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ABSTRACT

This document contains the proceedings of the China-Japan-U.S. Seminar on Mathematical Education that was held in 1993 in China. The focus of the Seminar was problem solving in mathematics education. The main purposes of the seminar were: to examine the present states of problem solving in school mathematics in China, Japan, and the U.S.; to explore classroom practices in problem solving; to examine existing data concerning problem solving in mathematics education research; to explore teacher training aspects of problem solving in school mathematics; and to provide a context in which scholars from the three countries could meet, talk, and pursue mutual interests and future interaction. Presentations include: (1) "Learning How to Integrate Problem Solving Into Mathematics Teaching" (Jerry Becker); (2) "Mathematics Education in Japan-Some Findings From the Results of the IEA Study" (Toshio Sawada); (3) "Some Remarks on Problem Solving" (Changping Chen); (4) "How to Link Affective and Cognitive Aspects in the Mathematics Class-Comparison of Three Teaching Trials on Problem Solving" (Nobuhiko Nohda); (5) "Critical Issues in Problem Solving Instruction in Mathematics" (Douglas Grouws); (6) "An Overview on Mathematical Problem Solving in China" (Dianzhou Zhang); (7) "Diversity, Tools, and New Approaches to Teaching Functions" (Jere Confrey); (8) "Mathematical Open-Ended Problems in China" (Zaiping Dai); (9) "The Place of Problem Solving in U.S. Mathematics Education K-12 Reform: A Preliminary Glimpse" (Joan Ferrini-Mundy and Loren Johnson); (10) "On Mathematics Education in Japan" (Yoshishige Sugiyama); (11) "Observations on China's Mathematics Education as Influenced by Its Traditional Culture" (Jun Li and Changping Chen); (12) "Outline of Textbook Authorization in Japan" (Toshiaki Sakama); (13) "Problem Solving and Lesson Reviewing-A CAI Teaching Approach" (Shen Yu, Pengyuan Wang, and Zhong Dai); (14) "Changing the Elementary Mathematics Curriculum: Obstacles and Challenges" (Susan Russell); (15) "On the Method of Guiding Exploration" (Benshun Xu and Jiying Li); (16) "Mathematical Problem Solving" (Yoshihiko Hashimoto); (17) "On the Problem Solving Teaching Pattern in China" (Xiaoming Yuan); (18) "High Achievement Versus Rigidity: Japanese Students' Thinking on Division of Fractions" (Yoshinori Shimizu); (19) "Revision of the Course of Study for Student Guidance in High Schools" (Azuma Nagano); (20) "Computer Literacy in Japanese High School Mathematics: Compare the Present-Day Mathematics Textbook with the Revised Math Textbook" (Isamu Kikuchi); (21) "On School Mathematics Curriculum Design" (Ersheng Ding); (22) "A Survey of Doing Sums on a Mental Abacus" (Dexiang Tang); and (23) "The Mathematics Olympiad and Mathematics Education in Qingdao, China" (Guoqing Guo). (JRH)

PROCEEDINGS OF THE CHINA – JAPAN – U.S. SEMINAR ON MATHEMATICAL EDUCATION

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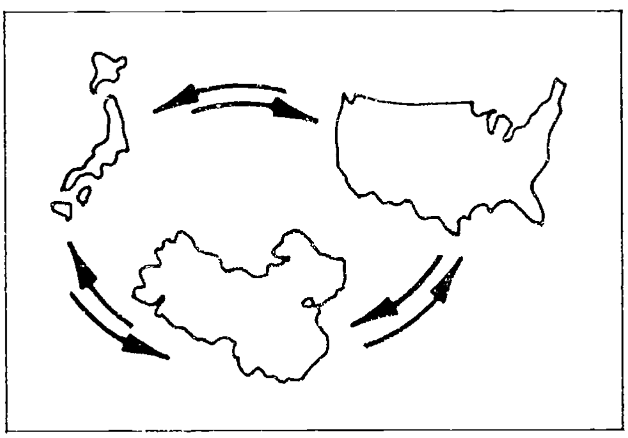

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PREFACE

Dianzhou Zhang, Toshio Sawada, and Jerry P. Becker, Co-Organizers

These are the Proceedings of the China - Japan - U.S. Seminar on Mathematical Education that was held October 4-8, 1993 in two sessions - the first session at East China Normal University in Shanghai, and the second in Weifang City, Shandong Province. The closing ceremony was held in Qingdao. The Seminar and these Proceedings mark the importance placed on mathematics education in the three countries, and on problem solving in school mathematics, in particular.

We believe, with all the delegates, that the Seminar was a success. Interesting papers and thought-provoking discussions filled the Seminar agenda. It was an important and enjoyable event held in excellent facilities, and the event marked the mutual and increasing interest by mathematics educators in all three countries in extending communication, exchange and cross-cultural collaboration in research.

We want to extend our heartiest appreciation to all the delegates who, through their paper presentations and the discussions, accounted for the quality of interaction during the Seminar. We need to also express appreciation to the Chinese National Science Foundation (CNSF), the U.S. National Science Foundation (NSF) and the Japan Society for the Promotion of Science (JSPS) which, through their funding programs, made this Seminar possible. We also express appreciation to Mr. Xue Mao Lin for his generous support of the Seminar.

No tri-national seminar can be successful without competent translators. In this respect, the Seminar was exceedingly fortunate to have Dr. Du Wei and Miss Qin Zhang as translators. Not only were they highly knowledgeable about the intricacies of translating, but they were friendly, amiable individuals who cooperatively worked patiently and tirelessly to smooth communication during the Seminar sessions. To both we extend our profound appreciation. We also need to thank the delegates themselves, all of whom prepared their papers in English; they also assisted with communication throughout the Seminar in a very cooperative manner.

This seminar was an important one and perhaps it is useful to describe its origin. The Co-Organizers have known each other for many years and have met on various occasions in China, Japan, the U.S., or elsewhere at professional meetings. In particular, they had a meeting at the Seventh International Congress on Mathematics Education (ICME-7) in Quebec, Canada and discussed the possibility of this Seminar. At that time, conversations were also held with other professionals about the appropriateness, timing, and content of the Seminar that would deal with problem solving. All agreed that such a seminar would be useful, as well as timely, and it was decided to seek support by submitting proposals simultaneously to the respective national agencies. The proposals were reviewed and recommended for support. There ensued preparation on all three sides covering a time period of one year, culminating in our Seminar at East China Normal University.

We need to express our appreciation to all the Chinese colleagues who, in a warm and friendly way, welcomed the U.S. and Japanese delegates to China and to the Seminar. We also thank the Chinese colleagues at Qufu Normal University who hosted the seminar participants while they were in Qufu. They were gracious, cooperative and helpful in numerous ways. We also want to express appreciation to Mr. Xuhui Li, a graduate student in the Department of Mathematics at East China Normal University, who, with his student colleagues, worked long, tedious hours to make sure that the Seminar was a success. To Ms. Joan Griffin goes our heartfelt appreciation for transporting the software disks to Microsoft Works and then organizing the papers and making edited changes. Her enormous energy and friendly competence in preparing the first draft was instrumental in getting the job done. To Ms. Connie Johnson goes our appreciation for typing further edited changes, proofreading and printing the final manuscript - without her competent assistance, the work would not have been completed.

It is our earnest hope that these Proceedings will be of interest to mathematics educators not only in the three countries, but to others around the world who share our desire to advance the cause of an improved mathematics education for children and students at all school levels.

Dianzhou Zhang | Toshio Sawada | Jerry P. Becker

January, 1996

SEMINAR PURPOSES

There has been considerable interest and a large number of activities in mathematics education in China, Japan and the United States in recent years. Mathematics educators in all three countries are exploring ways in which the teaching and learning of mathematics can be improved in all areas of the school mathematics curriculum. But the area of greatest interest, and the area in which mathematics educators of the three countries are focusing their attention, is problem solving. Accordingly, this is the focus for the present Seminar.

During the discussions between Chinese, Japanese and American mathematics educators, starting as far back as 1977 and continuing through 1992, a great and mutual interest was expressed in bringing mathematics educators in the three countries together to improve communication and propose further research. A joint China - Japan - U.S. Seminar seemed like an excellent manner by which to do this.

The main purposes of the Seminar were set as follows:

1. to examine the present states of problem solving in school mathematics in China, Japan, and the U.S.,
2. to explore classroom practices in problem solving in China, Japan, and the U.S.,
3. to examine existing data concerning problem solving in mathematics education research,
4. To explore teacher training aspects of problem solving in school mathematics, and overall
5. to provide a context in which scholars from the three countries can meet, talk, and pursue mutual interests and future interaction.

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THURSDAY, OCTOBER 7, 1993

Session III – Morning

Opening Ceremony (Weifang Session)

Session IV – Afternoon

Visit and inauguration of the Institute for Mathematics Education in Houzhen (China's first institute for mathematics education); visit to a school constructed with contributions of the Japanese people.

SATURDAY, OCTOBER 9, 1993

Session VII - Morning (Qingdao Session)

OPENING CEREMONY

Shanghai Session
East China Normal University

Monday, October 4, 1993

Professor Zhang:

Professors Becker and Sawada, colleagues and friends:

The U.S. - Japan - China Seminar on Mathematical Education now begins. On behalf of all the Chinese delegates, I would like to express our heartfelt thanks to all of you for your attendance and participation. It is our honor to have you with us. We are very happy to meet Professor Becker, Professor Sawada and many other old friends; meanwhile, it is also very nice to become acquainted with many new friends.

We express thanks for the support provided for the seminar by the National Science Foundation, both in the United States and China, and the Japan Society for the Promotion of Science. Due to this support, we are able to organize this joint seminar on mathematics education. As background and in preparation, the Co-Organizers had a short meeting in Quebec when ICME-7 was held there in 1992. "Problem solving" was singled out for the theme of this seminar. It is really a very important topic in which we are all interested.

China is a developing country, and the situation of mathematics education is also developing. When we reflect back on the past, Chinese mathematics educators have learned a lot from Japan from the early part of this century, and then from the United States after the 1920s, and the USSR had a considerable influence upon China in the 1950s. Now, we all are confronted with a changing world. Mathematics education is usually considered as a key factor in social development. Therefore, I think that this seminar is a significant event in the exchange between mathematics educators from the three countries.

The seminar will be divided into two sessions: the Shanghai Session and the Weifang Session. I hope you will enjoy your stay here at the East China Normal University in Shanghai.

LEARNING HOW TO INTEGRATE PROBLEM SOLVING INTO MATHEMATICS TEACHING

Jerry P. Becker

Southern Illinois University at Carbondale

Introduction

When mathematics educators think about problem solving, they probably think of George Polya and his widely known book *How to Solve It* (1945), his subsequent books on *Mathematics and Plausible Reasoning* : *Induction and Analogy in Mathematics and Patterns of Plausible Inference* (volumes 1 and 2, respectively, 1954) that deal with mathematical thinking and *Mathematical Discovery* (1962, 1965). Everyone agrees that Polya has had a significant influence on the thinking of mathematics educators about mathematical thinking. He has also had a similar influence on teacher education and pedagogy in the U.S. and elsewhere. Polya's works comprise a study of *heuristic*; in *How to Solve It* he outlined a framework for problem solving: *Understanding the Problem, Devising a Plan, Carrying out the Plan,* and *Looking Back.* With this framework came suggestions for solving problems. (For an in-depth and philosophical description of Polya's approach, the reader is referred to Schoenfeld (1987).)

To say the least, Polya's works provided the main ingredient for helping teachers at both the university and school levels to teach problem solving. They provided a large inventory of problems from several areas of mathematics. Teachers who studied from these books and worked the problems were challenged and learned a host of problem solving strategies and ways to think in tackling problems. In short, grappling with the problems had teachers and students *doing mathematics* - for many, perhaps, their first experience at it, rather than simply reading about it.

The way of thinking about problem solving due to Polya was far removed from the "drill and practice" in mathematics that was so characteristic of school and university textbooks and classroom teaching during the period from the 1950s into the 1960s. The advent of the "New Math" during the 1960s was consonant, in some ways, with Polya's ideas. Following the "New

Math," the *Back to Basics* movement evolved which is well-known as a setback in U. S. mathematics education. Indeed this movement was the first large-scale public-initiated intrusion into the domain of mathematics education. In response, the National Council of Supervisors of Mathematics (1977) recommended that problem solving be regarded as the main reason for studying mathematics in the schools. Shortly thereafter, the National Council of Teachers of Mathematics (NCTM) issued its widely disseminated booklet *An Agenda For Action* (1980). In it, the NCTM recommended that "problem solving be the focus of school mathematics in the 1980s." (p. 1) This recommendation was briefly elaborated in terms of curriculum development, the language of problem solving, teaching and the classroom environment, problem solving through applications, and a research agenda to address the nature of problem solving and developing effective problem solvers. (pp. 2-5) Subsequently, numerous publications appeared which cited, followed and advocated Polya's approach to problem solving. Both pre- and inservice teacher education programs were marked by Polya's approach.

But, in response to whether or not problem solving was dealt with in a serious way, Schoenfeld (1987) commented that

> Despite some serious effort, much of the problem-solving movement is largely superficial. For many, including the developers of mainstream textbook series, problem solving has been taken to mean adding a few trivial (and often one-step) word problems to the curriculum... For others, it has meant studying a few easy problem solving techniques in isolation, for example, looking for patterns. (p. 40)

Even the "new math" had little positive impact on raising students' problem solving prowess (Schoenfeld, 1987, p. 39). Schoenfeld further looked at Polya's descriptions of problem solving strategies from the perspective of people inexperienced with them (i.e., non-mathematicians) and concluded that not enough detail was provided for people to implement them. (p. 42) Further, Schoenfeld mentioned that

> "Research now indicates that a large part of what constitutes competent problem-solving behavior consists of the ability to 'monitor and assess' what you're doing as you work problems and to make the most of the problem-solving resources at your disposal. It also indicates that students are pretty poor at this, partly because issues of 'resource allocation during thinking' are almost never discussed. But there is evidence that when students get coaching in problem solving that includes attention to

such things - when they are encouraged to think about issues like "What are you doing?" "Why are you doing it?" "How will it help you solve the problem?" - their problem solving performance can improve dramatically." (p. 44)

Earlier, Kilpatrick (1985) mentioned similar ideas and commented that research shows that the solution of a complex problem requires (1) a store of organized knowledge about the content domain, (2) procedures which enable the solver to represent and transform the problem, and (3) a control system for selecting knowledge and procedures. (p. 28)

Well before mathematics educators raised questions, based on empirical research, that questioned whether heuristic instruction actually enhances students' problem solving performance, problem solving had already assumed the main theme of mathematics education in the U. S. in the 1980s. The momentum accelerated and it clearly reflected reliance on Polya's four-stage model. It was about this time that we began systematic work with inservice mathematics teachers in Southern Illinois - with a focus on teaching problem solving.

The Teaching Environment of Teachers in the Southern Illinois Region

Before describing our work with teachers, the reader needs to be aware of the environment in which our teachers teach. We have surveyed teachers and found the following characteristics:

- Teachers have very heavy classroom teaching assignments each day, usually with only one class period (of 6, 7, or 8 periods) free for preparation.
- Teachers commonly have 3-4 different class preparations each day.
- Teachers commonly work outside the normal school day grading assignments, developing tests, scoring tests, evaluating notebooks, etc.
- Teachers commonly have other extracurricular responsibilities after school hours (required or undertaken to earn extra salary).
- Teachers' lessons are frequently interrupted by "administrative intrusion": public address announcements, students called out of classes, students returning to classes, students leaving to go on field trips, etc.
- Teachers frequently have inadequate backgrounds in mathematics, especially for grades K-8 - their training commonly includes only one or two university mathematics courses and one "methods" course.
- K-8 teachers commonly express a fear of or anxiety about mathematics and they have frequently learned what mathematics they know solely through a professor/lecture approach, far different from the manner in which they are expected to teach.

- Teachers' incorrect perceptions or beliefs about the nature of mathematics are a consequence of their own inadequate experience with the subject.
- Teachers constantly feel pressure from the public's focus on students' poor achievement (e.g., National Assessments of Educational Progress (NAEP), international comparisons of student achievement, and state and local assessments of student learning).
- Teachers are besieged by new curricular regulations and policies handed down by local and state educational authorities.
- Teachers have to deal with the personal problems of students who carry the burdens of modern society with them into their classrooms.
- Teachers, in some cases, are given inappropriate teaching assignments, depending on their schools' circumstances and the needs of their schools - this sometimes leads to a loss of identity with mathematics, and they may experience a sense of isolation with too few opportunities to interact professionally with other teachers.
- Teachers experience fatigue - it is a dominant factor in teaching, along with a wearing down of the spirit.

As a consequence, creativity may be sapped, due to forces teachers perceive to be beyond their control. Nonetheless, we have found that they persevere and willingly work towards professional improvement. The point is, however, that there is this *reality* of both teachers and their schools which is part of the context in which our work with them must take place. In the final analysis, it is the shoulders of classroom teachers on which responsibility rests for improving mathematics teaching in the schools.

Brief Descriptions of Two Early Teacher Enhancement Projects in Mathematics at SIUC

The teacher enhancement work in which I have been involved began in 1983-84 with 40 teachers - 20 elementary (K-6) and 20 secondary (7-12). This was a small project in which we attempted to integrate problem solving into mathematics teaching. The project consisted of Saturday classes held on campus with an emphasis on exposition and on non-routine problem solving following Poyla's four-stage framework. Computers were also used in problem solving - programming in Logo (elementary) and Pascal (secondary).

In this work, we also attempted to follow the famous precept, paragraph 243, in the *Cockcroft Report* (Cockcroft, 1981) which states that mathematics teaching at all levels should include opportunities for

- exposition by the teacher,
- discussion between teacher and pupils and among pupils themselves,
- appropriate practical work,
- consolidation and practice of fundamental skills and routines,
- problem solving, including the application of mathematics to everyday situations, and
- investigational work.

Fletcher (1983) comments that the remarkable thing is that no one would say that any of the six opportunities above does not matter; "but if you remark that most mathematics lessons involve no more than two of the six headings, there is never any doubt in the mind of the listener as to which two you mean." (p. 70) Teaching characterized by concentration on exposition and practice at the expense of other areas he calls *excessively didactic*. (p. 70)

In our first project, in retrospect, the approach was *excessively didactic* and there was little evidence that we were successful at integrating problem solving into teachers' classrooms. Further, though teachers seemed to enjoy the work a great deal, the approach we used provided no really effective "bridge to the classroom" - there was no real connection made between work in the project and teachers' classrooms.

In the second project, in 1987-89, we attempted again to integrate problem solving into classroom teaching. This was a larger project in which sixty teachers participated in a four-week, on-campus residential institute: 15 from grades K-3, 15 from grades 4-6, 15 from grades 7-9, and 15 from grades 10-12. Again, the emphasis was on non-routine problem solving ala Polya and applications of mathematics. Computers were also used in problem solving as in the earlier project.

One important accomplishment of the project was the development of a "Community of Scholars" among teachers (K-12). In this informal seminar, teachers discussed *how to implement the problem solving approach in their classrooms*, NCTM's *Curriculum and Evaluation Standards*

For School Mathematics (1989/hereafter referred to as the *Standards*) which was released with great media attention, and other issues in mathematics education. In addition, teachers were required to keep an elaborate problem solving *Notebook* as a resource for implementing project work in their classrooms. Finally, each teacher was responsible for development of a problem solving *Teaching Unit* (*TU*). The procedure for this was to write the *TU*, revise it after staff critique, try it out with students, revise it again, try-out again by another teacher at the same grade level, and then revise it to final form for use by other teachers and for dissemination. The project provided follow-up of teachers in their classrooms as soon as the school year began, when the teachers began implementation of the summer work in their classes. Ten all-day Saturday "Community of Scholars" meetings on campus during the school year kept teachers in contact on an on-going basis.

We found that the problem solving *Notebook* was an extremely valuable resource for teachers. They had written all problems, solutions, heuristics used, extensions of problems and a record of discussions of problem solutions in their *Notebooks*. Similarly, the *TU* was valuable - it provided a "sense of ownership" and also a "bridge to the classroom." The "Community of Scholars" continued to grow after the summer, with teachers in communication with each other about mathematics and pedagogy; in effect, a network among teachers emerged which teachers found extremely valuable - it kept them in contact with each other and helped to remove the sense of isolation which was prevalent before the project.

The following summer in 1988, a new group of teachers - colleagues of the previous-summer teachers - was brought to campus for a similar institute. As a result, there were now two teachers in each school working together on implementation and providing mutual support in integrating problem solving into the curriculum. Altogether we now had nearly 100 teachers working together in the network trying to realize the projects' objectives.

Evaluation of the two summer programs (1987 and 1988) showed significant pre- to posttest improvement in teachers' attitudes towards problem solving and problem solving skills. Teachers reported that their students benefitted from the approaches emphasized in the summer programs. They also developed confidence in their ability to effect change in the curriculum. Through development of the *Notebook* and the *TUs,* there was a strong "sense of ownership" and teachers took great pride in their work, both in their classrooms and in assuming leadership roles in the reform movement that was beginning to emerge in the U.S. The "glue to the classroom" provided by the *Notebooks and TUs* and the Community of Scholars were clearly important ingredients in the projects. Evidence indicated that teachers were sensitized against using an *excessively didactic* approach in their teaching, about which we were careful in the projects. In this sense, we felt we made progress towards integrating problem solving into teachers' classrooms. Certainly we had invested a substantial amount of time, energy and resources and teachers reported positive results with their students.

Integrating Problem Solving Into Middle School Mathematics Teaching (Grades 6-8)

There were several developments on the "problem solving front" during the 1987-89 project. The first was the release of *Everybody Counts* (NRC, 1989) and *The Curriculum and Evaluation Standards* (NCTM, 1989). The former served as a preface to the latter; together they represented a strong and compelling case for emphasizing problem solving in school mathematics. *Everybody Counts* called for a transformation in both the content of the curriculum and instructional style; it further called for a focus on seeking solutions, not merely memorization of procedures; exploring patterns, not memorizing formulas; conjecturing, not simply doing exercises. (p. 84) The Standards called for similar emphases; in fact, it states four standards that should be emphasized at all grade levels: mathematics as *problem solving*, mathematics as *communication*, mathematics as *reasoning* and *mathematical connections.* The *Standards* also set forth an assumption about mathematics learning, namely, *what* we teach and *how* students experience it are the primary factors that shape students' understanding and beliefs regarding what mathematics is about. (p. 5)

The second development was a fairly wide recognition among mathematics educators, flowing from research, that students as well as teachers hold the following beliefs about mathematics (cf., McKnight (1987) and Schoenfeld (1991)):

- There is one correct answer to any problem.
- There is one correct way to solve any problem.
- Mathematics is passed to students from "above" (the teacher) for memorization.
- Mathematics is a solitary activity.
- All problems can be solved in 5 minutes or less.
- Mathematics is easy.
- School mathematics has little to do with the real world.
- Proof has nothing to do with mathematical discovery or invention.

The third development was the acquaintance we acquired with Japanese research on the "open-approach" to teaching mathematics. This teaching-evaluating research, largely unknown to U.S. mathematics educators, focused on development of problems (for use in teaching) that have a multiplicity of correct answers, a multiplicity of ways to solve a problem with a unique answer, or a multiplicity of problems formulated by students (problem posing) similar to one they have just solved (cf. Shimada (1977 - in Japanese), Becker and Shimada (1996), Nohda (1983), Nagasaki and Hashimoto (1984) and Takeuchi and Sawada (1985-in Japanese)). The aim was to provide students with experience in solving these "ill-conditioned" problems requiring, of students, no more than the repertoire of knowledge and skills they have already acquired as part of their education. Overall, Japanese researchers sought to make this style of teaching an indispensable part of school mathematics teaching (Becker and Shimada, 1996, p. 14). This teaching approach has students (Becker and Shimada, 1996)

- mathematizing situations,
- making good use of their knowledge and skills,
- solving the problem(s),
- seeing other students' discoveries and results,

• examining and comparing different ideas of different students, and

• modifying and further developing students' ideas.

Carefully developed problems and lesson plans that were tried-out were an important part of the teaching-evaluating research. Lesson plans were developed to "draw out" students' natural and different ways of thinking about problems. These different ideas then provided the substance for discussion, with the teacher as a facilitator. The teacher then plays a crucial role in summarizing the lesson. Further, Japanese researchers developed an assessment-of-learning approach that is interesting, useful and very different from traditional approaches to assessing student learning in the U.S.

Since the middle school years are the time when so many students are lost from mathematics in the U.S. (half of the students end their study of mathematics each year from grade 8 onwards), we decided to focus our next project at this level. Further, since the Japanese "open-approach" fit so compatibly with NCTM's first four standards, we decided to incorporate this approach, as a major component, into our teacher enhancement work with middle school teachers.

The fourth development was new curricular materials disseminated from Michigan State University (*Middle School Mathematics Project*/MGMP). The philosophy on which the MGMP was based mirrored recent findings in cognitive science, summarized by Resnick (1986) as follows:

> First, learners construct understanding. They do not mirror what they are told or what they read. Learners look for meaning and will try to find regularity and order in the events of the world, even in the absence of complete information. This means that naive theories will always be constructed as part of the learner process.
>
> Second, to understand something is to know relationships. Human knowledge is stored in clusters and organized into schemata that people use both to interpret familiar situations and to reason about new ones. Bits of information isolated from these structures are forgotten or become inaccessible to memory.
>
> Third, all learning depends on prior knowledge. Learners try to link new information to what they already know in order to interpret the new material in terms of established schemata. This is why students interpret science demonstrations in terms of their naive theories and why they hold onto their naive theories so long. The scientific theories that children are being taught in school often cannot compete as reference points for new learning because they are presented quickly and abstractly

and so remain unorganized and unconnected to past experience. (quoted from Fitzgerald, 1987, p. 13)

The MGMP materials consisted of five booklets titled *Mouse and Elephant*, *Spatial Visualization*, *Factors and Multiples*, *Probability*, and *Similarity*. These materials help students to develop a deep, lasting understanding of concepts and thinking strategies. They concentrate on a cluster of important ideas and their inter-relationships. Concrete models assist students in moving from a concrete stage to more abstract reasoning; i.e., students *abstract the concepts themselves* from their concrete experiences (emphasis added/Fitzgerald, 1987, pp. 16-17). Overall, the teaching strategy consists of three phases: first: introduce new concepts; second: exploration (students working individually or in small groups); third: summarizing (the teacher "pulls together" the ideas *of students*, facilitates discussion, and deepens understanding). Our view was that the MGMP materials complemented the "open-approach" to teaching problem solving (mathematics), fit nicely with the approach, and was also highly consistent with the recommendations in the *Standards*.

Finally, probability and statistics was recommended as a new and key content strand for all grade levels in the *Standards*. New materials developed at the Technical Education Research Center (TERC) become available in 1989-92 for the elementary grades, including early middle school (*Used Number Project*/UNP). In addition to dealing with important statistical concepts, we felt the pedagogy, here too, fit nicely with recommendations in the *Standards* and the "open-approach". The materials engage students in study of statistics (using real data), provide opportunities to model real mathematical behaviors like statisticians, and have them participating in (Russell and Corwin, 1989):

- cooperative learning,
- theory building,
- discussing and defining terms and procedures,
- working with messy data, and
- dealing with uncertainty. (p. 1)

We decided that we would incorporate the Japanese "open-approach" to teaching problem solving and the MGMP and UNP materials as major components in the instructional program of our project. They all represented approaches to teaching mathematics that are far from being *excessively didactic*; in fact, we felt that all would make a significant contribution to development of mathematical thinking abilities in teachers and their students.

We also decided that in the "open-approach" problem solving seminar, the faculty teachers would model the teaching in the same way that we wanted teachers to teach their students. In the two seminars for the MGMP and UNP materials, the teaching would similarly follow the teaching model fairly closely. But, in addition, we decided it would be important for : (i) teachers to observe the project staff teaching middle school *students* the same way they were being taught, using the same problems and materials, and (ii) have teachers develop lesson plans which they would use in teaching the middle school students.

So, we recruited three groups of middle school students (grades 6,7,8) and organized *Demonstration Teaching*. As an aside, it is noted that for U.S. teachers, teaching students with other teachers observing is not common; in fact, it is daunting and not easily accepted. However, we were able to work this out and, in the end, it was highly successful. We began a kind of "tradition," in this regard, and teachers became quite comfortable with it. The final component in the project was the "Community of Scholars" seminar. The seminar was organized in the same manner as in the 1987-89 project which worked so successfully.

With these five components, we began the new project with thirty middle school teachers in the summer of 1990. There was a four-week, intensive, residential institute on campus in the summer of 1990, followed by implementation of objectives in teachers' classrooms and continuing "Community of Scholars" seminars during the 1990-91 school year. Then, these teachers returned for a second two-week residential institute on campus in the summer of 1991. Here the instructional program included a continuation of the "open-approach" seminar and two new seminars - one with a focus on geometry and the other on problem solving using Logo. The same program was repeated for a new group of 30 teachers in 1991-92. We note here that one of the Japanese originators of the "open-approach," Professor Yoshihiko Hashimoto, came to SIUC to teach the second-summer seminar in both 1991 and 1992. This helped us to remain faithful to the "open-approach" and Professor Hashimoto made a very important contribution to our work.

Another comment needs to be added here. Since this was a residential institute, the teachers lived in a dormitory for four weeks. The dormitory had a large "commons area" for "Community of Scholars" seminar meetings and also a large workroom with blackboards, tables and chairs. There was room for computers (programming and word processing) and reference materials. Classrooms where seminars were held were located nearby as were the eating and recreation facilities. Thus, we had teachers living, eating and working together for four weeks. The facilities were necessary and excellent for the intensive work we had teachers doing.

Each of the 60 teachers developed an elaborate "open-approach" problem solving and probability and statistics lesson plan during the summer program. These were duplicated so each teacher had full sets of 30 problem solving and 30 probability and statistics lesson plans at the end of the summer, for use in implementing project work in their classrooms. They also used the MGMP materials in their implementation. Pairs of teachers developed a *Plan for Implementing* materials from the project near the end of the summer project.

In each year, the teachers' implementation was monitored fairy closely. Whole-day visits were made to their classrooms, during some of which the project director also performed some demonstration teaching. The latter proved to be quite novel and was very successful: a professor teaching school students is not common in the U.S. Visits were also scheduled with teachers' administrators, and each year in December and May, teachers submitted detailed reports of what lesson plans had been used and how they worked. At the end of each of the two years, a one-hour interview was scheduled with each teacher in her/his school. Here we assessed the extent to which teachers changed their teaching behaviors and how their students responded. During discussions with teachers' administrators and teaching colleagues, we assessed their reactions and those of parents to changes in our teachers' classrooms. Both administrators and parents showed nearly uniform acceptance, and frequently, outright enthusiasm for the changes underway.

The "Open-Approach" to Teaching Mathematics

Since the Japanese "open-approach" to teaching was such a prominent characteristic of the project, perhaps a little more elaboration is in order. The "open-approach" research was begun in 1971 (Shimada (1977/in Japanese) and Becker and Shimada (1996)) and later expanded (see Takeuchi and Sawada, 1985). Other researchers included Y. Sugiyama, Y. Hashimoto, N. Nohda, H. Kimura and others, including many classroom teachers.

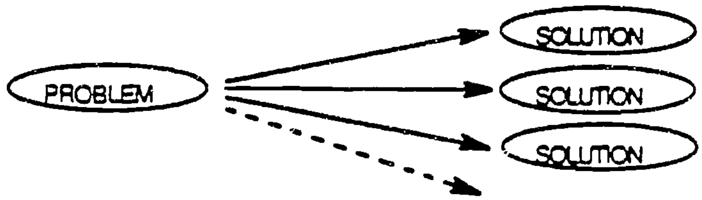
The "open-approach" engages students in mathematical inquiry through lesson plans developed around problems, as mentioned earlier (i.e., many different correct answers/responses in problem situations, many different ways to solve a problem which has a unique answer, and students formulating problems like one they have just solved. The figure below depicts what the author refers to as "Openness in Mathematics Education" reflecting the "open-approach" as we handled it in the seminar during the two summers.

1. ONE PROBLEM . . . . ONE SOLUTION (ANSWER)

WAYS

(Process is open)

2. ONE PROBLEM (OPEN ENDED) . . . . . SEVERAL OR MANY SOLUTIONS (ANSWERS)



WAYS

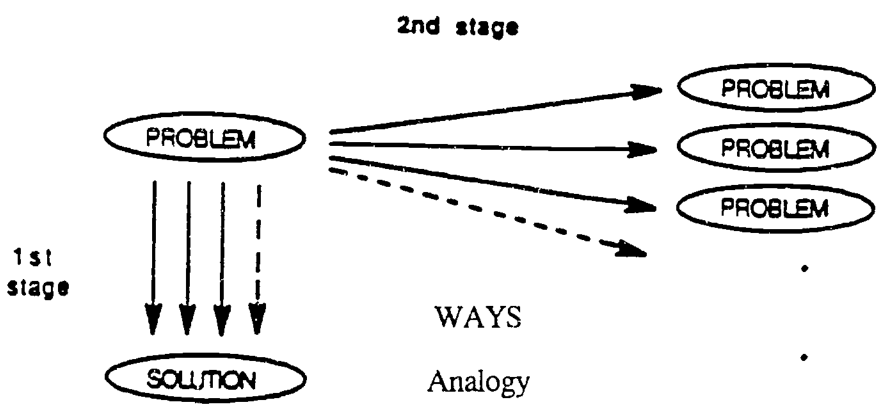
(End products are open)

A. Problems in which students find rules, relations, and patterns

B. Problems in which students find ways to numerically measure

C. Problems in which students find ways to classify

3. ONE PROBLEM . . . . SEVERAL PROBLEMS
("From problem to problem . . . . the developmental approach")



WAYS

Analogy

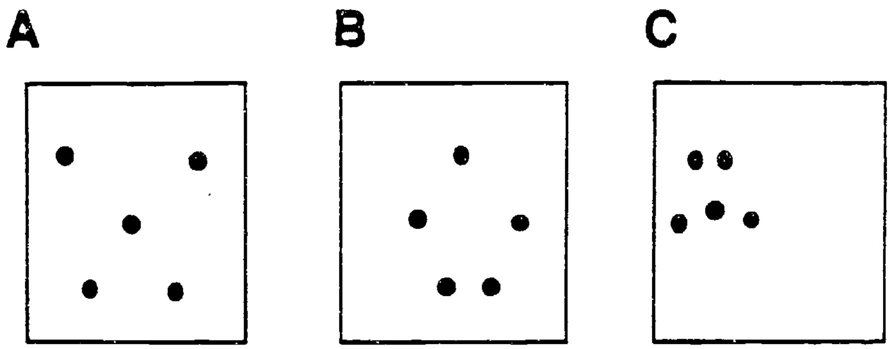
Generalization

(Ways to develop are open)

Openness in Mathematics Education

Examples of two problems used in the seminars are as follows:

Marble Problem (Shimada, 1977)



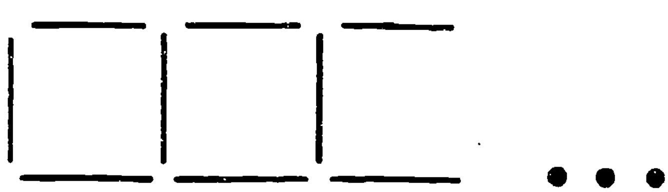
Three students A, B, C throw five marbles that come to rest as in the figures above. In this game, the student with the smallest "scatter" of marbles is the winner. To determine the winner, we will need to have some numerical way of measuring the "scatter" of the marbles.

(a) Think of this situation from various points of view and write down different ways of measuring the degree of "scattering".

(b) Compare the different ways of measuring. Do you think there is a "best" one? If so, what is it, and why?

Toothpick Problem (Hashimoto, 1987)

Squares are made by using toothpicks as shown below. When the number of squares is 8, how many toothpicks are used? Find your answer as in as many different ways as you can.



Later, after discussion of the problem and different ways for finding the answer:

Now, make up and write down as many problems as you can that are related to this problem.

The problems are of crucial importance in this teaching method. We used many problems from the Japanese research and we also formulated, tried out and developed problems of our own - including changing some from the textbooks used by teachers to "open-approach" types.

There is an obvious relationship between this approach to teaching and certain high priority goals of our current reform movement (i.e., the *Standards*). During the project, teachers also had a copy of the "Working Draft" of NCTM's *Professional Standards for Teaching Mathematics* (1991). The "open-approach," as we interpreted and used it, was highly consistent with the recommendations in the new *Professional Standards.*

Lesson plans are of crucial importance in using this approach. Accordingly, we adopted the following organization of lesson plans in the problem solving seminar (assume a 50-minute class period; cf. Becker et al. (1990) and Nagasaki and Becker (1993)):

I. Introduce the problem .... 5 minutes

II. Understanding the problem.................................... 5 minutes

III. Problem solving by students, working individually, in pairs or in small groups.................... 25 minutes

(Note: Here we draw on the students' *natural ways of thinking* about the problems.)

IV. Comparing and discussing (students put their solutions/ways of thinking on the blackboard for everyone to see)............................ 10 minutes

V. Summary by teacher... 5 minutes

Notes: (1) Of course, some problems we used required more than one period (e.g., 2-3 periods).

(2) Sometimes, during the last few minutes, we asked students to write down what they learned in the lesson.

(3) Homework may involve additional time to solve the problem, reflect on the problem, extend the problem, or solve other (perhaps related) problems.

(4) The times above, of course, are not absolute and may vary according to the circumstances of the lesson and teaching conditions. However, we developed some lessons designed to be carried out *in the prescribed time period* - our reason was to help teachers to become more conscious of managing the lesson carefully and using class time effectively and efficiently (which many teachers reported that they did not do regularly in their classes; in fact, many reported, after their experience with this approach, that much instructional time had previously been wasted).

Assessment of Learning

Japanese researchers developed an approach to assessment which we discussed in our "open-approach" problem solving seminar (c.f., Shimada, 1977/in Japanese and Becker and Shimada, 1996). Before describing it, a couple preliminary comments should be made. First, it is important in using this approach that teachers cooperate and collaborate in developing lesson plans. In particular, we found it important for them to solve problems together in order to generate as many different perspectives for approaching the problem(s) as possible; that is, generate many different correct answers, many different ways to solve a problem with a unique answer, or formulate many different problems. This helps the teachers to become *quite informed* about the problem situation (understand it), to develop teaching *confidence* and to be prepared to handle students' questions. Secondly, after teachers develop, say, many different correct ways to solve a problem, they then categorize the different ways according to the different *mathematical ideas* present in all the ways. This categorization plays a key role in doing assessment as is seen later.

There are four components or parts to the assessment, as follows (Shimada, 1977 - in Japanese and Becker and Shimada, 1996):

1. **Fluency**

   This is concerned with the *number* of different correct answers, diff rent solution approaches, or problems formulated by an individual student (or group of students). It is assessed *quantitatively* by simply counting.

2. **Flexibility**

   This is concerned with the *mathematical quality* of a student's (or group's) responses - how many mathematical ideas are discovered by the student (or group). It is assessed *qualitatively* mostly; however, it can also be assessed quantitatively by assigning points to a student's (or group's) responses.

3. **Originality**

   If a student (or group) develops a unique or original idea not found by other students (or groups) or makes as especially insightful observation, originality should be given a very high assessment (i.e., responses of very high mathematical quality should be acknowledged and recorded as a very high assessment).

4. **Elegance**

   This is concerned with the degree of elegance in a student's (or group's) *expression of their thinking in mathematical notation.* This may be difficult to assess objectively; nevertheless, it has potential as part of assessment.

Information or data for use in assessment can be collected, in our experience, in the following ways:

- Analyze student's' (or groups') worksheets. For example, after students have the problem presented to them and begin individual problem solving, they should write down their work (thinking) on their worksheet. These are then collected and later analyzed according to the approach outlined above. If students then form groups, a "measure" has been made of individual work and, in the group, they naturally share *their different ways of solving the problem* with other students.
- While individual students (or groups) are working, the teacher moves among them observing and listening to what they are writing and/or saying. This is not casual monitoring to keep students' behavior in check; rather, this is *purposeful scanning of students' work* that can be remembered and/or recorded by check marks on an assessment form. During this time, the teacher also selects particular students (or a representative of a group) to put particular problem solving processes on the blackboard, for discussion later.
- During the comparing and discussing part of the lesson, the teacher mentally notes observations and contributions of students for recording later.

Thus, there is opportunity for the teacher to collect important information for assessment purposes in at least these different ways. Further, instruction can be adjusted based on observations during teacher-students or students-students discussion. So, the assessment approach also serves as an important function to improve teaching *during* the lesson as well as to improve the lesson plan itself. Thus, it can be seen that this approach to assessment can contribute or lead to *teacher improvement* as well as to *curriculum improvement.*

Evaluation/Documentation of the 1990-1992 Project

The results of the project evaluation/documentation are summarized below (Becker, 1993):

A. Teachers' Feelings about the Project: A tabulation of teachers' feelings about the project in both 1990-91 and 1991-92 showed very high satisfaction on each of 16 items on a questionnaire, including a high general, overall evaluation of the project.

B. Reports of External Evaluators: Two professors external to SIUC in each of 1990 and 1991 visited the project, observed all seminars and many demonstration lessons, interviewed all staff members, and interviewed teachers individually or in small groups. Their oral and written reports showed clearly that teachers improved their knowledge of problem solving; skills in solving problems; attitudes towards mathematics, problem solving, and technology; identified with the spirit and letter of the pedagogy emphasized in the institutes; and that the objectives of the project were being achieved. They reported that this was a direct result of participation in the project. They also provided very useful suggestions/recommendations for follow-up during the academic year - their reports were very hard-hitting constructive critiques of project activities.

C. Pre and Posttesting and other Documentation [teachers used pseudonames on all measures]: Year 1 (1990/summer and 1990-91):(1) Results showed statistically significant change from pre-to-post on a 61-item Likert measure and subscales (Attitudes Towards Mathematics Problem Solving and Technology), with high Cronbach alpha's for the whole measure and each subscale; (2) results showed statistically significant change from pre-to-post on a Beliefs About Mathematics measure, with high Cronbach alpha's for each part; (3) results showed statistically significant change from pre-to-post on a MGMP Achievement measur with a high Cronbach alpha; (4) results showed a statistically significant change from pre-to-post on a Problem Solving Test and subscales, with Cronbach alpha's for the whole measure and each part; (5) results of Teacher Questionnaires, which were administered at the end of Summer 1990 and in Spring 1991 (covering all the components of the program), and Final Interviews were favorable and clearly indicated that a metamorphosis among participants had occurred. Further, results indicated that participants embraced the recommendations given in NCTM's two Standards. Since no funds were available and time was severely scarce for planning, no comparison group was used in the year 1 evaluation.

Year 2 (1991/summer and 1991-92): All the pre-post measures used in year 1 were used again in summer 1991, with only minor changes made. However, this time a comparison group of 25 middle school mathematics teachers was recruited and administered the pre- and post measures with exactly the same time between testing sessions. The ANCOVA analyses indicated that; (1) there was a statistically significant difference, favoring the project group, on the Problem Solving Test and subscales, with high Cronbach alphas for each; (2) there was a statistically significant difference, favoring the project group, on the whole Likert measure and subscales (Attitudes Towards Mathematics, Problem Solving, and Technology), with high Cronbach alphas for the whole measure and each subscale; (3) there was a statistically significant difference, favoring the project group, on the Belief About Mathematics measure, with high Cronbach alphas for each part; (4) there was a statistically significant difference, favoring the project group, on the MGMP Achievement measure with high Cronbach alphas for both groups; (5) results of both the Teacher Questionnaire and Final Interviews with teachers in their schools were again very favorable and, like year 1, indicated that a metamorphosis among participants had occurred. These results confirm that the summer projects had a significant impact on teachers and, further, the validity of the findings were enhanced by inclusion of the comparison group in the evaluation in year 2.

D. Follow-up of Teachers in Fall 1990 and Fall 1991. As mentioned earlier, all 60 teachers were visited in their classrooms following their participation in the summer institute. During these visits, from 1 to 4 classes were observed, the project director sometimes taught demonstration lessons, teachers were interviewed and discussions were held with administrators. It is clear that participants were committed to implementing the project materials in their classrooms and were enthusiastic about it. Moreover, they were discussing their work with other teachers in their schools, were conducting inservice institutes for other teachers and had acquired high profiles as teaching professionals throughout the region - they conducted workshops and gave talks at county, state and national meetings of teachers, many of whom had not done this previously.

E. Summary: A variety of measures and other forms of documentation were used to collect data on several dimensions of improving the middle school teaching of mathematics with these teachers. There were very clear indications from the results reported above that teachers benefited from project activities. Their knowledge, skills, attitudes, beliefs and classroom teaching changed in a positive direction. The results of the evaluation and documentation have both complemented and supported staff observations that the work had an impact on participating teachers, in a very significant and dramatic way. In fact, the analyses of 60- minute interviews held with each participant in their schools at the end of their academic year confirmed the findings reported above. At the same time, there are some programs aspects that need to be considered in future work: (1) there is a need to more carefully coordinate assignments given in the summer seminars - they need to be more evenly spread over the four-week institutes; and, it would be useful to make week 1, in comparison to weeks 2-4, a little less stressful in terms of work required of teachers; (2) in demonstration teaching, there is a need to provide more opportunity for participants following their teaching, to "debrief" with peers who observed the lesson; (3) teachers reported that the four-week institutes pressed them "to the limit," a fact about which we need to be mindful in the future; (4) for one of the two geometry courses, some participants reported that too much material was attempted by the instructor - less material with more depth was preferred; (5) in one of the two computer courses, some participants indicated that a closer connection to their classroom was needed.

FINAL NOTE: The formal involvement of teachers in the project concluded when they were joined by 80 other K-12 teachers and 35 administrators from throughout the Southern Illinois region for a two-day *Working Conference to Plan Future Work to Improve Mathematics Education in the Southern Illinois Region.* The Conference provided a clear indication of Needs of K-12 teachers, Goals for mathematics education, and a Plan of Action for the near future. The *Proceedings* of the conference has been widely disseminated throughout the Southern Illinois region. The Conference and *Proceedings* have contributed to formation of the Southern Illinois Mathematics, Science and Technology Network which will be instrumental in carrying our future activities throughout the region.

Closing Comments

We feel that we have made some progress in our projects by moving instruction: from teacher-centered to student-centered; from the teacher as an imparter of knowledge to the teacher as a facilitator of learning; from *excessively didactic* instruction to teaching which is much less so; and moving instruction to a meaningful context (in the last project especially) to which teachers can relate. We feel that teachers were at least beginning to understand the nature of what we are trying to accomplish, judging from their reactions in the projects and their reports of what subsequently happens in their classrooms. We feel that we have meaningfully been able to integrate problem solving in the classrooms of the teachers who participated in our projects and overall, we feel we "are on the right track."

The "glue-to-the-classroom" is essential; by this we mean that teachers need to *experience* learning the new teaching approaches themselves and they need to have curricular materials which they have studied and learned themselves in their hands, along with the pedagogy, in order to implement them in their classrooms and achieve what we hope will be change with a lasting effect. Modeling the teaching in the "open-approach," MGMP and UNP seminars was crucially important, we feel. Further, having teachers observe project staff (which included master classroom teachers) teach the materials to students helped to remove some of their skepticism; and, having teachers teach students using lesson plans they (themselves) developed also provided some of the final "glue." Choosing teachers in pairs from schools/school districts provided needed mutual support as they changed their teaching approaches. Also, teachers took great pride and had a "sense of ownership" in using materials they developed or learned to master in the project for use in implementing project objectives. Overall, we believe teachers were able to see *utility* in the project activities for their own classrooms. Thus, there is little room, generalizing from our experience, for lecturing. Being *excessively didactic*, as Fletcher mentions, is counterproductive - certainly this is one of the lessons learned in our work up to this time. Acquainting ourselves as project leaders with the reality of teachers' classrooms and schools, and then doing our work cognizant of that reality, helps to provide the "glue" in a very important way. The network of teachers who now professionally interact with each other on a regular basis and who continue to collaborate in improving their teaching is helping to further develop a base or *critical mass* that has potential for furthering our project objectives.

The crucial components in the "model" we have now developed, after a decade of work, seem clearly to be: (1) including instructional components of important mathematical content (e.g., the MGMP and UNP materials) and problem solving; (2) using the "open-approach" teaching methodology; (3) modeling desired teaching behavior when teaching teachers; (4) demonstrating that the new content and teaching approaches work with real students; (5) having teachers teach demonstration lessons; and (6) implementing an approach to assessment that is closely linked with and which reflects the goals of both the teaching approach and instructional components. To be sure, we need to further develop the "model" and improve it and that is on our agenda of "next steps".

Our plan is to next apply this model in a school district *with all teachers who teach mathematics in grade K-8*, and to further develop it in that context. Then we expect to apply the "enhanced model" in contiguously located school districts while further improving it. The area in which we hope to do this during 1994-97 is the fourteen school districts in the Metro-East area of Illinois across the Mississippi River from St. Louis.

Now let us return to Schoenfeld's (1987) article. The article provides an interesting and authoritative historical analyses of problem solving and Polya's works. He mentions that "there are both scientific and social components to issues in mathematics education" (p. 45). Further, at the scientific level some good work has been done to begin laying a foundation for a science of education. (p. 45) But so much remains to be done that there is not, at this time, anything substantial enough to show us, really, how we should be teaching problem solving from a scientific point of view.

On the social level, however, the issues are different and this is a crucial time for problem solving (Schoenfeld, 1987, p. 45). If we cannot begin to show some positive results from curricular work and change in the direction of mathematical thinking, the pendulum may swing again away from problem solving; but it doesn't have to be this way. If teachers, mathematics educators, cognitive scientists and mathematicians can work together to create curricula that are mathematically sound, psychologically reasonable and workable in the classroom, and which deal with both basic skills and development of mathematical thinking abilities, then we can avoid a pendulum swing back to basics again. (p. 46) While we cannot claim to have accomplished anything on such a grand scale in our work so far, at least we feel we have "made a dent" in working at the problem, perhaps with some potential for better and more significant change in the future.

Endnote

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MATHEMATICS EDUCATION IN JAPAN

- Some of the Findings from the Results of the IEA Study -

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National Institute for Educational Research

1. Introduction

In 1980-81, the Second International Mathematics Study (SIMS) was conducted in 20 countries including Japan. This study was carried out by the International Association for the Evaluation of Educational Achievement (IEA) in cooperation with each of the participating countries/systems. These national reports have already been published by National Institute for educational Research of Japan (NIER) in 1981(1), 1982(2), 1983(3) and 1991(4) in Japanese. The International reports on SIMS were issued in 1988(5) and 1989(6) .

The first major IEA study was concerned with mathematics achievement. The data from this survey were collected in 1964 in 12 countries, including Japan, and the main report was published in 1967(7). This study will be called the First International Mathematics Study (FIMS). It made public for the first time, throughout the world, the high level of achievement in mathematics of Japanese students.

The purpose of this paper is to discuss the level of mathematics achievement and the tendencies of Japanese students and mathematics education in Japan, based on the results of SIMS.

2. Curriculum and Mathematics Achievement

(1) Curriculum Analysis

The Second International Mathematics Study was carried out for 13-years-old students (Population A) and science-mathematics students directly before entrance into college or university study (Population B).

In Japan, we regarded students in the first grade of lower secondary school (7th grade) as Population A and students in the third grade of upper secondary school (12th grade) taking more than five mathematics lessons per week as Population B, and in each case, we studied about 8,000 students in 200 schools in 1980-81.

In this study we intended not to simply compare achievement among participating countries/systems, but to study each nation's mathematics curriculum and teaching methods and then study the relation between them and students' attainments.

When we compare the results of Japan with those of other countries, we must consider differences of curriculum between them. One of the indexes we can use to see the learning condition is "Opportunity to Learn." When students have many learning opportunities in mathematics, they will be able to get better achievement.

Table 1 Opportunity to Learn and Mathematics Achievement for Pop. A

(Lower Secondary School)		Japan		International	
Domain	(No. items)	OTL	p-value	OTL	p-value
Arithmetic	(46)	85%	60% (1)a	80%	51%
Algebra	(30)	83	60 (1)	73	43
Geometry	(39)	51	58 (1)	52	41
Statistics	(18)	76	71 (1)	57	55
Measurement	(24)	95	69 (1)	80	51
Total	(157)	77	62 (1)	69	47

Note: OTL: "Opportunity to Learn" rated by teachers
p-value: Average percentage of students correct responses by each area
a: Ranking among the 20 countries/systems

Tables 1 and 2 indicate percentages of teacher opportunity to learn and student achievement for lower and upper secondary students. Here "Opportunity to Learn" (OTL) means ratings by teachers of whether the content needed to respond to each item on the mathematics test had been taught that year, in prior years, or not at all, to their students.

According to the international means in Table 1, these are high means of opportunity to learn in arithmetic, algebra and measurement, but not in geometry, probability and statistics. And on algebra, the more opportunity the country gives to students, the better the achievement of the students, but in geometry no relation is seen between opportunity to learn and achievement, as reported. It seems that each country/system is confused about the content and teaching method of geometry. In Japan, except for geometry, there is high opportunity to learn in all the domains, and in each domain the Japanese students got the best scores in 20 countries/systems.

Table 2 Opportunity to Learn and Mathematics Achievement for Pop. B

(Upper Secondary School)		Japan		International	
Domain	(No. items)	OTL	p-value	OTL	p-value
Set, Function	(7)	94%	78% (2)a	70%	62%
Number System	(17)	80	68 (2)	76	50
Algebra	(26)	100	78 (2)	87	57
Geometry	(26)	90	60 (2)	68	42
Analysis	(46)	92	66 (2)	76	44
Probability	(7)	83	70 (2)	59	50
Total	(129)	91	68 (2)	75	48

Note: OTL: "Opportunity to Learn" rated by teachers
p-value: Average percentage of students correct responses by each area
a: Ranking among the 15 countries/systems

In the upper secondary school, there also is not so much opportunity to learn geometry, probability and statistics among the participating countries/systems. About the Japanese mathematics curriculum, it can be said that it is a relatively advanced and extensive curriculum compared to others, because it's percent of opportunity to learn is over 80% in all domains.

These are comparisons based on the "Implemented Curriculum" (i.e., the actual learning activity in each school). Against it we can take another way to compare using the "Intended Curriculum" in the Course of Study and textbooks, etc.

The IEA study applied the rates of opportunity to learn to the former, and the rates of appropriateness of items for the latter, which was determined in advance by what percentage of specialists in each country/system judged that each item was effective according to its curriculum. In a comparison on the "Intended Curriculum," we find that most parts of the mathematics curriculum of Japan is introduced one half or one year earlier in the elementary and lower secondary schools than in other developed countries like the U.S., England and France.

In most countries each curriculum will be reviewed in the next school year, but in Japan few schools introduce the spiral method, and in general, their curriculum is to teach intensively and efficiently within a school year. So they can introduce various parts of the curriculum earlier than other countries.

(2) Students' Achievement in Mathematics

We can consider each nation's educational level by considering the "Attained Curriculum" that is indicated by students' achievement and changes in their attitudes toward mathematics.

The international report showed the achievement of each country by the content domains because the situations of opportunity to learn are different from each other, but I tried to arrange them by making two bar graphs (Figure 1 & Figure 2), ignoring the degree of opportunity to learn.

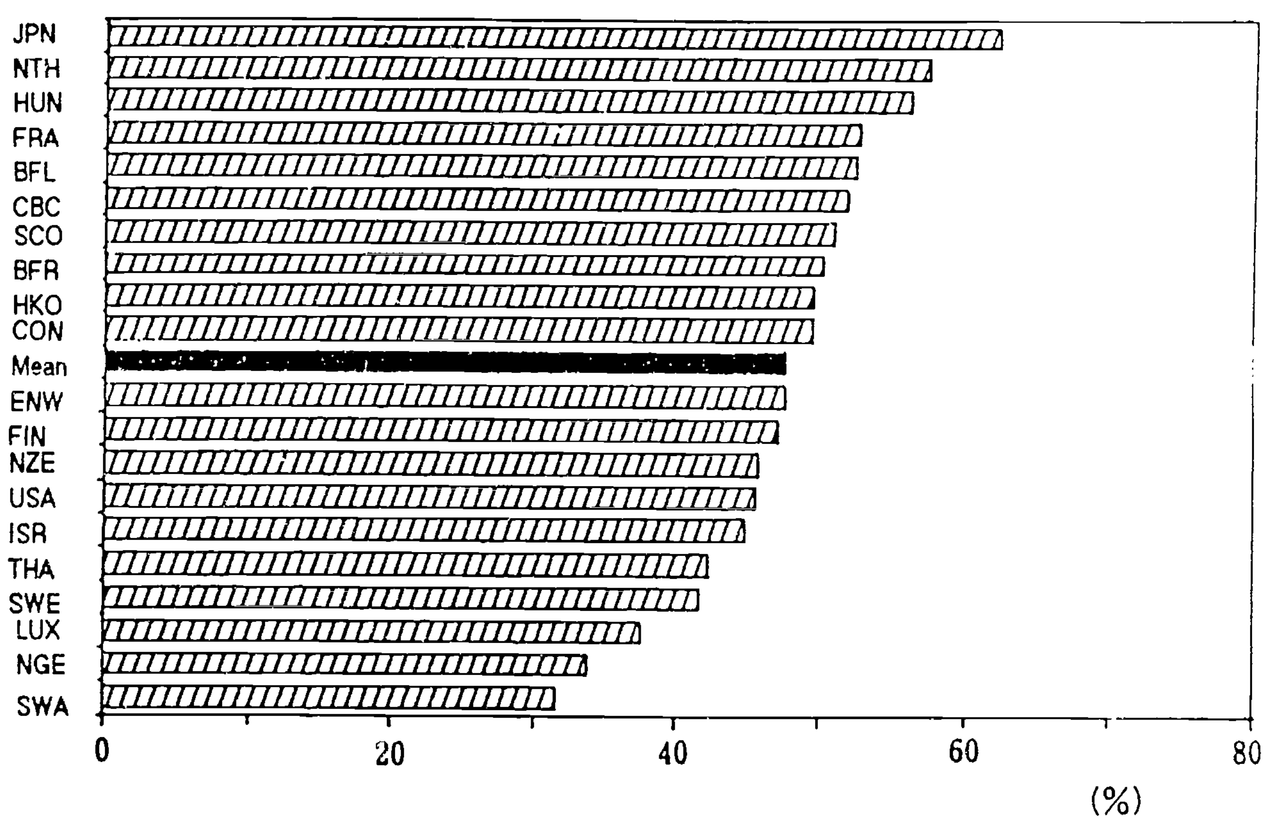


Figure 1 Achievement in Mathematics --- Pop. A ---

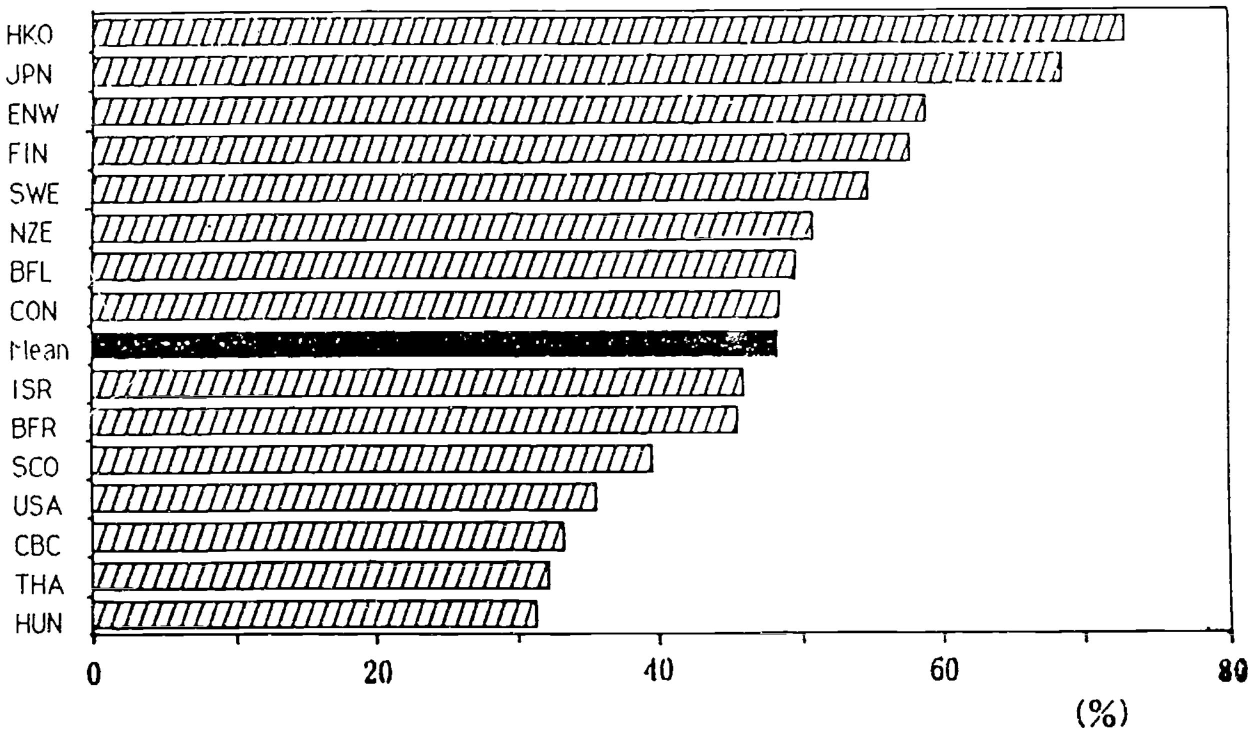


Figure 2 Achievement in Mathematics --- Pop. B ---

The results of mathematics achievement for 13-year olds (lower secondary school students) is shown in Figure 1. The result for Japanese students is superior to the others. And Figure 2 shows that Japanese students scored in second place, following Hong-Kong.

For the upper secondary school students, for Population B, it is a mistake to simply compare their scores. The reason is that there is a problem of the differences of curriculum and opportunity to learn, and moreover, we should be careful about the percentage of staying (i.e., the retentivity in Population B mathematics) which shows what percentage of all the same age cohort are in this population.

In Japan about 12% belong to this group. The percentage of retentivity is a high 50% in Hungary and 30% in Canada (British Columbia), and a low 6% in England, Hong-Kong and

Israel. Excepting these countries, retentivity is in the teens percentage. It can be said that the relative inferiority of Hungary's achievement, of upper secondary students to lower secondary students, is caused by this percentage of retentivity.

Next, I would like to show a comparison of each country's achievement by some common problems.

Example 1. 2/5 + 3/8 =

A. 5/13 B. 5/40 C. 6/40 D. 16/15 *E. 31/40

Example 2. A runner ran 3,000 meters in exactly 8 minutes. What was his average speed in meters per second?

A. 3.75 *B. 6.25 C. 16.0 D. 37.5 E. 362.5

Example 3.

1st row	1
2nd row	1 - 1
3rd row	1 - 1 + 1
4th row	1 - 1 + 1 - 1
5th row	1 - 1 + 1 - 1 + 1

What is the sum of the 50th row?

*A. 0 B. 1 C. 2 D. 25 E. 30

Example 4. For the equation $x^2 - 5x + 6 = 0$

A. There is no solution

B. There is exactly one solution

*C. There are exactly two solutions

D. There are exactly three solutions

E. There are more than three solutions

Example 5. P is a polynomial in x of degree m, and Q is a polynomial in x of degree n, with $n < m$. The degree of the polynomial $(P + Q)(P - Q)$ is

*A. $2m$ B. m^2 C. mn D. n^2 E. $m^2 - n^2$

Example 6. A radio-active element decomposes according to the formula

$$y = y_0 e^{-kt}$$

where y is the mass of the element remaining after t days and y_0 is the value of y for $t = 0$. Find the value of the constant k for an element whose half-life (i.e., time to decompose half of the material) is 4 days.

*A. $(1/4) \log_e 2$ B. $\log_e(1/2)$ C. $\log_2 e$ D. $(\log_2 e)1/4$ E. $2e^4$

Examples 1, 2 and 3 are for lower secondary school students and examples 4, 5 and 6 are for upper secondary school students. Table 3 shows the results of students' achievement in some countries.

Example 1 is a problem of fractional computation. This item had very high Opportunity to Learn ratings in most countries. However, the mean percent correct was only 58, with a range from 89 percent correct in Japan to 25 percent correct in Sweden. For this kind of problem, computation, Japanese students got very good scores, but for a word problem that requires the power of thought like example 2, the scores were not as good as expected.

Table 3 Achievement in the countries/systems by items

Country	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex .6
Japan	89%(1)a	37%(5)	41%(6)	97%(2)	34%(3)	56%(4)
Belgium(FL)	69	34	45	89	27	28
Belgium(FR)	68	34	40	93	19	23
Canada(BC)	68	25	37	88	14	24
Canada(ON)	62	23	39	87	13	30
England	42	29	39	94	17	64
Finland	41	28	34	98	39	54
France	72	34	47	-	-	-
Hong Kong	69	43	45	94	52	70
Hungary	66	46	37	84	7	13

Israel	65	29	48	95	8	11
Luxembourg	63	30	32	-	-	-
Netherlands	64	42	49	-	-	-
New Zealand	38	22	40	87	16	47
Nigeria	63	24	20	-	-	-
Scotland	55	29	37	81	4	16
Swaziland	44	17	13	-	-	-
Sweden	25	29	30	90	17	73
Thailand	45	35	29	78	10	32
USA	57	20	34	74	8	28

Note: - : did not participated in Pop. B
a : ranking among the 20 or 15 countries/systems

Example 3 received very low Opportunity to Learn ratings in most countries. On examples 4 and 6, each country got a high percentage of right answers, but for an applied question like example 5, Hong-Kong and Finland got better scores than Japanese students.

(3) School's Achievement in Mathematics

We can consider each school/class's educational level by comparison of the "Attained Curriculum" indicated by each school/class mean of students' achievement. This study was carried out in both populations with samples of approximately 200 schools/classes of 8,000 students each. The school sample was stratified by kind of school, size of school and population of the town, etc. for the sampled school. And then, one class was selected at random from within the selected schools.

Table 4 shows the distributions of school/class means for the above problems.

Table 4 Distributions of School/Class Mean

Item	p	sd	0-	10-	20-	30-	40-	50-	60-	70-	80-	90-	100
Ex. 1	89.	7.2	0	0	0	0	0	1	2	17	87	97	8
Ex. 2	38.	17.4	11	18	31	49	46	28	16	8	1	0	1
Ex. 3	41.	17.9	5	10	32	55	33	40	19	5	7	3	0
Ex .4	97.	6.1	0	0	0	0	0	1	0	2	15	23	149
Ex. 5	36.	22.9	28	20	29	26	28	26	11	16	6	0	0
Ex. 6	56.	25.8	6	10	18	17	21	27	21	26	19	14	11

Example 1 is a problem of fractional computation. This item had been previously taught in school. The school mean correct percentage was 89%, with a small range from 50% to 100%. For that kind of problem, computation, like example 4, the Japanese students got very good scores; but for a word problem that requires mathematical thinking like examples 2, 3, 5 and 6, and which had already been taught the previous school year, the scores were not as good as expected. School/classes distributions had very wide ranges. I think that teachers' methods and teachers' beliefs have affected students' achievement.

3. Change in Mathematics Achievement

(1) Comparisons between FIMS and SIMS

In the 20 years after the 1960s, in the decade that the First International Mathematics Study (FIMS) had been performed, great reform of mathematics education could be seen in many countries. In the first half of it, a modernization movement of mathematics education (the New Math Movement) was developed in each country and its revision was done in the second half. It was very interesting for me to study its results.

In the international report in 1989(6), the problems caused by a rapid rise of percentage of students entering upper secondary schools, the changes of mathematics curriculum in each country and the comparisons of results in each content area are stated in detail.

Table 5 shows change in achievement among the countries on FIMS/SIMS.

Table 5 Change in Achievement on FIMS/SIMS

	Population A			Population B		
Country	FIMS	SIMS	d	FIMS	SIMS	d
Japan	64%	64%	0%	64%	76%	12%
Belgium	56	50	-6	54	50	-4
England	52	44	-8	70	65	-5
Finland	43	45	2	52	63	11
France	50	55	5	-	-	-
Israel	56	46	-10	55	48	-7
Netherlands	48	54	6	-	-	-
Scotland	50	46	-4	55	42	-13
Sweden	37	37	0	54	58	4
USA	47	45	-2	32	38	6
Mean	50	49	-1	54	55	1

Note: - : did not participate in Pop. B
d : SIMS – FIMS

The results show that lower secondary school students began to get better results in the algebra domain, but worse in arithmetic, and though more students entered upper secondary school than that of the last study, the number of Population B students who took mathematics was not so large; and the results of fundamental understanding and technique were improved, but those of application were not good.

Here I want to consider the alteration of results of the 35 anchor items for lower secondary students and the 18 anchor items for upper secondary students for the countries who participated in both FIMS and SIMS.

Comparing the international average (mean) of FIMS and SIMS, there is no difference between them, but there is some difference in the results in each country. The countries whose results become better are France and the Netherlands for lower secondary students, and for upper secondary students, Finland, Sweden, USA and Japan. And the countries whose results became worse are Belgium, England, Israel and Scotland for both populations.

(2) Growth of Score between FIMS and SIMS for Pop. A in Japan

In Japan, the achievement of upper secondary students in SIMS was improved in all areas compared to FIMS. However, lower secondary students' achievement was almost the same.

Table 6 shows a comparison of the last study and this study of Japanese students with respect to the anchor items divided into non-verbal and verbal problems.

Table 6 Achievement in FIMS / SIMS : Pop. A

Area	(No. items)	FIMS	SIMS	Change
Overall	35	64%	64%	0%
Non-verbal	20	60	63	3
Verbal	15	69	65	-4

Note: No. : number of anchor items

According to this table, the total score is nearly the same as the last study but there are some differences by items; that is, while the results of simple calculation (non-verbal) improved by 3% from the first study, the results for verbal items, which requires reading ability and judgment, decreased by 4%.

Here we would like to compare the results of the first and second studies for Japan, for example 1 (Non-verbal problem) and example 2 (Verbal problem).

Ex. 1 FIMS: 84% SIMS: 89% change = 5%

Ex. 2 FIMS: 51% SIMS: 38% change = -13%

While the results for the computation item (Non-verbal item), like example 1, improved remarkably, the results for the verbal item, like example 2, declined generally. Though Japanese students' achievement was highly evaluated internationally, it can be said that there will be a big problem for education in the future, because students' results owe to only the ability of computational skill, but the results for questions that require thinking and application power declined.

Why did this occur? After the first study, there was a world-wide reform of mathematics education. That is to say, the 1960s-70s were the decades of a movement of the so-called "Modernization of Mathematics Education" (New Math Movement), and in this period some new concepts like "Set" were introduced in lower grades. And there was a problem in the USA and so on, of the decline of computational ability because much emphasis was laid on teaching these concepts; and, in fact, that was proved in the IEA's study. But the achievement of Japan was the reverse of it. It can be thought that the teaching and learning of mathematical thinking was not fixed, while teachers crammed students with knowledge and techniques due to the hard entrance examinations.

(3) Growth of Score Between Pre-test and Post-test for Pop. A

For students in Pop. A in Japan, a pre-test at the beginning and a post-test at the end of the school year were conducted, with 60 common items, in order to see whether achievement improved during the period. The change in percentage of correct answers is examined below for 32 of the 60 items which had already been taught in the previous year (at the elementary school level).

Table 7 shows the percentage of correct answers in the pre-test (X) and the post-test (Y), and the percentage of answers that were correct both in pre- and post-tests ($X \cap Y$).

Table 7 Change in Achievement between Pre-tests and Post-tests: Pop. A

Content	Pre-test (X)	Post-test (Y)	X∩Y
Arithmetic	54%	58%	37%
Algebra	59	63	41
Geometry	65	72	51
Statistics	59	63	43
Measurement	64	68	50
Overall	58	62	42

As an example, in the content of Arithmetic, the percentage of correct answers was 54% in the pre-test, whereas it was 58% in the post-test, an apparent increase of 4%. But only 37% of the students answered correctly in both the pre- and post-tests. In other words, Table 7 shows that 17% of the students who had answered correctly in the pre-test answered incorrectly in the post-test, while 21% who had had the incorrect answer in the pre-test had the correct answer in the post-test.

In this way, by judging from the real percentage of correct answers (the percentage of those who answered correctly in both the pre- and post-tests), we see that the percentage of those who gave a correct answer in the pre-test, but failed to do so in the post-test, is, respectively, 19% in Algebra, 14% in Geometry, 16% in Statistics, and 14% in Measurement. The corresponding figures for improved performance are 22% in Algebra, 21% in Geometry, 20 in Statistics, and 19% in Measurement.

Extrapolating from the results of these studies on the content which had already been taught to students, we may conclude that the value obtained by subtracting approximately 16% from the (apparent) percentage of correct answers should be regarded as the real percentage of correct answers. This value may also be regarded as an index of fixedness of an achievement to students.

Here examples of the items are given. The following are the problems for which the percentage of correct answers was low.

(a) "A runner ran 3,000 meters in exactly 8 minutes. What was his average speed in meters per second?" (This item was already cited earlier)

Pre-test (X) 33% Post-test (Y) 38%

Correct in both tests ($X \cap Y$) 17%

(b) "In a school of 800 pupils, 300 are boys. The rate of the number of boys to the number of girls is" (This item was already cited earlier).

Pre-test (X) 34% Post-test (Y) 31%

Correct in both tests ($X \cap Y$) 12%

(c) 1st row 1

2nd row 1 - 1

3rd row 1 - 1 + 1

4th row 1 - 1 + 1 - 1

5th row 1 - 1 + 1 - 1 + 1

\- - - - - - - - - - - - - - - - - -

What is the sum of the 50th row?"

Pre-test (X) 35% Post-test (Y) 42%

Correct in both tests ($X \cap Y$) 18%

(d) "30 is 75% of what number?"

Pre-test (X) 46% Post-test 47%

Correct in both tests ($X \cap Y$) 28%

These are verbal problems which require a complex operation of thinking, and are examples of items for which the performance was relatively low.

Next, examples of items for which the percentage of correct answers was high are given in the following.

(e) "2/5 + 3/8 = " (This item was already cited earlier.)

Pre-test (X) 84% Post-test (Y) 89%

Correct in both tests ($X \cap Y$) 78%

This is an easy problem involving calculation of fractional numbers, also with a high degree of performance.

(f)

"In the figure above, the length of *AB* is 1 unit. Which is the best estimate for the length of *PQ*?"

A. 2 B. 6 C. 10 D. 14 E. 18

Pre-test (X) 82% Post-test (Y) 87%

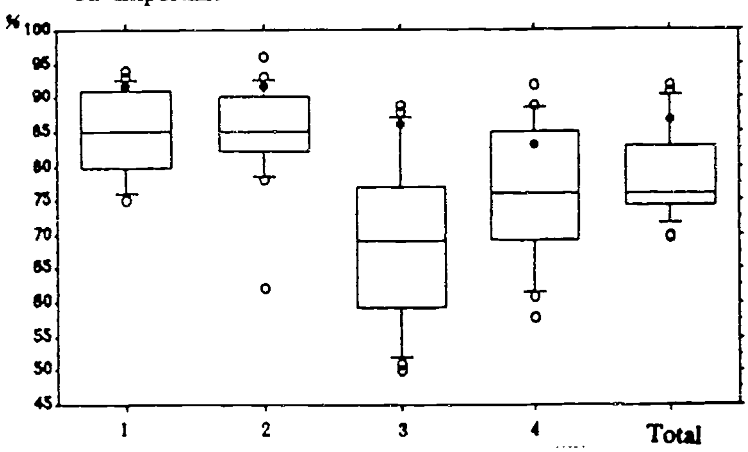
Correct in both tests ($X \cap Y$) 73%

These examples show that for problems which require a full understanding of their content for solution, that is, for such problems which belong to the verbal problem domain, the degree of fixedness is low.

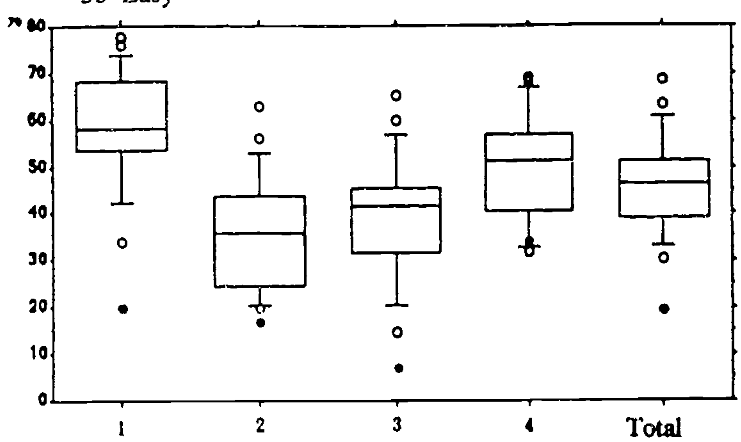
4. Attitudes toward Mathematics

In these studies, students were expected to give answers for mathematics tests and also for an inquiry questionnaire about their attitude toward mathematics. It included an inquiry questionnaire which consists of seven domains: "Mathematics as a Process," "Mathematics and Myself," "Gender Stereotyping," "Mathematics and Utility," "Calculators, Computers and Mathematics," "Home Support for Mathematics" and "Mathematics in School." Here I would like to consider students' responses on "Mathematics in School." This inquiry questionnaire requested each student to respond about various Mathematics in School items, whether it is important, or is difficult for him or her, and whether he or she likes it. They prepared 17 items for Population A and 20 for Population B as in the following examples: (1) checking an answer to a problem by going back over it, (2) memorizing rules and formulae, (3) solving word problems, and (4) solving equations. Figures 3a, 3b and 3c show the box-plot distributions for the above items. In the graphs, Japanese students are located in black points.

3a Important



3b Easy



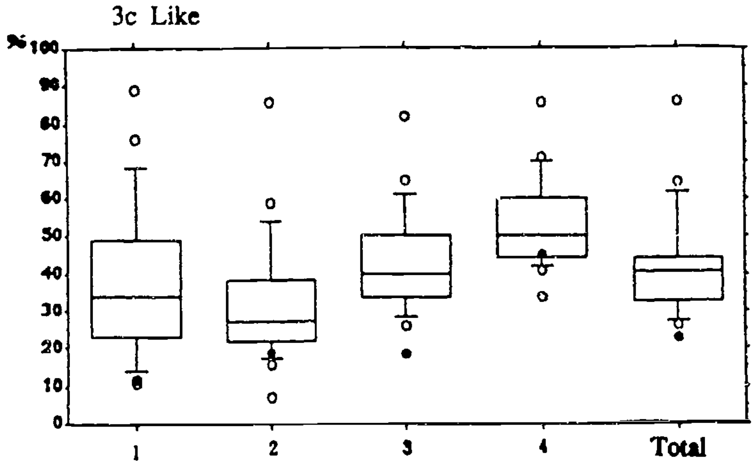


Figure 3 Mathematics in School

According to the results for the 20 countries/systems, many students had a favorable opinion of school learning in Nigeria, Sweden and Israel, and reversely, many students felt some difficulty in school mathematics and disliked it in Japan.

In the case of upper secondary students, the results were similar to those of lower secondary students in Japan. It showed that the index of the degree of importance of mathematics learning in school was average; but, on the contrary, there were many students who felt difficulty and dislike it. As a consequence of other inquiries about the results of the questionnaire, we can see that the proportion of negative responses is larger than other countries. And we need to be concerned about that.

5. Concluding Remarks

From the data and analyses, several generalizations can be made, particularly as they may bear on policy formulation for future goals. These are as follows:

(1) Although achievement in the fundamental techniques of calculation can be viewed, in general, as satisfactory, the attainment levels cannot be regarded as acceptable for problems which require a high degree of thinking and comprehension.

(2) Specifically, the achievement declined compared with the previous study for such problems as "ratio" and "proportion," which are taught in the upper grades of elementary school, and for verbal problems.

(3) It might be hypothesized that the introduction of the New Mathematics in the mid-1960's lowered the calculation ability of students in every country. In the case of our country, however, the contrary seems to be true: calculation ability improved in general, though the achievement levels in problems requiring a high degree of thinking and comprehension (which is stressed in the New Mathematics) was not as high as in the previous year.

(4) For Pop. A (students in the lower secondary schools), the change in mathematics scores over the one-year period was measured by pre-tests and post-tests. The data show that the degree of fixedness was low for verbal problems and for problems concerning ratio, proportion and percentage. From this, it is concluded that more attention must be paid to these matters and to the methods of teaching in the elementary school.

(5) As for the consequences of international comparisons about interest and attitude, it became evident that a lower proportion of students have a favorable opinion than in other countries, and that there is a problem in the learning method.

(6) As was recently found in the case of entrance examinations in our country, the improvement in calculation ability in this study may be regarded as due partly to "cramming" at some place other than the public school (like Juku--an informal outside school). However, in order to assure higher scholastic achievement, it is necessary to develop new methods of teaching this subject in the public school setting under conditions which are not so pressure intense.

In revising the Courses of Study, the Ministry of Education of Japan took account of anticipated changes in our society and the resulting changes in the life and attitudes of students. It intended to provide students with a sound foundation for lifelong learning. The basic aim of the revision of the Courses of Study is to ensure, keeping the 21st century in view, the development of student with rich hearts who will be capable of coping with the changes in our society such as internationalization in different sectors and the spread of international media.

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SOME REMARKS ON PROBLEM SOLVING

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Since the appearance of the American NCTM document *An Agenda for Action* in 1980, the idea that "Problem solving must be the focus of school mathematics" has been widely spread and heralded. This note will give some remarks about the problem of how to carry out the idea mentioned above effectively.

I. Lessons From China

In China we have had a long term tradition to require the students to do a great number of exercises in learning mathematics. The Chinese high schools commonly use one volume of mathematics text for each semester, and each volume contains about 300 exercises which are assigned to the students as homework. It means that the students have about 15 exercises per week as homework from the text. But in reality, the number of exercises which the students are required (by their teachers or parents) to do or voluntarily do is three to four times, and sometimes going up to seven times or more, the number contained in the texts.

This tradition has a deep rooted cultural background in our country. In fact, the ancient Chinese considered mathematics as some kind of techniques as revealed by the names of our ancient mathematics books, such as the famous *Nine Chapters of Arithmetics*, the original name of which (<九章算术>) really means "Nine Chapters of Calculation Techniques." And to a Chinese mind, the best way to grasp some kind of techniques is to follow the old proverb

"Familiarity engenders skill."

This proverb has such a strong influence that it leads some mathematicians to claim that in order to be good in mathematics, one has to do ten thousand exercises. This is practically the prevailing philosophy for learning mathematics in China. As the exercises in our school texts are not only routine ones, their doing should be considered as the activity (or part of the activity) of problem solving, and in this sense one can say that problem solving has gained emphasis in mathematics education in China.

And what is the result? As an advantage, it can be seen that our students gain some good training in doing mathematics, as shown in all kinds of competitions, domestic and international, including the International Mathematics Olympiad. But its price is rather high. At least two kinds of damage are caused by it. The first is that the students are overburdened and consequently can be hurt physically and mentally. The second is that it produces haters, but not lovers, of mathematics.

Many lessons can be learned from our case. For example,

* Our activity in problem solving must be greatly improved. As doing mathematics is not just doing textbook exercises, we must connect our school mathematics education more closely to the world outside of school, so as to help the students develop their ability to cope with real world problems, communication and appreciation of the predilection of mathematicians for quantities, shapes and patterns.

* The most important point is that we must recognize the fact that different students can have different temperament, different tendency, and different interest; therefore we must be content with moderate requirements in mathematics for most of the students while we certainly can and should help the mathematically able students to learn more.

II. Problems Versus Exercises

Some people make a distinction between problems and exercises by stating some characteristics of the problems, such as the following ones:

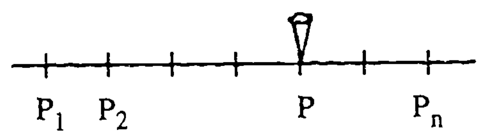
1. Problems are non-routine;
2. Problems have a real world situational background (or briefly, they are applicable).
3. Problems are such that for their solution, some research is needed. Especially, for example, they are over determined, under determined, open-ended, etc. (I like to call them non-ordinary).

This distinction has the advantage to emphasize letting the students experience some non-routine problems too. But in doing so, we must not go too far to consider non-routine as superior to routine, or applicable as superior to pure, or non-ordinary as superior to ordinary, because all kinds of problems have their merits in the instruction. Our problem is only that of when and how to use which. Moreover, in carrying out problem solving in instruction, our task should focus on the methodology side of the problem and not on its type. For example, the problem of finding the sum

$$1 + 2 + 3 + \ldots + 10$$

is just very routine, and it was Gauss' method which makes it fascinating. And the problem of finding the position of point P on a line, on which n robots are located, such that the sum of the distances from P to each robot P_i

$$\sum_{i=1}^{n} |PP_i|$$



be a minimum, is attracting, not because of its appearance of non-routineness, but because it is a simple and good example showing the effectiveness of induction for solving problems.

A point which I wish to argue especially here is that we must be careful in using the non-ordinary problems because, though they can be good for the mathematically able students, they may be bad for the average students. Let us consider, for example, the now becoming classical "Marble Problem" which can be formulated in the following way:

"Give a definition of denseness to a set, consisting of 5 points on a plane"

or, in a more intuitive way, as follows:

"Given two sets of (A) (B), each of which consists of five points lying on a plane as shown in Fig. 1. How to determine which set is denser than the other set?

Fig. 1

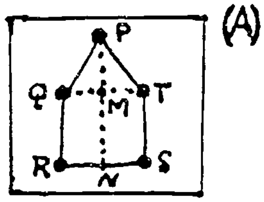
In certain publications, some answers to this problem are given (among others) as follows:

1. It (the denseness) is measured by the area of the pentagon with the given points as vertices.
2. It is measured by the length of the periphery of the above mentioned pentagon.
3. It is measured by the size of the circle which contains the given points.

But all these answers are problematic. For the answer 1, Fig. 2 shows that by definition, (A) is denser, while by intuition (B) is denser, where (A): The five points lie equidistantly on the whole diagonal of the square, and (B): The five points lie closely with four of them lying on the same diagonal while the other one lies outside the diagonal.

Fig. 2

For the answer 2, Fig. 3 shows that by definition (A) is denser while by intuition (B) is denser, where (A): Given points are P,Q,R,S,T with Q,R,S,T being the vertices of a square, while P lies on the line MN which bisects QT and RS, and $PM < MN$, and (B): The points Q, R, S, T are the same as before while P' lies on MN with $MP' > PM$.



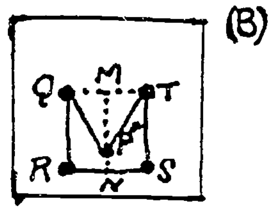
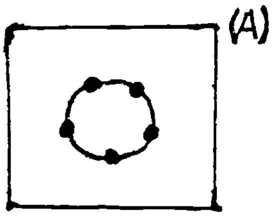


Fig. 3

For the answer 3, Fig. 4 shows again that by definition, (A) is denser while by intuition (B) is denser, where (A): The given points lie equidistantly on a circle with radius r, and (B): The given points lie closely together on a circle with radius $R > r$.



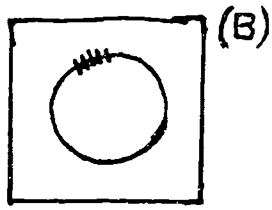


Fig. 4

If these complexities would be good for the able students in offering them an opportunity to do some research, they can probably cause psychological hardship to the average students.

III. About the "What" and "How"

If we conceive of "problem solving" as an approach of teaching and learning mathematics, then besides it we have to solve the problem of "what to teach and to learn in the schools." A story of mine perhaps can help to clarify the question. In a seminar at a university, someone raised the following problem

"Given n sticks of matches, find the number $f(n)$ of triangles which can be constructed with these sticks."

For example we have

n	3	4	5	6	7	8	9	...
$f(n)$	1	0	1	1	2	1	3	...

I spent a whole day solving this problem and found its solution in the following form

$$f(2n+1) = \sum_{k=0}^{\infty} [(n+1-3k)/2]$$

$$f(2n) = \sum_{k=0}^{\infty} [(n-1)-3k)/2]$$

for every positive integer n, where the expression $[x]$ means the positive integral part of the number x or zero, such as $[2.5] = 2$, $[1/2] = 0$, $[-2] = 0$.

My work gained some praise at the seminar, but a friend of mine commented afterwards that "you are wasting your time, you should spare it for better mathematics." My friend's comment reminded me of Dieudonne's address to the OEEC seminar at Royaumont in 1959 where he claimed that

"Euclid must go!"

Though I don't think that Euclid should go exclusively, I do appreciate very much Dieudonne's idea to let the students learn modern mathematics, i.e. the modern mathematical concepts, modern mathematical language and modern mathematical methodology, and not to waste time over such materials which, in Dieudonne's words, "has just as much relevance to what mathematicians (pure and applied) are doing today as magic squares or chess problems." Every time I reread Dieudonne's address, I do think that while we emphasize the importance of a problem solving approach, we must pay more attention to the problem of "what to teach and what to learn."

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HOW TO LINK AFFECTIVE AND COGNITIVE ASPECTS IN THE MATHEMATICS CLASS -- Comparison of Three Teaching Trials on Problem Solving

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1. Background

For about a hundred years, computational skills in mathematics have been highly emphasized in the elementary and secondary school mathematics curriculum in Japan. Computational skills have been emphasized in classroom lessons, and solutions using paper and pencil, the abacus and mental computation have been used to solve problems in the classroom activities. Japanese students have studied some skills in computational problems. It is well known that the Japanese students got higher scores on the international achievement tests during the first and second IEA studies of mathematics, and recently, the International Mathematical Olympiad. However, there are still many Japanese students in junior and senior high school who do not like mathematics and have lots of anxiety about it.

We would like to assert some tentative conclusions on the effectual link between affective and cognitive aspects in the use of computational tools in problem solving, in particular, the pocket calculator. Pocket calculators seem to be a more efficient way of bringing student's thinking into mathematical problem solving than paper and pencil. This report on the comparison of the three teaching trials on problem solving has some effectual outcomes in connection with the affective and cognitive aspects.

We conclude in asserting the value of the use of the pocket calculator in the process of understanding, solving and retention of the effect. This research suggests that higher-order thinking of mathematical problem solving is developed especially by the use of the pocket calculator. It is easier for students to operate on a complex number given using the pocket calculator since it is transformed into a more simple one. The value in the use of the calculator process on understanding, solving and extending the problem lies in its usefulness for the students to get the structure of the problem more easily. Also, it is useful for the students to forecast solutions and to make similar and general problems.

2 . Introduction

In 1989, the latest version of curriculum recommendations was published by the Japanese Ministry of Education (1989). Estimation is emphasized more in this version than in the former one. This curriculum was designed to help students have a good sense of numbers, and with a goal that students should become good "estimators." In this new curriculum, mental computation, paper and pencil computation, calculator use and estimation are given more attention. Opportunities are also provided for Japanese students to examine appropriate situations for estimation and checking the results of exact computations by estimating. All of the recommendations above are implemented in the elementary school mathematics curricula that began in 1992.

The value of teaching estimation and problem solving are commonly acknowledged and are clearly stated right now, both in the forthcoming Japanese mathematics curriculum and in the NCTM *Curriculum and Evaluation Standards.* These theoretical products concerning the processes of problem solving and computational estimation have many similar aspects and characteristics to the promotion of higher mathematical thinking (Reys, 1990).

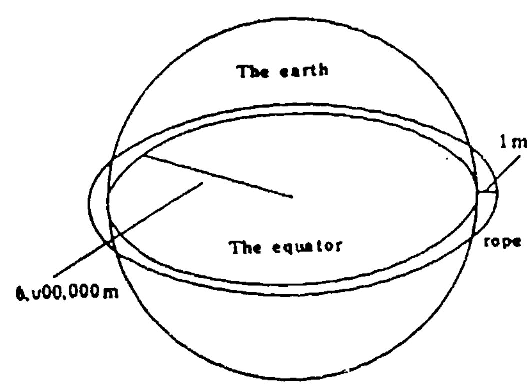
We have investigated the use of estimation, calculator and paper and pencil computation in situations of mathematical problem solving. There are some standpoints about estimation in its relation with other tools of computation. One point is that estimation is one of computation, the same as paper and pencil, mental and calculator computation. Another point is that estimation serves to compensate for calculator and paper and pencil computation. Furthermore, there is a standpoint that estimation itself involves problem solving performance. This is connected with the opinion that it is important to study the value of the use of estimation in mathematical problem solving. We have a supposition that estimation is useful to understand a problem and to make a plan, and useful to make a check on the way of solving the problem, and decision making and outcomes of executed plans. To examine these standpoints (i.e., the effect in the use of these alternatives and our supposition), this research was conducted.

3. Three teaching plans with computational alternatives in problem solving

Three teaching plans were done to study the effects of teaching trials by using estimation, calculator and paper and pencil computation as computational alternatives in a mathematical problem solving context. These lessons were conducted in a fifth-grade classroom of an elementary school in a small city near Tsukuba city on December 17, 1990. The problems and processes of the lessons were almost the same, except for the differences in the lessons as the result of computations gotten by estimation, by calculator and by paper and pencil. The lessons that have used estimation and a calculator were done by the same teacher and the other lesson was done by a different teacher.

Problem used in the lessons

> Imagine the earth as a big globe.
> Its radius is about 6378.136 km.
> If we tie a rope one meter above
> the equator of the earth, what is
> the difference between the original
> circumference and the length of the rope?



We chose the earth problem for the following reasons: We thought first that a true problem should satisfy the following three factors:

(1) Students should be able to understand the problem.

(2) Students should not be able to solve the problem by the use of a routine method.

(3) Students should have a desire to solve the problem.

Second, we thought that problem solving in a process and a problem solving process uses acquired mathematics knowledge and skills, which can be applied to solve the problem by an individual or in a group discussion. Furthermore, the problem used in these lessons was a thought experiment (Gedanken Experiment) that had some features as follows:

(4) The thought experiment had a process of idealization or abstraction.

(5) The thought experiment needs the conjecture of the results of problem solving before doing so.

(6) We could forecast some ways of solving the problem by trying the thought experiment.

We explained the three teaching plans to these teachers: The estimation lesson was to solve a problem by replacing any number with a simpler number, which was chosen by the students themselves. If estimation was taken a process to find an approximate answer to a problem, this lesson was an estimation teaching from a process to understand a problem, and a process to evaluate the decision made and also the outcome of the executed plan.

On the other side, we tried two lessons of solving the problem by using the given number which was not a simple number, about 6378.136 km. and used a calculator or paper and pencil to solve the problem. These two lessons were almost the same processes as the estimation lesson.

The specific characters of each lesson were as follows:

(a) Lesson emphasized that students use estimation.

1) To understand a problem by replacing a given situation with simpler numbers, that is an idealization of complicated situation into a clear situation by replacement.

2) To solve the problem with simpler numbers.

(b) Lesson emphasized that students use a calculator.

1) To get an answer easily by calculators.

(c) Lesson allowed students to compute by paper and pencil.

1) Let students compute numbers which come from real situations.

There were some common characteristics among these lessons such as a whole class teaching style and the lesson was mainly guided by the teacher, and sometimes the teacher asked each student some questions or opinions individually. In these lessons, the teachers asked students to forecast an answer and way of solution before they solved the problem, teachers tried to emphasize verification of a solution and teachers let students reflect on this process. Also, the teachers posed an additional problem.

4. Practical trials to teach the lessons in problem solving context

(1) Teacher explained the earth problem using a big terrestrial globe.

Students were asked to forecast the difference of the length, and their forecasts were as follows:

Table 1. Student's forecast (%)

	Calcu.(35)	Estima.(35)	Paper-P.(35)	Sum(105)
a) about 6000 km.	8.6	2.9	5.7	5.7
b) about 1000 km.	22.9	2.9	31.4	19.0
c) about 1000 m.	31.4	57.1	14.3	34.3
# d) about 6 m.	37.1	37.1	48.6	41.0
e) We can't guess	0	0	0	0

Students' forecasts in the paper and pencil class were excellent, but the other classes did not show such good performance of forecasting (see Table 1).

(2) After the students showed their forecast in each classroom, they were asked to solve the problem. This time, they recalled the formula

$C = 2 \times \pi \times R$ (C: Circumference, R: Radius, π: 3.14).

The estimation class got excellent scores on all items. The calculation class also had good scores on almost all items. But, the paper and pencil class did not understand the problem; thus, they did not get the expression, calculation and correct answer to the problem (see Table 2).

Table 2. Students expression, calculation and correct answer (%)

	Calcu.(35)	Estima.(35)	Paper-P.(35)	Sum(105)
f) expression	60.0	71.4	25.7	52.4
g) calculation	57.1	71.4	14.3	47.6
h) correct answer	68.6	71.4	11.4	60.0

In all classes, the students showed difficulties in dealing with this computation using the exact number. Therefore, the teacher in the estimation class allowed students to estimate and choose simple numbers to replace the original numbers. Students in this class chose numbers such as R = 6,000,000, 640,000, 1000, 10 meters and so on. Teachers in the other classes allowed the students to use other letters or special numbers. Students used a "box" or alphabetical letters.

The teachers are the ones who determine the correct answer, including both the expression and the calculation of problem solving in Japan. They emphasized the expression of equations of the problems. Students' expressions are shown in table 3.

Table 3. Characterization of student's expression (%)

Student's expression	Calcu.(35)	Estima.(35)	Paper-P.(35)	Sum(105)
i) synthesis ex.	51.4	34.3	11.4	32.4
j) analysis ex.	5.7	22.9	8.6	12.4
k) □, X expre.	2.9	0	2.9	1.9
l) special num.	0	14.3	2.9	5.7

Note: Synthesis expression: $(6{,}000{,}000 + 1) \times 2 \times 3.14 - 6{,}000{,}000 \times 2 \times 3.14$

Analysis expression: $6{,}000{,}001 \times 2 \times 3.14 = 37{,}680{,}006.28$

$6{,}000{,}000 \times 2 \times 3.14 = 37{,}680{,}000$

$37{,}680{,}006.28 - 37{,}680{,}000 = 6.28$ A. 6.28 m.

Letter expression: $(X + 1) \times 2 \times 3.14 - X \times 2 \times 3.14 = 6.28$

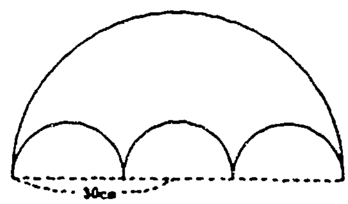
Special number/10 m.: $(10 + 1) \times 2 \times 3.14 - 10 \times 2 \times 3.14 = 6.28$

In students' answers of the problem solving used expressions, calculations and the correct answers, the estimation class was the most excellent and the calculation class was good. For example, the radius of the globe used by many were simple numbers like 6,000,000, 640,000, 1000 and 10 m. in the estimation class. Only one student in the paper and pencil class used the special number of the radius of the globe as 0 m., which astonished the teacher as well as the students in the class.

(3) At the end of these lessons, the teacher asked the students to solve the application problem of half circles as follows:

Application Problem

Compare the length of a half circle with three small half circles in the given half circle and check the correct item in the following three items:



m) Half circle is longer than three small half circles.

n) Half circle is shorter than three small half circles.

\# o) Half circle is the same as three small half circles.

Table 4. Students Answers to the Application Problem (%)

	Calcu.(35)	Estima.(35)	Paper-P.(35)	Sum(105)
o) correct answer	94.3	91.4	74.3	86.7

This problem was asked in order to check the post-effects of the teaching. Students in both the calculator and estimation classes understood and solved the application problem, while the paper and pencil class did not understand and solve it well, because a lot of students in this class did not explain the reason why "half circle is the same as three small half circles."

(4) As soon as the lesson was finished, the teacher distributed the Check List of Feeling about the lesson to students. The Check List of Feeling is to make a link between the affective and cognitive aspects in mathematics classroom activities - students' responses were as follows:

Check List of Students' Feeling After Lesson

Please read the following sentences, then mark the choice fitting your feeling. At the beginning of the lesson,

(1) Do you like mathematics lesson everyday?	Yes 3————2————1 No

After reading the task,

(2) Have you understood the task?	Yes 3————2————1 No
(3) Do you have an interesting the task?	Yes 3————2————1 No
(4) Have you solved the task?	Yes 3————2————1 No
(5) Have you forecast the answer before solving the task?	Yes 3————2————1 No

While you are solving the task,

(6) Please mark the item fitting your impression of the following:

(a) You can solve the task the same way as your forecast.

(b) You changed the way of task-solving while solving it.

(c) You failed to solve the task the same way as your forecast, thus having a hard time on it.

(d) others: please write your reasons, briefly.

At the end of the teaching,

(7) Do you have lots of ways of solving? Yes 3————2————1 No

(8) Are you satisfied with the lesson? Yes 3————2————1 No

(9) Can you understand the

explanations by the teacher? Yes 3————2————1 No

(10) What kinds of subject do you like in a lesson?

If you have some ideas or topics about the problem, please write them briefly.

Table 5. Numbers of student's choice to Yes (3) (%)

Student's Choice Yes	Calcu.(35)	Estima.(35)	Paper-P.(35)	Sum(105)
(1) Like Mathematics	28.6	14.3	37.1	26.7
(2) Understand Task	34.3	31.4	51.4	39.0
(3) Interesting Task	37.1	28.6	45.7	37.1
(4) Solve the Task	48.6	48.6	51.4	49.5
(5) Forecast the Task	45.7	40.0	31.4	39.0
(6) Solve Task as Thought	40.0	28.6	31.4	33.3
(7) Lots of Ways of Solving	45.7	25.7	54.3	41.9
(8) Satisfied with Lesson	54.3	40.0	31.4	41.9
(9) Understand Explanation	71.4	60.0	48.6	60.0

At the start of the lesson, lots of students in the paper and pencil class had a good feeling and a few students in the estimation class were satisfied with the task, but lots of students in the calculator class did not have any good feeling at all. During the process of problem solving, each class was in confusion. At the end of the lesson, the calculator class had excellent feelings towards the lesson, but the paper and pencil class was not satisfied with the lesson.

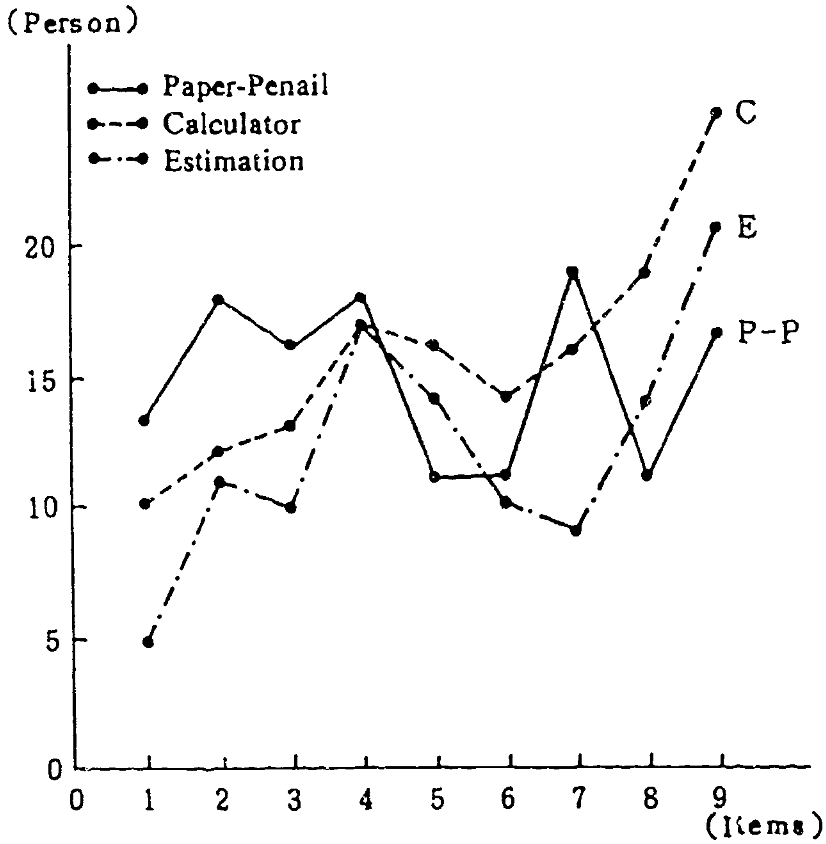
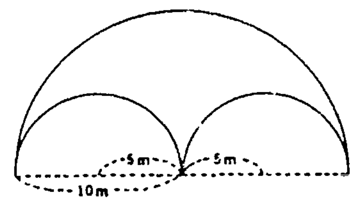


Figure 1. Numbers of student's choice to Yes (3)

(5) After we tried the practical lessons with computational alternatives in problem solving about one month later, the test dealing with the retention of the effect and students' ability to develop a similar problem was conducted.

Retention Problem

Compa. ͜ the length of a half circle with two small half circles in the given half circle. Check the correct item in the following three items:



p) Half circle is longer than two small half circles.

q) Half circle is shorter than two small half circles.

\# r) Half circle is the same as two small half circles.

Table 5. Students Answers to the Retention Problem (%)

	Calcu.(33)	Estima.(35)	Paper-P.(36)	Sum(104)
ı) correct answer	87.9	88.6	80.6	85.6

Extension Problem: Make as many similar problems to A as possible.

Table 6. Students Problem to the Extension Problem (%)

	Calcu.(33)	Estima.(35)	Paper-P.(36)	Sum(104)
s) all response	100	100	100	100
t) same problem	18.2	11.4	16.7	15.4
u) similar problem	30.3	11.4	11.1	17.3
v) general problem	24.2	28.6	13.9	22.1

Students in the calculator class were the most excellent in making similar and general problems, and the estimation class made many kinds of similar and general problems. Similar problem means the same kind of mathematical structure as the original problem, and general problem means different mathematical ideas from the original problem.

Many students dealt with calculator and estimation. In so doing, they were able to escape lots of computation, thus enabling them to make many similar and general problems from the original problem. The reason is that many students do not like long and difficult computations.

We could then assert some tentative conclusions on the effectual link between affective and cognitive aspects in the use of computational tools on problem solving, in particular, the pocket calculator. Pocket calculators seem to be a more efficient way in bringing student's thinking into mathematical problem solving than paper and pencil.

5. Conclusion

We conclude by sserting the value in the use of the pocket calculator in the process of understanding, solving and retention of the effect. This research suggests that higher-order thinking of mathematical problem solving is developed especially by the use of the pocket calculator. It is easier for students to operate on the complex number given by using the pocket calculator, since it is transformed into a simpler one. The value in the use of the calculator process on understanding, solving and extending the problem lies in its usefulness for the students to get the structure of the problem more easily. Also, it is useful for the students to forecast solutions and to make similar and general problems.

Furthermore, we conclude by asserting the value in the use of estimation in the process of understanding, solving and retention of the effect. As mentioned above, it is useful for students to replace the given number with a simpler number and to change to a simpler situation. The value in the use of estimation in the process of understanding, solving and extending the problem lies in its usefulness for the students to get the structure of the problem more easily. Also, it is useful for the students in forecasting solutions and making similar and general problems.

We want to close by using a quotation by Peter Griffin:

"Teaching takes place IN TIME and Learning takes place OVER TIME."

So, what are the relationships between teaching and learning? I am drawn back to the conjecture behind teaching takes place in time and learning takes place over time, that learning is a process of maturation of the learner. The teacher cannot perform this process for the learner. Rather, it is the atmosphere and environment created in the classroom by the actions of the teacher which can raise the awareness of the learner and shift attention in such a way as to stimulate this process of maturation in the learner.

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CRITICAL ISSUES IN PROBLEM SOLVING INSTRUCTION IN MATHEMATICS

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The importance of problem solving in the mathematics curriculum has been systematically acknowledged in recommendations for school mathematics in the last decade. Indeed, in America, the *Agenda for Action* report (NCTM, 1980) declared that problem solving should be the centerpiece of the curriculum and every reform-oriented report since then has advocated increased attention to problem solving. *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), for example, highlight problem solving as a core objective at every instructional level from kindergarten through high school. Given the prominence and attention problem solving has received, one is naturally led to ask what is actually done with problem solving in schools and how proficient are students in solving problems. I think it is fair to say that in the U.S., and perhaps world-wide, there is considerable dissatisfaction with problem solving instruction in schools (see for example, Dossey, Mullis, Lindquist, & Chambers, 1988; Dossey, Mullis, & Jones, 1993) and student progress in the problem solving domain has been disappointing; in fact, many educators would call it dismal.

The purpose of this paper is to discuss issues and make suggestions that hold potential for improving problem solving instruction and student learning. Special consideration is given to the role of teachers and to classroom context throughout the paper. I begin with a discussion of the meanings associated with the term problem solving and how they have influenced both instruction and research. I suggest an immediate and somewhat obvious solution to the confusion that results from the variety of meanings that exist. Then I make the case for beginning to realign thinking about problem solving to include more of an orientation toward mathematical thinking. The argument is based, in part, on the lack of progress flowing from current conceptions of problem solving, conceptions that give little regard to how it is taught and the context in which it is taught. Several potential reasons for inadequate, shallow student problem solving learning are then stated and used to identify three directions for study and change. These three directions are discussed using the labels: reformed instruction, authentic teaching, and instructional context. Then special attention is given to classroom culture including how it develops and its influence on students and learning. Finally, the importance of teacher conceptions in determining classroom culture is highlighted by summarizing results of two studies, a status study of teachers' conceptions of mathematical problem solving that I conducted and experimental work by Paul Cobb and his colleagues. Since terminology is important in communicating within any field of study as well as in advancing knowledge in the field, we begin with a discussion of the term problem solving.

Multiple Meanings of Problem and Problem Solving

It is not surprising that the term problem solving has taken on different meanings at different points in history, given the large shifts in emphases in education over the years. These educational movements have included basic skills instruction, integrated curriculums, and outcome based education to name a few. What is a surprise, and more of a concern, is that the term problem solving is currently used to represent quite diverse ideas in contemporary research literature and is used to convey quite disparate notions by researchers actively studying problem solving, particularly researchers from such disciplines as psychology and mathematics, disciplines that make important contributions to research in mathematics education. At one extreme, problem solving is sometimes taken to include situations that require little more than recall of a procedure or application of a skill. For example, Mayer (1985) using a traditional conditions-goal type definition of problem solving takes as problems for college students such tasks as writing an equation to go with the students-professor problem[1] (Soloway, Lochhead, & Clement, 1982) and the solving of linear equations in one unknown. At the other end of the continuum, problem solving is taken so broadly that it can almost be used synonymously with mathematical thinking (e.g., Schoenfeld, 1992).

[1] Write an equation to represent "There are six times as many students as professors at this university."

The lack of consistency of meaning of the term problem solving causes several difficulties. Whenever terms are used differently, problems of communication result. Thus, findings from one problem-solving study may initially appear to contradict findings from another study when in fact the results are not even amenable to comparison because they involve two very different phenomena. The problem is acute because there is a substantial research base in mathematical problem solving that should not be ignored but rather built upon to develop a theoretical framework to move the field forward in a productive way. In order to make sense of this research base one must take into account the meaning ascribed to problem solving in each study. On the surface it may seem important to alleviate this interpretation problem in the future by trying to reach consensus on the meaning of the term, but I suspect this would be difficult if not impossible to do, so an appropriate solution may be to strongly encourage researchers to carefully characterize how they use the term including a full verbal description of its meaning and numerous examples of problems that fit the conditions of the definition.

In passing, it is important to point out that the value of interdisciplinary research is now widely accepted. Early clarification of terms in research team activity may enhance their productivity. I say "early" because one assumes that eventually there is clear communication on such teams, but the sooner this is accomplished the better. In summary, it seems that at a minimum careful attention must be given to the use of the term problem solving in interpreting existing literature, in facilitating multidisciplinary research efforts, and in making future literature more useful.

Aligning Problem Solving With Mathematical Thinking

For many years a problem has often been conceived of as a situation where something is to be found or shown and the way to find or show it is not immediately obvious. That is, the situation is unfamiliar in some sense to the individual and a clear path from the problem conditions to the solution is not apparent. As psychologists often put it, there is some kind of blockage present that prevents reaching the goal state. Some researchers also prefer to indicate that there is no problem if the student has no desire or motivation to find the solution to the given situation. Definitions of problem solving that have the preceding characteristics, where there is a focus on moving from conditions to an end state, can be found in many places in the literature. For example, Polya (1962) indicates that "to have a problem means: to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable, aim." (p. 117)

Based on definitions like the preceding, problem solving is often loosely defined as solving problems. Making problem solving the centerpiece of a mathematics program or the focus of instruction thus becomes a matter of focusing on problems as a topic of instruction. This leads to the recommendation that the school curriculum include lots of problems. Sometimes the stress in this recommendation is on including certain types of problems. For example, the thrust may be on problems that reflect real world applications, or problems that will likely call for specific solution methods to solve them (e.g., make a diagram), or problems that introduce new concepts (e.g., Rachlin, Matsumoto, & Wada, 1992). Sometimes the emphasis is on incorporating a variety of problems.

My point is that considerable energy is, and has been, spent on endeavors to increase in appropriate ways the problem base in the curriculum. Furthermore, substantial programs of research have dedicated their efforts to studying how students solve problems in the curriculum and then using these insights to develop and test instructional programs. A classic program of research in this area was conducted by Lester, Charles, and colleagues (see for example, Charles & Lester, 1984). In their work specific strategies for solving problems were identified and taught to classes of students in circumstances where student growth could be compared to control situations. This research, along with the research of many others (e.g., Lucas, 1974), has shown that specific problem solving strategies such as guess-test, make a diagram, work backwards, make a table, and so on, can be successfully taught to students, that students learn them, that students use them when solving problems, and that this type of instruction results in improved student performance on a reasonably wide range of problems.

What we have shown to this point is that the traditional conception of problem solving, as moving from given conditions to something to be found or shown, has had both positive and negative outcomes. On the one hand, a large collection of problems have been compiled and useful research findings about strategy acquisition and instruction have been added to our knowledge base. On the other hand, the narrow conception of problem solving has restricted our thinking in both instructional and research areas and also resulted in significant shortcomings in our students. As Lester (1985) indicates, most problem-solving instruction not only does not enable students to use their heads, but in fact it does more harm than good." (p. 41)

Something important does indeed seem to be missing because many students are not able to respond to mathematical situations in ways that are expected and desired. For example, consider the following problem from the *Fourth National Assessment of Educational Progress* (Lindquist, 1989): "Suppose you have 8 coins and you have at least one of each of a quarter, a dime, and a penny. What is the least amount of money you could have?" (p.16) Only 23 percent of Grade 7 students and 43 percent of Grade 11 American students were able to select the correct answer in a multiple choice format composed of 4 choices. These are truly disappointing results.

The preceding illustration is not an isolated instance, in fact, it may be quite representative of how little sense making or reality testing students do when working within quantitative situations. For example, when students are asked to estimate answers to quantitative situations they often exhibit a lack of flexibility of thinking that is of concern. This lack of flexibility of thinking is not limited to American students, but seems to be common in other countries as well; see, for example, the study by Reys, Reys, Nohda, Ishida, Yoshikawa, and Shimizu (1991). The issue is not students' inability to interpret an answer, or be flexible in their thinking, or some other single behavior that might be remedied in the short term by some specific teaching strategy. Instead, the concern is that all these shortcomings, as a collection, indicate that student thinking is often shallow and without reflection when confronted with mathematical situations of a variety of types.

This missing component in students' problem solving ability has been discussed by researchers and educators and described in several ways. Some describe the missing link by indicating that we need to assist students in developing their mathematical thinking, others state that we need to empower students so that they can use the mathematics they learn, and still others declare that classroom instruction needs to focus more on sense-making and developing connections. A common theme in these suggestions is that while progress has been made in helping students solve problems, the learning about problem solving that occurs is either too narrow to apply to other situations or the learning has been accomplished in an instructional manner or instructional context that neither allows nor facilitates students using it in other situations. In some ways the situation relates back to the classic learning theory problem of the transfer of learning to new situations. I think, however, that this is too broad a characterization of the situation to be useful, and most researchers now have a scaled back notion of the extent of vertical transfer that can be expected, scaled back in comparison to the views of learning ,ychologists and theorists of past decades.

Recall that the position taken in this paper is that problem-solving learning under current instruction too often results in shallow, inadequate learning. That is, students are unable to apply what they have learned to new situations, and their responses to problems too often do not make sense. They are frequently insensitive to absurd answers, they are not inclined to interpret and work with situations mathematically. While the preceding descriptions are not fully developed, they should communicate the shortcoming that must be addressed by teachers and researchers. While there are no doubt many approaches to improving the situation, the aim of this paper is to focus on reformed instruction, authentic teaching, and instructional context as factors that hold significant potential for enhancing the kinds of student learning being advocated in current reform efforts.

DIRECTIONS FOR STUDY AND CHANGE

As will become clear in the following discussion, reformed instruction, authentic teaching, and instructional context are not to be considered independent dimensions of the teaching and learning process. Rather, they should be considered ideas that overlap in their domains yet each has distinctive focus with regard to change. Further, the ideas not only overlap but they are linked to one another and changes in one often cause changes in the other. The interrelatedness suggests that as each is studied the other must be taken into account.

Reformed Instruction

Reformed instruction, as used here, refers to problem solving instruction that takes account of recent advances in cognitive psychology. The role of student knowledge in the problem solving process is one place where cognitive science can, and perhaps should, influence teacher planning and classroom instruction. As we shall see, however, the specific ways to use these advances in improving teaching requires interpretation and judgment on the teacher's part. Further research is needed to provide guidance to teachers in this area.

Not too many years ago it was quite common to assume that if an individual were taught all the prerequisite knowledge needed to solve a problem, then positive transfer would likely occur and the student would find the solution to the problem. Prerequisite knowledge included definitions of terms in the problem, formulas relating problem conditions, the ability to represent problem conditions, and the necessary algorithms whether they be computational, algebraic, or other. Simply put, it was thought that if needed knowledge was in place, then problem solving success could be expected. Declarative and procedural knowledge, that is, "knowing that" and "knowing how" types of knowledge, are still deemed important in solving problems, but researchers are, as we shall see, becoming more aware of the importance of other factors as well. The reader interested in a more detailed discussion of declarative and procedural knowledge in a general setting should see Anderson (1976) and for a discussion of the relationships between the two types of knowledge in a mathematical settings should see Hiebert (1986).

The inclination to have declarative and procedural knowledge in place before beginning problem solving instruction still underlies some thinking about how to achieve success in problem solving, and probably guides most instruction on problem solving in schools. Current thinking by many researchers, however, now emphasizes that the presence of such core knowledge may be necessary for solving problems, but it is certainly not sufficient. What needs to be considered is how that knowledge is structured in the mind of the problem solver (see for example the work of Silver, 1979, 1981) and the kinds of access the student has to it. Not only is access important but how the knowledge is related to other knowledge also seems to be critical. At one time this notion of relatedness and the idea of meaning were considered under the general term, understanding, but research and theory building have moved the field forward and there are now have much more specific characterizations of understanding. These descriptions of understanding and the theories about the role of knowledge in the problem solving process, need rigorous research validation and classroom tests of their implications. Even in their present form, however, they provide useful insights for the classroom teacher seeking to improve problem solving instruction.

A framework for understanding learning and teaching for understanding that seems to hold considerable promise is one advocated by Hiebert and Carpenter (1992). Included in their model of understanding are detailed descriptions of how internal and external representation of ideas might be considered and discussions of how these representations may develop and be linked. They discuss understanding as part of a model of cognitive structure where ideas are represented as nodes of a network and relationships between ideas are thought of as connections between the nodes. Thus they describe understand as:

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (p. 67)

Some of the basic ideas in the Hiebert and Carpenter framework go back to the classic work of William Brownell, Henry Van Engen, and others, yet the framework makes an important contribution to the field through the detailed elaboration of the ideas in the framework and through the discussion of how it applies to many situations. For example, they show how the model can assist in helping us understand many existing result findings. Research findings on such topics as manipulative materials, where the results are mixed, become clearer when examined under the tenets of their theoretical framework.

The contributions of cognitive psychology are not limited to creating models of mathematical understanding. How students represent ideas mentally and in physical settings, and the relationship that develops between them, is another area where significant progress is being made. For additional details see the book by Janvier (1987). Exciting work is also being done using technology. For example, students can create various representations, such as an equation, table, and graph on a computer screen and then link them so that a change in one representation results in a change in the other. Further, the changes are shown instantaneously on the computer screen. Kaput (1992) provides a detailed discussion of this "hot linking" and discusses other areas where technology is providing interesting learning tasks that take account of advances in cognitive psychology.

How can information such as the preceding be used by teachers in the classroom? As mentioned previously, the theories are not directly translatable into classroom practice. However, research has shown that sometimes making teachers aware of research finding not only affects their practice but also student learning. The Cognitively Guided Instruction project (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989) is a good example of a research project that achieved positive results by providing teachers with research information. We discuss this project in detail in the next section. We conclude this section by hypothesizing that if teachers consider the importance of students developing connections between ideas as they plan instructional activities, then it may have a positive influence on what students learn and how they perform in the problem solving domain. Further, if teachers reflect and use other ideas from cognitive psychology such as research on representations of ideas, then positive outcomes may also follow. We now elaborate on the teacher's role in improving problem solving learning in classroom situations.

Authentic Teaching

Dossey (1992) points out that there is not a common view among mathematicians of what constitutes the nature of mathematics. Further, when mathematicians are confronted with questions about the nature of mathematics they often revert to discussing it as meaningless game played with symbols even though this view is not representative of what is happening when they are creating mathematics. Despite the lack of consensus among mathematicians, Dossey makes a strong case for reconceptualizing views of mathematics to consider it more a human activity that involves exploration, guesses, representations, generalizations, and arguments that convince.

In the education community there is strong sentiment for considering mathematics as a sense-making activity that is socially constructed and transmitted (Brown, Collins, & Duguid, 1989; Lampert, 1990; Schoenfeld, 1989). If this view is accepted then it is clear that students develop their sense of mathematics, and thus how they use mathematics, from their experiences with mathematics. Most of this experience occurs in the classroom and thus the instruction that teachers provide become crucial.

In too many classrooms, unfortunately, students' mathematical experience involves nothing more than doing endless sets of exercises where each exercise has one answer and there is one set way of doing each exercise in the set. Indeed, even problem-solving lessons designed to teach strategies do not escape this pitfall; that is, many times there will be an assignment where every task is designed to evoke a particular strategy, such as make a diagram or work backwards. This type of instruction does not foster sense making but rather stifles thinking of any kind, especially thinking that would be considered creative or novel. What is needed is teaching where the tasks teachers set for students require problem solving, where problem solving includes substantial mathematical thinking as previously discussed. In short, problem solving must become an approach to instruction rather than a topic of instruction.

What would a classroom in which the teacher is using a problem-solving approach to instruction look like? Actually a better question would be to focus on some of the characteristics of such instruction. Better because by its very nature we would expect classroom events in such an approach to follow student thinking and, therefore, detailing a prescription for using such an approach would be inappropriate. My view is that in a typical problem-solving approach to instruction one would see lots of exploration of situations, hypothesis generation, problem posing, multiple solutions and solution methods, arguments followed by justifications and verifications. In fact, the classroom would have much in common with what mathematicians do when they do mathematics (see Lampert, 1990 for further discussion of this position).

We conclude this section by discussing a research project, Cognitively Guided Instruction (CGI), where teachers moved from a transmission model of instruction to one that was more constructivist in nature (Carpenter et al., 1989). In this primary grade project, teachers were made aware of research findings about the strategies children use in solving simple addition and subtraction verbal problems but were not explicitly told how to use this information. They were also shown examples of children solving problems and discussed the variety of solution methods that naturally arise as children learn to solve problems. The impact of providing this information was substantial. Teaching behavior changed in the direction of spending more time on problem solving and teachers spent significantly more time than control teachers discussing student solution strategies as part of regular mathematics instruction. Student problem solving performance was also positively impacted and there was no loss in the development of basic skills.

This study makes a strong case for sharing research knowledge with teachers, changing teachers' view of mathematics, and the fact that if teachers create the right classroom environment then progress can be made in the directions called for in the current reform movement. The culture of the classroom is also an important component of change and is discussed in the next section.

Instructional Context

The least studied and understood factor associated with mathematics teaching and learning is the effect of context, yet it holds great potential for deepening our understanding of classroom practice and for helping develop a framework for improving student learning. I begin with a general discussion of context and then give special attention to how context affects student beliefs and behavior, how student beliefs may shape teacher behavior and development, how teacher conceptions are complex in both their nature and how they affect classroom culture, and finally examine a research study by Grouws (1991) that investigated teachers' conceptions of problem solving.

Context, the set of interrelated factors surrounding and often influencing a situation, has been recognized and shown to be important in many social science fields with anthropology being the discipline most often used to illustrate its value in interpreting events and behavior. Classroom context consists of many variables.

Factors outside of the school are part of the schooling context if they are in some way associated with student learning. One external factor that has shown considerable impact on the schooling process is the influence of mandated assessments. State mandated assessments have been found to influence the curriculum, how instructional time is used, and the nature of instruction. Romberg, Zarinnia, & Williams (1989) in a nation-wide survey of eighth grade teachers found that 70% of the teachers surveyed administered district-mandated tests to their students. Fewer than 20% of the teachers made no instructional changes as a result of the test. Romberg et al. summarized their findings about mandated tests by saying that almost all teachers use the test results to evaluate themselves or their students, that the majority of teachers change their use of instructional time based on the test results, and that nearly a third of teachers consider the style and format of the test when planning their mathematics instruction. This study demonstrates the power of context factors in shaping classroom instruction. It also speaks to the development of problem solving ability because as Romberg et al. (1989) point out, "The tests do exert an influence, and that influence is an entrenching of the emphases on basic skills and pencil-and-paper computation that the NCTM *Standards* recommends de-emphasizing." (p. 83) Thus, a strong argument can be made that changing the composition of required tests to include more problem solving items may be a needed context reform in order for mathematics classrooms to focus more on problem solving and developing mathematical thinking. The importance of context factors outside the school building is not limited to mandated testing as will become clear in the discussion of parental attitudes that follows.

Parental attitudes have been suggested as a major influence on how much mathematics students learn. Research has shown the existence of not only important achievement differences among American, Chinese, and Japanese students but also cultural differences that may contribute to the differences in learning. Stevenson, Lee, & Stigler (1986) found, for example, that 91 percent of American mothers in their sample judged that the school their child attended was doing an excellent or good job, while only 42 percent of Chinese mothers and 39 percent of Japanese mothers were this positive. In another part of the study mothers were asked to rank the relative importance of effort, natural ability, difficulty of school work, and chance in determining a child's performance in school by allocating a total of 10 points among the four categories. Interestingly, Japanese mothers assigned the most points to effort, and American mothers gave the largest number of points to ability. Stevenson et al. (1986) concluded that "the willingness of Japanese and Chinese children to work so hard in school may be due, in part, to the stronger belief on the part of their mothers in the value of hard work." (p. 697)

When context is considered with respect to schooling it also includes internal factors within the school environment such as school goals, classroom climate, the physical setting including availability of instructional equipment and materials, school policies and curriculum guides, administrators, and teachers' colleagues to name a few of the factors.

While many context factors may be important in coming to a better understanding of the teaching and learning of mathematics, determining the impact of any particular context factor in a school setting must involve more than logical analysis because in some situations what seems to be intuitively obvious is not necessarily true when examined in the light of research. Consider the often stated remark that administrators determine or strongly influence teaching practices in their schools. On the surface it makes sense and in some situations it may be quite true. Data from research, however, suggests that it is not true in many instructional situations; that is, teacher decision making at least in some domains may be quite independent of administrator influence. For example, Good, Grouws, & Mason (1990) surveyed 1509 teachers in elementary schools in 10 districts in three states about their grouping practices in mathematics. In their study the influence of school principals was not a factor that many teachers reported as influencing their decisions about whether or not to group students for instruction, when to group students, nor how to form instructional groups if and when they used them. Thus, the study casts doubt on the importance of some commonly cited context factors that purportedly influence teacher decisions about instructional grouping. On the other hand, it lends support to the importance of other context factors, such as time pressures, that do influence teacher decision making. Thus the study shows the importance of some external context factors, but reminds us not to accept generalizations about context without supporting research data.

We examine one more example of a commonly accepted generalization involving context that has not withstood the test of research scrutiny to emphasize the importance of research in validating the importance and contribution of context factors. There is a maxim that student teachers are greatly influenced by the context in which they do their student teaching, more specifically, that they are particularly influenced by the teachers in their school. However, in real school settings the situation is not that unambiguous. Brown and Borko (1992) point out from their review of the research literature, that when a student teacher is placed in a school where several teaching cultures are present, then it is unlikely that the placement will have much of a socializing effect on the student teacher.

The preceding discussion emphasizes the need for research validation of proposals concerning the importance of context factors in shaping the participants in the classroom (teacher and students) and classroom activity. The reader should keep this need for validation in mind as the influence of classroom culture is discussed in the section that follows.

CLASSROOM CULTURE

In this part, I make the case that a classroom culture develops through the interactions of the teacher and the students. Thus, the views and expectations that teacher and students bring to the situation affect the culture that forms. Then the development and influence of student beliefs in this culture are examined. Next, how student beliefs can influence teachers is discussed. This is followed by a look at how teacher conceptions influence classroom culture. Finally, because the research base for many parts of the argument are not in place, I conclude the section by examining a research study of teachers' conceptions of problem solving because it shows important variance that should be considered when studying classroom culture.

Characterizing the Term Classroom Culture

Each mathematics classroom assumes its own culture according to the unique knowledge, beliefs, and values that the participants bring to the classroom (Nickson, 1992). The students bring views of what one does in a mathematics class, judgments about how good they are at mathematics, and feeling about how well they like mathematics. The teacher brings to the class a view of mathematics, routines for teaching the class, expectations about what should be accomplished in the class, personal experience with learning mathematics, and either a like or dislike for the discipline. The preceding lists are not meant to be exhaustive, but rather to be suggestive. What teacher and students bring to the classroom situation has a role in the classroom interaction that takes place, and thus plays a part in the classroom culture that emerges. It should be clear at this point that no two classroom cultures will be exactly alike because they are composed of different students and teachers. Thus, when one talks about what a classroom culture should be like in order to promote learning, one is describing an idealized condition for purposes of communicating the desired characteristics of such a culture. I now discuss how student beliefs are influenced and shaped by classroom culture, where you will recall classroom culture is taken to include the shared meanings and beliefs that teacher and students bring to the classroom and that govern their interaction in it.

Development and Influence of Student Beliefs

Three points form the basis for our discussion of student beliefs. First, student beliefs and conceptions are influenced by classroom culture. Second, the beliefs and conceptions formed shape how students learn and what is learned. Third, student beliefs and conceptions shape teacher behavior.

We have already discussed one situation where classroom work promoted a particular view of mathematics. Recall the situation where the daily mathematics work was composed of doing endless sets of routine exercises. I will not illustrate this situation further because, unfortunately, most people have had direct experience with such classroom situations or have at least had it described to them. Such activity, day after day, undoubtedly has an influence on student thinking about what mathematics is about. It seems logical that students would soon begin to think that there is one way to do every mathematics task and that there is always one answer. In fact, there is data to support the prevalence of such a conception. Data from the *National Assessment of Educational Progress* (1983) indicate that 9 out of 10 students surveyed agreed with the statement, "There is always a rule to follow in solving mathematics problems." Interestingly, the survey also found that over one-half of the students thought learning mathematics was mostly memorizing. It seems logical to assume that the development of these beliefs about mathematics would develop, in part, through the interactions between teacher and students in the classroom, that is, from the culture of the classroom. We discuss how specific teacher conceptions and actions can influence classroom culture, and thus students, in more detail in a later section.

At this point it is appropriate to discuss how student beliefs and conceptions shape how student learn, and what they learn, because we have just shown how classroom culture molds student beliefs about mathematics. The case rests on suggesting that student beliefs about the nature of mathematics affects how they approach learning and thus what is learned and how it is learned. An illustration may clarify the argument. Consider a student who believes the following. Mathematics is mainly memorizing. If one understands mathematics then one should be able to do problems quickly. There is always a rule to follow in solving mathematics problems. It is likely that this student is attending to quite different things in instruction than a student who thinks of mathematics as an activity that requires thinking, that problems frequently require large blocks of time to solve, and that there are multiple ways to solve most problems. It is beyond the scope of this paper to detail situations of how these students would differentially respond, but I think the logic of the situation is sufficient that most would agree that it is at least plausible to assume that they would respond differently. In closing, the reader should note that the first student described is probably a fair representation of many students because it is based on responses to items found in the National Assessment survey previously mentioned. Next we consider how student conceptions can influence the teacher.

Influence of Student Beliefs and Conceptions on Teachers

The very idea that students beliefs and conceptions affect teachers and their teaching practice may seem quite unbelievable to some people, particularly people who see the teacher as the absolute ruler and authority in the classroom. It is quite clear, however, from the limited research that bears on this issue that students acting out their beliefs and conceptions do have a shaping influence on teachers.

Cooney (1985) studied a mathematics teacher during his preservice training and his first three months of teaching. Through a series of interviews he determined the meanings the teacher "ascribed to problem solving and to teaching more generally," and the teacher's perceptions of how implementation of his ideas played out in the classroom. The teacher believed strongly in the use of problems to stimulate student interest, particularly recreational problems. He further believed that "problem solving is the essence of mathematics" and a "teacher's chief responsibility is to motivate students." Classroom observation showed that as the teacher moved to implement these views, students in his algebra and geometry class were responsive, but he met with stiff resistance from students in his other three classes (Algebra II, General Mathematics, Geometry). Cooney characterized these unresponsive classes as lethargic and uninterested with several students in one class so disruptive that it seemed difficult for the teacher to maintain any continuity. The teacher, not unexpectedly, was frustrated by the situation and commented that "for all my efforts of explanation and good pedagogical techniques, I still don't know how to motivate students." Students were frustrated too. In one exchange where the teacher attempted to justify his inquiry based approach, one student responded, "We are not used to that [kind of approach]. It is a different style of approach than anything we have seen before. We reject it as a kind of culture shock." This situation clearly illustrates how the interaction between instruction and classroom culture can be problematic. For our purposes, it is important to note that the teacher in this study eventually abandoned his inquiry-based approach to instruction and his use of recreational problems and changed to a more traditional form of instruction that focused on covering the material.

It should be noted that while the impact of students on teacher beliefs about instruction, and how it should be conducted, was quite direct in the preceding research, it is likely that much more often the effects will be more subtle and the adjustments teachers make much less major. This makes detecting such changes through research much more difficult, but in spite of this difficulty, it is important that studies be conducted to better ground the idea that student beliefs, and their resulting actions, do influence teachers and their instruction.

Teachers' Conceptions and Their Influence on Students

The Cooney (1985) study previously discussed shows that the relationship between observed practice and teachers' professed beliefs are complicated and complex. There are, however, studies where there has been a close match between teachers' beliefs about teaching and their instructional practice (e.g., Grant, 1984). Thompson (1992) when discussing inconsistencies among studies makes a valid point when she indicates that

> teachers' conceptions of teaching and learning mathematics are not related in a simple cause-and-effect way to their instructional practices. Instead, they suggest a complex relationship, with many sources of influence at work; one such source is the social context in which the mathematics teaching takes place, with all the constraints it imposes and the opportunities if offers. (p. 138)

We have referred to some of these constraints that may cause teachers to circumvent implementation of their conceptions and beliefs. One of these was the pronounced effect of mandated testing on teacher planning and instruction. If one accepts that teachers conceptions in some cases directly affect instruction and in other cases the effect is mediated by context, then one question that surfaces is what are teachers conceptions and beliefs. We now turn our attention to one study of this situation which focused specifically on teachers' problem solving beliefs and conceptions.

Teachers' Conceptions of Problem Solving

As part of a larger study, Grouws (1991) examined teachers conceptions of problem solving. Twenty-five teachers drawn from eight junior high schools in a large Midwestern school district comprised the sample. This volunteer sample represented over 80 percent of the junior high mathematics teachers in the district. Together they taught 119 classes composed of more than 2500 students. The SES level of the schools in the district ranged from lower-middle to upper-middle class.

Using a pilot-tested set of questions, teachers were individually interviewed for approximately 50-55 minutes concerning their beliefs and teaching practices with special attention to problem solving. At the beginning of the interview teachers were reminded that although problem solving was important there was not a consensus about its meaning or how it should be taught. They were asked to be candid in their responses and reminded that all data collected were confidential. The discussions were audio-taped and later transcribed. The transcribed interviews were analyzed to identify patterns of responses and to detect relationships among the responses.

To help determine the teachers' definitions of problem solving, they were asked to state in their own words how they would define the term. Careful assessment of their responses showed they clustered into four distinct categories. Many of their responses clearly focused on types of problems whereas others centered on features of the problem-solving process. Four conceptualizations were identified: (1) Problem solving is word problems; (2) Problem solving is finding the solutions to problems; (3) Problem solving is practical problems; and (4) Problem solving is solving thinking problems; that is, requires thinking. In summary, teachers' definitions of problem solving could be categorized into four distinct classes, differentiated in three cases primarily by the type of problem mentioned and in the other case by a focus on the level of thinking required. Concerning problem types, some teachers classified problem solving as word problems. A variety of word problems were included but most could be solved in a step-by-step manner or by using a computational procedure. Another group defined problem solving to be those tasks that involved problems of any type, with no differentiation of complexity. Others felt problems had to be of a practical nature, applicable in the real world. The final group of teachers believed that problem solving had to require the student to think and not just apply some practiced procedure to find a solution. These findings clearly demonstrate that teachers do have different conceptions of what constitutes problem solving. This is important because, as was shown previously, teacher conceptions influence instruction, although the relationship is complex and mediated by the many factors that comprise classroom context. Nevertheless, the existence of different teacher conceptions should be taken into account in planning new research and in interpreting the results of existing research. For example, studies that report on the importance teachers ascribe to problem solving need to consider what each teacher meant by the term problem solving in each item of the instrument. This is especially true given the four different clusters of conceptions found for problem solving in the research summarized here.

I close on a positive note by mentioning the work of Cobb and colleagues. They have a program of research at the primary grade levels that emphasizes the construction of meaning in the experiences provided to children in mathematics class. Their instruction is inquiry-based and problem centered. Their program includes working with teachers and providing experiences to assist them in reflecting on their conceptions of mathematics and what it means to teach mathematics. The results have been positive in terms of the classroom culture that develops and in terms of the resulting student learning. For a detailed description of the program see Wood, Cobb, Yackel, & Dillon (1993).

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AN OVERVIEW ON MATHEMATICAL PROBLEM SOLVING IN CHINA

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Introduction

Chinese mathematical education has absorbed the quintessence of thinking from Japan, the United States and the Soviet Union in this century. Of course, China has its own tradition with many good characteristics and also some weaknesses. Therefore, it is appropriate to view mathematical problem solving in China both internationally and traditionally.

Historical Background

Mathematical problem solving is a most important characteristic of ancient Chinese mathematics. For example, the famous classical work *Nine Chapters of Arithmetic* that was completed in 200 B.C. consists only of 246 mathematical problems that were classified under nine types. At the same time, ancient Chinese mathematicians believed that solving a real-world mathematical problem or winning an intellectual game meant that one could find the answer by a computation program. This attitude tended to encourage a fragmented learning of skill in problem solving by separating problems into types, and particularly emphasizing the skill of quick and exact computations. This is the reason why the Chinese called mathematics the "Suan Xue (算学)," that means "the knowledge of computation," before World War II.

There exists an "examination culture" due to the traditional examination system in China. In 593 B.C., Sui Wen-Di, a Chinese King, selected some officers by using a nationwide examination. Thus, everyone had the opportunity to become a junior officer if they passed the nationwide mathematical examination. In 1888, 30 young men took part in an examination of Western mathematics, but only one of them passed and became an officer. Thus, the idea of "a test paper deciding one's future" and "the examination selects the elite only" has been a part of Chinese culture.

After 1900, Western mathematics was taught in all schools in China. However, the entrance examination was still important in the educational system. Mathematical problem solving then became almost equivalent to problem answering on examinations. In the earlier part of this century, we learned a lot from Japanese mathematics educators. Nevertheless, both Japan and China suffered from the pressure of keen entrance examinations (S. Shimada). Japanese examination problem books had a great influence upon Chinese mathematical education in the second half of this century. For instance, *The Dictionary of Geometry* and *The Dictionary of Algebra* written by Nagasawa have been popular references for Chinese teachers.

The Chinese mathematical curriculum and textbooks were influenced by the Soviet Union instead of the United States in the 1950s, so that the curriculum formed a rigorous, logical and pure deductive system. As to problem solving, we emphasize the skill of quick and exact computation and the rigor of deduction. In the meantime, we have neglected mathematical applications and modeling as well as intuitive thinking, because problems of these types cannot be used on the examinations.

When a Chinese delegation took part in the ICME-4 in 1980, we heard about a new mathematics teaching approach--problem solving; however, this new slogan was immersed in a tide of keen competition on the entrance examinations through the whole of the 1980s. Professor J. Becker gave a talk about problem solving in Shanghai in 1987, and Professor T. Sawada introduced the open-ended problem to us in Xian. These lectures and other work led to more attention to problem solving, but it was not put on our agenda until the 1990s.

Two Champions

Chinese pupils have performed well in many international competitions and assessments. There are two very important reports:

1. The Chinese team won first place in the International Mathematics Olympiads that were held in 1988, 1990, 1992 and 1993, respectively.
2. The International Assessment of Educational Progress (IAEP) published statistical data in

1992. In mathematical testing, the average percent correct of Chinese pupils was also listed in first place among the 21 countries and areas. Here is an incomplete listing:

China (Mainland)	80	Korea	73
China (Taiwan)	73	Switzerland	71
Soviet Union	70	France	64
Canada	62	United States	55
Spain	55	Brazil	37

These data may show that Chinese students can do the best in a timed written examination. Someone said that China seems to be a realm of routine problem solving. However, it is clear that Chinese mathematics teaching has its own weakness; for example, neglecting to cultivate students' creative abilities and paying less attention to mathematical applications.

Double-edged Sword--Keen Examination

As I mentioned above, there exists an "examination culture" in China. Obviously, it is impossible to abrogate the system and suspend the culture for a moment. In addition, the intense emphasis on the entrance examinations simultaneously plays both positive and negative roles.

On the positive side, the system

* encourages fair competition;
* compares achievement under one national standard;
* calls parents' attention to the mathematical performance of their children; and
* requires the students to solve examination problems quickly and exactly (as a result, mathematics teaching must emphasize basic mathematical skills).

On the negative side, the system

* places great pressure on the pupils;
* forces teachers to teach only what the examination demands;
* emphasizes memorization and programmed thinking in order to only answer the test questions.

In order to generalize the mathematical problem solving approach, we must make use of the positive functions of examinations; in particular, to reinforce basic mathematical skills. Chinese teachers believe that every pupil should solve a great number of routine problems--"Practice Makes Perfect." They used to say "We should treasure our childhood practices" (language practice, practice playing the piano, etc.), of course, including practice in mathematics.

Recent Progress

Accompanying the nine-year compulsory education requirement that is being carried out, most Chinese mathematics educators suggest that we replace "testing mathematics education" by "essential mathematics education;" i.e., to emphasize "mathematics for all" instead of "mathematics for some" (those who pass the entrance examination). However, a lot of teachers worry that pupils' skills will decline by reducing the amount of exercise. Of course, doing exercises to improve skills is very important, but alone it is by no means satisfactory. We must set a higher educational goal: from "exercise doing" to "problem solving." Since 1992, we are striving towards this goal and undertaking the following three things:

1. Innovation in Examination Problems

Examination problems, as a powerful influence, are able to guide mathematics teaching along a path of progress; but, of course, they can also guide it in a wrong direction. Thus, innovation in the mathematics problems of the entrance examinations is being promoted as an important way to emphasize mathematical problem solving. As the first step, the Education Committee of the Chinese Mathematical Society suggested to the National Examination Center that it add some problems of mathematical applications to the examination papers of the University and College Entrance Examination in 1993. Four applied mathematical problems were included on the examination this year. This measure has prompted school teachers to pay more attention to mathematical applications problems and modeling. (In the past ten years, no applied mathematics problems appeared on the National Examinations.)

2. Publishing a new *School Mathematical Problem Book.*

Chinese school teachers are used to solving routine problems; however, they don't know what a "mathematics problem" is. Routine problems are problems, and we do not intend to weaken the pupils' basic skills on routine problem solving. But, what we need to do is provide a number of valuable mathematics problems for use by school teachers. For this reason. the Research Center for Basic Education of the State Commission of Education entrusted us with the compilation of a mathematical problem book.

The Study Group For Mathematics Education completed the first book, *School Mathematical Problem Book (I)*, that includes 168 problems in which Chinese teachers may be interested. Some of them were selected from abroad and some were developed by us. We selected problems according to three guidelines:

(1) Challenging--different from routine problems;

(2) Close to the syllabus and the requirement of the examinations--different from pure intellectual games and puzzles;

(3) Easy to undertake--different from the Olympiad Competition Questions. (consult appendix for examples)

3. Organizing a Nationwide Seminar

In order to draw attention to the mathematical problem solving teaching approach, a nationwide seminar sponsored by the Research Center for Basic Education and the National Examination Center was held in August this year. A series of seminars are also being planned.

In the August seminar, we presented the 168 mathematical problems that are mentioned above to all participants and reported some experiences with problem solving teaching. However, the focus of the discussion was still on the relationship between the entrance examinations and problem solving:

(1) How can students' creative power and abilities in mathematical applications be assessed by a timed, written examination?

(2) Since solving an applied mathematical problem may require knowledge beyond mathematics, how can we evaluate students' mathematical abilities by these problems?

(3) Open-ended problems are very valuable; however, can they be put into an examination paper? Many teachers worry that we cannot evaluate solutions in a fair way due to a lack of standard answers for the open-ended problems.

(4) Bloom's theory of classification of the aims of teaching is very popular in China. If we were guided by Bloom's theory, then test papers would mostly cover the "knowledge domain," but then maybe there will be too much time on routine problem solving. Is Bloom's theory right or not?

(5) There are several papers reporting investigations on how pupils' abilities on routine problem solving transfers to general problem solving. We expect to bridge the psychological mechanism of this transition.

The theoretical investigation and classroom practice of problem solving were initiated in China only a few years ago, though we have more experience with routine problem solving. We need to learn a lot from our American and Japanese colleagues, to exchange ideas and information on problem solving, and to finally raise the general level of mathematical education in China.

APPENDIX

We collected 168 mathematical problems for the *School Mathematical Problem Book*. Here are some examples which may be of interest to Chinese teachers.

1. Nine robots are to perform various tasks at fixed positions along an assembly line. Each must obtain parts from a single supply bin to be located at some point along the line. Where should the bin be located so that the total distance traveled by all the robots is minimal?

   (This problem is selected from the *Curriculum and Evaluation Standards for School Mathematics* (NCTM). It has many reasoning features that are appreciated by

Chinese teachers).

2. There is a rectangular field. We will design a small garden so that the area of the garden is equal to half the area of the field. Please give your design.

   (This open-ended problem was provided by a participant in PME-17 which was held in Japan in August 1993. Chinese teachers show a special interest in it.)

3. A cube with six red faces will be divided into n^3 small equivalent cubes. How many small cubes will have three red faces, two red faces, one red face and no red face?

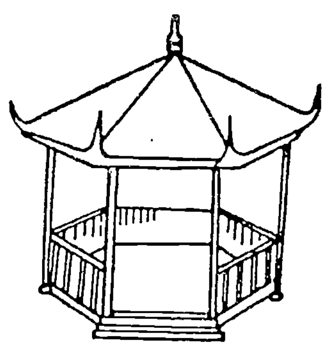
   (We think this problem is not only an intellectual game, but is also concerned with algebraic expressions:

$$n^3 = [(n-2) + 2]^3 = (n-2)^3 + 3 \cdot 2 \cdot (n-2)^3 + 3 \cdot 4 \cdot (n-2) + 8.$$

   Chinese teachers like problems like this that are also concerned with computation skill.)

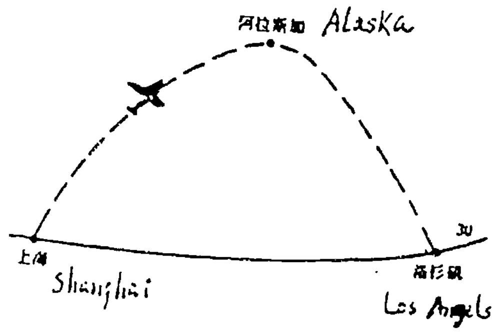
4. There is a hexahedral pavilion, whose two neighboring pillars are a distance 1.6 m. apart, and each pillar is 2.7 m. in height. The distance between the top point A of the pavilion and the ground is 3.9 m. Please find the plane angle between two neighboring triangles on the roof of the pavilion.

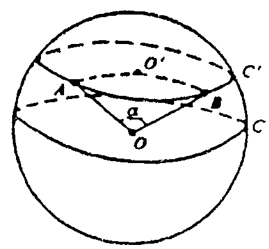
   (This is an ethnomathematics problem concerned with a Chinese style of construction.)



5. On the world map, Shanghai is near point *A* (with east longitude 120 and north latitude 60) and Los Angeles is near point *B* (with west longitude 120 and north latitude 60). Please find the length of arc *AB* on the large circle *OAB* and on the north latitude 60 circle *O'AB*, respectively.

(This problem is rewritten from news about an air accident in 1993.)





DIVERSITY, TOOLS AND NEW APPROACHES TO TEACHING FUNCTIONS[2]

Jere Confrey

Cornell University

Concerning educating, Ubi D'Ambrosio speaks of the fundamental importance of learning to respect the diversity of others. This conference, the China-Japan-U. S. Seminar on Problem Solving, that is attended by delegations from three different cultures, provides a unique opportunity to explore what it could mean to respect and learn from diversity in mathematics education. One component of respecting diversity is to be humble enough to realize that recognizing and understanding another's diversity, their differences from ourselves, is a difficult, albeit exciting, challenge. This applies equally to understanding children as to understanding those of different cultures, classes, races, genders or sexual preferences. Respecting diversity begins as we forego the arrogance implicit in believing that we possess universal insight. Since, in mathematics, claims of universality in our thinking arise frequently, overcoming arrogance is a serious issue for mathematicians and for mathematics educators. In this paper, I seek to establish the issue of diversity as a fundamental one, not only among the communities of learners of mathematics, but in the discussions of the epistemology of mathematics itself. I will make the argument that the suppression of diversity in the content of mathematics implicates the lack of diversity in the practice of doing mathematics since the two, practice and epistemology, are not separable. This argument will be made in the context of research on students' conceptions of functions.

The Gitksan Indians speak of male and female as two wings of a bird and remind us that a bird could not fly without both wings (Plant, 1989). Much of this paper is about reestablishing balance in mathematics education by creating stronger dialectics rather than by resorting to bipolar

[2] This paper is dedicated to the memory of Florence Velez, my first doctoral student who was investigating trigonometric functions within a Vygotskian perspective at the time of her death well before the rest of us anticipated the importance of such approaches.

opposition and hierarchical ordering. Dialectics make it clear that diversity only makes sense in light of commonalty and visa versa. For example, protecting biodiversity only makes sense in light of our shared eco-sphere. Recognizing our shared humanity is made far more likely once group and individual differences are acknowledged. Our frameworks for educating must find better ways to admit various viewpoints, competing or contrasting, without dismissing or failing to acknowledge our shared interests. Perhaps, the recognition of commonalty rather than a claim for universality will lead more effectively towards a non-arrogant approach to knowledge and foster mutual respect among the participants.

Epistemologically in mathematics, I make a parallel argument concerning the need for more attention to dialectics, such as the one between generalization and distinction. Using the treatment of functions as a case study, I seek to demonstrate that mathematics has over-stressed generalization to the detriment of distinction and difference. As a result, a public perception is that mathematics is a set of generalizations, axioms and subsequent theorems, whose generalizability is their defining quality. The discipline thus loses the recognition that contrast to the particular case is essential for an appreciation of the generalization. What is sought is the grandest generalization, the one least tainted by individual difference, and what emerges is a view of abstraction that heralds decontextualization and symbolic sterility, rather than a view of abstraction that recognizes the complexity in moving between the particular and the general to produce conceptual understanding. Along these lines, I will suggest that it is reflective abstraction, as a means of reflecting on the previous accomplishment and creating a genetic epistemological trail, that is the hallmark of knowing. It entails a synthesis of the evolution of the concept, including both distinction and generalization, and recognizing with each a loss and a gain. The knowledge we must seek is that which creates a lens for us to make sense of experience, not that which creates objects we can place on shelves in our mind's storeroom.

In juxtaposing a discussion of diversity and commonalty and a discussion of distinction and generalization, I am suggesting that there is a link between the elitism so evident in the practice of doing mathematics in the academy and the epistemology we legitimate in the portrayal of mathematics in teaching and learning. Inequality results from the suppression of legitimately viable diversity. Because this suppression is practiced regularly in our instructional practices, from kindergarten through post-secondary education in mathematics instruction, it is no wonder that these practices accumulate to act as a filter to participation in higher mathematics.

This, of course, begs the question of what is "legitimately viable diversity", and who determines this? I choose to respond to this question by providing an example of an exploration of functions which demonstrates a number of epistemological innovations demonstrated to us by our students, or found in the historical record. They are not typically a part of current curricula. These ideas will be very simple and accessible and will invite entry into important mathematics. So, the question is why are they not used in instruction? Perhaps it is due to a narrow view of mathematics, one which neglects context and privileges certain symbolically less accessible forms of representation. And, I will argue that in order to challenge this portrayal of mathematics, alternative frameworks need articulation; ones which are more democratic. In doing so, I will challenge current assumptions about abstraction and the role of concrete materials, and offer in its place an alternative theoretical description. As a result of this analysis, I will argue for the importance of a much more vigorous reform agenda at the secondary and post-secondary level.

The Context

In this paper, an example based on research on students' conceptions of functions is offered. The example will be cast within the use of a piece of software developed by my research group called *Function Probe©* and a problem from *Learning about Functions Through Problem Solving*.[3] The research on which the paper is based includes the design of the software since 1986, the use of the software in precalculus courses at Cornell University since 1990 (n = 150-200 per semester), a year-long study in a high school class in the Apple Classroom of Tomorrow in Columbus, three studies at the Alternative Community School in Ithaca and six doctoral dissertations (Rizzuti, 1991;

[3] This is available from Intellimation, Department Y4, 130 Cremora Drive, P.O. Box 1530, Santa Barbara, CA 93116-1530 1-800-346-8355.

Afamasaga-Fuata'i, 1992; Borba, 1993; Smith, 1993; Piliero, 1994; and Doerr, 1994).

In the history of mathematics, one sees two contrasting approaches to algebra which meet at the invention of calculus. They could be described as the algebra of change and the algebra of structure. The algebra of change was concerned with how to describe change as scientific enterprises entered the mainstream of mathematical thought. Describing motion is one example of this, and in it, variable is a description of variation. The algebra of structure, was more concerned with the solution to equations, the coding of arithmetic operations into generalizations with unknowns and the development of forms that led to solution types. This seems to be an older mathematics, with its roots in Indian and mid-Eastern civilizations. Its ties to arithmetic were strong and the variable was an unknown. Both developments of algebra had close connections to geometry. Our research group has been focusing on the algebra of change since my dissertation work on the history of calculus (Confrey, 1980) and incorporating in that study increasing referrals to geometric reasoning. Since the algebra of structure dominates in the schools in the United States, we have been challenging the emphasis on the manipulation of equations as the core of the secondary curriculum.

Our theoretical approach to functions involves four components: the use of contextual problems, the use of prototypical functions within functional families, multiple representations and transformations (Confrey & Smith, 1991). It should be stressed that the assumption we make is that tools mediate knowledge, and so, the tools one uses to explore this theoretical context have a significant impact on what is learned and is legitimated as knowledge. As a mediational influence, it is also the case that what is learned further influences one's use of the tool. Thus, some experience with the software would assist one in understanding the points in the paper. Since that isn't always possible, a frequent use of figures will be included.

The Example: A Ferris Wheel and the Trigonometric Functions

The conventional approach to functions in the United States defines a function as "a relation such that for each element of the domain, there is exactly one element of the range." For instance,

$f(x) = \sin(x)$ is typically introduced using the unit circle with a radius of one and a center on the Cartesian plane at (0,0). The $\sin(x)$ is defined as the function that maps the distance around the circumference of the circle from the point (1,0) in radians onto a vertical height off the axes, y, a magnitude.[4] Some prerequisite knowledge of right triangle trigonometry in defining sin as the ratio of the side opposite over the hypotenuse is assumed.

The conceptual hurdles implicit in this approach include students must be able to go from the unit circle where (x, y) represent the coordinates of points on the unit circle, to representing a function, $(x, f(x))$ where x is an input in radians and y is the same series of values as before, but now is related to the input of radians. Secondly, students must go from being able to calculate the $\sin(x)$ for special triangles (30-60-90) and the (45-45-90) to be able to imagine the $\sin(x)$ for all values [this is a transition from a point-wise to an across time view of functions, (Monk, 1989)]. Thirdly, using this approach, it is difficult for students to "see" the rate of change of the trigonometric functions, and as a result they tend to expect it to be linear (or proportional to the size of the angle as described between 0 and 90). Finally, this approach does not really encourage students to see $\sin(x)$ and $\cos(x)$ as tools for analyzing circular motion into its vertical and horizontal components.

An alternative approach which we find leads to better student invention and comprehension is to present the student with a problem which draws on their experiences and "creates the need" for the idea (Confrey, 1993b, 1994a). The problem we use is widely available and reads as follows:

Imagine you are riding on a Ferris Wheel that is 20 meters in diameter. At its lowest point, it is 2 meters off the ground. The Ferris Wheel makes one revolution every 24 seconds. You board the Ferris Wheel at the level of its hub or center and begin your trip traveling upwards. Your goal is to represent your height off the ground as function of time in several of the representations available on Function Probe© (See Figure 1).

4 Euler introduced radians to link trigonometric and exponential functions into a single family via the complex numbers(Euler, 1988). Since modern tools and textbooks use radians as the inputs to the sin function we are forced to introduce this symbolism prematurely and it is poorly understood by teachers and students alike.

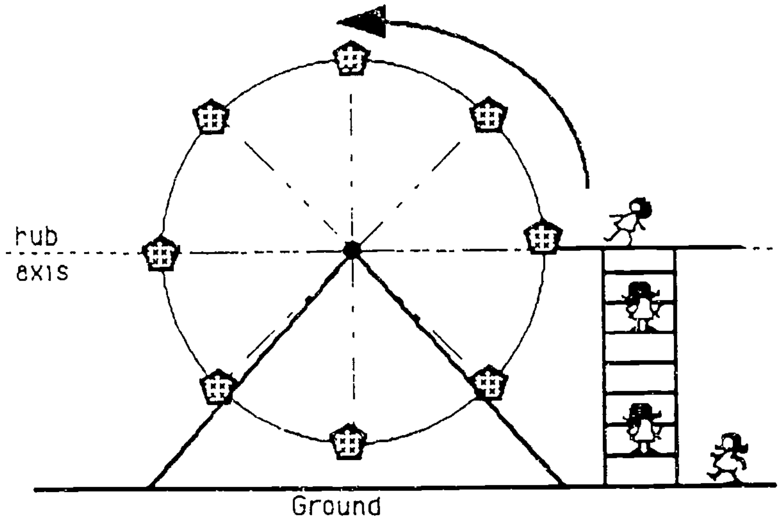


Figure 1. The Ferris wheel

I will describe a typical scenario of actions taken by a student who has been working with Function Probe for about two months. In the scenario that I will describe, the student enters the study of trigonometric functions having studied right triangle trigonometry, and having used the software to study linear, absolute value, greatest integer, and quadratic functions. She used graphs, tables, equations and a calculator and has seen how transformations can be applied as: 1) horizontal and vertical translations, 2) horizontal and vertical stretches (dilations); and 3) horizontal and vertical reflections and reflections about $x = y$ to build inverses.

A typical student will begin the Ferris wheel problem by putting in the values of the cardinal cases. In doing so, they will build the following initial table (See Table 1). They will easily figure out how to segment the 24 second cycle. It is possible that some of them will generalize to say that G = H + 12.

t	d	H	G
time (seconds)	angle (degrees)	height off hub (meters)	height off ground (m.)
0.00	0.00	0.00	12.00
6.00	90.00	10.00	22.00
12.00	180.00	0.00	12.00
18.00	270.00	-10.00	2.00
24.00	360.00	0.00	12.00

Table 1. Initial table entries

Now, in order to help the students consider how to interpolate in the table, we asked them the question, "how might you empirically investigate how the height off the ground changes as a function of time?" To do this, we advocate for having the resources available for students to build or investigate the following device pictured in Figure 2. Consider the outer rim and the movement of a flashlight mounted on it. The flashlight is hung from a point on the rim of the circle such that the flashlight remains horizontal as the wheel turns. The wheel is rotated by a motor and moves at a constant speed. The flashlight projects an image on a vertical screen. Students watching this can explore the motion and rate of change of the beam. Two interesting things can be witnessed relatively easy. The flashlight beam does not move up and down at a constant speed, but appears to remain longer at the ends of the paths and to move quickly through the center. And if the beam has any diffusion of its light, the circumference of the projected light also varies from broad to narrow. It too can be seen to stay longer at the broadest and narrowest periods and to move quickly through the intermediate stages.

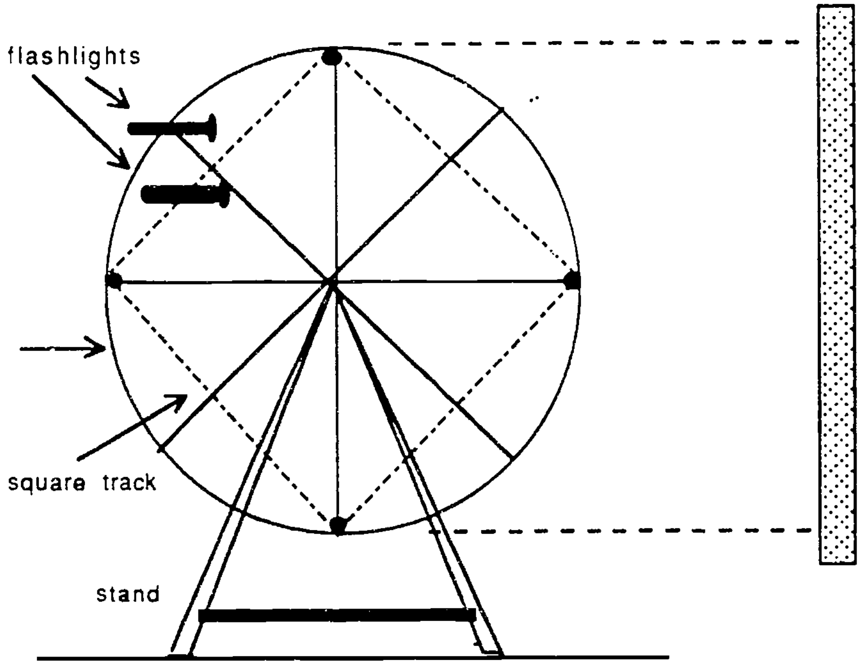


Figure 2. Diagram of a dual mechanical device for displaying motion on a screen: A wheel and a square track

This device allows students to relate what they see to their own experiences riding a Ferris wheel. They report apparently contradictory experiences; the Ferris wheel, they claim, moves at a constant rate. Yet the Ferris wheel feels faster and slower at different times. Is this just drama? Or, is it possible that they are feeling multiple rates: the rate of circular motion which is constant, the rate of motion forward and backward, and the rate of motion relative to the ground; note the the last two vary. The ground comes closer and goes away faster as the Ferris wheel passes through the height of the hub off the ground; the height relative to the ground changes most slowly at the bottom and the top. When going over the top, one can feel as if one will be flung forward out of one's chair. Is this experience mirrored when passing through the bottom?

It is my contention that such design-related, grounded activities are essential for students at all levels of mathematics. People erroneously believe that "manipulatives" are for children and that experts need only to think "abstractly." I will discuss the issue more extensively in the last section of the paper, but consider the following "variations" on the Ferris wheel as a device. Could you design a device that would move up and down proportionately to the time, but which was not simply a flashlight moving up and down a vertical shaft? These devices would produce heights off the ground that were proportionate to the constant rate of movement along the shape. Movement around the inscribed square on Figure 2 via a track achieves such a goal. Having students experience its contrast to the circular motion is a fruitful activity. Is this the same as saying that the height on the curve is proportional to its corresponding angle? If the answer is no, could one produce a shape who height was proportional to the angle?

Questions such as these are generated by the use of motion and curve making devices. Historically such devices were critical in the development of the algebraic representation of curves, their rates of change, and the notion of function. They forged a critical link between geometry and algebra, one that the development of algebraic notation in functional form (i.e., $f(x) = ...$) is suppressing. Interesting enough, it was the study of these forms that led Descartes to create algebraic geometry. To demonstrate the suppression of diversity that has occurred over history, one need only consider that the Cartesian plane with its perpendicular axes was not always used by Descartes. The selection of axes was made to allow him to investigate a curve and he would freely use skewed axes. In such a case, it was similarity relations rather than the Pythagorean theorem that became his intellectual tool. Drawing devices produced, however, equations that were not easily solved for y (therefore not put into $f(x)$ form) and also not easily classified into our current set of families. However, in its relationships between mechanical linkages and windings and geometric analysis, we have a rich historical source that has been lost to mathematicians and students alike. See Dennis & Confrey (1994); Dennis, Smith & Confrey (1992); and Dennis (in progress) for further discussions.

t	d	H	G=H+12
time(seconds)	angle (degrees)	height off the hub axis (m)	height off ground (m)
0.00	0	0.0000	12.00
2.00	30	5.0000	17.00
3.00	45	7.0711	19.07
4.00	60	8.6603	20.66
6.00	90	10.0000	22.00
8.00	120	8.6603	20.66
9.00	135	7.0711	19.07
10.00	150	5.0000	17.00
12.00	180	-0.0000	12.00
14.00	210	-5.0000	7.00
15.00	225	-7.0711	4.93
16.00	240	-8.6603	3.34
18.00	270	-10.0000	2.00
20.00	300	-8.6603	3.34
21.00	315	-7.0711	4.93
22.00	330	-5.0000	7.00
24.00	360	0.0000	12.00

Table 2. Ferris wheel data for familiar angular values

Returning to the original problem, the student fills in the triangles she knows from right triangle trigonometry (see Table 2). She experiences difficulty going from degrees, which she knows from the angles in triangle trigonometry, to radians, her normed measure of arclength. Again she is faced with the question of proportionality, is the change in angles proportional to the changes in arclength in radians? She wants to answer yes, but is uncertain having mis-predicted proportionality to exist in relation to height and angle. She tries two approaches. She opens a new graph window and puts the following values in the table (see Figure 3). She gets a straight line with a constant slope and an equation of $r = (2\pi/360)$ a, where r is radians and a is the angle. Now, confident of the proportionality, she proceeds to the calculator to build a button. She first thinks of how to change 360 into 2π. This is done by dividing by 360 and multiplying by 2π.

Changing the input to 90, she reasons that since she needs 1/4 of 2π to get her desired value, she should divide 90 by 360. She then builds a button in Function Probe called j1 that is defined as ?/360 * 2π = . This button will change her degrees column into radians[5]

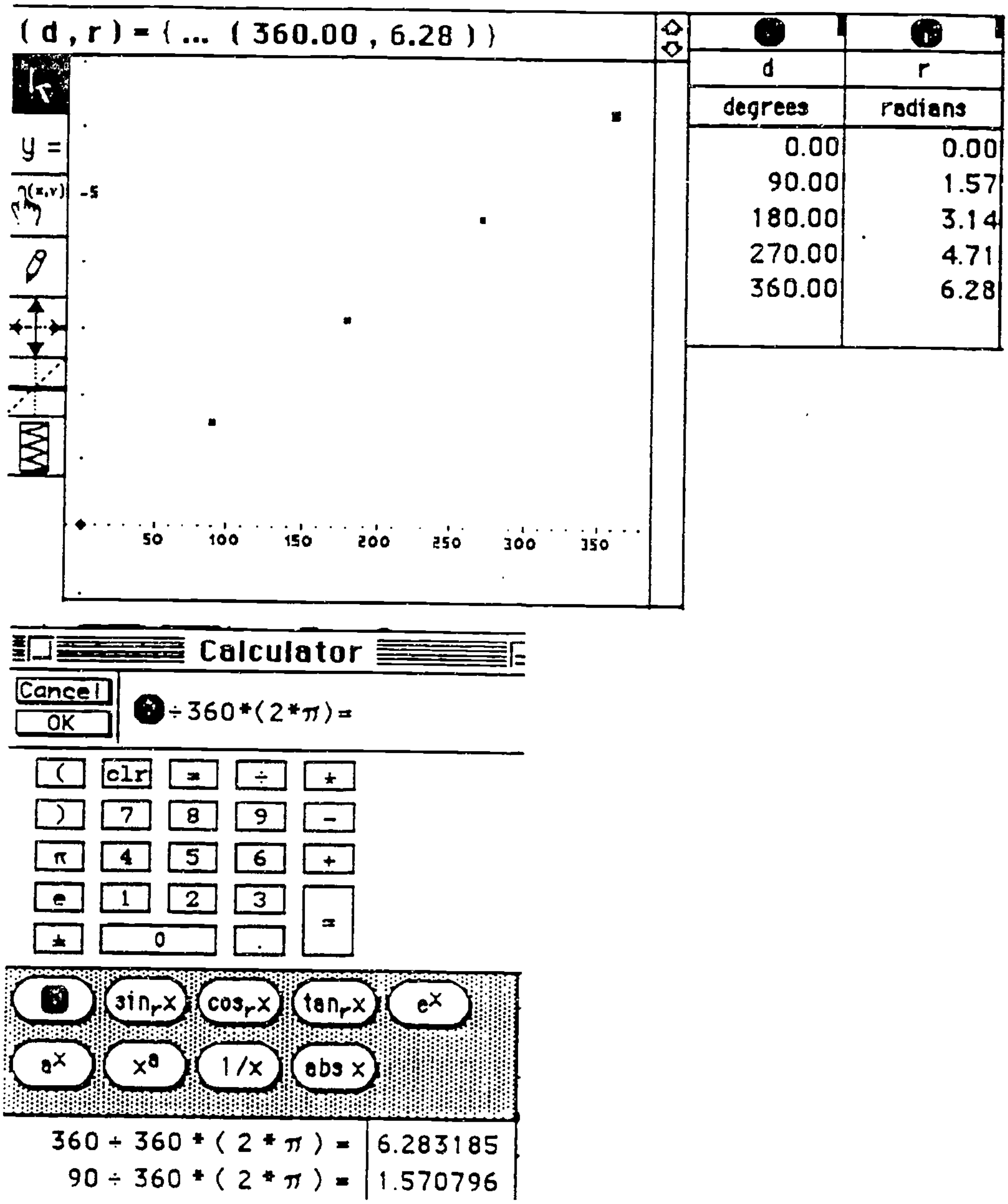


Figure 3

5 It is also possible in Function Probe to use a version of the sin function that takes degrees as its input. In the calculator, the buttons for sin(x) have a subscript of d or r ($\sin_d$ x or $\sin_r$x) indicating that they can take degrees or radians as input. To use these in the other windows, however, one has to build a button of the form jn, and then it can be called l inputting jn(x). We would suggest that such resources are an example of supporting student diversity and not very differe from allowing different bases for logs. In fact, the calculator in Function Probe also supports any base for logs making it two valued function and not limited to natural (base e) and common (base 10) logs.

She is then questioned about how to demonstrate the changes in the rate of change that she saw in the device by appealing to her data values in the table. Her goal at this point is also to find an equation to describe the height as a function of time so that she can extend her table to account for any time period she wishes. Seeing the changing rate is difficult in the table for the intervals of time are not equally spaced. She sends the data points to the graph to give her a first look. The shape of the graph convinces her that there is a varying rate, for she has studied linear functions and sees a difference between a line and the sawtooth curve that would have resulted if her [illegible] prediction was correct; the height was proportional to time during each six second interval.

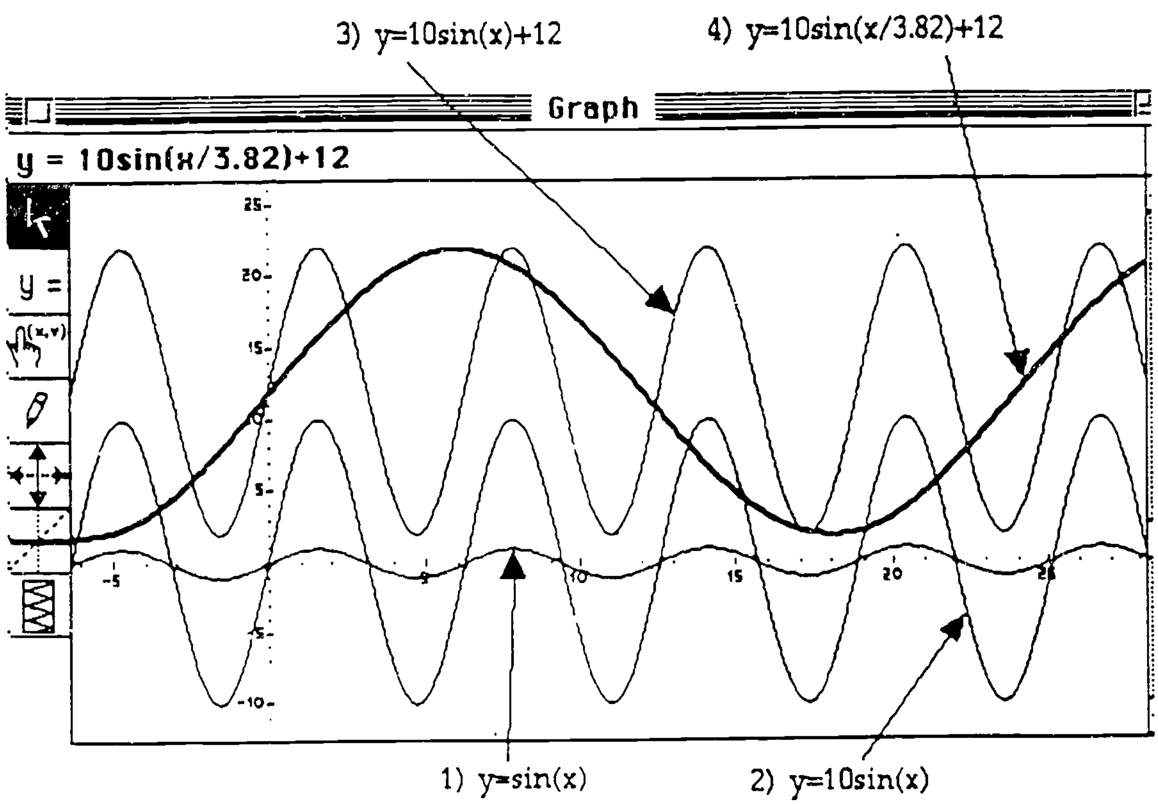


Figure 4

She now wants to tackle the problem of finding the equation so that she can generate y values for equal intervals on x. She recognizes the underlying prototype as $y = \sin(x)$. Since she is experienced in Function Probe, she decides to use a combination of visual and analytic approaches (See Figure 4). As with most students, she tends to see the amplitude as the easiest to understand and quickly inputs $y = 10 \sin(x)$. She knows she could have done this using the vertical stretch tool, by leaving the anchor line (the line of invariance) at $y = 0$ and stretching by 10, but she no longer needs to do it through graph manipulations and finds entering the equation to be a step towards meeting her goal. With some hesitation, she decides to translate the graph up by 12. This time she uses the translation tools so she can monitor the changes. Her attention is on the axis that horizontally divides the sin curve into two parts and changing it to a representation of the ground's distance from the curve. Now she faces the most challenging part of the problem for her, how to adjust the graph horizontally. She sees that her graph has a period of 2π and she wants a period of 24 seconds. She decides on a horizontal stretch since she wants the graph to have a longer period. She places her anchor line at $x = 0$ since her graph of $y = 10 \sin x + 12$ and the desired graph share a point there and she wants that link between the graphs to remain invariant. She knows that she doesn't understand horizontal stretches as well, so she works empirically and visually now, aiming for some further information that can help her reason about the equation.

When she stretches it, she gets the following equation: $y = 10 \sin (x/3.82) + 12$. She knows that it is a decimal approximation, but she hopes it can help her to figure out a more precise description. She recalls that pi is likely to be involved since she usually graphs trigonometric functions on a scale marked with pi. She also knows from her work with the greatest integer function that $y = \text{floor}(3x)$ produces steps of length 1/3 and $y = \text{floor}(x/3)$ produces steps of length 3. So she reasons if she wants a longer period, she ought to divide by the desired factor. She opens a new window to check this conjecture and plots $y = \sin 2x$ and its got a shorter period. She checks $y = \sin x/2$ and it has a longer period, that of 4π. She reasons that she wants a factor to divide by that changes her period from 2π to 24. So, she calculates what she gets by dividing 1 by 2π and multiplying by 24. This corresponds approximately to her stretch factor and she quickly inputs the equation and holds her breath until it passes through the desired set of points. She sighs in relief.[6]

The last question she has not yet resolved is how to demonstrate analytically that there is a changing rate of change. What's caused her problems is that the intervals for the time are uneven, so she can't calculate the rate over the same-sized intervals. She recognizes two choices, to go to the table and make two new columns, inputting her newly found equation, or to sample points from the graph. She chooses the first. From earlier in the course, she remembers that she used the difference command extensively to examine the rate of change of the quadratic and learned it produces an arithmetic sequence and a second application of it produces a constant difference. She uses the difference command on her trigonometric function, but all that is produced is another confusing set of values. She decides to send this to another graph just to see what it produces, but ΔG is between values, and cannot be plotted opposite t which she wants to use as her input. She goes to the table menu and nudges the column up putting the values directly opposite the starting values. She remembers doing this with the quadratic, but at this point she's not really sure of the argument for it. She sends it to her new graph window and sees something that looks roughly like another sin curve translated horizontally (Figure 5).

6 A second approach to this problem involves the use of the sampling tool. If she samples y = 10sin x +12 before the horizontal stretch in intervals of 1 and then stetches that sample and sends it to the graph, she will see that the same set of y values which haven't changed but the x values will now have intervals of 3.82. 1/3.82 = .26 which is her coeficient of horizontal stretch on her register.

t	d	H	G=H+12	w=ΔG
time(seconds)	angle (degrees)	height off the hub axis (m)	height off ground (m)	
0.00	0	0.0000	12.00	5.00
2.00	30	5 0000	17.00	2.07
3.00	45	7.0711	19.07	1.59
4.00	60	8.6603	20.66	1.34
6.00	90	10.0000	22.00	-1.34
8.00	120	8.6603	20.66	-1.59
9.00	135	7.0711	19.07	-2.07
10.00	150	5.0000	17.00	-5.00
12.00	180	-0.0000	12.00	-5.00
14.00	210	-5.0000	7.00	-2.07
15.00	225	-7.0711	4.93	-1.59

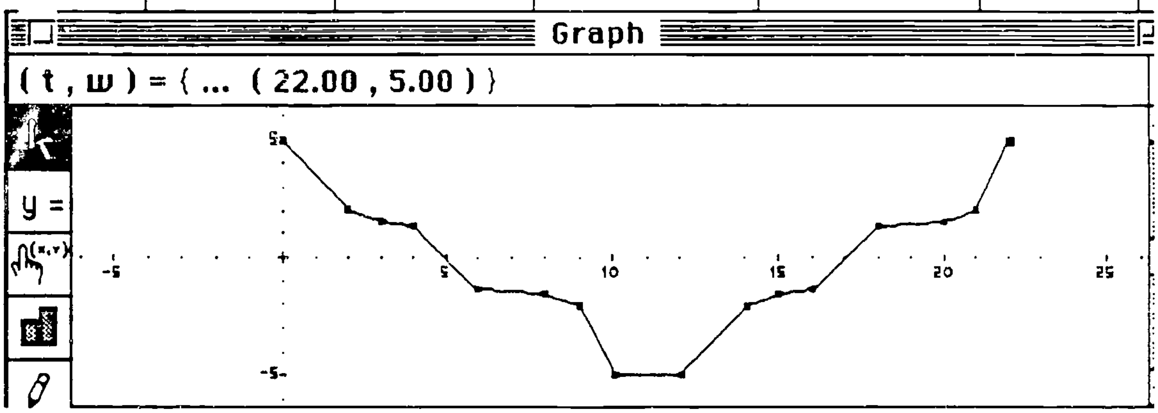


Figure 5

At this point, class discussion ensues and the scenario ends. In the next section of the paper I take one step back from the example and discuss how the use of context and multiple forms of representation enrich and broaden the conceptualization of "function" and how this, therefore, is an illustration of what I meant by the admission of "legitimately diverse points of view." In the final section of the paper I discuss how the dominant conception of academic mathematics inhibits reform due to its neglect of the role of context and tools and its narrow conception of abstraction. I offer, as well, the outlines of an alternative approach that can more effectively support reform at the secondary level and increase diversity in the participation in mathematics.

RETHINKING FUNCTIONS IN LIGHT OF STUDENTS' CONCEPTIONS AND HISTORICAL EVIDENCE OF DIVERSE APPROACHES

A question that arises from the research is what happens to the meaning of the term function when placed into a theoretical framework involving contexts, tools and the use of multiple representations. I would claim that a number of epistemological changes become necessary.

Changes Resulting from the Inclusion of Context. The use of the term, relation, in the definition of function needs examination. This term is really the heart of the definition, and one might notice that students are never asked to define a relation. Relation is treated as a kind of a priori term. A relation is, however, a built construct. Contexts can be useful in establishing a belief, a conviction on the part of students, that there is an underlying invariance that can be expressed, that a relation exists. For example, in the Ferris Wheel problem, the context gives meaning to the idea of periodicity quite quickly. The idea that the range is bounded between 0 and 22 meters is also discerned as part of the meaning of the term "relation." And, finally, establishing that for any point in time, an exact height off the ground can be calculated, and that the height will reflect the varying rate of change witnessed using the Ferris Wheel evolves from the context.

Contexts can also provide students with an experience of the action that creates the need for the application of mathematizing. The involvement of students in actions which are later transformed into mathematical operations, magnitudes and variables expresses a way in which context need not imply the use of everyday events and objects. Rather, context can imply students' engagement in activity that involves action and sensory experience. In our research, we have found that direct experience with balancing, bouncing, rotating, accelerating, or dropping is critical in establishing an effective use of context. Rate of change becomes particularly salient through context in that it is often felt in the action that is later "captured" by the function. In most of our research to date, we have relied on word problems, like the Ferris wheel to provide the context. More recently we have been experimenting with 1) curve drawing devices (Dennis, in progress); 2) demonstration devices (the Ferris wheel); 3) geometric constructions; 4) design projects; and 5) motion and other sensory detectors.

Contexts also add another dimension. Contexts balance the generalizations sought after in abstractions with the distinctions that enrich contexts. For example, although the general form $y = Af(Bx+C) + D$ provides view of transformations that can be applied across the families of functions, it is the contexts that really assist in helping students understand why in a quadratic function a horizontal stretch factor $(1/b)$ can be accomplished with a vertical stretch of b^2, or why in an exponential function, a vertical stretch is "equivalent" to a horizontal *translation*. In the Ferris Wheel problem, many students seem to see the vertical transformations of stretch, reflection and translations, as actions on fitting the sin function, and then see the horizontal actions as "changes in the scale on the x axis." I would suggest that context is a very useful aid in understanding the importance of these distinctions. Furthermore, I find that the students reason analogically comparing and contrasting contexts, and using problem names to describe and contrast new situations. This suggests that contexts allow the students to make better connections among the functional families

Changes from the Uses of Multiple Representations. There is much support in the literature for the use of multiple forms of representations (NCTM, 1989). Multiple representations are argued for the on the grounds that they allow for diverse approaches, provide students contrasting insights, and rely on different sensory impressions (Rubin, 1990). Our experience has been, however, that two critical issues are under-emphasized in the use of multiple representations First, the use of each individual representation needs to be rethought in light of the new tools and student conceptions. If one views the ultimate goal to be the presentation of the equation, the table or graph become secondary means to that end. Their individual integrity will remain underdeveloped and unexplored. In contrast, in a robust "epistemology of multiple representations" the difference between representations and their unique contributions will be recognized and strengthened. Each representation will be viewed as providing gains and losses of insight. This is a more fruitful way of working with multiple representations than to view them as contributing to the ascension towards one all-encompassing form, such as an equation.

Finally, when viewed this way, what becomes particularly important is the development of the story that weaves their uses together. In the software, Function Probe, a history is kept of the students' approaches. Recently, we have added a resource, Function Probe Recorder, which in addition to keeping a history of the use of the representations, allows students to capture a picture of any window at any point in time and to make a note about the process of problem solving (Haarer, in progress). By providing this resource, students have been able to share their different approaches productively and distinction as well as generalization is valued.

Each representation in Function Probe offers different insights on functions.

Table Relations: The use of the table in exploring functions has been of major importance in our research. Tables provide an entry to mathematical problems for many students. Data can be imported from physical tools or simulations or can be reasoned out by the students in relation to the context. Contrary to what is often described, I have found that students tend to select numeric values and formats and that this process is often invaluable in helping them to define the variables. Number patterns also help them to create operational connections. In our table, we have designed it specifically to promote the development of functions and as such it is the column rather than the cell that is the basic element. This differs from a spreadsheet. In building for column structure, the resources to explore rate of change (delta and ratio commands) support key insights into what it means to be a member of a functional family. Allowing for accumulation of a column provides precursors to integration ,just as rate of change promotes the transition to the derivative. This transition is strengthened by the ability to nudge a column up or down to allow students to treat an average sum or velocity as corresponding to a particular value in the domain. Interpolation is a key issue in the table in moving from the discrete to the continuous case and we allow a variety of ways to insert values. Finally, we built "link columns," so that two columns could be treated as related and then sorting or inserting values can be done on both at once. If two columns are linked without an equation, but instead by the action of filling each column, we call this "a covariation approach to functions."

Graphs: In the graph window, we have established a number of special characteristics.

Graphs allow students to work with the visualization of graph shape as a means to express functional relations. We have provided students a number of tools to support visualization. These include the sketching tool, and the three visual transformation tools. The visual transformations are designed to allow students to reflect around any vertical or horizontal line, to stretch from any vertical or horizontal line of invariance (the anchor line) and to experience these actions as reminiscent of the physical activity. The visual reasoning that develops from the use of these tools is in marked contrast to with the other representations. Rescaling on Function Probe is not done automatically for scaling and unitizing are an integral part of the understanding of functions. Thus, although scales can be saved, they must be specified in domain and range and the size of the units. Logarithmic scales are allowed on either axis and in any base. The graph window also provides three tools that aid in the transition from curve shape to numeric analysis. These include the bar graph, the point locator and the sampling tool. A slope tool is under design. These tools allow students to ground their examinations of accumulation and rate of change in the visual imagery.

Algebraic Notation is handled in both the graph and the table resources. After having withdrawn from an emphasis on algebraic notation in order to develop the other representations more fully, we are only now reconsidering the independent contribution of this symbolism to an understanding of functions. Algebraic notation provides a compact form of description that allows one to both input and act on individual values and to see a generalized form. It has become the defining factor in creating families of functions as one learns to distinguish the forms of equations. It allows one to apply arithmetic operations and rules (distributive, associative, commutative) and properties to the algebra of functions to decide on equivalence. It allows the variable to function as a description of variation across numbers and as a means to find an unknown. In future years, as we explore the use of Function Probe with middle school students, we hope to learn more about this representation's unique contributions.

Calculator: The calculator in Function Probe provides an interesting contrast to the other representations. Two characteristics have become prominent in its use. Firstly, it functions as a means to bridge numeric with functional reasoning. Students use it to get more accurate numbers. Secondly, it makes the functions act a tools in procedures. Students will carry out a procedure a few times and then build a button. This button captures the series of actions but then as it is graphed captures the sense of a function. By building the buttons, the students become more aware of the order of operations that can appear tacit in the algebraic symbolism.

With the calculator, sin and cosine become a linguistic primitive that is carried out by an underlying procedure and coded with a sign. Embedding the use of sin and cos in such a device can lead to their use as a command without necessarily understanding fully their underlying meanings. We witnessed this use in a study of an integrated math and physics class (Doerr, 1994) and describe such use as problematic only if the conceptual foundations are not developed over time. The calculator thus provides a linguistically rich representational from but must be used carefully to promote deep rather than surface competence.

Multiple Representations: These representational contributions then combine into what I have labeled *an epistemology of multiple representations*. By this term, I refer to the coordination and contrast among the different representations, the increasing sophistication of their use, and reflection on one's path in their use. When one combines multiple representations with context, one has the basis for a modeling approach to functions.

IMPLICATIONS FOR DIVERSITY IN STUDENT THINKING AND FOR REFORM AT THE SECONDARY LEVEL

In the United States, reform at the secondary and post-secondary levels has lagged behind reform at the elementary level. I wish to propose that the following four factors contribute to the resistance of the secondary and post-secondary program to change:

1) Technology and context are viewed as ancillary to mathematical thinking;
2) An excessive and narrow orientation towards abstraction alienates many students, causing them to fail;
3) Issues of equity and diversity have been viewed as secondary rather than as a primary

driving force behind reform; and

4) Constructivist instruction is interpreted as relevant only to young children.

Each of these assumptions secures the place of traditional teaching, and as such, slows down reform. In this final section of the paper, I would like to present the outlines of a theory of intellectual development that allows one to challenge each assumption, and by referring back to the Ferris Wheel problem demonstrate how such changes can lead to the improvement of mathematics education.

To view technology as ancillary to the development of mathematical thought is to isolate mathematics from much human and cultural activity. I would claim, instead that mathematics is a technology--its development suggests that it is inherently a tool to investigate human activities. Viewed this way, mathematics can be seen as a product of human intellectual activity. Though it may seek to capture patterns that we believe transcend human limitations, knowledge--that which we can claim to know--cannot achieve this transcendence; it is human knowledge, cast in terms of our understanding of the events, examined with our technologies, and communicated in our languages and forms of representation.

One approach I have found productive is to examine mathematics through a Vygotskian perspective. Vygotsky, in *Thought and Language*, analyzes conceptual development as the dialectic between the development of thought and the development of language. Thought, he argued, has its roots in the use of physical tools; and language has its roots in social interaction. These two components then interact to create the development of conceptual thinking. Analyzed this way, mathematics carries its character as both related to tools and language.

As demonstrated in the Ferris Wheel example, the mathematics of functions is influenced by the use of the computer software. New issues arise with the anchor line in stretching or with the difference command, and others become less problematic. The software tool creates the basis for communication. A language of stretches and shrinks, of fills and first differences, of building buttons and sampling points evolves around the use of the tool and influences the directions in which the students' proceed.

Vygotsky warned that in isolation from each other neither thought or language can become fully developed. He identified the "pseudoconcept" which occurs when a person learns to use a concept linguistically correctly, but lacks the development of thought to support it. He argued that the development of the pseudoconcept is a natural part of conceptual development, because language use can precede conceptual development, but warned that unless further development of thought takes place, conceptual development will cease. Further development involves the interplay between the spontaneous everyday rudiments of thought and the systematic and hierarchical aspects of scientific and analytic knowledge.

This analysis of the pseudoconcept is an apt warning to those of us in mathematics. Mathematics, when dislocated from its roots in the use of tools, can become the manipulation of abstract symbols. Excessive focus on the symbolism, without adequate attention to the development of students' spontaneous thought, can result in the development of pseudoconcepts which never ripen into concepts. Interestingly enough, Leibniz, the inventor of a powerful notation for calculus, recognized this danger. In creating his notation system, he sought to transform calculus into "mere child's play" and to allow it to "be done in the blink of an eye." In doing so, he created a notation that allows calculus to be taught to students who lack an understanding of the ideas that spawned it. Near the end of his life, Leibniz realized that in creating this remarkable notation system, he had contributed inadvertently to the decline of others' intellectual activity. In 1714 he wrote a lament, "One of the noblest inventions of our time has been a new kind of mathematical analysis known as the differential calculus, but while its substance has been adequately explained, its sources and original motivation have not been made public" (Child, 1920, p. 22). Ironically his own notational system is implicated by this lament.

The second component of resistance to reform, an excessive and narrow orientation towards abstraction. This entails a focus on language development without adequate attention to the development of thought. Vygotsky's dialectic requires the interplay between thought and language. Unfortunately, people have interpreted abstraction as the development of ideas increasingly distant from concrete, everyday, kinesthetic, or perceptual experience. Thus, symbolic algebraic presentation is considered more abstract, while graphs and tables are viewed as less abstract. Knowledge development is described as a spiral away from basic everyday experience into a world of abstract objects unrelated to activity. It is described as a spiral from process to object, to process, to object, etc. until one arrives in a mathematical reality where everything is abstraction (Sfard, 1994; Artique, 1992). (See Confrey, 1993c, for a criticism of this approach)

In contrast, I would argue that sophisticated mathematical thought does not entail leaving the realm of everyday activity but in forming rich connections with it. These connections may be in the form of actions, such as captured by the transformational tools, stretching, reflecting, translating, and not just in terms of physical objects, but they are no less connected to everyday activity. To describe this revised view of mathematical knowledge, I discuss the need for a dialectic between "grounded activity" and "systematic inquiry." Based in a revision of Vygotsky's thought and language dialectic, grounded activity includes the use of contexts, activity and tools to get a feel for an idea. Far from being just intuitive, this environment is complex, anchored in goal-directed activity and tangible results. Competence in it may be likened to craftsmanship or design expertise. Movement within grounded activity is based on example, experience, rules of thumb, and knowledge of one's resources. It can be likened to Donald Schon's descriptions of "thinking in action." Systematic inquiry, in contrast, describes ways of codifying that activity, of creating languages and symbol systems that allow one to move about logically and analytically, without reference back into the system of grounded activity. One can make predictions about the grounded activity because systematic inquiry entails creating an internally consistent logical system that models certain aspects of the grounded activity, a simplification or idealization. However, my claim is that robust intellectual development lies in the interplay, in the dialectic, between the two, not in alienation from grounded activity or in its portrayal as child-like, primitive or intellectually inferior.

The demonstration with the Ferris Wheel is an example of grounded activity. In watching the beam of light change, and relating that to the movement of the Ferris wheel, students gain experience with functions, trigonometric functions in this case, as tools for producing or analyzing movement. They see how one can transform circular motion into vertical motion in a plane and learn to relate the sine function as a mathematical description of the motion. Historically, the term "function" was originally defined as a tool. Leibniz described six functions as tools for examining a curve (Dennis & Confrey, 1994; Arnol'd, 1990). Seeing functions as tools anchors them as a form of grounded activity.

Using the dialectic between grounded activity and systematic inquiry as a guiding framework, one can use multiple representations to populate the dialectic. Some activities using different representations can be designed to be closer to grounded activity; others nearer systematic inquiry. For instance, a graph produced through the sequence of measurements taken using the Ferris Wheel device will be closer to grounded activity than one produced by the student using the transformational tools on Function Probe. Moreover, using an equation such as $y = 10\sin(x/3.82) + 12$ to produce a graph would be more closely aligned with systematic inquiry. One would need to be able to interpret the symbolic code for which the meaning of the symbols, the implied operations and the order of operations are all conventionally defined in a linguistic system. *The essential claim here is that all forms of activity, from those grounded in tangible actions to those systematized into a linguistic framework, contribute equally to mathematical understanding.*

When mathematics learning is idealized as a spiraling towards increasing planes of abstraction, and when it leaves the cultural activities of people behind, elitism becomes increasingly likely. The elite intellectual mathematician is portrayed as "freed" from the constraints of humanity and as transcending the impurities and imperfections of bodily demands, daily activities, cultural differences and political agendas. Although some mathematicians prefer to practice heavily abstract mathematics, and possess the linguistic facility to do so, many do not and draw extensively on metaphor, image, action and experience in carrying out their profession.

Mathematics, as cast by the academic world, is thus forged as an intellectual filter that creates and nurtures that view that only those with the most talent will survive. If one wanted to reinforce this view, then it is evident that one should continue to allow mathematics to remain inaccessible, cast it as a set of rituals into which only the elite can rightfully earn initiation and cleanse it of its cultural, and political character, so that any critique is nearly impossible.

Obviously, I do not believe that a conspiratorial group of academic mathematics is scheming to keep the field protected from the common person. To suggest this would be to suggest that mathematicians act intentionally to secure the elitism of the field, that they hide all traces of controversy, bias and imperfection. To a practicing mathematician it is clear that this would be caricature. However, it does not seem too farfetched to point out that the academic mathematics that dominates instruction has been practiced by an intellectual elite, and participation in that elite has been severely limited by gender and race. And, thus, it seems plausible that an intellectual drift in the directions described could have occurred as a result of the modes of operation of the academy. The claim is that the accumulation of a whole series of societal structures, from commercials to classroom practices, from tests to college entrance requirements, from funding to publication and tenure policies, from intellectual mentors to newly minted doctoral students, makes the system work. Changing this would require a major realignment.

More importantly, if this portrayal is accurate, what remedy would alleviate the problem? It is the purpose of this paper to suggest that if one puts the equity issue, the question "what would it take to achieve equitable access to the field?" first, a difference set of reform activities can emerge. One major contribution would be to acknowledge a broader set of activities as a part of mathematical activity, thus enfranchising more people to decide what should be a part of mathematics learning. A second goal would be to change the mathematics preparation from its orientation towards increasing abstraction to include a more diverse set of activities. The framework described in this paper, from grounded activity to systematic inquiry, is designed to do this.

In making the claim that mathematics and its instruction need to be revised to incorporate a dialectic between grounded activity and systematic inquiry, I am arguing that one can specify a view of mathematics that will support more equitable and widespread access, while maintaining curricular rigor. Such a view can be seen as <u>more</u> rigorous, in that it would demand multiple forms of argument across representations and would require one to seek and describe the genesis and evolution of an idea and its relationship to experience.

One can see how the Ferris Wheel example offers new challenges to us in the diverse realms of visualization, dynamic representation, the design of tools and so on. This is no intellectual desert, but a provocative and fruitful arena of challenge.

The final issue raised was that another obstruction to change at the secondary and post-secondary level is the belief that constructivist approaches apply only to young children.[7] Arguing for the dialectic between grounded activity and systematic inquiry challenges this assumption, in that the constructivist commitment to the role of activity and physical or kinesthetic experience is relevant. But this alone is not a sufficient to prove the relevance of the constructivist program to secondary reform. Constructivism also requires one to acknowledge the developmental progressions in learning. This requires one to seek multiple paths by which students can move towards increasingly sophisticated thinking. Viewing mathematics within a dialectic relationship between grounded activity and systematic inquiry does not imply that children must proceed from grounded activity to systematic inquiry. As studies in early childhood demonstrate, children from their birth are embedded in a world filled with expression, language, and activities and objects. They need not be limited to playing with objects before language acquisition, for they are already expressing their views. And they benefit tremendously from the exposure to language, to reading, to song before they ever can respond verbally to it.

Thus, a recognition of the role of constructivism entails the recognition that children's models

[7] In this discussion, I am ignoring the tensions between the Vygotskian and constructivist philosophies. I have written a number of articles on this question (Confrey, 1993a) and in the last, I present a means of unifying the theories.

may not mirror those of adults. These models must be assessed within a child's framework. As stated in the beginning of this paper, learning to see children's mathematics is a challenging and provocative activity and it can go a long way to overcoming the arrogance of adult mathematicians and mathematics e ucators. Children's perceptiveness is markedly underestimated in mathematics due to our own tendency to recognize linguistic forms but ignore their fundamental roots. Thus, constructivism also requires one to recognizes the epistemological significance of student invention and to reexamine and challenge one's own ideas in light of these inventions.

Starting to engage in intensive reform at the secondary and post-secondary level is essential if a more equitable set of participation patterns in mathematics is envisioned. However, that reform must include a deep reconceptualization of the issues raised in this paper: the role of context and technology, the reconstruction of abstraction, the prioritizing of equity, and the admission of constructivist views of teaching and learning.

CONCLUSION

In this paper, I have suggested that in order to support diversity, we need to reconceptualize our understanding of mathematics. Mathematics, I suggested, can be better viewed as a dialectic between grounded activity and systematic activity. Doing so allows us to admit the use of tools, contexts and multiple representations into the mathematical enterprise. As we do so, we will relinquish the over-reliance on generalization and create a more balanced view of generalization and distinction. These changes, I suggest, are necessary if we are to challenge a narrow view of abstraction and in its place recognize the importance of context and multiple representations. Finally, I have argued that doing so will be necessary if we wish to see a more diverse community participate in the practice of mathematics.

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MATHEMATICAL OPEN-ENDED PROBLEMS IN CHINA

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In mathematics education in China, solving problems has always been the focus of the people's attention. But it usually deals, by tradition, with routine problems and closed exercises. Such kinds of problems usually have integrated conditions, a clear strategy and a certain conclusion. Since the whole world has attached great importance to "problem solving," China has not only recognized it is necessary to introduce open-ended problems, but has also come to know their effect in the training of students' creative ability since the 1980s. But in China, the syllabi greatly limit mathematics teaching in the primary and middle schools. Every year the university entrance examination is also assigned by the State Education Commission in a unified way. This makes it clear that open-ended mathematical problems should be combined with what is stated in the syllabus, instead of ingenious problems for games.

Here are some examples:

1. Try pointing out the common points of the two algebraic expressions:

$$12a^2b^2c \qquad\qquad 8a^3xy$$

2. If there are twelve numbers (from 1 to 12) on the face of a clock, try to add negative signs to some numbers and make the sum of the above numbers zero.
3. Which numerical values can replace factorable (in the limits of integer) "a" and "b" to make the following polynomial?

$$(1)\ x^2+ax-18 \qquad\qquad (2)\ x^2+7x+b$$

4. What kind of figures can the sections of a cube be?

The following are the features of the problems:

A. The content is interesting and the students are familiar with it so that the whole class can participate in the experiment.

B. There are varieties of keys to the problems. Different keys can be concluded by students at all levels (high and low).

C. As they deal with non-routine problems, students can understand mathematical rules and their essence in the course of solving problems.

D. They are closely combined with the content mentioned in the syllabi.

The following is the teaching experiment on the second problem mentioned above.

1. Time: 9:00–9:45 a.m., June 15, 1993.
2. The experimental class: Class 3, Junior I, 47 students in it, Leidian Middle School, Deqing County, Zhejiang Province.
3. Procedure:

(1) *Introduction*: The teacher adds a negative sign at random to some number on the face of the clock:

$$(-12)+(-11)+(-10)+9+8+7+6+(-5)+4+3+2+1=2.$$

Since the key is not zero, the students are asked to readjust the signs in different ways in order to make the key zero.

(2) *Individualized learning*: The students are required to write out the keys on the paper in four minutes.

(3) *Collective discussion*: Conclude the work, with the help of the teacher.

A. As $1+2+3+\cdots+12=78$, the negative signs should be added before the numbers whose sum is 39.

B. The keys are antithetical; i.e., if the numbers (12,11,10,5,1) can be keys, other numbers (9,8,7,6,4,3,2) can also be keys.

C. Negative signs should be added to four numbers at least or eight numbers at most.

(4) *Individualized learning*: Write out the keys in four minutes after mastering the above patterns.

(5) *Varieties of exercises*:

A. If the odd numbers on the face of the clock are erased, only six numbers

(2,4,6,8,10,12) are left. Can the negative signs be added to some numbers to make the sum zero?

B. The nights and the days are shorter, on other celestial bodies, than the nights and days on the earth. Only nine numbers (1,2,...,9) are on the face of the clock. Can the signs be added to some of the numbers and make the sum zero?

(6) *Collective discussion*: Add the negative signs to (1,2,...,n) n numbers. Can we make the sum of the numbers zero?

4. Comment on the experimental lesson. In the same four minutes, each student can write out only 2.1 keys. But after summing up to regular patterns of problem solving, each one can write out 7.04 new keys. About 81% of the students believe that the lesson is interesting and helpful. Only 19% students believe it is tense, disorderly and unhelpful.

Opening Ceremony

Weifang Session
Shou Guang Hotel

Thursday, October 7, 1993

Professor Zhang:

My colleagues and friends:

After our two-day session in Shanghai and a visit to Qufu (the hometown of Confucius), enroute to Weifang, we now gather here in the lecture room of the Shou Guang Hotel, Shou Guang County, Weifang City, for the Weifang Session of our seminar. According to prior arrangements, a nationwide conference on mathematics education was held in Weifang yesterday. The sixty-two participants of that conference have joined us today.

Among them, there are many experts with a high reputation in China - for example, Zhong Shanji, Professor in Beijing Normal University. Let us give a heartfelt welcome to all of them.

Now, please allow me to introduce Professor Xue Maofang to you. He is not only a mathematics teacher in the Weifang Education College, but he is also an activist for reform in mathematics education in China. He established a fund, contributed to by his younger brother, Mr. Xue Mao Lin, a successful entrepreneur in the countryside. The fund is helping to establish many study programs in mathematics education, including the seminar and conference here. Tomorrow afternoon we will attend the inauguration of the Institute for Mathematics Education in Houzhen Town. It is the first private institute established for mathematics education in China. Professor Xue is the Deputy Director.

The goal of the nationwide conference that Professor Xue organized is to enable more Chinese mathematics educators to have opportunities to exchange ideas with mathematics educators from the United States and Japan. Just for this reason, we have the Weifang Session of the U.S. - Japan - China seminar.

It is my wish that both the seminar and the conference will be successful.

THE PLACE OF PROBLEM SOLVING IN US MATHEMATICS EDUCATION K-12 REFORM: A PRELIMINARY GLIMPSE[8]

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The purpose of this paper is to highlight issues about the position of problem solving in the current reform movement in mathematics education in the United States. However, a brief discussion of the context in which mathematics education reform is occurring seems an appropriate first step for this discussion. In addition, an overview of the reform effort is provided.

CONTEXT IN WHICH REFORM IS OCCURRING

Educational Policy in the U.S.: An Example of Diversity and Similarity

Control of educational policy in the United States is a function of local and state boards of education. There is no national curriculum for mathematics, and there are no uniform criteria for teacher certification. Funding for education comes from a variety of sources (e.g., income taxes, property taxes, sales taxes, and lotteries) and from various levels of government (i.e., local, state, and federal). As a result, each of the fifty states has its own education code, its own requirements for graduation from high school, its own process of adopting educational policy, and its own formula for funding education.

In spite of the autonomy of each state, there are many similarities that exist among the states' educational programs. This is particularly true in a discipline like mathematics which is often textbook driven--especially at the secondary level (grades 7-12). Other factors which tend to

8 Parts of this paper appeared in "Recognizing and Recording Reform in Mathematics Education: Focus on the NCTM Curriculum and Evaluation Standards for School Mathematics and Professional Standards for Teaching Mathematics" as part of the symposium, Multiple Perspectives on School-Based Reform of Mathematics, AERA Annual Meeting, Atlanta, GA, April 1993, and in *The Mathematics Teacher,* "Recognizing and Recording Reform in Mathematics: New Questions, Many Answers", J. Ferrini-Mundy and L. Johnson, March 1994. Material taken from project documenters' scenarios is indicated with an asterisk (*). Material from scenarios by: B. Whitley, T. Schram, T. Wood, J. Fisher, L. Moseley, G. Mills, L. Johnson, and P. Tinto. Preparation of this paper was supported by a grant from the Exxon Education Foundation, through the National Council of Teachers of Mathematics (NCTM). Any opinions expressed herein are those of the author and do not necessarily reflect the positions or policies of the Exxon Education Foundation or NCTM.

stabilize curricula among the states include content of the curriculum, tradition, standardized testing, and the limited time which teachers have to really get involved in change (Eisner, 1990). It is against this backdrop of diversity and similarity that we must view reform efforts in mathematics education.

A Professional Organization's Attempt at Reform in Mathematics Education

The National Council of Teachers of Mathematics (NCTM) is the most prominent professional organization for mathematics education in grades K-12 in the United States and Canada. Its more than 100,000 members are made up of classroom teachers, university mathematics educators, mathematics supervisors, and others involved in mathematics education. There are also institutional memberships, and many schools and universities are members. For seventy-five years, NCTM has served as a clearinghouse and resource center for all topics related to mathematics education. It publishes three monthly journals for each of the grade levels K-4, 5-8, and 9-12. Regional and national meetings are held for its members where new approaches to mathematics teaching and learning are introduced and where research is presented.

Reform efforts in K-12 mathematics had been piecemeal until NCTM saw the urgency and potential value, in the early eighties, in bringing together the thinking of the field into a vision about mathematics education reform. Broad based writing groups were formed to develop the two documents, *Curriculum and Evaluation Standards for School Mathematics* (1989) and the *Professional Standards for Teaching Mathematics* (1991). These documents are giving direction and meaning to the reform effort in mathematics education that is underway in the United States and Canada. The documents have been widely disseminated and discussed by the NCTM membership, and anecdotal evidence indicates that teachers of mathematics are seeking ways to enact the ideas contained in the *Standards* documents. These documents are also serving to inspire standards development in other disciplines. But there are a number of questions that are being raised as schools, districts, and states attempt to incorporate these *Standards* in changing their curriculum and pedagogy.

For classroom teachers, pressing questions include: What does it mean to implement the *Standards*? Are there examples of implementation that serve as models to be followed? Are my current classroom practices in accord with the *Standards?* Where is the curriculum that is consistent with the *Standards?* The documents also challenge policy makers to consider issues such as: Were the *Standards* developed to serve as a national curriculum in mathematics education for grades K-12? What are the financial requirements and the risks for making such changes in our schools? Can we implement parts of the *Standards* and ignore the rest? Will our teachers undertake such a sweeping change in content and teaching practices? What must we do to support teachers in these changes? Will students learn the mathematics they need to know? Can this be done quickly? How much will it cost? The general public is aware that the *Standards* documents exist, and now rightfully asks: Are the *Standards* being implemented? How is it working? Are students learning the mathematics they need to know? Researchers in mathematics education observing these efforts at reform are curious: What factors are enabling and hindering mathematics reform ? Can "transformative research" describe and explain the teaching and learning of mathematics occurring in times of change? How do schools move beyond general commitment to improved mathematics, toward specific strategies for change and innovation? What is sustaining and supporting mathematics reform? How do teachers, administrators, students, and parents associated with a school that is undertaking major shifts and innovation in mathematics react to, engage in, and feel about the process? What dilemmas, contradictions, and impediments to the reform effort do schools, teachers, and administrators face?

The mathematics education reform climate provides a rich and complex context for teachers, researchers, and learners of mathematics. There will be multiple perspectives, contexts, and interpretations for all of these issues. By deepening the discussion, we hope to advance efforts toward the improvement of mathematics teaching and learning.

The R³M Project

In an effort to address some of these questions, the NCTM Task Force on Monitoring the Effects of the *Standards* recommended in its final report (Schoen, Porter, & Gawronski, 1989) that NCTM "monitor their own and other activities designed to implement the *Standards* and to monitor (not conduct) a broader program of research and development" (p. 27). The Recognizing and Recording Reform in Mathematics Education (R^3M) Project began in 1992 as part of this monitoring work, with funding from the Exxon Education Foundation.

The goals of R^3M include:

- To measure the breadth and depth of knowledge about the *Standards* in various communities;
- To develop useful descriptions of teachers, classrooms, and children in settings where significant attempts at change in mathematics education, which seem to be consistent with the *Standards*, are underway;
- To describe the effects of this changed practice on classrooms and on children's learning of mathematics, in ways acceptable as evidence by teachers, policy makers, and the public;
- To increase understanding of the circumstances, forces, and situations in which change in the teaching and learning of mathematics occurs;
- To synthesize and disseminate insights and findings about contextual features that promote and hinder change in mathematics teaching and learning as envisioned in the *Standards.*

Current funding provides for collection of a modest amount of baseline data about the status of mathematics teaching and learning, called the *Landscape Scan* (Weiss, 1992), to be supplemented by synthesis of data available from other projects that provides a sense of the status of reform. The major focus of the project is to identify and study a series of interesting sites of reform, in an effort to learn about the change process and the interpretation of the ideas presented in the *Standards* documents in diverse contexts. This work involves selecting and visiting sites, and developing "documentaries" to tell the stories of the sites. The actual site visits, made by documenters representing classroom teachers, supervisors, and college and university teachers, began during the 1992-93 and 1993-94 school years. In most cases our sites are entire schools, with some instances of extension to full school districts. The context provided by the school district, as well as the particular examples found in individual classrooms, is important in all cases.

The R^3M project is still in its early stages. We are in the process of organizing and analyzing the very rich data developing from our site visits. As we progress on R^3M, we all are learning how important it is to recognize and highlight the complexity of significant change in mathematics teaching and learning. We hope that our project products can portray this complex context as we describe the efforts and accomplishments of school sites, struggling to bring about their visions of what school mathematics should be.

KEY ISSUES IN MATHEMATICS REFORM DISCUSSION

Images of Practice

Ball (1992) provides a helpful discussion of standards, informed by her own participation in the development of the NCTM *Professional Standards for Teaching Mathematics* (NCTM, 1991). She observes that "standards are intended to direct, but not determine, practice; to guide, but not prescribe, teaching" (p. 34). Porter (1989) advises that the best one should expect of standards is a "context of direction" for change. The *Standards* documents encourage teachers toward an unfamiliar and somewhat invisible version of practice. R^3M takes the position that multiple interpretations of the *Standards* are of great significance, and that these interpretations need to be described and shared.

Researchers and teachers both are calling for useful descriptions of practice, descriptions that provide images toward which they might aspire. Davis and Maher (1993) talk about the importance of enriching our collection of "assimilation paradigms" (p. 27). These assimilation paradigms might be construed as powerful, rich examples that can enlarge our wisdom about how to make change. Teachers are looking for examples of *Standards* pedagogy in which they "might move constructively in these [Standards-like] directions" (Boyer, 1990). Perhaps by providing practitioners with details of real and familiar situations, they might better see themselves participating in the reform process. The R^3M project hopes to develop such images.

We are crafting scenarios of examples that address issues such as: What happens when technology is regarded as central in a secondary school mathematics program? How does teachers' concern for students' knowledge of basic facts interact with pedagogy that is oriented toward inquiry and exploration? What results from collaboration with university faculty in mathematics reform? How are sites coping with standardized testing? What roles can be played by mathematics specialists? How do curriculum frameworks develop? As we continue to work with our data and to visit additional sites, we will enlarge this list of issues and will gain experience with what will be most compelling formats in which to share this information.

Interpretation or Implementation?

A view that *Standards* are intended to be *implemented* could lead to efforts which result in change at only a surface level. Reys (1992) describes a Parent-Teachers Association Meeting at his child's school where a parent asked the principal if the school was working with the *Standards.* The principal replied, "Yes, we did those last fall." "Doing them" meant devoting 20 minutes of a staff meeting to the document and distributing copies to all of the teachers. There is some danger in the current situation of falling into the change for change's sake that has characterized some past reform efforts. Fullan (1991) and Sarason (1991) both have warned about oversimplifying the change process and expecting that reform can result from piecemeal change. Mathematics classrooms can *appear* to be quite *Standards*- oriented, with calculators in evidence, students working in groups, manipulatives available, and interesting problems under discussion. One of the challenges we face in R^3M is to learn how to look beyond this evidence and gain some understanding, at a deeper level, of what is happening in these classrooms, and to articulate, as best we are able, how the teachers and students are experiencing these approaches.

Schroeder (1992) has expressed his dissatisfaction with the notion of *implementing* the *Standards.* He points out "Far from prescribing a plan for teachers to carry out, the standards portray teachers as active decision makers who are constantly monitoring their professional practice and frequently engaging in dialogue with other professionals." (p. 70.) Our preliminary finding in R^3M is that individual schools and groups of teachers are making conscious choices about the focus of their reform activities, often consciously or unconsciously well-suited to other contextual features, such as the nature of the school, community, or teaching staff. Within the focus they are developing interpretations of ideas proposed in the *Standards* and other reform resources. These interpretations are personalized, situated within the context of a site, and evolve as practice and experience informs the process. What we are seeing is better characterized as *interpretation*, than as implementation.

Validation or Catalyst?

An issue of interest for this project in particular is to better understand the role of the *Standards* documents in the reform process as it is unfolding in various sites. In many cases, it is possible that the *Standards* will serve as a validation and reinforcement for what a teacher or school district is already doing. This might be particularly true in sites that were committed to mathematics education reform prior to the release of the *Standards* documents. R3M is concerned with better understanding this phenomenon and learning how the existence of the documents will influence ongoing efforts. In other cases, the *Standards* seem to be functioning as a catalyst for reform and change. We are exploring whether the "stories of reform" in these two different contexts will turn out to have interesting differences.

METHODOLOGY FOR THE R3M PROJECT

The issue of how to look at the sites in a project of this sort is a complicated one. One approach might be to develop a checklist of *Standards*- like indicators and search for their occurrence. This could lead to a "*Standards* implementation score" for each sites. This line of thinking might appeal to certain publics interested in reform, but is inappropriate to the task of understanding the way that the documents are being interpreted. The perspective guiding this project is consistent with the philosophical intentions of the NCTM *Standards,* which seem to be based in constructivist assumptions. We recognized that those communities, schools, classes and teachers that we would visit were in a process of making sense of the *Standards,* or of more general reform discussion in mathematics, for themselves, and that this sense-making process would be visible through practice. A primary principle for R3M is that we hope to deepen our understanding of the site from the perspective of those involved in the setting; that is, to see and present the site's efforts from the site's point of view.

A set of five rather broad guiding themes were used to orient the documenters' visits. They emerge from the *Standards,* as well as from the various lines of thinking described earlier. We are trying to learn about:

1. The "mathematical vision" held by the people in the site.
2. The "pedagogical vision" held, relative to mathematics, by the people in the site.
3. How contextual features are influencing, both positively and negatively, the teachers' efforts to change their mathematics practice.
4. The way that the mathematical and pedagogical practices in his school are affecting students.
5. The evolution of the mathematics program in this school.

These broad guiding themes are explored through interviews with administrators, teachers, students, parents, and others; through observations of the classrooms and the school; and through examples of classroom materials and student work.

It Is Not a Contest, But a Study

Sites are eager to learn "how well they're doing" or "if the *Standards* are being implemented correctly." R^{3}M does not give seals of approval, judge correctness, or confirm that someone has found the right way. We are hoping to learn about many interpretations, to describe them in helpful ways, and then share them with practitioners who can use the stories to better predict and understand what they might encounter as they choose a particular interpretation. We looked for diversity among our selected sites. This diversity included the length of time change has been taking place, the cultural and geographical setting of the site, and varying degrees of commitment by staff to the change process. For example, some sites are just embarking on reconsideration of their mathematics programs, while others have been focused on mathematics for several years. Some sites are making incremental changes in an otherwise traditional mathematics program; others have initiated sweeping changes that encompass all aspects of mathematics learning and teaching. We selected sites because of their variation and not their uniformity.

Summary of Findings From Landscape Scan

In October of 1991, teachers of mathematics in grades K-12 from 121 schools in eleven states across the U.S. were surveyed (Weiss, 1992) about their attitudes toward teaching mathematics, their instructional practices, and their knowledge of the NCTM *Standards.* Basically, teachers at all levels are aware of the mathematics reform discussion, and express commitment to moving toward practice that values the tenets of reform. Their sense of how well prepared they are for this reform varies by grade level, as do their inclinations to use innovative methods and materials in their own classrooms.

PROBLEM SOLVING

"Mathematics as problem solving" is the first Standard in each of the three sections (K-4, 5-8, 9-12) of the NCTM *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). Although the discussion about the nature and position of problem solving is different in each of these three sections of the *Standards,* certain features are constant. At all three levels, there is an emphasis on: using problem-solving approaches to investigate and understand mathematical content; formulating problems from situations within and outside mathematics, developing and applying strategies to solve a wide variety of problems, verifying and interpreting results, and becoming confident in using mathematics.

This stance toward problem solving is in a sense the more modern version of a long-standing commitment, in the U.S. curriculum, toward making problem solving a central emphasis. Surveys done in the 1970s (Osborne & Kasten, 1980) revealed strong support for such conclusions as "problem solving should receive more emphasis in the school mathematics program during the coming decade." A number of interesting issues about problem solving were discussed in this context, including detailed descriptions of heuristics, problem solving as a means toward skills development, and problem solving as a means of introducing mathematical ideas. The issues about problem solving in 1980 are somewhat different from those under discussion today, at least as the NCTM *Standards* play a role in that discussion. A quick look at the 1980 NCTM *Yearbook* shows numerous references to Polya and his thinking, attention to various meanings of problem solving as a goal, process, and basic skill, heuristics, problem posing, textbook problems, problem solving strategies, and means for measuring problem solving ability. In today's reform, the issues relative to problem solving are framed in different language. The NCTM *Professional Standards for Teaching Mathematics* (1991) speak of worthwhile mathematical tasks, and these seem to be in the context in which problem solving, as well as development of understanding and skills, problem formulation, mathematical reasoning, and communication all occur. This broadened outlook on problem solving seems to raise certain new challenges, both in research and in practice.

While the role of the documenters in the R3M project was to be a non-evaluative one, it became clear on our site visits that teachers' perceptions of what it meant to implement the *Standards* varied widely. We saw some enthusiastic teachers who were fluent in mathematics reform language, and whose classroom practice was not necessarily consistent with their articulated views. Problem solving was not specified for documenters as a particular focus. Nonetheless, problem solving, in a general sense, has emerged in many ways as an important theme within many of the sites. Perhaps most striking in the preliminary data analysis is the diversity of interpretations of problem solving within the sites, the differing commitments to the various aspects of problem solving as indicated in the *Standards,* and the actual visibility of problem solving priority within the classrooms. In the Landscape Scan described previously, the issue of whether the *Standards* served as a catalyst or as a validation emerged as being related to problem solving. One interviewee noted: "There are many of the teachers who are changing [what they teach] because they want to add more problem solving focus to their curriculum. Some of these same teachers have not heard of the *Standards,* but they've heard that problem solving is important." The long-standing prominence of problem solving language in U.S. mathematics reform discussion is in some ways emerging as being synonymous with, or a shorthand description of, what is implied by the NCTM *Standards.*

"We are doing more problem solving," is a common response to the question, "What are you doing differently in your mathematics program?" Often the most frequently used terms are open to a variety of interpretations, and "problem solving" is no exception. A number of new issues seem to be arising in problem solving today: What is the place of the group? Is there a place for discussion in problem solving? What constitutes a "real" problem? What is the place of content organization within a problem-driven curriculum? What is the nature of feedback and evaluation? These are some of the questions we struggled with as we observed classroom instruction and interviewed teachers, students, and others instrumental in the school mathematics community. The following examples are intended to highlight some of the issues and interpretations we are seeing in the project relative to problem solving in the NCTM Standards context.

EXAMPLES ABOUT PROBLEM SOLVING IN PRACTICE

Problem Solving and the "Real" World:

The Standards and other reform documents make a case for "real world" problem solving. Our experience in this project is that this notion is interpreted many different ways. Although the Standards use much language to discuss meaningful mathematics, we have seen a number of situations where teachers' interpretation of "real world" seems to mean concrete in a more literal sense.

One teacher explained the importance of relating problems to the real world:

> I'm really trying to provide mathematics learning in a rich, problem-solving context and not have everything be so unreal. I'm trying to teach a lot with concrete materials before moving on to the abstract. I'm also trying to focus on the communication aspect of mathematics. . . So, I have them do a lot of writing and talking about what they are doing . . . I think I generally want them to learn something, to learn some kind of problem solving... *

She believes in "realness", defined as use of concrete materials, in communication, and in "rich context." Her actual classroom practice belies more of a mix of traditional computational work, and the problems chosen are not closely linked to compelling context in the students' worlds.

Then what follows are two "warm up" activities, placed on the overhead projector:*

1. Given the following sequence that continues in the pattern established, find the next two terms: 1, 3, 6, 10, 15, 21, . . .
2. In how many ways can 8 apples be placed in sacks with the same number of

apples in each sack?

Following the problem solving, the students were given a handout of problems dealing with addition and subtraction, and were directed to use colored tiles to solve the problems.

Other information indicated that this approach was somewhat new for the teacher involved, and represented a rather large shift in practice. What are the prospects, and what need might there be, for moving further into an approach to problem solving that pushes beyond these sorts of applications?

Problem-Driven Curriculum

> "Hear ye, hear ye hear ye! From this point on, I am Prince Navarro!" As the mathematics teacher passes out ribbon-enclosed decrees, he says, "See to it that you complete your tasks in a timely manner and with great pride, or I will see to it that you rot in the dungeon!" Each decree provides a diagram of a castle's exterior and the information that one gallon of paint can cover 550 sq. ft. Each group of three or four students is asked to find the total lateral surface area of the castle to be painted in celebration of the king's arrival; to advise Prince Navarro about how many cans of point to purchase; and to determine whether the prince has enough money to repair the roofs. *

This example from documenters' reports illustrates how one teacher introduced a problem-solving exploration to students in his class. Such an exploration is referred to as a *project*, in this secondary school. According to one of the mathematics department co-chairs:

> The process of incorporating projects into mathematics classes does not just mean giving students longer or more difficult problems. It is a change in the way students do mathematics and science. Since students work in groups to analyze a problem, brainstorm answers, break the problem into manageable pieces, and arrive at a solution to which all can agree, the process mimics many job situations. Since projects help students teach themselves and each other, the material learned becomes a part of the student, and this gives the student a stake in his own education. The fact that projects extend and apply the students' knowledge and require them to write clearly and explain their solution means that students remember what they learn and see new connections. The real-life applications of projects awaken many at-risk students to the fact that their education has some bearing on life outside the classroom.*

The view of problem solving that is embodied here is interesting in that the process focus is very salient. The teachers in this site have come together to develop projects as the centerpiece of their mathematics teaching, and this individual argues that problem solving skills such as analyzing, brainstorming, breaking down the problem, getting agreement on the solution, extending knowledge, explaining, and finding connections, are facilitated through the combination of projects and small group work. In this case, it would seem that the context has provided the teachers a way to think deeply about the elements of problem solving.

Place of Manipulatives in Problem Solving

Manipulatives are used in a variety of ways to assist students in problem solving. One documenter observed:

> In a geometry class, students were working cooperatively on a project which involved the application of geometric constructions. The task: Design a six-pack carton wrapper with a one-inch overlap. "This is to be done based on the measurements of one can." *

The manipulatives used included soda cans, English/metric rulers, scissors, calipers, calculators, planning sheets, wrapper paper, glue, tape, and markers. After the task was completed, students evaluated their own efforts.

We have seen many instances of problems posed to students with manipulative materials of various sorts as accompaniment. The actual role of the manipulative within the problem solving activity is not at all easily understood. For instance, in what context do manipulatives serve a role in helping children make sense of a problematic situation? In what context might they be superfluous? What challenges do teachers face as they incorporate these materials in their pedagogy? Can manipulatives function as "tools", not as "toys"? * How, and under what problem conditions, do manipulative materials support children's constructive mathematics activity or enable students to express their thinking to others?

Finding Problems Everywhere

In an elementary school, a teacher explained that mathematics has become an integral part of the entire school day, rather than a subject taught for forty-five minutes or so. One student told the documenters: "We always find a problem in something we do. Reading. [Our teacher] finds problems, and we have to solve it with a partner or something." Another student elaborated on the role of problems:

> Our teacher usually has something planned every time. She read a book yesterday, and there are problems in it, and she was reading; it was called *A Million Fish or Less*. It had problems in it, and we had to think of what the problem was and find out the answer. And now she made us think of a problem; and today you can work in a group; and today we have to tell her we came up with a problem; and we can give incomplete data so that people have to go search through the book and find it; and today we're going to ask kids, the class, the questions and whoever can stump the class will get a treat.*

How prevalent a by-product of a problem-oriented curricular and pedagogical approach is an inclination for students to see and find problems in their own worlds? Once this way of thinking about the world becomes part of the classroom norm, it is possible that the demands on the teacher's mathematical resourcefulness could become quite challenging.

Hard Choices: Problem Solving or Basic Skills?

A teacher says:

> I guess I'm a very traditional teacher. I've been teaching a long time, and I've seen a lot of different programs come and go; and I just find that in the last few years, when I think that it's good to be innovative and to offer the children a lot of different aspects and different ways of doing things; but we're finding a lot of basic skills are really slipping away....*

We found much evidence of teachers who are committed to experimenting in their pedagogy and to doing the best for their students, but who grappled seriously with the problem-solving "vs." basic skills dilemma. It is important to understand the ways in which teachers come to resolve and handle this dilemma and to write about and disseminate their resolution, because without providing this evidence, there is a strong possibility that problem solving could remain a Friday afternoon activity.

CONCLUSION

This project has the potential to represent a new role for a professional organization in which they ultimately will not have only been instrumental in the creation of national standards for curriculum, teachers, and students in mathematics education, but also would take some responsibility for describing the school interpretation of these standards. R^{3}M is intended to assist teachers of mathematics and local policy makers, in particular, through communication of the stories of the sites. Sharing useful information with these audiences, helping them understand what questions relative to the *Standards* are reasonable, and in learning more, as an organization, about what will count for these audiences, and adequate answers to the different questions they hold, are key elements of R^3M.

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ON MATHEMATICS EDUCATION IN JAPAN

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1. On the Excellence of Japanese Mathematics Education

In both of the first and second IEA research studies, the Japanese average scores were higher than those of any other country. In view of these remarkable results, Japan is presumed to have good mathematics education. However, I doubt that this is the case. It might be excellent in some respects, but I think that the higher achievement of Japanese students does not depend solely on mathematics education, but chiefly on the social conditions or on the general educational environment in our country.

The reasons are as follows:

- The Japanese educational level is higher in general, including in mathematics. For example, almost all Japanese (nearly 100%) are literate, and almost all are capable of reading newspapers.

- The situation is caused by the belief that education is the basis of the social development of Japan and of the prosperity of the individual, and by the social system in which people believe that those who have a higher academic background have a greater chance to live at a higher level. Thus, Japanese parents are eager to encourage their children towards a higher level of education; so, many Japanese parents have their children go to Juku, or some other additional education system other than school. This has also contributed to high achievement in the IEA research studies.

- The Japanese educational system has no repeaters. But many other countries have a system in which pupils repeat grades because they cannot learn some of the topics that are taught to other same-age pupils. In Japan, however, every pupil is expected to learn the same topics. I wonder whether, in other countries, repeaters might not have an opportunity to learn some of the topics that are taught at the grade level.

Beyond the reasons above, the quality of teachers is higher in Japan than in other countries. Japanese teachers are trained in universities, or in teachers colleges. Recently, the number of teachers that have a Master of Education degree is increasing. High quality teachers, of course, are expected to be able to teach lessons better.

Though Japanese students got high average scores in both the IEA studies, Japanese education has some problems. Our pupils achieve higher scores in calculation, but have lower achievement in solving problems. It is presumed that Japanese teachers place a lot of emphasis on doing calculations, but while the ability in calculation is important, the ability to solve problems is more important.

2. On the Use of Calculators and Computers in Learning Arithmetic

We Japanese have to reconsider our ideas about mathematics education in light of the technology that is now available. Calculators that display common fractions and carry out operations on common fractions are now available. Most are still expensive, but recently some calculators with these functions have been developed for elementary school use in Japan and are not so expensive. So, now is the time we have to consider whether or not we should use calculators in arithmetic. If we use them in the elementary schools, in everyday classes and also in examinations, then the aims or objectives of mathematics education might be changed.

One day I demonstrated a "fraction" calculator in a meeting of educational people. They were quite struck with the demonstration and said that we no longer need to give arithmetic lessons in school! Can this be true? This means that these people, and many others, think that the aim of arithmetic is to develop computational skills only. If so, the calculator could reduce the time for teaching arithmetic. Is this a better way for us than present practices? I don't think so. I think that there are at least two ways to cope with this problem.

First, we have to re-evaluate our idea of giving pupils paper-and-pencil computational skills so that they do not have difficulty when they have no calculator handy. (This is the position that we have taken up to now.) But is it reasonable not to use the calculator in these days when the technology is easily available? Second, we need to recommend the use of calculators in arithmetic classes and even on the examinations too. In such a case, we need not spend too much time to develop training skills. But we have to consider some other aspects of this situation.

First of all, we must re-evaluate how much time we devote to pupils' paper-and-pencil computational skills, and what degree of skill we demand. Even in the case of using the calculator, some degree of skill is still necessary. In any case, we must teach the meaning of an operation and its procedure in paper-and-pencil computations, and we might expect computational skill to some considerable degree. Secondly, we must consider the aims or objectives of arithmetic teaching in school. To date, we have put some emphasis on the skills of computation, and Japan had some success in pupils' achievement; but, we must have other more important purposes for teaching arithmetic. My opinion is that:

1) We have to place an emphasis on children understanding the paper-and-pencil calculation procedures, not simply on doing computation.
2) We have to develop the ability and attitude of children to make generalizations; to make guesses or hypotheses; to formulate and solve problems; to revise or improve findings; to make connections among things; and so on. These behaviors are not developed with the use of calculators alone.
3) We have to develop pupils' abilities to solve real-world problems using calculators, to use mathematics for understanding the real world, and to use mathematics to effectively express meaning and to solve problems. Problems from the environment are one example.

Japanese teachers might be placing emphasis on computational skills up to now, but we must now place more emphasis on the development of problem solving skills in the new age of technology use.

OBSERVATIONS ON CHINA'S MATHEMATICS EDUCATION AS INFLUENCED BY ITS TRADITIONAL CULTURE

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It is true that the mathematics curriculum should be designed on the basis of the research achievement in the fields of pedagogy, educational psychology, mathematics, etc., but the importance of the social background, the traditional culture and the ethno-mathematics of the country should not be neglected. Everyone, before going to school, has more or less been nurtured by the traditional culture and the ethno-mathematics of his or her own country. Therefore, it is plausible to argue that the modernization of mathematics courses will be accelerated if we can, in designing the curriculum, utilize the cream of traditional culture, make a rational connection with the ethno-mathematics, and absorb its advanced ideas which are nowadays still worthy of being popularized. This article attempts to observe, from the point of view of the Chinese traditional culture, the present mathematics education in China, focusing on its advantages and disadvantages and the origins thereof.

In talking about the influence of traditional culture, we should first of all clarify what we mean by "culture." Here we follow Clyd Kluckhohn's definition that the essence of a culture is composed of the notions and the value thereof handed down from generation to generation; i.e., of those formed in history and having survived the selection of history. Accordingly, the exploration of a culture is, in the final analysis, that of the evaluational tendency maintained by those people living in the culture. We have noted that China's traditional evaluation system is marked with four obvious tendencies: having favorable attitude towards tradition, authority, official rank and self-cultivation, which are still exerting great influence upon people's thinking and behavior, although they have only covered part of the evaluation system.

1. The influence of the favorable attitude towards tradition on China's mathematics education

The Chinese culture was originated in the Yellow River valley, and later on it gradually expanded down to the middle and lower reaches of the Yangtse River. The natural screen almost confined the living area of ancient China within a closed circle. Limited by undeveloped transportation, the ancient Hans were mainly in contact with the culture of those minorities around them. Compared with the culture of those minorities, the culture of the Central Plains at that time was well developed. Take ancient mathematics, for example, as far back as the later period of the primitive society, when our forefathers had already created the decimalism, and by the Spring and Autumn Period and the Warring States Period (770–221 B.C.) at the latest, "Suan Chou" (a bundle of bamboo chips created specially as counters) had been widely used in calculation. This advantageous number system and the calculating tool which was considered advanced at that time enabled Chinese traditional mathematics to get a series of achievements in calculation which could be ranked as first rate in the world. Mathematics, which was regarded as a nonessential skill in China, could achieve such great success, to say nothing of other fields. It was no wonder that the culture of the Hans was still venerated and upheld even during the time when China was governed by minorities. Not until the end of the Ming Dynasty and the beginning of the Qing Dynasty did China's traditional culture begin to encounter a great challenge. By this time, the Chinese had already formed their evaluational tendency towards the high respect for tradition and the faith in ancient conventions on the basis of the superiority and the sense of pride built up during the past several thousand years.

It is the classic work *Arithmetic in Nine Chapters* that has exerted the greatest influence upon the science of mathematics and its education in China. It has settled the traditional mathematics style that is very useful in application and calculation. One may ask why people today are still appealing for great attention to application in China's mathematics education, since the Chinese set great store by the tradition which lays stress on application. Before answering the question, we might as well take a scamper through *Arithmetic in Nine Chapters*. It illustrates solutions to more than two hundred questions about land area, exchange rate between goods, distribution by ratio, square root and cube root, solid volume (which is useful in building dams or piling cereals), profit and loss, line equation, and right triangle (Pythagorean theorem). Among them, some are out of date and have been eliminated while others, as to questions about length, area, volume, and road building, are still preserved and have occupied an important position in mathematics teaching in the schools. Therefore, we can argue that it is not the case that the Chinese nowadays no longer have high esteem for tradition. The fact is that now no new applied calculating methods have been introduced to replace the eliminated ones.

The Chinese had a long tradition to judge a person to be a learned man by his literacy works, and to be a hero by his martial skill. Consequently, mathematics was not taken seriously. *Arithmetic in Nine Chapters* was written in the form of "Question," "Answer" and "Algorithm," which can just meet the needs of those who merely want to apply mathematics instead of making a further study of its theoretic aspect. Since they can manage with "algorithm," it seems unnecessary for them to pay much attention to mathematics. In comparison with foreign students, the Chinese students are superior in calculating, and they would rather pay attention to familiarizing themselves with the skills that they have learned than to problem solving which may make it necessary for them to cultivate a kind of originality. This harmful tendency may be due to the Chinese mathematics style which is chiefly concentrated on the algorithm.

It cannot be denied that there is, to some extent, a connection between the monotone in China's mathematics curriculum designing and its out-dated content on the one hand, and the high respect for tradition on the other. The traditional school education in China always laid emphasis on unification, the unification of educational goal (to produce saints, worthies, gentlemen and officials), of educational contents (chiefly the classical works of Confucianism), and of teaching methodology (to read repeatedly and learn by heart). This traditional policy of education must be changed because it is no longer workable in a modern society in which school education is open to everyone.

Influenced by the favorable attitude towards tradition, China's present mathematics teaching still keeps the ancient mathematics tradition which was superior in calculation, and thereby students can have a good command of fundamental knowledge as well as skills, which is essential in developing their diverging thinking. We should not, however, over-emphasize the learning and training of fundamental knowledge and skills; otherwise we might choke the students' thinking and throttle their creative power. We should not, for instance, make our students spend too much time doing repeated exercises, as is now done in schools, such as in the consultation of the logarithmic table or the trigonometric table, the equality transformation, the calculation of the perimeter and the area of a plane figure, or of the surface area and the volume of a cubic figure, etc. Instead, we should make a kind of "spiral" arrangement which will gradually enable students to deal with them through calculus. In China's mathematics curriculum, too much time is also spent in the repeated training on trigonometric formulas and analytic geometry. Students would feel bored with this kind of repeated exercises because they can get the results by themselves as long as they are patient enough in dealing with them.

2. The influence of the favorable attitude towards authority on China's mathematics education

China's feudal society stood basically in the form of a unified social structure. To maintain the highly centralized feudal empire, it was necessary for the ruling class to adopt the concept of hierarchy which was like a pyramid, with the emperor as the highest authority. Consequently, a set of feudal ethics occupied a superior position in China's traditional education, which included the three cardinal guides (ruler guides subject, father guides son, and husband guides wife), and the five constant virtues (benevolence, righteousness, propriety, wisdom and fidelity). Long influenced by these feudal concepts, the Chinese have gradually formed their favorable attitude towards authority.

In the eye of the Oriental, the more one knows in a field, the more likely he ould become the authority in that field. It's of no importance whether he can put forward original ideas. This type of criterion of judgment results in the fact that examinations in China have always focused on remembering knowledge. Therefore, students would rather plunge themselves into a large number of exercises so as to pass examinations than try to draw inferences about other questions from one instance, or find new questions by changing the original conditions or conclusions. They would be satisfied with finding a way to the solution instead of seeking the best way.

Another reflection of the favorable attitude towards authority in education is the high respect for teachers' authority. Teachers often unconsciously put the whole class under their way. In class, students seldom have the chance to ask questions, or have discussions with their teachers and classmates. Even after they become college students, a few of them are able to select, compare and draw inferences when given a large number of materials or facts, let alone put forward original viewpoints.

The favorable attitude towards authority is advantageous in encouraging students to build up essential knowledge as the basis for the further comprehensive analysis and the cultivation of their creative ability. To inherit, however, is to develop. So the ancient admonition, "one should go from extensive study to intensive one and think while learning," remains instructive to mathematics education today. And teachers should have it as their duty to teach their students how to study by themselves, focusing on developing their self-confidence, initiative and independence.

3. The influence of the favorable attitude towards official rank on China's mathematics education

The patriarchy and its variant---hierarchy of authority is always very powerful and rigid in China. The several thousand years' tradition to worship patriarchy and hierarchy, and look down upon skills has made people come to the conclusion that, in the feudal society, only in the official rank could one achieve great success and attain in the end the object of becoming famous and bringing honor to one's ancestors.

How to secure an official position? Those who had the right to inherit could wait for inheritance, and those who were wealthy enough could buy their official positions. The ordinary people, however, had to study hard to follow the Confucian doctrine: "a good scholar will make an official." Children are usually too fond of play to concentrate themselves on study, so the Chinese parents have had a long tradition to urge their children to study with diligence. A typical example is the story of Mother Meng who removed her home three times so as to find a quiet place where her son could concentrate his attention to studies. Such notions as "a good scholar will make an official," "books contain wealth," etc., can still find their place in parents' minds nowadays, and the only difference is that they have broadened the original meaning of the term "Shi," that is, it not only means becoming an official, but also refers to leaving one's hometown in the countryside to become a permanent resident in a city, or working in an office. For these "official careers," people have to pass several entrance examinations. But at present the number of the students being enrolled in middle (secondary) schools or universities is limited - hence the fierce competition. In some cities like Shanghai, the competition for entering a university is not so fierce, but one would meet quite an intense competition if one wants to be enrolled in a famous university or major in a subject sought by many others. One may say that diligent study is out of date now. It is true, but don't you see that at the same time, many parents begin to invest in their children's preschool education? This indicates that, at first, parents all cherish the hope that their children can study well. They won't give up their hope unless after several years' study, it is impossible for their children to enter a higher school, and to continue them at school seems to have no foreseeable benefit. Some people now would like to work abroad, engage in business, or have a job in a city. Though they cannot thereby become "those who work with their brains" so as to be an official "to rule others," they can, instead, bring honor to their parents by earning a large sum of money to provide for their parents, just as the children of ordinary people did in the ancient time after they became officials. Therefore, it would be plausible to say that the intense popular interest in "going abroad," "engaging in business," or "working in a city" is also the reflection of the favorable attitude towards official rank.

Though it is significant that the favorable attitude towards official rank results in the parents' attaching great importance to their children's preschool education, it is necessary to point out that an inappropriate way of education may discourage their children's enthusiasm in their later studies. The Confucian doctrine "a good scholar will make an official" played a positive role at that time against the system of "enrolling officials by favoritism," but it is not advisable for the students today to study just for the purpose of obtaining an official position. If our headmasters and presidents of schools hold the same viewpoint, their schools are likely to unfairly focus on bringing up merely a few promising students at the expense of the ordinary students in the majority. The favorable attitude towards official rank is also one of the reasons for the slow development of the vocational technical education.

4. The influence of the favorable attitude towards self-cultivation on China's mathematics education

The traditional education in feudal China centered on morality. It laid stress on developing students' personality and ability of self-examination and self-restraint. This tradition has a great influence on the Chinese today. The Chinese, for example, would like to deal with things on their own. Most of them can steel their willpower and are used to studying behind a closed door. They do not have the habit to discuss with others, let alone start an argument.

According to the comparative research made by H. W. Stevenson and others, American teachers and parents usually relate the results in mathematics studies with students' intelligence, whereas the Asians have never attached importance to the intelligence difference among students while believing that the difference in mathematics achievements is connected with students' study attitudes and the time that they have spent in studies. As a matter of fact, the favorable attitude towards self-cultivation has shifted most of the responsibilities of study onto the students; i.e., they must rely on their own effort to get good results. Yang Hui, a Chinese scholar in the Southern Song Dynasty, worked out an "Outline for Arithmetic Exercises" for begi rs, in which he suggests that students should learn "the multiplication table" first, then turn to study multiplication and division which will last two months. After that they should learn addition, subtraction, and the transformation of multiplication and division into addition and subtraction, then spend ten da''s in calculating fractions and two months in reviewing, and finally learn extraction for a week and do exercises thereof for two months. Only with such fundamental knowledge can the students begin to learn *Arithmetic in Nine Chapters* and explore mathematical principles.

The mathematics teachers in China today usually believe that students can begin to drill on the subject after they have learned 60 or 70 percent of it. They can gradually understand it during the drilling. Take "the multiplication table" for example, the Chinese students are required to learn it in the primary schools, and to familiarize themselves with it to the point that they should be able to recite it with ease. Although most of them don't understand it while reciting, just like a little Buddhist monk reciting scriptures, these pithy formulas of the table can take root ever since in their mind, which they can benefit from all through their life. As to the meaning of the formulas, the students can gradually know it with the lapse of time. We don't think that it is sensible to abandon this way of learning; on the contrary, we should encourage it. We cannot, however, accept some ways of recitation. Take, for example, the principles of equivalent equation which are taught to the first-year students in middle schools. Though these principles are the basis of solving equations, what's the point of forcing the junior students to recite them while they have just met linear equations and know nothing about the solving procedure? None of them would be likely to multiply zero at both sides of the equation in solving a linear equation.

Every Chinese student knows the stories about "the Foolish Old Man who removed the mountains" and "an iron pestle which can be ground down to a needle." They firmly believe that quantitative change will finally lead to qualitative change. The precious Chinese spirit to study diligently and train hard is one of the important reasons why the Chinese competitors can often beat their rivals in international competitions. It's not sensible, however, to study mathematics through exercises alone. Constant reflection is necessary and much attention should be paid to study efficiency. Those who throw themselves into introspection and refuse to be in contact with others can hardly keep pace with the times.

As seen from the above observations, the Chinese traditional culture can have both a favorable and an unfavorable influence upon the development of today's education. How can we make the best use of the advantages and bypass the disadvantages, and how can we learn from the strong points of other countries, and then reform mathematics education in China and conform to the trend of the world and the spirit of the times? This is a very attractive and large topic to be explored by more people.

OUTLINE OF TEXTBOOK AUTHORIZATION IN JAPAN

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1. School Textbooks in Japan

In accordance with the provisions of the School Education Laws today, all elementary and secondary schools in Japan are required to use textbooks in the classroom teaching of each subject. As a principle, these textbooks must be authorized by the Minister of Education, Science and Culture. As a matter of fact, most of the textbooks currently used in schools are those published by commercial publishers and are authorized by the Minister.

Before World War II, textbooks compiled by the Government were used in elementary schools, and authorized textbooks were used in secondary schools. The current system of textbook authorization was adopted after the War and the Educational Reform. The "authorization" of textbooks means that, after examining "proposed" textbooks written and compiled by authors or publishers, the Minister approves those which are deemed suitable as textbooks for use in the schools. Authorization aims to encourage non-governmental bodies to exercise their own initiative and creativity in writing and compiling textbooks. It is also intended to ensure that schools use only appropriate textbooks.

2. Aims of Textbook Authorization

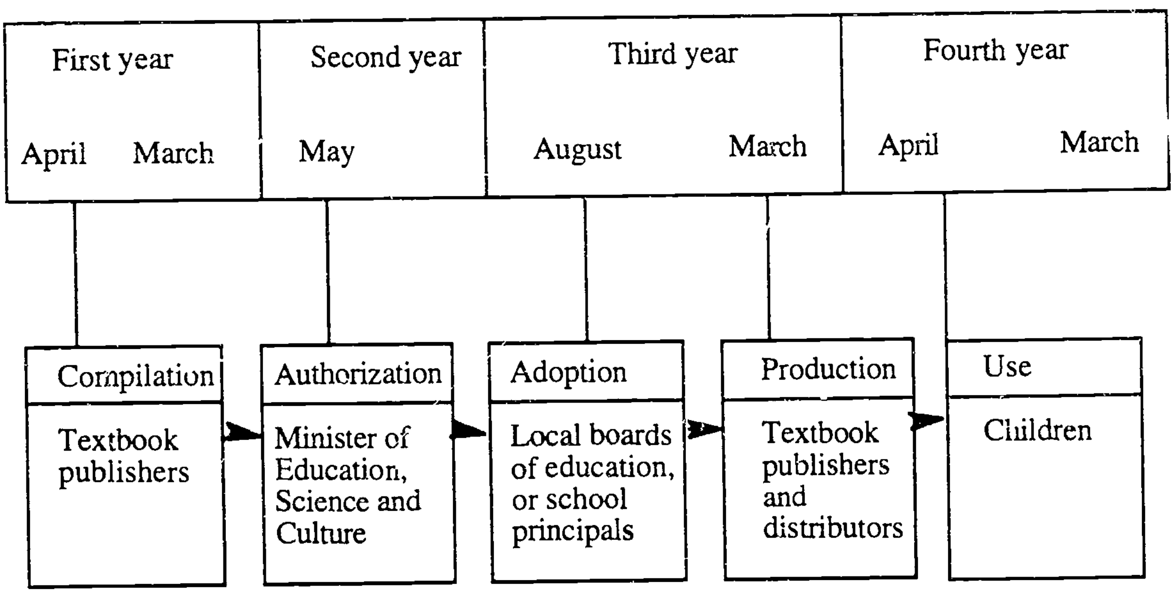
The Ministry sets the Course of Study (the national standards for the school curriculum) for each of the elementary, lower secondary, and upper secondary levels. The Ministry is responsible for the authorization of textbooks so that all textbooks may be compiled in accordance with the Course of Study. Through these activities, the Ministry aims to meet the demands for the improvement of educational standards throughout the country, for the provision of equal educational opportunity for all, for securing the appropriate content of teaching in the schools, and for ensuring neutrality in education.

According to the provision in the Constitution, that "compulsory education shall be free," the government has been supplying textbooks free of charge to all children in compulsory schools since 1963.

3. From Compilation to Distribution of Textbooks

It takes about four years from the start of the work of compiling a textbook to the time that it is distributed among school children. The existing systems for the compilation, authorization, adoption, production, and distribution of textbooks are outlined below. (Chart 1 illustrates the whole process.)

[Chart I] CYCLE OF TEXTBOOK AUTHORIZATION



(1) *Compilation of Textbooks*

School textbooks are written and compiled by non-governmental textbook publishers. They compile the textbooks, with their own initiative and creativity, in light of the provisions in the Course of Study for each school level. After compiling the books, they request the Minister of Education, Science and Culture to authorize these textbooks.

(2) *Authorization*

The Minister of Education, Science and Culture examines each of the "proposed textbooks" to judge whether it is appropriate for use in the schools.

(3) *Adoption*

The power to adopt textbooks to be used in local public schools rests with the local board of education that supervises these schools. On the other hand, the power to adopt textbooks for use in national or private schools rests with the principal of each of these schools.

(4) *Publication*

On adopting particular textbooks for different subjects, the local board of education or the principal of a national or private school submits to the Ministry the total number of copies of the respective textbooks that are needed for the municipality or for the school. The Minister then calculates the total of the demands for all textbooks, and then instructs individual publishers to print a specific number of copies of the respective textbooks.

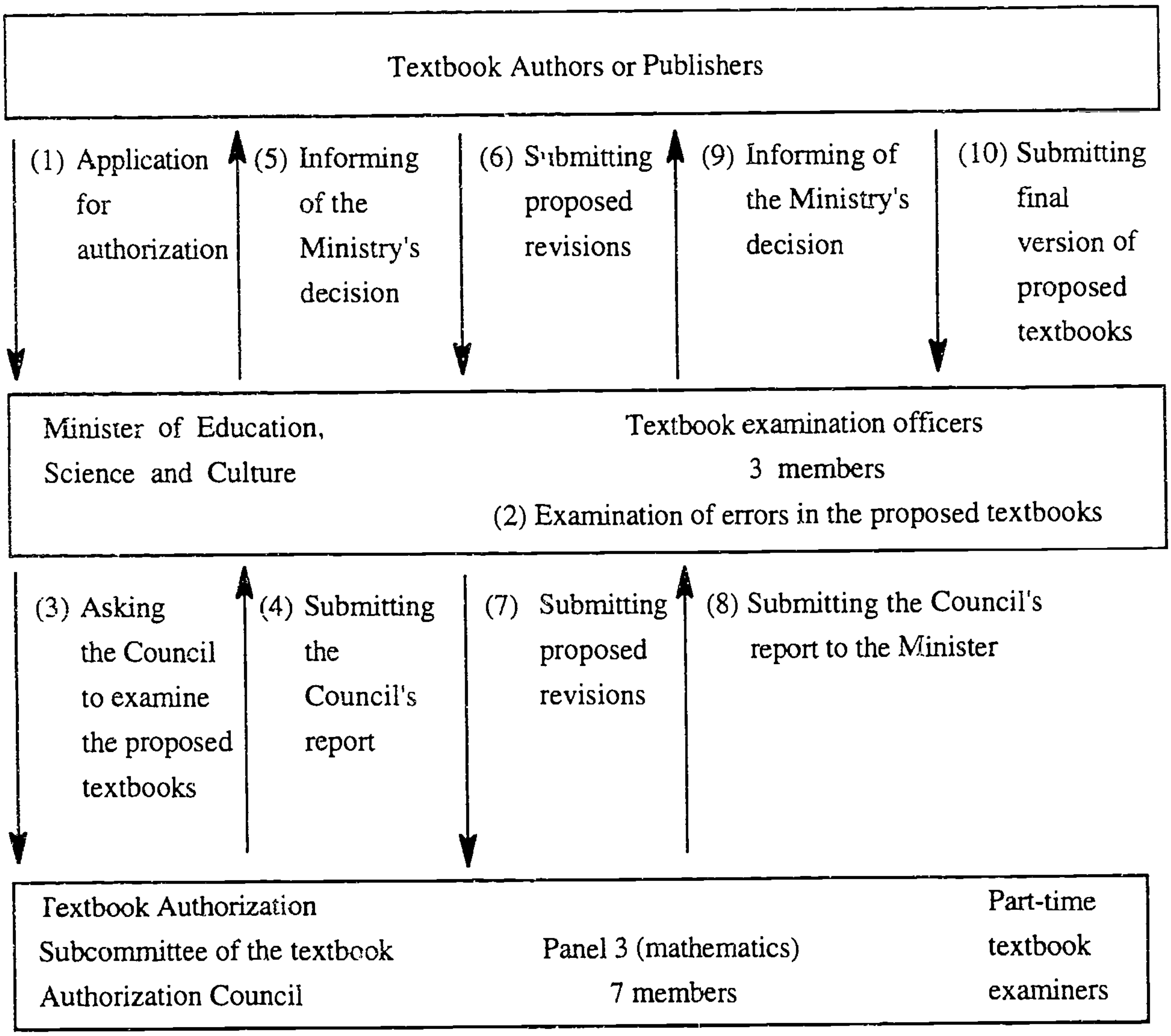
(5) *Use*

Textbooks are delivered to each school by the publishers and are given to all the children. At the present time, mathematics textbooks for compulsory schooling are published by six publishers (six textbooks); the texts for the upper secondary schools are published by twelve publishers (twenty three textbooks for each subject).

4. Procedures for Textbook Authorization

Chart II illustrates the procedures for textbook authorization.

[Chart II] PROCEDURES FOR TEXTBOOK AUTHORIZATION



On receiving applications from publishers for the authorization of their proposed textbooks, (1) the Minister instructs the textbook examination officers to examine the books for errors and mistakes in the content, and (2) then asks the Textbook Authorization Council to examine the appropriateness of each of the textbooks (3). The Council examines each of them in the light of the Criteria for the Examination of Textbooks laid down by the Ministry, and judges whether it is appropriate for use in the schools. The results of the Council's examination are reported to the Minister (4). On the basis of the report of the Council, the Minister approves or disapproves each textbook (5).

When the Council deems it appropriate to re-examine particular textbooks after the relevant revisions are made in their content, the Council defers its conclusions for these textbooks, and through the textbook examination officers, informs authors of its comments. Authors or publishers may revise the texts in accordance with the Council's comments and submit specific tables of revisions to the Minister (6). On receiving these tables, the Minister will ask the Council to examine the revised draft textbooks (7). The Council will report the results of its re-examination to the Minister (8). On the basis of the Council's report, the Minister will make the decision on the approval or disapproval of the revised drafts (9). On receiving the notice of approval from the Ministry for the textbook, the author or publisher will prepare the final version and submit a few sample copies of it to the Minister (10). After completing the whole textbook authorization process, the Ministry may make the proposed textbooks open to the public.

5. Criteria for Examining Textbooks

For the purpose of the examining whether each proposed textbook is appropriate for use in the elementary and secondary schools, the Ministry has set forth Criteria for the examination of textbooks for compulsory schooling and for the upper secondary schools.

The following Criteria are common to all subjects:

[Scope and Level of Textbooks]

(1) Every textbook for each subject should adequately deal with all items presented in the Course of Study, and should not deal with such items as are unnecessary.

(2) The content of every textbook should be relevant to the levels of the physical and psychological development of children.

[Selection, Construction, and Amount of the Content]

(1) Every textbook should choose the content in accordance with the Course of Study.

(2) Every textbook should be impartial with regard to political and religious aspects. It should be neither partial nor prejudiced to particular political parties or religious denominations, nor to their principles or beliefs.

(3) No textbook should carelessly present one-sided views.

(4) The amount and construction, and relations between the contents, should be appropriate.

[Accuracy of the Contents; and Appropriate Expressions]

(1) No textbook should contain wrong, inaccurate or inconsistent statements or expressions.

(2) No textbook should contain those expressions which may be beyond children's understanding, or which may be easily misunderstood by children.

(3) No textbook should contain inappropriate expressions.

Further, two special conditions for Mathematics are the following:

(1) No textbook should contain the content of higher grades with regard to the Course of Study.

(2) No textbook should incline to obtain knowledge of theorem or formula and skills of calculation.

Textbook examination officers of the Ministry (for Mathematics there are three persons) are selected from among those experts who have teaching experience at the university level or at other educational institutions. The Textbook Authorization Council consists of panels for each subject. The members of the mathematics panel (Panel 3) are four selected persons from professors at universities and three teachers at the elementary and secondary school levels. The part-time textbook examiners are appointed by the Minister. They are selected from professors at the university level and teachers at the elementary and secondary school levels.

The result of the examination by the textbook examination officers and the part-time textbook examiners is reported to the Council for its consideration. The Council makes the judgment as to whether each proposed textbook is suitable for use in the schools. After due consideration of the reports by the examiners and the results of the examination by the Council members themselves, and based on the recommendation of the Council, the Minister approves or disapproves each of the proposed textbooks. The various opinions of specialists and knowledgeable people are reflected to the consideration of the Council.

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Ministry of Education, Science and Culture

PROBLEM SOLVING AND LESSON REVIEWING --
A CAI teaching approach

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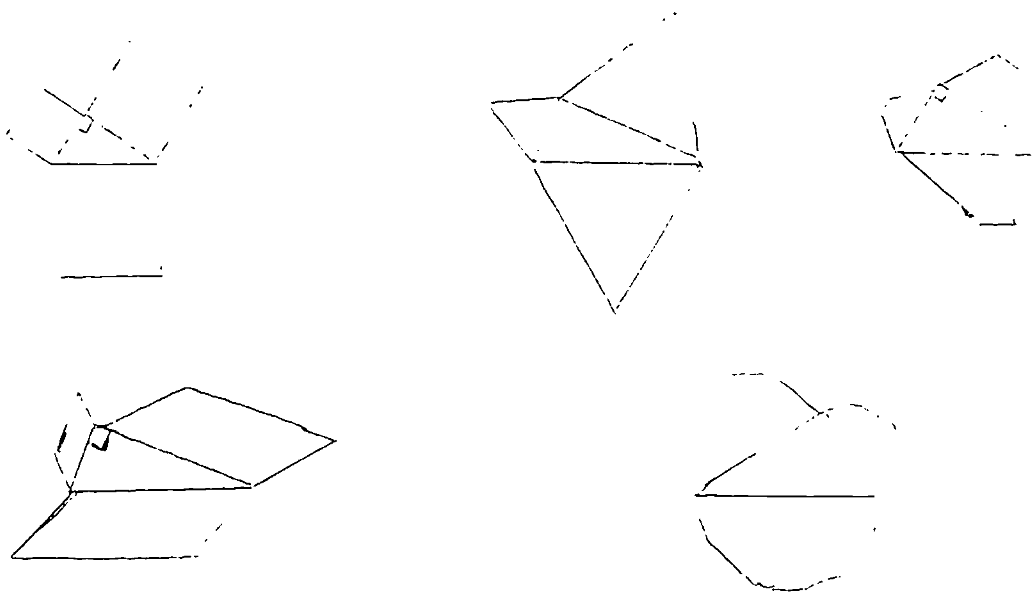
According to a traditional Chinese proverb that reviewing the past helps to gain new knowledge, combined with experiments and developments of school mathematics in CAI, we intend to investigate the relations between problem solving and lesson reviewing.

Recently, we developed the softwares CAI--1,2,3, that will be available to help students' review of the Pythagorean Theorem (Gou-Gu-Xuan Theorem). The software divided the reviewing process into three steps:

1. To prove the Pythagorean Theorem by means of the principle of "addition and deletion are compatible" as well as the congruent transformations.
2. To indicate various applications of the Pythagorean Theorem in the plane (space) configuration.
3. To transform the squares made by the three sides of a right triangle into other configurations, respectively, but without changing the area relationships of the three sides; i.e., the Gou-area plus Gu-area equal the Xuan-area.

For the purpose of reviewing the Pythagorean Theorem at the third step, we designed some problems by which pupils are able to gain new knowledge through the review process. For example, by using adequate rectangles, lozenges, parallelograms and trapezoids in place of the squares that appear in the Pythagorean Theorem, one can show that the Gou-Gu-Xuan area relation still holds. The software displays the changing process well.

Furthermore, we ask pupils to develop new patterns, such as Gou-Gu equilateral triangle, Gou-Gu general triangle, Gou-Gu semi-circle, Gou-Gu hoop, Gou-Gu pentagon and so on. In short, the essential difference between reviewing with problem solving and traditional reviewing is the Chinese proverb: "Reviewing the past helps to gain something new." Our work is an example in CAI only.



CHANGING THE ELEMENTARY MATHEMATICS CURRICULUM: OBSTACLES AND CHALLENGES[9]

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Problem-solving in the U.S. Elementary Curriculum: Some History

There has been an evolution in the definition of what an appropriate problem is for students ages 6-11. For many years in the U.S., problems for elementary age students consisted of disembodied numbers and operations. A typical textbook would consist of pages and pages of addition, subtraction, multiplication, or division problems. The underlying assumption seemed to be that if students did many, many problems of a certain type, they would understand how to do that kind of problem. Emphasis was on following one prescribed rote procedure for the indicated operation and getting the correct answer. Students were not expected to estimate or to use what they knew about the structure of the number system and the relationships among the numbers to help them solve the problems.

Textbooks have also typically included what we call "word problems" in the U.S.; for example:

Joan and Jere went to the beach. Joan found 19 shells and Jere found 12 shells. How many shells did they find altogether?

Some people identify such word problems as "problem solving." However, many of these word problems are nothing more than computation practice, just like the pages of numerical problems. When the central focus in the mathematics classroom is on using a prescribed procedure to find the correct answer, we find that students ignore the context in such word problems. They extract the numbers from the problem and use some operation on them—not necessarily the correct one. We have seen 8- and 9-year-olds who, rather than reading the problem, simply try every

9 The work reported in this paper, was supported in part by the National Science Foundation, Grant No. MDR 9050210. Opinions expressed are those of the author and not necessarily those of the foundation. This paper is based on an earlier paper prepared for the University of Chicago Mathematics Education Conference.

pos< ble operation with the numbers in the problem until they get an answer they think is reasonable. Rather than making sense of the problem by drawing a picture or making a model, they look for key words (such as "altogether" in the problem above) which will seem to indicate which operation is correct. Perhaps this is because they recognize that the problem is not a "real-life" application as it claims to be, but simply slightly disguised computation practice.

Educators who teach young children began to reassess the nature of the problems that young children encounter in their mathematics classes. Early childhood educators have long recognized that young children are engaged in the task of making sense of the world around them—sorting, classifying, naming, sequencing, comparing. A strong movement to make mathematics more "relevant" to students' own lives began to influence the nature of the problems given to students in the elementary grades.

Problems were formulated to be more "realistic" or to use "real data." However, when these efforts did not include reassessing the goals of mathematics education, even these efforts continued to generate what young students quickly recognized as phony mathematics. Consider, for example, a problem from a textbook in which students are given the length in miles of the world's seven longest rivers and are asked to find the average length of these rivers. While the data are "real" and the length of the rivers is potentially interesting to young students, the problem itself is silly. Why would we want to know the average length of these seven rivers? Students recognize that the problem is contrived—again in order to give them practice in demonstrating that they can use a particular algorithm.

The almost exclusive focus on learning rote procedures for operations with whole numbers, fractions, and decimals in the elementary grades led to several serious and unfortunate results. First, many students emerged from the elementary years with a dislike for mathematics that lasted into their adult lives. Their belief that they were not "good at math" led them to avoid taking mathematics courses beyond the minimum requirements. Mathematics became a barrier which filtered out far too many students from careers that require a good grasp of mathematics. Second, many students, including those who were successful in school mathematics, never developed an appreciation for the beauty, order, and pattern found in mathematics. They saw mathematics only as a way to solve individual problems, rather than as a *way of thinking* that involves making conjectures, finding patterns, examining the characteristics of mathematical objects, and using examples and counterexamples to test hypotheses about mathematical relationships. Third, students saw "school math" as a collection of arbitrary procedures disconnected from their own knowledge and experience. Students discarded their own sound intuitions and good number sense as they learned that their own thinking was not sought in the mathematics classroom.

In many elementary classrooms—and this is still true today—the place of the algorithm in the larger endeavor of doing mathematics became distorted. Memorization and use of particular algorithms became the whole aim and purpose in the mathematics classroom. Rather than developing a sound and deep understanding and appreciation of the number system, teachers and students believed that one was doing mathematics when reciting the chant,

$$\begin{array}{r} {}^{1} \\ 26 \\ +\underline{36} \\ 62 \end{array}$$

"6 and 6 is 12, put down the 2 and carry the 1; 1 and 2 is 3, and 3 is 6." It is well documented that when such algorithms are taught very early, without firm grounding in the structure of the number system, young children tend to focus on the procedures of manipulating individual numerals in a prescribed way. They no longer think about the *quantities* 26 and 36 and the relationships between them. Mistakes made through blind application of rote algorithms tend to go unnoticed by the students, and estimation is not used to predict or check results. When students learn to use algorithms in this way, we often see errors such as

$$\begin{array}{r} 27 \\ +27 \\ \underline{27} \\ 71 \end{array} \quad \text{or} \quad \begin{array}{r} 63 \\ -\underline{37} \\ 34 \end{array}$$

that result from misapplied or misremembered algorithms.

When students do pages and pages of similar addition problems, following the rules they had learned for addition, they are not engaged in problem solving, but in remembering and using an algorithm over and over. While I am certainly not making an argument here that algorithms are not useful—they are, of course, extremely useful tools—I am arguing that the blind, repetitive use of algorithms is not doing mathematics.

Towards a New Elementary Mathematics

During the last ten years, the mathematics education community in the U.S. has been reexamining the nature of mathematical problem solving for young children. With the publication of the National Council of Teachers of Mathematics' (1989) *Curriculum and Evaluation Standards*, the community has come together around new objectives for the elementary classroom. The focus in the elementary classroom is shifting towards an emphasis on mathematical reasoning and problem solving in a true sense—thinking mathematically in order to solve a problem that you do not know how to solve. In this view, what makes a problem a problem is that it is *problematic* for the person engaging in trying to solve it. Further, the *Standards* and other current reform documents (e.g., National Research Council, 1989, 1993) emphasize that in order to solve problems, students must learn to describe, compare and discuss their approaches to problems. Alternative strategies are valued, and multiple strategies—rather than a single, sanctioned approach—are encouraged. In order to learn, students must learn from each other, as well as from the teacher's questions. They must communicate about their mathematics.

Mathematics classrooms are changing. In the old style of an elementary mathematics classroom, students

- work alone
- focus only on getting the right answer
- record only by writing down numbers
- complete many problems as quickly as possible
- use a single, prescribed procedure for each type of problem.

In the mathematics classrooms many educators are now striving to create, students

- work together
- consider their own reasoning and the reasoning of other students
- communicate about mathematics orally, in writing, and by using pictures, diagrams, and models
- carry out one or two problems thoughtfully during a class session
- use more than one strategy to double-check.

Many elementary school teachers are eager to change their classroom practices in order to engage their students more deeply in mathematics. However, most elementary teachers have not themselves had sound mathematics training and experience. One of the biggest tasks we face in the U.S. is the development of elementary teachers in mathematics. One of the critical needs these teachers currently have is for new curriculum materials that can help them learn mathematics content and pedagogy as they are teaching their students.

New Curricula: Goals and philosophy

The National Science Foundation has funded about a dozen new curriculum projects to develop curricula at the elementary, middle school, and high school level. At TERC[10], we are working on one of these projects, a curriculum for kindergarten through grade 5 called *Investigations in Number, Data, and Space.* The major goals of this K-5 curriculum effort are to:

- offer students meaningful mathematical problems
- emphasize depth in mathematical thinking rather than exposure to a series of fragmented topics
- communicate mathematics content and pedagogy to teachers
- serve as a tool for radically expanding the pool of mathematically literate students.

[10] TERC is a nonprofit company located in Cambridge, Massachusetts that works to improve mathematics and science education. TERC's projects include research on children's understanding of mathematics or science, research on teacher development, and development and implementation of curriculum materials, staff development materials, and new software and technological tools for use in educational settings. The letters T-E-R-C no longer stand for anything.

The *Investigations* curriculum embodies an approach radically different from a textbook-based curriculum which leads students through 50-100 separate topics, most of which involve only basic arithmetic processes. Rather, this curriculum consists of a set of eight to ten units of work at each grade level. Each unit offers a set of connected investigations that focus on major mathematical ideas within the areas of number (including operations, computation, number patterns, and number theory), data collection and analysis, geometry, and the mathematics of change. Besides offering significant mathematics content, the investigations encourage students to develop flexibility and confidence in approaching mathematical problems, proficiency in evaluating solutions, a repertoire of ways to communicate about their mathematical thinking, and enjoyment and appreciation of mathematics. Because we see teachers as the primary audience for this curriculum, the materials are addressed directly to them and include notes on mathematical ideas and dialogues from classrooms designed to support teachers in learning more about mathematics and about children's mathematical thinking. The project has also developed assessment tools, videotapes for teachers, and computer environments that support this approach to mathematical investigation.

We want students to:

- *develop fluency in approaching mathematical problems.* Students must gradually acquire a repertoire of mathematical tools, processes, and approaches which they can use flexibly to solve problems. These include specific knowledge of number relationships, "number facts," and algorithms (these algorithms are developed by the children themselves), geometric relationships, ways of organizing and representing data using a variety of graphs, charts, tables, pictures, and concrete objects, knowledge of calculator use, and facility with mental arithmetic.
- *evaluate their solutions to mathematics problems.* In order to look at the reasonableness of their results, students must develop skills in estimation and have a solid foundation in the structure of the number system and the outcomes of arithmetic processes as well as experience with spatial relationships. We want students to know that an answer is correct

"not because the teacher says it is, but because its inner logic is so clear [National Research Council, 1989, p. 3]." This inner logic will only be apparent to the student who is well grounded in the structure of the number system and of geometric relationships.

- *communicate about their mathematical work.* In order to think about their mathematical work, students must keep track of their approaches and strategies for solving problems. "Keeping track" in mathematics is, we find, a process far from the experience of most elementary students. Without keeping track of their own work, they are unable to describe it, evaluate it, change it, or talk about it with others. We expect students to develop a large number of strategies for keeping track; through their writing, sketching, drawing, charting, graphing, students communicate with peers, with the teacher, and with their own thinking process.
- *enjoy and appreciate mathematics.* If students are to use mathematics, to continue studying mathematics beyond minimum requirements, and to maintain a lifelong curiosity about mathematics, they must come out of school with a sense of mastery of and appreciation for the power and beauty of mathematics. This affective component of mathematical learning in the elementary school is critical and is closely tied to the view of mathematics which is communicated to students in school.

We want to make sure that students are involved in investigations that involve number, geometry, and data. Traditional elementary curricula have included very little work with geometry or data, and we want to make sure that there is significant work in these areas at every grade level. We have also become interested in the mathematics of change—ideas that lead to calculus but that are accessible to young students—and are including a unit focused on change at each grade level as well.

Curriculum as Teacher Development

We see our curriculum as a vehicle for teacher development. The actual curriculum is not what we envision and write down, but what happens between students and teachers in the moment of teaching and learning. So, while part of our responsibility is to provide the material, the actual investigations in which students will participate (and this in itself is no easy matter), the other equally critical part of our responsibility is to open up that material to teachers, to invite them into both the mathematics and children's mathematical thinking.

The audience, therefore, for our materials, is teachers, not students. Our units are written *to* the teachers with many digressions about mathematics and about children's learning of mathematics. The responsibility is absolutely on the teachers to make this material work. If they fail, the material fails. On the other hand, by not making teachers partners in the past, we have made a grievous error. By not inviting teachers into mathematics, by attempting to make materials "teacher proof," because educators or mathematicians believed that classroom teachers were not smart enough about mathematics to teach it, not only have we denied the students a good mathematical education, but we have denied generations of elementary teachers—largely women—access to mathematics.

The Complexity of Apparently Simple Ideas

In order to open up mathematics to teachers of young children, our materials need to open up the complexity of apparently simple ideas to teachers.

A key issue in the elementary school for teachers—in all subjects, not just in mathematics—is that adults think that the ideas that are taught in these grades are simple. One of the factors that has made it so difficult for teachers to be recognized as professionals—and this is more true at the elementary level, where teachers are not subject matter specialists—is that everybody thinks it is easy to teach what students need to learn in these grades. After all, don't we all know how to count and how to add and subtract?

What is not understood by many people outside the schools, as well as by many inside them, is the complexity of these apparently simple ideas, both for the students and as they relate to mathematics. Here are three examples:

Elise, a sixth grader, was easily able to "find the average" of the grades on her spelling tests for the previous four weeks. However, when asked to answer the following question, "What would you have to score during the next four weeks to get an average of 90?" she was baffled. She said, "I know how to find the average but I don't know how you find the numbers that go into an average." While Elise is right, in some sense, that you do not know the particular set of numbers that "go into an average," further questioning revealed that she had no idea how to describe any possible data sets represented by this average. As adults we, in fact, most often meet averages in this way—we encounter the average, not the data. We must use our understanding of what an average represents to imagine what the data can be like. Elise's procedural understanding of average leaves her quite unprepared for dealing with the concept of average in the real world.

Gayle's third grade students were exploring multiplication patterns by skip-counting on a hundred number board. After students had counted by twos on the board—coloring in 2, 4, 6, 8 . . . up to 100—Gayle asked her students to describe the patterns they saw. "It's the even numbers," declared one child. "The even numbers—what can you say about even numbers?" After some further discussion about what "even" might mean, Jorge said, "the even numbers are the ones that have no middle." "No middle? Show us what you mean." Jorge came up and sketched three vertical slashes, circling the middle one. Underneath he drew two vertical slashes, not circling any. "See, three has a middle—it's not an even number. But two doesn't have a middle. It's even." Gayle asked, "What do you all think about Jorge's conjecture? Can an even number have a middle?" Students were soon busy building even and odd numbers using connecting cubes and exploring what the middles of these numbers might be. While as adults we might assume that what Ricky said was "obvious," these beginning third graders were genuinely engaged in deciding what the middle of a number like 26 might be. They were beginning an investigation of critical ideas about the structure of numbers, which might lead on to many kinds of conjectures about number relationships.

Carol's fourth grade students have certainly had many exposures to triangles during their four or five years in school. However, as Carol ran a discussion with her class which probed deeper into their knowledge, she found that many students had images of a prototypic triangle, usually equilateral and with one side parallel to the bottom of the page on which it was drawn, which restricted their views of the properties and relationships of triangles. For example, when she asked students to sketch various triangles with a perimeter of 12 centimeters, most students quickly drew an equilateral triangle with sides of 4 centimeters, but had great difficulty visualizing and sketching others. Some students sketched a triangle with sides of 6 centimeters, 4 centimeters, and 2 centimeters. Even when questioned hard about how this triangle would be constructed, many students insisted it could be done.

Illuminating Critical Mathematical Issues in the Curriculum

In all of these cases, teachers need information both about the mathematics itself and about the ways in which students grapple with the mathematics. What do we do about this in a curriculum? We can illuminate critical mathematical ideas. We can describe patterns of student learning, patterns of the ways in which students respond as they struggle with complex ideas. We can help teachers recognize ways in which we have seen many students respond, informal ideas that we have seen many students use, confusions that we have seen many students exhibit. Because we have extensive classroom data from our field tests, we are able to incorporate the experiences of many teachers and students into the final version of the materials through Teacher Notes (notes on the mathematical ideas and how students learn them) and Dialogue Boxes (examples of classroom interactions and issues that arise during them). For example, a Teacher Note called "Three Powerful Addition Strategies" is intended to help teachers (who themselves learned to add only by using the traditional "carrying" algorithm) become aware of other mathematically sound approaches to addition that their students may develop. Dialogue Boxes throughout the units give teachers examples of discussions we have recorded in classrooms where students are encouraged to share their computation strategies. These teacher materials offer glimpses into students' mathematical thinking, highlight critical mathematical issues that are likely to arise, and provide information about mathematical content that teachers may not have encountered or thought about deeply.

Through opening up the complexity of early mathematical ideas to teachers, we hope to engage teachers as researchers in their own classrooms. We hope our curriculum will help teachers to pay closer attention to what their students say and do as they are engaged in solving mathematical problems. For a classroom teacher, this most often means asking questions designed to illuminate the way in which students are thinking about a mathematical idea. By asking the question, "What can you say about even numbers?," Gayle showed interest in the deeper thoughts of her students. She wondered what the meaning underneath their words really was, and did not take for granted that all her students meant the same thing as she did when they used mathematical terms. Opening themselves up to the complexity of students' thinking can be disconcerting for many elementary school teachers. When they ask their students to think mathematically rather than simply repeat what they have been told, it often becomes clear that students know a lot less than the teacher thought they did. The teacher who conducted the conversation about triangles was appalled at how little her fifth graders knew. Further, the teacher begins to understand how truly heterogeneous her students are and how difficult it is to tailor learning experiences to meet all the needs in her classroom. Reading accounts drawn from other teachers' experiences and beginning to become familiar with patterns of student responses help not to make every student an isolated case.

What New Curricula Provide for Students

Besides providing new models for teachers, curriculum must, of course, provide substantive mathematical experiences for students. There are two needs in developing elementary curriculum. One is to find appropriate, engaging problems for children at this age. The other is to develop a pedagogy in which the emphasis is on the development of a *mathematical frame of mind.* The focus for young children, as in later mathematics, must be on thinking and reasoning mathematically.

Redefining work with number. If this is to be the case, work with number must be redefined and refocused in these new curricula. First, much more emphasis must be placed on developing a sound understanding of the structure of the number system and reasoning about number relationships based on this knowledge. For example, many children who have learned rote procedural approaches to solving problems solve the problem 1000 – 3 using the cumbersome method of "borrowing." We want children to reason from their knowledge of the place of 1000 as an important landmark in the number system. When students envision where 1000 is placed in the number system, they can easily count backwards from 1000 to 997.

Similarly, current research on young children developing their own strategies for addition and subtraction shows that children naturally add from left to right, dealing with the larger portions of the numbers first, rather than adding from right to left as we do when using the traditionally taught algorithm. Adding from left to right, students more readily retain a sense of the magnitudes of the numbers involved and are more likely to consider what a reasonable result might be.

Students make more use of prediction and estimation when they are encouraged to reason about numbers. For example, 8-year-old Anna reasoned in the following way as she solved the problem, "how many dollars do I need to give to the supermarket clerk in order to pay for potatoes that cost $3.45 and ice cream that costs $3.69?" Anna reasons, "Three dollars and three dollars, that's already $6.00. Then I round 69 cents to 70 cents. I know that 70 cents and 45 cents is already over a dollar. 40 cents and 60 cents is another dollar, so that's seven dollars, and five and nine go over, so I'll give her $8.00."

Finally, work with number should not be limited to work with operations. We want students to understand number as a way of describing relationships in the real world, but we also want them to encounter the purely mathematical (what patterns can you find in a 10X10 array of the numbers 1 to 100? How can you find the sum of all the counting numbers between 1 and n without adding each number? What is the tenth row in Pascal's triangle? What is the relationship between two consecutive square numbers?). Number relationships are in themselves fascinating objects of study. We want students to experience, appreciate, and be fascinated by the patterns of number and, on occasion, to catch glimpses of the quite surprising ways in which the purely mathematical turns out to fit reality (e.g., the occurrence of the Fibonacci sequence in nature).

Number theory offers a rich and accessible domain for exploration by young children. For example, second and third graders can become immersed in the study of the characteristics of odd and even numbers. In traditional mathematics classrooms, students learned to define even and odd numbers, but never spent time exploring how these numbers behave. In a third grade classroom working with our curriculum materials, students studied what happens when two even numbers are added together, or two odd numbers are added together. They drew pictures and used their knowledge of the structure of these numbers to develop statements that two even or two odd numbers would always result in an even number. When the question was posed, What would happen if you add an odd number to an even number?, students were eager to generate their own examples and develop informal proofs of their conclusions by referring to the structure of odd and even numbers. Most students reasoned from their knowledge that when the odd number is added to the even number, there is always "one extra" that is not paired, so that the result is an odd number. However, one student had a more unusual solution: "When we added two even numbers, the answer is even. When we added two odd numbers, the answer is even, too. So, when we add an odd and an even, the answers *have to be odd*, or else there wouldn't be any odd numbers."

By working with numbers in this way, students do real mathematical thinking—developing and testing mathematical conjectures, exploring the relationships among mathematical objects, using examples and counterexamples—not just solving a problem given by the teacher and coming up with the right answer.

Expanding the domain of mathematics to include work with data and geometry. While number has been the traditional center of the elementary curriculum, we believe that young students should also be doing substantial work in data analysis and geometry. Students in the elementary grades have experienced a very restricted view of mathematics as a discipline. To the elementary student, mathematics *is* arithmetic. Students in this age group can also be fruitfully engaged in collecting, representing, and describing data and in manipulating, visualizing, and reasoning about geometric objects.

Geometry, as the study of spatial objects, relationships, and transformations, is an essential component of mathematics. Geometric representations are essential for understanding such topics as functions and calculus (Balomenos, Ferrini-Mundy, & Dick, 1987) and, through measurement, geometry serves as a major source of practical applications of numerical concepts. As important as geometry proper is spatial thinking. Hadamard argues that much of the thinking that is required in higher mathematics is spatial in nature, and Einstein's comments on thinking with images are well known. Investigations in geometry and measurement provide opportunities for students to mathematically analyze their spatial environment, to describe characteristics and relationships of geometric objects, and to use number concepts in a geometric context.

Data collection and analysis is a critical skill in an information-rich society. From the earliest grades, students can collect, display, describe, and interpret real data so that they learn to become critical users of data and graphs. Students need to pose their own questions, collect data, critique and refine their own data collection methods, compare different ways of displaying their data, and, in the later elementary grades, learn to use appropriate statistical measures (Russell & Corwin, 1989; Russell & Friel, 1989). Research on students' understanding of statistical ideas in the elementary grades indicates that, just as in work with number, premature focus on memorization of definitions and algorithms (such as the algorithm for calculating the mean) undermines students' learning to make sense of a set of data. Just as students pull numbers out of "word problems" and manipulate them blindly, they pull numbers out of data sets and carry out calculations that no longer have meaning to them in terms of the data themselves (Mokros & Russell, in press). The elementary curriculum must include many opportunities for students to describe, analyze, and interpret a variety of data sets so that they begin to understand how data analysis can provide important information about a variety of populations.

Teacher Education: A Central Issue for Reform

In order for this reform of the elementary mathematics curriculum to work, we have a massive job to do in reeducating and supporting teachers as they attempt to expand and deepen the content of their mathematics program and to develop a pedagogy in which students are challenged to think mathematically (Ball, 1991; Russell & Corwin, 1993; Simon & Schifter, 1991). There is no question that we would prefer to see our curriculum used in the context of a strong, long-term staff development program. In fact, we know that a curriculum cannot provide all the necessary elements of such a program. One in particular—the opportunity for teachers to do mathematics together, at an adult level, on a regular basis, and to reflect with peers about their own learning of mathematics and its implications for their teaching—cannot easily be included in a curriculum. However, new curricula can be one of the tools that support teachers as they rethink their mathematics teaching. Insofar as these materials can invite teachers into mathematics and into the world of student thinking about mathematics, we believe they can help teachers open the doors of their classrooms to serious mathematics.

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ON THE METHOD OF GUIDING EXPLORATION

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§1 INTRODUCTION

To solve a problem, it is quite important to cultivate the student's ability for exploration. Scientific research is an activity of exploration. The nature of science lies in perpetual exploration (just like a search-light that always throws the light beam of research to the far unknown field), pursuing, trying, guessing and seeking. The pathway to research is rugged, rough and endless. Science is a glorious exploration of humankind's soul; that is to say, scientific cognition can't be separated from research thinking. To meet the needs of training talented students for national construction, it is necessary to cultivate the explorative ability of these students. In teaching modern mathematics, a coming agenda is to develop students' capacity to explore. We feel that this must be under the guidance of the teacher. How can the teacher guide students to explore? The student is consistently in the initial position in the course of open teaching, while the teacher is in the guiding position - the teacher must adroitly provide guidance according to the circumstances, neither taking on what ought to be done by the student, nor taking a laissez-faire attitude for the student to think in a way that digresses from the objective. The teacher must design the problem situation and build the relation towards the knowledge that the students must grasp. We think that the teacher has to infer the correct conclusions from the student's incorrect answers and also the main concepts from the student's confusing ideas in order to seek a reasonable way to proceed from disorderly complicated thinking and to ascend progressively from comparatively concrete perceptual knowledge to abstract rational knowledge. The topic, "An Ideal Method of Developing the Formula for the Area of a Trapezoid," is a designed example for "A Method of Guiding Exploration" for the student who has studied the equiform triangle.

§2 PHILOSOPHY OF DEVELOPING THE FORMULA FOR THE AREA OF A TRAPEZOID

The teacher, together with the students, devises the method of developing the formula for the area of a trapezoid. The teacher guides the students to think and respond, and then summarizes their thinking. The main steps of the teaching are explained as follows (*T* represents teacher, *S* represents student).

1. Analogy

T: (calls *S*1 by name) Please develop the formula for the area of a trapezoid.

*S*1: I forgot how to do this.

T: How about the formula for the area of a triangle?

*S*1: Build up another triangle with the same size and area to form a parallelogram, then reduce it by half, then you get the formula for the area of a triangle.

Now the teacher shows diagram 1, supposing the height to be h and the length of the base a, then the formula is derived as $1/2ah$.

T: Analogically.

*S*1: I remember! Build up another trapezoid with the same size and area to form a parallelogram, reduce it by half, then you get the formula for the area of the trapezoid.

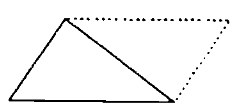

Diagram 1

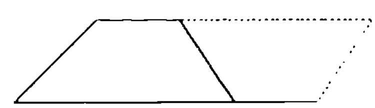

Diagram 2

Now the teacher shows diagram 2, supposing that the upper base is a, the bottom base is b, the height is h, and then its formula is $1/2(a+b)h$.

2. Abstraction and Generalization

T: The above process of solving the problem is finished because the solution of the problem involves seeking and recognizing the way by which the main body of mathematical activity shortens and eliminates the disparity and the gap between the known state and unknown knowledge, so as to recognize the mathematical object and transform the unknown into the known. The formula for the area of the trapezoid, as we know, is derived by transforming the unknown into the known and bridging the gap or dispelling the contradiction; however, a higher level as an ideal method may be reached through abstraction and generalization. Thus, we can find out their common ground by comparing and deriving the formula for the area of the trapezoid from the one for the area of the triangle.

S1: Either the triangle or the trapezoid is transformed into a parallelogram through a build-up process.

S2: The area formula is developed by reducing the parallelogram by half.

T: These two characteristics, as a method of solving the problems, are essential. Can you name the method?

S3: The name is the method of Build-up.

T: Can you characterize it at a higher level? (All students are silent in the class.)

T: What's the interrelation between the parallelogram, the triangle, and the trapezoid?

S2: It is the relationship between the whole and the part. The parallelogram is the whole relative to the triangle and the trapezoid, but the triangle and the trapezoid are parts compared to the parallelogram. Therefore, the problem of the triangle and trapezoid as parts, which can't be solved, may be transformed into one with the parallelogram as the whole. Because the entire problem has been solved, the "part problem" can be solved too.

T: Can you name the method?

S2: The name is the entire transformative method.

T: The method of solving problems by change through transformation from part to entirety is called the transformative method with common significance. Are there any other analogous mathematical formulas that you can think of?

*S*3: The formula for the area of a sector.

*S*4: ...

3. Sense Reversing Thinking

T: The above method is from the part to the whole. Can the area formula for the trapezoid be derived from the whole to the part.

*S*1: Transform a trapezoid into two triangles just like diagram 3. After the area of the separate triangle is evaluated, the sum of the areas of the two triangles is the area of the trapezoid.

*S*2: Transform a trapezoid into a parallelogram and a triangle, as shown in diagram 4. Their sum is the area of the trapezoid.

*S*3: Transform a trapezoid into two triangles and a rectangle, as shown in diagram 5. The sum of their areas is the area of the trapezoid.

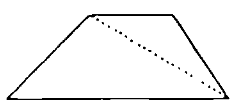

Diagram 3

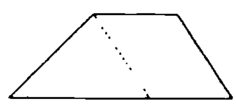

Diagram 4

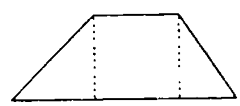

Diagram 5

T: Who can define the part transformative method?

*S*4: The method of solving a problem by changing from the whole to a part is called the part transformative method.

T: Can you draw a chart that shows the procedure for the solution of such a problem?

*S*5: Yes, as in diagram 6.

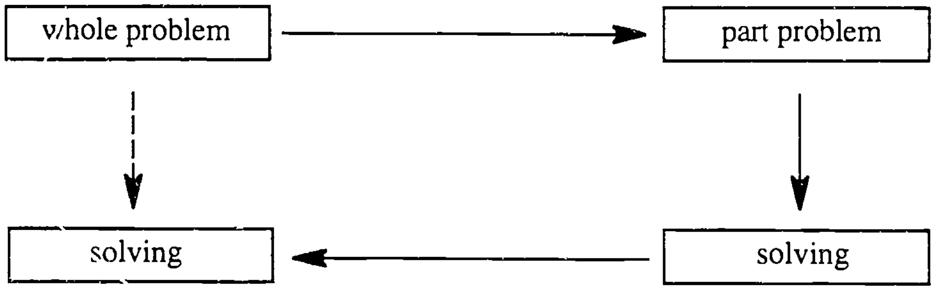

Diagram 6

T: This is a split-plot design by which the whole problem is transformed into a part problem. Thus, the part transformative method is called the split-plot design. The method of superposition is used to solve problems from the part to the whole. What is the relationship between the part and the whole transformative method from the viewpoint of the direction of thinking?

S6: The direction of thinking is just the opposite.

T: One is from the part to the whole, the other is from the whole to the part. If the whole transformative method is seen as thinking in the positive direction, then the part transformative method is...

S6: Thinking in the negative direction.

T: Can you give any other examples of thinking in the negative direction?

S7: An example is to operate on the formula $(a-b)(a+b) = a^2-b^2$ from left to right, then

$$(a-b)(a+b) \rightarrow a^2-b^2$$

is the thinking of the positive direction, while going from the right to the left is thinking in the negative direction.

T: Thinking in the negative direction is a thinking method by which the problem is handled contrary to a common process or from the opposite manner. If the problem is considered from the positive direction to the negative direction, it helps to develop thinking ability, especially to develop a creative thinking capacity.

4. Entire Transformative Method

We've already known that the part transformative method includes three deriving methods. Now we further consider the whole transformative method, which is the build-up method. How many methods are there besides those above?

*S*1: Build up a parallelogram, as shown in diagram 7; the area formula for the trapezoid is derived by way of the difference between the areas of the parallelogram and the triangle.

*S*2: Build up a triangle as in diagram 8, then the area formula for the trapezoid is developed by means of the difference between the areas of the bigger triangle and the smaller triangle.

T: Can you derive it in detail?

*S*2: Supposing the height of ΔEBC is h, then

$$S = S_{\Delta AED} - S_{\Delta BFC} = 1/2(h_1+h_2)-1/2h_1a \ =1/2(b-a)h_1+1/2bh$$

If $\Delta AED \sim \Delta BEC$, *then* $h_1 : (h_1+h) = a : b$ *and* $(b\text{-}a)h_1 = ah;$
then from the above, we have $S = 1/2(a+b)h$.

T: Very good! Here the principle of similar triangles is applied. Are there any other deriving methods?

*S*3: Build up a trapezoid into a rectangle, as in diagram 9. The formula for the area of the trapezoid is derived by way of the difference between the area of the rectangle and two triangles.

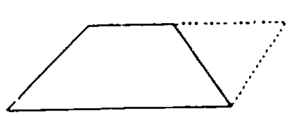

Diagram 7

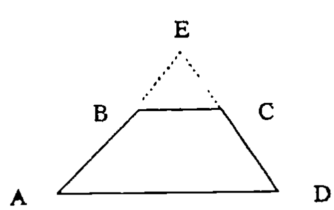

Diagram 8

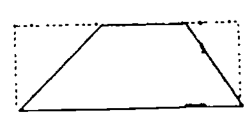

Diagram 9

5. Transformative Method of Equivalent Area

T: Besides the split-plot design and the build-up method, naturally we can consider the cut and complement method. How many methods are there regarding the formula for the area of a trapezoid that are derived by the cut and complement method?

*S*1: Cut down ΔGFD to build it up on ΔGEC, transform the equivalent area of the trapezoid *ABCD* into the area of the parallelogram *ABEF*, as in diagram 10.

*S*2: Cut down ΔEBC to build it up on ΔEFD, transform the equivalent area of the trapezoid into ΔABF, as in diagram 11.

*S*3: Build up ΔGAE on ΔGBF and ΔKDH on ΔKCJ, transform the equivalent area of the trapezoid into the rectangle *EFJH*, as in diagram 12.

*S*4: Build up the trapezoid *EBCF* on $\square GHDF$, transform the equivalent area of the trapezoid into that of the parallelogram *AEGH*, as in diagram 13.

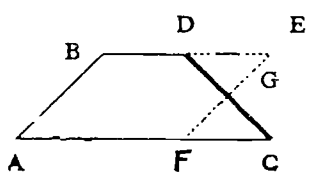

Diagram 10

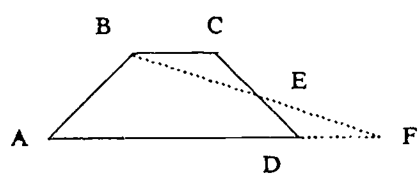

Diagram 11

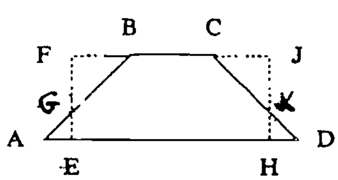

Diagram 12

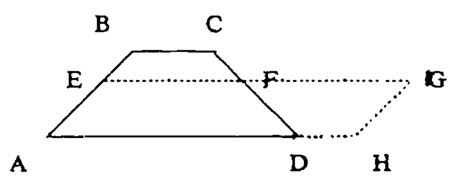

Diagram 13

T: Are there any changes of the area between the non-transformed and the transformed figures in diagrams 10-13?

*S*1: No.

T: That is to say, it is area-equivalent; then what can you name the cut and complement method?

*S*1: The method of equivalent area transformation.

T: Very good! The method of equivalent area transformation is a transformative method by which the problem is solved. It is further generalized to the method of geometric transformations and further to the RMI method (Note: see the appendix).

The above-mentioned method for the area of a trapezoid is also generalized abstractly to the principle of inter-build-up divergence; that is, a plane figure is moved from there to here without changing its area. If the figure is cut into several parts, the sum of the areas of the separate parts is equivalent to the area of the original figure; therefore, there exists simple equal relationships between the sum and difference of the areas of the figures moved before and after. This is quite an important principle in problem solution in figures in ancient China, and it is included in the book *Nine Chapters in Mathematics.* This indicates that ancient people already had a higher capacity to abstractly generalize the principle applied to solve a practical problem, by which the figures are transformed to each other, so as to reach the goal of transforming the unknown into the known.

6. Specialization

T: What method may be generalized from the deriving method for the formula for the area of the trapezoid, including the build-up, the split-plot design, and the cut and complement method?

*S*1: Transformation.

T: Very good! Can the lower level be compared with the transformation? (All fall silent in class.)

T: Think it over; what figures is the unknown trapezoid transformed into?

S: Triangle, parallelogram and rectangle.

T: What's the relationship between these figures and the trapezoid?

*S*3: A trapezoid may be changed into a parallelogram, and a rectangle when its two sections are in a special place; thus, they are the general and the special relationships.

T: Can a trapezoid be transformed into a triangle?

*S*2: Yes, when the short base of the trapezoid is 0.

T: That is to say, a triangle can be seen as a trapezoid in which the short base is 0, as a special case of the trapezoid. Thereafter, the general trapezoid may be transformed into special unknown figures, such as a triangle, a parallelogram and a rectangle. Then what is the method by which the formula for the area of the unknown trapezoid is derived?

*S*3: Specialization.

T: Right. The special method is an ideal guiding method, considering that a general object set is the union of small sets. What problems can be solved with the special method? This is your homework for after class.

7. Dynamic Thinking

T: Let's investigate what is common among diagrams 2,3,4,7,8,10,and 11.

*S*1: Their common ability is that an auxiliary line is drawn that intersects the line of the known trapezoid or its extended line, so as to reach the goal of developing the area formula for the trapezoid.

T: Very good! Can the auxiliary line in diagram 3 be changed into the auxiliary line in diagram 4?

*S*2: Yes, only rotate an angle of the auxiliary line in diagram 3.

T: Can it be transformed into the position on the auxiliary line of other figures?

S: Yes, the auxiliary lines in diagrams 2,4,7,8,10,and 11 may be drawn after the auxiliary line on diagram 3 is rotated and translated.

T: A straight line can be moved to any position in a plane by using a rotation and a translation. When this line is at a particular place, the auxiliary line is such as the ones in diagrams 2,3,4,7,8,10,11. If this auxiliary line is not in a particular place, but in the others, for instance, one shown in diagram 4, can the formula for the area of the trapezoid be derived?

S4: Supposing $BC=a$, $AD=b$, the height of the trapezoid is h, the height of the corresponding side BC on ΔFEB is h_1, $BE=a$, $DG=b_1$, and the height corresponding side DG on ΔHDG is h, then the area of trapezoid is:

$$S_{\Delta HCE}-S_{\Delta FEB}-S_{\Delta HDG}+S_{\Delta FAG}$$
$$=1/2(a+a_1(h+h_2)-1/2a_1h_1-1/2b_1h_2+1/2(b-b_1)(h-h_1)$$
$$=1/2(ah+ah_2+a_1h+a_1h_2-a_1h_1-b_1h_2+bh-b_1h-bh_1+b_1h_1)$$

Since $\Delta EBF \sim \Delta GAF$ and $\Delta DHG \sim \Delta CHE$, we have

$$a_1(h-h_1)=(b-b_1)h_1$$
$$b_1(h+h_2)=(a+a_1)h_2$$

Then, $S = 1/2(a+b)h$

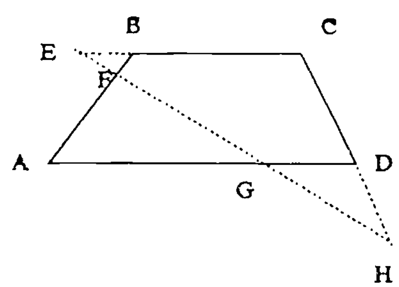

Diagram 14

T: How many methods are there by which the area formula for the trapezoid is derived, if classified according to the different positions of the auxiliary lines?

S5: There are countless methods.

8. Generalization

T: Can the auxiliary line in diagram 9 be put in a general place to derive the area formula for the trapezoid?

S1: As shown in diagram 15.

Draw $AE \parallel DF$, the extension line of BC line intersects points E and F, point G on line CD, then:

$$S = S_{\Delta ABF} + S_{\square AEFD} - S_{\Delta CDF}$$

$$= 1/2(a+CE)h + bh - 1/2(b+CE)h$$

$$= 1/2(a+b)h$$

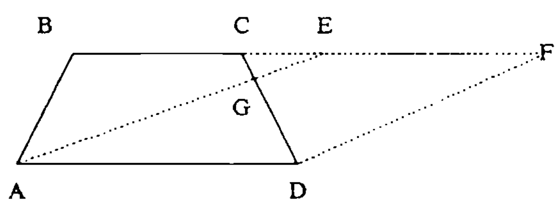

Diagram 15

Especially when the parallel lines AE and DF are perpendicular to the line AD, the auxiliary line can be drawn as shown in diagram 9.

§3 CONCLUSION

The mathematical approaches, such as abstraction, analytic, synthesis, classification, transformation, specialization, generalization and thinking in the negative and dynamic directions, that are used widely in the course of deriving the area formula for the trapezoid, are abundant and also possess significant common guiding features. It is practically proved that the effects on teaching mathematics are ideal in choosing such a typical example of the guiding explorative method The class was quite active, and this helps to train the wide-range, profound, nimble and critical thinking abilities of the students and to increase the students' mathematical concepts.

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Notes:

[1]: Working Unit: Research Center for Scientific Methodology of QuFu Normal University. Qufu, Shandong 273165 China

[2]: Working Unit: Agency for China Education News, Beijing 100034 China.

§4 APPENDIX

Approach of Inversion of Relation Mapping

The approach of an inverse relation mapping is quite a popular method of thinking. Its nature is the contradictory transformative approach by which a more difficult problem can be transformed into a problem that is handled more easily. Let's analyze two examples to generalize this method.

Method of Analysis

Descartes transformed a geometric problem into an algebraic problem by way of building up the coordinate system, and then he drew the geometric conclusion by means of an algebraic conclusion. From this, he raised a general model--a Universal Method to deal with the problem:

i Transform any problem into a mathematical problem

ii Transform any mathematical problem into an algebraic problem

iii Transform any algebraic problem into a solution of an algebraic equation

Because the problem of solving an equation may be regarded as solvable, various kinds of problems can be solved by the use of Descartes' Universal Method. Obviously, Descartes' conclusion is not totally correct because any method can be applied in only a certain scope, and not absolutely. The reduction idea raised here by Descartes is a heuristic. A problem that can't be solved or can't be easily solved may be reduced to a new problem that can be solved or can easily be solved. Analytic geometry, that was originated by Descartes, is a concrete application of the reduction principle. If the process of Descartes' solution of analytic geometry is expressed in frames, as shown in the diagram, it is easier for us to understand.

It is the method of analytic geometry that is applied to effectively study the conic section and the quadratic conicoid, so, the new discipline of analytic geometry has quickly been established.

$$\begin{tikzcd}
\boxed{\text{geometric problem}} \arrow[r, "\text{analytic expression}", "\text{build-up of coordinate system}"'] \arrow[d] & \boxed{\text{algebraic problem}} \arrow[d] \\
\boxed{\text{geometric conclusion}} & \boxed{\text{algebraic conclusion}} \arrow[l]
\end{tikzcd}$$

Diagram 1

Method of Logarithmic Computation

Similar to the method of analysis, the method of logarithmic computation is also a concrete application of the reduction principle. For instance, in order to compute $y = \sqrt[5]{0.6842}$ by way of logarithmic computation, the process is composed of three steps as follows:

Step 1 Compute logarithmically

$\lg y = \lg \sqrt[5]{0.6842} = 1/5 \ \lg 0.6842$

Step 2 Compute by use of a table of values

$\lg y = 1/5\ (0.8352-1) = 0.9670-1;$

Step 3 Compute the antilogarithm

$y = 0.9268$

The above computation process is expressed below in frames, as shown in diagram 2:

$$\begin{tikzcd}
\boxed{y = \sqrt[5]{0.6842}} \arrow[r, "\varphi", "\text{logarithm}"'] \arrow[d] & \boxed{\lg y = 1/5 \lg 0.6842} \arrow[d, "\text{compute by} \quad \text{table (look up)}"] \\
\boxed{y = 0.9268} & \boxed{0.9670 - 1} \arrow[l, "\varphi^{-1}"']
\end{tikzcd}$$

Diagram 2

Rough Generalization

The common point of the above two examples is to establish the corresponding relationship between two objects. In analytic geometry, a correspondence is established between plane-space, point-set and binary-ternary ordered sets of real numbers by means of the coordinate system, while plane point-set is a geometric structure. Therefore, this can reduce a geometric problem to an algebraic problem. In logarithmic computation, a correspondence is established between the set of positive real numbers and real numbers through symmetric functions, which realizes the reduction from a more complex operation (multiply, divide, involution, evolution) to a more simple operation (addition, subtraction, multiply, divide).

The set of mathematical objects that have a certain mathematical relation with each other, is usually called the relationship structures, so the transformation from complex to simple or hard to easy has a definite procedure, which is called a mapping relation. Since the corresponding relationship overall is applied twice in an opposite direction, if the correspondence from the original problem to a new problem is called a mapping, the correspondence of the solution of original problem is called an inverse mapping, expressed in diagram 3:

$$\begin{tikzcd}
\boxed{\text{original problem}} \arrow[r, "\varphi", "\text{mapping}"'] \arrow[d] & \boxed{\text{new problem}} \arrow[d] \\
\boxed{\text{solution}} & \boxed{\text{solution}} \arrow[l, "\varphi^{-1}"', "\text{inversion}"]
\end{tikzcd}$$

Diagram 3

To sum up, we may generalize the commonly significant inversion method of relation mapping, abbreviated as the RMI method. It means that " an original problem" after being transformed by a mapping ϕ into "new problem" that has been solved by inversion, can be solved.

In order to strictly express this method, some basic concepts must first be defined.

Some Basic Concepts

Mathematical Object. This refers to the mathematical concept involved in concrete mathematical theory; e.g., number, magnitude, variable, function, equation, point, line, surface, geometric figure, space set, computation are all mathematical objects. There are three features: determinacy, logic rationality and existence of a subjective background under a special condition. Therefore, fabricated concepts can't be regarded as mathematical objects.

Relationship structure. This refers to some mathematical relationship among the elements of set organization by some mathematical objects. Relationship structure refers to the relationship which may be exactly defined among the mathematical objects; e.g., algebraic relation, function relation, correspondent relation, consistent relation, non-consistent relation, and so on. Generally speaking, the structure of a mathematical relation should possess three prerequisites: one is that the object is only a mathematical object in the structural system; the second is that a connection among the objects should be a mathematical relation; and the third is that the structural system has logic rationality and logic deductivity to some significant extent.

Mapping. A correspondent relation that is established between two kinds of mathematical objects or elements of two mathematical sets is defined as a mathematical mapping. Particularly, if it is a one-to-one correspondence, it is called an invertible mapping.

Mathematical Formality. The variety of deductive reasoning, methods of proof and computational methods used in mathematics, is called mathematical formality, which consists of procedures completed through a finite number of steps.

Supposing ϕ is a mapping, it maps an element of the set $A=\{a\}$ into another set $A^*=\{a^*\}$, in which a^* represents a mapping of a which is called an inversion image, so

ϕA-----A^* $\qquad$ $\phi(a)=a^*$

If there exists an inversion mapping ϕ^{-1}, then $\phi(a^*)=a$. Supposing $P=\{\theta\}$ is a group of relations or computations which can be defined among the entire or part sets of A, thereafter $S=\{A,P\}$ organizes a relation structure. Supposing that S contains an unknown object X which is called an inversion image of the object under the action of ϕ, then $X^*=\phi(x)$ is called an object mapping.

Under the action of ϕ, A is mapped to A^*, the relation on A is mapped to $P^*=\{\theta^*\}$, S is mapped to $S^*=(A^*,P^*)$. S^* is called a relation structure of mapping, containing object mapping $X^*=\phi(x)$. Supposing x represents unknown object inverse image, then $x^*=\phi(x)$ represents a correspondingly an unknown object mapping.

Definable Mapping. If an unknown object x is mapped and is determined to be x^* from the structural system of mapping relations through the deterministic procedures, the mapping ϕ that transforms s to s^* can be called definable mapping.

Definition and Application of RMI

Now we can strictly describe the method of RMI:

Provided with the system of a relation structure $S=(A,P,X)$, including unknown object inverse image x, if both inverse and definable mapping ϕ to be likely found maps S to $S^*=(A^*,P^*,X^*)$, unknown object mapping $X^*=\phi(x)$ is determined through the inverse mapping ϕ^{-1}, the process is shown in diagram 4, and also may be simply expressed as

$$(S,X)\text{--}\overset{\phi}{}\text{--}(S^*,X^*)\text{--}\overset{\phi}{}\text{--}X^*\text{---}\overset{\phi^{-1}}{}\text{--}X$$

Therefore, the entire process of the RMI solution method is:

Relation--Mapping--Definable Mapping--Inverse Mapping---Solution

The method of RMI is applied widely in mathematics, such as in trigonometric substitution, variable substitution, geometric transformations, coordinate transformations, methods of complex numbers, vector methods, linear transformations, and logarithmic transformations. These are all special methods of RMI.

The key to solv problems by applying the RMI method lies in selecting a reasonable and effective mapping method. What is an effective mapping? It satisfies the following two conditions: one is that the mapping is not only invertible, but also definable; the other is that it plays the role in transforming relation of the inverse image into a mapping relation from complex to simple, hard to easy.

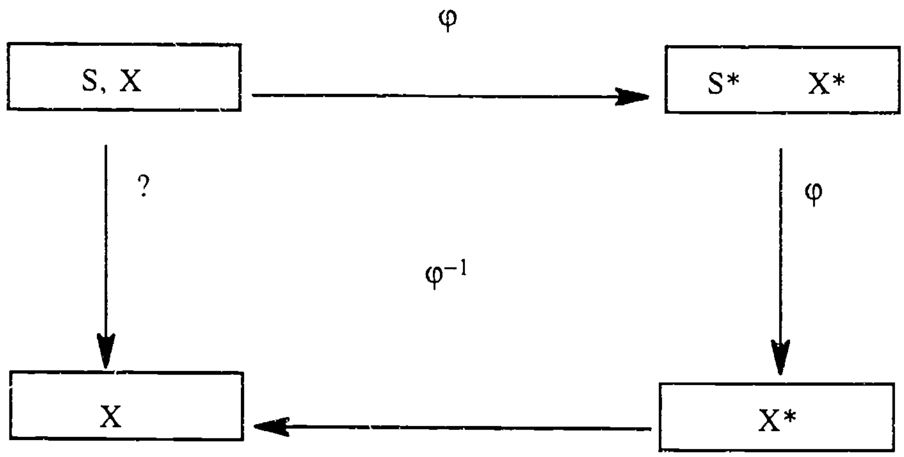

Diagram 4

MATHEMATICAL PROBLEM SOLVING

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1. Analyzing Problem Situations in Problem Solving

Much research on mathematical problem solving has been done in the last decade. Problem solving in Japanese elementary schools is introduced as one of the results of the U.S.-Japan Seminar on Mathematical Problem Solving (Hashimoto, 1989).

The following two aspects are pointed out there:

a. Recognizing and evaluating children's various ideas.

b. Discussions between the teacher and a child, and between the children.

The protocol in the classroom can be seen in the Nagasaki and Becker article in the NCTM Yearbook (1993). I think these two things should be emphasized even in secondary school mathematics. But there seems to be less emphasis the higher the grade level.

I will show an example of how to analyze a problem situation in problem solving:

How many parts can the plane be divided into by three lines?

The answers are 4, 6, and 7. If only 7 is required as the answer, add the term "maximum" to the original problem. The important thing is to analyze the problem situation. Generalization is carried out in the following process:

a. To change from a constant to a variable.

b. To delete a part of the conditions.

Regarding the original problem, there are three problems as follows:

Problem A: How many parts can the plane be divided into by three lines at the maximum?

Problem B: How many parts can the plane be divided into by n lines at the maximum?

Problem C: How many parts can the plane be divided into by n lines? (What is the number of regions formed by an arbitrary arrangement of n lines in the plane?) (Wetzel, 1978)

Problem B is a generalization of Problem A, and Problem C is a generalization of Problem B. The important thing in secondary school mathematics is to think generally about the problem after solving the problem.

2. From Open-ended Problems to the Developmental Treatment of Mathematics Problems

The following is a concrete example which was tried out with eighth grade students (13-14 years old).

Take a point P on the diagonal AC in parallelogram $ABCD$. Draw a parallel line EG to AD and HF to AB like Figure 1.

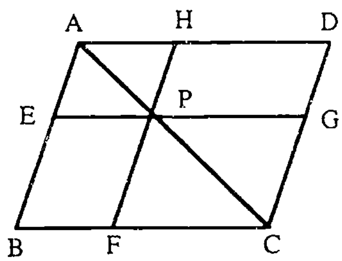


Fig. 1

Problem A: Find as many relations as possible among the segments, angles, triangles, etc. in Figure 1. There are many relations (answers). For example.

a. length of a segment: $AE = HP = DG$

ratio of a segment: $AE{:}EB = DG{:}GC$, $PH{:}PF = PE{:}PG$

b. size of an angle: $\angle DAC = \angle BCA$, $\angle EHF = \angle GFH$

c. area: area of parallelogram $EBFP$ = area of parallelogram $HPGD$; area of quadrilateral $EFGH$ = 1/2 x (area of parallelogram $ABCD$)

d. congruence: $\Delta AEP \cong \Delta PHA$, $\Delta ABC \cong \Delta CDA$

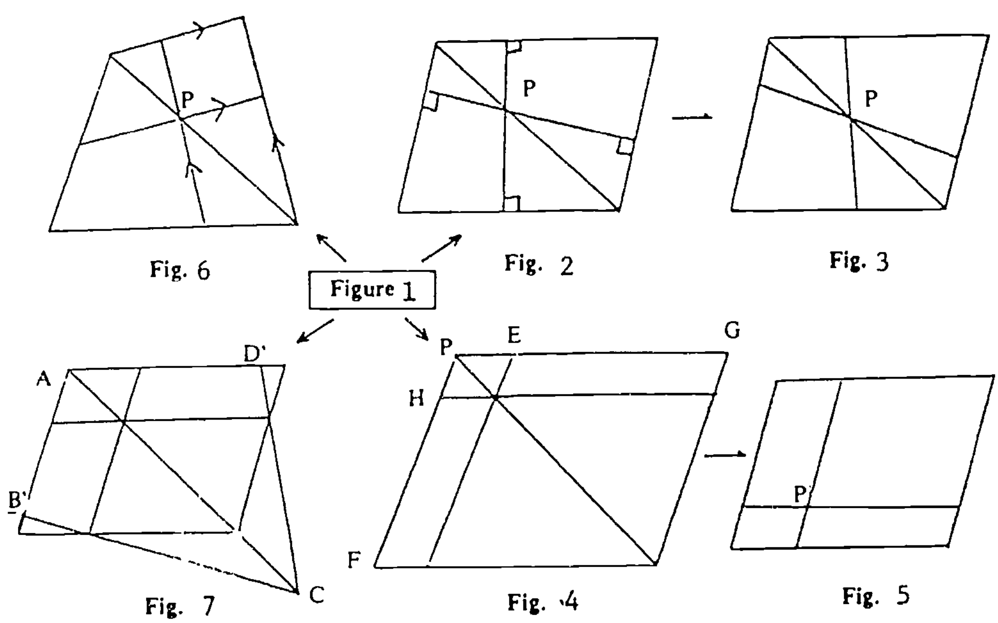
e. similarity: $\Delta AEP \sim \Delta CGP \sim \Delta ABC$

There are many correct answers in this problem. We call this an open-ended problem. An open-ended problem is one that allows several different solutions according to how students view the problem situation. (Shimada, 1977)

Let's pick up the following problem from among the answers given above:

> Take a point P on the diagonal AC in parallelogram $ABCD$. Draw a parallel line EG to AD and HF to AB as in figure 1.
>
> Prove that $PH:PF = PE:PG$

Problem B: Make up a new problem by changing parts of the problem. The analysis of problems was made by students (Hashimoto and Sawada, 1984).



Figures 2-7 are variations of Figure 1. The problems corresponding to these figures were made by changing parts of the problem or by using the converse of the given problem. The explanation of the problems is as follows:

Figure 2: Draw perpendicular lines by changing the part of parallel lines.

Figure 3: Draw arbitrary lines by changing the part of parallel lines. (Even in this case, the conclusion that $PH{:}PF = PE{:}PG$ is satisfied. Then one of the generalizations is satisfied, and students can learn the property between parallel lines and proportion.)

Figure 4: Changing how to take a point. For example, take a point on the extension of the diagonal.

Figure 5: Changing how to take a point. For example, take a point inside the parallelogram. (The conclusion is not satisfied in this case. Students can easily find a counter-example.)

Figure 6: Change a shape. (As the conclusion is not satisfied in this case, one of the generalizations is not satisfied.)

Figure 7: Consider the converse of the given problem. (The conclusion is not satisfied in this case because we can find a counterexample. In reality, students could not find it.)

This illustrates the developmental treatment of mathematics problems. Concrete examples of such problems can be seen in Japanese elementary and junior high school (7th, 8th, and 9th grades) mathematics textbooks.

In teaching the developmental treatment of mathematics problems, the teacher organizes the lesson according to the following scheme:

a. To solve a given problem,

b. To discuss the methods of and the solution to the problem,

c. To formulate new problems by changing parts of the given problem,

d. To propose new problems to the whole class (whole-class teaching is typical in Japan at all levels),

e. To discuss some of the new problems and classify them,

f. To solve common problems selected by the teacher or students,

g. To solve students' own problems.

3. Teacher's Attitude Toward Mathematics

I think it is very important for classroom teachers at all levels to carry out a lesson by using an open-ended problem or by the developmental treatment of a mathematics problem. This depends on the teacher's attitude towards mathematics.

The Japanese school mathematics curriculum is now changing; that is, at the elementary school level (1-6) since 1992, at the junior high school level (7-9) since 1993, and at the senior high school level (10-12) since 1994. These two kinds of mathematical problem solving should be encouraged by classroom teachers. I would like to point out that classroom teachers should have a chance to carry them out at least once or twice in each term (3-6 times a year). Judging from my experience over twenty years, this seems possible.

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ON THE PROBLEM SOLVING TEACHING PATTERN OF CHINA

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China is a country with rather fixed traditions in its educational culture. Influenced by traditions and economic developments, Chinese education has been following its own rules from the beginning to the end. Both educational content and methods are standardized. This kind of education of "Unity" not only effectively guarantees the handing down of Chinese national culture and national spirit from generation to generation, but also means that foreign culture is filtered by our traditions to fit our national situation, before it is spread out.

For the teaching of mathematics in China, The "Problem Solving Pattern" is exotic. Although it is a modern, advanced and effective teaching pattern, being exotic, it has to be transformed so as to become a kind of "Chinese Problem Solving Teaching," which can fit the basic situation of Chinese mathematics teaching. What is the basic situation of Chinese mathematics teaching? First we should look back on history.

I. History in Retrospect

China is a country which is good at mathematics as well as education. Before Western mathematics was introduced into China, mathematics in the Chinese people's mind was a kind of arithmetic---that is, calculating skills. In detail, it was the art of equation establishing and the art of computing in solving mathematical problems, where "art" means "method." The reason for this method, that is, the explanation for doing it this way, was called "reasoning" in ancient China. "Problem + calculating methods + reasoning" was the basic structure of traditional Chinese mathematics. Studying "calculating skills" meant studying "the methods and the reasons." Comparatively , the methods were more important. Since Chinese arithmetic focused on calculating methods, mathematics teaching in ancient China aimed at teaching and learning the methods. Problem solving was out of the question. Problems were just the support for exercises, not the objects to be solved.

From the 17th to the 19th century, the development of practical mathematics reached its high tide in China. Although it seemed that many problems needed to be solved in mathematics, the basic patterns of the problems were alike and were also very simple. So, you could deal with any mathematics problem as long as you grasped the calculating methods of the basic patterns. Because the content to study was not rich and not difficult, the basic content of mathematics teaching at that time focused on the teaching of calculating skills. The teaching method was a "Teach-You-to-Learn Pattern" which involved integrated teaching and practicing, just the way a master worker trained an apprentice.

After the 19th century, a great deal of Western mathematics was introduced into China, among which were algebra (which involved higher-level thinking), analytic geometry and infinitesimal calculus. At the same time, Western school education was also introduced. This was a revolutionary change for mathematics teaching, but the popular teaching method in school was the traditional "Teach-You-to-Learn Pattern." So, instead of being shattered and completely covered by modern education, the mathematics teaching pattern, that had a history of more than 2,000 years and that experienced the pounding of the reformation of ages, had both adaptability and vitality. It became one of the basic characteristics of modern mathematics teaching in China.

II. Present Situation

The basic situation of mathematics teaching in China possesses many characteristics, among which "Unity" is the most intrinsic one. The so-called "Unity" means teaching according to the same syllabi and the same teaching materials. In recent years we have changed the situation for "millions of students studying the same textbook," and different regions can now choose and compile their own teaching materials. But so far as one region is concerned, once the teaching material is decided on, teachers must then, based on the chosen material, adhere to the same teaching objectives, teaching requirements, rate of teaching and the examination paper. In this way, both the content and the form are standardized, so that the mathematics teaching in this region goes on regularly with a certain rhythm. But what we should notice is that although we strongly advocate the freedom of running schools and doing the teaching, the teaching objectives and requirements in the syllabus are becoming more and more detailed. The teaching objectives involve not only teaching students knowledge, but also cultivating their abilities and educating them in ideology. The teaching content and requirements are standardized in the ten stages of the three fields of cognition, feeling and practice. (The field of cognition contains four stages--knowledge, understanding, mastering and applying; in terms of feeling, we refer to motive, attitude, habits and emotion; as for the field of practice, it is divided into initial grasping and grasping.) Thus, restrictions are placed only to hinder teachers' initiative.

III. The Characteristics of Problem Solving in China

History and reality make Chinese teachers show their own characteristics in accepting and applying the Problem Solving Teaching Pattern. The main characteristics are:

1. They focus their attention on textbooks while choosing and designing problems.

   Although in applying the "Problem Solving Teaching Pattern," the selected problems are not necessarily required to closely follow the teaching material, they generally fit in with the content of the material in order to improve the effect of teaching. The introduction of the concept "minus" is a good example. Because they accept the concept "positive" through the prescription of the quantity of concrete things, students hope, more often than not, to learn "minus" in the same way. That is very hard to do. So, it is a good method to create a problem situation and to help them know and accept a new concept through solving problems.

   We can create a problem situation like this: On the blackboard hangs a model thermometer, in the middle of which the movable red paperboard stripe indicates the "mercury." In teaching, the teacher moves the "mercury" indicator to show the temperatures 0°C, 11.3°C and 3.4°C below zero, respectively. Then the teacher says, "In February, the average temperature in Guangzhou is 11.3°C. But in Beijing it is 3.4°C below zero. The question now is, how much higher is the average temperature in Guangzhou than in Beijing?" The students know that it is a problem of subtraction. But what would be the expression? 11.3 – 3.4? No. Then, 11.3 above zero – 3.4 below zero? That seems correct, but how do you calculate it? As a result, the idea of "minus" comes into being through the solving of the problem above. Simple as the problem may look, it possesses the following characteristics:

(1) It offers the necessary elements and forms to stimulate studying by the students:

(2) It provides the necessary elements and forms to induce the mathematical thinking and creative consciousness of the students;

(3) It provokes the students' desire to solve the problem;

(4) It helps to reach the teaching goal.

2. They do not seek novelty blindly.

Rather, they vary their teaching approaches, within the limits of the teaching requirements, to broaden the educational value of the problems. For example, "find the length of the diagonal of a rectangular parallelogram" is a conventional mathematical problem. According to the "Teach-You-to-Learn Pattern", the teaching of this problem generally follows five steps:

1) Introduce the concept of rectangular parallelogram through a concrete example;

2) State the definition of a diagonal;

3) Present the diagonal calculating formula--the theorem for the diagonal of a rectangular parallelogram;

4) Prove the theorem;

5) Apply the theorem to other examples.

In doing so, teachers are the operators while students receive knowledge passively, which has almost nothing in common with the "Problem Solving Teaching Pattern." But the difference is produced by the teaching method, not the problem itself. Actually, if the teacher could pose the problem in another way, accompanied by corresponding teaching methods, the problem would become an identical "problem solving" one, which is based on the teaching material.

Once, based on the teaching strategy of G. Polya, I used the problem solving pattern to give a lecture on "How to find the diagonal of a rectangular parallelogram" to the students of Class 3, Junior Three of the Middle School Attached to Shanghai Teachers' University. It had a good effect. Here I will briefly describe my teaching experience.

Problem Introduction

(Problem 1)

Teacher: Once I was attracted accidentally by a cuboid paper-pressing stone while I was reading at my desk. I thought that its volume could be obtained easily in mathematics by measuring its length, width and height and then multiplying them. But it was difficult to obtain its diagonal. Now I present the problem so that you can find the solution.

Teacher: Who can express the problem clearly? (That means to restate the problem in some form.)

Student: "If the length, width and height of a rectangular parallelogram are known, what is the length of its diagonal?"

(Problem 2)

Since they had learned the Pythagorean Theorem before, I had expected that their way of thinking for solving this problem would be: find the diagonal → give it a definition → look for a certain right triangle in which it lies → use the Pythagorean Theorem twice to obtain the formula. To my surprise, the students obtained the formula directly through analogy. (This is the out-of-control state of teaching which can be caused by the "Problem Solving Teaching Pattern.")

This, however, was just a phenomenon. Students might have used different analogical methods. They might have deduced $\sqrt{a^2+b^2+c^2}$ from $\sqrt{a^2+b^2}$ through figures by the analogy between a rectangular parallelogram and a rectangle, or deduced $\sqrt{a^2+b^2+c^2}$ directly from $\sqrt{a^2+b^2}$ without referring to figures. Actually, when I drew a rectangular parallelogram on the blackboard and asked the students to point out its diagonal, the student who had obtained the diagonal formula unexpectedly said that the segment DC' was the diagonal.

This shows that although the student had transplanted some knowledge, the cognition structure produced by the transplantation had some drawbacks. On the other hand, it presents, in one respect, that the "Problem Solving Teaching Pattern" is effective in exposing the students' thoughts. It is the exposure of their thoughts that makes the class atmosphere lively.

From producing the concept defining the diagonal of a rectangular parallelogram, and obtaining the calculating formula, other problems are derived. The derivation from the application of the theorem is called "nodal derivation." For example:

Problem 3: "Given the height of a cube, find the diameter of its circumscribed sphere."

Problem 4: "Given the heigh: of a regular quadrangular pyramid and the sides of its lower base, find the lateral edges."

Although Problems 3 and 4 are general ones, they can be solved through the "Problem Solving Teaching Pattern" because the same problem can become more interesting after being reformulated and also can become very standard through the teaching method design. For example, as designed by Polya, Problem 3 can be reformulated into Problem 3'.

Problem 3': Suppose that a small ball is hung in a cube. The ball and the cube share the same center. Inflate the ball to make it expand. At a certain moment, the surface of the ball touches the faces of the cube; a little later, it touches the vertices. Which values does the radius of the ball assume at these three critical moments? Problem 4 can also be reformulated in the same way. The reformulation of the problems is advantageous to the application of the "Problem Solving Teaching Pattern," but it is not essential. The "Problem Solving Teaching Pattern" in China is realized mainly through the teaching ideology and the riation of teaching methods and approaches, in which teachers play a very important role because in teaching, mathematical problems have no specific characteristics and can be used in the "Problem Solving Teaching Pattern" as well as other teaching patterns. In other words, whether a problem is used in the "Problem Solving Teaching Pattern" or not depends on the teachers' intentions and design. Only when a problem is used in the

"Problem Solving Teaching Pattern" by teachers, who design their teaching methods elaborately, can the teaching value be brought into full play.

The homologue of "nodal problem derivation" is "radiative problem derivation" - this means starting again from the original problem to derive a series of problems. For example, starting from the diagonal AC' of the rectangular parallelogram, we think about the following problem.

Problem 5: In rectangular parallelogram $ABCD\text{–}A'B'C'D'$, $B'C'=2a$, $B'B=a$, if the section with AC' in it intersects with lateral edges forever, when is the perimeter of the section the shortest? Find the shortest perimeter.

This problem can also be reformulated into a more interesting one as:

Problem 5': If an insect starts from the vertex A of rectangular parallelogram $ABCD\text{–}A'B'C'D'$ and crawls on its surface to C' along a straight line, how can it take the shortest distance?

This problem is related to the shape of the rectangular parallelogram and is open to a certain degree. It is a good example for problem solving teaching.

3. They pay attention to the summing up of the ways of solving problems. And they put the stress on the training of the logical thinking of the students.

China can be called a kingdom of problem solving. The reason for so much attention to problem solving in mathematics teaching is that solving problems not only can be used to examine students for their grasp of the mathematical concepts, but it can also help to develop their logical thinking and cultivate their thinking abilities. Mathematics is not equivalent to solving problems, but it cannot exist without problem solving. As a kind of problem solving pattern, solving problems shares the same significance and effect with general problem solving. But its effect can only be brought into full play through quantity accumulation and necessary summing up. That also suits the "Problem Solving Teaching Pattern." If a teacher trains students in solving problems without paying attention to the summing up of the methods, the result will be that the intelligence of the students cannot be brought into play, although they may have solved many problems. "Come up with a good idea," is what Polya says. But a good idea, as Polya said, cannot be gotten easily. In this regard, there is much in the Chinese training methods that can be used for reference, one of which is paying attention to the summing up of the methods and the development of logical thinking of the students. In practice, two points are worth mentioning: first, do not avoid the suitable repetition of problems of the same type; second, find laws in repetition. In recent years, some of the theorists in mathematical education and experienced primary and middle school teachers in China have made a summary of the ways of mathematical thinking most in use and th "strategic ideology tactics for problem solving," both of which are practically significant.

"HIGH ACHIEVEMENT" VERSUS RIGIDITY: JAPANESE STUDENTS' THINKING ON DIVISION OF FRACTIONS[1]

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1. Introduction

Japanese students have presented relatively high scores on various tests of mathematics achievement (e.g., NIER, 1981; Sawada, 1987). This is particularly true when the test items involve computation with numbers. For example, division of fractions is a topic in which most students have demonstrated mastery of the algorithm. A comprehensive nationwide survey of Japanese students' mathematical performance which was conducted by the Ministry of Education in 1984 showed that 93.2% of the 6th graders (N = 16,000) correctly answered the item "2/7 ÷ 3/4" [2].

Although it is important for students to know how to execute the "invert-and-multiply" algorithm for division by a fraction efficiently, they must also know why it works and how to verify that their answers are correct. Division by a fraction is, however, a typical topic in school mathematics as a "rule without reasons" (Skemp, 1976). For many students, computation with fractions may seem to involve a series of nonverifiable rules. Do Japanese students who have presented "high achievement" merely apply memorized rules? What are their responses if we ask them to explain their procedures?

The aim of this paper is to explore the aspects of students' thinking on division of fractions. This topic, though it might be hackneyed like "a negative times a negative is a positive," is interesting and allows us to examine the relationship between "conceptual knowledge" and "procedural knowledge" (Hiebert & Lefevre, 1986). As commonly used, conceptual knowledge refers to knowledge of the relationships and the interconnections of ideas that explain and give meaning to mathematical procedures. Procedural knowledge refers to mastery of computational skills and knowledge of procedures for identifying mathematical components, algorithms, and definitions. More specifically, procedural knowledge of mathematics has two parts: (a) knowledge of the format and syntax of a symbol representation system and (b) knowledge of the rules and algorithms that can be used to complete mathematical tasks. In the case of division of fractions, althoug , both procedural and conceptual knowledge are considered necessary aspects of understanding, one of them may be dominant in some situations.

In this study, I sought to use the conceptual/procedural distinction to examine the students' beliefs about mathematics and about mathematics learning. To do this, a non-standard but correct method of doing division was systematically introduced to the students to create a perturbation in their procedural knowledge and to examine its impact on their conceptual knowledge. By doing so, one can gain a window through which to see some of the "hidden" dimensions of teaching and learning mathematics; in particular, the students' beliefs and the "hidden" curriculum[3].

2. Method

In this study, two methods were used to explore the aspects of students' thinking on division of fractions: written tests (study 1) and semi-structured clinical interviews (study 2). These methods will be described briefly.

Study 1: In study 1, grades 6 (elementary school) and 7 (junior high school) students (N = 590) were asked to solve a problem which included a correct but unfamiliar procedure - the numerators and denominators are divided, respectively (e.g., 8/15 + 2/5=(8 + 2)/(15 ÷ 5) = 4/3). This procedure is called "Yoshiko's method" on the test sheet given to the students[4].

This written test was also administered to grades 10 and 11 (senior high school) students (N = ,26) in order to compare their performance with the sixth and seventh graders. Students' responses were classified by the "reasons" for their choices.

---

1. Find the answer to the following division of common fractions:

   8/15 + 4/5 (page 1)

---

2. To explain her method of computation, Yoshiko said,*"When I learned the multiplication of common fractions, I multiplied the numerators and the denominators respectively. So, in a similar way, I will try to divide the numerators and the denominators respectively."*

$$8/15 \div 4/5 = (8 \div 4)/(15 \div 5) = 2/3$$

However, Yoshiko was perplexed when she faced the next computation:

$$2/5 \div 3/4$$

What do you think about Yoshiko's method of division by fractions? Please circle one of the following items.

(1) Yoshiko's idea is correct.

(2) Yoshiko's idea is not correct.

(3) Although Yoshiko's idea is different from mine, her idea is correct.

(4) Other

Please write the reason why you chose the above item.

(page 2)

Study 2: After study 1 was conducted, a second study was needed to obtain a deeper understanding of the students' knowledge than could be gotten from analyzing the responses to written tests items. In study 2, about six months after the study 1, clinical interviews and instructional intervention were conducted with 16 selected students (7th grade) who had participated in study 1, to probe their understanding of division by a fraction.

Study 2 used two-person problem solving followed by interviews with the student pairs. The 16 students were paired according to their responses to the written test in study 1 as follows: Eight students who responded "Yoshiko's method is not correct" were interviewed in four pairs; four other pairs consisted of one student who answered "not correct" and one who answered "correct." These eight pairs were asked to solve a problem[5] similar to the problem on the written test in study

1 and were videotaped when they were working on the problem. The records were transcribed as verbal protocols and analyzed.

The interviews and instructional intervention in study 2 were based on the results of study 1. The interview questions were prepared by the author, taking the students' reactions to the written test into consideration. They included the following examples:

Student: "Yoshiko's method cannot always be applied."

Interviewer: "In the case of 2/5 ÷ 3/4, we can multiply both the numerator and the denominator of 2/5 to get 24/60. Then we can apply Yoshiko's method. What do you think of this?"

Student: "Yoshiko is not correct because '8/15 ÷ 4/5' means how many 4/5s are in 8/15."

Interviewer: "Don't you multiply the numerators and the denominators respectively, when you calculate multiplication of common fractions?"

Student: "Division by a fraction should be multiplication of the reciprocal."

Interviewer: "Can you explain the reason why you can get correct answers by doing so?"

In some cases, mostly in the first half of each interview session, these questions were intended for use in probing students' thinking. For example, when students said "Yoshiko had to turn upside down the numerator and the denominator, because division by a fraction should be multiplication by the reciprocal," the interviewer asked them "Can you explain the reason why you can get correct answers by doing so?"

In other cases, mostly at the end of the interview sessions, these questions were intended for use as instructional interventions. For example, when students said "Yoshiko's method cannot always be applied," the interviewer asked them "In the case of 2/5 ÷ 3/4, we can multiply both the numerator and the denominator of 2/5 to get 24/60. Then we can apply Yoshiko's method. We can apply her method to all fractions. Don't you think so?" Thus, follow-up interviews had the characteristics of both probing and instructional interventions.

At the end of interview sessions with each pair, students were asked to judge the "(in)correctness" of Yoshiko's method. These reactions were compared to those on the written test.

3. Results

(1) Results of Study 1

On the first part of the written test, most students got the correct answer (97% for 6th, 96.6% for 7th and 99.2% for 10th and 11th graders).

Table 1 shows the students' responses to Yoshiko's method at each grade level. As Table 1 shows, more than 71% of the 6th graders and more than 62% of the 7th graders judged the correct but unfamiliar procedure as a "wrong" procedure. By grade 11, these percentages drop to 35.8%; however, this still represents a significant proportion of the total group.

Table 1

Grades	6th	7th	10th	11th	Total
Correct	13 (4.3)	28 (9.7)	9 (15.3)	17 (25.4)	67 (9.4)
Not correct	215 (71.7)	181 (62.4)	29 (49.2)	24 (35.8)	449 (62.7)
Different but correct	60 (20.0)	56 (19.3)	14 (23.7)	15 (22.4)	145 (20.3)
Other	11 (3.7)	22 (7.6)	6 (10.2)	8 (11.9)	47 (6.6)
N.A.	1 (0.3)	3 (1.0)	1 (1.7)	3 (4.5)	8 (1.1)
Total	300 (100)	290 (100)	59 (100)	67 (100)	716 (100)

( ) indicates the percentage

The students who chose the response "other" could often reclassify their response into "correct" or "not correct" according to their comments on "Yoshiko's method." For example, one 6th grader commented that "Although Yoshiko's method is correct, it will be complicated and be in trouble with the second question. I prefer turning it upside down to Yoshiko's method." Another student said that "We cannot always apply Yoshiko's method to problems of division by a fraction. I think she has to invert and to multiply." These "other" students who mentioned the "(in)correctness" of Yoshiko's method were classified again as either "correct" or "not correct." Table 2 shows the results of this revision of Table 1.

Table 2

Grades	6th	7th	10th	11th	Total
Correct	76 (25.3)	95 (32.8)	24 (40.7)	34 (50.7)	229 (32.0)
Not correct	218 (72.7)	188 (64.8)	31 (52.5)	28 (41.8)	465 (64.9)
N.A. etc.	6 (2.0)	7 (2.4)	4 (6.8)	5 (7.5)	22 (3.1)
Total	300 (100)	290 (100)	59 (100)	67 (100)	716 (100)

( ) indicates the percentage

As Table 2 shows, about 73% of the 6th graders and about 65% of the 7th graders judged Yoshiko's method as "not correct." The reasons for their responses to "Yoshiko's method" were analyzed and classified by three persons. Once the categories were established, intercoder agreement was calculated to be more than 87%.

The categories of reasons given by the students who answered "not correct" are listed in the Table 3.

Table 3

A. Division by a fraction should be multiplication by the reciprocal

B. Yoshiko's method cannot always be applied

C. Yoshiko's method is contrary to the meaning of a division*

D. Divisions differ from multiplications

E. Yoshiko's method produces curious numbers**

F. Others

G. No Answer

* An example of "The meaning of a division" for the students was "8/15 ÷ 4/5 means how many 4/5s are in 8/15," which was derived from division of whole numbers.

** "Curious numbers" for these students were numbers like (2/5)/(3/4), because it was the first time they encountered to such numbers in this form.

Table 4

Grades		6th	7th	Total
	A	161 (61.2)	108 (52.9)	269 (57.6)
	B	66 (25.1)	36 (17.6)	102 (21.8)
	C	6 (2.3)	16 (7.8)	22 (4.7)
Categories	D	7 (2.7)	10 (4.9)	17 (3.6)
	E	2 (0.8)	3 (1.5)	5 (1.1)
	F	12 (4.6)	14 (6.9)	26 (5.6)
	G	9 (3.4)	17 (8.3)	26 (5.6)
Total		263 (100)	204 (100)	467 (100)

( ) indicates the percentage

Table 4 shows the frequency of each response. As seen in Table 4, one of the typical reasons why students judged "Yoshiko's method" as a "wrong" procedure was "because Yoshiko didn't turn it upside down and multiply" (61.2% for 6th graders and 52.9% for 7th graders). They would not allow Yoshiko to use an unfamiliar but correct procedure; still less did they try to explore the plausibility of her method.

The students who were classified into category B seem to be more thoughtful than those classified into category A. Some students in this category mentioned that "it's important for a method of calculation that it can be applied to any case." However, most of them added that "If I were Yoshiko, I would not do such a thing." They also never tried to explore the plausibility of Yoshiko's method. The students classified into category D were those who thought that multiplication and division were unrelated.

Next, the responses of the students who chose "correct" were also classified by using several categories. Sixteen categories were identified, which varied from "Yoshiko is correct because I could get the correct answer by using her method" to "If we find a common denominator, we can apply her method to any division by a fraction." Some students simply wrote their method (multiplication by the reciprocal) to calculate 2/5 ÷ 3/4, while others wrote the explanations for the algorithm.

Often observed were students who said "Although Yoshiko's method is correct, I prefer turning it upside down and multiplying to her method" (44 students out of 171 who chose "correct" (25.7%)). These students seem to have judged Yoshiko's method as "correct" by the fact that the first division (8/15 ÷ 2/5) using her method produced a correct answer. Some examples of other categories are shown below with the students' calculations.

a) *complex fractions*: Many students ($n = 25$) used complex fractions when they applied Yoshiko's method to the second division (2/5 ÷ 3/4):

$$2/5 \div 3/4 = (2/3)/(5/4) = (2/3) \times 12/(5/4) \times 12 = 8 \div 15.$$

b) *decimal fractions*: Some students ($n = 5$) mentioned transforming from a common fraction to a decimal fraction:

$$2/5 \div 3/4 = (2/3)/(5/4) = 2/3 \div 5/4 = (2 \div 5) \div (3 \div 4) = 0.4 \div 0.75 = 40 \div 75 = 8/15.$$

c) *common denominator*: Some other students ($n = 4$) found the common denominator;

$$2/5 \div 3/4 = 8/20 \div 15/20 = (8 \div 15)/(20 \div 20) = 8/15.$$

d) *"law of divisions"*: Some students ($n = 5$) described their explanation of the reason why they used "turning it upside down and multiplying," using a law of division. "Law of divisions" means some regularities in division such as "$a \div b = (a \times c) \div (b \times c)$";

$$2/5 \div 3/4 = (2/5 \times 4/3) \div (3/4 \times 4/3) = 2/5 \times 4/3.$$

(2) Results of Study 2

To summarize the results of the study 2, an overview of some of the major tendencies of the students will be given.

First, the four students pairs who answered "not correct" insisted on the "incorrectness" of Yoshiko's method. These students resisted instructional interventions. In the case of four pairs composed of two students of different opinions, "not correct" students at first were not so resistant to their partners. At the beginning of working on the problem, "not correct" students were engaging in the same tasks with their partners to find out how to apply Yoshiko's method to "2/5 ÷ 3/4" in the problem. They found some explanations for Yoshiko's method by finding such methods as "complex fractions" or "decimal fractions" (see the results of study 1). However, when the interviewer showed their writings in study 1, after they finished their problem solving, they returned to say "not correct."

In the end, only two of twelve "not correct" students were convinced to change their opinions in the process of two-person problem solving. And after both the two-person problem solving and the interviews, five students still held to "not correct."

Second, during the interviews, students' beliefs about mathematics and mathematics learning often became clear. For example, after they finished their problem solving, "student 1b" (a "not correct" student in "pair 1," composed of two students of different opinions) often mentioned a "rule." "The answer (gotten by the Yoshiko's method) is certainly correct. But there are two fractions, one to which we can apply Yoshiko's method and the other to which we cannot apply her method. Thus, the rule doesn't work. So, I think Yoshiko is not correct." Previously, on the written test, he had described "(Yoshiko is not correct) because there is a rule that division of fractions is to be calculated by exchanging the denominator and the numerator (but Yoshiko does not use this rule))." He was a typical student who mentioned a "rule without reasons."

Another student, 3b (one of the "not correct" students in "pair 3," composed of two "not correct" students) mentioned a "rule in the classroom," which compels her to follow the opinion of the majority in her class. "Isn't there something like a rule? It seems to be reasonable for me in the classroom that I follow whatever most classmates say or do, although I don't know the meaning." These statements, and others like them, reflect students' beliefs about and attitudes toward mathematics and mathematics learning.

Third, in study 2, a curious "students' logic" ($n = 5$) was identified during the instructional intervention. When they were requested to answer a question about their wrong justification, students often turned to the other points or the other "examples" in their explanations. Here is an excerpt from a protocol of interviews with student 4a (one of the two "not correct" students in "pair 4"). She kept insisting on the "incorrectness" of the Yoshiko's method as follows.

Interviewer: Finally, I ask you again, what do you think of Yoshiko's method?

Student 4a: Not correct. (Student 4b also nodded.)

Interviewer: Why is it?

Student 4a: "2/5 divided by 3/4" means neither a division of 5 by 4 nor 2 by 3. We have to divide one number, 2/5, by another number, 3/4. So, we cannot separate the denominator and the numerator.

Interviewer: Then, the success by Yoshiko as to both 8/15 and 9/14 were mere happenings?

Student 4a: Uhm...it might not happen...

Student 4b: Didn't we try other fractions?

Student 4a: It's something like...a statement that holds in the case of a cube but that doesn't hold as to a rectangular prism.

Interviewer: Then, Yoshiko's method can be applied in some cases?

Student 4a: Yes.

Interviewer: If we multiply both the numerator and the denominator of 2/5 to get 24/60 (writing on a paper), then we can apply Yoshiko's method. By doing like this, can we apply her method to any fraction?

Student 4a: Uhm...it seems to be right...but...the meaning of the number sentence is...if this ($8/15 \div 4/5$) means a word problem, "15 divided by 5" will be a nonsense phrase...

She seemed to behave quite "naturally" and to be "logical" in her arguments. But on some points, her reasoning was not consistent with her own argument. Thus, although these "logics" of reasoning differ from the logic in mathematics, it seems to represent the logic of these students.

4. Discussion

It would be hard to judge how representative the subjects in this study are of Japanese students. Nevertheless, their responses to "Yoshiko's method" probably reveals general trends. In this section, we will consider the results of this study from the following two points of view: predominance of the form over the reasoning and the question of students' validations of formal knowledge.

(1) Predominance of the form over the reasoning: rigid thinking of students

Many students tended to insist that "Yoshiko's method" was not correct because she did not use the "turn it upside down and multiply algorithm" for division by a fraction. They did not accept "Yoshiko's method" because it differed from their algorithm. Furthermore, many never tried to explore the plausibilities of her method. They preferred the "form" of the algorithm without taking account of the meaning. This "rigid applications of algorithm" (Gardner, 1991) was the striking feature of most students who chose "not correct." Gardner wrote

> The ways in which mathematics is customarily taught and the ways in which students learn conspire to bring about a situation where students perform adequately so long as a problem is stated in a certain way and they can therefore "plug numbers" into an equation or formula without worrying about what the numbers or symbols mean. (p. 161)

It is true that computational algorithms are useful precisely because they do not require reflection and that once they have been automatized, little mental effort is needed. Therefore, it is natural for students to perform an algorithm without taking account of its meaning "so long as a problem is stated in a certain way." It is important for them, however, when they encounter a problem stated in an unfamiliar way, that they explore it and think about the reason why it holds (or does not hold).

The rigidity of the students who chose "not correct," or their flat response to "Yoshiko's method," seemed to be shaped by their beliefs about and attitudes toward mathematics and learning mathematics. As one sixth grader wrote "Yoshiko is not correct because there is a rule to follow;" thus, students' attitudes were rule-oriented. These rule-oriented attitudes toward mathematics are well documented (e.g., Erlwanger, 1975; Schoenfeld, 1985). We cannot simply accept "high achievement" as evidence of solid mathematical thinking.

(2) Preservation of the logic: students' validations of formal knowledge

In the mathematics program in Japan, multiplication and division of common fractions are introduced in the sixth grade. Regarding the objectives of teaching these contents, the National Course of Study says:

> To help children understand the meaning of multiplication and division of fractions and become able to use them as well as to help them deepen their understanding of multiplication and division in general (JSME, 1990).

Thus, it is important for us to know the students' understanding of and underlying reasoning concerning division by a fraction if we are to succeed in our instruction.

Understanding how procedural knowledge and conceptual knowledge relate to one another is one of the major foci in mathematics education. Poor performance in school mathematics often can be traced to a separation between students' conceptual and procedural knowledge of mathematics (Hiebert & Wearne, 1986). However, this study suggests that we have to examine "good performance" from the same point of view in some cases.

Relating conceptual and procedural knowledge was in evidence in only a few cases, and even then the explanations and justifications were of limited quality. Building conceptual and procedural relationships may involve a complex process which need not require conceptual competence. Although the bulk of the theoretical arguments supports building meaning for procedures before practicing it for efficient execution, Noddings (1985) has suggested other possibilities. She distinguished three different domains for mathematical activities: "informal," "formal" and "metadomains." In particular, she recognized the possibility of approaches which allow for procedural information to provide an occasion for conceptual development. She wrote that

> ...we should abandon efforts to characterize mathematical learning as 'top-down' or 'bottom-up' and instead consider all questions concerning movement through the domains as context- or topic-bound (p. 123).

Further, she wrote that

> Students are taught to handle complex fractions by standard division, for example, but bright youngsters can certainly invent short cuts and, in doing so, they are led to explore the structural underpinnings of their inventions. (p. 126)

A few students in this study showed that they could explain and justify their procedures, including the "invert-and-multiply algorithm," in rather abstract ways (e.g., using a "law of division"; $a \div b = (a \times c) \div (b \times c)$). It seems to be possible to construct "a knowledge base for the procedure that is connected to a rich supply of knowledge about related procedures and concepts" (Silver, 1986) by moving into "metadomains," say, by emphasizing "laws of division" from the teaching of division of whole numbers.

This study suggests that we have to pay more attention to the "hidden" dimension of teaching mathematics when we analyze high achieving students' thinking in mathematics, in general, and on division of fractions, in particular. Finally, "students' logic," which was found in study 2, seems to have potential for further exploration.

NOTES

1. The research reported in this paper was partly supported by a Grant-in-Aid for Scientific Research (No. 05780146) from the Ministry of Education, Science and Culture. The author thanks Professor Jere Confrey for her comments on a draft of the manuscript.

2. According to the results of the IEA study, one division item of common fractions ("3/5 ÷ 2/7" given in the form of (3/5)/(2/7)) resulted in a mean of 38% correct across countries (Robitaille, 1989, p. 107).

3. As a result of their experiences in mathematics classrooms, students develop a set of beliefs about mathematics and about mathematics learning. For example, many students believe that mathematics is mostly memorization and that all mathematics problems can be solved, if at all, in a few minutes or less (See Schoenfeld, 1985). These statements, and others like them, reflect students' beliefs about and attitudes toward mathematics. These beliefs and attitudes have been shaped by their experiences in mathematics classrooms despite the fact that neither the authors of mathematics curriculum nor the teachers who taught the students had intentional objectives related to students' beliefs about and attitudes toward mathematics. Thus, the author uses the term "hidden."

4. "Yoshiko" is one of the most popular female names in Japan.

5. In this problem, another "example," 9/14 ÷ 3/7, to which we can apply "Yoshiko's method," was added.

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REVISION OF THE COURSE OF STUDY FOR STUDENT GUIDANCE IN HIGH SCHOOLS

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1. Correspondence for Student Diversification

Since 1990, more than 94% of Japanese junior high school graduates enter high schools every year. With this increased entrance rate to high schools, we find more students who cannot keep up with other students. It is becoming a social problem.

Due to social diversification, there are now so many different paths for students to choose after graduation that we have to meet these various new needs. Another serious problem is that more students are showing a tendency to dislike mathematics.

Considering such a situation, we propose a division in the teaching of mathematics. This will make it possible for teachers to make

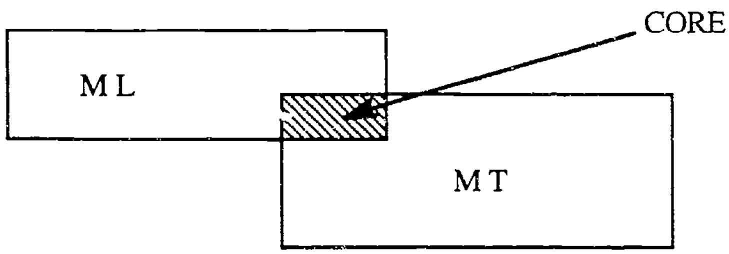
a. a curriculum for ML (Mathematical Literacy) as mathematics users, and

b. a curriculum for MT (Mathematical Thinking) as mathematics makers.

The common aim is to cultivate "mathematical intelligence." We propose to try to give students experiences to make a practical use of mathematics in the learning activities. In MT, we emphasize that these two fields are exclusive, but will supplement each other. Using computers will be a big help to make the study of these fields possible.

2. The Curriculum Structure will be COM

The goal common to M L and M T is "mathematical intelligence;" that is,



Where ML overlaps with MT is called the "core," which will be a required subject. The other parts are called "options," and these will be elective subjects.

The required subject is "Mathematics I" which includes

(1) quadratic functions

(a. knowing change)

(2) geometric figures and measuring

(b. measuring quantity)

(3) dealing with numbers of articles

(c. counting)

(4) probability

(d. estimating)

We will help students to know the basic usefulness and intelligence of mathematics in these studies. The curriculum structure is as follows:

Required:	Mathematics I	(4 credits)
Options:	Mathematics II	(3 credits)
	Mathematics III	(3 credits)
	Mathematics A	(2 credits)
	Mathematics B	(2 credits)
	Mathematics C	(2 credits)

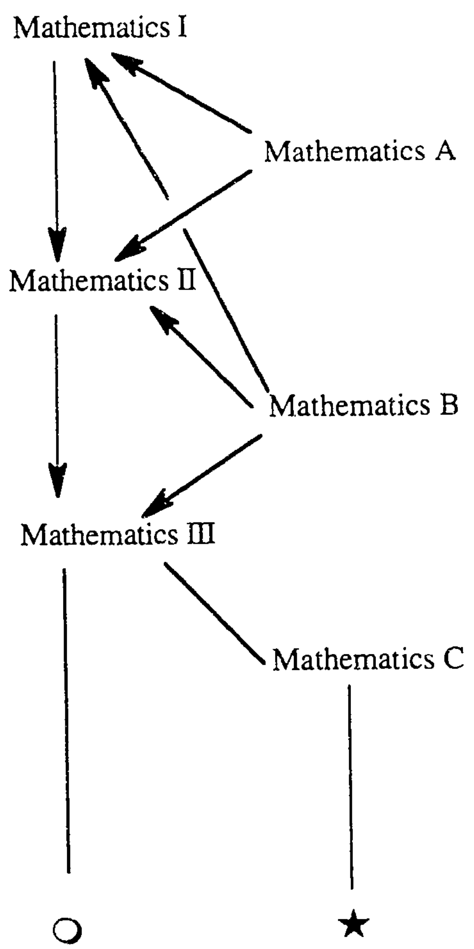
The options have three main functions, which are:

1) Remedial Option

2) Side Option

3) Advanced Option.

Teachers can choose any of these according to their situations. There are two choices in the elective subjects.



○ In this course, we attempt to expand and develop Mathematics I.

★ In this course, each subject offers 4 credits and students choose 2 credits from each of them.

3. Practical Use of Computers

Computers are now necessities in the information society. Also, in the circumstances of students, computers are very common and becoming a must in our everyday lives. So, it is necessary to use computers in the schools,

1) as a part of the means to study mathematics, and

2) in learning how to operate computer as users.

Mathematics A, B and C, mentioned above, include teaching materials using computers. Thus, we expect that students will better understand and develop their powers of reasoning, discovering, and creative thinking. Many vocational high schools offer program 2), learning how to operate computers as users. By 1996 all high schools in Japan will have sufficient numbers of computers for one complete class of students.

4. Problems in Preparing Curriculum

Every high school is now getting ready to teach according to the new manual for student guidance, and also preparing to put the new curriculum into action. The problems which teachers have are as follows :

Problem 1: The subjects of the entrance examination for colleges are not fixed.

Problem 2: The curriculum is restricted by other conditions.

Problem 1: Japanese college entrance examinations have a great influence on education. About 30% of high school students enter colleges, and people are getting more concerned about college education. So, the subjects tested in the college entrance examination heavily influence the high school curriculum. For example, the standard test required of all twelfth-grade students wishing to enter a university covers Mathematics I and II. Students who would like to take the science courses must take examinations covering Mathematics I, basic analysis, algebra and geometry, differential calculus, and integral calculus.

But these days some national colleges exempt mathematics entrance examinations. For these reasons it is getting very complicated in organizing the curriculum.

Problem 2: It is very difficult to keep 18-19 credits for mathematics, which we have now, for various reasons. For example, social studies needs more credits, and domestic science, physical education, and foreign languages also need more time. Additionally, students are coming to dislike mathematics. In spite of the fact that mathematics is becoming more important in society and that the demand for mathematics is increasing, we cannot give enough opportunity to students to study mathematics in school. The contradiction between the developing needs on the one hand, and the needs of mathematics on the other hand, is rapidly emerging. We are making efforts to solve problems relevant to this situation.

DATA 1

EXAMPLE OF MATHEMATICS TEXTBOOK

" MATHEMATICS I "(A required Subject)
(Completion Unit 4)

Content

" MATHEMATICS A "
(An Optional Subject)

Content

COMPUTER LITERACY IN JAPANESE HIGH SCHOOL MATHEMATICS - COMPARE THE PRESENT-DAY MATH TEXTBOOK WITH THE REVISED MATH TEXTBOOK

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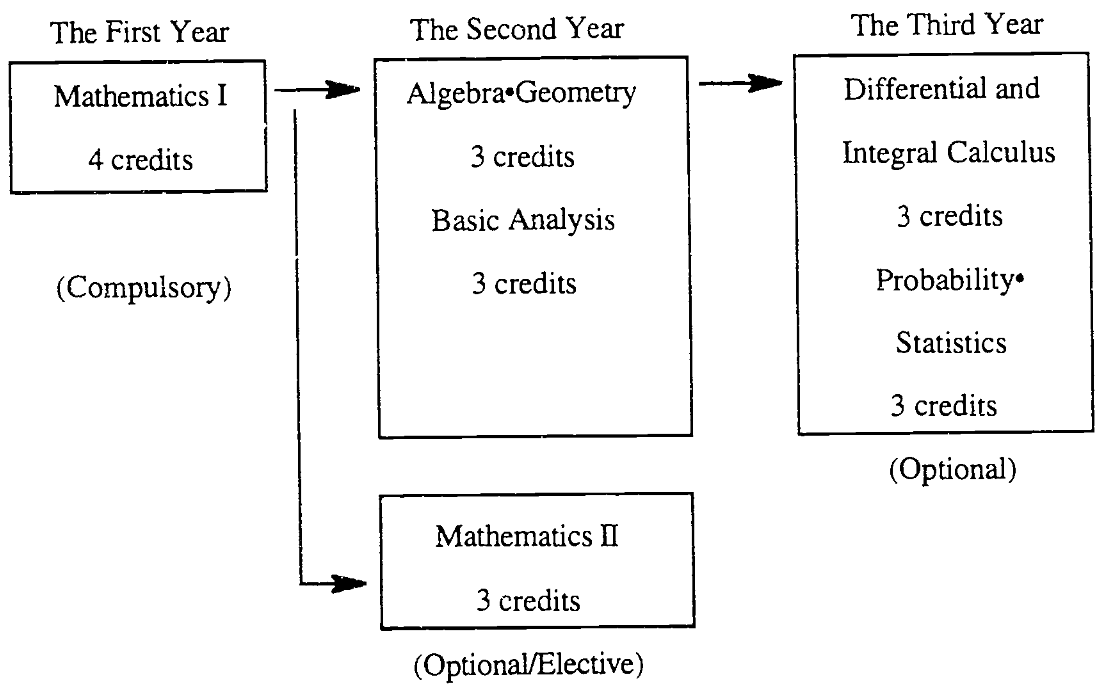
1. The Social Background of the Computerized World

Computers are now an indispensable part of our daily life, upon whose various aspects they have a great influence. As hardware came to have better performance and got smaller around 1982, personal computers became less expensive, thanks to mass production, and more familiar along with a lot of software packages. Some children use these for computer games and others utilize them as part of their own programs. Some companies and organizations have also introduced good PCs and software packages for various purposes: drawing up documents, performing calculations, using as a database, and so on.

As stated above, our society is becoming more and more computerized. In order to train students to be talented people that meet the needs of our modern advanced information-oriented society, more and more schools have been introducing computers since 1988. Especially in the last two years, almost all junior high schools and high schools have gotten enough computers for each student in one class to use to his or her heart's content.

The present-day curriculum (announced in 1978 and put into effect in high schools in 1982) requires teachers to instruct students in the basic principles of computer science in Mathematics II-- a subject taken mainly in the second year of upper secondary school; nevertheless, this curriculum is far from satisfactory as an educational system that should serve the needs of advanced information-oriented society. This is why the revised curriculum (announced in 1989 and implemented in high schools in 1994) contains a provision that computer science should be dealt with in junior high schools as well as in the high schools.

2. The Current Model in High School Mathematics (Announced in 1978 and Enforced in 1982)



* The number of credits is a standard. 1 credit is 35 lesson-hours or 35 fifty-minute periods of study throughout the year.

Mathematics II contains the basic principles of "Algebra•Geometry," "Basic Analysis,' "Probability•Statistics" and "Computers and Flowcharts." This subject is expected to be taken mainly in the second year, but most schools encourage students to take "Algebra•Geometry" and "Basic Analysis" because the college entrance exams have no questions concerning computers. The textbooks of Mathematics II are usually thick enough to contain lots of chapters that cover the content for 6 credits, but most schools take some chapters and leave others according to their local conditions.

Objective of Mathematics II

To help pupils understand fundamental concepts, principles and laws in broader fields of mathematics which follow the content studied in Mathematics I, and thereby to deepen their appreciation of the role played by mathematics in society.

The aim in the present-day Mathematics II makes no mention of computers.

Content of (6) Computer and Flowcharts

- Content

  (6) Computer and flowcharts

  a. functions of electronic computers

  b. algorithms and flowcharts

- Remarks concerning content

  (3) In (6), actual experience preparing programs for the computer, running them, and analyzing the results should be included.

How Textbooks Deal With the Current Curriculum

The Content of the Curriculum	Mathematics II (SANSEIDO)	High School Mathematics A (KEIRINKAN)
(1) Probability and Statistics (2) Vectors (3) Differention and Integration (4) Sequences (5) Various Functions (6) Computer and Flowcharts a. functions of electronic computers b. algorithms and flowcharts	Chap.1 Sequences p.16 Chap.2 Differention p.26 Chap.3 Integration p.24 Chap.4 Various Functions p.34 Chap.5 Vectors p.24 Chap.6 Probability p.30 Chap.7 Statistics p.32 Chap.8 Computer p.22 (1) computer (2) flowcharts (3) programing and calculation	Chap.1 Trigonometric Function p.14 Chap.2 Exponential and Logarithmic Function p.18 Chap.3 Sequences p.16 Chap.4 Differention p.28 Chap.5 Integration p.22 Chap.6 Vectors p.18 Chap.7 Permutation and Combination p.14 Chap.8 Probability p.28 Chap.9 Statistics p.22 Chap.10 Computer and Flowcharts p.21 (1) function of computer (2) flowcharts (3) flowchart of calculation (4) repetitive calculation
The Number of Pages	208	201
The Number of Flowcharts	13	14
The Number of Program	6 (BASIC)	1(BASIC)

Articles concerning Computers are common to the two textbooks.	assembly language, algorithm, arithmetic unit, storage units, machine language, line number, compiler-language output unit, set up, initial value, control unit. central processing unit, data, electronic computer, flowchart, input unit, address, programming, program, stored program system, branch, instruction, memory, loop	arithmetic unit, storage units, machine language, output unit, control unit data, flowchart, input unit, program, instruction memory, loop

3. The Predicted Model in the New Curriculum (Announced in 1989 and Enforced in 1994)

The First Year		The Second Year		The Third Year
Mathematics I 4 credits Mathematics A 2 credits	→	Mathematics II 3 credits Mathematics B 2 credits	→	Mathematics III 3 credits Mathematics C 2 credits
Mathematics I 4 credits Mathematics A 2 credits	→	Mathematics II 3 credits Mathematics B 2 credits	→	Mathematics III 3 credits

Mathematics I 4 credits Mathematics A 2 credits	→	Mathematics II 3 credits	→	Mathematics III 2 credits
Mathematics I 4 credits	→	Mathematics II 2 credits		

* Mathematics I is compulsory, but the others are optional.
* The number of credits is a standard. 1 credit is 35 lesson-hours or 35 fifty-minute periods of study throughout the year.

The new curriculum is implemented in the next school year, and it is almost impossible to foresee which type of the above four most schools will adopt. It must also be added that textbooks of Mathematics A, B, and C are all thick enough to contain the content for 4 credits. Twenty-three kinds of textbooks of Mathematics I and Mathematics A are published with 12 companies. The others have not yet been published.

Objective of Mathematics A

> As a broader content than "Mathematics I," to help students understand numbers and algebraic expressions, plane geometry, sequences or computation using computers, to encourage them to master basic knowledge and skills, and to develop their abilities to think and cope mathematically in dealing with various phenomena.

Mathematics A sets a clear goal about computers which Mathematics II makes no mention of.

Content of (4) Computation and Computer

- Content

  (4) Computation and Computer

  a. Operation of computer

  b. Flowchart and programming

  c. Calculation using computer

- Remarks concerning Content

  (6) As for the content (4)-b, the teachers should put their emphasis on helping students' understanding of the structure of programming, but only short programs should be treated. As for the content (4)-c, use of the computer should be at the level of using it for processing those computations that concern what students have learned in the lower secondary level or "Mathematics I"

How Textbooks Deal With the New Curriculum

The Content of the Curriculum	Mathematics A (SANSEIDO)	High School Mathematics A (KEIRINKAN)
(1) Numbers and Algebraic Expressions (2) Plane Geometry (3) Sequences (4) Computation and Computer a. Operation of computer b. Flowchart and programming c. Calculation using computer	Chap.1 Numbers and Algebraic Expression p.36 Chap.2 Sequences p.34 Chap.3 Plane Geometry p.36 Chap.4 Computation and Computer p.28 (1) function of computer (2) program in BASIC (3) calculation using spreadsheet	Chap.1 Numbers and Algebraic Expression p.50 Chap.2 Sequences p.38 Chap.3 Plane Geometry p.42 Chap.4 Computation and Computer p.47 (1) operation of computer (2) flowchart (3) calculation using computer
The Number of Pages	134	177
The Number of Flowcharts	4	9
The Number of Programs	1 1 (BASIC)	2 4 (BASIC)

Articles concerning Computers are common to the two textbooks.	symbol of operation, keyboard, storage, storage unit, row, line number, delete, repetitive statement, repetitive variable, summary sheet, integrated circuit, output, output unit, conditional statement, chip, central processing unit, display, flowchart jump statement, input, input unit, input statement, hardware, hard disk, spreadsheet, program, programming language, floppy disk, printer, BASIC, menu, sequence, worksheet	application software, operating system, keyboard, line number, computer system, output, initial value, direct mode, display, flowchart input, hardware, file, file name, program, programming language, program mode, branch condition, Basic, variable variable name, main frame infinite loop, unconditional branch, instruction word

Objective of Mathematics B

As more advanced content than "Mathematics I" and "Mathematics II," to help students understand vectors, complex numbers and the complex number plane, probability distribution, or algorithm using computer, and to encourage them to master basic knowledge and skills, and to develop their abilities to think and cope mathematically in dealing with various phenomena.

Content of (4) Algorithm and Computer

- Content
  - (4) Algorithm and Computer
    - a. Function of computer
    - b. Program of various algorithms
- Remarks concerning Content
  - (5) As for the content (4)-b, programming should be restricted to the level of the Euclidean algorithm and calculation of roots by iteration.

Objective of Mathematics C

Through using computers from the viewpoint of applied mathematical science, to help students understand matrix and linear computation, various curves, numerical computation or statistics, and to encourage them to master knowledge and skills, and to develop their abilities to think and cope mathematically in dealing with various phenomena.

Content of (2) Various Curves

- Content
  - (2) Various Curves
    - a. Algebraic expressions and geometrical figures
      1. curve represented by equation
      2. ellipse and hyperbola
    - b. Parametric representation and polar coordinate
      1. parametric representation and polar coordinate
      2. polar coordinate and polar equation
      3. various curves
- Remarks concerning Content
  - (3) As for the content (2), the teacher should help students observe and consider various curves by making use of computers and others, and become able to actually draw simple geometrical figures.

The new curriculum requires teachers to instruct students in computer science both in Mathematics B and in Mathematics C. The aim singles computers out for special mention.

4. Conclusion

Let us compare the new curriculum with the current one concerning computers. Mathematics A aims to train students to be proficient in operating computers, processing data with software packages and supplying computers with programs; on the other hand, Mathematics II contains only notational content about computers.

I insist that math teachers should seize the opportunity of the change of the curriculum to take the initiative in instructing students in computer science. Math teachers on the job may be most insensitive to the changes of society. We must no longer say nonchalantly that it's unnecessary to teach content that is not given in the entrance exams. Whether the new curriculum will be successful or not depends on our enthusiasm for the educational reform.

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ON SCHOOL MATHEMATICS CURRICULUM DESIGN

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The reform of mathematics education always focuses on the reform of the mathematics curriculum. Since 1958, I have participated in several projects of school mathematics curriculum design. During the period of more than thirty years, I have gradually found some rules of mathematics curriculum development.

There are three principle factors which impact, control and determine mathematics curriculum development. They are the demands from social, political and economical development, the demands from the development of mathematics, and the demands from the development of education. The development of mathematics curriculum is determined by the homogeneous integration of these three factors. This paper will interpret how to integrate them in order to improve the development of the school mathematics curriculum, based on the experiment of the "Experimental Textbook of School Mathematics" (briefly "Experimental Textbook" below).

I. The Demands From the Development of Society

Of all the impetuses which improve curriculum development, none is greater than that exerted by social development. The demands from social development are as follows.

1. *Help to achieve the goals of education.* Education should serve the Chinese socialist economic construction. Recently, Chinese society is being transformed from industrial into informational. More and more people will engage in management and the production of information, and the period of renewal of knowledge and the lifespan of professions are being shortened. In order to adapt to the rapidly changing society, we should highly enhance the quality and capacity of the people, so as to enable them to have the ability for lifelong learning.

2. *Practicality.* The school mathematics content should be useful. It may be used to solve practical problems in social production and in social life. It may be used to train students in

thinking. Thus, the curriculum content should be supplemented with some applied mathematics, such as computer literacy, statistics and theory of probability.

Mathematics is not only a tool for solving practical problems, but it is also used to train people in independent thinking, to foster students' mathematical quality; i.e. understanding the value of mathematics, confidence in their ability to do mathematics, abilities to communicate mathematically, and to reasoning mathematically.

3. *Ideology and educativeness.* Students should be developed in an all-round way--morally, intellectually and physically, with high ideals, and a deep love for the Motherland. Thus, the mathematics curriculum should introduce certain materials from the Chinese history of mathematics and should explain the curriculum content from a dialectical materialism point of view.

In order to meet the above demands, the "Experimental Text" takes the guideline: "to make the curriculum content fundamental, simple, and practical."

II. The Demands From the Development of Mathematics

1. *A close coordination as a whole of the most fundamental parts of algebra, geometry, analysis and the theory of probability.*

Algebra, geometry and analysis are widely applied to solve practical problems in related subjects. Therefore, the school mathematics curriculum should coordinate them as a whole. In the 1950s' and 1960s of this century, there appeared the design of algebraic structuralizing school mathematics curriculum, as well as some project of analyzing mathematics. However, they were not successful. One of the reasons for the failure may be that the design didn't meet the "close coordination" requirement.

2. *Appropriate implementation of some applied mathematics.* After World War II, applied mathematics is rapidly developing, many new branches have arisen and the scope for applications is quickly widening. The theory of probability, statistics, computer literacy and discrete mathematics are included in mathematics curriculum.

3. *The embodiment of systematicness.* According to the views of Bourbaki, every mathematics system can be reduced to three basic structures: algebraic, order, and topological. Hence, in order to meet the requirement from the structure of mathematics knowledge, the school mathematics curriculum should have a certain degree of systematicness and precision in logic.

A SURVEY OF DOING SUMS ON A MENTAL ABACUS

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There is an ordinary primary school, Minghe Primary School, in Cixi of Ningbo, which is located in Zhejiang Province in China. A very common teacher of arithmetic, together with some other teachers, has turned out a number of child prodigies who have "supernatural" abilities in calculation. This has caused a sensation throughout the world. They have been invited to visit various countries, including the U.S. and Japan, and those who saw their performance were so amazed that they turned up their thumbs in praise, saying "OK."

There were several primary school students doing an arithmetic problem mentally: 9370 + 4783 + 3254 + 61493 + 15879 + 97426 + 73421 – 86543 + 59384 + 2873 = ? In less than 25 seconds, they got the answer which were on the slips of paper in their raised hands: 241344. The answer was verified to be correct by a calculator. It was breathtaking to notice that scarcely had the problem been assigned, when the kids raised their hands with the correct answer. They didn't use a pen or an abacus; they only used their heads.

That was the result of employing the method of doing sums on a mental abacus. This can also be applied to multiplication with multi-figure numbers as well as the extraction of a root. The method can be summed up as "using an abacus in the head," as if the abacus has been established inside one's head, the figures being the beads on the mental abacus. Even a big, complicated sum can be worked out by this "psychological abacus" (mental abacus), and the answer is quickly and correctly obtained. Here is one more example. Before a problem like 25137 x 687 = ? is given, the kids will have 25137 x 687 = (25137 x 600) + (25137 x 80) + (25137 x 7). And after all the individual products are put together, they get the result. This complicated process is done by way of the "mental abacus."

Psychology proves that children at an age from 6 to 11 years are good at thinking in images and imitation. What they have done with their hands will leave a deep impression on them. So, the approach of teaching young students to do sums on a mental abacus conforms to this psychological characteristic of the young.

It generally takes three years to master doing sums on a mental abacus. The instruction can be divided into three stages: (1) The student is required to do calculations on the abacus - practice moving the beads while looking at them until the student is highly skillful in using the abacus. (2) The student is required to look at the abacus without touching the beads. By using the abacus mentally, the student finally acquires the ability to skillfully do sums as when the abacus is actually touched. (3) The student is required to imagine that there is an abacus in his/her head, and can read various sums on this "mental abacus" as if the beads were being moved. The student practices simple problems first, and then complicated problems, until finally it is possible to do any problem automatically.

This teaching method in arithmetic has evolved from the training conducted in a group of calculation-lovers on an abacus among the students. In their activities, they used to combine the three ways (using an abacus, a pen, and the head) in calculation. The inventor of the present way of doing sums on a mental abacus is none other than an ordinary teacher of arithmetic in Minghe Primary School, Mr. Wang Weida. He took great pains to persist in his experiment, and succeeded in enhancing the level of the time-honored skill in using the ancient Chinese abacus. He has opened up vast vistas for the mastery of the abacus calculation by many more ordinary people. This has also contributed to the reform of the teaching and learning in both primary and middle schools, breaking a new path for us to achieve a higher quality in teaching on a wide scale .

The practice of teaching students to do sums by means of a mental abacus has effectively enhanced the students' level of understanding and intelligence and their proficiency in calculation. The students have been more efficient in their study, better in thinking, more clever and deft, more will-powered and have been playing a better role in the main body of education. Now there are 2600 students in 53 study classes in Cixi, who are learning the method of doing sums on a mental abacus. The result of the experiment has proved that these students are better than the average ones as far as the level of intelligence and speed of doing sums are concerned.

We are justified to say that doing sums on a mental abacus has borne new fruit in exploiting the students' psychological factors and reinforcing practice in the field of education.

THE MATHEMATICS OLYMPIAD AND MATHEMATICS EDUCATION IN QINGDAO, CHINA

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The Mathematics Olympiad (MO) has already become an important part of mathematics education in China. It is a mass extracurricular activity which is loved by most of the middle school and primary school pupils. No doubt the Chinese Mathematics Competition was influenced by the former Soviet Union. When I visited the Moscow Lenin Normal University in 1990, I could feel that the mathematical education thought in China and Russia are much alike today.

In 1956, at the suggestion of Hua Logeng and Su Buqing, the two famous mathematicians in China, the senior middle school mathematics competitions were held in the four big cities (Beijing, Shanghai, etc.). In 1962 and 1978 the second and third mathematics competitions were set off. Eight cities and provinces initiated a mathematics competition which had an influence on the whole country, except Taiwan province. Since 1980, the nationwide Mathematics Competition has been held every year. In making the examination papers, we adopt the same method as the International Mathematics Olympiad (IMO). In 1985, the same kind of competition spread over the junior middle and primary schools. Among them the most important one is Hua Logeng Gold Cup Invitation tournament.

The Mathematics Olympiad of primary schools is the official competition accepted by the National Commission of Education in China. It is held on a large scale. The competitions are divided into two steps. Every school can send participants to the preliminary competition. After being selected, the participants are concentrated immediately into the cities or districts. Then they take part in the final competition. The examination papers are checked uniformly in cities and in districts, and the papers with high marks must be sent to the competition committee of the province to be verified and approved. From this year (1993), a general final is added. Two teams (6 players) can be organized by each province. Qingdao won the first place among 53 teams.

Since 1985, Chinese middle school students began to take part in the International Mathematics Olympiad and won the first place in 1989, 1991, 1992, 1993. In the past years, the practice of mathematics competitions has shown clearly that the competition has been useful to heighten the quality of mathematics teaching in the middle and primary schools.

The MO has furnished many useful tasks and experiences in problem solving. A professor of Moscow University once made a speech "What does problem solving mean?" Her answer is that "problem solving means to change the problems being solved into solved ones." In normal classroom teaching, we help students to solve conventional, routine, and familiar problems, and then to solve new, unfamiliar, and non-routine problems. However, the problems that appear in the mathematics competitions are problems with changing and developing forms. We encourage all pupils to join the mathematics competitions at the various levels. This includes the lower levels, just like classroom teaching.

The MO must take, as a basis, just the ordinary lessons. We put most of our energy into reformation of everyday teaching in Qingdao. In 1980, we launched a plan to enhance the quality in mathematics teaching on a large scale in the junior middle schools. The research is divided into six stages:

1. 1980-1981 Inquisition
2. 1981-1983 Commenting on one's teaching and choosing a good teaching model
3. 1984-1986 Running experimental classes
4. 1987-1988 Summing up and spreading the experience. Two successful models are used in the whole Qingdao City.
5. 1989-1992 Comparing and improving the model
6. 1993-1996 The second round for investigation.

Accompanying the rise in the quality of mathematics teaching, the achievement of the MO has been heightened year by year. The number of students who take part in the mathematics competitions now reaches over 20,000 students every year. Winning a lot of medals in the mathematics competitions is not just an occasional event. Similarly, Chinese students won first place in the IMO - this is also a result of improving the ordinary teaching of mathematics. However, a question has now been raised from the circle of Chinese mathematics education: "We wonder if Chinese pupils can achieve a high score in a written time-limited examination only, but what about their creative ability in mathematics?" We are not sure.

CLOSING CEREMONY

Remarks by Professors Sawada and Becker:

Colleagues and friends:

First we would like to acknowledge and congratulate the Chinese on the remarkable success of their students on the International Mathematics Olympiads.

In his talk earlier in Shanghai, Professor Chen Changping asked the question "Why is it a pleasure to meet people from far away?" And then he answered the question by commenting "Because it is a pleasure to meet and talk with people with similar interests." In this case, it is mathematics educators from three different countries, and the common interest is problem solving in mathematics. We concur with Professor Chen.

To make a meeting such as this really useful, adequate arrangements need to be made. The arrangements for our meeting have been handled by Professor Zhang Dianzhou in an excellent and outstanding manner. When we say arrangements, we mean much more than the physical arrangements, which have been superb. We mean arranging the intellectual and scholarly aspects of the program - that is, organizing a program in which participants write papers for, and present to, the seminar participants for consideration and discussion. All our colleagues join us in praising Professor Zhang for the contributions he has made towards setting a seminar agenda that has been full of ideas, analyses, opinions and discussion. The agenda has been so significant and the seminar so successful that we have already considered possible topics for a follow-up conference in the future.

The Japanese and U.S. delegates feel it is a great honor to be in China, and now here in Qingdao, for the China-Japan-U.S. Seminar on Problem Solving. Before coming, we were already aware of the fine traditions in mathematics and mathematics education in China. And now we feel that our respective delegations have learned a great deal more from the work of our Chinese colleagues in this Seminar.

Long before the Seminar, Professor Zhang communicated with us with respect to the scholarly content of the program and various logistical matters. He and his colleagues subsequently shaped a program for the Seminar that was excellent in all its aspects. And they also looked after other matters, such as our arrival in Shanghai, where we were met at the airport and welcomed in a warm and friendly manner. We were escorted to our lodging facilities and helped to get comfortably settled. We were made to feel welcome in every way possible by Professor Zhang and his colleagues in Shanghai, Qufu, Weifang and Qingdao.

The Seminar program and format was set in an exemplary manner when the Seminar first got underway in Shanghai. The Shanghai component was successful with interesting papers presented and useful, interesting and instructive reactions from the listeners. The overnight train journey from Shanghai and the visit to Qufu, which included sightseeing at the Confucious Palace and Mansion, and an excellent visit to Qufu Teachers College, are experiences that we will always remember. The Mathematics Conference in Weifang provided yet another dimension to the useful interaction that was begun in Shanghai. Here again, the conference was excellent in all respects, and we were especially honored to be part of the opening ceremony of the important Houzhen Institute for Mathematics Education.

Now we are all happy to be here at the Qingdao Teaching Research Center where we are having the closing session for our Seminar. Here, too, the traditionally warm and friendly Chinese welcome has been given and the Seminar has continued in the excellent fashion as in Shanghai and Weifang.

So, to Professor Zhang Dianzhou, his colleagues and to all our old and new friends, we say 'Thank you.' We hereby express our profound gratitude for the privilege and honor of participating in this Seminar and visiting the important Chinese centers of work in mathematics education.