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ABSTRACT

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A FIFTH GRADER'S ATTEMPT TO EXPAND HER RATIO AND PROPORTION CONCEPTS

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One fifth grade student, Martha, was encouraged to develop her informal ratio and proportion strategies during a six-month teaching experiment. In this paper, we describe the challenges Martha faced during the teaching experiment. The current study supports the claim made by Kaput and West (1994) that initial instruction on ratio and proportion which is based on children's informal strategies should be introduced as early as the third grade.

Ratio and proportion are important concepts in current mathematics curricula. Very often multiplication and division tasks in lower grades are presented in unit-rate form, which is a special form of ratio and proportion. In the middle grades, word problems involving equivalent fractions and fraction comparison can also be thought of as ratio and proportion situations. For example, to solve the task "Group A has 4 pizzas and 6 girls. Group B has 6 pizzas and 8 boys. Who gets more pizza, the boys or the girls?" (Adapted from Lamon, 1993), some students may draw pictures to figure out that in group A, each member gets 2/3 of a pizza, while in group B, each member gets 3/4 of a pizza. They can then compare these two fractions with the pictures. Other students may use ratio and proportion reasoning: "If I add 2 pizzas to group A, I would need to add 3 more people. So group A is like having 6 pizzas and 9 members. So, each member in Group B gets more pizza." The ability to recognize structural similarity, and the sense of co-variation and multiplicative comparisons illustrated in such a reasoning process are at the core of algebra and more advanced mathematics (Confrey & Smith, 1995).

Because of the importance of this topic in school mathematics, children's concepts of ratio and proportion have long been a focus of mathematics education research, and much has been learned about students' errors and difficulties in solving ratio and proportion tasks (Hart, 1984; Karplus, Pulos, & Stage, 1983), as well as different task variables which affect students' choices of strategies and performance (Harel, Behr, Post, & Lesh; 1991; Kaput & West, 1994).

But what are the roots of these difficulties? What arithmetic knowledge may be useful in developing the concepts of ratio and proportion? Vergnaud (1988) used the term "multiplicative conceptual field" to refer to "all situations that can be analyzed as simple or multiple proportion problems" (p.141). Mathematical concepts that are tied to those situations include, as pointed out by Vergnaud, multiplication, division, fraction, ratio, proportion, and linear functions. He suggested that students develop these concepts not in isolation, but in concert with each other over long periods of time through experience with a large number of situations. Therefore, research studies on children's ratio and proportion concepts need to consider also the other concepts that are a part of children's developing multiplicative conceptual field.

In this article, we report the findings from such an attempt. The analysis is based on data gathered from fifteen 70-minute teaching sessions with one fifth grade student, Martha, over a period of six months. The goals of this teaching experiment included a) encouraging Martha to develop her informal ratio and proportion strategies, b) documenting the nature of this developmental process, and c) analyzing how the development of her ratio and proportion knowledge might influence or be influenced by other constructs of the multiplicative conceptual field. Specifically, what challenges would Martha face and how would she overcome those challenges?

We were aware that a longitudinal teaching experiment was needed to fully study these questions. We hoped this case study of Martha would provide information that could be used in a much larger research program. Because of the space limitation, we will focus our discussion on one particular type of task—the missing-value proportion task.

Martha and Her Informal Strategy

Martha was a bright and confident fifth grader. She had quite sophisticated methods for solving missing value proportion tasks at the beginning of the teaching experiment. The following are two examples:

Episode 1: 6 quarters can buy 9 candies, how many candies can you buy with 14 quarters?

Martha's strategy and reasoning:

> Through doubling, Martha recognized that 12 quarters could buy 18 candies. Then she figured out that 2 more quarters would get her three more candies, because 6 quarters was like "3 sections of 2" and 9 was like "3 sections of 3." Then she added 3 to 18 and got 21.

Episode 2: 15 quarters can buy 40 candies, how many candies can you buy with 21 quarters.

Martha's strategy and reasoning:

> When the relationship between 15 quarters and the 6 more quarters was not apparent, Martha attempted to find the unit price - "how many candies can one quarter get?" for this particular problem. She first arranged these 15 quarters into five rows of three quarters. Through one-to-one distributing, she then assigned two candies to each quarter. With 10 candies left to be distributed among 15 quarters, she calculated 15 divided by 10 with paper and pencil and did not find the result 1 r 5 useful. Then she started to point in the air with her two right fingers. She pointed 8 times first and did not like the result (We interpreted that she was trying to put one candy by each two quarters). She started over and this time she pointed five times (She was trying

> to put one candy by each three quarters) and was pleased with the result. Then she started to place 5 cubes one at a time. She counted what she had left, thought a while, then repeated the same action with the remaining five candies. By identifying the relationship, "3 quarters for 8 candies," she added 16 to 40 and got 56 as her answer.

These strategies showed that Martha had a concept most researchers consider an important element of proportion concepts, "homogeneity." There was an implicit notion that a relationship existed between the number of quarters and the number of candies in the given condition and this relationship needed to be preserved between certain subsets of the quarters and certain subsets of the candies, thus 2 more quarters required 3 more candies. This approach was different from the one used by another fifth grader, Bruce, we worked with. To find the correct unit-price, he would try different unit price like "one quarter for one and one-half candy," or "one quarter for two and one-third candy" and iterate the amount over the number of quarters they matched the given condition.

One major objective of the teaching experiment with Martha was to help her extend her informal strategies to a variety of problem settings, larger numbers and difficult ratios. In order to achieve these goals, Martha needed to become more reflective to the mathematical nature of her informal strategies. That is, Martha needed to (a) articulate mathematically the goal of her trial-and-error based actions, (b) to give mathematical meaning of these actions, thus making the whole process more systematic, (c) to interiorize her physical actions so that they could be executed mentally without the sensory-motor actions, and (d) to generalize her actions across similar ratio and proportion situations. The following is a brief summary of the major challenges Martha faced when attempting to accomplish these tasks.

Articulating the Mathematical Meaning Behind the Operations

When the numbers involved in a problem were small, and/or a useful common factor between numbers could be identified, Martha used strategies similar to those described above to solve a wide range of ratio and proportion tasks. Neither the problem setting nor the semantic structure seemed to have much influence on her (The only exception was the tasks with enlarging or shrinking objects, which will be discussed later). However, when the numbers became large and/or the common factor could not be identified easily, paper-and-pencil computation became necessary. Martha was efficient with the procedural aspect of the computation, but frequently lost the direction of her solution method in the process of carrying out the computation procedure.

One source of difficulty came from her inability to articulate the mathematical meaning behind the operations within a problem context. For example, to solve the problem "12 quarters can buy 220 candies, how many candies can 3 quarters buy?" Martha quickly carried out the procedure of 220 divided by 12 and got 18 remainder 4. But was not sure what to do next. The interviewer asked Martha to give a meaning to her operation. The following conversation occurred:

M: You can get 18 candies with 4 remainder for 12 quarters.

I: No, you can get 220 candies with 12 quarters.

M: Oh, Yeah, you can get this many for... that's how many times 12 goes into 220. So, do I need that?

I: Yes, you do. But you also need to know what that means.

M: Okay, that is, that is how many times that goes into this, um, um, so, ... if 18 is how many, wait, there is 18 candies for one quarter.

It took some more probing before Martha identified that the remainder 4 meant there were four more candies needed to be distributed among the 12 quarters. She then had no difficulty in reasoning this situation with smaller numbers: Since 3 quarters was one fourth of 12 quarters. So, 3 quarters would get one extra candy in addition to the 54 candies (18 candies x 3) they got first. Martha's explanation showed that she had started making connection between the numerical operation and the physically activity of her strategies as described in Episode 2.

Even though we strongly believe the importance of explaining one's mathematics actions verbally, we recognize certain "problematic situations" which may occur in this process. For example, phrases like "2 quarters for 3 candies" or "2 candies per quarter" are commonly used in daily life which, we believed, facilitated the connection described above. Other ratio and proportion situations were harder to describe. For example, Martha was quick to identify the "7 for 2" information from the initial statement, "Fish A is 56 cm long and needs 16 pieces of food each day." But she needed assistance to verbalize the meaning, "7 cm long got 2 pieces of food." Furthermore, it was less natural to say "3 minute per mile" or "3 minute for each mile" than to say "3 miles per minute" which might have an effect on the choice of operation. The operation which involves norming "There are 3 sets of 2 pairs of socks" (Lamon, 1994) was the hardest to describe. But we also found that Martha learned quickly from her experience to verbalize a variety of ratio and proportion situations. Also with the verbalization, she was less likely to confuse the measure spaces in her computation.

Tasks Involving Similar Figures

Similar to the existing literature, Martha found the tasks involving enlarging and shrinking to be the most difficult ones. "Words" alone simply was not enough to communicate the ideas of shrinking/enlarging or similar figures. With the help of Anno's beautiful illustrations of "Magic Liquids" in *Anno's Math Game III* (Anno, 1991), as well as drawing and cutting of different geometric shapes, Martha was able to solve the following task in her unique way:

Episode 3: An object was 45 cm long and 15 cm wide. It becomes 105 cm long after applying the magic liquid. How wide will it become?
Summary of Martha's strategy and reasoning:

> Martha first drew 9 sets of 5 dots, and explained that 45 cm was like 9 segments of 5 cm. With the 60 cm difference in mind, she then tried to distribute the 60 cm differences among these 9 groups of 5 cm. She knew it would be at least 6 cm for each 5 cm "cause 9 times 6 is 54." Then she attempted to distribute the remaining 6 among 9 (groups of 5 cm). With strategy similar to that described in Episode 2, she figured out each group would get another 2/3 cm. So the growth for each 5 cm was 6 2/3 cm. "That's how much each 5 cm will grow," Martha explained. Since 15 cm equaled to 3 segments of 5 cm, there would be a total growth of 20 cm. So she knew the object would become 35 cm wide.

Even though Martha's strategy helped her identify the correct answer, it was hard to tell whether she had an image of stretching (Figure 1a) where the growth occurred uniformly at "each of the infinitely subdivisible parts of the smaller figure" (Kaput and West, 1994, p. 284), or her concept of change was more additive in nature as her language suggested (Figure 1b). We were also amazed by the observation that Martha treated this so called "continuous" situation as a "discrete" one. To help clarify the nature of stretching activity, the interviewer introduced the phrase, "each of the 5 cm grew into 11 2/3 cm" and used the rubber band to simulate the stretching. Both of these seemed to have some effects on Martha's thinking. Later on, Martha would figure out the amount of change by dividing the new length by the old length. and multiplying the result by the old width to solve similar types of tasks.

(a) (b)

Figure 1

Larger Numbers or Difficult Ratios

The most difficult tasks for Martha were the ones with the second and/or the third quantity smaller than the first quantity, <u>and</u> the numbers involved were less familiar. Furthermore, this difficulty was apparent across a wide range of ratio and proportion situations. Because this phenomenon was identified toward the end of the teaching experiment, we were not able to study it as fully as we would like to. Nevertheless, we would like to offer our tentative findings for further discussion.

First of all, Martha was not comfortable working with three-digit (or larger) numbers. She had no difficulty carrying out the written algorithm quickly and correctly but was unable to verbalize the meaning of the operation even when the problem setting involved buying and selling. Second, Martha did not have enough experience dealing with non-unit fractions directly. Her favorite distributing strategy (as seen in Episode 2) only created unit-fractions and this strategy was less effective when the numbers involved were large. Even though sharing, dividing, and folding are common daily experiences which may help interpret the fractional situations, more carefully developed activities are needed to provide experience with non-unit fractions. One potential activity we used in the study was the weighed sharing, for example, "We bought 150 little toys with 60 dollars. I paid 24 dollars, you paid 36 dollars. How many toys should I get and how many toys should you get?"

Discussion

Our interaction with Martha and the other fifth grader, Bruce, (Lo and Watanabe, 1993, 1994) helped us see the conceptual bases of the formal or informal ratio and proportion strategies. For example, in order to use the unit-price approach meaningfully, a student has to have a solid understanding for division, rational numbers, homogeneity relationship, etc.

However, this does not mean that the instruction on ratio and proportion should wait until children have mastered the four basic operations with both the whole numbers and fractions as it is currently done in school. Our study indicates that children can develop sophisticated ratio and proportion strategy as long as they have a good understanding of numbers and operations which are frequently used in their daily lives. The attempt to generalize such strategies to larger or fractional numbers gives rise to the need to develop a more sophisticated understanding of numbers and operations. The current study supports the claim made by Kaput and West (1994) that initial instruction on ratio and proportion based on children's informal strategies should be introduced as early as the third grade.

Furthermore, our analysis indicates that the process of extending the meanings of four operations from single-digit to multiple-digit numbers should not be taken lightly as "applying analogy." Frequently, students lose their number sense while focusing all their attention on carrying out the computational procedures. Martha's difficulty with the meaning of whole number operations may help to explain why many students do not conserve operations when rational numbers are involved. The problem is really much deeper rooted.

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