DOCUMENT RESUME

ED 389 582 SE 057 225

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TITLE A Proposed Constructive Itinerary from Iterating Composite Units to Ratio and Proportion Concepts.

PUB DATE Oct 95

NOTE 10p.; Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (17th, Columbus, OH, October 21-24, 1995). For entire conference proceedings, see SE 057 177.

PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC01 Plus Postage.

DESCRIPTORS *Cognitive Structures; *Concept Formation; Elementary Education; Grade 2; Grade 5; Grade 7; Interviews; Mathematics Education; *Multiplication; *Ratios (Mathematics)

ABSTRACT

In this article, the meanings students construct and the conceptual advances they make as they deal with ratio and proportion problems are described. The study cites episodes with a second grader, two fifth graders, and three seventh graders. A critical factor in students' comprehension of and solution to ratio and proportion problems is their explicit recognition of the action that links composite units. Critical transitions in students' constructive itineraries are highlighted, showing that an essential component of these transitions is students' development of related concepts and their integration of that conceptual knowledge with ratio and proportion reasoning. (Author/MKR)

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A PROPOSED CONSTRUCTIVE ITINERARY FROM ITERATING COMPOSITE UNITS TO RATIO AND PROPORTION CONCEPTS

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In this article, we attempt to describe the meanings students construct and the conceptual advances they make as they deal with ratio and proportion problems. We argue that a critical factor in students' comprehension of and solution to these problems is their explicit recognition of the action that links composite units. We highlight critical transitions in students' constructive itineraries, arguing that an essential component of these transitions is students' development of related concepts and their integration of that conceptual knowledge with ratio and proportion reasoning.

Conceptual Milestones

Iterating Composite Units

Multiplicative thinking is the foundation on which students construct notions of ratio and proportion. Steffe (1988) has argued that the key to students' meaningful dealings with multiplication is the ability to iterate abstract composite units. This involves taking a set as a countable unit while maintaining the unit nature of its elements. For example, suppose a student is asked "If there are 9 groups of 3 blocks, how many blocks are there?" If the student can solve this problem by coordinating two number sequences, he or she has established an iterable composite unit. That is, the student counts: 1 group is 3, 2 groups is 6, 3 is 9, 4 is 12, 5 is 15, ..., 9 is 27.

Extending the Thinking

Once students are able to iterate composite units, they can extend their multiplicative thinking to ratio situations. Episode 1 describes how a second grader who regularly iterated composite units in solving multiplication problems extended his multiplicative schemes to ratio situations.

Episode 1. The interviewer made a bundle of 5 white and 3 red sticks and asked how many of the same kind of bundles would be behind his back if he had 10 white sticks. JB recited "5, 10," then answered 2. With the same bundle, the interviewer asked how many whites there would be if there were 12 reds. JB figured "You need four bundles to get 12 reds. Then 5, 10, 15, 20." Hence, JB coordinated the iteration of composite units of 5 and 3.

Iterating "Linked Composites"

In Episode 1, JB extended his coordination of a counting-by-1 scheme with another counting scheme to coordinating counting-by-1 with two other counting schemes. That is, before these examples, JB had constructed counting sequences in which he coordinated a counting-by-1 scheme with, for example, a counting-by-3 scheme.

1 group	2 groups	3 groups	4 groups
3 objects	6 objects	9 objeects	12 objects

But in Episode 1, he extended this counting scheme to construct a "linked composite" counting sequence:

1 group	2 groups	3 groups	4 groups
3 of object A	6 of object A	9 of object A	12 of object A
5 of object B	10 of object B	15 of object B	20 of object B

JB was able to iterate a composite consisting of a composite of 3 linked together with a composite of 5. He solved the ratio problem by analyzing the iteration of linked composites. In Episode 1, he first iterated the 3 until he got 12 to determine how many of the 3-to-5 composites there were. He then iterated 5 that same number of times.

It is our contention that the type of thinking that JB exhibited in Episode 1 can serve as a foundation for future meaningful dealings with ratio and proportion. In fact, the strategy of iterating linked composites used by JB (often called a build-up strategy) is used widely and successfully by older students (Hart, 1984; Kaput & West, 1994; Lamon, 1994). In Episode 2, we see a seventh grader making this iterated linked-composites thinking more sophisticated. JR used the same reasoning as JB in Episode 1, except that JR used division to find the number of linked composites, and multiplication rather than skip-counting to find the answer. Kaput and West (1994) call this an abbreviated build-up strategy.

Episode 2. At a dining room table, there are 3 serving utensils for every 2 plates. If there are 10 plates, how many serving utensils are there?

JR: I got 15.

Int: What were you thinking?

JR: Well, I used the 2 and 10. I divided the 10 by 2 and got 5.

Int: Why?

JR: Well for every 10 plates, I got 15 utensils. There are two plates in a set and there are 5 sets. For every set, there are 3 utensils. So for 5 sets, 2 plates and 3 utensils, so 3 times 5 is the number of utensils in the number of sets.

The Transition from Iteration to Multiplication and Division

It is essential to determine what enables students to make the transition from solving ratio problems by iterating linked composites to using multiplication and division. The work of the two students below, who just completed fifth grade, suggests some elements of this transition.

EB dealt with linked composites in ratio problems by iterating, making drawings, and using the operations of multiplication and division. When she used the former two methods, the interviewer often asked her if there were other ways she could solve the problems, hoping EB would see how the use of operations could shorten the iteration process. But EB often struggled with her use of the arithmetic operations.

Episode 3. Mitch paid $4.50 for 5 computer disks. How much did he pay for a dozen?

> EB iterated linked composites, reasoning that 10 disks cost $4.50 + $4.50 = $9.00. But to find the cost of the additional 2 disks, she simply divided $4.50 by 2. (We saw this same mistake by a ninth grader.) However, EB noticed that she made a mistake, so she divided $4.50 by 5 to get $.90: "One disk would be 90 cents. Then another one, plus them together and it would be $1.80. So it would be $10.80." Significantly, even though EB found the price per disk, she used it only to find the cost of the left over 2 disks; she was still iterating linked composites. When asked if there was another way to solve the problem, "now that you've done this division," EB did not know until the interviewer asked how much each computer disk cost. EB: "Do $4.50 ÷ 5, you'd get $.90; then do $.90 ¥ 12."

As the next episode illustrates, EB also needed to explicitly conceptualize linking the two composites to make sense of ratio problems. In particular, she had difficulty conceptualizing the linking action in unfamiliar contexts.

Episode 4. If you can exchange $3 for 2 pounds, how many pounds can you exchange for $21?

> EB initially said that you'd get 5 pounds for $6: "You always get 1 less." The interviewer then asked questions to make explicit the pairing of $3 for every 2 pounds: How many pounds for the first $3? How many pounds for the next $3? So how many pounds for $6?... Although EB was able to answer the questions correctly, she focused on patterns in the separate sequences, first noting the differences between successive values in the linked composites, "It's minus 2, minus 3, minus 4, ..." then differences between the differences in the two sequences "It's always 3 then 2." Indeed, EB seemed to see two separate sequences; the unfamiliarity of the context prevented EB from seeing the problem in terms of the action of exchanging $3 for every 2 pounds.

EB's ability to use operations with linked composites seemed to involve three essential components. First, she needed to explicitly conceptualize the repeated action of linking the two composites to make sense of ratio problems. Second, she needed to have sufficient understanding of the meaning of multiplication and division so that she could see their relevance in the iteration process. Third, and finally, EB needed to have sufficiently abstracted the iteration process so that she could reflect on it, then reconceptualize it in terms of her knowledge of the multiplication and division operations.

CR also dealt with linked composites in ratio problems by iterating, but attempted to shorten the iteration process in the episode below.

Episode 5. (Problem 1) There are 3 boys for every 4 girls in Mrs. Smith's class. If there are 28 students in the class, how many girls and how many boys are there?

> CR: Well 4 plus 3, so 7 altogether. There are 28 students. 7 times 3 is 21. And you need 28. So another 7 on to 21 would equal 28. That's 4 different groups of 4 girls and 3 boys. For every group there's 3 boys and 4 girls. So you have 3 times 4 which equals 12 boys, and 4 times 4 which equals 16. So 12 boys and 16 girls.
>
> (Problem 2) Suppose in a large class, there are 4 girls for every 6 boys. There are 250 students altogether. How many boys and how many girls are in this class?
>
> CR: I know that there are 4 girls and 6 boys, and that equals 10. There are 250 students in the class. And so to make 100, that's 10 of them. So double that to make 200, that's 20 of them. And then to make 50, that's 5 of them. So that would be 25. Int: 25 what?
>
> CR: 25 groups of 10, groups of 4 girls and 6 boys. You take 6, and times that by 25 (she does the computation 25 times 6). So there's 150 boys in the class.

In these problems, CR curtailed the iteration process by using known multiplication facts to aid her in determining the total number of iterations. She then correctly multiplied the relevant composite unit by that total. This curtailment required CR to sufficiently abstract the iteration action so that she could reflect on it and anticipate that the result of several iterations could be captured by a known multiplication fact. After CR completed Problem 2, the interviewer queried her about other ways to solve the problem. CR mentioned guess and check, then division; but it wasn't immediately obvious to her how division could be used to solve the problem. CR also admitted to getting confused by division. However, she solved several subsequent problems by dividing with a calculator.

Extending Linked-Composite Sequences beyond Whole Numbers

One of the major accommodations that students have to make to the multiplicative scheme employed by JB and JR occurs when the numbers do not "divide evenly." For instance, Lesh, Post, and Behr (1988) report a seventh grader enlarging a 2x3 rectangle by doubling the lengths of the sides to produce a 4x6 rectangle. However, when asked to enlarge this rectangle so that the base would be 9, the student responded that doubling would make the base 12. So he added 3 to 4 because 3 had to be added to 6 to get 9. This student was unable to find a way to appropriately alter his linked composite scheme to deal with this new situation, so he switched to an additive scheme. Two other seventh graders, TM and JR, however, were able to make proper accommodations to their linked pair iteration schemes, although their strategies were quite different.

Episode 6. In hot chocolate, for every 2 cups of milk, one needs 4 teaspoons of cocoa. If a person has 5 cups of milk to make hot chocolate with, how many teaspoons of cocoa are needed?

TM: There are 2 cups for every 4 teaspoons, so this is a ratio, proportion kind of thingy. Two goes into 4, 2 times. Four goes into 8, 2 times, and there is a fifth cup left, so divide 2 by 2 to get 1. So 10 is the answer. Int: How did you get that?

TM: O.K. There are 2 cups of milk, and 2 cups of milk. That's 4 cups of milk. We need 5 cups, so one left over. The first 2 cups of milk is 4 teaspoons. The second 2 cups is 4 more. That's 8 teaspoons so far. There is 1 cup left over. 1 cup is 1/2 of 2 cups. What's 1/2 of 4 teaspoons? That's 2 teaspoons. 8+2 is 10.

JR: I divided 2 cups of milk and 4 teaspoons by 1/2, and got 2 teaspoons. I divided by 2 to come up with 1/2 of it.

Int: Half of what?

JR: I needed to know 1/2 of it because 2 doesn't go into 5. I divided it so I could find out what 1 cup was. Half of 2 is 1 cup; half of the cocoa is 2 teaspoons. So, 5 cups is 10 teaspoons.

Because 2 does not divide 5, TM returned, in essence, to a skip-counting approach: 2 cups of hot chocolate for 4 tablespoons of cocoa, 2 more cups for another 4 tablespoons. TM then altered his strategy; instead of adding a full 2-to-4 linked composite, he added half of such a composite, getting 1 cup with 2 tablespoons. In essence, he started to extend his iteration scheme beyond whole-number increments, so that instead of making whole-unit increments of a 2-to-4 linked composite, he made a half-unit increment of this composite.

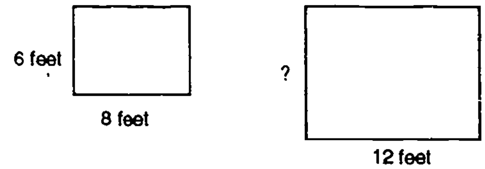
JR's method, on the other hand, addressed the problem by recalibrating the 2-to-4 linked composite to make it a 1-to-2 composite. He divided the 2-to-4 composite by 2 so that the 2-cup component of that composite evenly divided 5 cups. JR seemed to anticipate what unit he needed by envisioning the whole iteration sequence (i.e., by embedding his linked composite in a whole sequence). In the episode below, JR did a similar thing, but gave it a new interpretation. He kept dividing by 2 until he again obtained an increment unit of 1. But this time he interpreted the final ratio as a unit ratio.

Episode 7. Mr. Short is 4 large buttons in height. Mr. Tall is similar to Mr. Short but is 6 large buttons in height. Measure Mr. Short's height in paper clips and predict the height of Mr. Tall if you could measure him in paper clips. Explain your prediction.

JR: I took half this [6 paper clips] which is 3, then he [Mr. Short] is half of 4 buttons, which is 2. The 3 and the 2 are the same thing. Then I divided the 3 by 2 to get 1.5. I needed to figure out the number of paper clips in a button; 1.5 times 6 = 9.

That JR's method included creating new increment units seems to be verified by his strategy use in Episode 8.

Episode 8. These rectangles are the same shape, but one is larger than the other. Explain how you would find the height of the larger rectangle.



JR: I got 9. These two are the same proportions. Everything is the same other than the size.

Obs: Why do you say that?

JR: If you blew this up and made it bigger, or shrink it, it would be the same size.

Obs: How can it be the same size?

JR: It would be the same size as the bigger one, if you blew up the smaller one. So I took this rectangle [the 6 by 8] and divided all the sides by 2; also I multiplied this [the sides] by 3 and that's the same size as this [the larger rectangle].

Obs: What sides did you get?

JR: I took the smaller box, I got 3 and 4, then I multiplied this by 3, so you get 9 and 12.

Obs: Why did you divide by 2?

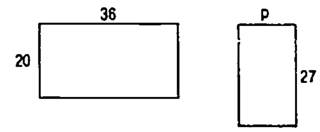
JR: Because I knew that if I divided by 2, I could find the missing side. The smaller rectangle [6 by 8] I could find the missing side, then of the larger rectangle [9 by 12] if I multiplied by something, and I knew I could do this.

In this episode, JR saw that 8 did not evenly divide 12. So he divided the 8 by 2 to get 4, which divides 12; so the iteration sequence included the target 12. He then saw that it takes three 4s to get 12, and concluded that it takes 3 of the 3s to get the desired side length. Also, he seemed to be able to use his thinking about stretching and shrinking to help him reason through this problem, especially with the difficult interpretation of what he got when he first divided by 2 (a "smaller box"). Essential to JR's last step seem to be numerical transformations that stretch and shrink rectangles, while preserving their shape and the ratio of the lengths of their sides.

Proportional Thinking

Students have achieved proportional thinking when they see how to numerically transform the terms in one ratio to the corresponding terms in an equivalent ratio, when they see that the same transformation applies to corresponding terms of equivalent ratios.

Episode 9. Find the value of p in these similar rectangles.



JB (sixth grade) writes the following:

$$\frac{20}{36}=\frac{p}{27}$$

JB then solved the problem by figuring that you get 27 from 36 by dividing by 4 then multiplying by 3, so you must do the same to 20: $20 \div 4 = 5, \times 3 = 15$.

Obs: How did you know to do this?

JB: They're equivalent fractions.

In Episode 9, JB has made the equivalence of ratios explicit. He performs a more complex numeric transformation on the elements of the first ratio to get the second, a natural evolution of the type utilized by JR.

Episode 10. Sue can walk 15 miles in 5 hours. How far can she walk in 3 hours?

JB (seventh grade) writes: $\frac{15}{5}=\frac{x}{3}$ $\frac{3}{1}$

JB: You have to multiply 5 by 3/5 to get 3, so x is 3/5 times 15; so it's 9.

So JB has extended his thinking from the year before to be even more sophisticated. He combined the two operations of multiplying and dividing into the single multiplication by a fraction. He could even extend this thinking to irrational numbers.

Obs: How far can she walk in $\sqrt{2}$ hours?

JB: Because it's 3 over 1, you multiply 1 by $\sqrt{2}$ to get $\sqrt{2}$, so you multiply 3 by $\sqrt{2}$.

JB, who was in algebra when this interview occurred, also used cross multiplying to find answers to some proportional problems.

Cross Multiplying

Solving proportional equations by using cross multiplying requires the use of structural operations from algebra, which is a difficult step for most students to make. Thus, students are likely to make sense of this strategy only when they understand such operations in algebra.

Conclusion

In our proposed constructive itinerary for ratio and proportion, students move from iterating single composites to iterating linked composites to solve ratio problems. They progress to using operations with linked composites when they have sufficiently abstracted the iterative process so that it can be connected to already firm conceptualizations of multiplication and division. They also extend the iterative process from whole number to fractional increments. Students make the transition to proportional reasoning as their focus shifts from implementing the iterative process to reflecting on the numerical operations that transform one ratio to equivalent ratios. (This shift may be strongly connected with their emerging knowledge of fractions and equivalent fractions.) The final step occurs as students apply structural operations from algebra to classical proportional equations. In all cases, transitions to more sophisticated thinking occur as students reflectively abstract their current ratio schemes, taking them to a higher level in which they can be integrated with knowledge of other relevant mathematical concepts.

References

Hart, K. M. (1984). *Ratio: Children's strategies and errors.* Windsor: NFER-Nelson.

Kaput, J. J., & West, M. M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Ed.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 235-287). Albany: SUNY Press.

Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Ed.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 89-120). Albany: SUNY Press.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Ed.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Ed.), *Number concepts and operations in the middle grades* (pp. 119-140). Reston, VA: National Council of Teachers of Mathematics.