a range of sources. Observing students participating in a small-group discussion may contribute valuable insights related to their abilities to communicate mathematically. Interviews with individual students will complement that information and also provide information about students' conceptual and procedural understanding. Students' journals are yet another source that can help teachers appraise their students' development. Teachers can also learn a great deal from closely watching and listening to students during whole-group discussions.

As they monitor students' understandings of, and dispositions toward, mathematics, teachers should ask themselves questions about the nature of the learning environment they have created, of the tasks they have been using, and of the kind of discourse they have been fostering. They should seek to understand the links between these and what is happening with their students. If, for example, students are having trouble understanding inverse functions, is it because of the kinds of tasks in which they have been engaged? Is it related to the ways in which the group has explored and discussed ideas about functions and their inverses? Although it may be that the students lack prerequisite understandings, it could also be that this is a difficult piece of mathematics or that the teacher needs to consider alternative ways to help students "unpack" the ideas. Or, if students quickly give up when a direct route for solving a problem is not apparent, teachers must consider how the experiences that students have been having and the environment in which they have been working may not have helped them to develop the perseverance and confidence they need. Teachers need to analyze continually what they are seeing and hearing and explore alternative interpretations of that information. They need to consider what such insights suggest about how the environment, tasks, and discourse could be enhanced, revised, or adapted in order to help students learn.

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