DOCUMENT RESUME

ED 389 724 TM 024 224

AUTHOR	Pashley, Peter J.
TITLE	An Alternative Three-Parameter Logistic Item Response Model.
INSTITUTION	Educational Testing Service, Princeton, N.J.
REPORT NO	ETS-RR-91-10
PUB DATE	Feb 91
NOTE	39p.
PUB TYPE	Reports - Evaluative/Feasibility (142)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	Equations (Mathematics); *Guessing (Tests); *Item Response Theory; *Maximum Likelihood Statistics; Models
IDENTIFIERS	Asymptotic Distributions; Four Parameter Model; *Three Parameter Model; Two Parameter Model

ABSTRACT

Birnbaum's three-parameter logistic function has become a common basis for item response theory modeling, especially within situations where significant guessing behavior is evident. This model is formed through a linear transformation of the two-parameter logistic function in order to facilitate a lower asymptote. This paper discusses an alternative three-parameter logistic model in which the asymptote parameter is a linear component within the logit of the function. This alternative is derived from a more general four-parameter model based on a transformed hyperbola. (Contains 7 figures, 1 appendix of likelihood equations and information functions, and 13 references.) (Author)

RESEARCH REPORT

AN ALTERNATIVE THREE-PARAMETER LOGISTIC ITEM RESPONSE MODEL

Peter J. Pashley

Educational Testing Service
Princeton, New Jersey
February 1991

An Alternative Three-Parameter Logistic Item Response Model

Peter J. Pashley

Educational Testing Service

February 1991

Abstract

Birnbaum's three-parameter logistic function has become a common basis for item response theory modeling, especially within situations where significant guessing behavior is evident. This model is formed through a linear transformation of the two-parameter logistic function in order to facilitate a lower asymptote. This paper discusses an alternative three-parameter logistic model in which the asymptote parameter is a linear component within the logit of the function. This alternative is derived from a more general four-parameter model based on a transformed hyperbola.

Introduction

Birnbaum (1968) introduced the three-parameter logistic (3PL) item response model in a contributed chapter in Lord and Novick (1968). Since then, this particular formulation has become a standard for investigators who wish to include a lower asymptote in their latent trait models. The logistic function was chosen as an alternative to normal ogive models (Lord, 1952), due to its more convenient mathematical properties.

Simpler item response theory (IRT) models, such as the one-parameter logistic (1PL) or Rasch model (Rasch, 1960); and two-parameter logistic (2PL) model (Birnbaum, 1957), have also been proposed, studied, and widely used. In addition, many models which do not conform to the common assumptions underlying most IRT have been investigated. Examples of these include Samejima's (1979) models for nonmonotonically increasing item response curves; Bartholomew's (1980) full-information factor analysis for multidimensional data; and Masters's (1982) partial credit model for polychotomously scored items. However, besides the three-parameter normal ogive, no other alternative latent trait model which is similar in shape and underlying assumptions to the 3PL has been extensively investigated.

This lack of research into alternatives might be due in part to the fact that the 3PL has served the testing community very well. The three parameters are easily interpretable and the model's links with the 1PL and 2PL are obvious. Also, the application of this model to actual test data has been facilitated by the availability of associated computer programs, such as BILOG (Mislevy & Bock, 1982) and LOGIST (Wingersky, Barton, & Lord, 1982).

From an analytical point of view, however, the 3PL is quite difficult to work with; and as a consequence, many basic properties have not to date been completely explored. For instance, the 3PL has yet to be shown to be (or not to be) a consistent model. While simulation studies have recently taken the place of analytical investigations in many situations, a good theoretical basis will always be preferred over a finite number of empirical observations.

These analytical difficulties can also translate into practical problems. For example, while simultaneous confidence bands are easily found for the 1PL and 2PL given abilities (Hauck, 1983), this is not the case for the 3PL (Lord & Pashley, 1988). Parameter estimation problems have also been experienced, especially with regard to the lower asymptote (Mislevy & Stocking, 1989).

In an effort to provide an alternative avenue of research, this paper presents a new three-parameter logistic function which possesses many of the same modeling characteristics of the *standard (Birnbaum) 3PL* but with a distinctly different formulation. In particular, the asymptote parameter can be written as a linear component within the argument of a logistic function. This alternative is derived from a more general four-parameter model based on a segment of a transformed hyperbolic curve. These two models will be referred to as the *hyperbolic 3PL* and *hyperbolic 4PL*.

Notation

As mentioned above, the standard 3PL can be written as a linear transformation of a two-parameter logistic function. A common form of this model is

$$P(\theta) = c + (1 - c)\Psi[1.7a(\theta - b)] \ , \tag{1}$$

where $P(\theta)$ represents the probability that an examinee with ability θ will answer a specific item correctly; a, b, and c are item discrimination, difficulty, and lower asymptote (or pseudo-guessing) parameters, respectively; the constant 1.7 is a scaling factor; and

$$\Psi(z) = \frac{e^z}{1 + e^z} = \frac{1}{1 + e^{-z}} \tag{2}$$

is the logistic function.

Associated with the logistic function is the logit transformation, denoted by $\lambda(\theta)$, which can be expressed as

$$\lambda(\theta) = \ln\left[\frac{P(\theta)}{1 - P(\theta)}\right]. \tag{3}$$

The space defined by this transformation, called the logit space, provides an alternative modeling environment to the usual probability correct space.

Logit Space

IRT models which are logistic functions can be conveniently written in the form $P(\theta) = \Psi[\lambda(\theta)]$. For example, the logit transform of one form of a 2PL model is

$$\lambda(\theta) = 1.7a(\theta - b) \ , \tag{4}$$

which is simply the argument of the corresponding logistic function.

Unfortunately, in the case of a standard 3PL which is a linear transformation of the 2PL (in the probability correct space), a logit transformation does not yield such a simple expression. To illustrate, the following is one form of a logit transform of a standard 3PL (P. W. Holland, personal communication, August 27, 1990):

$$\lambda(\theta) = \ln\{1 + \exp[1.7a(\theta - b) - \ln c]\} + \ln\left(\frac{c}{1 - c}\right). \tag{5}$$

This expression can be rewritten to yield an alternative formula for the standard 3PL in the probability correct space (C. Lewis, personal communication, August 31, 1990):

$$P(\theta) = \Psi\left\{\ln\left[\frac{c + \exp[1.7a(\theta - b)]}{1 - c}\right]\right\}. \tag{6}$$

An example of a standard 3PL item response curve and its corresponding logit transform with associated asymptotes is shown in Figure 1. In all cases, this function approaches the logit transform of the 2PL (which is a straight line) in the upper tail and the logit transform of the c parameter in the lower tail. Also note that the curve is not symmetric with respect to these two asymptotes in the logit space.

Insert Figure 1 about here

The Hyperbolic 4PL

There are many well-known curves within the field of analytical geometry which exhibit shapes similar to those of the logit transformed standard 3PL. One in particular, the hyperbola, also possess comparable asymptotic properties. Because of these features, the hyperbola was chosen as a basis for developing an alternative to the standard 3PL.

A general equation for a hyperbola is given by

$$\frac{Z^2}{s^2} - \frac{W^2}{r^2} = 1 , \tag{7}$$

where Z and W denote the axis coordinates; and s and r are parameters which define the shape of the curve. An example of a hyperbola is shown in Figure 2.

Insert Figure 2 about here

Two transformations and a constraint are needed in order for this curve to resemble the logit transformed standard 3PL illustrated in Figure 1. The first transformation is a rotation defined by

$$\begin{aligned} Z &= Y\cos(\alpha) - X\sin(\alpha) \text{ , and} \\ W &= Y\sin(\alpha) + X\cos(\alpha) \text{ ,} \end{aligned} \tag{8}$$

where X and Y denote the new coordinate system, and

$$\alpha = \tan^{-1}\left(\frac{s}{r}\right). \tag{9}$$

Substituting (8) and (9) into (7) yields

$$Y = \frac{sr}{r^2 - s^2}\left[X + \sqrt{X^2 + (r^2 - s^2)}\right]. \tag{10}$$

This manipulation rotates the curve until its upper left asymptote is horizontal, thus providing a lower asymptote and ensuring that the resulting model is monotonically non-decreasing. Note that embedded in Equation 10 is the constraint that only the upper segment of the curve will be considered (i.e., the part of the curve which lies above the W axis in Figure 2).

The second transformation is a translation needed to center the curve properly. This is achieved through the following reparameterizations:

$$\begin{aligned} X &= \theta - h \text{ , and} \\ Y &= \lambda(\theta) - k \text{ .} \end{aligned} \tag{11}$$

The result of inserting (11) into (10) is the logit of a new four-parameter logistic item response model:

$$\begin{gathered} \lambda(\theta) = f\left[(\theta - h) + \sqrt{(\theta - h)^2 + g}\right] + k \text{ ,} \\ f > 0,\ g > 0. \end{gathered} \tag{12}$$

Three examples of this hyperbolic 4PL, with associated asymptotes, are shown in Figure 3. In this case, the three curves differ only with regard to the parameter g. This parameter reflects how quickly the curve approaches its asymptotes. As g approaches zero, the curve tends toward the asymptotes and coincides with them when g equals zero.

Note also that in the logit space, the model is symmetric with regard to the asymptotes, unlike the standard 3PL.

Insert Figure 3 about here

Relationships between the hyperbolic 4PL parameters f, h, and k, and parameters a, b, and c from the standard 3PL, obtained by equating their respective asymptotes, are given by

$$f = \frac{1.7}{2}a \ ,$$

$$h = b + \frac{1}{1.7a}\ln\left[\frac{c}{1 - c}\right], \text{ and} \tag{13}$$

$$k = \ln\left[\frac{c}{1 - c}\right].$$

These relationships indicate that f may be regarded as a slope parameter; h is similar to a difficulty parameter; and k may be thought of as a lower asymptote parameter, as it is the logit transform of c. As in the case of the standard 3PL, the hyperbolic 4PL approaches the form of the 2PL as k approaches $-\infty$ (or as c approaches 0); and is equivalent to the 2PL when $k = -\infty$. Note that the hyperbolic 4PL parameter g is not included in the equations of the asymptotes, but rather, as indicated above, determines how quickly the curves approach their asymptotes.

The Hyperbolic 3PL

In order to reduce the hyperbolic 4PL model into a three-parameter form, one parameter, or a combination of parameters, needs to be constrained in some way. Consider Figure 3 once more. Note how the parameter g affects the slope of the curve in the probability correct space. In the case of the standard 3PL, the amount of slope is determined mainly by the c parameter. If the hyperbolic 3PL is to behave in a similar fashion to the standard 3PL, one constraint option is to let g be a function of f, and denote it by $g(f)$. This is a reasonable option, as Equation Set 13 indicates, since f may be regarded as a slope parameter that is related to the standard 3PL a parameter when the respective asymptotes have been equated. Given this constraint, the logit of the hyperbolic 3PL can be written as

$$\lambda(\theta) = f\left[(\theta - h) + \sqrt{(\theta - h)^2 + g(f)}\right] + k \; . \tag{14}$$

One specification which simplifies this model definition is $g(f) = f^{-2}$. Then Equation 14 reduces to

$$\lambda(\theta) = f\,(\theta - h) + \sqrt{[f\,(\theta - h)]^2 + 1} + k \; . \tag{15}$$

Note that this constraint, or any others considered in this paper, does not affect the monotonically non-decreasing nature inherent to the hyperbolic 4PL, nor does it preclude the 2PL and 1PL as a submodels.

A comparison of the hyperbolic 3PL defined by Equation 15 and a standard 3PL is shown in Figure 4. The parameters for the hyperbolic 3PL were derived from the standard 3PL values using Equation Set 13, in order to ensure that both curves had the

same asymptotes. As was the case for the hyperbolic 4PL, the logit is still symmetric with respect to the asymptotes.

Insert Figure 4 about here

A comparison of the two models in the more familiar probability correct space is shown in Figure 5, using the same parameter values as in Figure 4. While these two item response curves are fairly similar, a better match can be obtained by optimizing the fit of one to the other. This can be accomplished by sampling systematically points from the standard 3PL (with given parameter values) and then performing a nonlinear logistic regression to determine the most appropriate hyperbolic 3PL parameter values. Three examples of such model fitting are found in Figure 6.

Highly discriminating standard 3PL items were found to be the hardest to replicate. Even in these cases, though, the two curves were found to be for the most part within .01 of each other. This compares favorably to the maximum difference between the three-parameter normal ogive and logistic functions (when using a constant of 1.7), which is .01.

Insert Figure 5 about here

Insert Figure 6 about here

For estimation purposes, at least within a maximum likelihood framework, practitioners usually rely on the associated likelihood equations and information matrices. Those related to the form of the hyperbolic 3PL defined by Equation 15 are derived in the appendix.

Other Formulations and Associated Models

Various other hyperbolic 3PL definitions can be formulated through a variety of model constraints and/or reparameterizations. To illustrate this point, three different hyperbolic 3PL model specifications are given in this section. Note that only the logit formulas are given, since in all cases $P(\theta) = \Psi[\lambda(\theta)]$.

Example 1. Consider the model derived from using the links to the standard 3PL found in Equation Set 13 in order to reparameterize Equation 15:

$$\lambda(\theta) = \frac{1}{2}\left[1.7a(\theta - b) + \sqrt{\left[1.7a(\theta - b) - \ln\left(\frac{c}{1 - c}\right)\right]^2 + 1}\,\right] + \frac{3}{2}\ln\left(\frac{c}{1 - c}\right). \tag{16}$$

This model formulation uses the parameters common to the standard 3PL but retains the shape of the hyperbolic 3PL.

Example 2. Setting the hyperbolic 4PL parameter g (Equation 12) to zero yields a connected line segment curve in the logit space, as illustrated in Figure 7. The form of the model may then contain a square root or absolute value sign:

$$\begin{aligned}\lambda(\theta) &= f(\theta - h) + \sqrt{[f(\theta - h)]^2} + k \\ &= f\big[(\theta - h) + |\theta - h|\big] + k \ .\end{aligned} \tag{17}$$

Note that this function is not continuously differentiable at $\theta = h$.

Insert Figure 7 about here

Example 3. The model specification given in Equation 15 can be reparameterized by incorporating the lower asymptote parameter k, along with the parameters f and h, into a new intercept term to form

$$\lambda(\theta) = \beta_0 + \beta_1\theta + \sqrt{(\beta_2 + \beta_1\theta)^2 + 1} \ , \tag{18}$$

where

$$\begin{aligned}\beta_0 &= k - fh \ , \\ \beta_1 &= f \ , \text{ and} \\ \beta_2 &= -fh \ .\end{aligned} \tag{19}$$

This parameterization yields a slightly simpler form compared to Equation 15.

This is, of course, only a small sample of the many reformulations which are possible. The potential advantages and disadvantages of the various model specifications will depend on several factors; including estimation considerations, parameter interpretability, and model fit.

Conclusions

While there has been a recent trend toward new item types, including free-response and interactive formats, the multiple-choice question is still the most common item type in use today within standardized tests. These items typically exhibit differing difficulty and discrimination characteristics. Due to their constrained nature, they also possess a lower probability correct threshold since examinees with little or no ability still have some nonzero chance of responding correctly by guessing. These lower item response curve asymptotes may also differ across items and do not depend strictly on the number of choices available to the examinee.

For test developers, investigating these three inherent characteristics of individual items can be very useful in the construction of tests. The standard 3PL model incorporates these three characteristics explicitly through its three item parameters. As such, the standard 3PL has become one of the most investigated and implemented IRT models.

On the other hand, the particular formulation of the standard 3PL (i.e., a shifted 2PL in the probability correct space) makes this model difficult to work with from a analytical standpoint. In order to provide an alternative model formulation, but with similar modeling characteristics as the standard 3PL, this paper has introduced the hyperbolic 3PL whose logit is based on a transformed hyperbolic curve. A more general four-parameter model was first presented, then a simplifying constraint was applied to produce a three-parameter version whose parameters have direct links with those of the standard 3PL.

These links between the hyperbolic 3PL and standard 3PL were established under conditions in which the respective asymptotes had been equated. After using these linking formulas, the actual curves in either the probability correct or logit space can still differ substantially. However, if the hyperbolic 3PL is optimally fitted to the standard 3PL, the result is usually within a .01 difference between the function values. Often the two curves which lie virtually on top of each other, especially when the discrimination parameter is in the low to moderate range.

In addition to being able to mimic the shapes of the standard 3PL, the hyperbolic 3PL possess other properties in common with its predecessor. These include a monotonically non-decreasing nature; and having the 2PL and 1PL as submodels.

One advantage of the hyperbolic 3PL is that its lower asymptote can be expressed as a linear component of the logit. As can be seen from Equation Set 28 in the appendix, this means that the number of examinees answering an item correctly is a sufficient statistic for this pseudo-guessing parameter, if all other parameters are known.

In addition, the hyperbolic 3PL item response curve is symmetrical with respect to its associated asymptotes in the logit space, a feature not enjoyed by the standard 3PL. This particular characteristic may stabilize related estimation procedures (C. Lewis, personal communication, June 19, 1990), especially in the case of the lower asymptote. Plans for testing this hypothesis through the implementation of this model in an IRT computer package have been set forth.

References

Bartholomew, D. J. (1980). Factor analysis for categorical data. *Journal of the Royal Statistical Society, Series B, 42,* 293-321.

Birnbaum, A. (1957). *Efficient design and use of tests of a mental ability for various decision-making problems.* (Series Report No. 58-16). Randolph Air Force Base, TX: USAF School of Aviation Medicine.

Birnbaum, A. (1968). Some latent trait models and their use in inferring an examinees's ability. In F. M. Lord, & M. R. Novick, *Statistical theories of mental test scores* (pp. 397-479). Reading, MA: Addison-Wesley.

Hauck, W. W. (1983). A note on confidence bands for the logistic response curve. *The American Statistician, 37,* 158-160.

Lord, F. M. (1952). A theory of mental test scores. *Psychometric Monograph, No. 7,* Psychometric Society.

Lord, F. M., & Novick, M. R. (1968). *Statistical theories of mental test scores.* Reading, MA: Addison-Wesley.

Lord, F. M., & Pashley, P. J. (1988). *Confidence bands for the three-parameter logistic item response curve.* (ETS Research Report No. 88-67). Princeton, NJ: Educational Testing Service.

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika, 47,* 149-174.

Mislevy, R. J., & Bock, R. D. (1982). *BILOG: Maximum likelihood item analysis and test scoring with logistic models for binary items* [computer program]. Mooresville, IN:

Scientific Software.

Mislevy, R. J., & Stocking, M. L. (1989). A consumer's guide to LOGIST and BILOG. *Applied Psychological Measurement, 13,* 57-75.

Rasch G. (1960). *Probabilistic models for some intelligence and attainment tests.* Copenhagen: Danish Institute for Educational Research.

Samejima, F. (1979). *A new family of models for the multiple-choice item* (Research Report 79-4). Knoxville, TN: Department of Psychology, University of Tennessee.

Wingersky, M. S., Barton, M. A., & Lord, F. M. (1982). *LOGIST* [computer program]. Princeton, NJ: Educational Testing Service.

Appendix

Likelihood Equations and Information Functions

This appendix contains the usual likelihood equations and information functions often used in conjunction with maximum likelihood estimation. The more general formulas, from which equations specific to the hyperbolic 3PL model were derived, were taken from Lord (1980). The form of the hyperbolic 3PL used here is from Equation 15, and so the probability of examinee j (with ability θ_j) correctly answering item i, denoted by P_{ij}, can be written as

$$P_{ij} = \Psi\left\{f_i(\theta_j - h_i) + \sqrt{f_i^2(\theta_j - h_i)^2 + 1} + k_i\right\}, \tag{20}$$

where i ranges from 1 to n items, and j ranges from 1 to m examinees.

Case 1: known item parameters

Consider first estimating an ability θ from an examinee's performance on n items, all of which have known item parameters. Note that the subscript j has been dropped in this case to simplify the notation.

The first partial derivative of P_i with respect to θ is given by

$$\frac{\partial P_i}{\partial \theta} = f_i\left[\frac{f_i(\theta - h_i)}{\sqrt{f_i^2(\theta - h_i)^2 + 1}} + 1\right]P_iQ_i, \tag{21}$$

where $Q_i = 1 - P_i$.

The general form of the likelihood equation that can be solved to yield a maximum

likelihood estimate of θ is

$$\sum_{i=1}^{n}\left(\frac{x_i - P_i}{P_i Q_i}\right)\frac{\partial P_i}{\partial \theta} = 0 \ , \tag{22}$$

where $x_{ij} = 1$ if examinee j answered item i correctly, and zero otherwise. Using (21), the specific form of this likelihood equation for the hyperbolic 3PL can be expressed as

$$\sum_{i=1}^{n}(x_i - P_i)f_i\left[\frac{f_i(\theta - h_i)}{\sqrt{f_i^2(\theta - h_i)^2 + 1}} + 1\right] = 0 \ . \tag{23}$$

The general form of the test information function, denoted by I_θ, is given by

$$I_\theta = \sum_{i=1}^{n}\frac{1}{P_i Q_i}\left(\frac{\partial P_i}{\partial \theta}\right)^2 \ . \tag{24}$$

Again using (21), a specific form of this test information function for the hyperbolic 3PL is

$$I_\theta = \sum_{i=1}^{n} f_i^2\left[\frac{f_i(\theta - h_i)}{\sqrt{f_i^2(\theta - h_i)^2 + 1}} + 1\right]^2 P_i Q_i \ . \tag{25}$$

Case 2: known ability parameters

Now consider estimating the parameters f, h, and k for a single item based on results from m examinees whose abilities are known. In this case the subscript i has been discarded in order to simplify the notation.

The first partial derivatives of P_j with respect to item parameters f, h, and k are given by

$$\frac{\partial P_j}{\partial f} = (\theta_j - h)\left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1\right]P_jQ_j \ ;$$

$$\frac{\partial P_j}{\partial h} = -f\left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1\right]P_jQ_j \ ; \text{ and} \tag{26}$$

$$\frac{\partial P_j}{\partial k} = P_jQ_j \ .$$

The associated likelihood equations have the general form

$$\sum_{j=1}^{m}\left(\frac{x_j - P_j}{P_jQ_j}\right)\frac{\partial P_j}{\partial \xi} = 0 \ , \tag{27}$$

where ξ is an arbitrary item parameter. Then the corresponding likelihood equations that must be solved simultaneously for f, h, and k, can be derived as

$$f\!: \ \sum_{j=1}^{m}(x_j - P_j)(\theta_j - h)\left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1\right] = 0 \ ;$$

$$h\!: \ \sum_{j=1}^{m}(x_j - P_j)f\left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1\right] = 0 \ ; \text{ and} \tag{28}$$

$$k\!: \ \sum_{j=1}^{m}(x_j - P_j) = 0 \ ,$$

by using (26).

An element of an information matrix for item parameters, denoted by $I_{\xi\gamma}$, has the following general form:

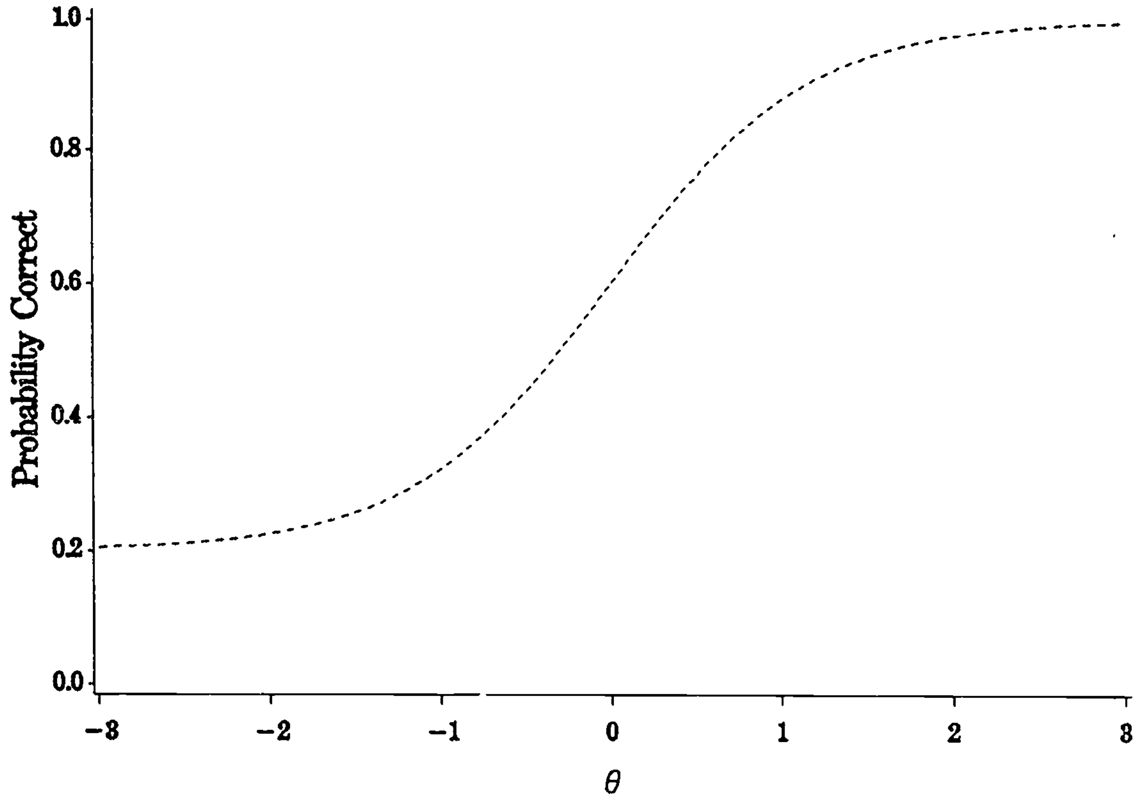
$$I_{\zeta\gamma} = \sum_{j=1}^{m} \frac{1}{P_j Q_j} \frac{\partial P_j}{\partial \zeta} \frac{\partial P_j}{\partial \gamma} \, , \tag{29}$$

where ζ and γ are arbitrary item parameters. Again using (26), the specific elements of the information matrix corresponding to the parameters f, h, and k, can be derived as

$$\begin{aligned}
I_{ff} &= \sum_{j=1}^{m} (\theta_j - h)^2 \left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1 \right]^2 P_j Q_j \, , \\
I_{hh} &= \sum_{j=1}^{m} f^2 \left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1 \right]^2 P_j Q_j \, , \\
I_{kk} &= \sum_{j=1}^{m} P_j Q_j \, , \\
I_{fh} &= \sum_{j=1}^{m} f(\theta_j - h) \left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1 \right]^2 P_j Q_j \, , \\
I_{fk} &= \sum_{j=1}^{m} (\theta_j - h) \left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1 \right] P_j Q_j \, , \text{ and} \\
I_{hk} &= \sum_{j=1}^{m} f \left[\frac{f(\theta_j - h)}{\sqrt{f^2(\theta_j - h)^2 + 1}} + 1 \right] P_j Q_j \, .
\end{aligned} \tag{30}$$

FIGURES

Figure 1. A standard 3PL curve (with parameters a = 1, b = 0, and c = .2) plotted in the probability correct and logit spaces. The associated asymptotes (solid lines) are included in the logit space.


1.0
0.8
0.6
0.4
0.2
0.0
Probability Correct
−3
−2
−1
0
1
2
3
θ

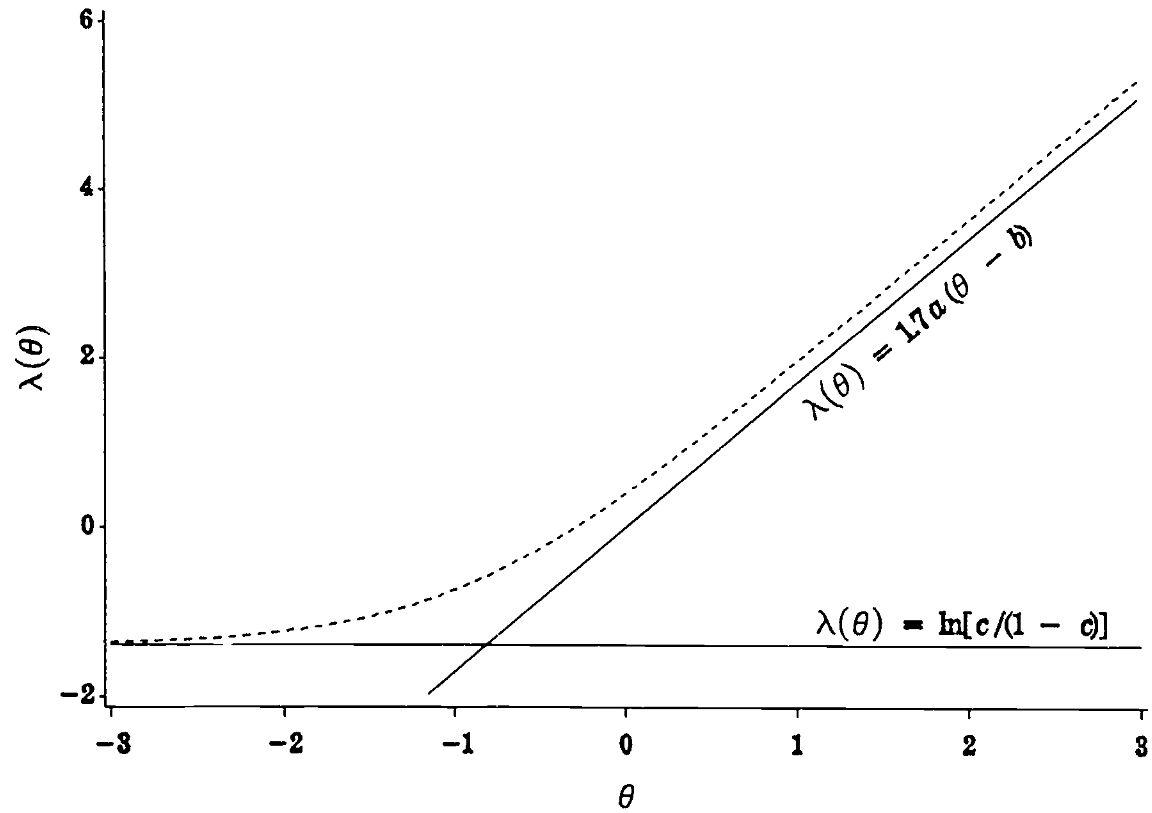

6
4
2
0
−2
λ(θ)
λ(θ) = 1.7a(θ − b)
λ(θ) = ln[c/(1 − c)]
−3
−2
−1
0
1
2
3
θ

Figure 2. An example of a hyperbola (dotted lines) plotted with asymptotes, which are dependent on the parameters s and r, within a Z and W coordinate system.

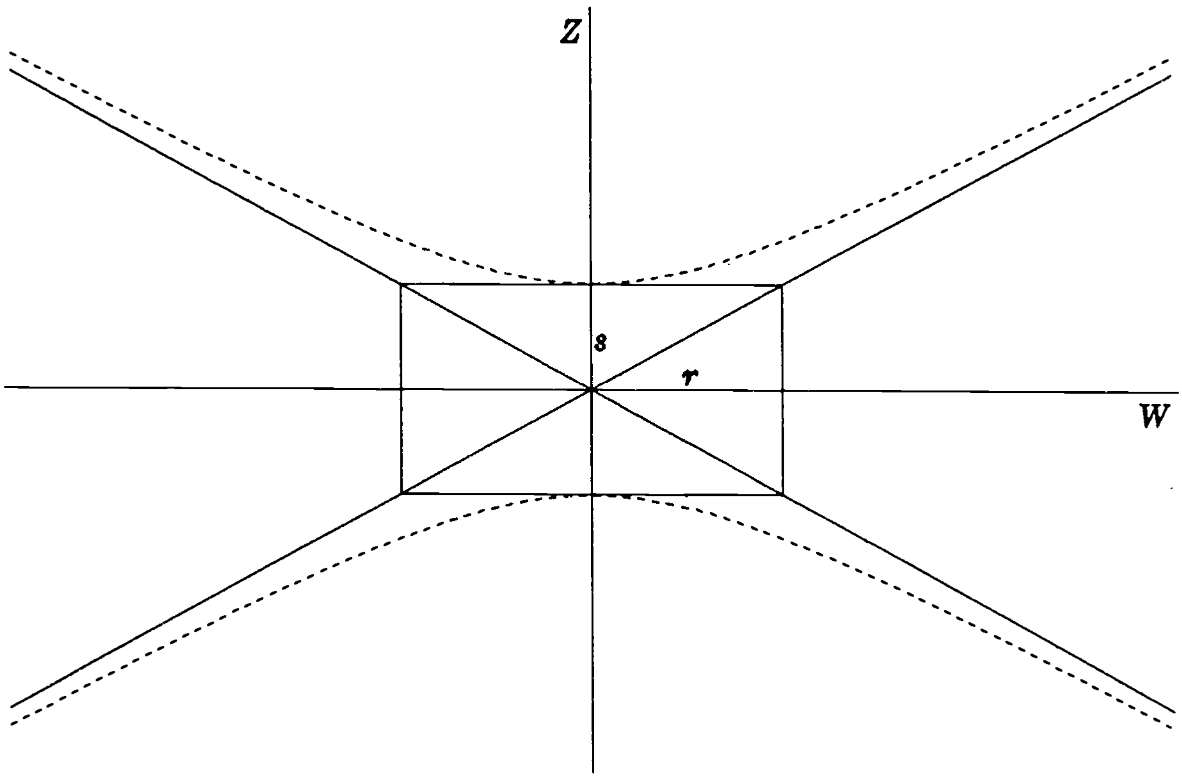
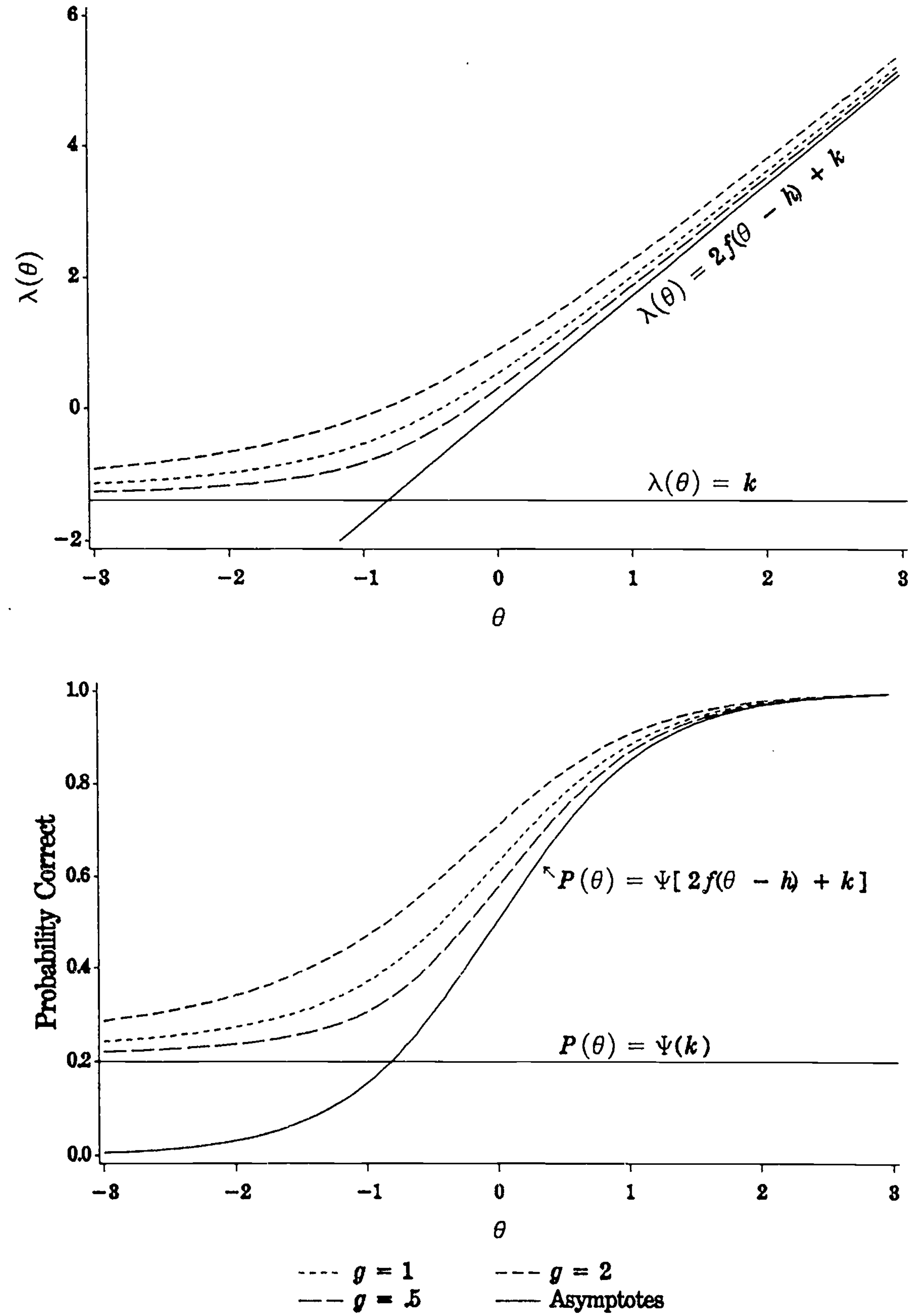
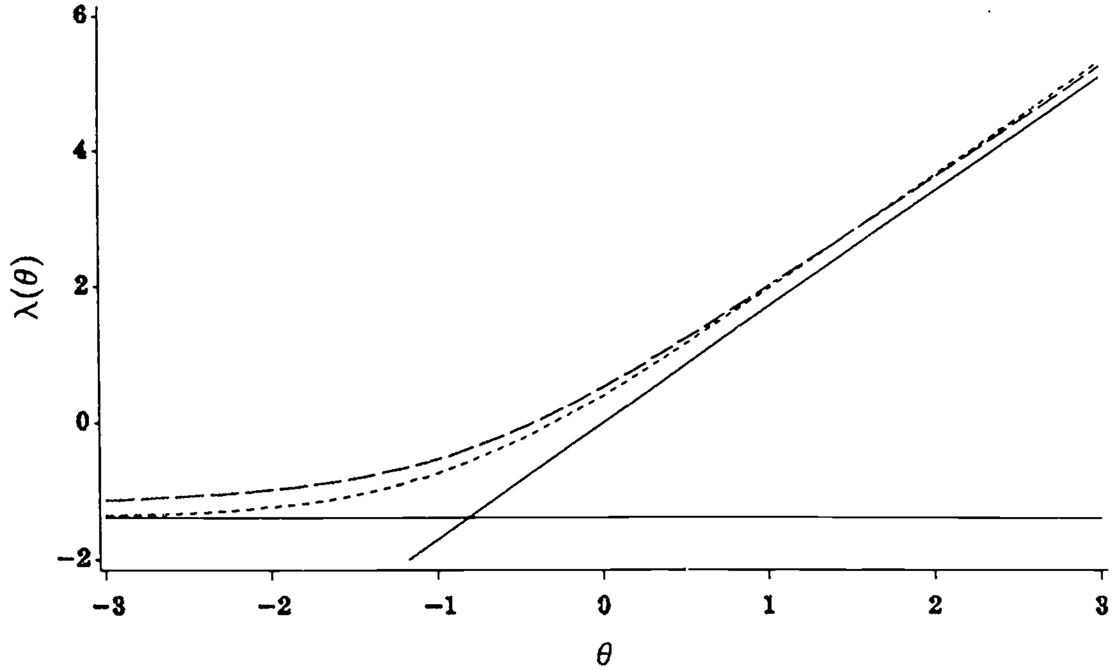

Z
s
r
W

Figure 3. Three hyperbolic 4PL curves (with parameters f = .85, h = -.82, k = -1.39, and g = 1, 2, and .5) and associated asymptotes plotted in the logit and probability correct spaces.



6
4
2
0
−2
λ(θ)
λ(θ) = 2f(θ − h) + k
λ(θ) = k
−3
−2
−1
0
1
2
3
θ
1.0
0.8
0.6
0.4
0.2
0.0
Probability Correct
P(θ) = Ψ[2f(θ − h) + k]
P(θ) = Ψ(k)
−3
−2
−1
0
1
2
3
θ
g = 1
g = 2
g = .5
Asymptotes

Figure 4. A comparison in the logit space of a standard 3PL with a hyperbolic 3PL where both curves have the same asymptotes.



---- Standard 3PL (a=1, b=0, c=.2)
— — Hyperbolic 3PL (f=.85, h= −.82, k= −1.39)
—— Asymptotes

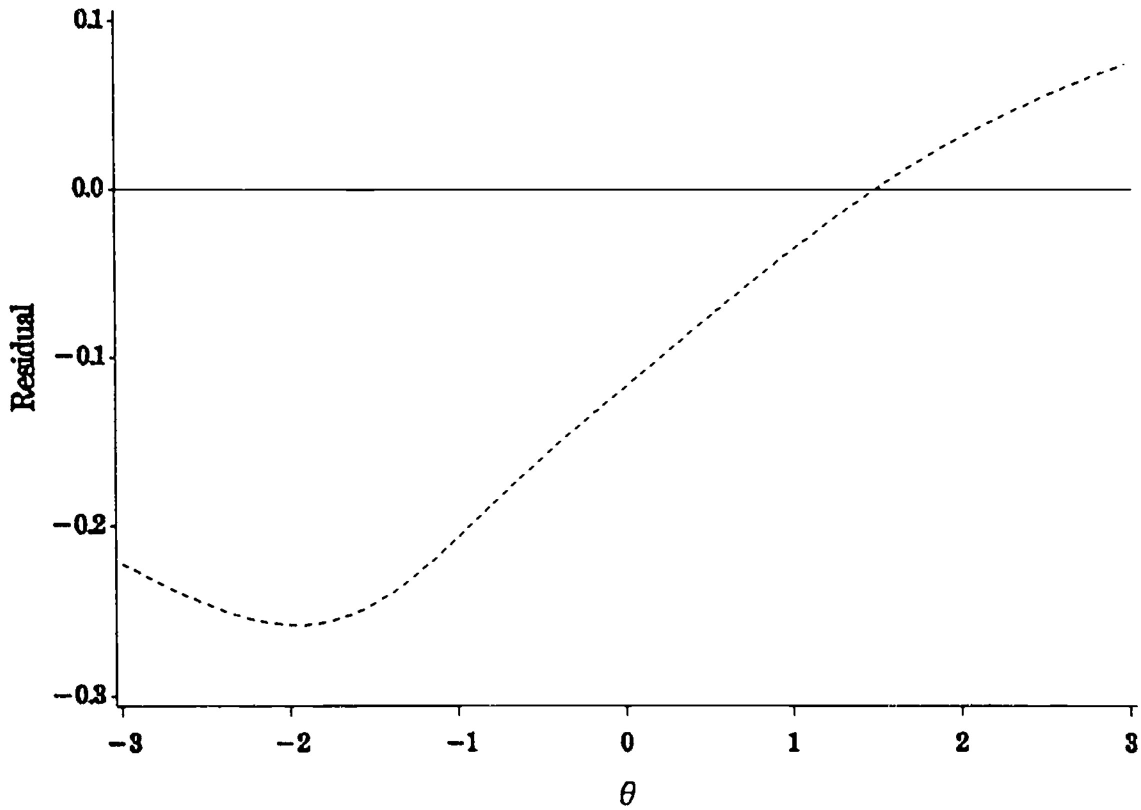
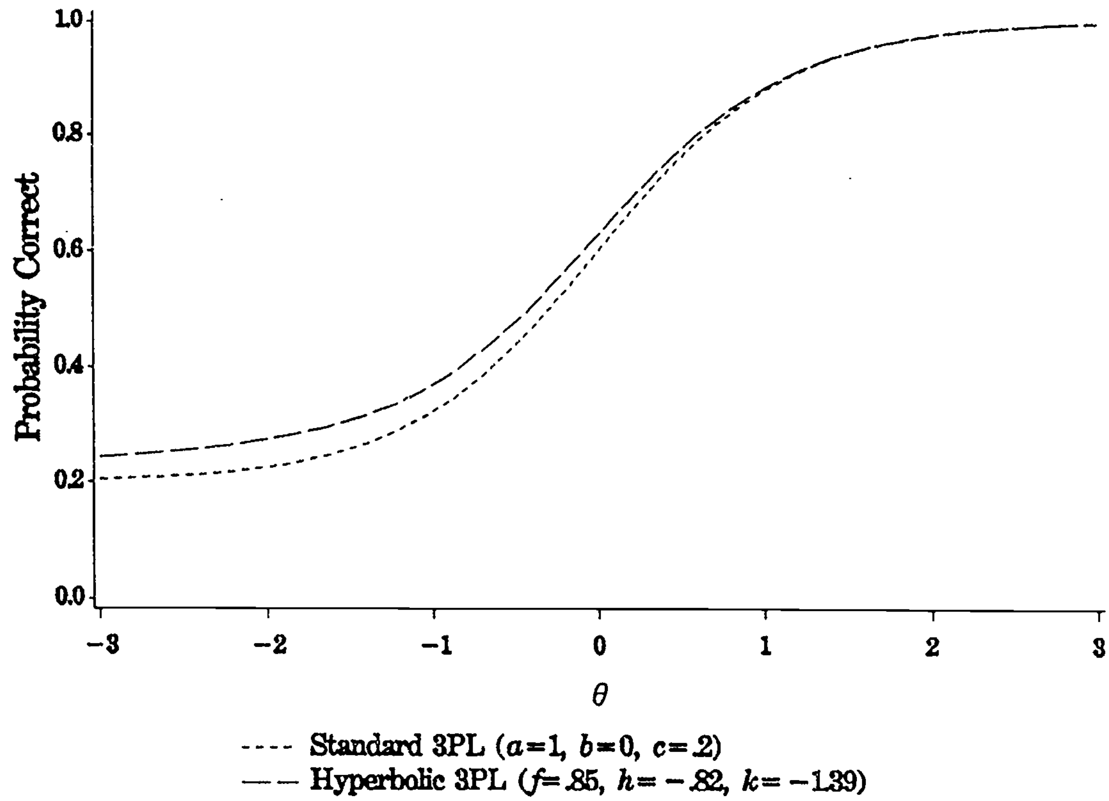


Figure 5. A comparison in the probability correct space of a standard 3PL with a hyperbolic 3PL where both curves have the same logit space asymptotes.


Probability Correct
1.0
0.8
0.6
0.4
0.2
0.0
−3
−2
−1
0
1
2
3
θ
Standard 3PL (a=1, b=0, c=.2)
Hyperbolic 3PL (f=.85, h=−.82, k=−1.39)

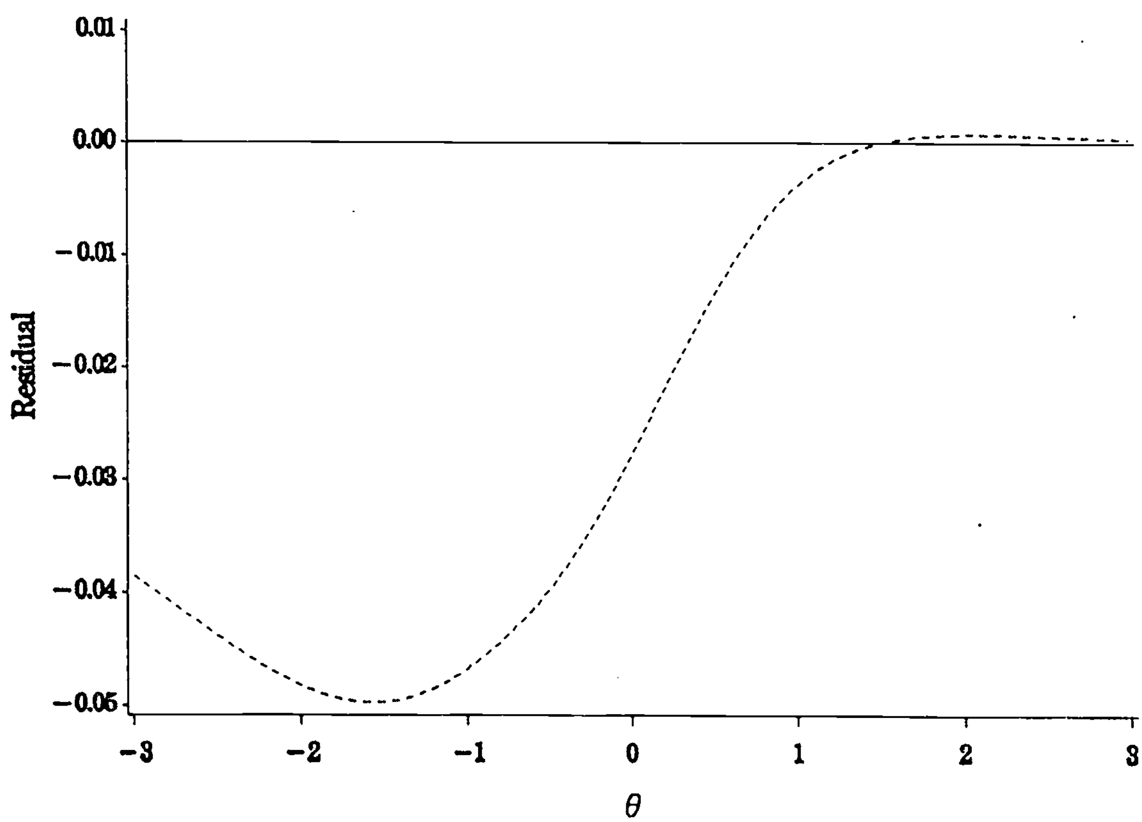
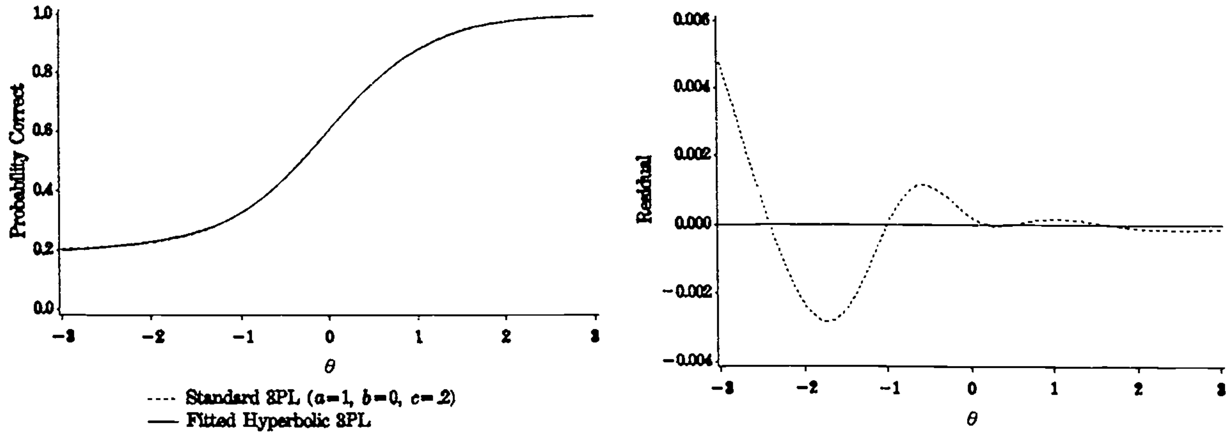
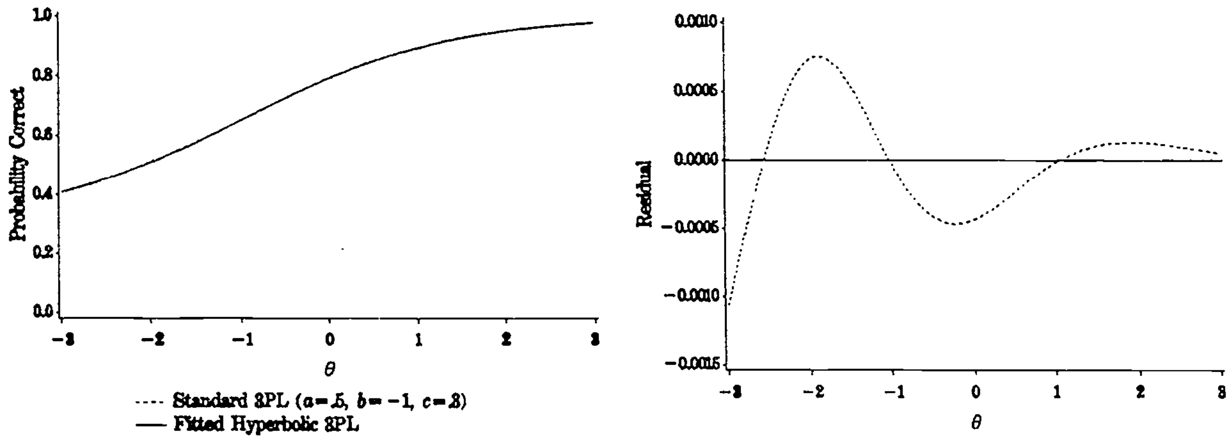

Residual
0.01
0.00
−0.01
−0.02
−0.03
−0.04
−0.05
−3
−2
−1
0
1
2
3
θ

Figure 6. Item characteristic curves and corresponding residual plots resulting from fitting the hyperbolic 3PL to three different standard 3PL specifications.


Probability Correct
θ
Residual
θ
---- Standard 3PL (a=1, b=0, c=.2)
— Fitted Hyperbolic 3PL


Probability Correct
θ
Residual
θ
---- Standard 3PL (a=.5, b=−1, c=.3)
— Fitted Hyperbolic 3PL

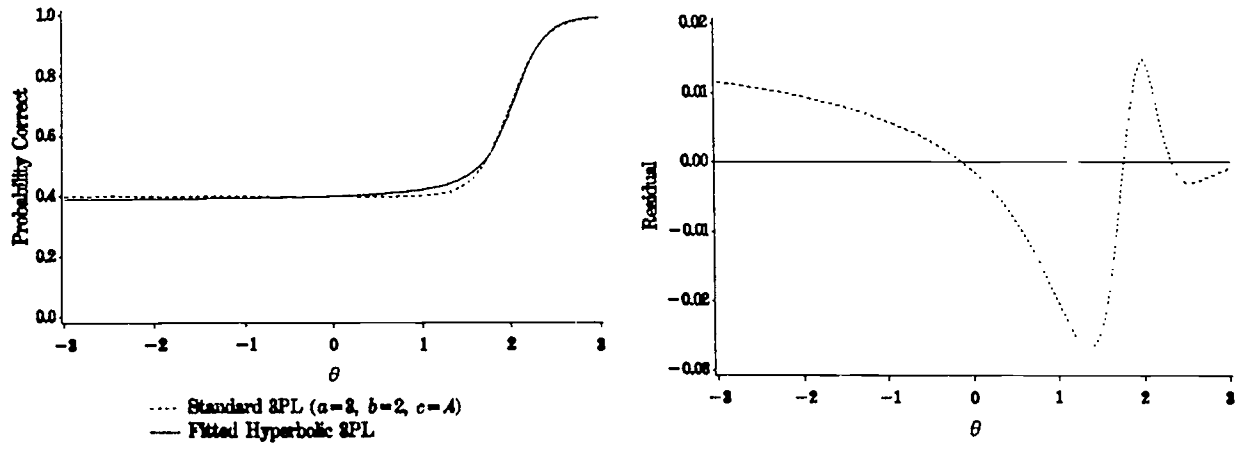
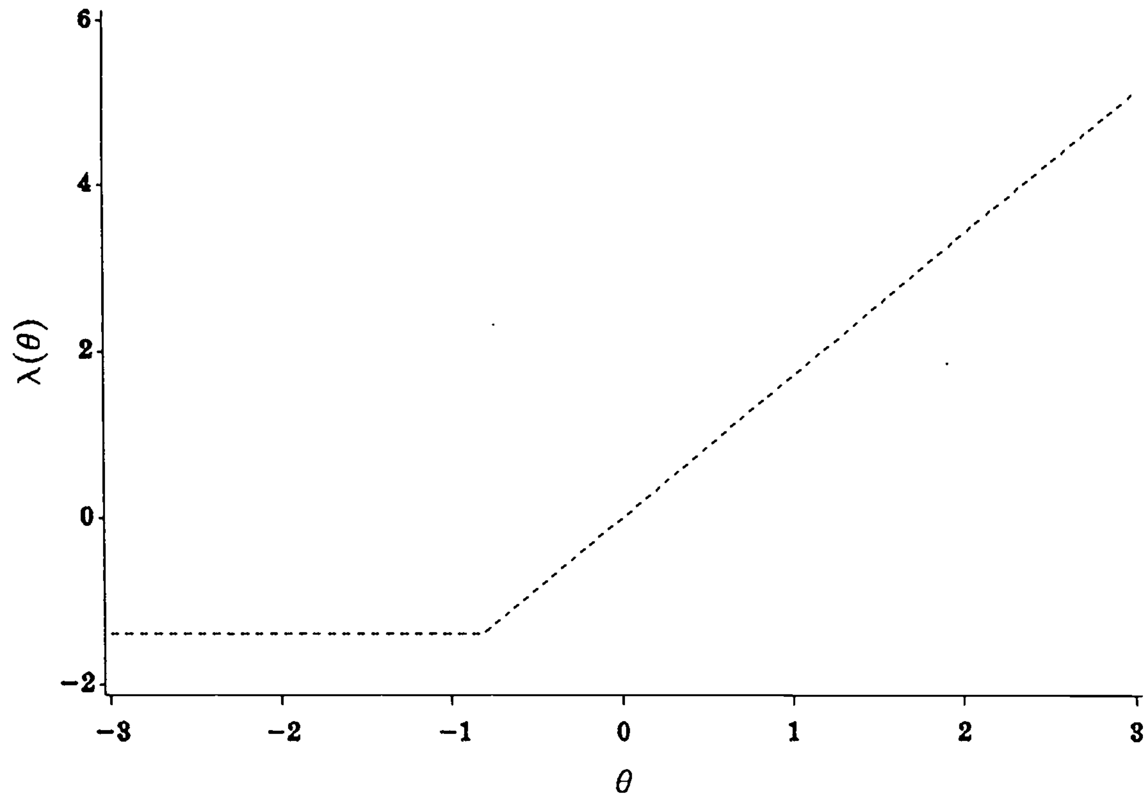
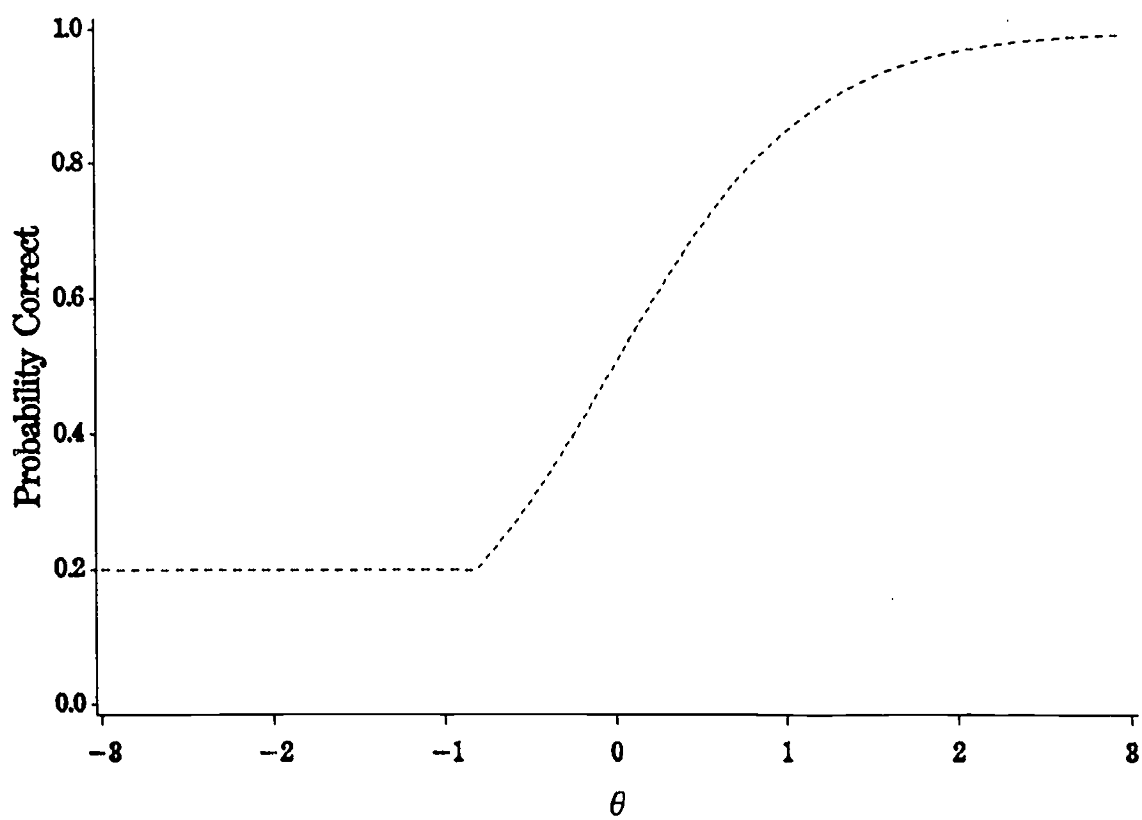

Probability Correct
θ
Residual
θ
···· Standard 3PL (a=3, b=2, c=.4)
— Fitted Hyperbolic 3PL

Figure 7. An example of an asymptotic hyperbolic 3PL curve (with parameters f = .85, h = -.82, and k = -1.39) plotted in the logit and probability correct spaces.


6
4
2
0
−2
λ(θ)
−3
−2
−1
0
1
2
3
θ


1.0
0.8
0.6
0.4
0.2
0.0
Probability Correct
−3
−2
−1
0
1
2
3
θ