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ABSTRACT

Middle school prealgebra students (n=157) learned about functions in a 20-day unit that emphasized: (1) representing problems in multiple formats, (2) anchoring learning in a meaningful thematic context, and (3) discussing problem-solving processes in cooperative groups. They produced smaller pretest-to-posttest gains on symbol manipulation tasks, such as solving equations, and larger gains in problem representation tasks, such as translating word problems into equations, tables, and graphs, than did a comparison group taught in the standard way. Although the groups did not differ in their pretest-to-posttest gains in calculating correct answers for word problems, the treatment group produced a larger gain in using mathematical representations while solving word problems than did the comparison group. The same pattern of results was obtained for lower-achieving students and language-minority students. Implications for cognitive theory and educational practice are discussed. Contains 72 references. (Author)

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THE ROLE OF MULTIPLE REPRESENTATIONS IN LEARNING ALGEBRA

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Abstract

Middle-school prealgebra students learned about functions in a 20-day unit that emphasized (a) representing problems in multiple formats, (b) anchoring learning in a meaningful thematic context, and (c) discussing problem solving processes in cooperative groups. They produced smaller pretest-to-posttest gains on symbol manipulation tasks such as solving equations and larger gains in problem representation tasks such as translating word problems into equations, tables, and graphs than did a comparison group taught in the standard way. Although the groups did not differ in their pretest-to-posttest gains in calculating correct answers for word problems, the treatment group produced a larger gain in using mathematical representations while solving word problems than did the comparison group. The same pattern of results was obtained for lower-achieving students and language-minority students. Implications for cognitive theory and educational practice are discussed.

THE ROLE OF MULTIPLE REPRESENTATIONS IN LEARNING ALGEBRA

In spite of a growing consensus for reform of mathematics education (National Council of Teachers of Mathematics, 1989; California State Department of Education, 1992), there has been (Resnick & Ford, 1981) and continues to be (Grouws, 1994) a lack of methodologically sound studies on how to promote students' construction of mathematical understanding in a school setting. This project was designed to contribute to this needed research base by determining the cognitive consequences of redesigning a prealgebra unit based on math reform principles. In particular, we examined whether students can learn to be more effective in the ways they use symbols, words, and graphics to represent mathematical problems involving functional relations. This introduction reviews the rationale for reforming mathematics education, the case for emphasizing multiple representations, and the design of a mathematics unit based on reform principles.

Rationale for Reforming Mathematics Education

The transition from arithmetic to algebra is a notoriously difficult one (Booth, 1989; Herscovics & Linchevski, 1994), particularly for Latino students and for students with less skill in basic mathematics (Mestre & Gerace, 1986). During the junior high school years, the mathematics curriculum can serve as a filter that prevents students, including a disproportionate number of Latino students, from completing enough mathematics to qualify for college admission (Oakes, 1990; University of California Latino Eligibility Task Force, 1993). If the goals of equity and achievement set by the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989) as well as the Mathematics Framework for California Public Schools (California State Department of Education, 1992) are to be met, major changes are needed in the junior high school curriculum and the supporting instructional techniques.

A major contention of the mathematics reform movement is that prealgebra courses focus mainly on symbol manipulation skills, such as how to solve equations, but do not emphasize the underlying problem representation skills, such as understanding what a word problem means. This lack of expertise in problem representation creates difficulties as students attempt to transition from arithmetic to algebra. For example, beginning algebra students have difficulty in linking graphical and tabular forms of representation to algebraic forms of representation (McCoy, 1994). Techniques for understanding mathematical representations are seldom directly covered in mathematics classes (Kieran & Chalouh, 1993) and lack of this understanding underlies many of the misconceptions that impede student progress in algebra (Booth, 1989; Matz, 1982).

The Case for Multiple Representations

A growing research base in cognitive psychology, cognitive science, and mathematics education points to the crucial role of problem representation in mathematical problem solving (Campbell, 1992; Charles & Silver, 1989; Ginsburg, 1983; Grouws, 1992; Resnick & Ford, 1981; Mayer, 1992; Schoenfeld, 1985; Wagner & Kieran, 1989). This research base has prompted the call for instruction that fosters multiple representations concepts of mathematical concepts--and this study is a response to that call (California State Department of Education, 1992; National Council of Teachers of Mathematics, 1989, 1991). For example, the California State Department of Education (1992) recognizes the issue of multiple representations as a key factor in the transition from elementary school arithmetic to high school mathematics. Multiple representations are designated as one of three unifying ideas in the middle school mathematics curriculum, in addition to proportional relationships and patterns and generalizations. The Mathematics Framework for California Public Schools (California State Department of Education, 1992, p. 124) states:

> Mathematical representations are powerful tools for visualizing and understanding problem situations, communicating mathematically and

> solving problems. Too frequently, students are exposed only to a narrow range of mathematical representations centered on numbers and literal symbols for numbers. They need to see that it is easier to understand important mathematical ideas when these ideas are presented in several ways and that different representations bring out different aspects of the data.

In addition to learning different representations, students need to learn to coordinate and move between representations in order to become effective problem solvers. The Framework (California State Department of Education, 1992, p. 124-125) states:

> Moreover, it is not sufficient to illustrate specific methods of representation one at a time. These kinds of representations need to be used together to provide complete and balanced mathematical analyses.

Researchers have analyzed the process of solving a word problem into several cognitive phases (Mayer, 1989): problem representation--in which a problem solver constructs a mental representation of the situation described in the problem; solution planning--in which a problem solver determines the arithmetic and/or algebraic steps required to solve the problem; solution execution--in which a problem solver carries out each arithmetic and/or algebraic procedure required to solve the problem; and solution monitoring--in which a problem solver monitors his/her computations to catch errors and to move from one step to another. Although a major difficulty of students involves representing the problem (along with using that representation in planning and monitoring), most instruction tends to emphasize solution execution (Kintsch & Greeno, 1985; Mayer 1989, 1992; Riley, Heller & Greeno, 1982).

Success in solving algebra word problems depends both on problem representation skills for the problem representation phase of problem solving (as well as for planning and monitoring) and symbol manipulation skills for the solution execution phase of problem solving. Problem representation skills include constructing and using mathematical representations in words, graphs, tables, and equations. Symbol manipulation skills include being able to carry out arithmetic and algebraic procedures. Assessments of mathematics achievement of students in the United States reveals that they perform better on tasks requiring symbol manipulation, such as arithmetic problems, than on tasks requiring representation skills, such as drawing conclusions from word problems (Dossey, Mullis, Lindquist & Chambers, 1988). In traditionally taught classrooms, new mathematical topics are usually introduced by teaching the symbol manipulation skills and subsequently the use of such of skills in a problem solving context, most typically word problems. The lower performance on tasks requiring representation skills may derive from the students' lack of opportunity to explore the structure of new kinds of problems before being asked to apply new symbol manipulation skills. Thus students are being asked to solve problems before they really understand them. In junior high school algebra units, students are asked to write and solve equations before they have a chance to explore the functional relationships that underlie the problems.

Research on cognitive strategy instruction is emerging as potential source of ideas for reforming the way that students learn problem representation skills across a wide variety of subject domains including reading, writing, and mathematics (Lochhead & Clement, 1979; Pressley, 1990). Attempts to teach problem representation strategies for mathematical problem solving have focused on teaching students ways to translate the words of a problem into other modes of representation using diagrams, pictures, concrete objects, the problem-solver's own words, equations, number sentences, computer programs, and verbal summaries (Cardelle-Elawar, 1992; Lewis, 1989; Riley & Greeno, 1988; Willis & Fuson, 1988) and embedding learning within a familiar context (Haneghan, Barron, Young, Williams, Vye & Bransford, 1992). Mayer (1987) has reviewed successful techniques such as asking students to restate the goal or givens in a problem, to indicate relevant and relevant information in a problem, to draw or select a drawing that corresponds to a statement in the problem, to write or select an equation or number sentence that corresponds to a statement in the problem, to sort problems into categories based on problem type. Finally, research on expertise indicates that strategies for building mental representations and metacognition are best learned within the context of specific situations rather than as general principles (Chi, Glaser & Farr, 1988; Ericsson & Smith, 1991; Sternberg & Frensch, 1991).

Reports from mathematics programs designed to enhance the performance of minority students also seem to confirm the importance of problem representation skills. It is an explicit goal of the mathematics reform movement to make advanced mathematics more accessible to a wider range of students including language minority students and students who have low basic skills (California Department of Education, 1992; Silver, Smith & Nelson, 1995). However, simply placing all students into algebra at the beginning of high school, even algebra courses which incorporate some instructional changes in alignment with the mathematics reform movement, can lead to widespread failure instead of a widening of opportunity (Risacher, 1994). Students may benefit from an introduction to algebra at the junior high school level as a problem solving activity with an emphasis on all phases of the problem solving, including problem representation. For instance, the Algebra Project introduces each new mathematical concept by proceeding through a five step instructional model which begins with concrete experiences with the concept to a model or drawing and at the end to mathematical symbols (Moses, Kamii, Swap, & Howard, 1989). The Algebra Project has been very successful in enabling minority students to pass screening tests and to earn placement in algebra at the beginning of high school.

Instruction on problem representation skills may be particularly effective for language minority students as well (Brenner, 1994; Cardelle-Elawar, 1992) because they must devote attentional resources to low-level linguistic processing. Given that the amount of attention that a student can apply to a mathematics problem is limited (Sweller, 1994), it is a major disadvantage to have to allocate most of one's attention to linguistic processing. For these students especially, practice in how to construct and coordinate mathematical representations is crucial; once a problem is understood, then the student can allocate his or her full attention to developing and monitoring a plan for how to solve it.

Although much has been written about the importance of making prealgebra courses more meaningful for students, there is a need to back up the call for reforms with classroom-based studies which assess their effectiveness. For example, in a classic review of the role of concrete manipulatives in mathematics instruction, Resnick and Ford (1981, p. 126) observed that "the structure-oriented methods and materials have not been adequately validated by research, and we know little from school practice about the effects of the curriculum reforms upon the quality of children's mathematical learning." Unfortunately, more than a decade later, there is still a need to meet Resnick mad Ford's challenge to provide valid tests of mathematics reforms, particularly in the area of prealgebra mathematics. For example, in a review of current thinking on reform efforts at making functions more understandable, Kaput (1993, p. 279) concluded that "research beyond that reported in this volume is urgently needed" and Williams (1992, p. 324) lamented that lack of research forces educators to base instructional decisions by default "on what we can do rather than what we should do." The current study contributes to this needed research base on school-based tests of mathematics reforms. In particular this study focuses on the effects of a reform-based program for a critical part of the mathematics curriculum--namely, middle-school prealgebra topics--and a critical population of students--namely, at-risk language-minority students.

Our research project implements ideas derived from cognitive strategy research, but differs from that research base in several ways. The instructional time frame of our study (one-month) is longer than most of those studies and takes place in intact classrooms. Our study also incorporates many kinds of representations rather than studying whether one or two representations facilitate understanding. Unlike the direct instruction that takes place in many cognitive studies, our study incorporates more student-centered instructional strategies derived from recent mathematics reform documents (National Council of Teachers of Mathematics, 1991). Our study also differs from most other studies which have suggested or demonstrated the success of implementing mathematics reform (e.g. Carpenter, et al., 1989; Cobb, et al., 1991; Moses, et al. 1989; Silver, Smith, & Nelson, 1995) because it focuses on the students' learning of one particular topic--functions and basic algebraic notation. This enables us to better trace the impact of particular instructional decisions on the development of one area of student reasoning rather than employing more global measures such as standardized test results or problem solving skill.

Rationale for Research on Functions

We focus on lessons about functions because it is a central topic in mathematics that is relatively understudied (Herscovics, 1989; Kaput, 1989; Leinhardt, Zaslavsky & Stein, 1990; Romberg, Carpenter & Fennema, 1993). For example, Romberg, Carpenter and Fennema (1993, p. 1) note that "functions are among the most powerful and useful notions in all mathematics" and Yerushalmy & Schwartz (1993, p. 41) state that "the function is the fundamental object of algebra and ... it ought to be present in a variety of representations in algebra teaching and learning from the onset." As early as the 1920's reform-minded scholars recognized that "without functional thinking there can be no real understanding or appreciation of mathematics" (Breslich, 1928, p. 42), and that functional relations "occur in real life in connection with practically all the quantities with which we are called upon deal in practice" (Hendrick, 1922, p. 165). Similarly, the current mathematics reform movement recognizes that "the concept of function is an important unifying idea in mathematics" (National Council of Teachers of Mathematics, 1989, p. 154). Yet, in spite the need to understand how to help students learn about functions, Leinhardt, Zaslavsky and Stein (1990, p. 54) note that "actual studies of teaching [functions] at either the elementary or secondary level are quite rare" and Romberg, Carpenter & Fennema (1993, p. 1) lament that "the learning and teaching of functions are understudied in comparison to other areas of mathematics instruction."

The concept of a function is notoriously difficult for many students to understand, perhaps because it is presented in abstract algebraic form rather than in a concrete, practical context. Romberg, Carpenter & Fennema (1993, p.2) argue that the "abstractness of the algebraic expressions and the variety of transformations is such expressions have proved difficult for many students to fathom." In an analysis of the content of mathematics textbooks used in the United States, Demana, Schoen & Waits (1993) found that less than 3% of the page space is devoted to graphical representations, so that "most American junior high students have not been ... expected to learn how to construct a graph" (p. 17) nor "to make global or qualitative interpretations of graphs" (p. 19). Philipp, Martin and Richgels (1993, p. 249) observe that in existing textbooks, "graphs are not integrated with other topics" and "graphical representations of functions played a minor role in the algebra curriculum of the past." A survey of 3000 middle school students by Hart (1981) revealed that "translating a functional relationship between data pairs into algebraic symbols was one of the most difficult of representation tasks for students" (Kieran, 1993, p. 203). Student misconceptions lead to a tendency to misinterpret graphs (Kerslake, 1977), to view all functions as linear (Karplus, 1979; Lovell, 1971), to misunderstand the role of scale (Vergnaud & Errecalde, 1980), to think that a variable stands for a single number to be determined (Demana, Schoen & Waits, 1993), and to view graphs as a set of discrete points rather than a continuous relation (Kerslake, 1981; Leinhardt, Zaslavsky and Stein, 1990). For example, Kerslake (1977, 1981) asked middle-school students to describe the journey represented in a graph that sloped upward, then sloped downward, and finally sloped upward again with time on the x-axis and distance on the y-axis. The overwhelming majority failed, often describing the picture of a "mountain" presented in the graph rather than its meaning.

Based on such evidence, Kieran (1993, p. 199) argues for teaching "the notion of a function as a dependency relation in a practical situation" rather than the more formal definition of a function as correspondence between two sets. Swan (1982) has shown that mathematics textbooks emphasize skills such as tabulating, plotting, and reading values from graphs in abstract contexts. Kieran (1993, p. 200) notes that "the consequences of emphasizing exclusively these skills are that students lose sight of the meaning of task, rarely meet graphs other than those of straight lines, and get little practice at interpreting graphs in terms of realistic situations."

In contrast to the lack of attention paid to concrete representations of functions in the past, the current approach advocated by educational reformers is to emphasize practical and concrete contexts of functional relationships in school mathematics (Karplus, 1979; Thorpe, 1989). Demana, Schoen & Waits (1993, p. 36-7) argue that "mathematical ideas introduced in the context of realistic problem situations are appreciated by students" so "the curriculum needs to be infused with realistic problems to help all students learn to value mathematics." The Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989, p. 154) state that "the study of functions should begin with a sampling of those that exist in the student's world" and "students should have the opportunity to appreciate the pervasiveness of functions through such activities as describing real world relationships that can be depicted by graphs." For example, Phillips, Burkhardt & Swan (1982) developed a computer program, Eureka, to help students see the relation between water level in a bathtub and variables such as turning on or off the tap, putting tl e plug in or out, putting a man in the tub or taking him out, and having the sing or not sing. They learn to interpret a graph consisting of time on the x-axis and water level on the y-axis, by discussing it in terms of a concrete situation.

Mathematics educators commonly analyze the representations of functions into three categories--tables consisting of ordered pairs of values, graphs consisting a pictorial presentation, and equations consisting of algebraic notation (Burkhardt, 1981; Romberg, Carpenter & Fennema, 1993; Swan, 1982; Verstappen, 1982). Kieran (1993, p. 230) notes that there is "a renewed emphasis on integrating various representations of functions, such as graphical, algebraic, and tabular." The National Council of Teachers of Mathematics (1989, p. 154) state that students in grades 9-12 should be able to "represent and analyze [functional] relations using tables, verbal rules, equations, and graphs; translate among tabular, symbolic, and graphical representations of functions." Williams (1993, p. 329) summarizes one of the major themes of research on functions is "the importance of being able to move comfortably between and among the three different representations of function: algebraic, graphical, and tabular." To this list we have added verbal representations as another primary way in which students should be expected to understand functions. This means being able to identify functional relationships as encoded in word problems as well as being able to explain in words the functional relationships represented in the other representations.

In addition to learning about functional relationships, it is important for students to learn about the varying uses of algebraic notation. The NCTM Curriculum and Evaluation Standards for students in grades 5 to 8 state:

> Students need to be able to use variables in many different ways. Two particularly important ways in grades 5-8 are using a variable as a placeholder for a specific unknown, as in n + 5 = 12, and as a representative of a range of values, as in 3t + 6. (National Council of Teachers of Mathematics, 1991, p. 103.)

The first of the uses of algebraic notation mentioned in the quotation above, in which a variable stands for one unknown, reflects how students are expected to use equations to solve simple word problems. The second usage of algebraic notation, in which a variable can represent a range of values, is less common in traditional curricula. In this usage algebraic notation can be seen as representing patterns of relationships or generalizations of relationships. Based upon this usage of algebraic notation, students can use variable expressions to generalize a pattern in a table or a formula to represent the process for finding the area of a geometric figure.

Designing a Mathematics Reform Unit for Prealgebra Students

A team of mathematics teachers and educational researchers developed a 20-day instructional unit on functions based on four math reform principles: (a) Instead of emphasizing symbol manipulation, we emphasize problem representation skills. In particular, students learn to construct and coordinate multiple representations of functions, including expressing functions in words, tables, graphs, and symbols. (b) Instead of teaching problem-solving skills in isolation, we anchor them within a meaningful thematic situation. In particular, we anchor instruction within a unifying context about choosing the best pizza provider for the school cafeteria so that students can use their everyday knowledge to help them understand the nature of functional relations between two variables. (c) Instead of focusing solely on the product of problem solving, we emphasize the process by which problems are solved. Through cooperative learning situations and cognitive modeling by teachers, we allow students to discuss their own problem-solving processes and to recognize that there can be many alternative methods. We also require students to write about their mathematical understandings in a variety of formats. (d) Instead of providing direct instruction in how to construct and use mathematical representations in problem solving, we emphasize a guided-discovery approach in which students are encouraged to explore different representations and to develop their own understanding of each one. At times the students are provided with explanations of the conventions of graphing or review on the structure of equations, but they are not given a prescribed method for dealing with the problems that form the core of each day's activity.

The unit focuses on a central concept in prealgebra courses, namely the concept of a functional relation between two variables. A functional relation refers to a lawful pattern involving two variables, in which the value of one variable can be computed by applying the functional rule to the value of the other variable. Figure 1 exemplifies four modes of mathematical representation of functional relation--verbal, tabular, graphic, and symbolic. In a verbal mode of representation, the relation is expressed in words such as in a number sentence. In a tabular mode of representation, the relation is expressed as two corresponding columns of numbers with each column representing a variable and each row representing a pair of numbers that satisfies the function. In a graphical mode of representation, the relation is expressed as a line or bar graph, with the x-axis representing one variable and the y-axis representing the other variable. In a symbolic mode of representation, the functional relation is expressed as an equation with one unknown representing one variable and another unknown representing the other variable.

Insert Figure 1 Here

The major goal of the instructional unit is to help students become more effective mathematical problem solvers by helping them develop two cognitive skills that are essential for success in algebraic reasoning--namely, translating and applying mathematical representations of functional relations. In translation, the student converts one mathematical representation into another, such as taking a function that is expressed in tabular form and expressing it in verbal form as a simple sentence. In translating among relations, students must build one-to-one correspondences between the elements in each representation--the name of variable 1 (e.g., volume in ml), the values of variable 1 (e.g., 1, 2, 3...), the name of variable 2 (e.g., mass in g), the values of variable 2 (e.g., 1.5, 3.0, 4.5, ...), and the relation between variable 1 and variable 2 (e.g., variable 2 is one and half times as much as variable 1). For example, the heading of column 1 in a table corresponds to the x-axis label in a figure, the heading of column 2 in a table corresponds to the y-axis label in a figure, the values in column 1 of a table correspond to the numbers along the x-axis in a figure, the values in column 2 of a table correspond to the numbers along the y-axis of a figure, and the pairings of the numbers in a table correspond to the points in the line in a graph. In application, the student works with one or more mathematical representations to draw a conclusion or compute an answer to a question. Application involves interpreting the implications of a mathematical representation of a functional relationship.

Each of the lessons in our unit involves activities in which students create or translate among verbal, tabular, graphical, and/or symbolic modes of representing a functional relation, and each of the lessons in our units involves activities in which students discuss the implications of various representations of functional relations. For example, in one of a series of lessons about a "computer malfunction", students see an invoice listing the quantities of flour, mozzarella cheese, olive oil, pepperoni, and mushrooms shipped on each of the last five orders, along with listings of the actual quantity received of each of these five ingredients, as shown in Figure 2. In the accompanying instructions, students are asked to work in groups to determine the pattern in the errors, to represent the pattern in a table, graph, chart, or equation, and to write a letter describing the pattern in words.

Insert Figure 2 Here

Other lessons involved using formulas for area of pizzas within the context of examining the advertising claims of competing pizza companies, determining the allowable relations among the amounts of various ingredients for constructing pizzas to meet nutritional requirements involving the total amount of fat, and using tables and graphs to determine the profit and loss in various pizza businesses.

The Traditional Curriculum

Reviews of commonly used junior high school textbooks and discussions with junior high school teachers revealed that the mathematical concept underlying the word problems in prealgebra lessons was that of functions. However, the ideas about algebraic notation and symbol manipulation presented in these lessons were distinctly different from our curriculum. Variables were presented as a simple unknowns and the goal of manipulating equations was to find the value of that variable. Although teachers reported developing the concept of function during other parts of the curriculum in the form of tables and 'number machines', this concept was not explicitly linked to the algebraic notation introduced in the textbooks. Most student work in the chapters on algebra was devoted to learning how to solve equations and secondarily to writing equations to represent word sentences and word problems. Notably missing from this approach is an attempt to teach students to use algebra as a way of representing general relationships.

Comparison of the Experimental and Traditional Curricula

Although the content of our unit--functions, variable expressions and one variable equations--is the same as conventional instruction, we emphasize different cognitive skills and employ different instructional methods than conventional instruction. In particular, our unit focuses on problem representation skills--namely, translation and application, by having students use algebra to represent mathematical relationships in conjunction with other representations of problem situations including tables, graphs, pictures, and diagrams. The problems in the unit were typically either open-ended or amenable to multiple solution paths. In contrast, conventional coverage in textbook emphasizes symbol manipulation skills--namely, how to solve equations or how to solve simple word problems. In addition, while our unit employs instructional methods based on anchoring and collaboration, conventional instruction relies mainly on drill and practice in mathematical procedures.

At the end of our unit we expected the students who received the experimental unit to show greater improvement of their understanding of functional relationships as demonstrated by their ability to make and interpret tables, graphs, and equations. We expected about equal improvement in both classes on performance on standard algebra word problems. The comparison classes would have more practice at this kind of problem because it was an major part of their textbooks lessons and could be solved by fairly rote methods. The treatment classes would not have the experience with word problems, but would have developed more general problem solving skill from dealing with the open-ended problems in the unit. The comparison classes would be expected to be equal or better at solving equations because this was the focus of the textbook chapters, but was only reviewed in passing in the experimental unit.

Method

Participants and Design

The participants in the study were students in seven intact prealgebra classes at three junior high schools in a small urban area in Southern California. The schools ranged in size from approximately 680 to 800 students. The ethnic composition of these schools varied with a range of from 38% non-Hispanic whites to 67% non-Hispanic whites, as reported by the school district. Most of the nonwhite students were Latinos of Mexican origin. The statistical analyses are based upon test results from 157 seventh- and eighth-grade students who were present for all pretesting and posttesting sessions. One-hundred and one of these students were in 4 classes that participated in a representation-based unit on functions (i.e., treatment group) whereas 56 students were in 3 classes taught by the same teachers using the conventional textbook lessons on algebra (i.e., comparison group). All of the students were in their first year of pre-algebra and the same textbook was used at all three schools. 'At the time of the study, all of the students had finished an introductory chapter on algebra which covered translating expressions and sentences into variable expressions and equations, solving equations through inverse operations and basic facts and one variable word problems. The teachers were fairly new teachers with between four and six years of teaching experience. All of them had incorporated components of new instructional strategies into their teaching including more open-ended problem solving and more extensive use of calculators. But the textbook and traditional instructional practices still formed the core of their mathematics program.

In light of our interest in preparing Latino students for algebra instruction, we identified a subsample of 40 students who indicated on a questionnaire administered prior to the study that they spoke Spanish as a first language or had received instruction in Spanish in mathematics. Of these Spanish-speaking students, 25 were in the treatment group and 15 were in the comparison group. All of the students included in the study were fully bilingual in both English and Spanish. In light of our interest in less skilled students, we also identified a subsample of 61 students who correctly answered 50% or less on eight basic problems requiring solving simple equations on the pretest. Of these 61 less-skilled students, 34 were in the treatment group and 27 were in the comparison group.

Materials

The materials consisted of a battery of four pretests, a battery of four posttests, an instructional booklet, and a teacher's booklet[1].

The Equation Solving Test was created to assess students' ability to apply algebraic and arithmetic procedures, so it contained problems that depend on symbol manipulation skills but not on problem representation skills. It contained 8 items such as shown in the top portion of Figure 3.

Insert Figure 3 Here

The Word Problem Solving Test was created to assess students' ability to calculate numerical answers to two-step word problems, so it contained problems that could be solved using either symbol manipulation skills or problem representation skills. It contained 4 items such as shown in the second portion of Figure 3.

The Word Problem Representation Test was designed to assess students' ability to create and coordinate multiple representations of word problems using words, equations, tables, and graphs, so it contained problems that required using problem representation skills but not symbol manipulation skills. Given the central role of problem representation skills in this study, the Word Problem Representation Test contained 14 items covering a variety of representation tasks. Four of the items required the student to translate a verbal statement of a functional relation into an equation such as the top two problems in the third portion of Figure 3. Five of the items were based on the statement, "At Speedy Delivery Service, the cost to deliver a package is $2.00 plus an additional $.50 per pound." and required the student to complete various tasks such as shown in the bottom of the third portion of Figure 3. Five items were based on the statement, "Monica can ride her bicycle at a rate of 1 mile every 5 minutes" and contained the same kinds of tasks as the previous example.

The In-depth Word Problem Solving Test was designed to produce information concerning the strategies that students used to solve a word problem, so it allowed us to determine the extent to which students correctly used multiple representations as they solved word problems. It consisted of one word problem, shown in the bottom of Figure 3, along with instructions to solve the problem, including, "Show your work to the solve the problem", "Circle your answer", "Draw a diagram, chart, table, or graph to represent the problem", and "Write an equation to represent the problem."

The posttests consisted of parallel versions of each test, using problems of the same form as the pretest but with different numerical values. Each test was typed on 8.5 x 11 inch sheets of paper, with space on each sheet for the students' answers.

The instructional booklet consisted of 77 pages broken down into 19 lessons for approximately instructional 20 days. The lessons were printed on 8.5 x 11 inch sheets and were bound in a three-ring binder. Each lesson consisted of a statement of the day's activities, homework, and warm-up activities. As a final goal, the unit asked students to make a decision about which of three pizza companies should be allowed to provide pizza for the school cafeteria. Lesson 1 involved a taste test in which students sampled several pizzas, collected data to characterize student preferences, and constructed graphs. Lessons 2 through 5 involved a computer malfunction task in which students looked for patterns of errors in order forms and invoice sheets for a pizza maker, completed tables using a variable expression as a guide, generated graphs and tables based on the malfunction, and wrote about their interpretations of various graphs and tables. Lesson 6 involved a pizza delivery game in which students used variable expressions to determine the correct destination. In Lessons 7 through 10 students learned about formulas for area within the context of an advertising problem. Lessons 11 through 14 focused on nutrition in which students generated equations expressing the fat content of various pizzas. In lessons 15 through 18 students used tables and graphs in solving problems about profit and loss in various pizza businesses. Finally, during the last lesson, which was designed to take two instructional days, students designed and wrote a final report concerning which pizza company should be selected to operate in the school cafeteria.

The teacher's booklet was identical to the instructional booklet except that each lesson contained supplemental pages indicating needed materials, key concepts, answers, estimated time for each activity and suggestions about how to use the lesson.

Procedure

Three of the four teachers who agreed to participate in the study taught two sections of the same prealgebra class, so for each of these teachers one of their classes was randomly selected to be taught using the treatment materials (i.e., treatment group) and the other was taught in the conventional manner using the standard textbook (i.e., comparison group). The fourth teacher taught only a treatment class. Prior to teaching the treatment unit, the participating teachers attended a workshop aimed at exploring how to present the unit. Each teacher was given a teacher's booklet, student booklets for all treatment students, and a set of handheld calculators. Special materials necessary for teaching the unit such as oversized graph paper and jar lids were supplied by the researchers. A member of the research team was available throughout the project to support teachers. Teachers received a stipend for participating in the study.

Prior to the instructional unit, all students took the pretests at their own rates during a two-day period. Then, the students received approximately 20 days of mathematics instruction on algebra, equations and functions based on a multiple-representation approach (treatment group) or a traditional approach (comparison group). Following the instructional unit, all students took a series of corresponding posttests at their own rates during a two-day period.

Although the research team supplied the materials and activities, each teacher made the day to day decisions about how to teach the experimental unit. The heterogeneous cooperative groups used in the treatment classes were organized by each teacher. The teachers also used their own discretion about when to extend a lesson to more than one day, which topics to skip and when to assign the homework assignments. Thus there were slight differences in which concepts were emphasized in each class.

The teachers also made all of the instructional decisions in the comparison classes. As in the experimental classes, this resulted in some differing emphases. All of the students covered the algebra concepts, but in the time between the pre and post tests, two of the classes covered a chapter on graphing while the third class covered a chapter on measurement and area. These same topics are treated in the experimental unit. This means that the comparison classes had experience with the representations included in the experimental unit but as separate topics, not in relation to algebra.

Results and Discussion

Overall Results

Figure 4 summarizes the proportion correct by the two groups on the pretests and posttests for three of the tests--the Equation Solving Test, the Word Problem Solving Test and the Word Problem Representation Test. We expected the comparison group to do equally well or better on the Equation Solving Test because the treatment lesson did not emphasize symbol manipulation skills whereas the comparison lesson did. The treatment lessons emphasized representation skills, which are not relevant to the Equation Solving Test, whereas the comparison lessons emphasized symbol manipulation skills, which are relevant to the Equation Solving Test. Consistent with these expectations and as shown in the left panel of Figure 4, the comparison group produced a significantly greater pretest-to-posttest gain (from 55% to 76% correct) on the Equation Solving Test than did the treatment group (i.e., from 65% to 69%),

$t(155) = 3.30$, $p < .01$. Thus, there is no evidence that instruction emphasizing problem-representation skills has stronger influence on students' equation solving behavior than does instruction emphasizing symbol-manipulation skills. If accuracy in solving equations is the goal of instruction, then conventional methods of instruction appear to be more effective than methods which emphasize multiple representations of word problems.

Insert Figure 4 Here

We did not predict that the treatment group would show a greater pretest-to-posttest gain than the comparison group on Word Problem Solving Test because word problems can be solved using either rote or meaningful methods (Mayer, Lewis & Hegarty, 1992). Although we expected the treatment students to learn to use different methods than the comparison students, we did not expect the groups to differ in their improvement in overall accuracy. As shown in the middle panel of Figure 4, the pretest-to-posttest gain of treatment group (i.e., from 62% to 76% correct) did not differ significantly from the gain of the comparison group (i.e., from 59% to 71%), $t(155) = .48$, $p =$ ns. It should be noted that the comparison groups had more practice with word problems so this portion of the test is a more direct measure of what was taught for the comparison group.

The overall results based on the traditional skills of solving word problems as well as on solving equations offers no evidence to support the use of reform-based methods of instruction, largely because these tests can be successfully completed using symbol manipulation skills and arithmetic skills taught in earlier years.

However, our study is based on the premise that students' future progress in the transition from arithmetic to algebra depends on their construction of a new set of skills--which we ca . problem representation skills. Because problem-representation skills may be underrepresented in conventional mathematics curricula, we developed a unit that emphasizes these skills.

Therefore, we predicted that, in contrast to the foregoing results on other tests, treatment students would show a greater pretest-to-posttest gain on tests of problem representation than would conventional students. As predicted, the treatment group gained significantly more from pretest to posttest (i.e., from 34% to 46% correct) than did the comparison group (i.e., from 29% to 33%) on the Word Problem Representation Test, $t(155) = 2.49$, $p < .01$. These results are summarized in the right panel of Figure 4. This test demonstrates that the treatment students learned better to understand several different representations for functions as shown by their ability to complete tables and graphs and that they learned to translate between representations as shown by their ability to write equations from words and to make graphs from tables.

We also examined the development of problem-representation skills within the context of the In-depth Word Problem Solving Test, in which students were encouraged to create multiple representations of problems as a way of solving a word problem. Figure 5 summarizes the proportion of correct answers and the proportion of correct representations produced by the treatment and comparison groups on the pretest and posttest versions of the In-depth Word Problem Solving Test. The pretest-to-posttest change for generating the correct answer for the treatment group (i.e., from 53% to 62% correct) was identical to that of the comparison group (i.e., from 39% to 48% correct), $t(59) = .00$, $p =$ ns. In contrast, the treatment group produced a greater pretest-to-posttest gain (i.e., from 34% to 65% correct) that the did the control group (i.e., from 34% to 25%) in creating appropriate mathematical representations (e.g., table, graph, or equation) to help understand the problem, $t(155) = 4.14$, $p < .01$. These results are consistent with the idea that the comparison group learned to use different methods to solve word problems than did the treatment students. In particular, the treatment group was more likely to use appropriate tables, diagrams, or equations than was the comparison group whereas presumably the comparison group was more likely to use rote symbol manipulation methods than was the treatment group.

Insert Figure 5 Here

Thus, if we had focused solely on dependent measures that did not tap problem-representation skills we would not have found any evidence to support math reform efforts; in contrast, when we focused on dependent measures that did tap problem-representation skills we found encouraging evidence for reform-based methods of instruction.

Results for Language Minority Students

An important focus of this study concerns the role of prealgebra courses as a filter for language-minority students. We were particularly interested in whether reform-based instruction emphasizing multiple representations of mathematical concepts would produce benefits for language-minority students. One potential impediment to employing reform-based methods for language minority students is that reform-based instruction requires the use of language more than conventional instruction. However, given the low success rate of language-minority students in college-preparatory mathematics, they seem to require the benefits of reform-based instruction in prealgebra courses. In short, an argument against using our treatment unit is that reform-based instruction relies more heavily on language than does conventional instruction whereas an argument in favor of using it is that reform-based instruction provides better opportunities to learn how to use mathematical language and representations than does conventional instruction.

The major question addressed in this section is whether or not Spanish-speaking students benefit from the treatment unit in the same way as the overall sample of middle school students. In order to address this question, we examined the pretest-to-posttest changes for the 40 language minority students in our sample including 25 treatment and 15 comparison students. Figure 6 summarizes the proportion correct by the two groups on the pretests and posttests for three tests.

The language-minority students produced the same pattern of performance as did students in the overall sample. On the Equation Solving Test, the comparison group produced a significantly greater pretest-to-posttest gain (from 44% to 59% correct) that did the treatment group (from 58% to 59%), $t(38) = 2.11$, $p < .05$. On the Word Problem Solving Test, the pretest-to-posttest gain of treatment group (from 52% to 69% correct) did not differ significantly from the gain of the comparison group (from 51% to 67%), $t(38) = .19$, p = ns. In contrast, the treatment group gained significantly more from pretest to posttest (i.e., from 32% to 46% correct) than did the comparison group (i.e., from 29% to 24%) on the Word Problem Representation Test, $t(155) = 2.96$, $p < .05$.

Insert Figure 6 Here

Figure 7 summarizes the pretest and posttest scores for producing the correct answer and producing a correct representation on the In-depth Problem Solving Test by the Spanish-speaking students in each group. On the In-depth Word Problem Solving Test, the pretest-to-posttest change on generating the correct answer for the treatment group (i.e., from 28% to 48% correct) did not differ significantly from that of the comparison group (i.e., from 7% to 33% correct), $t(38) = .06$, p = ns. In contrast, the treatment group produced a greater pretest-to-posttest gain (i.e., from 12% to 40% correct) that did the control group (i.e., from 20% to 7%) in using appropriate mathematical representations (e.g., table, graph, or equation) to help understand the problem, $t(38) = 2.97$, $p < .05$. This pattern is consistent with the performance of the overall sample of all students.

Insert Figure 7 Here

Results for Lower Achieving Students

The final analyses in this study focus upon a sub-sample of lower achieving students. The mathematics reform movement has the goal of enabling virtually all students to take more advanced mathematics courses. This means that even lower achieving students will be expected to take algebra and other courses that have traditionally been restricted to college prep students. Since many students in our sample could be tracked into general mathematics courses when they enter high school, we felt it was important to determine if the instructional approach used in this study holds as much promise for the lower achieving students as the higher achieving students. Some educators feel that these students are unlikely to benefit from a curriculum that is based upon open-ended problem solving. In addition, the foregoing analyses for the total sample can be criticized on the grounds that students in the treatment group scored higher on some of the pretests than did students in the comparison. In order to address this problem, we conducted an analysis on the data for lower-achieving students, that is, students who correctly answered half or less of the 8 problems on the pre test version of the Equations Solving Test. This sample yielded 34 treatment students and 27 comparison students.

Figure 8 summarizes the proportion correct by the two groups on the pretest and posttest for three types of tests. The lower-achieving students produced the same pattern of performance as did students in the overall sample. On the Equation Solving Test, the comparison group produced a significantly greater pretest-to-posttest gain (i.e., from 26% to 68% correct) that did the treatment group (i.e., from 29% to 52%), $t(59) = 2.77$, $p < .01$. On the Word Problem Solving Test, the pretest-to-posttest gain of treatment group (i.e., from 54% to 75% correct) did not differ significantly from the gain of the comparison group (i.e., from 42% to 66%), $t(59) = .32$, p = ns. In contrast, the treatment group gained significantly more from pretest to posttest (i.e., from 34% to 48% correct) than did the comparison group (i.e., from 29% to 30%) on the Word Problem Representation Test, $t(59) = 2.69$, $p < .01$.

Insert Figure 8 Here

Figure 9 summarizes the pretest and posttest scores for producing the correct answer and producing a correct representation on the In-depth Word Problem Solving Test by the lower achieving students in each group. On the In-depth Word Problem Solving Test, the pretest-to-posttest change on generating the correct answer for the treatment group (i.e., from 35% to 56% correct) was not significantly different from that of the comparison group (i.e., from 41% to 48% correct), $t(59) = .75$, $p = \text{ns}$. In contrast, the treatment group produced a greater pretest-to-posttest gain (i.e., from 29% to 62% correct) than did the control group (i.e., from 26% to 15%) in using appropriate mathematical representations (e.g., table, graph, or equation) to help understand the problem, $t(59) = 3.08$, $p < .01$. This pattern is consistent with the one produced by the overall sample of all students.

Insert Figure 9 Here

Thus it can be seen that the students who begin the experimental unit with the lowest level of algebra skills are as likely to benefit as their higher performing peers. In addition, their performance improves in the same ways showing that they have gained an increased understanding of the functional relationships that form the core of the instructional unit.

Conclusion

Implications about Learning

Taken together these results demonstrate qualitative differences in the learning outcomes produced by two different instructional treatments. The cognitive consequences of traditional instruction focusing on symbol manipulation are reflected in improvements in students' ability to solve equations and perform arithmetic computations. The cognitive consequences of learning through problem solving involve improvements in students' ability to represent functional relations in words, equations, tables, and figures and to translate among these representations. These results are consistent with the idea that problem representation skills--such as cognitive strategies for forming and coordinating multiple representations--are learnable in a classroom context (Brenner & Moseley, 1994; Lewis, 1989). More importantly, students who learn through a problem solving approach are able to transfer problem representation strategies to new situations such as word problems.

We interpret the test results presented in this paper to demonstrate that students learn not only to use different representations, but that they also gain a greater understanding of the nature of functional relationships. When students are asked to create a representation such as a table, graph, chart or diagram to help them solve a problem, as they were in the In-depth Word Problem Solving Test, they are engaged in several sequential cognitive activities each of which requires some understanding of the underlying relationship. They must first abstract the relationship from the verbal text. They must then transform it into another form which highlights different characteristics of the relationship. For instance, to make a table to represent a word problem the student must first determine the relevant information, i.e. variables, that will determine the rows and columns of the table. Then a series of ordered pairs must be generated from the data provided in the word problem. In our scoring protocol, a student who made a table which only contained information already given in the problem did not gain credit for making an appropriate representation.

It could be argued that the experimental classes were given more experiences with the alternative representations and therefore we have merely measured familiarity with the conventions of making tables, graphs, etc. However, the comparison classes were also given extensive experience with certain representations during the month covered by this study. Two of the classes did a textbook chapter on making graphs. The other class worked on measurement topics related to area including diagrams, a topic that was also covered in the experimental unit.

In addition, both our pre test data and other research projects have shown that junior high school students are fairly adept at understanding tables. Any of these representations were acceptable in our scoring protocol as long as they reflected the structure of the relationship in the word problem. In addition to having instruction in some of the representations, the comparison students also made some improvement on the test which required them to interpret the various representations, demonstrating that they do understand at least some aspects of these representations. The critical element that seemed to be missing for the comparison classes was the recognition of the relationship expressed in the initial verbal presentation of the problem. Thus, although the comparison students often tried to make a representation, they usually failed to capture the underlying relationship. Over the course of the study the ability of the comparison students to make a representation actually decreased form 34% correct to 25% correct although they had solved many word problems during this time. The results from the other tests in our study suggest that they used arithmetic to solve the problems. In contrast, the treatment group achieved a 65% score by the time of the post-test.

Given the superiority of the treatment classes on the representations tasks, it is somewhat surprising that these classes did not show a corresponding superiority on the problem solving tasks. One of the limitations of written test data is that it can not easily reveal the processes by which students solve problems. Thus we don't know if the treatment students actually used the representations in their problem solving or not. In addition, it is quite probably that the treatment group's understanding of the representations was somewhat fragile. Other researchers (e.g. Moschkovich, Schoenfeld, & Arcavi, 1993) have amply demonstrated the mathematical complexity inherent in the topics addressed in the experimental unit.

Differences in instructional emphases may also be part of the reason that both groups improved in their problem solving performances, but neither group outachieved the other. The model of problem solving which informed this project (Mayer, 1989) makes a clear distinction between phases of the problem solving process. The experimental unit emphasized the problem representation phase while the traditional classes emphasized the problem solution phase. Other phases of the problem solving process such as planning were not directly addressed in either class. Although it might be possible to give a somewhat more balanced emphasis to the overall process of problem solving in the experimental unit, other research has shown that even students in college mathematics courses lack the general problem solving skills to deal with complex mathematical problems (Mayer, Lewis, & Hegarty, 1992; Schoenfeld, 1985).

Educational Implications

Symbol manipulation skills and word-problem representation skills are cognitive prerequisites for success in algebra, yet traditional instruction may focus on symbol manipulation skills at the expense of representation skills. The present study demonstrates how a prealgebra unit can be redesigned successfully to emphasize problem representation skills. This may prepare students better for future study of algebra because many new high school mathematics courses (e.g. Kysh, 1991) also incorporate a greater emphasis upon the representation of functions which would be provide continuity for students between prealgebra and algebra.

The mixed results on our achievement measures parallels that found in many other studies which have attempted to implement substantial changes in mathematics curriculum and instruction (Cobb, et al. , 1991; Henderson, 1995; Hirschhorn, 1993; Sallee, 1992). Measures most related to the project goals, such as tests of problem solving, mathematical reasoning or problem representation show clear advantages for the classes which have received the new program. However, more traditional measures of achievement such as standardized tests or tests of computation such as our Equations Solving Test and Word Problem Solving Test show mixed results. Most typically other studies have found that the experimental classes are about equal to the comparison classes on the traditional measures, with some variation by site or test.

At least two distinct explanations account for this type of result. The first is that there are definite trade-offs to be made when new materials and goals are added to a mathematics curriculum. The added components take time, both for classroom instruction and for teacher preparation. Thus traditional skills may receive less direct attention, although they may be somewhat incorporated into the new materials. For instance, the students in our treatment groups spent very little time solving equations and they had none of the detailed explanations give to the comparison classes about how to solve equations when different operations are involved. The second explanation concerns the implementation of the new curriculum and instruction. Teachers and students need time to learn to operate optimally with new curricula and instructional methods. With more experience teachers may be able to achieve a better balance in the presentation of topics.

Our results are promising vis-à-vis the equity goals set by the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989) as well as the Mathematics Framework for California Public Schools (California State Department of Education, 1992). Representation-based instruction was effective for Spanish-speaking students and for lower achieving students despite the greater language demands of the curriculum and the de-emphasis upon methods of direct instruction. This confirms the work of other mathematics education researchers that new curricula and teaching methods can be effective (Henderson, 1995; Silver, Smith, & Nelson, 1995) at least at the middle-school level for a diverse range of students.

Although the mathematics education reforms recommendations have been broadly endorsed, there is a need to build a research base concerning the cognitive consequences of learning via reform-based methods. This project contributes to that research base by exemplifying how to implement one of the basic math reform goals--fostering student understanding of mathematical representations.

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Footnote

1. We also videotaped some of the classes and administered in-depth interviews to a subsample of the students, but these data have not been analyzed and are not reported in this paper.

Figure Captions

Figure 1. Four modes of representing a functional relation.

Figure 2. A computer malfunction problem.

Figure 3. Four types of mathematics tests.

Figure 4. Proportion correct by two groups on pretests and posttests for three types of tests: All students.

Figure 5. Proportion correct by two groups on pretests and posttests for two types of scores on the in-depth word problem-solving test: All students.

Figure 6. Proportion correct by two groups on pretests and posttests for three types of tests: Language minority students.

Figure 7. Proportion correct by two groups on pretests and posttests for two types of scores on the in-depth word problem-solving test: Language minority students.

Figure 8. Proportion correct by two groups on pretests and posttests for three types of tests: Lower achieving students.

Figure 9. Proportion correct by two groups on pretests and posttests for two types of scores on the in-depth word problem-solving test: Lower achieving students.

<table>
<tr><th>Verbal mode</th><th>Tabular mode</th><th>Graphical mode</th><th>Symbolic mode</th></tr>
<tr>
<td>The mass (in grams) of a material is one and half times as great as its volume (in milliliters).</td>
<td>
<table>
<tr><th>Volume (ml)</th><th>Mass (g)</th></tr>
<tr><td>1</td><td>1.5</td></tr>
<tr><td>2</td><td>3.0</td></tr>
<tr><td>3</td><td>4.5</td></tr>
<tr><td>4</td><td>6.0</td></tr>
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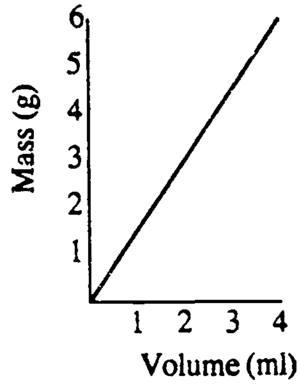
<td>
</td>
<td>$M = 1.5 \times V$
(where M is mass in grams and V is volume in milliliters)</td>
</tr>
</table>

Name________________________ Date______________________ Period_________

Lesson #2--The Computer Malfunction

Directions:

Restaurants do not always have all the necessary ingredients they need to make certain types of dishes. When Rodolfo's needs some items it calls a supplier, such as National Baking Supplies, and orders them by phone. The order is then processed by the company's computer. In a few days, the goods arrive in the mail with an invoice attached to the box. The invoice tells the store the type and amount of each item that was ordered over the phone. The employees of Rodolfo's have physically counted each item and marked a sheet labeled "receiving goods" to indicate the amount that was actually received by the store.

It seems that the computer at National Baking Supplies is making some errors, because the two lists are not the same. This means that Rodolfo's did not receive the quantities that they ordered. Your job is to convince the company (National Baking Supplies) that their errors are not random mistakes, but regular patterns.

First you need to identify the pattern of errors that was made for each different ingredient. To do this make a different table for each ingredient which shows how much was ordered and how much was received. You will have five tables. You can consult with your group about the tables but each student should make a copy of all tables to keep in the notebook.

Then, working with your group, write a letter to National Baking Supplies that describes the error your group found for one of the items. Show your results to the company using one or more of the following: a chart, table, graph or algebraic expression that will help you convince the company that their computer has made a consistent error. Your group will write one letter working together. You may be asked to read this letter to the class.

National Baking Supplies

P.O. Box 60609 Frozen Pan, Minnesota

INVOICE

Quantity Ordered	Description	Shipped	Unit Price
4	Flour (10 lbs.)	6-1-94	2.00
2	Mozzarella Cheese (5 lbs.)		5.00
20	Olive Oil (1/2 gal.)		1.00
1	Pepperoni (1/2 lb.)		3.00
3	Mushrooms (1 lb.)		6.00
10	Flour (10 lbs.)	7-1-94	2.00
5	Mozzarella Cheese (5 lbs.)		5.00
12	Olive Oil (1/2 gal.)		1.00
7	Pepperoni (1/2 lb.)		3.00
1	Mushrooms (1 lb.)		6.00
13	Flour (10 lbs.)	8-1-94	2.00
0	Mozzarella Cheese (5 lbs.)		5.00
32	Olive Oil (1/2 gal.)		1.00
10	Pepperoni (1/2 lb.)		3.00
5	Mushrooms (1 lb.)		6.00
0	Flour (10 lbs.)	9-1-94	2.00
6	Mozzarella Cheese (5 lbs.)		5.00
44	Olive Oil (1/2 gal.)		1.00
3	Pepperoni (1/2 lb.)		3.00
10	Mushrooms (1 lb)		6.00
20	Flour (10 lbs.)	10-1-94	2.00
10	Mozzarella Cheese (5 lbs.)		5.00
8	Olive Oil (1/2 gal.)		1.00
5	Pepperoni (1/2 lb.)		3.00
4	Mushrooms (1 lb)		6.00
	Total--		xxxxxx

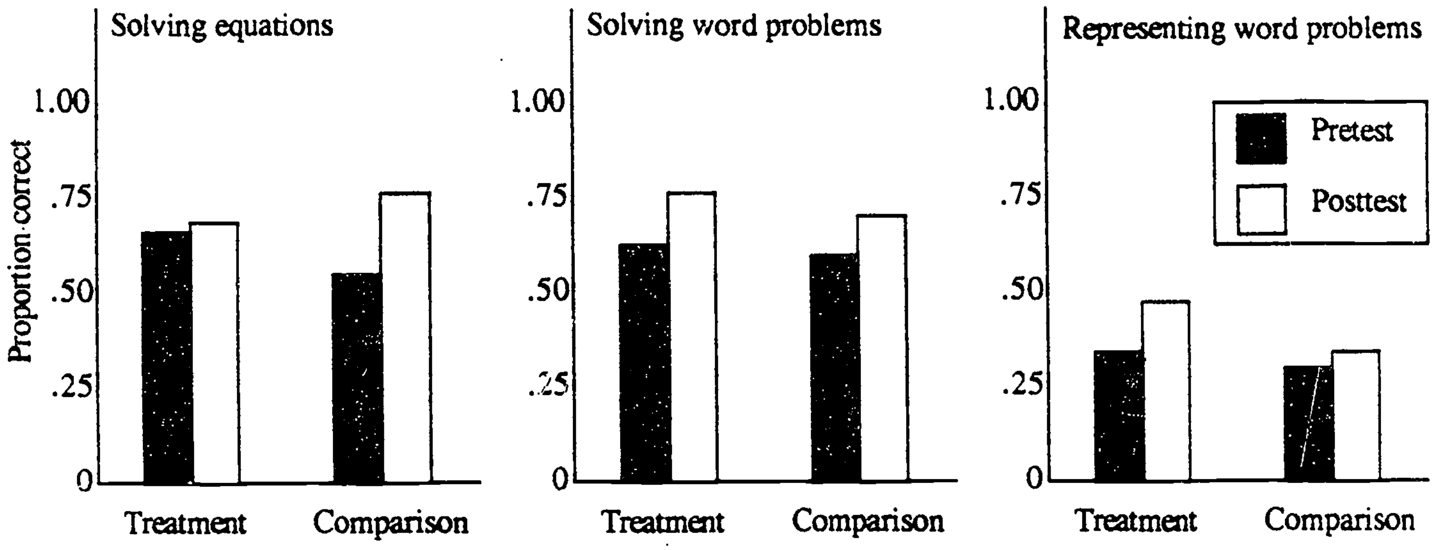
Receiving Goods

	Date Rec'd	Quantity
Flour (10 lbs. bags)	6-8-94	11
Mozzarella Cheese (5 lbs.)		7
Olive Oil (1/2 Gallon)		5
Pepperoni (1/2 pound)		5
Mushrooms (1 Pound)		8
Flour (10 lbs. bags)	7-7-94	17
Mozzarella Cheese (5 lbs.)		13
Olive Oil (1/2 Gallon)		3
Pepperoni (1/2 pound)		35
Mushrooms (1 Pound)		2
Flour (10 lbs. bags)	8-7-94	20
Mozzarella Cheese (5 lbs.)		3
Olive Oil (1/2 Gallon)		8
Pepperoni (1/2 pound)		50
Mushrooms (1 Pound)		14
Flour (10 lbs. bags)	9-8-94	7
Mozzarella Cheese (5 lbs.)		15
Olive Oil (1/2 Gallon)		11
Pepperoni (1/2 pound)		15
Mushrooms (1 Pound)		29
Flour (10 lbs. bags)	10-7-94	27
Mozzarella Cheese (5 lbs.)		23
Olive Oil (1/2 Gallon)		2
Pepperoni (1/2 pound)		25
Mushrooms (1 Pound)		11

Test Instruments

Type of Test	Name of Test	Sample Items
Traditional	**Equation Solving Test** 8 items	B = 5(H-5) If H = 163, find B
Traditional	**Word Problem Solving Test** 4 items	At McDonald's, workers earn $6.00 per hour. Workers at Wendy's earn 50 cents less per hour than workers at McDonald's. If you work for 8 hours, how much will you earn at Wendy's?
Experimental	**Word Problem Representation Test** 14 items	Write an equation for the following sentence: Two less than the product of X and 9 is 16.
		Write an equation for the following sentences, using S for Sally's age and M for Maria's age: Sally is 3 years younger than Maria.
		At Speedy Delivery Service, the cost to deliver a package is $2.00 plus an additional $.50 per pound. Fill in the missing values in the table below: (Table with missing values given) If the number of pounds is N, then write an equation that will give the total cost. TOTAL COST =________________ Draw a graph that shows the relation between the number of pounds and total cost (Graph with labeled axes given)
Traditional	**In-depth Word Problem Solving Test** 1 item Request for answer	Mary Wong just got a job working as a clerk in a candy store. She already has $42. She will earn $7 per hour. How many hours will she have to work to have a total of $126? Show your work to solve the problem. Circle the correct answer.

Experimental	Request for representation	Draw a diagram, chart, table or graph to represent the problem. Write an equation to represent the problem.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Solving word problems
Representing word problems
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.50
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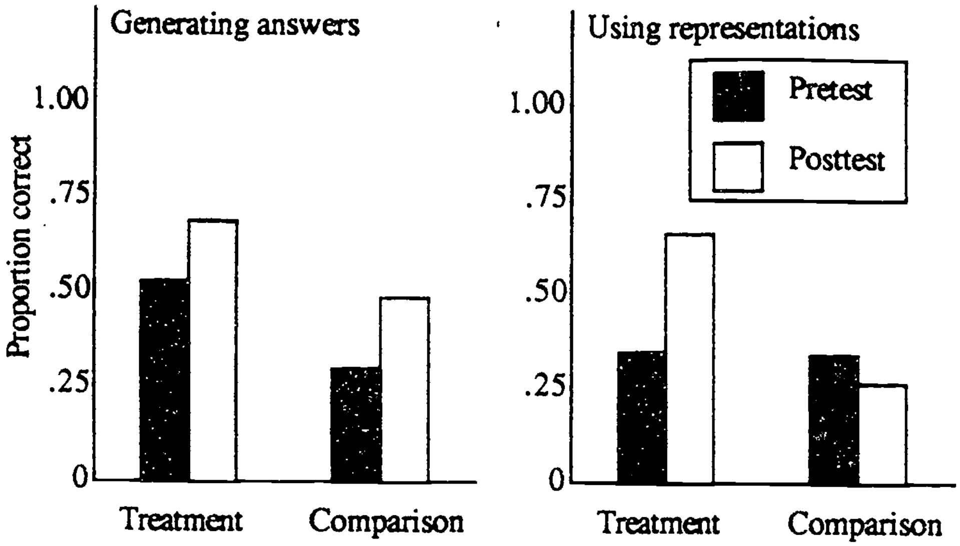
0
Proportion correct
Pretest
Posttest
Treatment
Comparison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Using representations
Pretest
Posttest
Proportion correct
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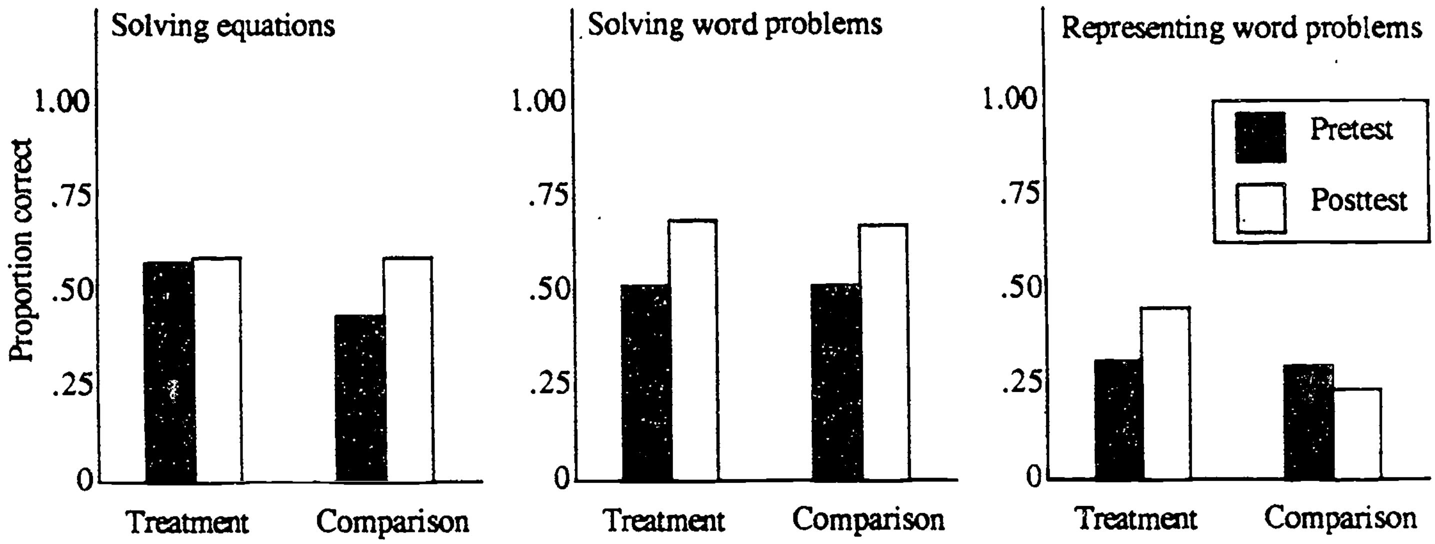
Treatment
Comparison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Treatment
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Pretest
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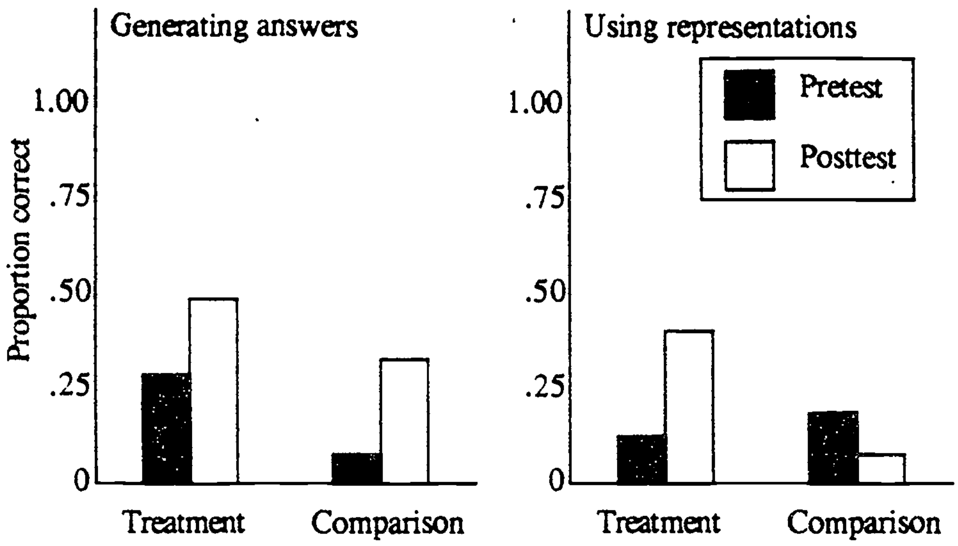

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Pretest
Posttest
Proportion correct
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.75
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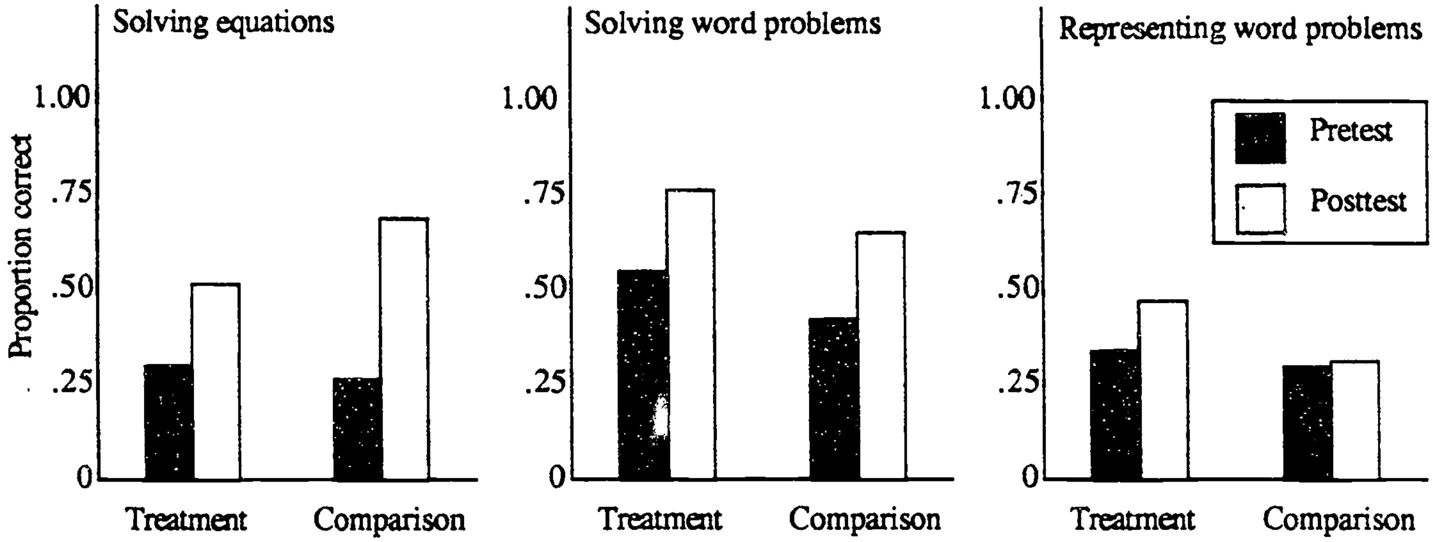
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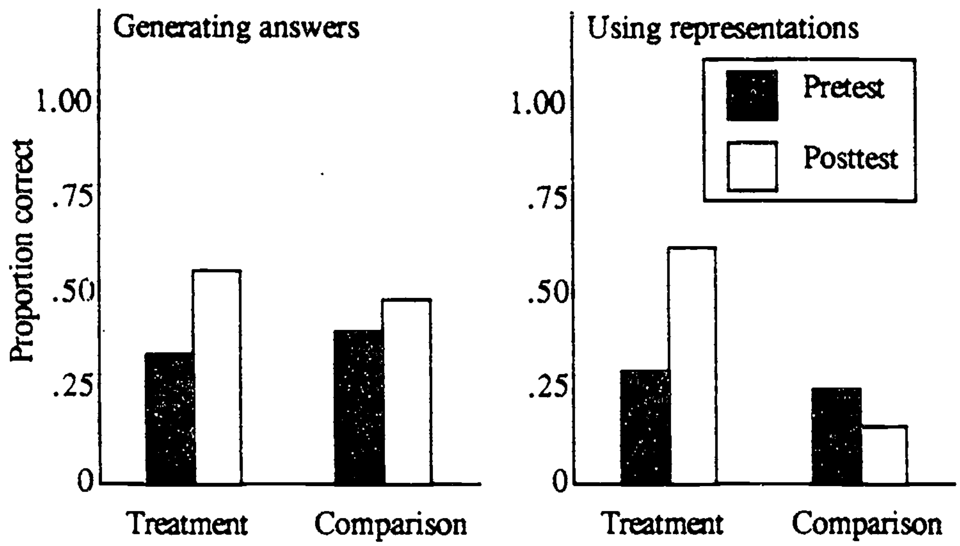
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