DOCUMENT RESUME

ED 396 931 SE 058 595

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TITLE Students' Difficulties with Proof by Mathematical Induction.
PUB DATE Apr 96
NOTE 21p.; Paper presented at the Annual Meeting of the American Educational Research Association (New York, NY, April 8-12, 1996).
PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *College Students; *Concept Formation; Higher Education; High Schools; *Mathematical Concepts; Problem Solving; *Proof (Mathematics); *Secondary School Students
IDENTIFIERS *Mathematical Induction

ABSTRACT

The cognitive difficulties encountered by 40 high school and 13 college students beginning to learn the proof technique of mathematical induction were investigated. Students provided data in the form of proof-writing and proof-analysis tasks followed by interviews to clarify their written responses. Both groups of students had significant difficulties with the proof technique, both procedurally and conceptually. The evidence suggests that mathematical content knowledge played a significant role in difficulties. Many students focused on the procedural aspects of mathematical induction far more often than on conceptual aspects. The evidence collected suggests that examples play a critical role for many students in verification, for insight, and as patterns for how proof should be conducted. There was some evidence of students trying to reason mathematically using everyday reasoning. Difficulties predicted by earlier studies with proof by mathematical induction, other forms of proof, and problem solving were confirmed. (Author)

Students' Difficulties with Proof by Mathematical Induction

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Paper presented to the Annual Meeting of the American Educational Research Association, New York, 1996.

Abstract

The cognitive difficulties encountered by 40 high school and 13 college students beginning to learn the proof technique of mathematical induction were investigated. Students provided data in the form of proof-writing and proof-analysis tasks followed by interviews to clarify their written responses. Both groups of students had significant difficulties with the proof technique, both procedurally and conceptually. The evidence suggests that mathematical content knowledge played a significant role in difficulties. Many students focused on the procedural aspects of mathematical induction far more often than on conceptual aspects. The evidence collected suggests that examples play a critical role for many students in verification, for insight, and as patterns for how proof should be conducted. There was some evidence of students trying to reason mathematically using everyday reasoning. Difficulties predicted by earlier studies with proof by mathematical induction, other forms of proof, and problem solving were confirmed.

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Introduction

There are several common proof techniques in mathematics; among these is mathematical induction[1]. As such, mathematical induction is a part of the curriculum in high schools and colleges. This technique is particularly valuable in developing the theoretical foundations of computer science.

The contributions of the previous research on mathematical induction (Brumfiel, 1974; Dubinsky, 1986, 1989; Dubinsky & Lewin, 1986; Ernest, 1984; Lowenthal & Eisenberg, 1992; Moore, 1990; Movshovitz-Hadar, 1993; Solow, 1990) inform our specific understanding of some of the difficulties that students encounter in terms of procedural and conceptual knowledge, and mathematical resources required when doing mathematical induction. Ernest (1984) provided the most comprehensive analysis of this proof technique. He identified algebra skills, knowledge

[1] Mathematical induction is one of the common proof techniques that can sometimes be applied to establish the truth of sequences of mathematical statements (Solow, 1990). Lowenthal & Eisenberg (1992) stated the theorem implied by the technique of mathematical induction as follows: if a statement $S(n)$ is true for the starting value of the sequence, say $n=1$, and its truth for $n = k$ implies its truth for $n = k+1$, then the statement is true for all whole numbers. My interpretation of proof by mathematical induction is as follows: the truth of the sequence of mathematical statements is argued, first by showing that one or more specific, initial instances of the statement are true (i.e., the base cases), and second, by showing that when the truth of one or more of these statements is assumed, that another of the statements can be derived (i.e., the inductive step), The indirect nature of the proof comes from the argument that the truth of any of the mathematical statements can be derived by recursive application of the inductive step, starting with the base cases.

of implication and logic, and the ability to do a proof in the correct form, as important factors. The relative importance of these skills is not clear from his article, nor is the relationship to cognition. In support of Ernest, however, Dubinsky and Lewin (1986) and Dubinsky (1986) confirm the need for students to be skilled in mathematical logic, the function concept, and the method of mathematical induction. For them, a hierarchy of conceptual understanding was measured.

Moore (1990) noted that little research had been done on the learning of mathematical proof and found justification for the need for additional research to identify cognitive difficulties. Moore looked at students' beliefs and ability to problem solve and found that students are often unprepared for the demands of mathematical proof. Although mathematical induction was a topic in the mathematics course observed by Moore, it was given little attention in the descriptions.

Recent research related to mathematical proof provides ways to increase our understanding of difficulties with mathematical induction. Polya clearly made a connection between proof and problem solving by identifying "problems to prove" as one of two problem types (1945). A further connection to problem solving was made (Polya, 1954) by dividing the cognitive activities in proof between plausible and demonstrative reasoning, where problem-solving skills are used to informally discover a proof that is later demonstrated formally.

Any problem must be understood, not only for the relevant information in it, but for connections to relevant mathematical knowledge. Lester and Kroll (1990) provide a widened definition of understanding. According to their definition, the question being posed must be considered along with the problem's conditions and variables. Lesh and Akerstrom (1982) and Silver (1982) report that understanding is increased with problems that are on familiar and meaningful topics. It is, therefore, reasonable to expect that problem understanding and familiarity are factors in students' difficulty with proof.

Going beyond problem understanding, Schoenfeld (1985) provides a comprehensive model for studying problem solving in terms of cognitive and affective variables: mathematical resources, heuristics, metacognitive control mechanisms, and belief systems. Schoenfeld's model provides an avenue for understanding students' difficulties with mathematical resources in terms of: (a) the domain-specific knowledge for interpreting symbols and the procedural knowledge required for doing proofs, (b) the mathematical connections the learner has made in his or her knowledge, and (c) the informal knowledge possessed by the learner and applied to problem solving. Schoenfeld's second variable, heuristics, is the general knowledge one possesses that guides actions in doing new problems. The third variable, metacognitive control, is the ability to allocate resources and to monitor one's progress. According to Schoenfeld, planning and monitoring are essential elements of control. The fourth variable, belief systems, assumes that beliefs about oneself and mathematics influence success in mathematics. In addition to beliefs, McLeod (1992) and Piaget (1981) suggest that emotional factors influence mathematical problem solving.

In addition to the research literature on problem solving, the research topics on heuristics in doing proof, students' requirements for convincing evidence, and the development of reasoning ability provide clues to students' difficulty with proof. For instance, it has been suggested that working forwards is used by higher level students (Hart, 1994). A work-forward strategy is certainly more consistent with our notion of proof as demonstrative reasoning where the argument starts with what is known and reasons toward the unknown. However, a strategy of working backwards is consistent with the plausible reasoning in finding a proof through problem-solving. It also seems consistent with a tendency for people to be predisposed to confirm a theory (Kuhn, Amsel, and O'Loughlin, 1988). Since proof is a form of reasoning, the literature related to everyday and mathematical reasoning provided explanations for students' responses. Additionally, examples seem to play a significant role in convincing evidence. Porteous (1990) found that children will provisionally accept the truth of a mathematical statement from examples, and further, Moore (1990) reported that students want and need illustrations of concepts and definitions through worked examples. These factors were studied to determine their role in students' difficulties with mathematical induction.

In Summary

There are four missing components from previous studies on proof by mathematical induction. First, the prior experiences of students have only been given limited attention. Second, although conceptual understanding and procedural knowledge have been studied by other researchers, conceptual, procedural, and applications knowledge have not been investigated as separate aspects of understanding. Third, the resources needed to be successful with proof by mathematical induction have not been analyzed relative to cognitive factors involved in problem solving. Fourth, findings from research on proof- related topics and on everyday reasoning have not been applied to proof by mathematical induction.

Research Questions

Specifically, this study pursued an answer to the following question: What kinds of cognitive difficulties do high school and college students encounter when they are beginning to learn the proof technique of mathematical induction?

Three additional questions were also considered: (a) what depth of understanding did the sample of students have of the procedures and concepts in a proof, (b) how was performance in proof by mathematical induction related to mathematics backgrounds, and (c) what differences did high school algebra students and college computer science students exhibit in their learning?

Unique Contributions to the Study of Proof

There are four contributions to understanding students' difficulties with proof by mathematical induction not found in the research literature. First, both high school and college students answered essentially the same tasks. The second contribution is that students were given both proof-analysis and proof-writing tasks. Third, the analysis of student responses was placed in a context of difficulties predicted by literature on problem solving as well as on proof in general. Fourth, I have made connections to students' attempts to mimic everyday reasoning when analyzing proofs by mathematical induction.

Methodology

This study was conducted in the first half of 1995 at a high school and university in an Indiana town of about 60,000 residents. Participants were recruited from two university classes of a lower-division, computer science course in the foundations of digital computing and two high school mathematics classes of honors algebra. The high school algebra course was equivalent to second-year high school algebra. Participants were given a brief description of the value of the study and offered $5 as incentives.

Students

Most of the 40 high school and 13 college students were from middle-class families. Both high school and college students were recruited so that different mathematical backgrounds and different levels of academic achievement could be compared. Most of the high school students were sophomores and all of them had either completed high school geometry or its equivalent, or were currently enrolled in a geometry course . The mathematics backgrounds of the college students differed widely, ranging from some high school and college mathematics to completion of high school calculus and courses beyond college calculus.

Data Collection

Volunteers completed two phases of data collection. Phase 1 was completed after approximately two weeks of instruction on mathematical induction. The questionnaire included: (a) questions about the students' mathematics backgrounds (b) a proof-writing task, (c) four proof-analysis tasks, (d) questions about the tasks, and (e) three general questions about proof and mathematical induction. Written responses to Phase 1 tasks provided the primary source of evidence for this study. Phase 2 of data collection consisted of individual interviews to clarify and elaborate the students' written responses.

The proof-writing task of Phase 1 was an algebraic identity involving the summation symbol and factorials. The proof-analysis tasks were four arguments that looked like proofs by mathematical induction: (a) with a missing base case, (b) that relied on one base case rather than the required two base cases so that the inductive step was not a general argument, (c) that reasoned with specific values of the mathematical statement rather than providing a generalized inductive step, and (d) that was correctly reasoned but contained non-trivial algebra involving the square-root symbol. The proof-analysis tasks were based on the work of Movshovitz-Hadar (1993).

Following each proof-writing and proof-analysis task in Phase 1, five questions were asked. Specifically, students were asked: (a) if there was anything about the mathematical statements that was not understood, (b) if they were already familiar with the problem, (c) about their confidence in the correctness of the proof, (d) about their confidence in the truth of the mathematical statement being argued, and (e) about planning and monitoring for the proof-writing task and if the argument was a proof by mathematical induction for the proof-analysis tasks. Phase 1 questions concluded with three general questions: to describe how mathematical induction works, to state when this proof technique can be applied, and to indicate whether a proof can be correctly done in more than one way.

Phase 2 of data collection began a few days after Phase 1. After a preliminary analysis of data collected from the first phase, 21 students were interviewed to clarify and elaborate on their written answers. The criteria for choosing students for interviews was one of the following: (a) recent graduates from high school, (b) poor or excellent backgrounds in mathematics, or (c) answers that were ambiguous or intriguing. The prompts for the open-ended interviews were students' answers to questions in Phase 1 of data collection. Individual interviews were audio taped for subsequent analysis.

Data Analysis

Written answers and responses during personal interviews were analyzed in three categories: (a) student performance, (b) knowledge about mathematical induction, and (c) difficulties with this proof technique.

Student performance was evaluated with researcher-developed scoring rubrics for the proof-writing and proof-analysis tasks. For the proof-writing task, students' proofs were scored based on successful completion of the base case and progress in doing the inductive step. For the proof-analysis tasks, the scoring rubric was based on students' ability both to identify whether the argument was a proof by mathematical induction and to explain why they thought so. Most students' answers fit easily within the scoring criteria. The few exceptions that arose were assigned a category based on the researcher's judgment. The overall score given to each student was this study's measure of performance.

Data was analyzed to identify aspects of students' procedural, conceptual, and applications knowledge. Procedural knowledge was demonstrated by recognizing a missing base case, identifying the need for a generalized inductive step, recognizing correctly argued proofs, and identifying the elements of a proof by mathematical induction. Conceptual knowledge was demonstrated by identifying the need for multiple base cases and conceptual describing mathematical induction. Applications knowledge was demonstrated with a correctly written proof and with examples of other statements where this proof technique might be applied. Students needed only to provide a minimal amount of evidence for these criteria.

All data were examined for evidence of students' difficulties within one week of data collection for written responses and within one day for recorded interviews. The data was repeatedly analyzed by the researcher in several independent sessions to provide consistency of interpretation.

Results

The primary question for this investigation was to characterize students' difficulties with the proof technique of mathematical induction. Secondary questions to be addressed by this study involved depth of understanding and characteristics of performance.

A Characterization of Difficulties

Students' difficulties with proof by mathematical induction were predicted in: (a) mathematical resources, (b) conceptual understanding, (c) convincing evidence, (d) everyday reasoning, (e) metacognitive control, (f) heuristics, (g) meaningfulness, (h) procedural knowledge, and (i) affective factors. As predicted in the literature, one additional category emerged from the data collected: difficulty associated with everyday reasoning.

Mathematical resources. Students were found to have many difficulties with mathematical content and deductive logic, especially the high school students.

Lack of mathematical content knowledge resulted in many difficulties with doing proof-writing and proof-analysis tasks. In many cases, students were unable to operate with symbols and to use algebraic procedures; the ones that appeared in tasks were a subset of what students might have mastered in high school or college courses. Specifically, students had difficulties due to incorrect representations of the proof-writing task, faulty calculations, wrong interpretations of algebraic statements, and inability to follow algebraic steps. Table 1 characterizes the types of difficulties students had with mathematical content and the effects they had in being able to write and analyze proofs.

Mathematical Content	**Effect on performance**
Summation symbol	Unable to make progress in proof writing: the mathematical statement was not represented correctly
Factorial symbol	Unable to make progress in proof writing: the base case was incorrectly calculated
Definition of divisibility	Unable to correctly analyze arguments: either being unsure whether the definition meant evenly divisible or not understanding the algebraic representation given
Definition of variable	Unable to correctly analyze arguments: x was interpreted to mean a constant
Algebra	Unable to make progress in proof writing: the representation of (n+2)! was incorrect Unable to correctly analyze arguments: either 10^{k+1} and $(10)10^k$ were not recognized as being equivalent or the base case was believed to be incorrect

Table 1. Specific difficulties with mathematics content

High school students found the level of mathematics difficult; much more difficult than problems demonstrated in class by their teacher. As one particularly frustrated high school student put it,

> We learned a couple of really simple proofs, then you throw these at us. Do you really think we're going to understand? It's like teaching a 10 year old how to play basketball, then put him in the NBA the next week! How do you think he's gonna do!

Rules for deductive logic also were difficult for many high school and college students in three ways. First, reliance on informal rules of logic seemed to interfere with an ability to look at the logic of given arguments. For example, 11 of the students said that the falsehood of the mathematical statement indicated that the proof was wrong. Although this is a correct inference on their part, it does not explain the flaw in a mathematical argument. There were other personal rules of inference: one college student believed that an incorrect proof implied that the mathematical statement was false, another college student expressed some discomfort with the idea that a true statement could have a faulty proof, and one high school and one college student believed that the truth of the mathematical statement implied that the proof was correct. These personal rules of inference seem to indicate that many students make a strong connection between the truth of a mathematical statement and the accuracy of its proof.

Second, students had trouble connecting the base case of a proof with the validity of the hypothesis of the inductive step. Students needed to realize that the base case is required to establish the truth of the hypothesis for the inductive step. About half of the 13 college students and almost all of the high school students—30 out of the 39 who completed the task—did not recognize a missing base case in a proof-analysis task. Further, only 5 college students and 2 high school students recognized the need for more than one base case. Many students did not appreciate the dependence of the inductive step on the base case, as 11 high school students expressed suspicions about a proof technique that relies on assumptions.

Third, students had difficulty with the requirement that the inductive step must be a general argument that applies to all cases of the statement. Over one third of the college students and about two-thirds of the high school students did not fully appreciate that the argument of a proof must be generalized, rather than rely on special cases.

Conceptual understanding. Most students did not describe mathematical induction in terms of its concepts. In proof writing, most students relied exclusively on procedures to convey a convincing argument, and did not explain what was being shown. Only 4 college and 5 high school students made any reference in their written proof to what was being shown by the proof. Poor performance often resulted from the understanding that if the proof-analysis task followed the general form of a proof, then it was a proof by mathematical induction. One high school student put it this way, "Yes, it started with an equation and proved it true through a number of different steps or assumptions."

A focus on the form of a proof over its substance was clearly in evidence. Showing difficulty with conceptual understanding, some students expressed a lack of confidence in the proof technique. One high school student said, "When I do induction, I don't believe it's true." Another high school student was more specific, "No, it is just by algebra and some general common sense." These two students did not appear to appreciate the convincing argumentation in a proof by mathematical induction.

Convincing evidence. There was evidence that examples played roles in providing convincing evidence for students. Examples were found to be important to students: (a) to verify statements, (b) to gain insight into how to go about doing a proof, and (c) as a guide in recognizing or doing a proof (as a template).

First, many students verified the truth of mathematical statements with examples other than the base case. Seven of the 13 college students and 13 of the 40 high school students used examples to verify the truth of the mathematical statements presented in the data collection tasks. Seven students used examples to verify more than one of the statements in the research tasks. One of the high school students said that examples give him confidence that statements are actually true.

Second, some students used examples to gain insight into how to go about doing a particular proof by mathematical induction. In interviews, 3 college students indicated that examples provide a case to follow in proving mathematical statements.

Third, examples seemed to be particularly important as templates. Students used examples to decide whether the arguments in proof-analysis tasks were proofs. For example, one college student performed a generalized inductive step for the proof-analysis task that was shown as a specific case. However, the template role of examples was subject to error. In many cases, students identified surface features of proof-analysis arguments in making their decisions. For example, 4 high school students made decisions about proof-analysis based on "adding the same thing to both sides" of an equation.

Everyday reasoning. When investigating students' resources in deductive logic, evidence of everyday reasoning applied to mathematical situations was found. Three specific instances of reasoning that resembled everyday reasoning were reported. One high school student drew from her everyday reasoning developed from school experience when she decided that the proof-analysis arguments were true because whoever wrote them was smarter than she. Second, in a response not unlike one trying to confirm a theory (Kuhn et al., 1988), one college student redefined the domain of a proof-analysis argument to try and justify it as a proof. Third, one student reacted to his experience with mathematical induction and said in an interview that he thought he may have used mathematical induction all his life, but he did not elaborate further. I interpret this response to mean that there is something about mathematical induction that makes it seem like some aspect of his everyday reasoning. These examples may be indications that students try to associate the logic of proof to their everyday reasoning.

Based on previously-discussed evidence that students develop informal rules of inference about a proof-like argument from the truth of the mathematical statement, it appears that students make rules from limited experience to guide their actions in mathematical situations, just like they would be expected to do in everyday reasoning situations.

Metacognitive control. Confirming the findings of Moore (1990), there was little evidence of planning or reflecting. Four college students and 5 high school students specifically stated that they did not see the need to plan before starting the proof-writing task. Only one high school student said she reread through one of the proof-analysis tasks in order to better understand the mathematics and one college student said that his false starts in proof-writing were due to a lack of elegance or were not getting him closer to a solution.

Heuristics. A difficulty was predicted by Hart (1994); namely, that low-level students would use a work-backwards strategy more often than high-level students. In confirmation of Hart's finding, college students with experience in mathematical induction preferred the work-forwards strategy and high school students who had the highest performance scores—9 or above out of 20—also chose a work-forwards strategy.

However, college students used a work-backwards strategy more often, but a work-forwards strategy was used more often by high school students. Of the 11 college students who attempted the inductive step of the proof-writing task, 8 used a work-backwards strategy. For the few high school students who had a recognizable solution strategy when attempting the proof-writing task, the preference for working forwards mirrored the way similar problems had been shown in class.

Work-backwards strategies were subject to two kinds of difficulty. First, through inattention to detail, 4 college students arrived at what appeared to be a circular argument, having dropped one side of their equations. The reasoning appeared to start and end with the same mathematical statement. To be mathematically correct when working backwards, an identity should have been reached. Therefore, this approach to writing a proof may be prone to the misconception about mathematical induction reported by Ernest (1984) that you assume what you want to prove.

Second, a work-backwards approach can obscure proof as an argument where the truth of an unknown is reasoned from known or assumed truths. In order to make the meaning clear, students should explicitly state that the steps in working backwards can be reversed. I asked 3 college students who had worked through the proof-writing task using a backwards strategy what had been shown and none of them was able to explain why the proof had been completed. To one of them, the procedure had become so automatic that its meaning had been lost.

Meaningfulness. Several students reported that they did not have enough time to comprehend the new material on proof. This was a frustration primarily expressed by high school students. Familiarity with the mathematical statements in the research tasks was not a factor in difficulty as far as the data showed, but only a limited portrayal of familiarity could be derived from the evidence collected.

Procedural knowledge. High school students seemed to have much greater difficulty with procedures than college students. No high school student completed the proof-writing task; 24 scored 4 or below (out of 20) on the performance measure. In contrast, half of the college students successfully wrote a proof and most received a performance score of at least 9.

Affective factors. From the evidence collected, neither the high school nor the college students studied expressed a belief that there is only one right way to do a proof. Although affective factors can cause difficulties, I could find only indirect evidence that affective factors were causal. Instead, it certainly was the case that negative attitudes and feelings accompanied difficulties with proof.

A Characterization of Understanding

Most students were struggling to gain an understanding of the three aspects of knowledge. However, results of the researcher-developed criteria revealed trends in students' understanding. As shown in Table 2 below, more participants satisfied the criteria for procedural knowledge than for either conceptual or applications knowledge. Further, high school students were less likely to provide criteria for knowledge about procedures than did college students. This evidence seems to indicate that the students tended to focus their cognitive attention on procedures rather than on concepts or applications.

Knowledge:	No. of college students satisfying all criteria	No. of HS students satisfying all criteria
Procedural	7 of 13	5 of 40
Conceptual	3 of 13	none
Applications	3 of 13	none

Table 2. Students providing all criteria for knowledge about mathematical induction

An example of a high school student's description of mathematical induction illustrates the procedural nature of students' thinking,

> First you show that the statement is true for the first number P_1. Then you assume it is true for any given number k and show that you can get to the next number P_{k+1}.

College students' descriptions tended to be more conceptual in nature; for example, one college student's description was as follows:

> 1. Prove base case. 2. Prove that for <u>any</u> arbitrary starting point, if that point gives a true value then the next consecutive point also gives a true value.

These examples illustrate the best descriptions; some students' descriptions were put in more algebraic terms and many were not mathematically correct.

The 3 college students who provided evidence to satisfy the criteria for conceptual knowledge also satisfied criteria for procedural knowledge. This evidence seems to indicate that a hierarchy of knowledge may exist, as Dubinsky and Lewin (1986) and Dubinsky (1986) have suggested, but the evidence was too limited to make any conclusions.

A Characterization of Performance

Researcher-developed scoring rubrics measured students' performance on proof-writing and proof-analysis tasks. As expected, overall performance was related to students' mathematics background, as Figure [illegible] (below) shows for the college students. For comparison, high school students' average performance measured 4.9 out of 2[illegible].

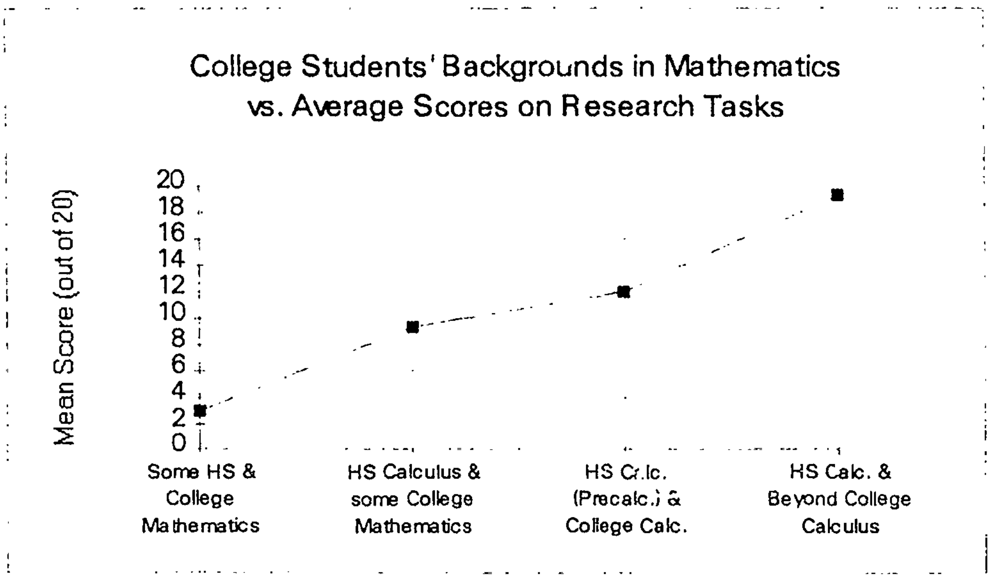

Figure 1. Mathematics backgrounds vs. scores on research tasks

When performance scores for each task were presented as distributions for the high school and college students separately, the distributions showed that the college students scored zero or 4 (out of 4) more frequently than they scored the median score of 2. Showing weaker performance than the college students, the high school students scored zero most often. When scores of zero were removed from the analysis, the gap in performance between high school and college students narrowed dramatically, but still, the college students outperformed their high school counterparts.

Summary

Both high school and college students had significant difficulties with the proof technique, procedurally and conceptually. A primary source of difficulty was attributable to a lack of mathematical content knowledge. The evidence collected suggests that examples play a critical role for many students in verification, for insight, and as patterns for how proof should be conducted. Some students showed evidence of trying to reason mathematically using informal reasoning. In general, students focused on the procedural aspects of mathematical induction far more often than on conceptual aspects. Difficulties predicted by earlier studies relating to procedural knowledge, conceptual understanding, use of a backwards or forwards strategy, and metacognitive control were confirmed.

Discussion

From the evidence in this study, the following aspects of difficulty were most critical to performance: (a) knowledge of mathematical symbols and content, (b) the ability to identify the steps in a proof, and (c) the need for and connection between the base case and the inductive step of a proof. Five aspects were suggestive of being important causal factors: (a) insufficient formal mathematics backgrounds, (b) the use of a work-backwards or work-forwards strategy, (c) inability to recognize substantial elements in examples of proof, (d) flawed generalizations that acted as prototypes for evaluating other proofs, and (e) attempts to use everyday reasoning and to generate informal rules of inference.

Limitations

There were four limitations of this study. First, the number of students studied did not allow any generalizations to be made. Second, the research tasks were pilot-tested only on college students so that the mathematical content may have been too difficult for many of the high school students. Third, data was collected in the early stages of instruction on mathematical induction. Therefore, the evidence collected did not reflect learning that occurred later. Fourth, the instruction given to the high school students differed substantially from that given to the college students. In their courses, college students received instruction on deductive logic. In contrast, high school students had no instruction in deductive logic and had somewhat less class time devoted to the study of mathematical induction before the study started. Therefore, the comparisons between high school and college students studied must be qualified.

Implications for Instruction

The results of this study suggest three responses by practicing teachers of courses where proof techniques are an introductory topic. First, since proof requires sufficient mathematical content knowledge to be successful, instructors should not assume mastery of content knowledge when demonstrating examples or constructing examinations. Otherwise, excessive cognitive burden may interfere with students' ability to show mastery of proof techniques.

Second, examples act as templates for action and for verification. Because of this, students should be given proof-analysis tasks to confirm the important features of a proof, to gain insight for what action to take next in a proof, and for conceptual understanding, including when mathematical induction breaks down.. In demonstrated examples, the instructor can help students by explicitly and carefully pointing out important aspects of what is being demonstrated (e.g., what parts of an example demonstrate proper form). Further, both proof-analysis and instructor-demonstrated examples can help make up for a lack of student-generated examples.

Third, students have difficulties with proof by mathematical induction as documented here and elsewhere, especially Dubinsky and Lewin (1986), Ernest (1984), Lowenthal and Eisenberg (1992), Moore (1990, 1995), and Movshovitz-Hadar (1993). I expect that instructors of courses where proof is an introductory topic who are aware of common difficulties can better monitor students' progress when analyzing in-class questions and homework problems, as well as when designing classroom lessons.

Suggestions for Further Study

I have three general recommendations for further study. First, I believe that there are many factors associated with difficulty, not just those directly considered for this study. Therefore, other studies on students' difficulties with proof by mathematical induction are warranted. Second, issues of frequency, potency, and persistence of difficulties were not in the scope of this study, but are important questions. Third, the question of how mathematical induction can most effectively be taught is worth considering. How does the curriculum available to teachers address students' difficulties? What is the relative amount and order of emphasis that should be placed on conceptual versus procedural knowledge for students learning mathematical induction? Finally, for students learning mathematical induction, how can the form and procedures of a proof be presented so as not to obscure its conceptual nature?

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