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ABSTRACT

This paper presents an analysis of the different types of meanings that an individual may assign to a collection of algebraic symbols depending on the mathematical context in which the symbols are presented and the mathematical knowledge possessed by that individual. Four contexts for the Quadratic Theorem are used to illustrate the ways in which generalization and abstraction develop the meaning of algebraic entities by changing the focus from process to structure. (Author)

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Mathematical Contexts and the Perception of Meaning in Algebraic Symbols

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MATHEMATICAL CONTEXTS AND THE PERCEPTION OF MEANING IN ALGEBRAIC SYMBOLS

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This paper presents an analysis of the different types of meanings that an individual may assign to a collection of algebraic symbols depending on the mathematical context in which the symbols are presented and the mathematical knowledge possessed by that individual. Four contexts for the Quadratic Theorem are used to illustrate the ways in which generalization and abstraction develop the meaning of algebraic entities by changing focus from process to structure.

Research investigating students' construction of mathematical ideas can be enriched by including analyses of the mathematical structures under study. It is important for researchers to be aware of the "implicit, unspoken assumptions about the nature of the concepts being considered" (Tall, 1992, p. 508). Behr et al. (1994, p. 124) recommend a "deep, careful, and detailed analysis of mathematical constructs both to exhibit their mathematical structure and to hypothesize about the cognitive structures necessary for understanding them." This paper uses an analysis of the mathematical concepts embodied in the Quadratic Theorem to investigate mathematical structures and processes involved in the development of algebraic thinking.

The Quadratic Theorem: If $a \pm 0$, then $ax^2 + bx + c = 0$ is equivalent to

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Quadratic Theorem is used to solve equations. As with many other theorems, it expresses an abstract symbolic problem-pattern, "$ax^2 + bx + c = 0$" (if $a \pm 0$), and gives a corresponding solution-pattern,

$$\text{“} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \text{”}$$

This theorem aptly illustrates how the language of algebra can be used as a highly effective medium for expressing mathematical thoughts. However, the meaning that is assigned to such a symbolic sentence depends upon the knowledge of the reader and the mathematical context in which the sentence appears (Sfard and Linchevski, 1994). Four different contexts related to the Quadratic Theorem are presented to illustrate how the perception of meaning may vary according to the kind of mathematical constructs an individual is prepared to notice.

Context 1: Quadratic Formula

Evaluate $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for a = 5, b = 2 and c = -3.

The problem-pattern, "$ax^2 + bx + c = 0$," plays no role in this example. Only the second half of the theorem, known as the Quadratic Formula, is given. Here the collection of symbols performs the role of a formula, which represents a different type of conceptual entity than that of the whole Quadratic Theorem. In such a "plug in" problem the context is numerical. The reader only needs to interpret the symbol sentence as a set of directions for computing a number.

Context 2: Problems appear only in a limited, simplified quadratic form.

Use the Quadratic Theorem to solve the following:

a) $5x^2 - 7x - 12 = 0$.

b) $10 - 2x^2 + 7x = 0$.

c) $3x^2 = 6x + 14$.

d) $x(x + 1) = 7$.

e) $dx^2 + kx = c$.

This context requires the reader to focus attention on the pattern of coefficients in the problem-pattern of the theorem, a pattern that is ignored in Context 1. These five problems emphasize the concept of parameters and the role of symbols as placeholders (which are also called dummy variables). Although the computational process is the same as in Context 1, the numbers (or expressions) to which the process applies must first be identified. The reader must be able to perceive that the problem-pattern "$ax^2 + bx + c = 0$" represents a generalization that is common to the collection of symbols in each equation. Only equation (a) utilizes the theorem's given left-to-right alphabetical order. The other equations require interpretation of what the problem-pattern is intended to represent.

Equations (a) through (d) can also be solved numerically (instead of algebraically) simply by graphing the two component expressions of each equation and noting the x-values where they are equal, without any reorganization or identification of parameters. The numerical approach avoids using the theorem, and thus avoids the necessity to discriminate between the different symbolic roles of "a," "b," and "c."

The role of the problem-pattern in the Quadratic Theorem is to abstractly describe the type of problem to which the theorem applies. In Context 2 this type is distinguished by the appearance of "x^2" and "x" in each equation, using the particular symbol "x." As such, these equations are a very specific representation of the abstract problem-pattern. This application of the Quadratic Theorem does not require the reader to regard the symbolic "x^2" as representing the operation of squaring as opposed to the result of that operation applied to "x."

Context 3: Problems where the squaring does not apply to an unknown "x."

a) In the Law of Cosines, solve for b: $c^2 = a^2 + b^2 - 2ab \cos C$.

b) Solve for x: $\sin^2 x = \sin x - .2$ [given the ability to solve "$\sin x = c$"]

c) Using a graphics calculator, graph: $y^2 + 3xy + x^2 = 14$.
[When equations must be entered in the form "$y = ...$"]

In the Quadratic Theorem "x" is just as much a dummy variable as "a," "b," or "c." The role of "x^2" in the problem pattern is to represent squaring (the operation) applied to *any* expression, not just to "x." The problems in Context 3 require a shift in understanding of the nature of the conceptual entities represented by the variables in the given Quadratic Theorem. In Context 2 the signifiers (x^2 and x) directly represented that which they signified. Even though the quadratic nature (the squaring) of the equations was apparent, it did not need to be the focus of attention since the theorem could be applied through a one-to-one matching of patterns of symbol strings.

In contrast, in Context 3 the algebraic symbols in the Quadratic Theorem must be perceived as representing sequences of operations, not just strings of similar symbols. Although "x^2 " may represent a number, the purpose of "x^2" in the theorem is now seen as representing squaring, even if it is not "x" that is squared. For example, in part (c) "y" plays the role of "x" in the theorem and "$x^2 - 14$" is represented by the symbol "c." To recognize that the Quadratic Theorem is relevant in Context 3 it is necessary to regard squaring as an object divorced from a particular symbolic representation.

The quadratic nature of the three equations can no longer be determined by a direct correspondence to specific symbols in the problem-pattern of the Quadratic Theorem. For example, in equations (a) and (c) squaring may appear more than once. In equations (a) and (b) it is not "x" that is squared and in equation (c) "x" does not represent the unknown. It may be particularly difficult to recognize the relevance of the Quadratic Theorem to graphing the equation in equation (c).

Context 4: A textbook's statement of a theorem.

a) The Quadratic Theorem.

b) The Theorem on Absolute Values: $|x| < c$ is equivalent to $-c < x < c$.

In Context 4 the theorem itself is the focus of attention. Meaning is assigned according to the symbolic *structure* of the theorem, which contains paired equations or inequalities, rather than through the interpretation of symbols within individual equations. As a conceptual entity, a theorem is perceived as describing *when* it may be used (with its problem-pattern) and *how* it may be used (with its solution-pattern). The collection of symbols in a theorem conveys information about the abstract family of problems to which the theorem applies and the problem-solving process the theorem describes, rather than about the end results of using such processes. This shift in perception represents a level of abstraction above that used in the preceding contexts which were focused at a parametric and operational, rather than a structural level.

Conclusions

The four contexts illustrate how different types of meaning can be assigned to the same collection of algebraic symbols according to the nature of the mathematical entities for which these symbols act as signifiers. Context 1 represents the use of algebraic symbols as a way to convey a generalization about a particular pattern of arithmetic computations. In Context 2 symbols are used to identify a single family of equations to which a single solution-process applies. The Quadratic Theorem is perceived as a description of the way in which this family can be represented and manipulated rather than as a process that is executed.

Context 3 requires an *expansive generalization* of the concept of a quadratic family of equations. This type of generalization extends existing cognitive structures rather than changes them (Tall, 1991). In this context the objects to which the operation of squaring applies need no longer be fixed unknown numbers represented by "x," but can also be variable quantities expressed by other algebraic expressions. This use of dummy variables where "x^2" can represent "y^2" and "c" can represent "x^2 - 14" does not have a parallel in English or other languages. The dummy variables in this context take on meaning for their ability to represent operations as objects.

In Context 4 the theorems describe certain types of problems and how to solve them by using an abstract problem-pattern/solution-pattern structure. This structure represents an abstraction of the operations used in previous contexts to solve specific families of equations.

What is different in each of the four contexts is the way that the collection of algebraic symbols representing the Quadratic Theorem is perceived. However, shifting perceptions is not a simple matter. Expansive generalizations, which create more complex contexts for conceptual entities, may perform an important role in preparing students to move to a new level of abstraction. If "we inadvertently present simplified regularities which become part of the individual concept image, [these] deeply ingrained cognitive structures can cause serious cognitive conflict and act as obstacles to learning." (Tall, 1989, p. 37)

Shifts in perception that involve conceptual reorganizations take place through the process of abstraction. According to Sfard (1991, p. 18), "First there must be a process performed on already familiar objects, then the idea of turning this process into an autonomous entity should emerge, and finally the ability to see this new entity as an integrated, object-like whole must be acquired." Students at a lower level of mathematical abstraction will not perceive the higher-level objects (Sfard and Linchevski, 1994). The new objects are apparent only when one has made an appropriate abstraction and shifted to a new perceptual focus.

The four contexts also exhibit another property of algebraic entities. Collections of symbols may be perceived operationally as processes or structurally as objects (Sfard, 1991). The specific abstractions that are required to shift perception from Context 1 to 2, and from Context 3 to 4 illustrate how meanings assigned to collections of symbols shift from one of using processes to one of studying the structure of these processes. According to Sfard, there are "differences between these two modes of thinking [that reflect different] beliefs about the nature of mathematical entities. There is a deep ontological gap between operational and structural conceptions" (p. 4).

The examples discussed in this paper illustrate the range of mathematical entities that may be perceived within the same collection of algebraic symbols and how specific contexts can elicit a procedural or a structural interpretation of these entities. Such an analysis has been used to formulate research tasks to study students' abilities to use particular algebraic constructs (Sfard and Linchevski, 1994; Teppo and Esty, 1994).

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