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AUTHOR Rachlin, Sidney L.

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ABSTRACT

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# The Development of Problem-Solving Processes in a Heterogeneous Eighth Grade Algebra Class

**Sidney L. Rachlin**

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(17th PME-NA, Columbus, OH, October 21-24, 1995)

# THE DEVELOPMENT OF PROBLEM–SOLVING PROCESSES IN A HETEROGENEOUS EIGHTH GRADE ALGEBRA CLASS

Sidney L. Rachlin, East Carolina University

The primary purpose of this study is to evaluate the application of a process approach (Rachlin, Matsumoto, and Wada, 1992) for the teaching of algebra with a heterogeneous class of eighth grade students. The assessment is conducted by identifying the processes used by (above average, average, and below average) algebra students in solving standard and nonstandard problems ranging across a content × process × form matrix — (integers, fractions, polynomials) × (generalizations, reversibility, flexibility) × (expression, equation). To give some perspective to the analysis, the processes used by the eighth grade students are compared to processes identified by Wagner, Rachlin, and Jensen (1984) using the same series of interview tasks with ninth grade students in Georgia and Alberta.

## Mathematical Form and Content in Algebra

Regardless of what content society ascribes to problem solving and algebra, there is a need for research on the learning and teaching of the curriculum at two levels — that of the students and that of the teachers. The algebra project of the University of Hawaii provides one example of how federal, state, and local funding have combined to support a decade of research on the design of an algebra curriculum.

Much of the content of elementary algebra appears in one of two forms: in *expressions* (combining or simplifying terms, operations on polynomials, operations on rational expressions, etc.) or in *equations* (solving equations and inequalities, graphing of functions, solving systems of equations, etc.). Both of these forms rely upon the use of *variables* (literal symbols: $x, y, z, \ldots$) for their written expression. The algebra tasks used in this study were designed to probe students' conceptual and operational understanding of variables, expressions, and equations.

## Content

The mathematical content considered in this study included all of the topics in a typical Algebra I text, with the notable exception of the "standard" algebra word problems. Although all tasks in the study were presented to students in a verbal format, it was felt that the age/coin/mixture/distance problems of elementary algebra involved special translation problems that went beyond the scope of this project. As mentioned earlier, explicit consideration was given to including in the interview tasks problems involving the content areas of integers, rational numbers, and polynomials, as well as the operations of addition/subtraction and multiplication/division.

## Psychological Processes in Learning Algebra

A basic premise of this study was that the learning of algebra, beyond the level of rote memorization of formulas and algorithms, can be regarded as a kind of problem-solving process. That is, even the application of formulas to "routine" textbook exercises involves some degree of problem-solving activity on the part of most students, at least initially. Thus, in addition to considerations of mathematical form and content, three well established problem-solving processes were used to guide the development of interview tasks — reversibility, generalization and flexibility (Krutetskii, 1976). Standard problems are used as foundation tasks upon which the process tasks are constructed. For example, the following sample tasks involve polynomials operations and equations:

| | **Algebraic Expressions** | Equations |
|---|---|---|
| **Standard** | Multiply: $(2a + 3)(2a - 3)$. | Solve: $4a^2 - 9 = 0$. |
| **Reversibility** | Find the binomial which multiplied by $2a - 3$ equals $4a^2 - 9$. | Find an equation whose solutions are ±f(3,2) . |
| **Ability to Generalize** | Find 2 binomials whose product is a binomial ... a trinomial ... has 4 terms ... has 5 terms. | Find a quadratic equation whose solutions are proper fractions. |
| **Flexibility** | Can you find the binomial which multiplied by $2a - 3$ equals $4a^2 - 9$, another way? | Solve: $4a^2 - 9 = 0$.<br>Solve: $4(a+1)^2 - 9 = 0$.<br>Solve: $4(2a+1)^2 - 9 = 0$. |

## Reversibility

Krutetskii (1976) describes reversibility as "an ability to restructure the direction of a mental process from a direct to a reverse train of thought." For example, in the expression $a + b = c$, we might be given values for $a$ and $b$, and be asked to find a value for $c$. The reversibility of this addition incorporates three variations: where the values of $a$ and $c$ are given and the value of $b$ is to be found, where the values of $b$ and $c$ are given and the value of $a$ is to be found, and where the value of $c$ is known and both $a$ and $b$ are to be found. To possess complete reversibility of addition of whole numbers, children should be able to solve problems involving all three variations: $5 + b = 7$, and $a + 2 = 7$, and $a + b = 7$.

Correspondingly, a student who possesses complete reversibility of addition of polynomials should be able to solve the following three problems:

1) What polynomial added to $5x^2 + 3xy$ equals $3x^2 + y^2$?

2) The trinomial $2x^2 - 3xy + y^2$ added to what polynomial equals $3x^2 + y^2$?

3) Find two polynomials with at least one non-similar term such that their sum is $3x^2 + y^2$.

## Generalization

Krutetskii (1976) considered the ability to generalize mathematical material to be on two levels: first, the ability to subsume a particular case under a known general concept and second, the ability to deduce the general from particular cases, to form a concept. This notion of generalization is commonly reflected in the ordered series of exercises found in most mathematics texts in which increasingly more complicated extensions of a form are made. For example, the following series of polynomials, algebraic fractions, and real numbers provides a generalization for addition:

1) Find three integers whose sum is $-2$.
2) Find two polynomials whose sum is $5x^2 + 2x + 4$.
3) Find two fractions whose sum is $\frac{3}{8}$.
4) Find two fractions whose sum is $\frac{2x+7}{8}$.
5) Find two real numbers whose sum is 12.

Krutetskii's (1976) second level of the ability to generalize mathematical material is the ability to deduce the general from particular cases. For example, students' concepts of a difference of two squares are examined through their discussion of open-ended tasks such as: Find two binomials such that their product is a binomial.

## Flexibility

Flexibility was identified by Krutetskii (1976) as the ability to switch from one level of thinking about a problem to another. Flexibility can be shown either within or across problems. Within problem flexibility refers to the ease with which a student switches from one method of solving a problem to another method of solving the same problem. How students perceive a problem shapes the approach that they will use to solve the problem. The various solution paths which a student selects establish the structure for the problem. For example, the task "What number divided by 24 equals $\frac{3}{4}$?" has a wide variety of appropriate structures depending on the way in which the task is perceived; e.g., as equivalent fractions, a proportion, a division problem, an equation, etc. A student who is unable to solve this problem as a division problem because of a lack of skill in operations with fractions may still solve it by thinking of the problem as a proportion. Interview tasks such as the following are used to investigate students' alternative ways to solve the same problem:

a. Solve the following equation for $x$: $7 - 5x = 32$.
b. Solve the equation above another way.
c. Write an equation like the equation above that has a solution of $-4$.

The ability to switch from one approach to another, more efficient, approach is a question of degree. Across problem flexibility refers to the degree to which a successful solution process on a previous problem *fixes* a student's approach to a subsequent problem. Many students solve the following equations without seeing a connection between them:

Solve each of the following for $x$:

a. $2x = 12$

b. $2(x + 1) = 12$

c. $2(5x + 1) = 12$

## Design Of The Study

This investigation of the development of problem-solving processes in elementary algebra replicates the methodology of an earlier pair of studies conducted in Athens, Georgia with eight ninth-grade algebra students and in Calgary, Alberta, involving four ninth-grade algebra students. The data obtained in these earlier studies is used to represent a norm for *traditional* algebra programs. The present study contrasts the 92 hours of interviews from these studies with over 70 hours of interviews collected from ten high, average, and low achieving eighth grade algebra students in four heterogeneous classes.

### Participants

A total of 4 boys and 6 girls (4 with above average, 4 with average, and 2 with below average achievement levels) were selected from four Algebra I classes taught by the same junior high school math teacher in a suburb of the greater Denver area. As an experiment, *all* eighth grade students in the school were enrolled in a concepts of algebra course. This course covered the content of beginning algebra using the text *Algebra I: A Process Approach* (Rachlin, Matsumoto, and Wada, 1992). All teachers using the text, including the special education teacher and a substitute teacher participated in 45 hours of inservice preparation for teaching by a process approach. Since the decision to participate in this experiment was not made until May of the year preceding the project, no effort was made to prepare the seventh grade students for taking algebra in eighth grade. After completing the concepts of algebra course, the students were tested to determine which students would be permitted to use the concepts of algebra course for their algebra credit and which students would follow this course with a year of traditional algebra. At the end of the year students who failed the concepts of algebra course were asked to take the Orleans-Hanna Algebra Readiness Test to determine if they were prepared to take algebra in ninth grade. With the exception of three students who refused to take the test, all students tested were measured as ready for algebra.

## The Hawaii Algebra Curriculum

The University of Hawaii Algebra Learning Project designs instructional materials and methods to help students of *all* ability levels develop problem-solving processes as they learn algebra. A basic premise of the project is that if we are to meet the literacy needs of the future, students must do more than memorize formulas and get answers. They must learn to think mathematically and communicate their thinking. Hence this project alters the sequence of algebra content and instructional methods to foster the development of understanding.

The Hawaii Algebra Curriculum is:

- is based on research into how students think and learn.
- offers a problem-solving approach to algebra. Concepts are introduced through problem situations.
- allows students to construct their own methods to solve mathematical problems. There is more than one right way to solve a problem.
- offers students non-routine tasks to encourage the development of problem-solving processes such as reversibility, flexibility, and the ability to generalize.
- promotes open-ended inquiry appropriate for individual differences in any classroom.
- allows time for students to grasp concepts, make generalizations, and refine their skills.

Hawaii Algebra has been identified as a promising practice by the Laboratory Networking Program at the U.S. Department of Education's Office of Educational Research and Improvement. The program grew out of research into how students tackle problems. Students were given problems to solve and asked to think aloud as they tried to find different solution paths. The research confirmed the project's belief that students differ in the time they need to grasp a new topic.

The algebra curriculum has been redesigned to include many open-ended questions. As students discuss their approaches to solving homework problems, they gradually internalize the process of algebra. To allow time for this development, students are given one or two nonroutine problems from a topic every day for three to eight days, thus working simultaneously on several concepts each day. From then on, a topic is treated as a skill and reinforced through practice exercises in later problem sets.

## Interview Tasks

This study provides a replication of two conducted ten years earlier by in Wagner, Rachlin, and Jensen (1984). The tasks and procedures used in the present study mirror those used earlier. The two populations are very different — the prior studies were conducted with students who took algebra by ninth grade, while the present study includes the total eighth grade student population. To better understand the nature of the role of a process approach in developing algebraic thinking, the processes used by the eighth grade students are compared to processes identified with ninth grade students in Georgia and Alberta.

The interview procedure was adapted from one used by Rachlin (1982) and refined by Wagner, Rachlin, and Jensen (1984). The tasks were given to a student one at a time, with only one problem on a sheet and plenty of room for the student to write. The students were asked to *think aloud* as they attempt to solve each problem. If the students lapsed in their verbalization, they were encouraged to tell what they were thinking. If a student appeared to be having a lot of difficulty, hints were provided. At first the hints were general (*What are you trying to find? What's giving you a problem?*), but if the frustration continued the hints increased in specificity. The hints ranged from pointing out a particular error to directed teaching of a new generalization, concept, or skill.

The interviews were flexible in design. No two interviews were alike. On the one hand, a problem was rarely left incorrect or incomplete. On the other hand, if the interviewer noticed something of interest in a student's response, the interviewer created new questions to follow the direction of the student's thought.

## Analysis of data

The analysis of this study is too lengthy to be included in the confines of this abbreviated report. Transcripts of the experimental data have been coded and contrasted with the normed data provided by the earlier studies. What is unusual at this site is the attempt to have *all* eighth grade students (including special education students) taking algebra in un-tracked classes. The results from the study provide qualitative evidence of the strengths and weaknesses of this approach. Copies of the full report will be distributed at the presentation and are available from the author upon request.

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