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ABSTRACT

Graphs without a time axis, such as velocity-versus-position graphs, offer interesting possibilities for exploring graphing and motion. Relations depicted by these graphs are not limited to functions. Interviews with a high school student named Olivia, who uses a motion detector to create such graphs, indicate that she uses thought experiments as well as physical experiments to determine which graphs are logically impossible. Her encounter with the constraints of the velocity-versus-position graph leads her to imagine a group of several "Impossible" figures that she could never create using the motion detector, which becomes a crucial part of learning about this type of graph and about the relation of velocity to position. (Author/SW)

IMPOSSIBLE GRAPHS

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Graphs without a time axis, such as velocity vs. position graphs, offer interesting possibilities for exploring graphing and motion. Relations depicted by these graphs are not limited to functions. In this paper, we describe interviews with a high school student named Olivia who uses a motion detector to create such graphs. While exploring which graphs are logically impossible, she encounters the constraints of the velocity vs. position graphing environment, which we argue are a crucial part of learning about this type of graph and about the relation of velocity to position.

Introduction

In any mathematical representation, there are things that are impossible to do. In a graph of distance vs. time, for instance, the graph line cannot come back to the left, because time cannot go backwards. This constraint arises from the formal properties of distance vs. time graphs, and from the way we understand distance and time. Students of mathematics are generally discouraged from considering the impossible cases in any representation; they are encouraged, instead, to consider cases of possible graphs, since those are the cases they will likely encounter. However, we argue that for any type of graph, considering the impossible graph shapes and trying to understand why these graphs are impossible is an important aspect of learning about the constraints of the graph, and thus the logic that governs how one moves in that graphical space.

In this paper, Olivia, a senior in high school, uses a motion detector to create graphs of velocity vs. position (v vs. p). These graphs are an important part of dynamical systems modeling (Tufillaro, Abbott, and Reilly, 1992), in part because they represent the state of a moving object (given by its position and velocity) as a single point on the graph, creating a compact representation of a system's behavior, in which the relationship between the velocity and position of an object determines which graph shapes are possible. For example, the simple harmonic motion of a weight bouncing up and down on a hanging spring, assuming no damping for simplicity, could be represented as a sine wave on a position vs. time graph that evolves to the right for as long as the motion lasts, or as an ellipse on a v vs. p graph, drawn over and over as the weight continues to bounce:

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Figure 1

As we see in Figure 1, the constraints on graph shapes in velocity vs. position space are different than those we are accustomed to in temporal graphs. A velocity vs. position graph can double back, because position can decrease as well as increase. However, if the position increases, the velocity must be positive, so the graph line must lie above the x-axis; similarly, if distance decreases, the line of the graph must lie below the x-axis. Thus it is possible to create the ellipse in Figure 1, but it can only be created in a clockwise direction, so that the line is above the x-axis while it extends to the right, and below the x-axis as the line comes back to the left. The realm of impossible graphs is thus more complex than in the case of temporal graphs, where any figure that doubles back on itself or has a perfectly vertical line in it is impossible to make. It provides a rich territory for exploring the distinctions between possible and impossible graphs, and for investigating motion from a new perspective.

As Olivia determines which graphs are possible and which are impossible in velocity vs. position space, we learn about how she distinguishes between the two. In some cases, Olivia does "thought experiments," in which she traces out a graph shape on the computer screen at the same time as she describes the physical motions needed to create the graph. In this way, she finds that some graphs are not physically realizable. Olivia also uses physical experiments with the car to find out what is possible, but experimentation doesn't always provide the final answer for her. We have found that the relationship of her physical experiments to the rules she constructs about what are possible graphs is more complex than the relationship described in textbooks as "the scientific method". Olivia's thought experiments and her physical experiments both help her to distinguish possible from impossible graphs, and she uses both of them in unusual ways that can help us understand how to make sense of the realms of both impossible and possible graphs.

The Interviews

Olivia was in 12th grade at a Boston-area public high school at the time of the interview, had a strong background in science and math, and felt competent in these subjects. Olivia was one of five students in this study, each of whom was interviewed for five hour-long sessions, using individual teaching experiments (Cobb and Steffe, 1983). The interviewer, Tracy Noble, posed some pre-determined problems to the students, but also encouraged them to explore questions of their own whenever possible.

Tracy and Olivia spent the first interview playing a game in which they made drawings to represent their motions of a hand-held toy car. At the start of Olivia's second interview, Tracy introduced the motion detector to her, with the minimal explanation necessary for Olivia to start using it. The motion detector senses the distance from itself to the nearest object in its path, and the software (MacMotion™) uses this information to compute the velocity of the object in real time (See Figure 2).

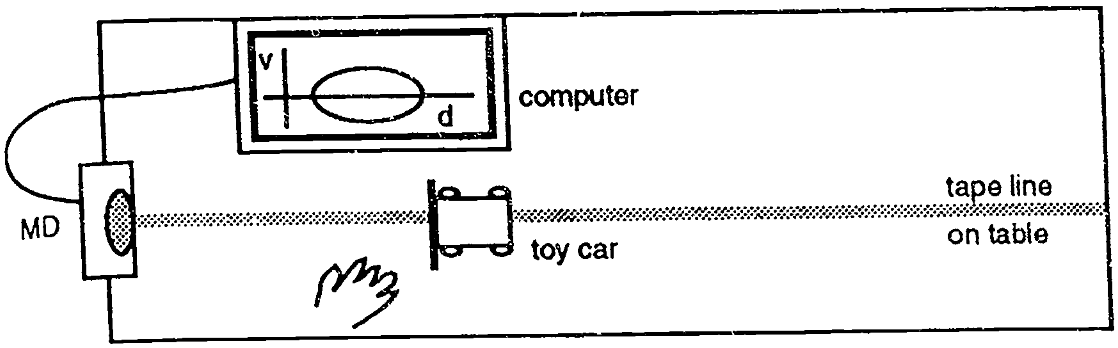

Figure 2

Episode 1 - Direction of Motion on a Graph

Tracy and Olivia spend about 20 minutes using the motion detector to make velocity vs. position graphs of the car's motion before Olivia makes a graph which is a large oval, half above and half below the x-axis (See Figure 3), and tries to determine where on the graph her motion started. [In this figure, Olivia's gestures with the cursor are represented in the left column, and her associated utterance is shown in the right column]

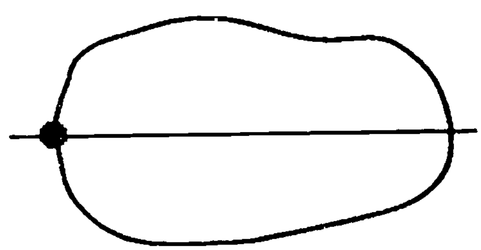

Figure 3.

Notes on Episode 1: In trying to determine how she started her motion, Olivia does a thought experiment in which she imagines moving from left to right along the bottom half of the graph: "Did I start here [left-most point of the oval] and zoom off [tracing with the cursor the bottom half of the oval, from left to right]". Olivia "fuses" the graph and the motions of the car on the table in her language, speaking about "zoom[ing] off," while moving the cursor on the computer screen and also referring to her motion of the car on the table (Ochs, Jacoby, and Gonzales, 1994; Nemirovsky, Tierney, and Wright, 1995). She uses her movement of the cursor along the graph to try to imagine the physical situation that would create the graph, and finds a contradiction, "because to go further away you have to be going [tracing from left to right along top half of oval] a positive velocity away from it [the motion detector]," but she had been moving the cursor from left to right along the bottom half of the oval, where the velocity is negative. Olivia's thought experiment allows her to imagine creating a graph in a direction that would not have been possible, and to understand why it is not possible.

Episode 2 - The Vertical Line, an Impossible Graph

After a few minutes more of discussion, Olivia and Tracy make several more velocity vs. position graphs, and Tracy suggests organizing the shapes in a table with three columns. Olivia and Tracy fill in the "Easy" column with "oval," "waves," and "crazy shapes." In the Difficult column, they place "circle" and "horizontal line" (See Figure 5). They leave the third column blank.

Olivia asks, "Vertical line was pretty easy, wasn't it?" and she finds a way to make two vertical lines show up on the computer screen, by removing the car from the range of the motion detector, seemingly creating a huge velocity peak, but actually creating a graph that does not represent the car's motion. Olivia quickly realizes this, and attempts to make a vertical line while keeping the car in the motion detector's range, moving it toward and away from the motion detector quickly, creating the following graph:

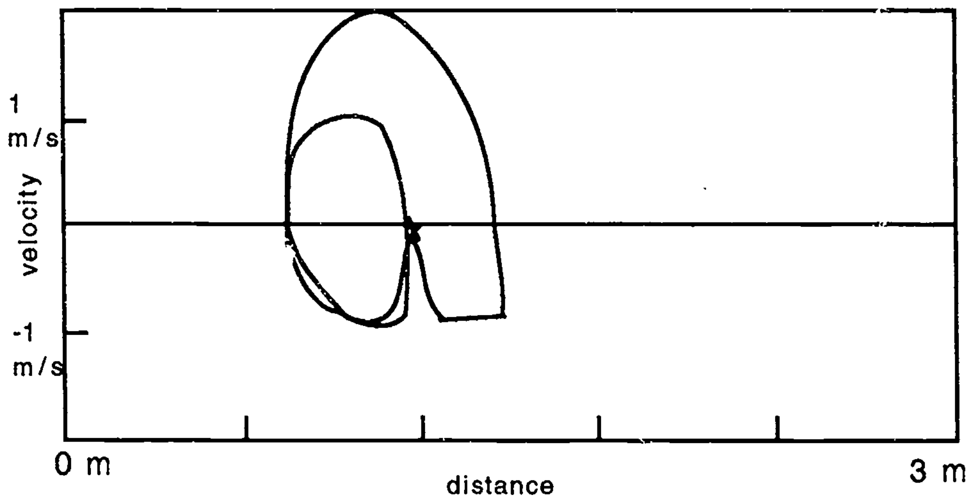

Figure 4

Olivia:	Its—The problem is you have to go very quickly in no, very quickly in no [pause] distance, which is impossible unless; no, it's impossible.
Tracy:	[pointing to the nearly-vertical sides of the ovals] What about this—these? What would you call these guys here [the nearly-vertical sides]?
Olivia:	They're pretty close to vertical but they're not actually [vertical], I mean, they can't be [vertical]. They're just very quick.

Tracy then adds the title "Impossible" to the third column of the table, and Olivia places "vertical line" as the first entry in this column.

Notes on Episode 2: This is the first discussion in which Olivia uses the term "impossible" to describe the making of a particular shape. She has encountered graphs which were difficult to create before, but the vertical line has a new quality: it disobeys a rule she constructs for this graphical space: the rule that you can't go "very quickly in no distance."

When Tracy asks Olivia about some of the nearly-vertical lines of her graph, Olivia responds by saying that "they can't be" vertical lines. Even if Olivia saw a vertical line produced at this point, this statement suggests that she would not believe that it was both truly vertical and truly representing the motion of the car. She has determined that an actual vertical line would be impossible, trusting her sense of the logic of this graphical space more than she trusts the mechanics of the motion detector and graphing program. This is a case which does not fall into the typical model of trusting an experiment to determine the validity of an idea or theory.

Episode 3 - Table of Shapes

Throughout the rest of this interview, and part of her next interview, Olivia fills in the table of figures even further. The final table is represented below in Figure 5:

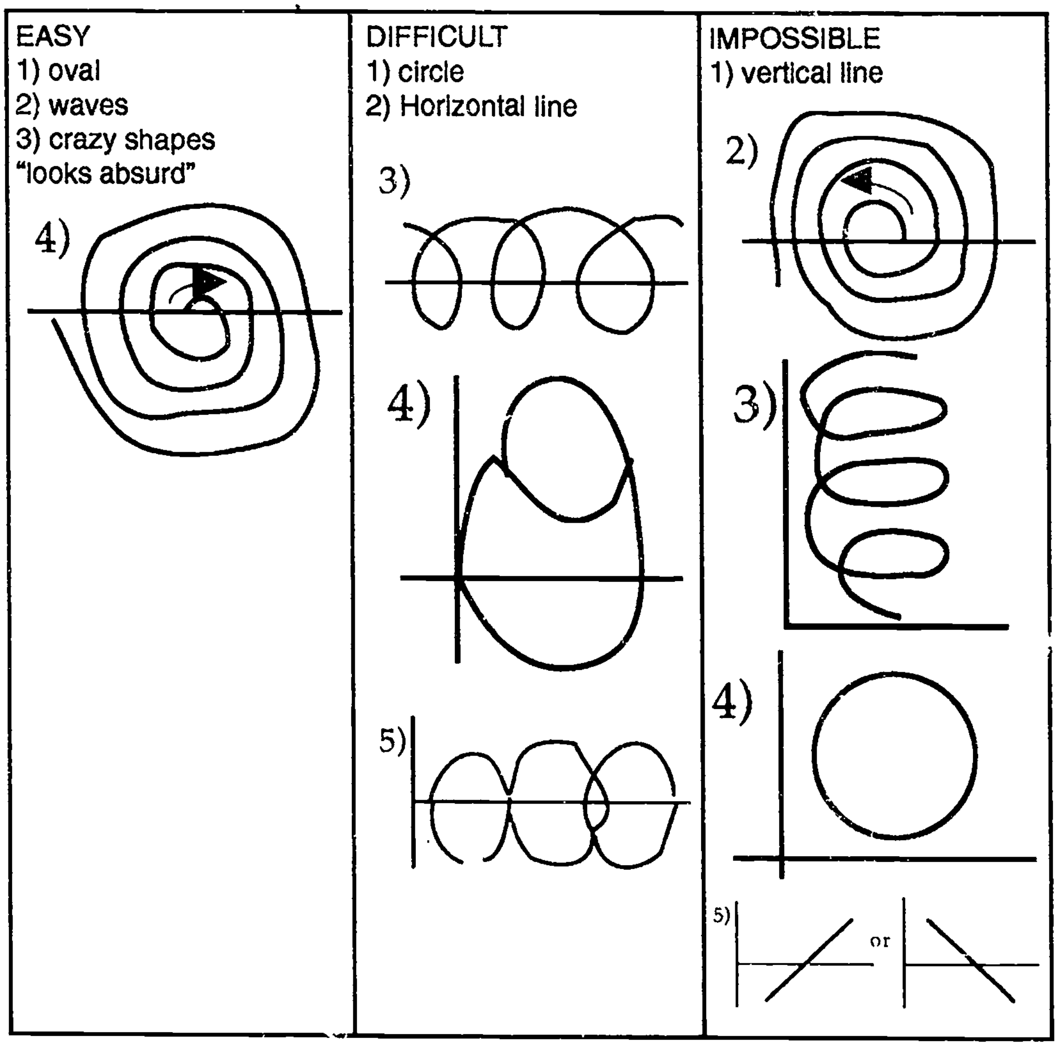

Figure 5

Olivia uses her rule that an object has a positive velocity going away from the motion detector and a negative velocity going toward the motion detector to deem the counter-clockwise spiral impossible ("Impossible" #2). She also develops a new rule that "anytime you want to look at something [a graph] which doubles back on itself, it has to go below the horizon [x-axis]," that is, no doubling back (coming back to the left) without crossing the x-axis first. Olivia uses this rule to classify the vertical spiral (#3) and the circle above the x-axis (#4) as impossible.

Conclusions

Finding the ways in which she cannot move causes Olivia to articulate some rules about how she *can* move, which help her to understand the logic of this graphical space and how to move within it. Far from limiting Olivia's exploration, her encounter with the constraints of this graph leads her to imagine a group of several "Impossible" figures that she could never create using the motion detector.

Olivia uses a number of resources to make sense of the possible and impossible graph shapes she imagines. Her experiences of moving in front of the motion detector and trying to understand the resulting graphs, have helped allow her to describe a graph in a way that "fuses" the graph shape and the motion needed to create it. Thus, Olivia's tracing out of a graph shape works as a thought experiment involving both her tracing a particular graph shape on the screen and her imagining a particular motion of the car. This ability to "try out" a graph shape without actually performing the motions becomes an important tool for distinguishing between possible and impossible graphs. Olivia's fusion of graph shapes and the motions needed to create the shapes also helps her to create a set of rules that govern how one can move in the graphical space, based on the logic of physical motion. Olivia's confidence in these rules is sometimes stronger than the confidence she shows in physical experiments themselves, contrary to some textbook descriptions of the scientific method, in which theories "are accepted only so long as they are consistent with all observed facts" (Shortley and Williams, 1971, p. 2). Olivia's experience of exploring impossible graphs is valuable to her in large part because of the way she experiences a graph shape, the quantities that are graphed, and the motion that would make the graph, all as parts of a whole experience that blurs the distinction between physical event and representation.

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