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ABSTRACT

This paper argues that operations with negative numbers should be taught using a curriculum that is grounded in algebraic geometry. This position is supported by the results from a study that compared the conceptual understanding of grade 9 students who received the Algebra Project transition curriculum to a control group of grade 6 gifted students who received a traditional introductory algebra course. The overall scores on an open-ended examination showed that the Algebra Project students, who performed lower than the traditional students at the beginning of the year, had surpassed the traditional group by the end of the year. Further examination of the students' problem-solving strategies revealed that the Algebra Project students had developed an understanding of integer addition and subtraction, based on vector operations, while the traditional group of students still exhibited confusion from the use of the conventional sign rules. The study results show how all operations with integers can be made more intuitive to students by providing them with physical experiences that correspond to vector operations in space/time coordinates. These results not only reinforce the view that all students should have the opportunity to learn the important ideas of mathematics, but that all students need to learn the traditionally "higher-order mathematics" that provide geometrical grounding for abstract algebraic concepts. (Contains 19 references.) (Author)

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Abstract

This paper argues that operations with negative numbers should be taught using a curriculum that is grounded in algebraic geometry. This position is supported by the results from a study that compared the conceptual understanding of students who received the Algebra Project transition curriculum to a control group of students who received a traditional introductory algebra course. The overall scores on an open-ended examination showed that the Algebra Project students, who performed lower than the traditional students at the beginning of the year, had surpassed the traditional group by the end of the year. Further examination of the students' problem solving strategies revealed that the Algebra Project students had developed an understanding of integer addition and subtraction, based on vector operations, while the traditional group of students still exhibited confusion stemming from the use of the conventional sign rules.

Introduction

> Their houses are very ill built, the Walls bevel, without one right Angle in any Apartment; and this Defect ariseth from the Contempt they bear for practical Geometry; which they despise as vulgar and mechanick. . .. I have not seen a more clumsy, awkward, and unhandy People, nor so slow and perplexed in the Conceptions upon all other Subjects, except those of Mathematicks and Musick. (Jonathan Swift, 1726)

Through Gulliver's description of the people on the floating island of Laputa, Swift is criticizing the mathematicians of his time who he felt were too preoccupied with purely abstract studies. This criticism certainly applied to the use of negative numbers, which were considered to be "fictitious" and "absurd," but were used anyway as "nice playthings" that made the operation of subtraction possible in all cases (Klein, 1908; Freudenthal, 1983).

Arbitrary sign rules are particularly troublesome for the educationally disadvantaged students who make up a large portion of the public school population in the Western regional area, which includes California, Nevada, and Arizona. These children, who are already unfamiliar with the culture and language of the American classroom, have little chance grasping the subtle shifts that occur in the meanings of symbols as mathematics becomes more advanced. Swift did not provide us with a description of how the children of Laputa learned their country's system of abstract mathematics. Research in our own time, however, has led to the establishment of a set of pedagogical standards that recognize that students in the upper elementary and middle grades are beginning to "develop their abilities to think and reason more abstractly," and need engaging, concrete learning experiences that provide them "the means by which they construct knowledge" (National Council of Teachers of Mathematics, 1989, p. 68). Unfortunately, for most of these students, the algebra they encounter is the algebra of Swift's time, which introduces them to seemingly arbitrary rules for performing operations with signed numbers.

This paper argues that the conceptual barriers that students encounter when they are introduced to integer operations and the "game of signs" (Moses, Kamii, Swap, & Howard, 1989, p. 433) can be avoided by providing students with physical experiences that correspond to vector operations. Geary (1995) postulates that young children have innate navigational skills that underlie the eventual development of basic geometrical knowledge. These intuitive navigational skills provide a natural foundation for vector operations within two and three dimensional coordinate systems. Instruction aimed at introducing students to algebra should provide students with experiences that require them to abstract more complex meanings from their own navigational knowledge. The purpose of this paper is to support this position by examining the conceptual understanding of students who received this type of curriculum, namely, the Algebra Project Transition Curriculum, developed by Robert Moses (see Moses, Kamii, Swap, & Howard, 1989). The study compares the learning outcomes of a group of students who received the Algebra Project transition curriculum, during their introductory algebra course, to a control group who received a traditional rules-based course. The results of this study are prefaced by an overview of the history of negative numbers and a description of the Algebra Project transition curriculum.

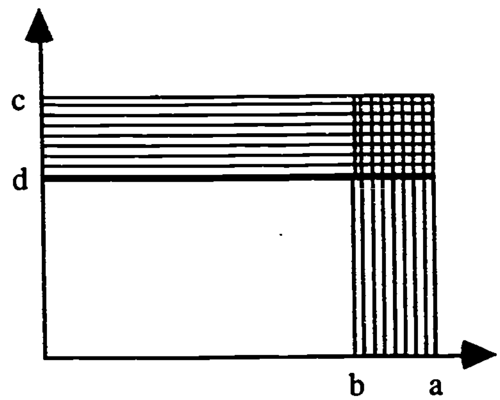
Historical Overview of the History of Negative Numbers

The language of mathematics, like the natural languages, has evolved unsystematically (Davis & Hersh, 1981). The symbols that represent the various functions and operations have survived more "by accident than design" (Cajori, 1893, p. 139). Mathematical symbols, like words, often are associated with multiple meanings that are contextually based. The minus sign (–) can mean subtraction, or it can indicate that a number is "negative" with respect to a relative zero point. Double minus signs signify the inverse of subtraction—addition. Mathematics teachers need to be aware of the multiple meanings of conventional notation and must convey the subtle differences in the underlying mathematical concepts to their students (Lochhead, 1991). An understanding of how mathematical concepts and their associated symbols have developed historically provides a starting point for unraveling the subtleties of this complex system.

The rules for operations with signed numbers were established centuries before negative quantities were recognized as legitimate (Thomaidis, 1993; Cajori, 1928). Indian manuscripts from the 7th and 8th centuries reveal that Hindu mathematicians had developed a formal set of rules for subtraction that had been adapted from the Greek geometrical theorems. Figure 1 presents the classical reasoning, based on the area of a rectangle, for one of the earliest records of the "sign rules," found in an algebraic text by Brahmagupta:

> Positive divided by positive, or negative by negative is affirmative... Positive divided by negative is negative. Negative divided by affirmative is negative. (Boyer, 1968, p. 242)

Figure 1
Geometrical Reasoning for Sign Rule: "A Negative Times a Negative Equals a Positive"



Given a>b & c>d, Find Area of Rectangle (a–b)(c–d):

$(a-b)(c-d) = ac - ad - bc + bd$

The Hindus also recognized that quadratic equations have two roots, and that often one or both roots can be negative. However, even though they recognized the existence of negative roots, they did not accept them as valid. For example, regarding the equation: $x^2 - 45x = 250$, with roots, $x = 50$ and $x = -5$, the mathematician Bhaskara wrote, "The second value is in this case not to be taken, for it is inadequate; people do not approve of negative roots" (cited in Cajori, 1893, p. 93).

Although negative numbers were not accepted as legitimate by mathematicians, the sign rules continued to appear in algebra texts. In 1494, the Italian Pacioli (who uses an m for the minus symbol) wrote:

> 10 m 2 equals 8; this means that if 10 m 2 is multiplied by 10 m 2 the result is 64; if however, the cross multiplication is applied, we obtain 10 multiplied by 10, namely 100, then 10 twice multiplied by m 2, which gives m 40, which together give 60; thus it becomes evident that m 2 multiplied by m 2 should give the number 4. (Thomaidis, 1993, p. 77)

A similar proof, is found in the work of Vieta (1591):

> When the positive name (nomen adfirmatum) of a magnitude is multiplied by a name also positive of another magnitude, the product will be positive, and when it is multiplied by a negative name (nomen negatum), the product will be negative. (Thomaidis, 1993, p. 76)

The plus (+) and minus (–) symbols appeared for the first time in a 16th century German textbook on commercial arithmetic written by Johann Widmann (Smith, 1923). Widmann used the symbols as signifiers for addition and subtraction and as qualifiers for excesses and deficiencies in merchandise. The German symbols competed with the Italian abbreviations for plus and minus (p̃ and m̃) throughout the 16th and 17th centuries, with the + and – finally winning out. There is no particular reason why this set of symbols survived while the other did not. In the words of Cajori (1928), "It appears, indeed, as if blind chance were an uncertain guide to lead us away from the Babel of languages" (p. 245).

From the 16th century on, negative numbers, designated by the minus sign (–), began appearing in algebraic expressions. However, they were still treated as fictitious numbers—they "were seen," but "they were not admitted" (Cajori, 1893, p. 93). Michael Stifel, in his *Arithmetica integra*, used negative coefficients to derive the formula for quadratic equations. However, although Stifel was familiar with the rules for operating with negative numbers, he still rejected them as roots to equations, calling them "numeri absurdi" (Boyer, 1968, p. 310).

During the 17th and 18th centuries, mathematicians began expanding the properties of algebra to include negative solutions to equations while carefully preserving the fundamental laws that govern operations with positive numbers. Negative values still had a secondary status to the concrete positive values (Freudenthal, 1983). However, there were a few mathematicians who began to think about what they physically represented. Girard (1629), in his *Invention nouvelle en l'algèbra*, was one of the first to describe negative numbers as having a direction opposite to positive numbers. He wrote, "The negative in geometry represents a retrogression, where the positive is an advance" (cited in Boyer, 1968, p. 336).

René Descartes objected to the abstract nature of algebra and geometry. He felt that every geometric proof called for some "ingenious twist" that was only useful to "exercise the mind"; and, that algebra was an "art full of confusion and obscurity" (cited in Kline, 1972, p. 308). Descartes' interest in applying mathematics to science led him to develop an "algebraic geometry" that was a combination of the best of both disciplines. In his *La Géométrie*, he showed how algebra could be used to formulate geometric constructions and how geometry could be used to graphically represent algebraic expressions. Although Descartes viewed negative numbers as "false," those who "stood upon his shoulders" expanded his coordinate system to include physical representations of negatively directed magnitudes. Of this achievement, Freudenthal (1983) wrote:

> The negative numbers would have remained a nice plaything, and the operations, motivated by algebraic permanence, rules of a game, which could have been fixed in another way, were it not that geometry had seized upon them. (p. 450)

Following Descartes, two substantially different approaches to algebra existed. The first primarily viewed algebra as a formal system of equations. The second employed algebra in a science that sought to describe objects in space and time (Novy, 1973). Throughout the 18th century, the writers of algebra textbooks followed the formal, abstract tradition and their mathematics became increasingly algorithmic. There were dissenters, however, who voiced concern particularly about the difficulty presented to students by the ambiguous use of the minus sign. Some of these individuals tried to introduce alternative systems that represented the qualitative and quantitative aspects of numbers with different symbols. These attempts were never very successful, however. Friedrich Schmeisser summarizes the difficulty he faced trying to introduce new symbols into the traditional language of mathematics:

> The use of the signs + and –, not only for opposite magnitudes . . .but also for Addition and Subtraction, frequently prevents clearness in these matters, and has even given rise to errors. For that reason other signs have been proposed for the positive and negative. . . . Since in our day one does not yet, for love of correctness, abandon the things that are customary though faulty, it is for the present probably better to stress the significance of the concepts of the positive and additive, and of the negative, and subtractive, in instruction, by the retention of the usual signs, or, what is the same thing, to let the qualitative and quantitative significance of + and – be brought out sharply. (cited in Cajori, 1928, p. 247)

To this day, the algebra that is presented to students in their introductory courses is the rules-based algebra of the 18th century. Only those students who survive this course and gain admittance to the higher level mathematics courses ever receive instruction in analytic geometry where the abstract algebra is reinforced by operations with vectors in space/time coordinate systems. The presentation of mathematical concepts is locked into the historical order in which they were developed; and, uses the symbols that have become established by tradition. The natural development of mathematical knowledge from innate navigational knowledge argues in favor of challenging our pedagogical customs and developing curriculum that, following Descartes, uses geometry to represent algebraic expressions.

Description of the Algebra Project Transition Curriculum

Robert Moses developed the Algebra Project to provide students with a conceptual path that would aid their transition from arithmetic to algebraic. He realized that students often struggle, not because they are incapable, but because they are unprepared for the illogical shift that occurs in the meaning of mathematical symbols and the seemingly arbitrary rules that they are faced with at this juncture. In particular, Moses could see that students are confused when the familiar minus symbol (–), that had meant "take away" in arithmetic, suddenly acquires new meanings in the context of algebra. This insight led him to develop a transition curriculum that recognizes students' innate navigational skills and uses them to develop the geometrical concepts that underlie the addition and subtraction of displacements (vectors) in a coordinate system. Through this approach, Moses hoped to eliminate "the game of signs" and the perception that mathematics is merely "the manipulation of a collection of mysterious symbols" (Moses, Kamii, Swap, & Howard, 1989, p. 433).

The Algebra Project transition curriculum is organized into four units that build upon the initial experience of taking a trip. In the first unit, students are introduced to the concepts of direction, displacement, and equivalence. In the second unit, students create a new metaphor for subtraction that is based on comparing the end points of pairs of displacements. Throughout the third unit, students develop an understanding of relative coordinate systems and the nature of integers as displacements that have both quantitative (magnitude) and qualitative (direction) significance. In the fifth and final unit, students formalize the concept of integer addition as a sequence of consecutive displacements (Program Evaluation Working Group, 1994).

The Algebra Project curriculum uses a five-stage learning process that supports students as they construct symbolic representations of mathematical concepts from their experiences. The students are provided with an experience in the first stage. During the second stage, students create pictorial representations of their experiences. They are committed to writing in the third stage. In the fourth stage, students regiment and structure their language descriptions. Finally, in the fifth stage, students develop and use abstract symbols to describe the mathematical concepts behind their experiences.

The remainder of this paper is devoted to the presentation of the results of a study that examined how students' conceptual understanding of integer operations can be enhanced by instruction that uses the Algebra Project transition curriculum.

The Study

This study examined and compared students' conceptual understanding of integer operations before and after receiving an introductory algebra course that either incorporated the Algebra Project transition curriculum or a traditional instructional approach. An assessment task, featuring a set of open-ended problems, was given to the Algebra Project students and to the students in the control group at the beginning and at the end of the 1993/94 school year. The open-ended nature of the problems made it possible to compare the students based on the differences between their pretest and posttest scores and on their demonstrated understanding of the underlying mathematical concepts.

Method

Subjects. This was a quasi-experimental study because it was not possible to randomly assign students to groups. The students who were used attended two of the three schools that were implementing the Algebra Project in the Southern California area during the 1993-94 school year. The experimental group were the students of the only "veteran" Algebra Project teacher in the area at the time (this teacher had been teaching the curriculum for three years). The remaining Algebra Project teachers had just received their training and varied in their level of commitment to the curriculum. One of these teachers, who was uncertain about the suitability of the curriculum for his more advanced students, agreed to participate in the study and allow his algebra students to became the control group.

The Algebra Project students differed from the students in the control group in several ways. The control group was comprised of sixth graders who had been identified as "gifted" and placed in a high ability mathematics track. Their school was located in a predominantly White, middle class, suburban neighborhood. The Algebra Project students were from an inner-city high school and were taking algebra for the first time in the ninth grade. These students had not had any pre-algebra training and were generally not expected to do well in the algebra sequence. Approximately half of these students were Black and the other half were Hispanic. Both the experimental and control groups had a higher proportion of girls than boys. The ethnic, age, and gender characteristics for the two groups of students are summarized in Table 1.

Table 1
Student Characteristics

Characteristic	Algebra Project students	Control group
Sample size	39	44
Average age	14 yrs., 3 mos.	11 yrs., 4 mos.
Gender		
Boys	10	18
Girls	29	26
Ethnicity		
Black	19	4
Hispanic	19	6
White	1	28
Asian	0	6

Procedure. During the year, the students in the experimental group received all four units of the Algebra Project transition curriculum. The students in the control group received a traditional introductory algebra curriculum that presented them with the conventional rules for operating with signed numbers. All of the students were assessed on their knowledge of integer operations at the beginning and the end of the school year (the instrument used was designed by the Program Evaluation and Research Group in 1992). The students were presented with five unsolved equations and were instructed to make up story problems for each one. The students were then asked to solve the equations and explain their steps.

The students' written responses to each of the five problems were evaluated based on a four-point rubric. If students did not attempt a solution, they were given a score of zero. They received a score of three if their story problems would produce the given equations and if their solutions were complete. Intermediate scores went to students based on the severity of their mistakes. To maximize reliability, the students' final scores were assigned after two readings of the problems by each of the authors. During the first reading, the authors worked independently to evaluate all of the students' work. During the second reading, the authors worked together, focusing on the solutions where there was a scoring discrepancy. The merits of these responses were discussed until the authors were able to reach agreement on the final score. The reliability of the instrument was computed following a principle components analysis. This analysis showed that the five test items loaded onto two principle components. The addition problem loaded onto its own principle component, and the problems involving subtraction all loaded on the other. This is attributed to the fact that nearly all of the students could rely on previous learning to create successful story situations for the straight addition problem. Therefore, the reliability of the instrument was computed for the four problems that incorporated the minus sign. Thus, the reliability of the instrument, computed as a Cronbach alpha, was found to be .60.

Students' understanding of integer operations was assessed in two complementary ways. First, differences in the groups' pretest and posttest scores were analyzed using a repeated measures analysis of variance. This procedure is necessary in situations where it is not possible to randomly assign students to groups. The procedure controls for group differences by separating the error due to consistent individual differences from the errors that determine the significance of the between group and interaction effects (Estes, 1991). Following the quantitative analysis, the content of the students' solutions was examined to identify differences in problem-solving strategies that could explain the differences between the groups.

Results

Overall, students from both groups benefited from the instruction they received (the within group test effect for the combined group of students was significant, $F_{1,81} = 140.58$, $p < .0001$). More importantly, the significant interaction ($F_{1,81} = 33.77$, $p < .0001$) indicates that the two groups responded differently to the instruction they received. The control group demonstrated a greater understanding of integer operations at the time of the pretest. However, by the time of the posttest, the Algebra Project students, as a group, outperformed the control group. These results are depicted in Figure 2 and summarized in Table 2. A second analysis was undertaken to illuminate the reasons for the significant interaction.

Figure 2
Group By Test Interaction

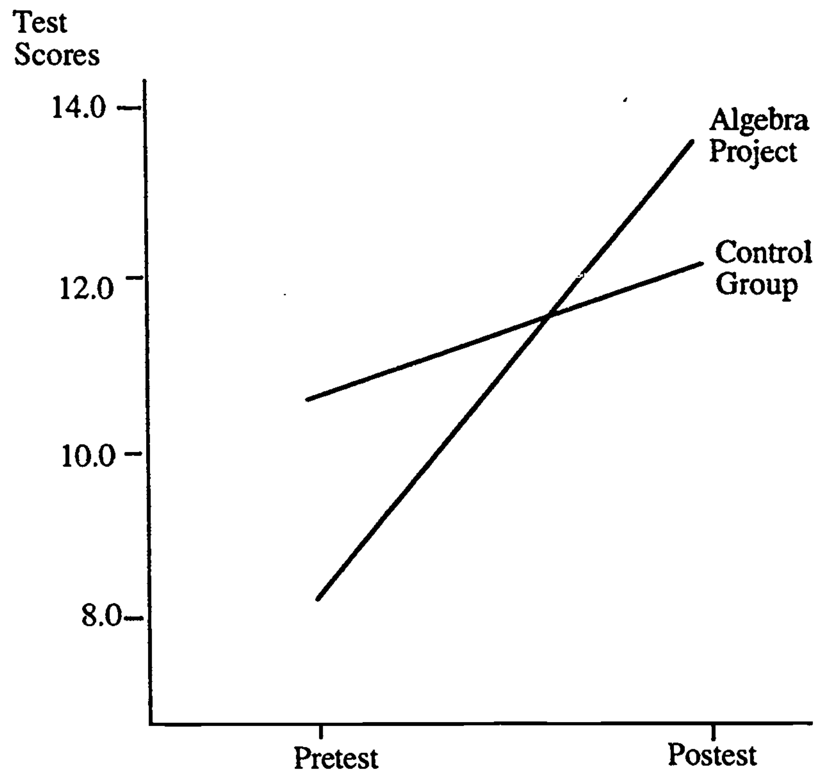


Table 2
Results of Repeated Measures' Analysis of Variance

Source	df	SS	MS	F
Between subjects				
Groups	1	5.17	5.17	0.91
Subjects within groups	81	461.09	5.69	

table continues

Source	df	SS	MS	F
Within groups				
Pre/post test	1	565.95	565.95	140.58***
Group x pre/post test	1	135.95	135.95	33.77***
(Pre/post) x subjects				
Within groups	81	326.09	4.02	

***$p<.0001$

The five equations presented to the students varied in their level of difficulty. The first two problems (3 + 4 = ? and 7 – 3 = ?) were the easiest and could be solved based on students' understanding of arithmetic operations with whole numbers. The third problem (–2 + 6 = ?) began to draw students out of the natural number system and required at least an intuitive idea about the geometric meaning of negativity. In the fourth problem (–2 – 4 = ?), students needed to recognize that the minus symbol is used to signify both qualitative and quantitative information. The final equation (–1 – –5 = ?) was conceptually the most difficult because, not only did it present students with the ambiguity of the minus sign, it also required them to create a story situation that corresponded to the subtraction of a negative value. Table 3 summarizes the percentage of students in the Algebra Project and control groups that were able to successfully complete each of the five problems at the time of the pretest and the posttest.

Table 3
Percentage of Students Who Solved The Problems Successfully on the Pretest and the Posttest

	Algebra Project students		Control group	
Problem	Pre %	Post %	Pre %	Post %
3 + 4 = ?	98	100	100	98
7 - 3 = ?	64	97	93	98
-2 + 6 = ?	18	89	66	84
-2 - 4 = ?	5	62	18	39
-1 - -5 = ?	3	72	7	14

The first two problems ($3 + 4 = ?$ and $7 - 3 = ?$) posed little difficulty for students in either group. Students could automatically draw on their understanding that addition and subtraction is the union and disunion of sets of common objects Nearly all of the students created addition stories that combined sets of "things" (e.g., ducks, apples, dollars); and subtraction problems where a number of "things" were stolen, eaten, or lost from an initial set. Most of the students who were unsuccessful with these problems either did not attempt them or did not finish them. In the second problem, one student lost points because he ignored the minus symbol, and a few more lost points because they created situations involving unrealistic physical quantities, such as –3 rings, –3 dollars, and –3 sisters.

Problems 3, 4, and 5 ($-2 + 6 = ?$; $-2 - 4 = ?$; and, $-1 - -5 = ?$) were more difficult for the students because they could no longer fall back on previous learning to create story situations for them. The types of errors that students made on these problems have been broken into three general categories:

1. Minus. Student solutions fell into this category if they either displayed confusion over the quantitative/qualitative significance of the minus sign; or, they ignored the minus sign completely. Because this type of misconception varied slightly for each of the three problems, examples of student work that is representative of the types of errors that were made are presented in the Appendix.

2. UnReal. Student solutions fell into this category if they used unrealistic physical situations in their stories.

3. NoTry. Student solutions fell into this category if they either did not attempt or complete a given problem. Solutions also were placed in this category if the students tried to get around creating a story situation by writing a version of the following: "My teacher put the problem, $-2 + 6 = ?$, on the blackboard. Can you solve it?"

Over 400 student errors were classified into one of these three categories. The percentages of errors tabulated by problem, group, and time of test are summarized in Table 4. This table reinforces the statistical analysis by showing that (a) more students, in both groups, attempted solutions at the time of the posttest; (b) the percentages of errors for the control group shifted from the NoTry category to the UnReal category between test administrations; and (c) the percentage of errors produced by the Algebra Project students decreased from the pretest to the posttest.

Table 4
Percentage of Total Errors By Test, Student Group, and Problem

Test	Problem	Minus %	UnReal %	NoTry %	Total %
Pre	-2 + 6	0	3	1	4
	-2 - 4	5	1	4	10
	-1 - -5	5	4	5	14
	Total	10	8	10	28
Post	-2 + 6	0	2	0	2
	-2 - 4	4	4	1	9
	-1 - -5	5	10	1	16
	Total	9	16	2	27
Pre	-2 + 6	7	5	1	13
	-2 - 4	7	2	3	12
	-1 - -5	4	3	4	11
	Total	18	10	8	36
Post	-2 + 6	0	1	0	1
	-2 - 4	3	1	0	4
	-1 - -5	3	1	0	4
	Total	6	3	0	9

Differences in the context of successful solution strategies also were evident at the time of the posttest. Students who were successful in the control group displayed their mastery over the sign rules and the commutative property, as evidenced in the following two examples:

Example 1: (–2 – 4 = ?)
If Fred loses 2 pencils one day, and 4 pencils the next day, how many pencils has he lost in all?
(–2) – 4 also can be said as (–2) + (–4). (You do this by adding the additive inverse).
–2 pencils + –4 pencils = –6 pencils.

Example 2: (–1 – –5 = ?)
Sally bought 5 pieces of bubble gum at the candy store. On the way home, she chewed one piece. How many pieces of gum does Sally have left?
By adding the additive inverse (–1) – (–5) can equal (–1)+ 5.
–1 + 5 (or 5 – 1) = 4 pieces of bubble gum.

Successful students from the Algebra Project group, on the other hand, displayed an understanding of the comparison metaphor that they had learned to use to subtract displacements. Examples of the use of the comparison metaphor are illustrated in the following two representative samples of student work:

Example 1. **(–2 – 4 = ?)**
I lost 2 tennis balls at the court. At my house I lost 4 tennis balls. How many tennis balls did I lose in all?
–2 – 4 = –6. 2 tennis balls lost and 4 tennis balls lost equals 6 tennis balls lost. I lost 6 tennis balls in all.

Example 2. **(–1 – – 5 = ?)**
I went to the beach and lost one pair of shoes compared *to* Susie who lost 5 pairs. How many pairs of shoes did Susie lose compared to me?
–1 – –5 = 4. –1 pair *compared to* –5 pairs are equal to 4 pairs. So she lost 4 more pairs than me.

Discussion

In the rules-based teaching model, negative numbers often are introduced to students as equally spaced points to the left of zero on a number line. Addition and subtraction are portrayed as forward and backward jumps, and negative results occur when the number of jumps backward are greater than the number of jumps forward. The usefulness of the number line begins to break down when it is extended to adding and subtracting negative numbers. The best that this teaching model can do is introduce an animated creature (a rabbit, a turtle or a marching man, for example) that can jump forwards and backwards and spin around (see Çemen, 1993; Thompson & Dreyfus, 1988). Subtraction using this imaginary creature becomes confusing because it is not intuitively grounded.

In a traditional course, after the students have been presented with number line teaching models, they are usually introduced to the sign rules. The difficulty that students have in understanding the meaning of these seemingly arbitrary rules was evident in the posttest performance of the control group.

In the Algebra Project teaching model, arrows are used to represent displacements (vectors). Equivalent classes of arrows represent displacements that are equal in magnitude and direction, but can be moved anywhere in a coordinate system. Opposite displacements are represented by arrows that have the same magnitude but are opposite in direction. Figure 3 depicts two classes of equivalent displacements that are opposite in direction. Coordinate systems are relative, and have moveable zero points, or benchmarks. Addition is modeled by a sequence of consecutive displacements. Subtraction in this model means to compare the positions of the end points of a pair of displacements. Figure 4 illustrates addition and subtraction of two pairs of displacements. The Algebra Project students' performance on the posttest revealed their mastery of the concepts of addition and subtraction of vectors. They were not confused by the sign rules because they did not have to learn them in this physically intuitive approach. Mastery of vector addition and subtraction provided both immediate and long-range benefits for these students. Not only were they prepared for the more abstract algebraic concepts that they would encounter later on in their course, but they had also become familiar with one of the fundamental units of calculus, analytic geometry, and physics.

Figure 3
Equivalent Classes of Displacements or Vectors (the +a displacements are opposite to the -a displacements)

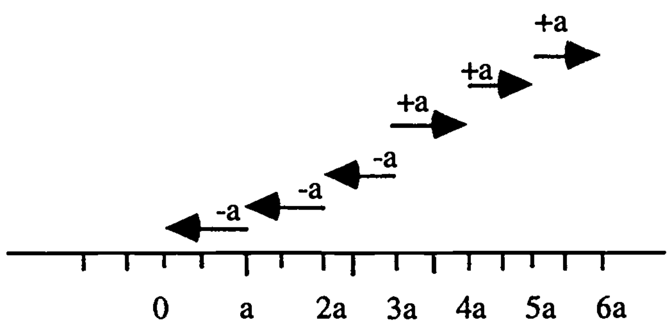
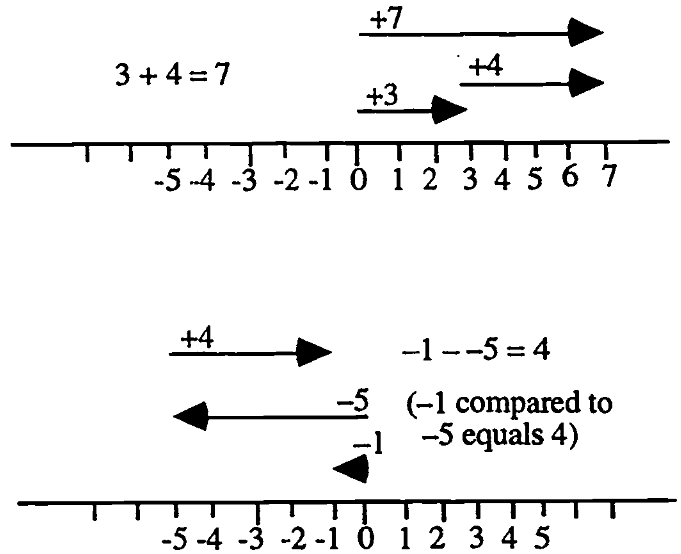


Figure 4
Addition and Subtraction of Pairs of Displacements



Conclusion

> Earlier I insisted on how important it is that mathematics should be closely tied to reality when it is to be learned. No other approach can in general guarantee a lasting influence of mathematics on the learner. . . .What is unrelated to our living world fades away from memory. (Freudenthal, 1973, p. 405)

The mathematical teaching models that are prevalent today follow the historical development of concepts and the symbols that represent them. This traditional, historical ordering of topics is in many ways out of sync with the modern view of how mathematical learning develops. The fact that the students in the Algebra Project group were from an educationally disadvantaged background, and that many of them had learned English as a second language, reinforce the strengths and advantages of this approach.

The study results show how operations with integers can be made more intuitive to students by providing them with physical experiences that correspond to vector operations in space/time coordinates. These results not only reinforce the view that *all* students should have the opportunity to learn the important ideas of mathematics (National Council of Teachers of Mathematics, 1989), but that all students *need* to learn the traditionally "higher-order mathematics" that provide geometrical grounding for abstract algebraic concepts.

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Appendix

Table A-1
Representative Examples of Student Misconceptions Concerning the Various Contextual Meanings of the Minus Sign in the Performance Task Problems

Misconception	Student example problem and solution
	–2 + 6 = ?
Qualitative significance of minus symbol ignored, although student remembers something about "sign rules."	We had 2 negative answers on the test and 6 positive answers all together there were how many answers and which did we have more negative or positive. We had 8 problems on the test all together and there were more positive than negative – 2 +6 = 8
Student combines sets and offers an operational "rule" for tacking the minus symbol back on the solution.	There's 2 pencils in the floor. Somebody threw 6 more pencils in the floor. What is the total of pencil in the floor. 2 pencils add 6 more pencils it all add up to ⁻8. if there's a negative - first and a positive second + it equals -.
	–2 – 4 = ?
Student overlooked qualitative significance of first minus sign, then subtract 2 from 4.	Jhon lost 2 baseball cards, but he only had 4. How many baseball cards does he have left. –2 – 4 = 2. If Jhon had 4 baseball cards and lost 2 he has 2 baseball cards left.
Student overlooked qualitative significance of first minus sign, and misapplied the commutative property.	I had four peaces of paper. My friend took 2. How many did I have left. If you had four peaces of paper and your friend took 2 you would half 2 peaces of paper left.

Note. Student work (under Student example problem and solution column) has not been edited. It is presented as the students originally wrote it.

table continues

Misconception	Student example problem & solution
−2 − 4 = ? (continued)	
Student needed to create an initial quantity that made it possible to have something to subtract 2 from.	The store had 10 dollars. Then the store had −2 dollars. 4 more dollars got stoln from the register. How money did they have left? Change the − sign to a + sign and change the positive four to a negative four. The answer is −6 dollars are left.
Problem: −1 − −5 = ?	
Student subtracted −1 objects from −5 objects to obtain −4 objects.	I ate −1 apples and my friend ate −5 apples. How many more apples did he eat than me? −1 − − 5 = −4. −1 apples subtracted by −5 apples equals −4. He ate −4 apples more than me.
Student's story involved a loss, but magnitudes of numbers were added.	You lost 1 ball then you lost 5 more. How many balls are you in debt? 6 balls
Student offered sign rules to justify attaching a minus sign to solution.	Erik didn't have 1 pear. His sister didn't have 5. How many didn't they have in all? −1 − −5 = −6. It's like adding 1 + 5, except you put a negative sign in front of them.

Note. Student work (under Student example problem and solution column) has not been edited. It is presented as the students originally wrote it.

Table A-2
Percentage of Total Errors By Test, Student Group, and Problem

Test	Problem	Minus %	UnReal %	NoTry %	Total %
Pre	-2 + 6	0	3	1	4
	-2 - 4	5	1	4	10
	-1 - -5	5	4	5	14
	Total	10	8	10	28
Post	-2 + 6	0	2	0	2
	-2 - 4	4	4	1	9
	-1 - -5	5	10	1	16
	Total	9	16	2	27
Pre	-2 + 6	7	5	1	13
	-2 - 4	7	2	3	12
	-1 - -5	4	3	4	11
	Total	18	10	8	36
Post	-2 + 6	0	1	0	1
	-2 - 4	3	1	0	4
	-1 - -5	3	1	0	4
	Total	6	3	0	9