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ABSTRACT

This paper discusses three dimensions of algebra reform: breadth, integration, and pedagogy. Breadth of algebra includes algebra as: generalizing and formalizing patterns and constraints; syntactically-guided manipulation of formalisms; study of structures abstracted from computations and relations; study of functions, relations, and joint variation; and cluster of modeling languages and phenomena-controlling languages. Also discussed are research supporting algebra reform and how research can lead the practice of teaching algebra in new directions. Three phases of reform (short, intermediate, and long term) are discussed using examples of current research projects. Short term reform is seen as first attempts that leave course structures in place, but which contain significant enrichments, such as technology. The second or intermediate phase of reform centers on the integration of algebra into the middle school and offered to all students. The third, long term phase of algebra reform involves full integration of the development of the many forms of algebraic reasoning across all grades with the learning of important mathematics. Contains 108 references. (MKR)

A Research Base Supporting Long Term Algebra Reform?

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A RESEARCH BASE SUPPORTING LONG TERM ALGEBRA REFORM?[1]

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1. Defining and Situating Algebra Reform[2]

Before discussing any research base supporting algebra reform, we must address some prior questions:

> What kinds of reform, what kinds of algebra, and reform on what time scales?

But even before discussing what kinds of reform and algebra, we should acknowledge why algebra reform is so widely called for. Where are we coming from?

1.1. Recent and Current Practice: The Base-Line

School algebra in the U.S. is institutionalized as two or more highly redundant courses, isolated from other subject matter, introduced abruptly to post-pubescent students, and often repeated at great cost as remedial mathematics at the post secondary level. Their content has evolved historically into the manipulation of strings of alphanumeric characters guided by various syntactical principles and conventions, occasionally interrupted by "applications" in the form of short problems presented in brief chunks of highly stylized text. All these are carefully organized into small categories of very similar activities that are rehearsed by category before introduction of the next category, when the process is repeated. The net effect is a tragic alienation from mathematics for those who survive this filter and an even more tragic loss of life-opportunity for those who don't.

It would be easy to mistake this cryptic description for a deliberately harsh and cartoonish denigration of actual practice, but, unfortunately, it is reasonably accurate for the great majority of students studying algebra in the U.S. today, *especially as experienced by those students.* (Watch them, listen to them, and examine their errors. What is the race or income of those whose lives are most likely to be damaged?) Some of these activities might be described by teachers or other adults as, say, "expression simplifying," "equation solving," "or problem solving." Some others might describe them as "function rewriting," "function comparisons," or "modeling," respectively. Others might describe them as operations in and applications of ratio ıal or algebraic functions over the rationals or reals. But most

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[2] This paper will focus on school algebra in the United States, on the assumption that its larger features are shared by our PME-NA neighbors and that reform efforts in the US are of interest to our immediate neighbors.

students see little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test. Most arithmetic and calculus are experienced similarly, while teachers burn out by the tens of thousands annually trying to teach the unwilling the unwanted. Well-meaning policy makers are now requiring the algebra medicine for all students, since, at least when viewed from a distance, it seemed to have a salubrious effect on some students. The widely appreciated political rhetoric "We can't afford to waste a single person" is now colliding with a curriculum that, in fact, wastes millions. Algebra has been transformed in the national consciousness from a joke to a catastrophe.

1.2 Three Phases of Reform

A potential for confusion exists regarding the kinds of reform possible or desirable over different time scales. What may seem radical as a proposal for immediate implementation appears less so in the context of a longer term picture. Hence we will discuss three overlapping phases of reform, short, intermediate, and long term. Short term, over the next two or three years, involves carrying out ongoing changes in existing curricula - the use of graphing calculators in existing Algebra I and II courses, for example. Intermediate term, covering the period from the late 1990s through the first few years of the next century, involves implementing the larger middle school and other reforms currently nearing completion of their first editions. The third, long term, phase begins during the early years of the next decade and involves deep restructuring of the curriculum that makes room for important new content and flexibility, especially at the secondary level. Indexing phases of reform temporally ignores the fact that change moves unevenly across the land, so that one phase may be well underway in one location while its predecessor is in full swing elsewhere. My comments will focus more on the longer rather than the shorter term - the genuine and significant influence of research on practice is inevitably long term. Short term connections between research and practice are usually closely related to evaluation of one or another innovation or theoretical perspective.

1.3 Three Dimensions of Reform

To clarify the nature of the reforms to be discussed, and implicitly predicted, I offer three dimensions in which to measure change:

(1) *Breadth* - breadth of conceptions of algebra coherently implemented;

(2) *Integration* - curricular integration of algebra with other subject matter; and

(3) *Pedagogy* - movement towards a more active, exploratory pedagogy, particularly exploiting electronic technologies.

The Breadth dimension refers to the many forms of algebra and algebraic reasoning and the ways that they cohere: algebra as generalizing, abstracting and representing; algebra as the syntactically defined manipulation of formal objects; algebra as the study of structures abstracted from computations; algebra as a modeling language or as a cluster of related languages; algebra as the study of functions, relations, and joint variation; algebra as means of controlling physical or cybernetic events, including simulations. These will be elaborated below. The Integration dimension, curricular integration of algebra strands with other subject matter, is meant to include both mathematical and non-mathematical subject matter. Taken together, Breadth and Integration enable a large scale restructuring of the curriculum that removes algebra as a costly pair of high school courses, and when coupled with restructuring of other subject matter into more longitudinally coherent strands, make space in the secondary school curriculum for the new mathematics needed by students of the next century - space also needed for curricular innovation and exploration that is absolutely impossible today.

The Pedagogy dimension has relatively little directly to do with algebra in strictly mathematical terms as a received cultural artifact, but *everything to do with the way that algebra is experienced by students.* Without improvement in this dimension along the lines described in the NCTM *Professional Teaching Standards*, (NCTM, 1991) for example, change in the other dimensions will be meaningless.

"Reform" in the usual modern sense, perhaps deriving from the 19th century notion of "progress," implies improvement relative to some value-norms, and I take the three dimensions to be ordered in some sense appropriate to each: more Breadth and more Integration are presumed to be better, as is a more student-active-reflective Pedagogy. There is no clean separation among the phases of reform to be described, and most reform efforts vary in their progress across dimensions. Furthermore, the dimensions th· mselves are not entirely independent - increased Breadth serves Integration, and vice-versa, while improved Pedagogy serves both. Lastly, different implementations of the "same" reform can vary, especially in the Pedagogy dimension (Romberg, 1981; Romberg, 1983). Folks seeking non-intersecting categories, orthogonal dimensions and linear orderings will not find them in realistic appraisals of educational change in such a sprawling domain as algebra - at least not in this paper.

2. Three Dimensions of Algebra Reform

2.1. The Breadth of Algebra: Five Aspects of Algebra

Despite the fact that we all use one word "algebra," there is no one algebra, no monolith. Instead, we need to make sense of a richly interwoven tapestry of constructs and processes that both serve and constitute mathematics. The analysis offered here is somewhat finer than that used in the NCTM Algebra Document (in preparation), but, I believe, consistent with it - where the NCTM Document refers to "themes," we refer to "aspects" - although when attending to how they develop in students' minds or appear in curricula, we also refer to them as "strands."

Talk about mathematics often slips between mathematics as implicitly shared cultural artifacts— objects, procedures, relations independent of any individual— as when we talk about learning functions, polynomials, factoring, ring theory, linear algebra, and so on— and mathematics as ways of thinking— generalizing, specializing, abstracting, computing, analogizing, justifying, and so on. To describe algebra requires mixing both types of talk. Finally, characterizing algebraic reasoning in terms of the types of mathematical objects involved is inadequate - students may be working with matrices or with integers mod 7 in clock arithmetic (Picciotto, in preparation), for example, in entirely concrete, arithmetic ways rather than algebraically. On the other hand, they might be reasoning quite abstractly while using specific numbers, perhaps only orally, with no writing (Bastable & Schifter, in preparation).

The first two aspects of algebra embody "kernel" features of algebraic reasoning that infuse all the others, the middle two amount to centrally important mathematical topics, and the last addresses algebra as a web of languages. All the aspects should be regarded as loosely spun and richly interwoven—they are by no means separate. And each has different roots in human cognitive and linguistic powers and draws on different kinds of experience, particularly in its primitive and emergent forms among younger children.

2.1.1. [Kernel] Algebra as Generalizing and Formalizing Patterns & Constraints, especially, but not exclusively Algebra as Generalized Arithmetic Reasoning and Algebra as Generalized Quantitative Reasoning

Generalization and formalization are an intrinsic feature of much mathematical activity, and the mathematical systems and situational contexts in which generalization and formalization can be done are unlimited. I suggest that there are two sources of generalization and formalization: reasoning in mathematics proper, and reasoning in situations based outside mathematics, but subject to mathematization. The particular forms described below, arithmetic and quantitative, differ in exactly this fundamental way: generalizing in arithmetic (numerical patterns, arithmagons, etc.) begins within a mathematical system, (often) the system of integers, their properties and operations, where understanding of the mathematical structures plays the core constraining role; quantitative reasoning is based in mathematizing situations and offers a different basis for generalizing and formalizing, where understanding of the semantics of the situation plays the core constraining role.

Both the means and the goal of generalizing is to establish some formal symbolic objects that are intended to represent what is generalized and render the generalizations subject to further reasoning, perhaps aided by computation - where the computations are at least temporarily guided by syntax and patterns associated with the formal system rather than what is formalized. Acts of generalization and gradual formalization of the constructed generality must precede work with formalisms - otherwise the formalisms have no source in student experience. The current wholesale failure of school algebra has shown the inadequacy of attempts to tie the formalisms to students' experience after they have been introduced. It seems that, "once meaningless, always meaningless." We now turn to the two prime candidate sources for generalization and formalization in school mathematics.

Algebra as Generalized Arithmetic Reasoning. An enduring theme in algebra education, with roots in 18th-19th century views of the subject (Pycior, 1981; Sfard, 1995), regards algebra as a language that encodes the general rules of arithmetic, particularly rules concerning the operations. It has proven itself to be attractive as a factor in curriculum design because it explicitly builds on what students presumably know (arithmetic), helps generalize that knowledge, helps build a more general ability to generalize in the process, and exploits the rich intrinsic structure of the integers as a context for pattern development, formalization and argument - for example, how many reasonable conjectures might one make concerning combinations of consecutive integers? Linchevski (1995) put it thus: "Algebra with Numbers and Arithmetic with Letters: A Definition of Pre-Algebra" (Summary Report to the ICME-7 Working Group on Algebra, 1995). Work along the same lines by Bastable and Schifter (in preparation) offers rich examples of second to fourth grade students generalizing and discussing generalizations of arithmetic relations based in specific cases, where formal representations are not used, but where generality is at the heart of the activity and discussion. This is one set of examples that points the way to building depth in arithmetic, serving the Integration dimension of reform. Other types of activity involving arithmagons and numerical patterns, as examples among many possible, provide contexts for extending this strand of algebra towards simultaneous equations and beyond (Bell, 1995; Romberg & Spence, 1995; van Reeuwijk, in preparation). It forms the major part of some recent attempts to begin the study of algebra in the early middle grades (Curcio, 1994).

Algebra as Generalized Quantitative Reasoning. As defined by Thompson (1993; 1995), Thompson & Smith (in preparation) a person is thinking of a quantity when he/she is thinking of a quality of some aspect of a situation that he/she regards as measurable (or countable)— length, density, mass, age, velocity, numbers of red marbles, area, rate of inflation, and so on. Such conceptual acts may or may not involve the actual assignment of numerical values to the quality involved via the use of some unit of measure or counting. Quantitative reasoning might also involve abstract quantities, such as in determining "how many 3's in 15" (where the quality is simply "size") by, for example, counting how many units of 3 need to added together to yield 15. Thus this aspect of algebra can be thought of as encompassing the Generalized Arithmetic aspect. I argued (Kaput, 1995), and Thompson & Smith (in preparation) argued that quantitative reasoning is superior to arithmetic in opportunities to build algebraic reasoning. It draws more fully on different forms of experience, including growth and change, can be more oriented towards the expression of relationships for purposes of inference rather than merely towards computations of values of quantities, and, unlike arithmetic-based activity, it involves a more direct link to physical and cultural experience. Indeed, a closer look at the history of algebra from this perspective suggests that this is where algebra started: a review of the historical "algebra" problems dating back to Arabic algebra reveals them to be quantitative reasoning problems, not arithmetic problems (Katz, 1995). Nonetheless, and despite their concreteness, they served as bases on which general, more algebraic solutions could be (and were) built.

In thinking of algebra both as generalized arithmetic and as generalized quantitative reasoning, it is important to keep in mind that the generalizing does not start with elementary school mathematics and end there, leading to algebra. Generalizing is a continuing activity that can occur at the most sophisticated levels of mathematics (e.g., in algebraic number theory or advanced mathematical modeling) where the qualities being defined and measured might be subtle economic constructs, such as elasticity of demand or the impact of the Fed's interest rate on fluidity of capital.

2.1.2. [Kernel] Algebra as Syntactically-Guided Manipulation of (Opaque) Formalisms

The tremendous power of formalisms is behind the prodigious development of modern science and technology (Bochner, 1966). For example, when one computes the derivative of $(3x^2+2)^{1/2}$ using the Chain Rule, one is exploiting the formalisms developed by Leibniz (Edwards, 1979). Indeed, the word "calculus" refers precisely to this feature - applying rules to calculate with symbols without regard to what they might refer to. When dealing with formalisms, whether they be traditional algebraic ones or more exotic ones, the attention is on the symbols and syntactical rules for "manipulating" them (changing their form). However, it is possible to act on formalisms semantically, where one's actions are guided by what you believe the symbols to stand for. To clarify, consider two ways of solving the equation $3x-2=10$: One way is semantically guided— in this case by reasoning within the numerical conceptual system represented by the formal equation. It is usually approached as an inverting process. One thinks something like "If I take 2 away from 3 times a number, I get 10. So 3 times the number must be 12, so the number must be 4." The syntactically guided approach treats the symbols as objective entities in themselves, and the conceptual system of rules applies to the system of symbols, not what they might stand for. In this case, one applies a rule for adding 2 to both sides of the equation, to get $3x=12$, and then one divides both sides by 3 to get $x=4$. And often these rules come to be thought of as applying to the symbols as physical objects "move the -2 to the right hand side and change its sign."

As noted, much of the traditional power of algebra arises from the internally consistent, referent-free operations that it affords. For an historical discussion of the loosening of referential constraints, see (Kaput, 1994, pp. 101-103). Many (e.g., Cuoco, in preparation) take syntactically guided computations on formalisms to be the essence of algebra. However, as already noted, neither the formalisms nor the actions on them can be viably learned without some semantic starting point where the formalisms are initially taken to represent something in the student's experience. Furthermore, this referential relation is best anchored in the act of generalization from the semantics of the domain represented by the formalisms.

2.1.3. [Mathematical Topic] Algebra as the Study of Structures Abstracted from Computations and Relations

Acts of generalization and abstraction give rise to formalisms that support syntactic computations that, in turn, can be examined for structures of their own, usually based in their concrete origins. This aspect has some roots in the 19th century British idea of algebra as universalized arithmetic (Kline, 1972) but also can draw on structures arising elsewhere in students' mathematical experience—for example, in matrix representations of motions of the plane, in symmetries of geometric figures, and in manipulations of letters in words. These structures seem to have three purposes, (1) to enrich understanding of the systems that they are abstracted from, (2) to provide intrinsically useful structures for computations freed of the particulars that they once were tied to, and (3) to provide the base for yet higher levels of abstraction and formalization. While this aspect in the past has been reserved for elite students at the college level, some now call for earlier introduction for the majority of students (Cuoco, in preparation; Picciotto, in preparation; Picciotto & Wah, 1993, March).

2.1.4. [Mathematical Topic] Algebra as the Study of Functions, Relations and Joint Variation

Fey (1984) recalls the long history of attempts to use the idea of function as an organizing principle for the mathematics curriculum, including and especially algebra. Schwartz (Schwartz & Yersulshamy, 1990) and Yerushalmy & Schwartz, (1993) have offered an analysis of how studying the idea of function and its several standard representations can simplify and organize the confusing algebra curriculum confronted by today's students and teachers, while Dubinsky and colleagues (Breidenbach, et al., 1993) and Thompson have analyzed its conceptual growth in individuals. As a product of generalization, the idea of function has roots in causality, and joint variation (Freudenthal, 1982; van Reeuwijk, in preparation) and hence permeates the sciences. Examples of young students developing this idea have been offered by Tierney & Monk (in preparation), and middle school curriculum materials embodying this point of view have been produced by Connected Math Project, TIMS. On the other hand, functions used in the context of less temporally mediated phenomena, such as occurring in arguments involving divisibility of products of consecutive even integers (where the underlying variable works to carry generality more than it works to carry covariation), the idea of covariation may be less salient, and attention focuses on the generality of the patterns being expressed. When coupled with the ideas of iteration and recursion in computational media, functions feed into the idea of dynamical system (Devaney, 1989; Sandefur, 1990). This strand grows out of and intertwines with the Generalized Quantitative Reasoning strand.

2.1.5. Algebra as a Cluster of Modeling Languages and Phenomena-Controlling Languages

Modeling Languages. Some would argue (e.g., Freudenthal, 1983) that modeling is the primary reason for studying algebra. The generalized quantitative reasoning aspect can be regarded as part of a larger modeling aspect that extends to include the rapidly widening collection of notation systems that are used to represent and visualize phenomena of all sorts. These support the new forms of visualization and reasoning associated with dynamical systems, deterministic chaos, and generally the modeling of nonlinear phenomena. Of special interest is how "algebraic" are the various notation systems? One way to approach this question is to ask how do they express generality, and how do they support syntactically guided manipulation? Some are pictoric, some are coordinate-based, while others are character-based. The computer medium now supports operations on virtually any notation system; for example, one can systematically adjust the color scales of a color-coded temperature map to help reveal patterns, or one can overlay such a map with a topographic map, etc. Is this modeling in the classic sense that developing a differential equation for describing the motion of a falling body is modeling? Of course, many, perhaps most models have functions at their core, so as a curricular and cognitive strand it weaves intimately with the previous strand.

Languages that Create and Control Physical and Cybernetic Phenomena. In modeling, we begin with phenomena and attempt to mathematize them. But computers now enable us to reverse this referential relationship in interesting ways - by creating simulation phenomena within the computer medium (Kaput, 1994) and by driving physical devices (Nemirovsky, 1994). In these cases, one usually cycles repeatedly between the phenomena, wherever they happen to be located, and the notations that give rise to them. In recent work (Kaput, in preparation) we are also able to import phenomena into the computer via standard MBL systems and compare them with algebraically generated phenomena. For example, one can "walk" a certain velocity graph that controls the motion of a character in a simulation, and then create algebraic functions that control another character whose motion can be compared with your motion as they "walk" side by side. In these sorts of environments new relationships between algebra and physical phenomena are possible. Lastly, computer languages, beginning with FORTRAN, then BASIC and more recently Logo (Grant, Faflick & Feurzeig, 1971; Noss, Hoyles & Sutherland, 1993; Papert, 1980) and ISETL (Dubinsky, 1991; Dubinsky & Leron, 1994) amount to algebraic formalisms within which one can create or experience explorable and extensible mathematical environments. Nor do these languages need to be alphanumeric, e.g., Function Machines (Feurzeig, 1993). As has been noted (Kaput, 1986; Noss et al., 1993), these computer environments change in fundamental ways the relations between the particular and the general, and hence the nature of mathematical experience available to students, including and especially means of argument and justification.

2.2. Integration of Algebra with Other Subject Matter

As we all know, and for many good reasons both cognitive and practical, the NCTM *Curriculum and Evaluation Standards* (NCTM, 1989) put a premium on "connections." Integration and connections can take place at several different levels:

> Within-mathematics connections between different representations of given mathematical objects such as functions, or between different areas of mathematics, as between algebra and geometry involving, for example, traditional analytic geometry or connections between matrices and transformations of the plane.
>
> Connections between mathematics and subject matter from other mathematical sciences such as computer science, probability, or statistics.
>
> More distant connections usually involve mathematics in modeling situations developed within the structures and from the perspectives of other disciplines, in the physical, life and social sciences, as well as in business, medicine and engineering.

Pedagogical Power. To the extent that algebra can be learned while learning other subject matter, not only is its power appreciated, but its power is learned. And importantly, learning of the other subject matter is enhanced - how much "science" is learned in grades K-8 as vocabulary, or, more recently, as collections of interesting phenomena, without any quantitative content (AAAS, 1994).

Curricular Efficiency. We can no longer afford to teach academic subjects one at a time, end-to-end. We need to exploit the compounding effect of connecting algebra with other subject matter: the algebraic languages reveal the common structures across domains. Building algebra in different domains can reveal the similarities of the underlying ideas while simultaneously strengthening understanding of the structures, exercising the associated procedural skills, and enhancing appreciation of mathematics' power.

Curricular Depth. But perhaps even more importantly, this last observation applies to much of the mathematics that now appears in K-8: *How much could the mathematics of the pre-high school grades be enriched, deepened and made more coherent if, at every turn, questions of generality and extension were raised and pursued* (Bastable & Schifter, in preparation)? To raise such questions inevitably invites algebra as a means for expressing generality and abstraction, and for reasoning within these expressions.

Longitudinal Coherence - From Layercake Filter to Coherent Strands. Algebra is not only a powerful filter of students, but it is also a barrier preventing access to powerful ideas. As now structured, algebra courses lie between elementary mathematics and calculus - the mathematics of change - and all the fields that use calculus. Historically, only a small minority of students cross this barrier, but current work in the SimCalc Project indicates that the mathematics of change may be an ideal site for the learning of algebra, a notion implicit in the growth and change theme of the NCTM Algebra Document (in preparation). Integration, coupled with Breadth, are critically important dimensions of reform.

2.3. Changes in Pedagogy

The distinction between curriculum and pedagogy is a slippery one, especially when one departs from a description of mathematics as a received cultural artifact as represented in books and other media, and instead discusses mathematics as constructed or experienced by individuals. Nonetheless, for analytic purposes it is useful to distinguish between descriptions of acts of teaching and their surrounding circumstances on one hand, from the shared objects, procedures, relations, forms of reasoning, and notation systems that we expect students to learn on the other. Desirable pedagogies have been set forth in the NCTM *Professional Teaching Standards*, (NCTM, 1991) for example, and, for brevity's sake will not be repeated here except to note that it is possible to achieve surface forms of valued pedagogies while failing entirely to engage students with significant mathematics. We often hear that changes in curriculum without changes in pedagogy are empty changes. But the reverse is at least as true, perhaps because it may be easier, especially at the lower grade levels, where teachers are often more equipped to grow pedagogically than they are to grow mathematically. An implication is that growth in pedagogy and growth in mathematical power need to be intimately linked in the kinds of teacher education that will move practice along the three reform dimensions.

3. Research Supporting Algebra Reform

3.1. Research Associated with the Breadth Dimension: Mapping Algebraic Thinking in Its Full Diversity

Obviously, the aspects, especially when thought of as strands, interweave complexly. Mapping these connections, especially how they grow in students' minds under various instructional approaches, is an important research agenda for long term algebra reform. Acknowledging the real complexity and breadth of algebra in our research and how algebra may emerge in students' own language and action, particularly in diverse forms, is an important step towards research of relevance to long term reform that respects the diversity of both the students who need to learn algebra and the many ways they will use it (Confrey, 1995, Dennis & Confrey , 1995). Steps in this direction are necessarily made by the large curriculum development projects in outlining curricula, and these can serve as starting points, e.g., (Romberg & Spence, 1995).

3.2. Traditional Research Supporting and Informing Current Practice

Research and curriculum are, as parts of a larger integrated social and cultural system, intimately, albeit complexly, connected. And, as noted, the forces now at work pushing reform of algebra emanate at least as strongly from the larger society as they do from education researchers, a fact not uncommon historically (Howson, et al., 1981, chapter 1). To the extent that they share a common vision of school mathematics, curriculum and research each helps define the other. This has been especially true in the case of the deficit model "disaster literature," where student shortfalls in learning, "misconceptions," and so on (e.g., Kaput & Sims-Knight, 1983; Kuchemann, 1981; Kuchemann, 1984; Matz, 1982; Sleeman, Kelly, Martinak, Ward & Moore, 1989) are in large part a measure of the impact of the current or recent curriculum, although this is seldom suggested in the research reports, which seemed to take for granted the basic shape of existing curricula. On the positive side, in studies of what students can learn, researchers' visions of school algebra have extended well beyond what typically appears in current courses. However, some researchers have studied the learning of symbol manipulation (Davis, Jockusch & McKnight, 1978), especially learning within computer environments (Chaiklin, 1989; Feurzeig, 1986; McArthur, Stasz & Zmunidzinas, 1990; Sleeman, 1982; Sleeman, 1984; Sleeman et al., 1989), where the subject matter fits reasonably well with the formal side of today's curriculum, although the organizations offered by researchers tend to be much more principled than those embodied in the textbooks.

Prior research also tended to treat algebra one aspect at a time. A significant amount of earlier research, particularly research emanating from other countries (Soviet Studies in Mathematics Education, 1976), was directed towards algebra as generalization, especially generalized arithmetic (Bell, 1995), or formal argumentation (Davydov, 1975, 1990). Some research viewed algebra as a modeling language (de Lange, 1987). Another line of research has investigated students' development of understanding of concepts of function (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Dreyfus & Eisenberg, 1984; Dubinsky & Harel, 1990; Eisenberg & Dreyfus, 1994; Thompson, 1994) and the different representations of functions (Goldenberg, 1988; Romberg, Carpenter & Fennema, 1993; Yerushalmy, 1991). Again, it is worth emphasizing that this research did not strongly affect practice in the U.S., which has been tightly defined by commercial textbook series for "Algebra I & II" dominated by a few major publishers.

Integration has traditionally taken the form of algebra applications in the form of "word problems" rather than in the larger senses described above. And, since these researchers by and large shared the curricular assumption that ability to use algebra is reflected in ability to solve such problems. much research, far too extensive to be cited here and extending well into the psychological sciences, focused on learning how to solve word problems of various types. This research helps only indirectly in the current reform effort, because the current reform no longer shares this curricular assumption. Research centered on pedagogy is perhaps best exemplified by Rachlin (1981), who shows how far one can move along the pedagogy dimension with the current content.

3.3. The First Phase of Reform: Short Term

First attempts at reform leave the larger course structures in place, but can be characterized as significant enrichments, inevitably using electronic technology, of existing courses. These enrichments give much more prominence to and encompass a wider set of applications; utilize the production, comparison and manipulation of functions in linked numerical, graphical and symbolic forms; and usually engage students in conjecture and exploration using the interactive technology. I would judge these attempts as relatively low in the Integration dimension since they share the feature of reforming algebra where it already appears in the grade 8-12 curriculum, leaving the algebra as isolated from other subject matter except as it may be incorporated into problem-applications. In terms of the Breadth dimension it is a significant move towards inclusion of a functions-view of algebra, forced in part by the input-expectations of the electronic devices used. These same devices support multiple, linked representations of these functions—largely defined symbolically, of course—and hence support within-mathematics progress in the integration dimension . Also, depending on the case at hand, generalization and the expression of generality play an increased role in the Breadth of algebraic experiences offered.

Much of this work is the product of innovation by individual teachers or the use of slightly modified texts or supplementary materials (usually associated with graphing calculators). Obviously, much variation is embedded in this category, especially in the Pedagogical dimension. Nonetheless, especially as the technology supports exploration and active learning, significant movement along the Pedagogical dimension tends to occur. However, movement in all these dimensions is limited by the presence of the traditional constraints of the courses in which the innovation is taking place.

3.4. Research Supporting and Informing the First Phase of Reform

A very revealing dissertation study of a short term Algebra II reform effort led by an individual teacher at a progressive private school has been provided by Slavit (1994). The teacher was extremely competent by all standard measures, the students were committed to learning, and the classroom circumstances were near-optimal for use of graphing calculators. We would rate him "high" on the Pedagogy dimension (he was a Presidential Award winner). Many teachers and mathematics educators would envy this teacher's situation and applaud his and his students' achievements, which were considerable. However, his students were afflicted with most of the limitations of concept image of function reported by Vinner (1983; Vinner & Dreyfus, 1989), particularly as revealed by problems involving functions that were not described in algebraically closed form. What of typical students and teachers working under sub-optimal conditions? While the teacher's efforts and achievements were impressive, certain key elements of the curriculum remained unchanged; for example, functions were almost always described in algebraically closed-form (except on a revealing assessment), the course was sandwiched in a rather traditional sequence, and the problems and activities were usually textbook-brief (with a few exceptions) and made relatively little use of real data (physical or otherwise), not unlike findings from another pair of dissertation studies (Rich, 1990; Teles, 1989) and well known work by Heid (Heid & Kunkle, 1988) and others. It is important, both for fairness and for our analysis, to note that none of these factors was within the teacher's (or researchers') direct control. They await the next phase of reform, and, in fact, define the boundary between Phases 1 and 2.

3.5. The Second Phase of Reform: Intermediate Term

The second phase of algebra reform centers on the integration of algebra into the middle school very much in the spirit of the first level of reform, but with two important differences: (1) the algebra is integrated into a larger curriculum, and (2) as middle school mathematics, it is intended (by its authors) to be offered to *all* students. Again, considerable variation exists in this category, particularly in the role and types of applications. Generally, however, the curriculum and the activities tend to be structured in larger pieces than Phase 1, and the algebra tends to emerge from the activity and contexts in which students work. Furthermore, materials are usually structured according to topic strands, with algebra used to express generalizations and abstractions within these strand topics (Romberg & Spence, 1995). Thus considerable movement along the Integration dimension is achieved in Phase 2.

Algebra as a means of modeling and generalization is increased, the place of functions and their multiple representations is preser ..—if not increased to include non-traditional diagrammatic and pictorial notations (Romberg et. al, 1995)—and some of the materials broaden the subject to include some formal, structural aspects of algebra as arise in the contexts of matrices and clock arithmetic. Hence further movement along the Breadth dimension is achieved.

In the Pedagogical dimension, even more movement occurs, since much material is open-ended by design, involves students working in groups, and in some cases involves students designing and producing artifacts (Goldman, 1994). The level of pedagogical change has, in some reports, reached the limits of traditionally educated teachers' ability to adapt.

Most of this work is connected to ongoing curriculum development projects that will not be widely available until 1996 or 1997, with the exception of UCSMP, whose newer editions began to appear in the mid 1990s, and which is distinguished by its K-12 comprehensiveness. Phase 2 seems likely to dominate the end of this decade and the early part of the next. Because of the shift of the focus of these innovations to middle school, many of the constraints of existing secondary school structures are loosened. However, the resulting changes at the secondary school are unclear, except that much of the Phase 1 activity will be inappropriate for those students who will have progressed through Phase 2 materials in middle school. Hence Phase 2 reform is more clearly defined at the middle school level than it is at the secondary school level, a fact that is likely to yield considerable difficulty in transition between Phase 1 and Phase 2.

3.6. Research Supporting and Informing the Second Phase of Reform

Most of the research about Phase 2 has taken the style of research-based formative evaluation of curriculum materials and the school-based implementation process because the innovators are either researchers themselves, or are affiliated with researchers.

3.7. The Third Phase of Reform - Long Term

This phase of reform has not yet begun in the U.S. (to my knowledge), although, as argued below, the ingredients needed to begin are available. It involves full integration of the development of the many forms of algebraic reasoning across all grades with the learning of important mathematics. In this phase algebra is treated less as a subject in its own right (with exceptions noted shortly), and more as a general, ubiquitous means for creating, expressing and operating on generalizations and abstractions, as a medium for modeling, and as a set of computer based languages to create as well as model phenomena. It serves a wide variety of purposes, making sense of the quantifiable and structural aspects of experience in the context of modeling and in other mathematics. It is also a medium for creating new mathematics and reorganizing old mathematics (including concepts of number and operations on numbers). Algebraic reasoning, and the various notational systems, conventional and otherwise, grow organically and gradually, developing as they are needed, with technology likewise introduced gradually as needed. At certain junctures, however, consolidation and some practice are required, perhaps as long as a few months, but not a full course. The exception could be mathematical electives at the secondary level, where particular aspects of algebra may be explored more fully, (e.g., linear algebra, or algebraic structures) (Cuoco, et al., 1995). Computer technology supports just-in-time learning that enables students to learn specific skills when they are needed. In this phase of reform, algebra enhances and provides coherence to the learning of other subject matter strands—the mathematics of number and quantity, of space and dimension, of data and uncertainty, of growth and change (including growth and change in other sciences such as physics and biology). of data structures, and so on. Algebra disappears both as a set of isolated courses and as a set of intellectual tools, in the sense that for the carpenter, when in use the hammer becomes an extension of the arm (Polanyi, 1958). The different aspects of algebra become habits of mind, ways of seeing and acting mathematically—in particular, ways of generalizing, abstracting and formalizing across the mathematics and science curricula, including curricula leading to the world of work. The new freedom from the constraints of the historic high school mathematics curriculum is exploited to include mathematical electives such as dynamical systems and nonlinear modeling (Sandefur, 1992), combinatorics, number theory, non-Euclidean geometry, and so on, studies not currently present in school curricula. A market for innovation is incentivised and mediated by telecommunication technologies that enable individuals to offer instructional materials to geographically dispersed students on a royalty basis.

Relative to content, this under specified and utopian-appearing scenario is not too far from the approach to algebra taken in certain other countries, (e.g., the Netherlands, Russia, and elsewhere). However, I would suggest that the particulars in the U.S. may very well be substantially different from those that have evolved in other countries, especially given that computer technologies are a powerful ingredient operating in Phase 3 but not strongly present today. Glimpses of details are provided in the new NCTM Algebra Document for Algebra in the K-12 Curriculum (in preparation) where algebra is depicted as a K-12 enterprise touching all aspects of mathematics. While provision is made for practice and consolidation, the implicit pedagogy is student-centered, with active exploration, conjecture, verification and student authorship of mathematics and models emphasized throughout.

3.8. A First Pass at Organizing Research Supporting and Informing the Third Phase of Reform

The research basis of this approach certainly does not exist today, although the issue has been discussed as early as the 1930s (Slavit, 1994) and thirty years later in the mid 1960s (Davis, 1964 ; 1984). Below I will attempt to point to research that seems to offer promising starting points. This research largely involves younger children since I believe that the early grades will initially and necessarily be the locus of greatest change in algebra instruction, leading to even larger changes at the secondary level later. Secondly, we need to revisit and extend research in the learning of specific subject matter, especially at the foundational levels, in order to find where and how opportunities to generalize and abstract can be exploited, that is, opportunities to learn and use algebra. Thirdly, we also need to look closely at research and development work in other countries where algebra learning has been integrated with other learning, and where the approaches seem to be in line with what seem appropriate for students of this country.

3.8.1. Beginning the Strands in Elementary Mathematics

Early work has taken the form of documenting opportunities for generalizing and formalizing in arithmetic (Bastable & Schifter, in preparation), and in quantitative reasoning (Confrey, 1994; Confrey & Smith, 1995; Thompson, 1994; Thompson, 1995; Tierney & Monk, in preparation). Additional work, based on new curricula, has shown children capable of handling formal symbolism (Romberg, et al., in press), building abstract formal structures in geometry (Lehrer & Danneker, in preparation), and handling complex interpretation of graphs (Russell, et al., 1995; Ainley, 1995). An important feature of much early work is the subtle and oral rather than written character of children's early attempts to generalize. Since they have not developed symbolism to represent their generalizations, they must use natural language and the many oral strategies for expressing generality developed in daily communication (Mason, in preparation). Hence those who would study these activities as opportunities for the development of algebraic reasoning need a sensitive eye and ear. And furthermore, teachers who would nurture the development of algebra as a means to express generality likewise would need to be sensitized to create as well as identify such opportunities. Fortunately, foundations for such work already exist in the research of those who have studied the development of arithmetic reasoning (e.g., Carpenter, Fennema, & Peterson, 1987;

Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Carpenter & Peterson, 1989; Fuson, 1990; Sandefur, 1990; Steffe & Cobb, 1988) as well as in the study of the associated teacher development (Cobb, Wood, & Yackel, 1990; Cobb, Wood & Yackel, in press; Fennema et al., 1989; Gravemeijer, 1992; Schifter, 1994).

An important alternative to the oral expression of generality and an accompanying move to formal expression occurs in computer environments, especially in situations such as Logo programming (Harel, 1991; Lehrer, et al., submitted; Noss, in preparation; Noss et al., 1993), where the formal expression is intrinsic to the production of a dually-layered visible artifact—the Logo program and the outputs of that program. Another context involves the control of simulations, where students need to set algebraic parameters as part of the process of exploring the phenomena of the simulation (Kaput, in preparation). For example, when controlling the motion of synchronized swimmers in a pool, the students must determine how to distinguish between a positional and a temporal head start; furthermore, in some circumstances they must deal with as many as 20 coordinated swimmers, each of whom is to be offset in their initial position by a fixed distance from the swimmer to their left, say. In this case, to achieve efficient and systematic control of the swimmers begs parametric thinking, where each swimmer's motion is a particular function of time, but where the functions themselves vary systematically across the swimmers. We are currently developing simulation environments to scaffold this kind of thinking among 5th-7th graders.

3.8.2. Approaches in "Algebraically Successful" Countries

Perhaps the best, and surely the most available example, of a curriculum that approximates the vision sketched above is that developed by the Freudenthal Institute in the Netherlands. This curriculum contains no algebra courses, but is rich in algebra experiences beginning in the early grades. A distinguishing feature is the repeated application of the principle of "progressive formalization," whereby students' productions are gradually shaped into more formal systems over time, all in the context of realistic applications.

Another example of active early development of student algebraic reasoning and argumentation is offered in the work of the Russian mathematics educator Davydov (1990). A comparison study of the rather dramatic impact of Davydov's approach has been made by Morris (1995).

4. How Can Research Lead Practice in New Directions?

4.1. General Strategies: Embed Knowledge in Shared Artifacts

One way the insights of disciplined inquiry find their way into practice is by being embedded within artifacts—curricula, tools, and explicit pedagogies associated with these—just as medical research leads to drugs, apparatus, and therapies. The process of reification of knowledge in widely usable tools and representations is a primary means for the distribution of that knowledge (Latour & Woolgar, 1986; Pea, 1993). This is exactly the approach taken by the Dutch (Gravemeijer, 1992). It seems likely to me that such systemic approaches are likely to have the greatest long term impact, partly due to the changing economics of R&D work (Lesh & Lovitts, 1994), and partly due to the dramatic increases in connectivity afforded by electronic networks that will allow distributed collaborative efforts involving many researchers working together at a distance (Hunter, 1993, Fall; Hunter & Goldberg, 1994). Another traditional way not to be ignored is through policies and vision statements such as the various NCTM standards statements, especially the NCTM Algebra Document (in preparation), and MSEB vision statements.

4.2. Changes in Perspectives on What Constitutes Algebra Research: Switching the Duck for the Rabbit

The foreground/background switch that I have advocated for algebra's place in the school mathematics curriculum needs to be matched with a corresponding switch in the way we approach research in the development of algebraic reasoning. Much of the research will need to be based in the learning of the subject matter that gives rise to the use of algebra. Not only does this imply that we need to study the processes of generalizing and notating that generality in basic arithmetic and quantitative reasoning, but also in the context of other major subject matter strands (the mathematics of space, change, data, and so on). For example, in my own work, we are examining how the mathematics of change, including the basic ideas of calculus, can be the site for algebra learning in the latter elementary and early middle school. This changes deeply the traditional prerequisite relationship between algebra and calculus, and moves it closer to the historical relationship, wherein they co-evolved (Kaput, 1994).

4.3. Methodological Changes: Testbeds for Longitudinal Study

I have already noted the need for larger studies coordinated with materials development and teacher development. In addition, we need long-term studies extending over four or more years, following both a cohort of students and a group of teachers as they evolve under circumstances that differ in major ways from today's practice. This calls for a testbed approach, wherein one or more sites participate in material development, evaluation and research for an extended period in an ecologically authentic context that involves practitioners throughout (Hawkins, 1994). The process of dissemination may also take the form of such a site becoming a specially supported resource on the World Wide Web that can act to support teachers and graduate students at other sites, perhaps sharing clinical data. Just as abrupt, late, isolated algebra may be a curricular strategy that deserves to be abandoned, the same might be said of brief, narrow, and isolated laboratory algebra research, especially as a primary strategy.

References

AAAS. (1994). *Benchmarks for Science Literacy*. Washington, DC: American Association for Advancement of Science.

Ainley, J. (1995) Re-viewing graphing: Traditional and intuitive approaches. *For the Learning of Mathematics*, *15*(2) June, 10-16.

Bastable, V., & Schifter, D. (in preparation). Classroom stories: Examples of elementary students engaged in early algebra. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Bell, A. (1995). Purpose in school algebra. *Journal of Mathematical Behavior, 14*, 41-73.

Bochner, S. (1966). *The role of mathematics in the rise of science*. Princeton, NJ: Princeton University Press.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247-285.

Carpenter, T. P., Fennema, E., & Peterson, P. L. (1987). Cognitively guided instruction: The application of cognitive and instructional science to mathematics curriculum development. In I. Wirszup & R. Streit (Eds.), *Developments in school mathematics education around the world*, (pp. 397-417). Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-532.

Chaiklin, S. (1989). Cognitive studies of algebra problem solving and learning. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, (pp. 93-114). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Wood, T., & Yackel, E. (1990). Classroom as learning environments for teachers and researchers. In R. Davis, C. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics*. Journal for Research in Mathematics Education Monograph No. 4, (pp. 125-146). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Wood, T., & Yackel, E. (in press). Discourse, mathematical thinking, and classroom practice. In N. Minick, E. Forman, & A. Stone (Eds.), *Education and mind: Institutional, social, and developmental processes*. New York: Oxford University Press.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning*, (pp. 293-332). Albany, NY: SUNY.

Confrey, J., & Smith, E. (1995). Splitting, covariation and their role in the development of exponential functions. *Journal for Research in Mathematics Education*.

Confrey, J. (1995). Student voice in examining splitting as an approach to ratio, proportions, and fractions. In L. Miera, & D. Carraher (Eds.). *Proceedings of the 19th Annual Conference for the Psychology of Mathematics Education*, Vol. 1, Recife, Brazil, 3-29.

Cuoco, A. (in preparation). Early algebra and the structure of calculations. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Cuoco, A., Goldenberg, E. P., & Mark, J. (1995). Habits of Mond: An organizing principle for mathematics curriculum. *The Journal of Mathematical Behavior.*

Curcio, F. (Ed.). (1994). *Curriculum and evaluation standards for school mathematics: Addenda series, grades 5-8: Patterns and functions.* Reston, VA: National Council of Teachers of Mathematics.

Davis, R. (1964). *Discovery in mathematics: A text for teachers.* Menlo Park, CA: Addison-Wesley.

Davis, R. B., Jockusch, E., McKnight, C. (1978). Cognitive processes in learning algebra. *Journal of Children's Mathematical Behavior,* 2(1), 10-320.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* Norwood, NJ: Ablex Publishing Corporation.

Davydov, V. (1975). The psychological characteristics of the "prenumerical" period of mathematics instruction. In L. P. Steffe (Ed.), *Soviet studies in the psychology of learning and teaching mathematics,* (Vol. 7, pp. 109-206). Chicago, IL: The University of Chicago Press.

Davydov, V. (1990). *Types of generalization in instruction.* Reston, VA: National Council of Teachers of Mathematics.

de Lange, J. (1987). Mathematics, insight and meaning: Teaching, learning and testing of mathematics for the life and social sciences. Unpublished dissertation, Rijksuniversiteit Utrecht, Utrecht, The Netherlands.

Dennis, D., & Confrey, J. (1995) Functions of a curve: Leibniz original notion of functions and its meaning for the parabola. *The College Mathematics Journal, 26*(3) May, 124-131.

Devaney, R. (1989). *An introduction to chaotic dynamical systems.* (Second Ed.). Redwood City, CA: Addison-Wesley.

Dreyfus, T., & Eisenberg, T. (1984). Intuitions on functions. *Journal of Experimental Education, 52,* 77-85.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking,* (pp. 95-123). Dordrecht, The Netherlands: Reidel.

Dubinsky, E., & Harel, G. (1990). *The nature of the process conception of function.* Purdue University, West Lafayette, IN.

Dubinsky, E., & Leron, U. (1994). *Learning abstract algebra with ISETL.* New York, NY: Springer-Verlag.

Edwards, C. (1979). *The historical development of the calculus.* New York, NY: Springer-Verlag.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformation. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education. I,* (Vol. 4, pp. 45-68). Providence, RI: American Mathematical Society.

Fennema, E., Carpenter, T. P., & Peterson, P. L. (1989). Teachers' decision making and cognitively guided instruction: A new paradigm for curriculum development. In K. Clements & N. F. Ellerton (Eds.), *School mathematics: The challenge to change*, (pp. 174-187). Geelong, Victoria, Australia: Deakin University Press.

Feurzeig, W. (1986). Algebra slaves and agents in a Logo-base mathematics curriculum. *Instructional Science, 14*, 229-254.

Fey, J. (1984). *Computing and mathematics: The impact on secondary school curricula*. Reston, VA: NCTM.

Freudenthal, H. (1982). Variables and functions. In G. v. Barneveld & H. Krabbendam (Eds.), *Proceedings of Conference on Functions*, (pp. 7-20). Enschede, The Netherlands: National Institute for Curriculum Development.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: D. Reidel.

Fuson, K. (1990). Conceptual structures for multi-unit numbers: Implications for learning and teaching multi-digit addition, subtraction, and place value. *Cognition and Instruction, 7*, 343-403.

Goldenberg, E. P. (1988). Mathematics, metaphors and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behavior, 7*(2), 135-173.

Goldman, S. (1994). *Middle school mathematics through applications project (MMAP): Second year progress report to the national science foundation*. Palo Alto, CA: Institute for Research on Learning.

Grant, R., Faflick, P., & Feurzeig, W. (1971). *Programming languages as a conceptual framework for teaching mathematics. Volume 2: Logo teaching sequences on numbers, functions and equations* (Report No. 2165). Cambridge, MA: Bolt, Beranek and Newman.

Gravemeijer, K. (1992). Educational development and developmental research in mathematics education. Unpublished paper, State University of Utrecht, Freudenthal Institute, Utrecht.

Harel, I. (1991). *Children designers. Interdisciplinary constructions for learning and knowing mathematics in a computer-rich school*. Norwood, NJ: Ablex.

Hawkins, J. (1994). Cognitive Science Conference Report. Available from author, Educational Technology Center.

Heid, M. K., & Kunkle, D. (1988). Computer generated tables: Tools for concept development in elementary algebra. In A. Coxford (Ed.), *The ideas of algebra, K-12*, (pp. 170-177). Reston, VA: National Council of Teachers of Mathematics.

Howson, G., Keitel, C., & Kilpatrick, J. (1981). *Curriculum development in mathematics*. Cambridge, England: Cambridge University Press.

Hunter, B. (1993, Fall). Collaborative inquiry in networked communities. *Hands On (TERC Newsletter), 16*(2), 1.

Hunter, B., & Goldberg, B. (1994). *Learning and teaching in 2004: The big dig*. Paper prepared for the Office of Technology Assessment. Available from the first author at Bolt, Beranek and Newman, Inc. Educational Technologies Dept., Cambridge, MA.

Kaput, J. (1986). Information technology and mathematics: Opening new representational windows. *Journal of Mathematical Behavior, 5*, 187-207.

Kaput, J. (1994). Democratizing access to calculus: New routes using old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving*, (pp. 77-156). Hillsdale, NJ: Erlbaum.

Kaput, J. (1995). Long-term algebra reform: Democratizing access to big ideas. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initative Colloquium*, (Vol. 1, pp. 33-49). Washington, DC: U.S. Department of Education.

Kaput, J. (in preparation). *Year 2 SimCalc NSF Annual Report*. Available from author.

Kaput, J., & Sims-Knight, J. (1983). Errors in translations to algebraic equations: Roots and implications. In M. Behr & G. Bright (Eds.), *Mathematics learning problems of the post secondary student*, Special Issue of *Focus on Learning Problems in Mathematics*.

Katz, V. (1995). The development of algebra and algebra education. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initiative Colloquium*, (Vol. 1,). Washington, DC: U.S. Department of Education.

Kline, M. (1972). *Mathematical thought from ancient to modern times*. New York, NY: Oxford University Press.

Kuchemann, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics concepts: 11-16P*, (pp. 103-119). Oxford, England: Alden Press.

Kuchemann, D. (1984). Stages in understanding algebra. *Journal of Structural Learning, 8*, 113-124.

Latour, B., & Woolgar, S. (1986) *Laboratory life: The construction of scientific facts*. Princeton, NJ: Princeton University Press.

Lehrer, R., & Danneker, D. (in preparation). Springboards to algebra. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Lehrer, R., Horvath, J., & Schauble, L. (manuscript submitted for publication). Developing model-based reasoning.

Lesh, R., & Lovitts, B. (1994). *Priorities for Research*. National Science Foundation Program for Research in Teaching and Learning, Washington, DC.

Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behavior, 14*, 113-120.

Mason, J. (in preparation). Invoking children's mathematical powers of mathematical thinking. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. H. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems*, (pp. 25-50). New York: Academic Press.

McArthur, D., Stasz, C., & Zmunidzinas, M. (1990). Tutoring techniques in algebra. *Cognition and Instruction, 7*, 197-244.

Morris, A. K. (1995). Development of algebraic reasoning in children and adolescents: Cultural, curricular, and age-related effects. Unpublished doctoral dissertation, The Ohio State University.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (in preparation). *Algebra in the K-12 Curriculum* (Discussion Draft). Reston, VA: Author.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and velocity sign. *The Journal of Mathematical Behavior,13*, 389-422.

Noss, R. (in preparation). Programming and mathematics: Equation or inequity? In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Noss, R., Hoyles, C., & Sutherland, R. (1993). *Teachers' characteristics and attitudes as mediating variables in computer-based mathematics learning*. Paper presented at the Proceedings of the Fourteenth PME Conference, Mexico City.

Papert, S. (1980). *Mindstorms*. New York, NY: Basic Books.

Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Salamon (Ed.), *Distributed cognition*, (pp. 47-87). New York: Cambridge University Press.

Picciotto, H. (in preparation). A proposal for new directions in early mathematics: Operation sense, tool-based pedagogy, curricular breath. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Picciotto, H., & Wah, A. (1993, March). New algebra: Tools, themes, concepts. *Journal of Mathematical Behavior, 12*(1).

Polanyi, M. (1958). *Personal knowledge*. Chicago, IL: Chicago University Press.

Pycior, H. M. (1981). George Peacock and the British Origins of Symbolic Algebra. *Historia Mathematica, 8*, 23-45.

Rachlin, S. (1981). Processes used by college students in understanding basic algebra. Doctoral Dissertation, University of Georgia, Athens, GA.

Rich, B. (1990). The effect of the use of graphing calculators on the learning of function concepts in precalculus mathematics. *Dissertation Abstracts International, 52*, 835A.

Romberg, T. (1981). The field-based basis of quantitative procedures for the study of schooling. In B. Tabachnick & T. Popkewitz (Eds.), *The study of schooling: Field-based methodologies in educational research and evaluation*. New York: Praeger.

Romberg, T. (1983). A common curriculum for mathematics. In G. I. Fenstermacher & J. I. Goodlad (Eds.), *Individual differences and the common curriculum*, (pp. 121-159). Chicago, IL: NSSE.

Romberg, T., et al. (in press). *Mathematics in context: A connected curriculum for grades 5-8*. Chicago, IL: Encyclopedia Britannica Educational Corporation.

Romberg, T., Carpenter, T., & Fennema, E. (Eds.). (1993). *Integrating research on the graphical representation of functions*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Romberg, T., & Spence, M. (1995). Some thoughts on algebra for the evolving work force. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initiative Colloquium*, (Vol. 2, pp. 177-192). Washington, DC: U.S. Department of Education.

Russell, S., et al. (1995). Investigations 3rd/4th grade interpreting graph units. Palo Alto, CA: Dale Seymour Publications.

Sandefur, J. (1990). *Discrete dynamical systems: Theory and applications*. Oxford: Clarendon Press.

Sandefur, J. (1992). Technology, linear equations, and buying a car. *Math Teacher article, 85*(7), 562-567.

Schifter, D. (1994). Voicing the new pedagogy: Teachers write about learning and teaching mathematics, *Center for the Development of Teaching Paper Series*. Newton, MA: Education Development Center.

Schwartz, J., & Yerushalmy, M. (1990). *The function supposer*. Pleasantville, NY: Sunburst Communications.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14*, 15-39.

Slavit, D. (1994). Algebra instruction using the graphing calculator and its effect on students' conceptions of functions. Unpublished doctoral dissertation, University of Delaware.

Sleeman, D. (1982). Assessing aspects of competence in basic algebra. In D. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems*, (pp. 185-200). New York: Academic Press.

Sleeman, D. (1984). An attempt to understand students' understanding of basic algebra. *Cognitive Science, 8*, 387-412.

Sleeman, D., Kelly, A. E., Martinak, R., Ward, R., & Moore, J. (1989). Studies of diagnosis and remediation with high school algebra students. *Cognitive Science, 13*(4), 551-568.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Soviet Studies in Mathematics Education, Vol. 6. (1976). Reston, VA: NCTM.

Summary Report to the ICME 7 Working Group. (1995). *Journal of Mathematical Behavior, 14*.

Teles, E. J. (1989). Numerical and graphical presentation of functions in precalculus. *Dissertation Abstracts International, 51*, 777A.

Thompson, P. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, J. Kaput, & A. Schoenfeld (Eds.), *Research in collegiate mathematics education*, (Vol. 1,). Providence, RI: American Mathematical Society.

Thompson, P., & Smith, J. (in preparation). Quantitative reasoning as a foundation for the development of algebraic reasoning. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208.

Thompson, P. W. (1995). Quantitative reasoning, complexity and additive structures. *Educational Studies in Mathematics.*

Tierney, C., & Monk, S. (in preparation). Children's reasoning about change over time. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

van Reeuwijk, M. (in preparation). Algebra & realistic mathematics. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293-305.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(5), 356-366.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning, 7*, 42-57.

Yerushalmy, M., & Schwartz, E. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating Research on the Graphical Representation of Functions,* (pp. 41-68). Hillsdale, NJ: Lawrence Erlbaum Associates.