DOCUMENT RESUME

ED 409 162 SE 060 296

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|---|---|
| AUTHOR | Brinker, Laura |
| TITLE | Using Structured Representations To Solve Fraction Problems: A Discussion of Seven Students' Strategies. |
| SPONS AGENCY | National Science Foundation, Arlington, VA. |
| PUB DATE | Mar 97 |
| NOTE | 28p.; Paper presented at the Annual Meeting of the American Educational Research Association (Chicago, IL, March 24-28, 1997). |
| PUB TYPE | Reports - Research (143) -- Speeches/Meeting Papers (150) |
| EDRS PRICE | MF01/PC02 Plus Postage. |
| DESCRIPTORS | Algorithms; Arithmetic; *Cognitive Processes; *Concept Formation; Educational Strategies; *Fractions; Grade 4; Grade 5; Intermediate Grades; *Knowledge Representation; Mathematics Activities; Mathematics Instruction; *Problem Solving; Ratios (Mathematics) |

ABSTRACT

This paper describes how a target group of seven students in a combined fourth and fifth grade mathematics class used structured representations to solve fraction problems situated within various realistic contexts. Emphasis is given to the ways in which students' thinking about rational number concepts influences and is influenced by the students' use of two structured representations. The first structured representation was a set of fraction strips the students used as a manipulative. The other structured representation was a ratio table, a pictorial model used flexibly by most of the students. Findings indicate that most of the target students did not connect their symbolic procedures to the underlying concepts, particularly when they tried to write formal addition sentences using the fraction strips. The findings also suggest that the ratio table afforded more opportunities for the students to apply their informal knowledge of fractions as quantities. Contains 28 references. (DDR)

# Using Structured Representations to Solve Fraction Problems: A Discussion of Seven Students' Strategies

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Paper presented at the annual meeting of the
American Educational Research Association
Chicago, IL March 1997



The research reported in this article was supported, in part, by the National Science Foundation. The opinions expressed are those of the author and not necessarily those of the Foundation.

The author wishes to thank Thomas P. Carpenter and Edwin M. Dickey for their thoughtful critiques of earlier drafts of this paper.

This paper describes how a target group of seven students in a combined fourth- fifth grade math class used structured representations to solve fraction problems that were situated within various realistic contexts. Emphasis is given to the ways in which students' thinking about rational number concepts influenced and was influenced by their use of two structured representations. One of these representations is a set of fraction strips that was used as a manipulative by many of the target students. The other structured representation, a ratio table, is a pictorial model that was used flexibly by most of the target students.

Representations

Although the word *representation* is used frequently and with some variability in the mathematics education literature, one definition that broadly defines this term is that of a *relationship* between *what* (ie., object or idea) is being represented and *how* (ie., model or picture) it is being represented (Norman, 1993; Kaput, 1987; Palmer, 1978; Goodman, 1976). From the students' perspective, the representation could be their reconstruction or drawing of a contextualized problem situation or their interpretation or use of a representation that was structured by somebody else.

In both of these cases, the meanings that the students attribute to these representations influences how they use them to solve problems. For example, a problem asks students which amount of a candy bar means that more has been eaten, 3/4 of the candy bar or 4/5 of the candy bar. One solution might be to draw two rectangles the same size, divide one into fourths, shade three sections, and divide the other one into fifths and shade four of them. Two students may make this representation of the context but interpret it in different ways. For example, one student might look at which rectangle has more of it shaded and reason about the problem strictly from the picture. Another student may reason that one fifth left over is less than one fourth leftover and therefore 4/5 is the larger amount. This strategy required mathematical reasoning beyond the representation. Likewise, different strategies are possible if students use a set of fraction strips or fraction bars that are structured by somebody else to represent the problem context.

Rational Numbers

Many of the research studies that have examined students' thinking about rational number concepts have considered the role of structured representations in their strategies for solving problems (Behr, et al., 1983, 1988; D'Ambrosio & Mewborn, 1994; Hunting, Davis, & Pearn, 1994; Kerslake, 1986; Mack, 1990). Results of some of these studies showed that students' use of pictorial and other structured representations enhanced their ability to make sense of rational number concepts and solve problems (Behr, et al., 1983; Mack, 1990). Results of other studies illustrated the potential for misconceptions about rational number concepts when students used certain structured representations in algorithmic ways or solely as a means to get an answer (D'Ambrosio & Mewborn, 1994; Kerslake, 1986). In both the D'Ambrosio & Mewborn and the Kerslake studies, students' misconceptions were documented at a given point in time.

Other research on students' rational number knowledge examined the ways that they used fraction symbols to solve problems (Mack, 1988, 1990; Smith, 1995). Some of the results indicated that students were able to connect their informal strategies to fraction symbols and algorithms. However, in Mack's (1990) individualized teaching experiments, most of the sixth grade students in the study did not initially relate their strategies for solving fraction problems in a real-world context to the same problems involving fraction symbols that were devoid of context.

This study attempted to document the meanings that students attributed to the fraction strips and ratio table as they used them during the course of instruction of a fraction unit that lasted approximately four weeks. Interviews were conducted by the researcher before and after instruction of the unit. All of the students in this study had prior instruction of fractions at the symbolic level.

The classroom teacher in the study utilized a reform middle school mathematics curriculum called *Mathematics in Context: A Connected Curriculum for Grades 5-8* (MiC) (National Center for Research in Mathematics and Science Education & Freudenthal Institute, in press). MiC was developed to meet recommendations set forth by NCTM *Standards* documents (1989, 1991, 1995). It was also designed in the spirit of *Realistic Mathematics Education* or RME, an approach to mathematics instruction and learning developed by Dutch researchers at the Freudenthal Institute in The Netherlands.

## RME

The RME approach is based on Freudenthal's ideas of mathematics as a "human activity" in which the learning of mathematics is characterized by an active process of "mathematizing everyday reality" and teaching as a process of guided reinvention (Gravemeijer 1992, 1994). Mathematizing reality refers to the process of using contexts in which specific mathematical concepts are produced. Guided reinvention refers to the interactive role the teacher plays with students' own constructions and the underlying mathematics (Gravemeijer, 1992). Representations serve as intermediaries between students' understanding of the contexts and the more formalized mathematical concepts.

Gravemeijer (1992, 1994) characterizes representational or tool use within RME as a "bottom-up" approach. He contrasts this approach with typical top-down approaches in that students' inventions/interpretations serve as the starting point for making sense of the representations. Whereas materials such as Cuisenaire rods or fraction strips are considered concrete representations of rational number concepts from a top-down approach, within RME, this may not be the case. It is in the meanings that students give to these types of materials that determines the level of concreteness of abstractness of them. Often times, within RME, the representations are initiated by the students or are negotiated between the students and the teacher (e.g., ratio table in Streefland's 1991 study).

## MiC

MiC is comprised of forty individual units, ten at each grade level, across four content strands: number, algebra, geometry, and statistics. Within the number strand, emphasis is placed on helping students expand their sense of numbers and the relationships between both symbolic and non-symbolic representations of rational numbers. Non-symbolic representations such as fraction strips and the ratio table are provided within the written materials and are all introduced within contexts that are familiar to students. Thus, the introduction of these representations could be considered "bottom-up" even though they were not initiated by the students (Cobb, Yackel, & Wood, 1992; Cobb, 1994; Gravemeijer, 1992, 1995).

### Instruction

In this study, instruction focused on providing students opportunities to solve problems within the context of small group work and share their strategies with the entire class. The classroom teacher made efforts to avoid providing explicit instructions on how students should use the structured representations, in particular, the fraction strips. Rather, students were encouraged to elaborate their thinking about how they used or, in some cases, why they did not use the structured representations in their solution processes.

## Methodology

The interpretivist perspective taken in this classroom- based case study provided a framework for considering "the processes by which meanings are created, negotiated, sustained, and modified within... human action" (Shwandt, 1994, p. 120). The primary data sources were from pre- and post- unit interviews with students and small group and whole class discussions. The role of the researcher was that of a participant-observer (Glesne and Peshkin, 1995).

### Participants

#### Students

Seven fourth grade and 20 fifth grade high-ability students (14 girls and 13 boys) from a suburban elementary school in the midwest participated in the study. There was one minority student in the group, a male Indian student. None of the students were on free or reduced lunch. All 27 students were identified for the Talented and Gifted math class on the basis of teacher recommendations and on their scores on an end-of-year written test that was normally given to students at the end of fifth grade.

Classroom Teacher

Mrs. Hedges was in her fourteenth year of elementary school instruction at the time of this study. She was also in her second year of implementing MiC. She participated in the field-testing of the entire fifth grade curriculum the previous year and was familiar with most of the 10 fifth grade units. She did not participate in any of the institutes given by MiC staff prior to using the curriculum. She did, however, plan instruction of the units with other teachers in her school who had also participated in the field-testing of the curriculum.

Mrs. Hedges also participated in a pilot study conducted by the researcher prior to this study. The purpose of the pilot study was to examine students' responses to questions involving their use of the fraction strips. The ratio table had not yet been included in *Some of the Parts* at the time of the pilot study. Following the pilot study, minor changes related to the introduction of the fraction strips were made and the ratio table was added.

MiC unit *Some of the Parts*

*Some of the Parts* (Van Galen, et al., 1996) is the first fraction unit in the fifth grade MiC curriculum. The problems in this unit were designed to build on students' informal knowledge of partitioning continuous objects and identifying part-whole relationships. The main goals of the unit were for students to use equivalent forms of benchmark fractions within a context, use fractions to describe the relative magnitude of quantities, order and compare fractions, and use a variety of informal strategies to add, subtract, multiply, and divide fractions. Even though students were introduced to formal fraction symbols, they were not expected to use formal procedures or master standard algorithms as a result of solving problems in the unit.

Fraction Strips

The fractions strips were designed to serve as a model of the measurement context that is introduced near the beginning of *Some of the Parts*. Figure 1 shows a reduced picture of a set of fraction strips as they appear in the unit.

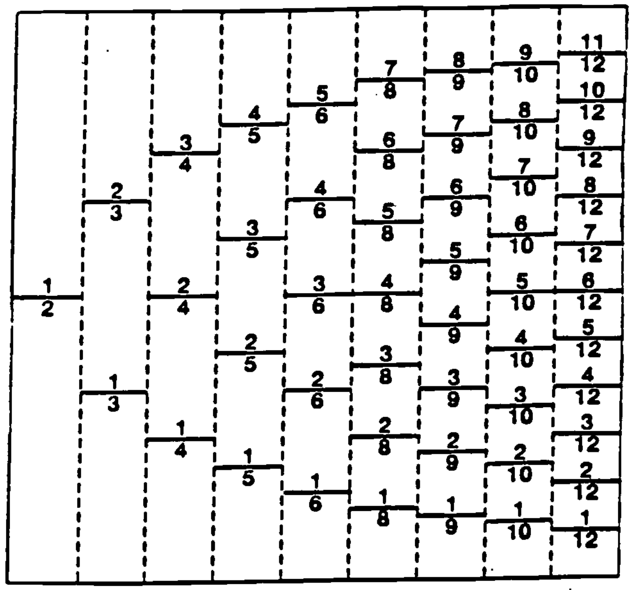


*Figure 1.* A set of fraction strips introduced in *Some of the Parts*

Prior to the introduction of the fraction strips, students are asked to estimate the sums of fractional quantities of coconut milk. Questions addressing the notion of equivalence implicitly appear early in the unit. The students are asked to make fractional markings on unmarked tin cans of the same size to help them with their estimates. The fraction strips are then introduced as representations of the marked tin cans. Students are then asked to use the fraction strips to find additional fraction sums.

Ratio Table

The ratio table is introduced midway through *Some of the Parts* within the context of increasing and decreasing the number of servings of different recipes. The ratio table is organized such that the ingredients are listed in rows and the number of servings and the corresponding amounts of each ingredient are listed the same column. Figure 2 shows an example of a ratio table from *Some of the Parts.*

| | | | |
|---|---|---|---|
| Number of pizzas: | 4 | | |
| Cups of bread crumbs | $\frac{1}{3}$ | | |
| Cans of tomato sauce | 1 | | |
| Pounds of ground beef | 1 | | |
| Teaspoons of dried oregano | $\frac{1}{2}$ | | |
| Number of olives | 2 | | |
| Tablespoons of mozzarella cheese | $\frac{1}{4}$ | | |
| Tablespoons of shredded cheddar cheese | $\frac{1}{4}$ | | |
| Pimiento slices | 4 | | |

*Figure 2.* A ratio table introduced in *Some of the Parts*

Subsequent columns can then be used to list new amounts of ingredients as the number of servings is increased or decreased.

Planning for Instruction

The classroom teacher and researcher began planning for instruction of *Some of the Parts* six months prior to the study. One planning session occurred the summer before the study was conducted. During this time, changes to the unit following the field test and also what the purpose of the study would be were discussed. Mrs. Hedges agreed that students should given opportunities to explore how the fraction strips and ratio table could be used to solve fraction problems without a great deal of direct instruction about how they should interpret them mathematically. The rationale for this decision was to allow students to interpret them based on their understandings of the contexts.

Interviews

Individual interviews with all 27 students were conducted by the researcher prior to and following classroom instruction of a unit on fractions. The audio-taped interviews were used to complete field notes about the strategies students used to solve a series of fraction problems. During interviews prior to instruction, students were not provided with any structured representations. Fraction strips, similar to the ones used during instruction were provided during the interviews conducted after completion of the fraction unit.

Tasks

Table 1 shows a selected list of problems that were given to each student before and after instruction of the fraction unit.

Pre-unit Tasks

| Problem | Problem Type |
|---|---|
| Carlita and Robert each buy peppermint sticks that are all the same size. Carlita eats 3/4 of her peppermint stick. Robert eats 4/5 of his peppermint stick. Who ate more peppermint stick? | Comparison |
| Qiana uses a measuring cup that only has marks to show 1 cup, 2 cups, 3 cups, etc. She thinks for most recipes, estimation is good enough. Show how much 2/3, 1 3/5, and 1/2 + 3/4 would be on the measuring cup. | Quantitative (Fraction Sums) |
| To make Sloppy Joe sandwiches for 3 people, Bob needs a half pound of ground beef. If Bob decides to make enough Sloppy Joe sandwiches for 15 people, how many pounds of ground beef will he need? | Ratio |
| The number of cups of tomato sauce needed to make the Sloppy Joes is 3/4 cup. If Bob decides to triple the recipe, how many cups of tomato sauce will Bob need? | Operator |

Post-unit Tasks

| Problem | Problem Type |
|---|---|
| Keisha and Kenneth each make a loaf of bread. The two loaves of bread are the same size. However, Keisha divides her loaf of bread into 12 equal slices and eats 5 of them, and Kenneth divides his loaf of bread into 10 equal slices and eats 4 of them. Who ate more bread, Keisha or Kenneth? | Comparison |
| Remember when you were asked to use your fraction strips to write at least 10 fraction sentences? Someone in the class wrote the sentence 2/5 + 1/4 = 2/3. Another person in the class wrote the sentence 2/5 + 1/4 = 13/20. Do you think 2/3 and 13/20 are equal? If so, why? If not, which one do you think is correct and why? | Fraction Sums |
| A fruit punch recipe that serves 10 people requires 1 1/3 cup of lemonade. Trudy wants to make enough to serve 75 people. How many cups of lemonade will she need? | Ratio |

*Table 1.* Pre- and Post- unit tasks

These problems were selected to parallel the types of problems that would be used during instruction. The Fraction Sums problem was added to the post-unit interviews in order to give students opportunities to reflect on their use of the fraction strips during instruction.

Selection of Target Students

Prior to instruction, six students were identified on the basis of their responses to the pre-unit interview questions. A seventh student was added during the study because of the later realization of the importance of the student's work. The names of the target students were not revealed to Mrs. Hedges until after the study.

Daily Lessons

There were 17 days of instruction, not including individual interviews. All classroom observations were audio-taped and used to complete field notes. Fifteen of the 17 lessons were led by the classroom teacher, Mrs. Hedges. Two of the lessons were led by the researcher. One of these was due to the absence of Mrs. Hedges. On the other occasion, Mrs. Hedges requested that the researcher lead the class discussion related to the use of the ratio table. Planning sessions continued on a weekly basis throughout the study.

Typical daily lessons were as follows. Students were randomly placed in groups of four. Mrs. Hedges would begin class by discussing upcoming problems that students should work on that day. Students would then begin working on assigned problems in their groups. During this time, the researcher rotated among the groups and would spend approximately twenty minutes with the groups that the target students were in. Data was collected on the individual target students on approximately three out of every five days.

One or two days each week were devoted to allowing students to share strategies with the whole class. Several sets of overhead transparency fraction strips were made available for students to use to show their solutions. The groups were changed mid-way through the study and prior to the introduction of the ratio table. The format for the daily lessons did not change.

## Results

This section describes the ways in which the fraction strips and ratio table were used by the target students to solve fraction problems during small group work and whole class discussions. Results of pre-unit interviews provided evidence of students' initial representations of the problem contexts. In some cases, the invented representations used by students during pre-unit interviews influenced their use of the structured representations introduced in the unit. Results of post-unit interviews provided further evidence of how students were able, in most cases, to apply a wider variety of strategies to their use of the ratio table to solve fraction problems than to their use of the fraction strips.

### Initial Strategies

The target students' initial strategies for solving fraction problems fell on a continuum of heavy reliance on a pictorial representation of the context to use of symbols only. Two pre-unit problems, the quantitative comparison and the ratio problem, are discussed since the students' representations of these contexts were similar in structure to the representations provided in the fraction unit. On the quantitative comparison problem, two of the target students reasoned directly from their drawings of the context. For example, Anon drew two rectangles that were approximately the same size, divided one of the rectangles into fourths and shaded three sections and divided the other rectangle into fifths and shaded four of them. He reasoned that the two amounts were equal since, according to his drawing, the two quantities looked equal. He did not attempt to reason beyond his drawing.

One of the target students who did not make a drawing of the context, Rachel, solved this problem by reasoning with the leftover amounts. She argued that 4/5 is larger since 1/5 leftover is smaller than 1/4 leftover. One other target student utilized this strategy.

The other three target students used symbolic strategies. Only one of them, Wayne, used a correct symbolic strategy. Wayne found a common denominator, wrote equivalent fractions, and compared the numerators to solve this problem.

On the ratio problem, three of the target students used a building up strategy that is similar in structure to the ratio table to correctly solve this problem. For example, Julie solved this problem by first writing down five 3's. She then paired off two sets of two 3's to get two whole pounds with one pound leftover. The other four target students solved this problem by multiplying 5 and 1/2.

Strategies with the Fraction Strips

The discussion in this section will focus on how the students used the fraction strips to find addition sentences. Two dominant approaches to interpreting and using the fraction strips were documented. One approach was to manipulate the fraction strips by lining them up end-to-end to find fraction sums. The other approach involved using the strips to find equivalent fractions in order to write several different fraction sums. This section will also highlight several of the target students' attempts to resolve the conflict between answers obtained with the fraction strips and the answers they obtained using symbolic strategies. During the first two weeks of instruction when the fraction strips were introduced, six of the seven target students were in a group with at least one other target student. The other target student was grouped with three other non-target student.

Prior to the introduction of the fraction strips, students were given problems from *Some of the Parts* that involved implicit notions of equivalence and estimating fractional quantities. For example, one set of problems asked students to divide rectangular fruit tape (what*we* know as fruit roll-ups) into different numbers of equal pieces. For these problems, students cut out several rectangular strips of paper of the same size and divided them into two, three, four, five, six, and eight equal pieces, respectively. Students are then asked questions such as, "Which sets of pieces of fruit-tape could be used if three students want to share one fruit-tape?". Most students were able to answer that the fruit-tapes that were divided into three and six pieces could be used.

Another set of problems required students to estimate combinations of fractional quantities such as 2/3 can, 1/2 can, 1/3 can, 1/4 can, 1/6 can, and 1/8 can that could be used to fill a whole can of coconut milk without going over. They are then asked to use these same quantities to find combinations that fill a whole can of coconut milk exactly. Upon completion of these problems, students were not given direct instruction on how to use equivalent fractions to find fraction sums.

Following these problems, students were given the set of fraction strips shown on page 6 and asked to cut them out. They were then given an example (in *Some of the Parts* ) of an addition sentence, 1/2 + 1/3 = 5/6 and asked to explain why this sentence was correct. Most of the students could use their strips to show why the sentence was true. Students were then asked to come up with ten other addition sentences using their fraction strips. (Note: the MiC unit only asked them to come up with six.)

Figure 3 shows an example of how students used the fractions strips to compute the sum 1/2 + 1/3 = 5/6. This strategy, hereafter called the end-to-end strategy, was initially used by most of the students in the class to verify this sentence. The end-to-end strategy involves lining up the second fraction strip at the appropriate division line of the first fraction strip.

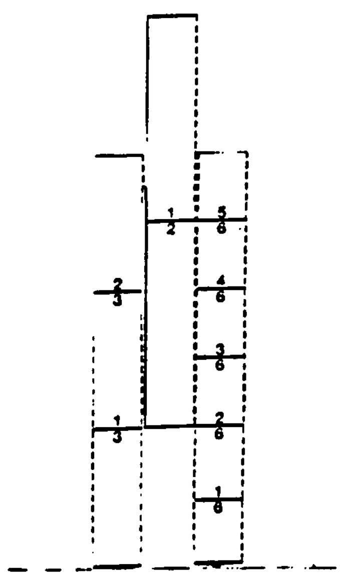


*Figure 3.* End-to-end strategy used to add fractions

When using the fraction strips with fractions that have smaller denominators, the end-to-end strategy was successful for many of the students who used it, including Julie from the target group. However, several other target students attempted to use the end-to-end strategy with larger denominators and those for which the equivalences were not as obvious. Discrepancies arose when they compared their answers using the fraction strips to the answers that were obtained using the standard algorithm. The next excerpt highlights a discussion between two target students, Rachel and Jay, and another non-target member of their group, Iven, when they shared their addition sentences with the whole class. Jay had used the fraction strips to write the sentence 2/5 + 2/8 = 5/8.

| | |
|---|---|
| Iven: | 2/5 + 2/8 equals 26/40. |
| Jay: | Huh??!! How did you get that? |
| Iven: | I did it...I did it..I didn't reduce it enough to get 5/8. |
| Rachel: | (uses the strips) 2/5 +2/8 = 5/8. (lesson 1/18/96, p. 5) |

At this point, Rachel and Jay both thought that their end-to-end strategy with the fraction strips had produced the correct answer. When asked to explain his solution, Iven described the process of finding common denominators, etc., to get 26/40.

After this lesson, both Rachel and Jay discontinued using the fraction strips to write fraction sums. On a homework assignment given after this lesson, students were asked to determine which sentence was correct, 1/5 + 1/4 = 4/9 or 1/5 + 1/4 = 9/20, and explain their reasoning. This was the result of a similar discrepancy in answers from another group. On this assignment, Jay used a symbolic strategy of finding a common denominator to determine which sentence was correct.

Similarly, Rachel relied on a symbolic strategy for finding common denominators when further discrepancies came up in a class discussion the following day. The following excerpt highlights the class discussion following a student's presentation of the sentence 2/5 + 3/8 = 7/9 using the fraction strips.

| | |
|---|---|
| T: | Are there any comments from the class....? |
| Rachel: | Um, I don't think its correct because both 8 and 5 have to go into 9 equally and they don't. |
| T: | Ok, Rachel is saying both 8 and 5 have to go into 9... |
| Iven: | Um, I don't think so because maybe its just reducing..maybe they found um one of the higher common denominators and then they reduced it to get their numbers... |

Rachel: Its reduced and both of them should go into it...they have to have a common denominator....(lesson, 1/23/96, p. 1)

At this point, Rachel had abandoned her use of the fraction strips to find fraction sums. Instead, she relied on the procedure of finding a common denominator which Iven had used the previous day. However, she did not attempt to relate this symbolic strategy to the end-to-end strategy she used with the fraction strips. Rachel later pointed out that the fraction strips are not accurate when using fractions with larger denominators. Neither Rachel nor Jay attempted to use the fraction strips following these lessons or during the post unit interviews.

Anon did not abandon his use of the end-to-end strategy. However, he did continue to use the fraction strips to write addition sentences that were incorrect or, at best, close approximations of the correct sum. Figure 4 shows Anon's arrangement of the fraction strips to write the sentence 2/3 + 3/5 = 1 1/18.

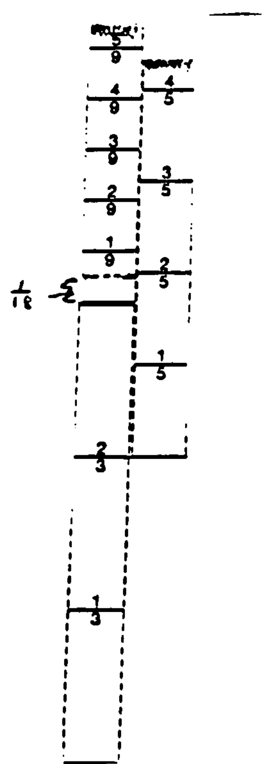



*Figure 4.* Anon's strategy for adding 2/3 and 3/5

When Anon shared this solution with the entire class, several students, including Rachel and Wayne from the target group, disagreed with his strategy and used symbolic procedures to find the sum of 2/3 + 3/5. Anon, however, continued to assert that his solution was correct based on what his representation showed, which is similar to his solution for the quantitative comparison problem during the pre-unit interviews.

Two of the target students did not use the end-to-end strategy. Both of them appeared to apply the concept of equivalence to their use of the strips. For example, Wayne, during one small group discussion, arranged the strips based on common division marks such as halves, fourths, and eighths, and fifths and tenths. Figure 5 shows Wayne's arrangement of the fraction strips. He used this arrangement to write several fraction addition sentences.

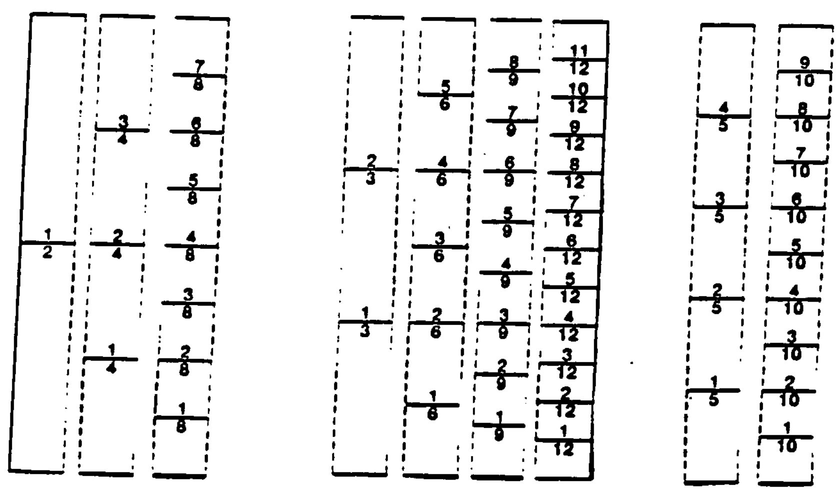


*Figure 5.* Wayne's arrangement of several fraction strips.

To summarize, with the exception of two students, most of the target students did not apply the concept of equivalence to their strategies with the fraction strips. In particular, target students such as Rachel, had explicitly used the concept of equivalence prior to being introduced to the fraction strips. In addition, several target students disregarded the fraction strips in favor of other, mostly symbolic, strategies. Only two target students, Anon and Julie, continued to use the end-to-end strategy to find fraction sums with the fraction strips.

Strategies with the Ratio Table

The introduction of the ratio table occurred during the last two weeks of instruction of the fraction unit. This pictorial representation was introduced within the context of increasing the number of servings of different recipes. The concept of ratio was emphasized as well as the usefulness of the ratio table for organizing calculations with fractions. The students were randomly assigned to different groups. This time, four of the seven target students were in groups with one other target student.

Wayne, as with his use of the fraction strips, was able to reflect on the meaning of the ratio table beyond the actual calculations. The following excerpts highlight his understanding of the ratio table.

| | |
|---|---|
| T: | ...I'm thinking of other situations in life where it might be useful to use a recipe. |
| Wayne: | Situations in your head... |
| T: | Situations where you have to multiply and divide even without being able to do it... |
| Wayne: | If you didn't want to do it in your head. (lesson 1/24/96, p. 2) |

| | |
|---|---|
| T: | Ok, so you and your mom talked about it [ratios]...what do you think it means? |
| Wayne: | I think it means that if you have a certain amount...um like um of two things...you would multiply the things...like in a recipe, you would multiply it and you would be creating a ratio table because you are multiplying many things but its [difference in ingredients] staying the same. (lesson 1/30/96, pp. 1-2) |

The other target students did not articulate the concept of ratio with their use of the ratio table as Wayne had but, with the exception of two target students, showed a great deal of flexibility in their use of the ratio table to calculate products of fractions and mixed numbers. In particular, their knowledge of doubling and halving fractional quantities facilitated their use of the ratio table. When more difficult fractions and mixed numbers were introduced, several of the target students extended their use of the ratio table to calculate more complex products. Many of these strategies exemplified their implicit use of the distributive property in their solution processes.

Most of the target students used doubling and halving strategies to find the new amounts of ingredients for the new serving amounts. Since most students were solving the recipe problems from *Some of the Parts* without much difficulty, Mrs. Hedges suggested that more difficult problems should be given to the students. Figure 6 shows two additional problems that were given to students during the last two weeks of instruction.

1. Below is a list of ingredients for *Frozen Fruit Cups*. The amount of ingredients listed makes four servings. Determine the amount of each ingredient that would be needed for 25 servings.

| | | |
|---|---|---|
| Number of Servings | 4 | |
| Cups of Fresh Strawberries | 2/3 | |
| Cups of Sugar | 1/4 | |
| Cups of Nonfat Plain Yogurt | 1 1/3 | |
| Number of Ice Cream Cones | 4 | |

2. Below is a list of ingredients for Laura's version of *Frozen Chocolate Crunch*. The amount of ingredients listed makes six servings. Determine the amount of each ingredient that would be needed for 28 servings.

| | | |
|---|---|---|
| Number of Servings | 6 | |
| Cups of Whipping Cream | 1 1/4 | |
| Cups of Chocolate Sauce | 5/12 | |
| Tablespoons of Almond Brickle Chips | 2 1/2 | |
| Teaspoons of Vanilla | 1 1/5 | |

*Figure 6.* Supplementary ratio table problems

These problems extended the students' thinking about operations with fractions as exemplified in several of the target students' strategies. Julie used the distributive property to multiply 1 1/5 teaspoons of vanilla by four to determine the amount needed for 24 servings for a recipe that normally serves six. When asked how she would multiply 1 1/5 by four, Julie responded, "I've got it, you just take one times four and one times 1/5".

Rachel and Anon showed flexibility in their use of the ratio table to determine the number of cups of chocolate sauce needed for 28 servings in the *Frozen Chocolate Crunch* recipe. Anon reasoned that each of the amounts could be multiplied by five to determine the amount for 30 servings and then the amount for two servings could be subtracted from each of those amounts. Anon, like Julie, used the distributive property to determine the number of cups of whipping cream needed for 30 servings and for 2 servings, 1 1/4 times 5 and 1 1/4 times 1/3, respectively. Rachel suggested a different strategy for determining the amount of each ingredient needed for 28 servings. The following excerpt highlights the reasoning she used.

> ...I'm gonna divide all these by six and then you would have what it would be for one and then you would add what it would be for one to what it would be for six and then you would have 7 and 28 is divisible by 7. (lesson 1/26/96, p.3)

Dixie, Wayne, and Julie also demonstrated flexibility in their strategies with these two supplementary problems.

Of the seven target students, only two target students tended to incorrectly apply additive strategies in their use of the ratio table to solve recipe problems. For example, to figure the number of cups of bread crumbs needed to serve 24 people when 1/3 cup of bread crumbs are needed to serve 4 people, instead of adding 1/3 six times or multiplying 1/3 times six, Jay added 6 and 1/3 and wrote 6 1/3 in the 24 column.

To summarize, five of the seven target students used the ratio table flexibly to solve the recipe problems in *Some of the Parts*. In addition, these students were able to extend their strategies to recipe problems involving operations with non-benchmark fractions and mixed numbers. When asked to show more than one solution, most target students were able to do so quite readily.

<u>Post-unit Strategies</u>

Responses to the fraction sums and the ratio problem are highlighted in this section. A set of fraction strips similar to the ones used during instruction were made available during the interviews. There were no ratio tables explicitly drawn on the problem sheets that were given to students. In general, only one of the target students, Julie, used both the fraction strips and the ratio table to solve fraction problems during the post-unit interviews. Anon used the fraction strips on the fraction sums problem and reasoned that 2/3 and 13/20 were the same.

The other five target students did not attempt to use the fraction strips to solve any of these problems. In particular, the fraction sums problem was used during these interviews to elicit students' thinking about the conflicting answers that arose during class discussions. However, most of the target students used either their own drawings of the problem context or a symbolic strategy. When asked how students in class may have gotten two different answers, Rachel and Wayne, both of whom used standard algorithms to verify 13/20 as the correct answer, reasoned that the answers were close. However, none of the target students related their symbolic strategies to the end-to-end strategies that were prevalent during instruction.

Three target students drew a ratio table to solve the ratio table. Figure 7 shows Dixie's solution to the ratio problem.

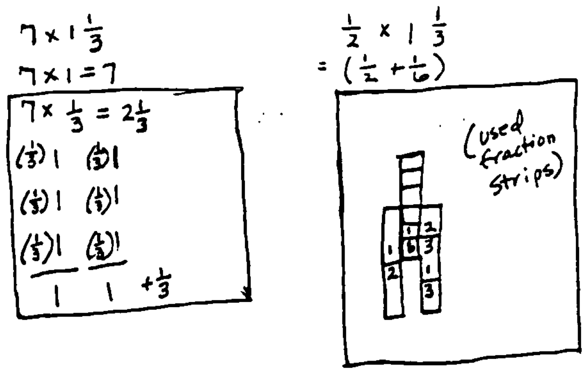
| Servings | 10 | 70 | 5 | 75 |
|---|---|---|---|---|
| Lemonade | 1 1/3 | 9 1/3 | 2/3 | 10 |

*Figure 7.* Dixie's solution to the post-unit ratio problem

In this solution Dixie implicitly used the distributive property in her solution with ratio table. She multiplied 1 1/3 by 7 by multiplying 1 times 7 and 1/3 times 7 to get 9 1/3. She then halved 1 1/3 to get 2/3, and then added 2/3 to 9 1/3 to get 10.

Julie used a combination of her own invented representation, the fraction strips, and the ratio table to solve the ratio problem. She used her own drawing to multiply 7 and 1 1/3. She used the fraction strips to multiply 1 1/3 and 1/2. (She added 1/6 and 1/2 using the end-to-end strategy.) She used the ratio table to organize the answers she obtained from these two steps.

Figure 8 shows Julie's solution.




| people | 10 | 5 | 75 |
|---|---|---|---|
| cups of lemonade | $1\frac{1}{3}$ | $\frac{2}{3}$ | 10 |

$7 + 2\frac{1}{3} + \frac{2}{3} = 10$

*Figure 8.* Julie's solution to the post-unit ratio problem

Even though the other four target students did not explicitly draw ratio tables to solve the ratio problem, several of them used a strategy that fit the structure of a ratio table. For example, Rachel used a strategy that involved doubling the cups of lemonade as she doubled the number of people, ie., 10, 20, 40, 80 and 1 1/3, 2 2/3, 5 1/3, 10 2/3. To get the amount for 75 servings, she subtracted half of 1 1/3 or 2/3 from 10 2/3 cups to get 10 cups.

## Discussion

The attempt to utilize a "bottom-up" instruction approach in this study invoked a variety of uses and interpretations of the two structured representations by the students. In tracing the development of the seven students' strategies for solving fraction problems over several weeks, the delicate nature of their knowledge about fractions became evident even though this was considered a high-ability group of math students. Most of the target students did not connect their symbolic procedures to the underlying concepts, particularly when they tried to write formal addition sentences using the fraction strips. They seemed to be more successful solving problems when they relied on their informal strategies such as with their use of the ratio table.

### Fraction Strips as a Manipulative

Two of the target students used the fraction strips to reflect on the concept of equivalence.

Thus, when asked to write several addition sentences using the fraction strips, they were able to do so quite easily by substituting equivalent fractions for the different addends. Most of the other target students struggled to give mathematical meaning to the fraction strips. They did not attempt to reason about the mathematics beyond the answers that the end-to-end strategy produced. Several factors should be considered with this statement. One obvious factor was their prior knowledge and experiences with fraction operations. Clearly, several of the target students had learned to add fractions with unlike denominators using standard algorithms that they did not understand. It also appeared that these strategies did not emphasize equivalence. Therefore, if they could not relate their symbolic strategies to the concept of equivalence, it seems reasonable to assume that they would have difficulty relating either the concept of equivalence or common denominators to their end-to-end strategy with the fraction strips.

Another factor to consider was the context within which the fraction strips were situated. They were introduced as a representation of a measurement context in which students were asked to estimate fraction sums and not necessarily compute exact answers. Other studies have shown that providing students with opportunities to approximate sums using their knowledge of benchmark fractions can actually enhance their ability to understand and use symbolic procedures (Mack, 1988; Smith, 1995). The introduction of the fraction strips in this study appeared to interfere with these opportunities. Many of the target students immediately perceived them as tools that could be manipulated to obtain different addition sentences. Thus, they did not need to reason beyond them to produce fraction sums.

When discrepancies arose between the two different methods (fraction strips and standard algorithm) for finding fraction sums, several of the target students abandoned the fraction strips completely in favor of the symbolic strategies. While specific discourse analyses were not conducted in this study as has been done in other studies of socio-mathematical interactions (e.g. Cobb, 1995), it is possible that the discussions with a non-target student, Iven, influenced Rachel and Jay's decision to abandon their end-to-end strategy with the fraction strips. It is also worth considering their perception of symbolic algorithms, in general, as being the "best " strategies for doing mathematical computations.

Ratio Table and Informal Strategies

In general, the ratio table afforded more opportunities for the target students to apply their informal knowledge of fractions as quantities. Whereas, there were two basic strategies used with the fraction strips (end-to-end and finding equivalent fractions), there were countless strategies used with the ratio table. Doubling and halving and additive strategies were often used by students. Two target students incorrectly applied additive strategies to their use of the ratio table on several occasions. However, the other target students appeared to conceptualize ratio through their use of the ratio table which enhanced their ability to correctly apply the additive and multiplicative strategies they already knew. Also, their solutions to the ratio problem during the pre-unit interview were similar to the way they used the structure of the ratio table when it was introduced during instruction.

Other studies have also illustrated the utility of having students organize fractions in a ratio table (Streefland, 1991; Scarano & Confrey, 1996). In Streefland's (1991) study, the ratio table was constructed as a result of negotiations between the teacher and the students for a more efficient way to represent equivalent fractions. The fact that the ratio table was introduced to students within the curriculum materials did not seem to inhibit their ability to make sense of them and use them to solve fraction problems.

Curricular Considerations

While RME has been characterized primarily from a research and development perspective to mathematics instruction and learning, MiC is a curriculum project that will be published as written materials to be used in middle school mathematics classrooms. Gravemeijer (1994) points out the differing goals in the following discussion:

> In developmental research, knowledge gain is the main concern. The focus is on building theory, explicating implicit theories. In curriculum development, especially in textbook development, other priorities hold; there are deadlines to be met, and any elbowroom that the deadlines permit

will be used to improve the product, not for reflection (p. 450).

As a result of this study, other revisions were made to the unit and additional suggestions were made in the teacher's guide. In particular, in sections where the concept of equivalence is implicit, suggestions are provided to help the teacher make this concept more explicit.

Instructional Considerations

Both the researcher and the classroom teacher thought that the notion of equivalence would be evident to students, especially since careful attention was given to enlarging the entire set of fraction strips. However, with two exceptions, the notion of equivalence was not considered by the students. The classroom teacher expressed reservations about the whether or not the students would be able to use the ratio table effectively. However, the students clearly showed a better understanding of the utility of the ratio table than they did of the fraction strips.

The argument has been made that the concept of equivalence is foundational to students understanding of higher level rational number concepts (Carpenter, et al., 1995). While there was implicit attention to the notion of equivalence in the curriculum unit *Some of the Parts* , the results of this study suggest that more explicit attention should have been given to the equivalent fractions when the fraction strips were introduced. Also, instruction should have focused on helping students utilize both equivalence and the quantitative nature of fractions to make sense of their answers. Instead, class discussions tended to focus on the symbolic strategies.

Julie's success with both the fraction strips and the ratio table is worth noting. Her knowledge coming into the study was the weakest of all of the target students. However, she also showed less dependence on the use of symbolic strategies than the other target students. Thus, her strategies with the fraction strips were more concrete in nature and more effective. She was able to use both fraction symbols and the representations to solve fraction problems. She also did not abandon using them to solve problems as others had done. Furthermore, she was able to solve problems that were quite complex in nature, particularly the ratio problem during the post-unit interview. This lends further argument to giving students more time with these materials prior to the introduction of symbolic strategies.

Final Thoughts

Representations that are designed to help students mediate between the context and the underlying mathematical ideas are becoming more prevalent as teachers make problem solving curricula the core of their mathematics instruction. Future studies should continue to explore pedagogical strategies that will enhance students' abilities to make connections between non-symbolic representations of the contexts and symbolic algorithms. Also, the role of invented procedures for operations with fractions is important to consider with respect to intermediary representations. Representations that afford more opportunities for students to invent procedures that make sense to them, even if they are not concrete (in the sense that they cannot be physically manipulated), may in fact contribute more to their understanding of the mathematical concepts that underlie the symbolic algorithms than representations that are designed only to illustrate a mathematical concept in a "concrete" manner.

## REFERENCES


Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes.* (pp. 92-106). Orlando: Academic Press.

Behr, M. J. & Post, T. R. (1988). Teaching rational number and decimal concepts. In T. R. Post (Ed.), *Teaching mathematics in grades K-8: research based methods.* Boston: Allyn and Bacon, Inc.

Carpenter, T. P., Fennema, E., Franke, M. L., Empson, S. B., & Jacobs, V., (1995). Teaching and learning of rational number: Proposed framework for the CGI inservice program. Wisconsin Center for Educational Research: U. of Wisconsin - Madison.

Cobb, P. & Bauersfeld, H. (1995). *The emergence of mathematical meaning: Interaction in classroom cultures.* New Jersey: Lawrence Erlbaum Associates.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for research in mathematics education.* 23 (1), 2-33.

Cobb, P. (1994). Theories of mathematical learning and constructivism: A personal view. Paper presented at the symposium on trends and perspectives in mathematics education, Institute for Mathematics, University of Klagenfurt, September, 1994.

D'Ambrosio, B. S. & Mewborn, D. S. (1994). Children's constructions of fractions and their implications for classroom instruction. *Journal of research in childhood education.* 8 (2), 150-161.

Glesne, C. & Peshkin, A. (1992). *Becoming qualitative researchers: An introduction..* White Plains, New York: Longman.

Goodman, N. (1976). *Languages of art.* Indianapolis: Hackett Publishing Company.

Gravemeijer, K. (1995). Symbolizing from a developmental-research perspective. Paper presented for the symposium *Symbolizing, Communicating, and Mathematizing*, Nashville, September, 1995.

Gravemeijer, K. (1992). Mediating between concrete and abstract. In T. Nunes & P. Bryant (Eds.), *How do children learn mathematics.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Gravemeijer, K. (1994). *Developing realistic mathematics education.* Freudenthal Institute, Utrecht, The Netherlands: Technipress, Culemborg.

Hunting, R. P., Davis, G. & Pearn, C.S. (1994). Engaging children's whole number knowledge for conceptual advancement in rational number learning using a computer-based tool. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, 1994.

Kaput, J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics.* (pp. 19- 26). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Kerslake, D. (1986). Fractions: Children's strategies and errors. A report of the strategies and errors in secondary mathematics project. England: NFER-Nelson Publishing Company, Ltd.

Mack, N. (1988). Using estimation to learn fractions with understanding. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April, 1988.

Mack, N. K. (1990). Learning fractions with understanding; Building on informal knowledge. *Journal for research in mathematics education.* 21(1), 16-32.

National Center for Research in Mathematical Sciences Education & Freudenthal Institute. (Eds.). (in press). *Mathematics in context: A connected curriculum for grades 5-8.* Chicago: Encyclopedia Britannica Educational Corporation.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (1995). *Assessment standards for school mathematics.* Reston, VA: NCTM.

Norman, D. A. (1993). *Things that make us smart: defending human attributes in the age of the machine.* Reading, MA: Addison-Wesley Publishing Company.

Palmer, S. E. (1978). Fundamental aspects of cognitive representation. In E. Rosch & B. Lloyd (Eds.), *Cognition and categorization.* (pp. 262-304). Hillsdale, New Jersey: Lawrence Erlbaum Associates, Inc.

Scarano, G. H. & Confrey, J. (1996). Results from a three-year longitudinal teaching experiment designed to investigate splitting, ratio and proportion. A paper presented at the Annual Meeting of the AERA, New York, April, 1996.

Schwandt, T. A. (1994). Constructivist, interpretivist approaches to human inquiry. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research.* Thousand Oaks, CA: SAGE Publications, Inc.

Smith, J. P. (1995). Competent reasoning with rational numbers. *Cognition and instruction,* 13 (1), 3-50.

Streefland, L. (1991). *Fractions in realistic mathematics education: a paradigm of developmental research.* The Netherlands: Kluwer Academic Publishers.

Van Galen, F., Wijers, M., Shew, J. A., Cole, B. R., Brendefur, J., & Brinker, L. (in press). Some of the parts. National Center for Research in Mathematical Sciences Education & Freudenthal Institute. (Eds.). (in press). *Mathematics in context: A connected curriculum for grades 5-8.* Chicago: Encyclopedia Britannica Educational Corporation.

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