DOCUMENT RESUME

ED 385 437 SE 056 573

AUTHOR	Lacampagne, Carole B., Ed.; And Others
TITLE	The Algebra Initiative Colloquium. Volume 2: Working Group Papers.
INSTITUTION	Office of Educational Research and Improvement (ED), Washington, DC.
REPORT NO	ISBN-0-16-048075-2
PUB DATE	May 95
NOTE	231p.; For volume 1, see SE 056 572.
AVAILABLE FROM	U.S. Government Printing Office, Superintendent of Documents, Mail Stop: SSOP, Washington, DC 20402-9328.
PUB TYPE	Collected Works - Conference Proceedings (021) -- Viewpoints (Opinion/Position Papers, Essays, etc.) (120)
EDRS PRICE	MF01/PC10 Plus Postage.
DESCRIPTORS	*Algebra; Conference Proceedings; *Educational Change; Elementary Secondary Education; Higher Education; *Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; Teacher Education
IDENTIFIERS	*Reform Efforts

ABSTRACT

This volume presents recommendations from four working groups at a conference on reform in algebra held in Leesburg, Virginia, December 9-12, 1993. Working Group 1: Creating an Appropriate Algebra Experience for All Grades K-12 Students produced the following papers: (1) "Report" (A. H. Schoenfeld); (2) "Five Questions About Algebra Reform (and a thought experiment)" (D. Chazan); (3) "Algebra and the Democratic Imperative" (R. B. Davis); (4) "Realism(s) for Learning Algebra" (R. Hall); (5) "Algebra, The New Civil Right" (B. Moses); (6) "Issues Surrounding Algebra" (E. Phillips); (7) "Is Thinking About 'Algebra' a Misdirection?" (A. H. Schoenfeld); and (8) "Thoughts Preceding the Algebra Colloquium" (Z. Usiskin). Working Group 2: Educating Teachers, Including K-8 Teachers, to Provide These Algebra Experiences produced: (1) "Report" (A. Buccino); (2) "Educating Teachers to Provide Appropriate Algebra Experiences: Practicing Elementary and Secondary Teachers--Part of the Problem or Part of the Solution?" (C. Gifford-Banwart); (3) "Educating Teachers for Algebra" (A. Buccino); (4) "Experience, Abstraction, and 'Algebra for All': Some Thoughts on Situations, Algebra, and Feminist Research" (S. K. Damarin); (5) "Educating Teachers, Including K-8 Teachers, to Provide Appropriate Algebra Experiences" (N. D. Fisher); (6) "On the Learning and Teaching of Linear Algebra" (G. Harel); and (7) "Algebra: The Next Public Stand for the Vision of Mathematics for All Students" (H. S. Kepner, Jr.). Working Group 3: Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce produced: (1) "Report" (S. Forman); (2) "Algebra, Jobs, and Motivation" (P. Davis); (3) "To Strengthen Technical Education Systematically" (J. G. Greeno); (4) "Thoughts About Reshaping Algebra to Serve the Evolving Needs of a Technical Workforce" (R. Lesh); (5) "Algebra for the Technical Workforce of the 21st Century" (P. D. McCray); (6) "Some Thoughts on Algebra for the Evolving Work Force" (T. A. Romberg & M. Spence); and (7) "Algebra: A Vision for the Future" (S. S. Wood). Working Group 4: Renewing Algebra at the College Level to Serve the Future Mathematician, Scientist, and Engineer produced: (1) "Report" (J. Gallian); (2) "Some Thoughts on Teaching Undergraduate Algebra" (W. D. Blair); (3) "Toward One Meaning for Algebra" (A. Cuoco); (4) "Some Thoughts on Abstract Algebra" (S. Montgomery); and (5) "Suggestions for the Teaching of Algebra" (W. Y. Velez). Appendices include the conference agenda; Conceptual Framework for the Algebra Initiative of the National Institute on Student Achievement, Curriculum, and Assessment; and a participant list. (MKR)

The Algebra Initiative Colloquium

Volume 2

The Algebra Initiative Colloquium

Volume 2

Papers presented at a conference on reform in algebra
December 9-12, 1993

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May 1995

Ordering Information

Copies of this publication are available from the Government Printing Office. For further information, call their Order Desk at 202–512–1800.

For sale by the U.S. Government Printing Office
Superintendent of Documents, Mail Stop: SSOP, Washington, DC 20402-9328
ISBN 0-16-048075-2

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Introduction

Carole B. Lacampagne

Sponsored by the U.S. Department of Education's Office of Educational Research and Improvement, the Algebra Initiative Colloquium was held December 9-12, 1993, in Leesburg, Virginia (see the Agenda, appendix A). The Colloquium addressed specific issues in the algebra curriculum and its teaching and learning. Algebra was chosen from among the many subject areas of mathematics because algebra is the language of mathematics; it is central to the continued learning of mathematics at all levels. Moreover, although several groups are already at work on the reform of algebra, there has been little dialog bridging educational levels. The Algebra Initiative Colloquium fostered such dialog (see the Conceptual Framework for the Algebra Initiative of the National Institute on Student Achievement, Curriculum, and Assessment, appendix B).

Fifty-one distinguished algebra teachers, mathematics education researchers, algebraists, and mathematics experts from federal agencies attended the Colloquium (see List of Participants, appendix C). In addition to hearing and discussing the plenary and reactor addresses, participants were assigned to one of four working groups to debate the issues and, where possible, to come up with recommendations. The working groups and their foci are listed below:

Working Group 1: Creating an appropriate algebra experience for *all* grades K-12 students;

Working Group 2: Educating teachers, including K-8 teachers, to provide these algebra experiences;

Working Group 3: Reshaping algebra to serve the evolving needs of the technical workforce; and

Working Group 4: Renewing algebra at the college level to serve the future mathematician, scientist, and engineer.

Prior to the Colloquium, working group participants wrote short papers on their group's focus area and shared papers within the working group, thus getting a head start on Colloquium deliberations. These papers, revised after the Colloquium, together with recommendations from each working group and a summary of Colloquium discussions appear in this volume. Plenary and reactor papers appear in the companion publication, *The Algebra Initiative Colloquium, Volume I*.

A short document for teachers and policymakers was prepared on the basis of recommendations from the Colloquium. This document, *Algebra for All: A Lever, not a Wedge*, will soon be available through the National Institute on Student Achievement, Curriculum, and Assessment.

Summary

Carole B. Lacampagne

Sponsored by the U.S. Department of Education, The Algebra Initiative Colloquium generated a number of questions, concerns, and recommendations to promote the reform in algebra across the educational gamut, K through 16.

According to Lynn Steen, "The major theme of this conference is very simple: Algebra is broken but nonetheless essential." From this premise, several subthemes/questions/concerns emerged that might be considered in setting an algebra agenda. These include:

- Algebra, the new civil right;
- The chasm that separates K-12 and 13-16 algebra education;
- A new algebra curriculum and pedagogy for preservice teachers and support and practice for current teachers;
- Algebra for the technical work force;
- The story line of algebra; and
- The brick wall.

Algebra, The New Civil Right

As Robert Moses so eloquently put it, "Algebra is the new civil right." Working Group 1 (Creating an appropriate algebra experience for all grades K-12 students) calls algebra "... the academic passport for passage into virtually every avenue of the job market and every street of schooling." Moreover, since students from non-Asian minority groups are less likely to have obtained this passport than others, algebra becomes an equity issue.

Working Group 1 recommended that all students have significant experiences in algebra before the end of grade 8 and that these experiences be the equivalent of current ninth-grade algebra. They also recommended that these experiences be strands that flow through K-8 mathematics.

Algebra for all is a bold statement. As keynote speaker Victor Katz pointed out, algebra has a history of being an elitist subject meant for the education of future priests or leaders. Many of the story problems found in current high school algebra texts date back to antiquity and are artificial, not real-life problems. If we are to implement algebra for all, we must overcome this longstanding elitist tradition.

A question that plagued Colloquium participants was, "How do we insure that 'algebra for all' is not 'dumbing down' algebra?" The mathematical community as well as parents of college-bound students will and should demand sound preparation in algebra for the college bound. We will be faced with building an algebra curriculum and pedagogy that will support the needs of all students.

Integrating algebra into the K-8 mathematics curriculum was also a topic of considerable interest to Colloquium participants. Such integration would eliminate algebra from its current gate-keeper function, for algebra would be learned over a period of 8 years. The college death knoll—failing algebra—would no longer exist. Moreover, many countries already integrate algebra into their K-8 curriculum through the use of a strands approach; that is, each year a strand of algebra as well as strands of arithmetic and geometry are taught. Several curriculum projects in the United States have developed or are developing such an approach. These curriculum projects need to be tested in classrooms to prove their effectiveness, and promising practices need to be disseminated.

Several participants were concerned that we do not know enough about how children and adults develop algebraic concepts. More research is needed on how algebraic concepts are developed. Findings from this research should then be considered when designing algebra curriculum.

Looking to the future when algebra could be folded into the K-8 curriculum raised several questions:

- What do we do with ninth-grade mathematics once algebra has been learned by eighth grade? How do we prepare for such change?
- How do we influence public policy—state legislators, boards of education? State legislators and boards of education must adjust their ninth-grade mathematics requirements to the new curriculum. How do we get parents to buy into this new alignment?

The Chasm That Separates K-12 and 13-16 Algebra Education

It became disturbingly apparent as the Algebra Initiative Colloquium progressed that mathematicians involved with research in the learning and teaching of algebra at the K-12 level seldom talk with those involved in algebra at the 13-16 level.

Working Group 4 (Renewing algebra at the college level to serve the future mathematician, scientist, and engineer) suggested several areas in which further dialog was needed between K-12 and 13-16 people to:

- Lay the basis in high school for some important ideas from linear and abstract algebra;
- Find new ways to engage students in linear and abstract algebra;
- Develop places across the entire algebra experience to expose students to the use of proof; and
- Search for the "big themes" which run throughout the algebra experience.

Initiatives for getting this dialog going are already under way, thanks to the groundwork laid at the Colloquium.

A New Algebra Curriculum and Pedagogy for Preservice Teachers and Support and Practice for Current Teachers

Reform in the school algebra experience necessitates a change in the algebra experiences of pre- and in-service mathematics teachers. Questions wrestled with by Working Group 2 (Educating teachers, including K-8 teachers) to provide these algebra experiences included:

- How do we help elementary school teachers gain the knowledge, pedagogical skills, and desire to integrate algebra into the K-8 curriculum? How can we prepare parents and communities for this new approach?
- How can we encourage mathematics and mathematics education faculty at the college level to model the pedagogy we wish to see prospective teachers use in the schools?
- What experiences with mathematical modeling should pre- and in-service teachers have in order to teach algebra effectively?

Working Group 2 believed that all teachers of pre-college algebra (K-12) need an in-depth understanding of numeracy and quantitative reasoning. They should also have a working knowledge of how to use technology in instruction and how to access real-world examples that employ algebra. They should possess a belief in the value of mathematics and a commitment to the development of algebraic thinking for all students.

Algebra for the Technical Work Force

Working Group 3 had several recommendations for reforming the K through 14 mathematics curriculum and teaching, including that all students should study the same mathematics through grade 11, with algebra playing a significant part in the curriculum. Moreover, they saw mathematical modeling as a central or organizing theme for school mathematics. These recommendations led to such further concerns as:

- What should be the common mathematics curriculum required of students through grade 11?
- What alternative mathematics courses should be taught in grade 12?
- How do we mount a serious study of what algebra/mathematics is used in technical jobs?
- How do we remediate the remedial mathematics programs in our colleges?
- How do adult learners best learn algebra?
- Currently, we teach content, then application. If mathematical modeling were our focus, we would teach applications then content on a need-to-know basis. Can or should we embark on such a radical change?
- How can we begin to work with the vocational/technical education community to provide a meaningful algebra experience in all its programs?

The Story Line of Algebra

At the Colloquium, participants had trouble identifying the "story line" of algebra. Specifically, Bob Moses challenged us to describe what (high school, linear, or abstract) algebra is and why a student should study it. If algebraists and teachers of algebra cannot explain to each other what algebra is all about, how can we expect to engage prospective students, their parents, and the community in the study of and support for algebra? Participants felt that the mathematical community must be able to answer such questions and to communicate these answers to the public at large.

The Brick Wall

Working Group 4 spent much time deliberating on how to confront the "brick wall"; that is, how to help students through their first proof course (usually abstract algebra). They discussed the possibility of starting a dialog, perhaps in a journal like *The American Mathematical Monthly,* on how to help students over (around) the brick wall and on sharing concrete examples and problems in abstract algebra among those who teach abstract algebra.

The big problem that this group wrestled with was how to reconcile the need for formal proof at the college level with the trend to downplay formal proof in the schools in favor of communicating ideas and understanding.

Participants at the Algebra Initiative Colloquium made a good start in identifying problems in the learning and teaching of algebra at all levels. They made several recommendations to help solve these problems. But many questions they raised remain unanswered. It is hoped that federal agencies working with leaders in the mathematical community will provide leadership in renewing algebra to meet the needs of the 21st century.

WORKING GROUP I

Creating an Appropriate Algebra Experience for All Grades K-12 Students

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Report of Working Group 1

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This report is written in the context of a series of conferences and reports, spanning better than a decade, urging the reconceptualization of what is taught as algebra in this nation's schools. (See, e.g., the reports Algebra for Everyone [1990] and Algebra for the Twenty-first Century [1992] from the National Council of Teachers of Mathematics [NCTM].) To pick just one group whose recommendations are similar to ours, we note that the NCTM's Task Force on Algebra urged the development of a national consensus vision statement with regard to algebra. The working definition of the Task Force is that "Algebra is a study of patterns/relationships and functions which uses a variety of representations including verbal, tabular, graphical, and symbolic. These representations make it possible to: use technological tools effectively; communicate, analyze, and interpret information; formulate and solve problems by collecting, organizing and modeling data; describe important patterns of behavior of families of functions; [and] recognize, interpret, and use discrete and continuous relationships." The Task Force's vision statement included the following non-negotiable principle: "A national goal that every student will graduate from high school with the algebraic skills and knowledge necessary to function in our technological society;" also that "algebra will become a K-12 strand" (rather than being isolated in a single year's instruction, typically ninth grade). [Reference: Betty Phillips' conference paper, page 10.] We begin this report with an affirmation of those statements of principle.

All Students Must Study Algebra

There is a new literacy requirement for citizenship. Algebra today plays the role that reading and writing did in the industrial age. If one does not have algebra, one cannot understand much of science, statistics, business, or today's technology. Thus, algebra has become an academic passport for passage into virtually every avenue of the job market and every street of schooling. With too few exceptions, students who do not study algebra are therefore relegated to menial jobs and are unable often even to undertake training programs for jobs in which they might be interested. They are sorted out of the opportunities to become productive citizens in our society.

The study of algebraic ideas must pervade the curriculum. It must not be localized into one or two intense, decontextualized courses in symbol-manipulation techniques during middle and secondary school. Rather, as elaborated in recommendations 1 and 2, the study of algebra must begin early in the curriculum, and increase consistently through the years.

Recommendation 1: All students should have significant experiences in algebra (totaling at least one year of work) before the end of grade 8.

This recommendation represents a significant departure from current practice, and a significant challenge. It follows, first, from the goal statement above. In the most recent National Assessment (1992) about 81 percent of 17-year-olds reported having taken at least a semester of algebra in school. To provide the relevant skills for the rest of the students (largely those who have been shunted into courses in remedial mathematics, or those who drop out of school) requires that we reach them earlier in their school careers. But this is not "only" an issue of equity; it is an issue of providing relevant skills for all students, when they are ready to learn them. While the traditional algebra course may seem to represent, for reasons of familiarity, the way things "must be," the fact is that such a course is an American anomaly: the rest of the world has not had a separate, 1-year, high school course in algebra. Indeed, in other countries, where (traditional) algebra is taught earlier, more students seem to be successful. Because algebra is very much a language, like other languages it is better learned earlier and harder to learn when one is older. In addition, when students start earlier, the learning of algebra can occur over a longer period of time, with all the advantages that greater time exposure allows. The technical aspects of current algebra courses that serve as barriers for students, such as very complicated symbolic manipulations, can (and should) be done with the help of technology. We have ample evidence that the use of contextual situations makes it easier to approach algebra. Furthermore, if one begins earlier, the unfortunate role of algebra as a sorter will necessarily be diminished, because all students are in school and there is less tradition of separating students.

Exactly how these experiences should be designed is open to study. Mathematics curricula in other countries and current NSF-funded curriculum projects can provide a start, but some more specific curricular efforts in algebra at the K-8 level would be appropriate.

What Algebraic Understandings Should Students Develop?

Recommendation 2: All students need to learn the following aspects of algebra:

- ***The representation of phenomena with symbols and the use of these symbols sensibly;***
- ***The use of variables to describe patterns and give formulas involving geometric, physical, economic, and other relationships;***
- ***Simple manipulations with these variables to enable other patterns to be seen and variations to be described;***
- ***The solving of simple equations and inequalities and systems by hand and of more complicated equations and inequalities and systems by machine; and***
- ***The picturing and examination of relationships among variables using graphs, spreadsheets or other technology.***

If we describe mathematics as the study of numerical, geometric, logical, and structural patterns, then the overarching reason for the study of algebra is that algebra is a powerful language in which patterns from all of mathematics are described. It is a powerful means of expression that reflects and helps to structure the use of logic and abstraction, which are valuable mathematical "habits of mind."

A major aspect of algebra is the representation of a situation in algebraic or other symbolic form, and operating on the symbols in a way that makes sense—both with regard to the symbols themselves, and with regard to the situations being represented. The development of symbol sense, and of representational capacities, are among the main goals of algebra instruction. Symbol sense includes such ideas as: 2n is twice n; if you are of age A, the age of your parent who is 25 years older than you is given by A + 25; if this section contains n words and the next section contains m words, then they contain m + n altogether; when area = length X width and the width is multiplied by some quantity, so is the area. At more advanced levels, it includes having an understanding of notions such as commutativity, not only in symbolic terms but in meaningful applications—for example, in noting that certain universal replacements made while using a text processor are not commutative, while others are, and acting accordingly.

The patterns of algebra (should) begin in the early elementary school when students (should) learn that one can add two numbers in either order and obtain the same sum; in algebraic language, that $a + b = b + a$. This algebraic description of the general pattern has three advantages over verbal or visual descriptions. First, it looks like the specific instances $2 + 3 = 3 + 2$ or, later, $12 + 38 = 38 + 12$. Second, it is shorter, with no unnecessary words or symbols. Third, the algebraic objects themselves—the variables a and b—can be manipulated to relate old rules and to form new rules. For example, the Distributive Property $a(b + c) = ab + ac$ can be extended to the multiplication of binomials $(a + b)(c + d) = ac + ad + bc + bd$, and both properties can be used to do mental arithmetic.

Because calculators can and should do virtually all of the complicated arithmetic that students have been asked to do in schools, knowing these patterns (e.g., knowing when addition and multiplication can be done in any order, or knowing when two divisions can be combined into one) is critical for intelligent use of these machines.

The patterns continue through elementary school when students (should) learn formulas for the perimeters, areas, and volumes of common figures, and when they learn that the physical and economic world around them is described in the language of mathematics. Area formulas require a knowledge of squares and square roots; volume formulas require cubes and cube roots; compound interest and population growth require the algebra of exponents more generally; the descriptions of orbits of thrown or propelled objects involve quadratics. All of these help the student understand that the world around them has regularities with predictable consequences, and help the student learn to live in that world.

It is often the case, when dealing with formulas, that some of the quantities are known and others are not. The length and area of a building may be known, but not its width. The temperature in °F might be known but not in °C. We may know some positions of a batted ball, and want to know how high or how far or how long it traveled. We may wish to know, under certain assumptions, when the world population will reach a certain level. We may have several constraints on a situation, and wish to know whether they can all be satisfied. These lead to the solving of equations, or of systems of equations, or of inequalities. Today, paper-and-pencil techniques are used in schools to solve these equations, but computer software and sophisticated calculators already exist that enable such equations to be solved automatically. We expect that by the end of the century the price of such tools will be such that it will be economically feasible for all students to have access to them in schools, and that the tools will be user-friendly enough to make them attractive for purchase by virtually all students. But whether or not these tools are universally available, the use of algebra to obtain new information from given data is an exceedingly important and powerful reason for its study.

As soon as one has a formula relating two or more quantities, natural questions occur. What will happen if one of the quantities is changed? For instance, if the area of a planned building is to be a certain quantity, and we are wondering what its dimensions should be, what happens as we consider larger and larger lengths? Or what happens to the velocity of a batted ball over time? Or what happens to the real value of money put in a savings association when there is a certain amount of interest paid but also there is inflation? We often wish to know what happens if we change a variable just a little. Algebra is naturally associated with the study of functions and other relations between variables. It provides a language by which change and variation are easily described. Historically, this connection is so powerful that though it took thousands of years from the time that algebraic problems were first considered until the development of algebra like that we use today, it took less than another hundred years, from the 1590s until the 1680s, to go from algebra to the development of calculus simultaneously in two different countries.

The development of spreadsheets and graphing technology has made it possible to store and access numerical and geometric information in a way unheard of a half century ago. In small and large businesses, people examine relationships between costs and profits by storing the information in spreadsheets, by changing one variable and seeing what happens to others. Population, or sports statistics, or results of polls can be graphed and examined. We used to think of functions as abstract entities, and it was appropriate that they be introduced in the later years of high school mathematics. But today it has become possible to see them, and to manipulate them both numerically and graphically, and as a result they are concretized and far more accessible to younger students. They are so concrete that some believe that it is through functions that one should learn algebra. Note that the use of such media, and the development of algebraic understandings grounded in them, represent a two-way street: operating on spreadsheets or with graphing programs can help students develop understandings of algebraic ideas, and understanding the algebraic ideas provides a structure for understanding how to work with the media.

When one looks at these reasons, it is difficult to explain why any student should not learn at least this much algebra.

We believe the above ideas should be mastered by all students by the end of tenth grade. In order to do so, much of this algebra should be given concentrated attention before the end of eighth grade. It is also the case that there are concepts (e.g., estimation and optimization) and processes (such as modeling and cooperation) that need concentrated attention. For this to take place, the algebra will have to be packaged differently than it has before. It will have to be done over a longer period of time, and with many more connections to other branches of mathematics. But there is other algebra to which all students should be introduced before the end of tenth grade, and as many students as possible should have opportunities to study in more depth.

Recommendation 3: All students should be introduced to operations and their properties in the various number systems (such as whole numbers, integers, real numbers, and complex numbers), and on objects other than numbers (such as sets, matrices, transformations, and propositions).

The storing of data in arrays such as those found in spreadsheets is not new. Newspapers have for a long time given weather, stock, and sports information in rows and columns. These arrays, when viewed mathematically, are called matrices. The mathematics of these arrays—one aspect of linear algebra—is surprisingly fruitful. It is related to the solving of systems of equations and to the changes in size and shape of geometric figures, with applications as diverse as to the efficiency of delivery systems in business and weather forecasts. It is not by accident that virtually every business major in college needs to take some linear algebra. Linear algebra begins with the arrangement of information in rows and columns and the solution of systems of two linear equations in two unknowns such as has been traditional in early algebra study. One can then progress up a ladder of abstractions, to the point where substantial mathematical power is seen. The groundwork for such mathematics, including matrix operations and transformations they represent, should be established in the high school algebra curriculum.

It is critical also that students realize that algebra studies patterns among patterns, and that its variables may stand for objects other than numbers or points. The study of commonalities in the structures of arithmetic can begin in the early elementary grades, when a student realizes that addition and multiplication are commutative but subtraction and division are not. It should continue through the grades, as ideas of inverse operations and inverse functions, and identities are studied both in arithmetic and with transformations in geometry. The union and intersection of sets in geometry, systems of equations in algebra, and their relationships to the logic of propositions provide still another example of the many commonalities that exist among diverse mathematical objects. The structures themselves—fields, groups, and so on—we believe to be most appropriately studied at the college level, but the language of properties (such as the properties of whole numbers, integers, real numbers, and complex numbers) and the emphasis of common features of diverse mathematical systems is appropriately a school level experience.

Recommendation 4: The mastery of a smaller number of paper-and-pencil manipulative skills should be expected of all students. Symbol manipulation software and calculators should be used for other manipulative skills, perhaps including skills not now accessible or easily accessible with paper-and-pencil techniques.

It should be noted that we are not calling for an "easier" or "watered down" algebra experience for students; rather, we are calling for a more appropriate experience, which may well mean dealing with more complex, but more meaningful, topics than are currently taught. Currently, there are topics taught even in early algebra that we feel are not necessary for all students to know because they do not fit any of the major reasons for studying algebra. These include, for example, the paper-and-pencil long division of polynomials, extensive operations with rational polynomial expressions, and manipulations with nth roots beyond $n = 2$. Also, students should not be asked to spend extensive amounts of time mastering paper-and pencil manipulations that can more easily be done by computer or calculator.

Recommendation 5: Both short-term and long-term strategies are needed to ensure that all students study a significant amount of algebra.

The recommendations made above require a careful, long-term plan for the development of a revitalized and meaningful algebra program. Ultimately, we envision a more intellectually coherent, more accessible, and more powerful algebra experience for all students. However, the development of such a program will take time—and there are students in the current program who cannot wait until the new program is fully in place.

Those students need help now, on the basis of what we do know. Hence, we propose a two-pronged strategy: (a) the development of a long-term approach for the restructuring of K-12 mathematics, with a deeply embedded strand of algebraic ideas as discussed above; and (b) the pursuit of short-term improvements in current courses called "algebra" for those students who are currently in the pipeline. In the short run, current courses can be made more meaningful, and the experience of some algebraic ideas can be embedded substantively into the middle school curriculum.

Recommendation 6: A much expanded research base is required to make continued progress on recommendations 1-5 above.

Much of the progress of the past decade has been undergirded by increased understandings of the processes of thinking and learning—understandings of the kinds of active engagement required for mathematics learning, and of the inadequacy (for the vast majority of students) of instruction in disembodied symbol manipulation. We have, for example, made some progress on conceptualizing the kinds of connections students need to make among various representations (verbal, tabular, graphical, symbolic) to be flexible and competent users of algebra. We have an improved, although still sketchy, understanding of the cognitive underpinnings of competence in algebra, and of the kinds of instructional practices that support the development of such competence. Some progress has been made on the uses of technology in algebra (though this is a moving target!). However, we need to know much more about these issues; also about what constitutes "symbol sense" and how it is developed; about the roles and balance of "rote skills" and machine-performed computations; about the development of modeling skills and the character of mathematical abstraction; about the successes and difficulties of various approaches to teaching algebra (including attempts in other countries), and about the use of mathematical and algebraic thinking, often unrecognized, in work settings. Likewise (though we have dealt with this theme only implicitly, because it is the province of Working Group 2), we need to know much more about the creation of effective instructional environments, and the kinds of teacher understandings required to foster algebraic thinking in such environments.

Five Questions About Algebra Reform (and a Thought Experiment)

Daniel Chazan
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1. Why Does High School Mathematics Fulfill a Gate Keeping Role in Our Society? (assuming that mathematics fulfills such a role)

I find Philip Cusick's idea of the "egalitarian ideal" important in this context—that schools will provide each student with an opportunity for social, political, and economic equality. Our society at least pays lip service to the desire to actualize the potential of a wide range of our population, yet college is a limited resource and is not universally accessible. Criteria for access to continued education can't be obviously unfair. There needs to be some sort of seemingly objective measure. The valuation of mathematics has a long history in western thought. It is seen as a difficult subject of study and one which is some sort of index of intellectual capability.

As mathematics educators, it is important to acknowledge that we benefit from this gate keeping role. This role makes mathematics important politically and helps justify much of the funding that is now available to us. Art educators, for example, are finding the current budgetary constraints much more problematic. At the same time, there are unfortunate side effects of the identification of mathematics with gate keeping and access to college. In particular, it makes it difficult to think about the mathematics education of students who don't end up going to college. Serious attempts in such a direction are hampered by concern that such a mathematics education will systematically exclude identifiable groups (by race, class, gender, ethnicity) from a college education and thus contravene the egalitarian ideal.

2. Why Has the Traditional Set of Algebra Courses Become So Central in the High School Mathematics Curriculum?

I believe that the centrality of algebra is a result of its (hierarchical) connection with calculus and our adherence to the egalitarian ideal. Arguably, calculus is seen as an important part of the background of a literate graduate of U.S elite institutions of higher education. Collegcs that don't insist on calculus for the most part seem to do so apologetically with some sort of argument about the lack of preparation of incoming students. Skill at algebraic manipulations is seen as a prerequisite for calculus. In many institutions, students who are not ready for calculus must do prior coursework without receiving credit.

Though I do not have any scholarly insight, it is interesting to speculate about how calculus came to have this role in college education. Clearly, there are some fields for which calculus is an important technique (faculty in these fields thus have concerns about changes in algebra and calculus curricula). However, I think that the importance given to calculus is an historical artifact of the felt achievement that calculus represented at the time it was initially being developed and colleges were being created. These ideas were thought to be the most powerful ideas created to that point. At the same time, people were deciding what to include in a college education.

The combination of the importance of calculus at the college level and the egalitarian ideal suggests that it is important, at least in theory, to provide a wide range of students with access to mathematical preparation which would allow them to enter calculus successfully (though much of the mathematics taught at colleges is remedial from this point of view). Algebra is seen as the necessary preparation for calculus. Thus, high school in particular should prepare as many students as possible for calculus by taking algebra. There must be "good" reasons for putting students in mathematics classes which do not include such preparation.

When thinking of algebra specifically, much of what is taught in traditional algebra I and algebra II courses is a comfort with the manipulation of symbols. The main reason to acquire this comfort is to enable the student to do further mathematics (mainly calculus). These manipulations are taught by modeling and by rote, and the appropriateness of particular manipulations for particular instructions (simplify, factor, solve) is accomplished by pointing and repetition (ostention). The unfortunate effect of this situation is that as mathematics teachers, we are always focusing on what mathematics comes next and not how the mathematics we are teaching can be of value for people in their lives. Again, students who do not go to college end up being poorly served.

One intriguing idea proposed by Jim Kaput at the conference is to change our model of curriculum drastically from a layer cake approach to a strands approach. Thus, he proposes removing calculus as the capstone of high school mathematics by integrating calculus into the K-10 mathematics curriculum as the study of the mathematics of change. Study of change could then be used as a context for developing algebraic understandings. Algebra would then be a part of all students' elementary and middle school experience.

3. What Might Be Important Criteria for Proposals for Changing the Algebra Curriculum? (if there actually is such a possibility)

As someone who teaches an Algebra I class with students who in large percentages will both go on to some postsecondary education and who will work upon graduation from (or leaving) high school, I would like to be able to serve both types of students. I would like to prepare students who might like to go to college in such a way that they can go to college. I would also like to teach students who are not headed to college in such a way that they will feel that their interests, concerns, and life trajectory are being acknowledged and planned for.

At the same time, I don't want to be making the decision for my students about which category they fall into. I'd also not like to have school counselors making those decisions. Finally, I am also concerned about having some of my students making those decisions without adult guidance and without access to a broad range of information. Interestingly, at our conference, university mathematicians also expressed a concern about the preparation of students for linear and abstract algebra at the college level.

4. Is it Possible to Meet Students' Needs by Teaching a Single Curriculum to a Heterogeneous Group of Students in a Single Class? If So, What Might Be a Conceivable Basis for Such a Curriculum? (It's a setup! I'm sure there must be other approaches as well.)

I'm not sure if it is possible to do so, but I think that the benefits of having such a curriculum justify an exploration of any possibilities that people can suggest. Such a curriculum would allow teachers to fulfill their responsibilities to the egalitarian ideal and also teach meaningful mathematics to students who are not headed to college.

For the past 3 years, I have been teaching Algebra I as a course focused on learning how to express and study mathematically, relationships between two varying quantities—an independent variable and a dependent one (similar in many respects to the kinds of problems presented by Alba Thompson). We spend a lot of time learning to read and write tables, graphs, and expressions (symbols) which represent actual quantities in situations (ones that I design, one that students design, an ones which we find in exploring mathematics in the world of work). We also study tables, graphs, and expressions in the context of pure mathematics.

I see such an emphasis as valuable in helping students who are going on to calculus develop intuitions and skills which will be useful as they continue to study functions. After talking with Guershon Harel, I think developing an algebraic habit of mind (a phrase which Al Cuoco kept raising) which insists on understanding what it is that the symbols are standing for is a helpful prerequisite for other more advanced mathematics courses like linear algebra and abstract algebra. I also think that such of course is of value to students who do not go to college. Tables and graphs are cultural artifacts that are widely found in our society (more widely than literal symbols, though literal symbols are becoming more important in interactions with computers). These artifacts are not necessarily self-evident. There is a lot to learn about reading these representations.

5. What are Some Obstacles to the Creation of Such a Curriculum? (I'm assuming that there are many more than the one indicated below, for example, what are the ramifications for college math curricula.)

The traditional Algebra I course focuses on developing skills at the manipulation of symbols. Most of the manipulations are done in a "pure" math context, without reference to situations. Students are rarely pressed to find contexts which would lead to strings of algebraic symbols. Word problems, as the representative of situations in the traditional course, are seen as a place to apply equation solving. Tabular or graphical solutions are not developed or stressed.

As a teacher, I have found it challenging to take the agenda of quantities in situations seriously. There is a huge intellectual challenge here—to construct an understanding of how mathematics helps us appreciate aspects of the relationships between quantities which otherwise are not accessible. I've been looking for where tables, graphs, and symbols show up and how they help me understand calculations (algorithms) which people compute repetitively and the relationships between quantities which such algorithms signify.

To close this paper and give an example of the assertions in the previous paragraph, I'd like to share attempts to think about the purposes and values for learning to simplify linear expressions. I'm not sure whether the understandings expressed below suggest that learning about such simplifications is important for people not going further in mathematics (maybe there are other more important things), but it is an attempt to provide such a rationale. (Again, it's a set up. I've tried to choose the hardest thing i could find.)

Forms of Linear Expressions

Linear functions of one variable are functions for which there is a constant rate of change, in other words, there is a constant change in the value of the dependent variable for equal sized increments in the independent variable. Traditionally, a series of "properties" are valued in describing such functions and differentiating them one from another: whether the function is decreasing or increasing; the rate of change, or slope; the value which corresponds to 0, or the y intercept; and the value of the independent variable which results in an output of 0, or the x intercept. All of these properties can be read off suitably prepared graphs, tables, or expressions. I'll focus below on the relationships between forms of linear expressions and these four properties.

Linear functions come in many shapes and sizes, for example:

$$-350(t - 3) + 700;$$
$$3x - 27;\ 3(x - 9);$$
$$4-(3-(2-(1+x)));\ \text{and}$$
$$3x+4+7x+3x+2$$

Traditionally, one form is considered to be the privileged, simplified form. This form mx+b, sometimes called the slope intercept form, cannot be made simpler. The knowledgeable reader of expressions can, without substituting values, read off the values of the slope and the y intercept of an expression in this form. For the purpose of simplifications, particularly of rational expressions, students are also asked to factor expressions in the mx+b form and put them in what is commonly known as the factored form m(x-r).

I will not attempt an exhaustive analysis of all possible forms, or investigate how different forms arise in a variety of practices. Rather I will concentrate on three particular forms, all of which can be considered simple in that they cannot be simplified further without the use of the distributive rule, that is there are no like terms which can be combined. In the chart that follows, these forms are named to highlight the information each presents to the knowledgeable reader.

	mx + b slope/y intercept	m(x-r) slope/x intercept	$m(x-x_1) + y_1$ point/slope
increasing/ decreasing	can be read off	can be read off	can be read off
slope	can be read off	can be read off	can be read off
y intercept	can be read off	must be computed	must be computed
x intercept	must be computed	can be read off	must be computed

In some ways, the point/slope form can be considered the most general one, where the other two are specific cases. The point slope presents information about some point (x_1, y_1) whereas the other two forms present information about "special" points. Thus, these three forms differ in the particular points which they identify most easily for the knowledgeable reader.

To me, this difference becomes interesting when working on word problems or in understanding how algebraic expressions represent a situation. The alternative forms can be considered as different mathematical descriptions of the situation. For example:

> You take out a car loan (at 0 percent financing) and then repay it by paying the same amount each month. Here are three mathematical descriptions of the balance that you owe and how it changes over time.
>
> Balance of account (time) = 215t -4300
>
> Balance of account (time) = 215(t - 10) -2150
>
> Balance of account (time) = 215(t - 20)

In particular, in cases like this one, where time is the independent variable, the descriptions can be thought of as descriptions from a privileged moment in time, written with the information available at that moment. Thus, 215t -4300 indicates the balance at the start of the loan; 215(t - 10) -2150 indicates that $2150 are owed at the 10-month mark; and 215(t - 20) indicates that the loan was repaid after 20 months.

Symbol Manipulation

The traditional algebra curriculum has been roundly criticized for an emphasis on "meaningless" symbol manipulation where students have no idea why they are rewriting expressions or how to judge when their answers are valid. As a result, current reform efforts (e.g., UCSMP) have tried to minimize or remove symbolic manipulation wherever possible. The UCSMP Algebra I text de-emphasizes factoring. The Algebra II text limits the amount of attention to rational expressions.

Appreciating that different symbolic forms provide different perspectives on the same function, symbolic manipulation takes on a different perspective. These manipulations (in the case of linear functions, the distributive rule) can help us demonstrate to ourselves that two different expressions are indeed the same (in the sense of producing the same output for the same input). In addition, and perhaps most importantly, manipulation of symbols can sometimes lead to expressions that provide new insight into the situation which they describe. Switching the form of the expression may highlight information about a different point on the graph, a different aspect of the situation.

Notice, however, that symbol manipulation can be avoided. Knowledgeable readers of the three forms described above might not need to use the distributive rule to carry out a manipulation. They might be able to create alternative expressions for the same function simply from information provided in a table of values. Thus, if the goal is to switch from one known form to another, traditional symbol manipulation—using the distributive law—is not necessary, though the law justifies why such a manipulation works.

Conclusion

At the conference, Bob Moses made a series of powerful statements about students' need to learn whatever symbol sense is necessary to assure themselves economic access (a moving target). As others (like Rogers Hall, Jim Greeno, and Alan Schoenfeld) argued, at the present time, we do not know much about the facility with algebraic representations that current jobs/professions require (let alone future ones). There is also much that we do not know about helping students become fluent with these representations (literal symbols, tables, and graphs). All of which suggests that we have a large agenda on which to work.

Algebra and the Democratic Imperative

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The evidence is unmistakable: in the past dozen years or so, the United States has become more of a bimodal society in terms of family income levels, and all the life opportunities and constraints that that implies. We are becoming a two-caste society. (See, for example, Danziger & Gottschalk 1993; Hacker 1992; Phillips 1990; Wilson 1987. For other dimensions of the problem, see also Lemann 1991.) Clearly, this is not solely attributable to education, but it is equally clear that education does play an important role in this result. And, within education, the subject that is sometimes called ninth-grade algebra plays a particularly decisive part. Acquire a mastery of ninth-grade algebra and many possibilities open up to you, but they will be unavailable if you do not learn ninth-grade algebra. This creates one more aspect of a caste system: those who know algebra versus those who do not. We will pay a formidable price if we ignore this situation (see, for example, Moses, in press).

It is not equally clear how to address the problem. There are constraints on all sides, limiting possible strategies for creating solutions. For some careers, a certain level of proficiency in algebra is a true necessity, but what this level actually is is in doubt, and many stated requirements probably go beyond anything that is actually needed (see, for example, Noddings, in press). Existing instruction in algebra is often remarkably ineffective, not merely as mathematics education but as education itself (see Davis, in press). Better routes toward the learning of algebra need to be created, but even here there is not complete agreement on desirable directions for improvement, and there is an inadequate research base for deciding, especially if one aims for the relatively new goal of student understanding.

Perhaps above all, there is the structural peculiarity of U.S. curricula, virtually unique in the entire world, of employing the so-called layer-cake approach: algebra in grade nine (or perhaps in grade eight), followed by a year of Euclidean synthetic geometry, which is then followed by an additional year of intermediate algebra, and so on. Most other nations arrange for students to study algebra beginning in, perhaps, grade two, and continuing on through grade 12. Clearly, a layer-cake approach, by making algebra an event that occurs at a definite, pre-specified time, maximizes the risk that any particular student may fail in this event; the risk would be far less if "learning algebra" were an on-going activity spread over many years, as the learning of language inevitably is. (Indeed, as Rogers Hall has pointed out, when one looks carefully at the expressive power of algebraic symbolism, the similarity to natural language linguistics is quite striking, and comparisons with language learning may not be unreasonable. The meanings are complex and subtle, even if the notations themselves are limited and orderly. Perhaps learning algebra ought to take a considerable number of years.)

It is hard to escape the conclusions that U.S. curricula would avoid the "ninth-grade algebra" problem—the fact that this specific school subject denies many students the opportunity to continue further in mathematics--if the study of algebra began in, say, grade two, and continued throughout the remaining years of education, as it does in most nations today (this is sometimes called the strands approach—different strands, such as algebra, geometry, and measurement, extend over the years of schooling; there is no question of which comes first or which comes second, etc.). Such a change may be hard to bring about, which suggests a two-pronged approach: make whatever improvements can be made, for a short-term remedy, but realize from the outset that more fundamental change is really needed, and begin immediately to build toward this deeper change.

There is some foundation to build on, but more is needed. For what we do know, consider the evidence acquired in other countries and, since at least the 1950s, also in some schools in the United States (see, for example, Davis 1985). But if we intend to continue our modern emphasis on meanings and understanding, we need to build a much broader research base. Younger children can think profitably, even creatively, about algebraic ideas, but their understandings must be developed carefully and gradually. (As two examples, from among a great many, consider the case of a class of fifth-graders who were unwilling to accept the statement

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

as a true statement because, they argued, "for subtraction you have to have a common denominator"—thereby confusing, as children often do, the truth value of a statement with the use of some specific algorithm. Mathematics, for them, means "doing what you are told, and doing it the way you were told to do it"—for them it does not deal—as it does for mathematicians—with truth, falsity, and implication. This can be a very important difference. (See, for example, Davis 1988) As a second example, many researchers report, independently, that fifth-grade students who are able to speak about specific situations in almost-algebraic language—for example, "If the numerator is 1, you just multiply the denominator by 24"—often experience great difficulty when they try to write this in something closer to algebraic notation (see also English, in press).

But perhaps this analysis provides grounds for optimism: there may be a sensible direction for remedying the situation in a basic way, if we have the will to make a commitment to doing so—and if we do what is needed in the areas of research and teacher education, and in the area of creating appropriate curricula.

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Realism(s) for Learning Algebra

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By setting out to redefine the role of algebra across the mathematics curriculum, the Algebra Initiative has a huge agenda, and a great deal might ride on the kinds of consensus that emerge from this group. In writing about an "appropriate algebra experience" across the K-12 curriculum, I want to focus on the role that realism can or should play in learning algebra. But first I should describe the position from which I approach this question.

I teach graduate courses at the intersection of cognitive science and mathematics education that are part of an interdisciplinary program in a school of education. My own training is in computer science and psychology, where I have studied (a) how (or whether) we can make sensible computational models of human reasoning and learning and (b) what are good representational media for learning and doing mathematics. I have ended up looking very closely at what people—some good and some less good at solving algebra problems—do when they use representations to coordinate understanding and computation. It turns out that people's representational practices in mathematics are much more complicated than I had imagined when I started trying to write computer programs to model their activity. That's good for pushing along research on cognition, and my own view is that developing and testing curriculum materials provides a way to study representational practices in places that can make a difference.

Two things, then, about how I approach this agenda. First, my background qualifies me to drone on about the mathematical properties of programming languages or processes constructed out of these languages, but my comments should not be read as coming from a mathematician or a teacher or mathematics. Second, I want to position my work pretty clearly within the political and technical agendas that Kaput and Lacampagne have sketched for the working groups. From my perspective, teaching in a school of education and doing research on representational practices in mathematics, the need for more and better education in algebra or new forms of computing technology might seem like an unalloyed good thing. I don't think this is necessarily true, however, and want to use this paper to think about where to push forward in either area. In particular, I will argue that how we treat pedagogical realism has a lot to do with what we think mathematics is, why algebra is important within mathematics, and how we should invest resources in changing mathematics education.

What is a Mathematical Application?

As with Elizabeth Phillips' list of definitions for algebra, I've heard lots of variations on what applied math problems are about:

- Oh, *those* problems (school alumni);
- Problems we never do at the end of the chapter (algebra students);
- Word or story problems (psychologists);
- Math problems with colorful pictures (a particularly caustic graduate student); and
- Where you don't learn about using mathematics, that comes later (alumni).

Before pulling apart historical views of particular kinds of applied mathematics problems, I want to focus on a prior question: What kind of realism do we value in teaching people about mathematics, and in particular about ideas that are central to algebra? Since part of my life over the past 2 years has been caught up in an NSF-sponsored Middle School Mathematics Through Applications Project (MMAP 1992) at Stanford University and the Institute for Research on Learning, I will draw examples from my own research and from the ongoing experience of developing particular kinds of applications, working with teachers to implement these units in diverse classrooms, and trying to determine how things are going.

Reasons for Realism

I won't worry much here about arguing for realism in teaching mathematics, particularly since I think we should treat what "real" means as a serious design problem. It should suffice to point out that most versions of individual or social constructivism (Cobb et al. 1992; Walkerdine 1988; von Glasersfeld, Suchtung 1992), psychological or sociological accounts of mathematical knowledge and learning (Bloor 1976/1991; Kitcher 1984), and reformist writing about the mathematics curriculum call for students to encounter mathematical ideas in realistic situations. There are many, not entirely compatible reasons for wanting realism: familiar constraints in the domain of application, the requirement that one find and structure an open-ended problem for solution, or engaging student interest. Any of these could be important in working to overcome the bad news about algebra that Kaput's position paper outlines.

Realism as a Comparison

For discussion purposes, I'll suggest a general frame for unpacking what we could mean by realism by transforming adjectives like "authentic," "applied," or "realistic" into a more explicit statement like "activity X in school is like activity Y in Z, in the following ways...". This is a familiar proportional analogy,

X : school :: Y : Z

By thinking about realism as a comparison that places components from a source domain (activity Y in setting Z) in correspondence with a target domain (activity X in a school setting), we can bring into relief the importance of relations between mathematical activities and the settings in which they are pursued. Kaput (opening talk) emphasizes this clearly by showing cognitive scientists and educators working to convert images of mathematical work outside school (complex problems, tools, skill, and communication between people) into images of mathematical activity in school. By asking us to be explicit about what parts of these images are being placed in correspondence, the analogy may help us see what's at stake in claims about realism for the algebra curricula, at least from the perspective of people who design and evaluate these materials.

Treating realism as an analogical comparison, several issues immediately pop to the foreground.

1. Applications are not just tasks (X's or Y's) but are activities that can have complex relations to the settings in which people pursue them. This assumes, of course, that we see an application as more than a pedagogical vehicle for delivering a mathematical problem (discussed below).
2. It is not just that activity X is like activity Y, although this may be hugely important. Instead, we also need to worry about whether the relation(s) between X and school are like the relation(s) between Y and Z. These could include relations of accountability, ways of talking and writing about mathematics, conventions for being right, done.
3. There are many activities in situ (Y/Z pairs) that could provide areas of applications, and choosing among them requires deciding (a) what kind of

mathematics we value and (b) what resources we are willing to deploy in schools.

4. The mathematical structures that apparently underlie activity in situ can appear similar for us, as designers, across school and application setting. But why should we assume these structures will be similar for students and practitioners in the area of application, a critical element in ideas about transfer of training (Singley and Anderson 1989; Greeno, Smith, and Moore 1993)?
5. What are we saying about mathematics, its uses, and students' access to different forms of after school life when we carry one of these proportional analogies into instruction?

These questions go beyond the scope of a short conference paper. In my view, answering them in ways that are productive for the learning and teaching of algebra will require research that examines possible relations between math in school and at work as an empirical problem. To make the issues more concrete, however, I will consider two instances of "mathematical applications" that may help to exercise the framework.

Story Problems as Vehicles for Mathematical Structure

Mathematical "applications," as they typically appear in algebra textbooks over the past two decades (e.g., Mayer 1981), have had an almost universally recognizable form:

(CLOSURE)

Tom can drive to Bill's house in 4 hours and Bill can drive to Tom's house in 3 hours. How long will it take them to meet if they both leave their houses at the same time and drive toward each other?

This is a story or word problem typical of many algebra textbooks, and one that many people find very difficult to solve, since rates are presented as hypothetical events (i.e., can drive to Bill's house in 4 hours) rather than constant values, and the distance between Tom and Bill's house is not given. Now, this kind of problem produces very "real" difficulties for algebra students and their teachers, as evident in the following transcript excerpts from research that contrasts how people with different levels of achieved mathematical competence (i.e., levels of schooling) use different forms of representation to solve these problems (Hall et al 1989; Hall 1990). K is an algebra student, taking courses in a community college in the hopes of matriculating into the University of California system. R teaches algebra in a community college, though K is not one of his students.

K starts work on the problem with a drawing that weakly organizes given quantities. After constructing "my little dirt table," her voice becomes progressively more tentative, the pauses between her utterances elongate, and she wonders whether she is missing something to "plug into that... table."

K: Tom can drive to Bill's house in 4 hours, and Bill can drive to Tom's house in 3 hours. How long will it take them to meet if they both leave their houses at the same time and drive toward each other? So we've got...Tom driving... Let's put Bill's house here (draws box at left), and Tom's house here (box to right). And... Ok, so Tom's gonna drive to Bill's house... in 4 hours (draws a left directed segment labeled 4 hrs)... it takes him 4 hrs to drive there. Bill can drive to Tom's house in... 3 hrs (draws right directed segment labeled 3 hrs)... He's got a better car (both laugh). How long will it take them to meet if they leave their houses at the same time and drive toward each other? Alright... my little dirt table here (draws a 2 X 3 table, labels columns d, r, t)... Alright so we've got... Bill driving... is that him going to his house? ... That's 3 hrs. Wait, let's see, Bill... wait, that's Tim's house (laughs) but Bill's driving there? Oh well, doesn't matter, Tim is driving... Let's see now, I'm confused (mumbles while reading text). In any case they're driving toward each other, it doesn't matter where they're starting from actually.

Rogers: Um hmm.

K: Ok... umm... So Tim can drive in 4 hrs. Here's Tim (writes T as row label)... and Bill (writes other label)... So Tim... 4 hrs (writes 4 in Tim's rate cell)... We don't know the distance. Bummer! (laughs) Ok, he drives in 3 hrs (writes 3 in Bill's rate cell)... (long pause) How long will it take... for them to meet... if they leave at the same time and drive toward each other? (long pause)

Rogers: What you thinkin'?

K: I'm wondering if this is the right table, if its a distance table, because I don't have all the information... that you would normally plug into that... table. I'm trying to think if there's another... um... no TRW form doesn't work. That's the... the work...

Now for R, who regularly teaches algebra and has a widely reported reputation for making algebraic concepts accessible to students. He also starts with a drawing, but distinguishes clearly between rates as hypothetical events (e.g., a trip in 4 hours) and the circumstances of collision that the problem asks about.

R: Tom can drive to Bill's house in 4 hours, and Bill can drive to Tom's house in 3 hours. How long will it take them to MEET if they BOTH leave their houses at the same time and drive toward each other? Oh, I like that, that's a neat problem! And I don't at the moment have any idea how to solve it.

Rogers: OK.

R: So I'm just gonna draw a picture. This is Tom's house and this is Bill's house, and Tom drives to Bill's house in 4 hours, so if he were going this way, it would take 4 hours to complete the trip (draws and labels directed segment). If Bill, on the other hand, is going toward Tom, it takes him... 3 hours for the trip (draws opposite directed segment and labels it). Now the DISTANCE is fixed. I'll write myself a note, distance is fixed (writes "d = fixed").

R: (reading) How long will it take them to meet if they both leave their houses at the same time and drive toward each other? So we've got a situation in which one guy is going this way and another guy is going this way (draws a second diagram, below first, segments don't close) and they meet somewhere. It won't be in the MIDDLE, because this guy takes longer to travel the same distance, so he will not have... again, the amount of time they travel is the same, if they both leave and then they meet, there's a good joke that goes along with that.

These reactions are typical of what I call a gap between certainty and precision: both K (student) and R (teacher) understand the situation presented by the problem perfectly well, but they are momentarily (sometimes, much longer) at a loss for a way to go about finding a precise value for the length of time it will take the drivers to collide. The flurry of overlapping representational forms (talk, bodily activity, and writing or drawing) that ensue make for a fascinating exploration of how various material and social resources can structure mathematical cognition (Hall 1993), but I want instead to focus here on how these problems present a particular form of pedagogical realism.

Gaps between certainty and precision are a hallmark of applied problems in conventional mathematics texts, where assumed familiarity with the problem situation (i.e., relatively well-developed common-sense reasoning) is thought to anchor and guide students' developing understanding of mathematical notations and operations. Drawing this problem across the frame of our proportional analogy (X : school :: Y : X), however, we find an activity (two travelers instantaneously reaching uniform speeds) in a physical setting (a road laid out in a straight line between two private residences) that could not literally be familiar to anyone.

The problem is familiar to both K and R, but it is familiar as a school problem—i.e., within our proportional analogy, this kind of realism maps the school domain back onto itself (Greeno 1989). To their credit as adaptive participants in school math, people learn to deal with these kinds of problems in classroom and testing situations. However, once at work trying to solve such problems, they find it difficult to squeeze any meaningful constraints out of the situation that will help them organize mathematical inference or calculation. Rather than providing a "realistic" context for traveling in vehicles, this story problem and others like it simply provide a vehicle for delivering a mathematical structure.

As was pointed out in the keynote address to Initiative working groups, it is important to realize that mathematical forms have a history, and I will argue that the CLOSURE problem is just such a form. I would add that, once these forms are inserted into the capital intensive process of producing textbooks and assessments, they have an historical momentum that is difficult to elude, whatever our beliefs about "authentic" mathematics. In the case of algebra story problems, their history goes back (at least) to the earliest known printed mathematics textbook in the West, the *Treviso Arithmetic* (Swetz 1987), used to train young men (and only men, apparently) seeking careers as "computers" in the expanding mercantile economies of Europe. Within the *Treviso* we find a couple of couriers who appear to be the ancestors of Tom and Bill:

> The Holy Father sent a courier from Rome to Venice, commanding him that he should reach Venice in 7 days. And the most illustrious Signoria of Venice also sent another courier to Rome, who should reach Rome in 9 days. And from Rome to Venice is 250 miles. It happened that by order of these lords the couriers started their journeys at the same time. It is required to find in how many days they will meet, and how many miles each will have traveled. (*Treviso Arithmetic*, in Swetz 1987, p. 158)

Machiavelli, at the tender age of nine, had yet to enter government service, and contemporaries only slightly older (aged 12 to 16) were facing story problems very similar to those found in algebra texts we use today. According to Swetz, even these problems had little to do with actual mercantile practices in the late 1500s, but readers of the *Treviso* were encouraged to study these problems and associated calculation schemes (e.g., the "rule of three things") to acquire essential skills,

> As a carpenter (wishing to do well in his profession) needs to have his tools very sharp, and to know what tools to use first, and what next to use, etc., to the end that he may have honor from his work, so it is in the work of the Practica. (*Treviso Arithmetic*, in Swetz 1987, p. 101)

The role these kinds of problems might play in exercising generalizable algebraic skills has not escaped psychologists, who have taken up story or word problems as malleable experimental materials for investigating transfer and analogical inference (Anderson and Singley 1989; Dellarosa-Cummins et al. 1988; Hall et al. 1989; Reed et al. 1985). There is also a line of work that takes the underlying structure of word problems as a basic conceptual domain for understanding quantitative relations (Carpenter et al. 1984; Kintsch and Greeno 1985; Vergnaud 1982, 1983), and these structures have been shown to be learnable when made explicit to teachers and students (Carpenter et al. 1989).

While these studies bring cognitive science and educational practice together in interesting ways, they generally take realism to be an unanalyzed property of good experimental or instructional materials. However, stretching story problems over a proportional analogy that questions realism as a relation between school and some other place, can we find any image of authentic mathematical activity that these forms bring into mathematics classrooms or testing situations? Rather than accepting the historical momentum of these problems as curricular forms, we might ask what other forms of realism could be productive in school mathematics?

Design Projects as Settings for Mathematical Work

As an alternative approach, I want (a) to sketch a different kind of mathematical application, (b) to analyze briefly the kind of realism that it is designed to provide for learners, and (c) to consider how this relates to "big ideas" in algebra that might take root in elementary and middle school mathematics.

This example is taken from field trials in an NSF-sponsored curriculum development project for middle school mathematics (MMAP 1992), where an explicit aim is to create mathematical activities in classrooms that have specific relations to the mathematical activities of professionals in design-oriented work (e.g., architectural design). Curriculum units are developed with teachers, who spend time with math practitioners in field sites during a summer practicum, help to write curriculum activities and design software with our project staff, and then work with us to implement these units during the following school year. All these applications units involve design activities in areas like living/working space, ecological models of interacting species, or secure encoding schemes. In the example that follows, I focus on early field data from one of these units, called the Antarctica Project.

In MMAP classrooms using the Antarctica Project, students work in groups on an extended project (4 to 10 weeks) where they design models of living/working space using computer based design tools (the ArchiTech environment) within a repeated cycle of activities. During each cycle, students analyze a design problem (e.g., the needs of an NSF-sponsored biological research team wintering over in Antarctica), they construct a model that provides a partial solution to their analysis of the problem (e.g., a structure with sufficient lab space and adequate heating for an Antarctic winter), they evaluate this model in light of new or elaborated constraints (e.g., providing storage space for redundant supplies and equipment), and then they repeat the cycle with an updated model and design problem. The interface of a computer-based simulation environment (ArchiTech) provides students with specific representational systems (e.g., two-dimensional scale models and a rudimentary database of basic quantities that can be used to describe these models) and a set of tools for changing, recording, or analyzing the quantitative behavior of the models they construct. By porting data from the design tool into a simplified spreadsheet environment, related values generated during analysis or evaluation of a model can be organized in tabular or graphical form, for presentation or comparison with alternative models from other student groups.

The following memo is a Request for Proposals handed to students at the outset of their design project:

FROZEN SCIENTIFIC GROUP - MEMO 1

TO: Expedition Design, Inc.

FROM: Frozen Scientific Group
2550 Hanover Street
Palo Alto, CA 94304

RE: Request for Design Proposal

This memo is a request for a design proposal. We are asking for proposals from several companies, and will accept the proposal that best meets our needs at the most reasonable cost.

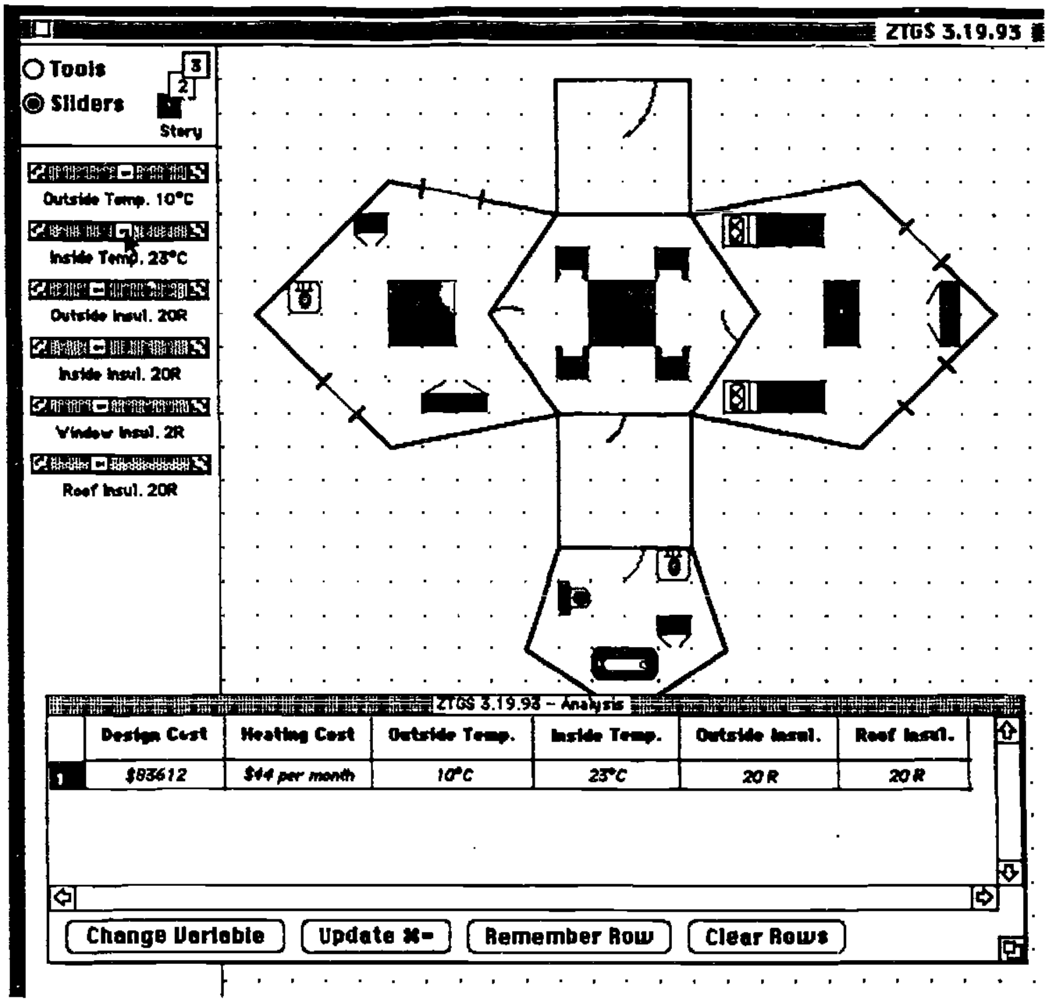
We need a design for a scientific research station on the Antarctic coast. The site for the station is a small flat field of dry rock 17 meters wide and 30 meters long. The station should provide lab space, housing, and recreational facilities for four scientists, who will be studying phytoplankton. Our project will use the station for two years. Our funders are particularly concerned that the design we accept is energy-efficient. Your proposal should reflect that concern, while still maintaining overall reasonable costs. Attached you'll find a short description of life in Antarctica, which should give you a better idea of our requirements.

Your design proposal report should include a floor plan of the design, along with a proposed budget for building and heating costs over the two years. Of course, you will want to include materials that explain why your design best meets our needs.

We look forward to receiving your proposal.

Students are provided with documentary sources about the continent of Antarctica, the sometimes tragic history of its exploration, the rigors involved in wintering over at coastal and central research stations, and why these settings provide such an attractive site for biological and atmospheric research activities. Students are also encouraged to expand their research and analysis activities using additional resources provided by their teachers, school or community libraries, or parents (e.g., in some field trial sites, adults who work in architectural design visit the classroom and discuss the work they do).

Students spend several weeks iterating through cycles of analysis, design, and evaluation, using the ArchiTech environment shown below and a simplified version of a commercial spreadsheet package.



This screen shot shows a floorplan-in-progress from a group of four seventh-grade students, just starting to use the environment. This software is intended to resemble computer aided design (CAD) tools used by architectural engineers and has been developed to support units like the Antarctica Project by a MMAP team of teachers, educational researchers, and technologists.

Three modes of activity are supported in this interface: (a) design tools (not shown in the screen shot) allow students to construct a scalable, multistory floorplan model out of local materials like walls, windows, doors, and different types of furniture; (b) analysis tools (sliders to the left of the floorplan in the screen shot) allow students to vary a collection of quantitative parameters that determine global properties of the model such as inside or outside temperatures and the insulation rating of various surfaces; and (c) a display window (floating window at the bottom of the screen shot) allows students to selectively examine the value of named quantities for temperature, spatial features of the design, the cost of materials, and heating cost. Values displayed for a particular, parameterized version of a student model can be saved in a tabular display and, along with names for quantities, these can be copied into a spreadsheet environment for constructing and examining more complex quantities (e.g., total cost to build and heat per unit of floor area).

We have found that middle school students energetically engage in the process of constructing models of living space (in the Antarctica unit and other design problems) and, with guidance provided by teachers and supporting curriculum materials, their design activity can be used to draw out relevant mathematical questions (e.g., exploring relations between the perimeter and area of a closed figure) that students can then investigate using these same design and spreadsheet tools.

The following transcript illustrates the character of group work in this environment, drawn from the same group of seventh graders asked to work on a different design problem, which we piloted as a performance-based assessment at the end of the Antarctica Project. They were given an existing design for a dormitory in Seattle, Washington and asked to redesign the facility under overall cost constraints for the much colder climate of Duluth, Minnesota. G and Z are boys who in this excerpt take responsibility for using the mouse (the primary input device for ArchiTech) and a hand-held calculator (although given the option, no group of students used a spreadsheet available as a background process during this task). T and S are girls, and while not at the controls of either device (ArchiTech or the calculator), they have argued for reducing building costs since this working session began.

G: Is this (heating cost) 42?

T: No, (points at display window) you do 38 times...

G: Twelve.

Z: Times... Oh, no, no, no. Its, = = um... 12 times 25 ... right? I think its 300. Yeh, that's right. Times 38.

??: That's PER MONTH!

G: If its $42 dollars it'd be 12 ... (looks down at design brief)

Z: (sharp intake of breath) Its WAY more. These cheapskates! We're gonna have to make the (points to design cost in AW) design costs less.

T: I know. How about, this is a cheap college!

??: Its $14 over.

S: (looks from calculator to screen) Maybe (littler windows).

Z: How about we make, delete THAT door (pointing). We don't need two windows, I mean two doors.

S: (leaning in) We don't (inaudible).

T: We don't need that many windows either!

Z: Yeh, all right (points at floorplan). Delete a door.

Without unpacking more complex analyses of these students' modeling activity in relation to task structure and representational media available in different design environments (Berg, Chiu, and Hall, to appear), I want to draw attention to several preliminary findings about this kind of interactive work.

- Cycles of activity in design work (i.e., considering alternatives, making a change to the structure, and evaluating its consequences) appear to be organized, repeating at a surprising (for us, at least) rate of every 30 seconds or so in the groups we have been studying. We take this as evidence of high levels of engagement on the part of some students.
- Activity at the interface (pointing, running the mouse) are important in group work but do not determine what happens in the final group solution, since students who appear to be at the periphery of the action can play a determining role in proposing or evaluating changes to a final design.
- Talk and inference about quantities and their relations are strongly dominated by the interface features available to students (e.g., selecting named quantities, setting their values, and reading off outcomes from the display window).
- More systematic investigation of relations between quantities (e.g., outside wall/roof insulation and the long run cost to build and heat a structure over

time) sometimes occur during the course of design activity. However, these spontaneous mathematical investigations are rare, so our curriculum needs to deliberately propose activity structures that call for them.

Returning to the proportional analogy used to think about the *realism* of algebra story problems in the preceding section, my analysis of an "application" like the Antarctica Project is that we are able to bring a complex image of the relation between mathematical activity and design work outside of school (Cuff 1991; Gantt and Nardi 1992; Henderson 1991; Nardi and Miller 1990, 1991; Suchman and Trigg 1993) into a middle school classroom in a way that preserves aspects of the workplace that we think are likely to be important in learning about mathematical problem solving. At present, these aspects include:

- Mathematical problems emerge in functional contexts (i.e., mathematics is for something that students are otherwise working on).
- Social relations of accountability are built into the activity, in this case a competitive bid structure across student teams and natural divisions of labor within groups according to student interests and skills.
- Traditional symbolic notations for managing quantitative relations are mixed with a heterogeneous collection of tools (e.g., CAD and spreadsheet software) that present a variety of representational systems (e.g., linked scale models, a rudimentary database of named quantities, tabular arrays, and linked graphical displays).
- The extended scope of these design projects, by comparison with traditional mathematical applications, provides for sustained work on problems that have more than one solution and encourages students to treat their thinking as a valuable historical product.

These are complicated claims, and we have quite a ways to go before presenting a rigorous empirical evaluation of their validity. However, we feel this approach is certainly richer in possibilities for realism than disembodied travel on imaginary roads in the service of delivering a single mathematical structure (e.g., the CLOSURE problem). As we learn more about mathematical activity in actual work settings, another line of research that moves beyond the scope of this paper (Hall and Stevens, to appear), the aspects of "realistic" mathematical activity we try to reconstruct inside classrooms will no doubt change our view of what the curriculum should contain and how we should evaluate our students' learning.

Mathematics Incarnate

I want to end this short paper on realism with another analogy. Consider for a moment all the mathematical activity that goes on across this planet on some particular day, say a typical weekday but not over the holiday season. Let's say this corpus is made up of lots of little parts, problems solved here, arguments made there, demonstrations upheld or overturned in various places where people find the need to argue from necessity about this quantity or that relation. If we think about this corpus as an organism for a moment, the active body of mathematics incarnate, what parts of it do we really understand? At an assembly like this Algebra Initiative, or in the research and teaching that many of us do, which parts of the organism are we talking about, how broad is the scope of our analysis, and how much do we understand about how to reproduce this organism?

When I think about this honestly, I have to believe that we are really only scratching at the surface of the organism, maybe at the toenails of the body of mathematics, as it actually unfolded on our typical weekday. Some would prefer to think that we are working closer to the head of the organism, maybe studying how higher cerebral functions play out. But in any case, our view of what mathematics is, how it works, and how it changes is necessarily partial, usually static, probably self-interested, and even sometimes nostalgic. If there is a body of mathematics in daily activity, it keeps moving, it is shaped by what people actually do, and the ways in which they are able to share their work. My hope is that we will get better at finding and using images of real mathematical activity when teaching our children what this body is about and how it is changing.

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Algebra, The New Civil Right

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I would like to put out some ideas which face us, face the country, and also, of course, face this group. I find it ironic that mathematicians should be so centrally placed in a national issue. It seems that history has done you a disservice. It has put to you a task for which you are not prepared. Mathematicians would be the last people that I would turn to to organize the country. And yet, that's what it seems you have to do.

One way I think about the situation we're in is that we have this kind of moving of the plates which is associated with earthquakes. These plates move and sometimes they lead to these earthquakes and houses fall down and people get dislocated. So it seems that we are living in a time when two technological plates are rubbing up against each other, living through some of the earthquake-type phenomenon which has resulted from this—the movement from industrial technology into a new information age based upon computer technology.

There is a lot of social dislocation which is happening because of this kind of earthquake-like phenomenon. At least, that is how I see it. So, it's again ironic that the new technology puts mathematics and science into front and center and, therefore, requires that mathematicians play a role in stemming this social dislocation for which they were not prepared. How shall we characterize this? One way that I think about it is that there is a literacy issue for citizenship. I view myself as working in a tradition in the Civil Rights Movement which is not really well known. How many people here saw "Eyes On the Prize" or some segment?

Well, "Eyes On the Prize" is a documentary of the Civil Rights Movement. It was a 6-hour documentary dealing with the Civil Rights Movement in the early sixties put on PBS about 3 or 4 years ago and is replayed every year. You really need to see it. It usually

*Taken from his address at the SUMMAC II Conference held on November 6, 1993.

comes on in February, Black History Month time, but other stations do it at different times around the year. So you need to keep an eye out for it.

"Eyes On the Prize" puts forth a certain myth about the civil rights history. I think of it as our first visual history book. It's a product of the new technology. It puts forward the idea that we might have different kinds of people being historians; that the person with the camera will also be an historian of the future; that the new technology allows us to record enormous amounts of visual data in small spaces, and opens up the issue of "Well, what is a history book?".

I mention it because I took issue with Henry Hampton who developed "Eyes On the Prize." I said, "Look, Henry, there is a part of the history that you are not telling." Henry's point of view was that if someone didn't capture it on film at the time in which it happened, then it didn't happen That is, you couldn't tell about it. You couldn't put it in an interesting way on film to a mass audience. So, the "Eyes On the Prize" visual history book tells one myth about civil rights history. And it is a myth which deals with the history of great campaigns, March On Washington; Birmingham; the Voting Rights March in Selma, Alabama; Albany, Georgia; Freedom Summer—big campaigns of the Civil Rights Movement—and the person who came to symbolize such campaigns, Dr. Martin Luther King.

There is another myth about that history, and I consider myself to be a part of a legacy of that myth. That's a myth dealing with the organizing aspect. The part that did not get on film. Remember in the 1960s the nation was cutting its teeth with its TV programs. So, you had three major networks, and they were learning how to do TV. They learned using the Civil Rights Movement and its campaigns. But they did not pick up the organizing efforts which undergird those campaigns, and that's the tradition that I came out of.

Ella Baker, who was the person who helped found Dr. King's organization, sort of provided the model for us. The idea that leadership could be found in and among what we call grassroots people. That it was important to keep working with grassroots people to help develop the leadership from among them. That is the tradition that I came out of in the Rights Movement. It's an organizing tradition. It's a tradition which tries to stake out some problem around which there is consensus and builds to see if there is a way to find a solution to the problem.

In those days the issue was the Right to Vote, the question was Political Access, and associated with both of these was a literacy question around reading and writing. In these days there is another issue which is math and science literacy. It is associated with, not political access, but economic access. At the center of it as it is constituted in our society are you folks, mathematicians, and the question about algebra. I look at the work I am doing today as a continuation of the work that we did in the sixties. That is, certain people in Mississippi were serfs, people who were living in serfdom on plantations. They basically had no control over their lives—their political lives, their economic lives, their educational lives. So, within our industrial society we had this sort of microcosm of serfdom that we permitted to thrive, and the movement used the vote and political access to try to break it up.

It seems to me that we are growing these serf-like entities or neighborhoods within our cities today. We have within our midst, I think, a process of criminalization of our neighborhoods. I find an analogy to that with what we found in the Mississippi Delta plantations. We learned some things that we could do to change Mississippi. One of the things we learned was that if somehow there was a consensus that everyone agreed that we should do item "A," this consensus provided some base for strategy and action to try to work our way out of the problem. What everyone agreed to in Mississippi was that the vote would help.

So for a short period of time, all of the people who were acting to try to change Mississippi agreed to work together on a common program to get the vote. That enabled us to get resources from around the country to come and work with us, because they could all work on the same program.

Now, it seems to me there is a similar type of agreement today around math. That is, everyone agrees that if we can teach these children this mathematics, and let's suppose we agree about what the mathematics is, but if we can teach these children this mathematics, then we ought to. There seems to be universal agreement about this, that if we can do it, we ought to do it.

That's a basis. If we can get some consensus about how to work this, and granted, there is the issue that Paul Sally raised about "keeping the math honest" and the issue about "a truly objective standard," I think those were his words. But I think there is an opening here if somehow we can get consensus, and you people here are critical to the fashioning of such a consensus. So, I don't want to let you off your task. That is, it is really critically important that you fashion a consensus. It isn't fun and games. And no one else can do it. For good or for bad, this ball has been tossed in your laps. It isn't something I would have chosen to do. As I said, the mathematicians really are not prepared for this job. Your training as mathematicians didn't prepare you to organize.

But, the first job of the organizer is to flesh-out a consensus. You cannot move this country, unless you have a consensus around which you are going to move them. The country's too big, too huge, too diverse, too confused. Now, that's part of what we learned in Mississippi. We learned, and I am saying we learned it on the ground...running. We learned that if we could fashion a consensus which everyone agreed about, then we could get resources and develop strategies to try to work our way out of this situation.

Now, that's one thing, this legacy and a parallel between it and the situation we were in Mississippi, and I am just trying to show you how I look at this, these issues, and the right to vote; and political access; and political freedom; and this situation we have in the country; and the idea of citizenship which now requires, not only a reading/writing tool but a math/science tool.

The subject of math literacy and economic access, that is how are we going to give hope to the young generation. I think of them as imploding. Los Angeles exploded for a brief second and everyone got concerned, concerned about the trials and all of that. But those of us who live in these neighborhoods, we are watching them implode all of the time, everyday. They are imploding. The violence and the criminalization is people eating each other up inside. There are all the issues about band-aid solutions, about how do we patch up, how do we build more jails, how do we put more police on the street—working at the problem from the back end to try and keep it manageable—keep the lid on. But working it from the front end to try and put something in place which we know has to be put in place if there is going to be some light at the end of this tunnel rests with us. The front end of this problem rests with the people who hold the key to the mathematics education of the youngsters.

That's a new problem for you. It's a new problem for the country. The traditional role of the mathematicians has been to find the bright young math potentials and bring them to your universities and help them become mathematicians and scientists. It hasn't been a literacy effort. There is a difference between doing projects and doing systemic change. As a country, we don't know how to do systemic change. We don't have any track record. We can't point to any school system where we put through systemic change around the math education in that school system.

How are we going to do that? That's the issue that confronts us. I think the first step is to try and get hold of that as our issue. That's our problem and it is related to the much larger problem which is facing the country because it's in part going through this technological shift which is shifting the ground out from under us. My generation, we grew up with the metaphor of *E pluribus unum*: Out of many one. If you remember, those of you who saw "Eyes on the Prize," the metaphor at the beginning of each series is there are black people marching and then they change into the American flag. That's the prize. The American flag.

We were the last generation that had our eyes on that prize. The Civil Rights Movement was the last movement in this country to believe in the melting pot. That we were to create in America the identity of this American who is fashioned from all of the different peoples, somehow a new person, an "American" was to be created. The generation coming up now, they don't have that metaphor. They don't have any metaphor. That's part of their problem. That's part of our problem. They are a generation which has to create a metaphor of what it means to be an American. Does an American speak Spanish and no English? Does an American speak Chinese and no English? Is it possible to take many different peoples and find some common identity? That is, if we are going to retain our different cultural heritages, is there some unity in all this diversity? Is there enough there to hold the country together? It's a different question that this generation has to wrestle with. And it's a question that's driven by the same forces that are driving math to be a central element of school education along with reading and writing.

These are very deep problems. They are not going to be solved over night. But the question is, "Is there some strategy that this group has for making sure its contribution, which really now turns out to be central, counts?"

There is discussion about process and the National Council of Teachers of Mathematics (NCTM) *Standards*. Irvin Vance said that one of the good things about NCTM *Standards* is that they raise the question about math for everyone. The technological shift is also a shift from technology that deals with physical work to technology that deals with mental labor, mental products. You have in the industrial technology, machines trying to routinize physical labor. But computers aren't doing that. Computers are dealing with products of the mind, forcing the issue of critical thinking, because you are no longer trying to get the kids to learn how to do the "purple sheets," as Irvin said, "the drill and kill." Computers are forcing on education the issue that it has to produce graduates who can think in a critical way with quantitative data. This brings in process and something like the NCTM *Standards*.

So, we are not going to escape from this issue about process. But are we going to be able to handle it? Because if you say that, well they have to get to the content at some time. Who are we going to look to to tell what that content is, if not this group here? But do you have a consensus about it? Do you have some idea about what that content needs to be so that the children on the receiving end are viable? We are not just talking about professionals and jobs, we really are talking about democracy and citizenship. What is the content of the democracy going to be in this country?

These are heavy issues to lay on a group of mathematicians. But it seems again that history has played this trick on us and put mathematicians in this sort of critical place around the question of the democracy of this country, because this math/science tool is really assuming as important a place as reading and writing assumed in the old dispensation. And those people, we know who didn't have that tool, they really were not citizens.

Just think about Washington, DC. On the TV last night I think they said the murder toll had gone up to over 400 and nobody blinked an eye. They have these schools and the principals are saying, "Can you really expect me to secure this school?". It's not conversation about education, it's conversation about this criminalization of the schools.

I should say a word about the Algebra Project because I am able to come here and talk to you like this because we have this Algebra Project.

My family had been living in Tanzania for about 6 or 7 years, fleeing the political events of this country. Three of our children were born there and our youngest daughter, Malaika, was born in June of '76 about a month after we came back. We wanted to put our children in the public school system, but we wanted the school system to work for the children. So, within that arrangement my job was to look after their math. So, I undertook to working with the children as they went through the grade school years with their math.

We have two girls and two boys. Maisha is the oldest and Omowale, who is next, is here with us. I have to warn you he plays basketball.

We had a long discussion this morning about athletes. He is also trying to do a little math while he is here at GW (George Washington University). But there is an issue about coaches and math which surfaced this morning. Let me just say, when that issue surfaced, people were saying that the reason they use coaches to teach math is because they *need* math teachers. I stood up and gave a comment, an elliptical response that no, the reason they use coaches to teach math is because they think that they *don't* need math teachers. So, therefore, you can use a coach to teach math.

I was serious. Part of the reason we are in the trouble we're in is because we don't train elementary school teachers to be math teachers. So we use coaches.

Anyway, Omo and I have slugged it out over the years about doing his math and whether he had to be in the same math course that the other guys on his high school basketball team were in. Why should he be in a different math course than the rest of his teammates? If it was okay for them to take a certain course, why wasn't it okay for him?

So the Algebra Project grew as a family project. We slugged it out in the family. We're still slugging it out.

Then it got into the school system when my oldest daughter was ready to do Algebra, and they weren't offering it, so in a nutshell, I went into the classroom. I was able to do that because I had a McArthur Fellowship that came through at that time in 1982. I used it to teach Algebra. I went into the classroom for 5 years with that fellowship and taught Algebra as a parent volunteer to Maisha and Omo and then Taba and Malaika. To all my children and all of their classmates. Out of that then came the question of who takes Algebra, who gets access to Algebra, and how do we address the question of math literacy. Out of that came the Algebra Project which now has spread around the country and is trying to take root in different cities around the country.

I suppose if I think about it, I would think of it in a metaphor that the project is sort of like a young kid who is trying to stand up and is teetering and falling down a little and getting back up, falling down a little and getting back up. What I hope is that the project has the same kind of perseverance that makes young children keep getting back up. And then the same kind of perseverance that makes them eventually walk. So they keep walking until they learn how to walk. It doesn't really matter how many times they fall down, they keep getting up and walking.

Probably part of the reason that happens is they have a lot of people around them who are also walking. Unfortunately, we don't have that in the Algebra Project. That is, there are not a lot of projects around which are looking at this issue of literacy and how you make systemic change in schools. I am hoping that the project will have the kind of perseverance that young people have so that it will keep standing up every time it falls and eventually learn how to walk.

In the Algebra Project we do training. As Paul Sally, who helps us in Chicago, reminds us, there are two types of training that we need to do. There is training in specific curriculum, and for us we have our own little curriculum in which we do some training. Then teachers need training in the background of mathematics. Paul has helped on this second issue; he has his own history, as you know, of training teachers. But he has also been working specifically with Algebra Project teachers in Chicago. I would like to raise this issue with you because there is a specific question: How can the mathematics community relate to a project such as the Algebra Project?

Paul has two courses which he has developed, one on number theory and one on geometry. He offers each in 10 sessions, across the spring semester to middle school teachers in the Algebra Project. He offers it to other teachers as well. These are courses which are trying to address the issue of how middle school teachers fill out their mathematics background.

Part of what is driving this for the teachers is that the Algebra Project raises for these teachers the need to fill out their mathematics background in order to successfully engage the Algebra Project in the classroom. So the question is: Can we develop across the country a network of people in the mathematics community who would work on this kind of issue? In this instance, we are trying to pull together a seminar in the summer where we get mathematicians who come to the University of Chicago to sit down in a seminar with Paul and look at the issue: How can we develop courses or procedures or methods for doing the second kind of training with middle school teachers, in this case, in the Algebra Project? That is some kind of network around the country. So this is one of the projects that the Algebra Project is trying to do. I think that I will stop here.

Questions from the Audience

Question

Is the Algebra Project about Algebra or is it preparing students to be able to succeed in Algebra and, if so, what is it in the students that you are reaching to that the rest of us haven't been?

Answer

What is Algebra? When Omo was a sophomore and he was in Cambridge High School and he was taking Algebra II, of course he had taken geometry as a ninth grader because he had passed the citywide test for Algebra after his 8th grade, he came to me and he said: Why do I have to do Algebra II because all the other players on the basketball team are not doing Algebra II? Why do I have to do this?

We had it out head to head that this is what you have to do. You need to come out at the end of high school with your college basketball scholarship true, but you've got to be ready to go into college with the kind of background in these kind of subject areas.

There has got to be a product on the other end from my point of view. In other words, there is a way for the project to fail. If the project doesn't get students who go through the college-prep math sequences and come out on the other end and enter college ready to do mathematics for which they can get college credit, then it fails.

To do that, we are making an intervention at the 6th-grade level. For historical reasons, it began at the 6th-grade level. I think of the intervention as an intervention which is saying, if a student can count and if we can get their attention, then we can get them on this college-prep track. The hardest thing is to get their attention. But if we can figure out a way to do that, then there is a way to get them on the track where they can get through their middle school years and get ready to do the college prep math sequence in high school. That's the goal. That's what we are setting out as a goal. We are saying that this is a goal for all of our students. That is, there's got to be a floor. It's not the ceiling. It's the floor. We're not saying anything about the 2 million that I just heard we have in this country who are gifted and talented students, and we are not doing right by them.

I am not saying anything about what should be done for the 2 million. It's not the ceiling, it's the floor. We are trying to say we need to put a floor under all these students and the floor, as I mentioned, has to do with these enormous issues about citizenship and democracy.

Content-wise what we have said is that at the 6th grade we are trying to help students get a more general notion of number. So if you just look at what is driving the 6th grade curriculum, it is the idea that in arithmetic, students have a question in their mind about numbers which roughly is a "How many?" question. They pick it up when they learn how to count, how many fingers, how many toes, and we're trying to put another question in their mind about number which roughly is a "Which way?" question in a directional sense.

Having a more complex set of questions around their number question, we are trying to change their metaphor for subtraction and addition. So instead of having a "take away" metaphor for subtraction based on a "How many were left?" question, we want a metaphor for subtraction which is based on a comparison in opposite directions. How do we get direction into their subtraction concept and into their addition concept?

I really don't care whether people say that's Algebra or not Algebra, that's what we are trying to do in the 6th grade. We are trying to get them to make a shift in their number concept and in their subtraction and addition concepts. Then we have some other stuff we try to follow that up with: multiplication and division in the next grade.

This floor, the college-prep math sequence, is itself a moving target. But that's okay. The image I have here is that if you are trying to catch a bus that is moving, you just can't stand still and as it passes by try to grab hold. You'll loose an arm in the process. What you've got to do is as that bus is coming, you start running. And if you get close to the speed of the bus, then you can hop on.

The students today have got to hop onto one of those college-prep math sequences that are out there now. If they don't, that other one you guys are trying to think of as the one that should be in place for the 21st century will whiz right on by them. They won't have a chance in the world of getting on to that.

So in our program, what we are saying is that we need to get the kids ready for whatever is out there, they need to lock in on the idea that they should do the college-prep math sequence, and they keep locked in on that like some laser beam. However that sequence changes, however it evolves, whatever kinds of transmutations it goes under, they stick with it. And they pass the word on that this is what you've got to do.

We're just saying that this is the floor. It's got to be done. Currently 11 percent of the students in the country finish pre-calculus and 3 percent of those do calculus. We are saying whatever that is, that's the minimum. Everybody needs to do that. The country has to take seriously that this is now a literacy effort. It's not creaming. Everybody has got to do that.

So, here you come to a great crisis in belief. Who believes that all these kids can do this? The only thing that has given us some play is that we stepped in there with a little piece of curriculum at the 6th-grade level and said hey, these kids can do this much. And, if they can do this much then they can think of themselves and we can get them moving to do this much more. So that's my answer to that question. I'm not sure if it's an answer.

Question

In short, is there some elaboration on this which is available in published form, either some materials that you have developed or something that someone has written so that we can learn more about what it is that you are doing on the ground?

Answer

I will give out a phone number, but be warned we are not heavily funded. So, our office staff in Cambridge is really bare bones. The number is (617) 491-0200. That's the phone number of Algebra Project, Inc., which is a little small group at our central office. There is some literature which we will be glad to send you. We don't have back logs of the student text. There is a student workbook and text. What we do is negotiate with school systems who want to do the project so they get a copy and then they print it themselves. I don't know what would happen if we got an order for a large number of those textbooks. We are not really equipped to handle that.

There is another problem which is serious and on us. What happens to these kids in high school? It is the universities that have whatever kind of contact is going on with the high school teachers of mathematics. We are not even thinking at this point about trying to look at high school curriculum or training high school teachers. We can't do it. We're not equipped to do it. So, how can the university communicate with mathematicians? How can they help with this issue of working the high schools? Those are two really concrete issues that I think this group could help us with.

Question/statement

First, just a little story which perhaps might amuse you, when I told my husband that I was coming here to listen to you and have lunch he said with a smile, "You know that man is upsetting a whole neighboring state: Mississippi." He grew up for decades with Arkansas second to the bottom in education, literacy, per capita income, whatever. The saying was "Thank God for Mississippi." He said, "Bob Moses, he's going to upset this apple cart!," by the extraordinary work that you are doing in Mississippi. So that was a backhanded compliment.

Answer

Well, we might work in Arkansas.

Reply

Well, that was his next statement. Please come to Arkansas. Being a non-mathematician, one of the things that I have truly admired and respected about your work is that you have taken mathematics, through your work in using non-mathematical methods and areas and disciplines, and have made math more a part of a larger learning process. I was just wondering whether you had any plans to work with that non-mathematical world in a greater way to make it even more an integrated kind of studies.

Answer

There is an effort. We have a curriculum which we are getting ready to try to pilot which is called an African Drum and Ratio's Curriculum. The idea in this curriculum is that children learn certain competencies on the African drum, and some of the African drum family. Then they try to abstract certain mathematical concepts out of these practices. We are targeting that for 4th and 5th grades. In doing that we are thinking that this could drive a larger integrated curriculum. There is the obvious area of selecting some African or African/Caribbean culture which becomes a part of the study at that grade level.

Then there is the issue such as information systems. That is, the drum is an ancient system of transmitting information over distance. Therefore, it opens up the idea of a larger look at what are the different ways in which information is carried today as a part of a larger interdisciplinary structure. There is also the question of rhythm, which is so prevalent in music, and also in science and in life in general. So, how would you use this as a way to drive an interdisciplinary curriculum at that grade level around those concepts?

We are trying to see if we can get some funding to actually do a training this summer. We figure we need at least 6 to 8 weeks to do a training to try to develop a basis for that as a pilot.

This raises the general question about how to teach math. There was the question this morning about process and the NCTM *Standards*. If you think about epistemology and how to go about teaching math, I think we inherited a way of approaching it which says that we should start with what is simple. The problem is that most of what is taken as simple is also very, very abstract. The idea of building the mathematics curriculum around what is simple and abstract is one approach. That is not the approach which we are taking.

The approach that we are taking is along the other axis. That is, if you think of simple as opposed to complex, and abstract as opposed to concrete, we are looking to develop curriculum which finds the right level of complexity in concrete events. We are not building the curriculum around concepts which are simple and abstract. We are trying to build a curriculum around events which are concrete and complex. And the question is the right level of complexity.

So, when you look at this as a jumping-off point for curriculum, then you are in very different terrain. The question then is: What are the concepts that you want to teach and what are the events which can be the jumping-off point for those concepts? It is in that sense that we are trying to engage the non-mathematical world, as you call it, in a very general sense in this curriculum process. We are just exploring this. We have just, sort of, gotten our toes wet about how to go about developing this kind of curriculum.

It would seem that one of the advantages in doing this, or approaching curriculum this way, is that you have a chance at getting the student's attention. Then the issue is: If you can get their attention, how do you move them through to get the mathematics out? What we have done is maybe unique.

I came across the literature of the philosophy of mathematics in the writing of Quine. Some of you know Quine, he is a math logician at Harvard, Emeritus now. In his writing he talks about mathematical logic as involving a regimentation of ordinary language. So, he speaks about regimented language and regimenting ordinary language to develop the language of mathematics. There is this famous debate between Quine and Alonzo Church. Church is a math logician at Princeton, or at least he was when the debate was going on. They were arguing about the existential quantifier. Church was taking the position that the existential quantifier acquires its meaning from the axioms in which it is embedded. Quine was taking the position that the existential quantifier acquires its meaning from the logician who had in mind the ordinary language there is something such that and that's how it gets its meaning.

I look at that discussion of Quine and Church as the 20th century version of the 19th century discussion between Hilbert and Frege. Frege, who was the chief logician in the 19th century, wrote to Hilbert that the axioms of geometry are consistent because they are true. Hilbert wrote back and said, "My God, for as long as I have been teaching, I have said exactly the opposite. The axioms of geometry are true because they are consistent." It was Hilbert's point of view about mathematics that won out. Church then inherited Hilbert. So the meaning of the existential quantifier is embedded in the axioms. Quine is inheriting the 20th century version of Frege. Quine doesn't believe truth as Frege saw it. He has his own version of truth, but he is saying that meaning is attached to language and discourse and so forth.

The version of math that is taught to the kids in the school began after Sputnik in 1957. I was teaching at Horace Mann when Sputnik broke out. Right after Sputnik in 1958, I used to go down to Columbia University when Professor Fehr was holding forth, teaching students math and so forth, and I was a teacher from Horace Mann going down there to do courses. I was looking through the School Mathematics Study Group [materials] and Max Beberman's mathematics reform effort, the Madison Project and the Syracuse project, and I was taking all of these back to my classroom at Horace Mann and doing them with the kids. Looking back on all of that work, that work was predicated on the line of thought coming from Hilbert and Church and those people on the simple and abstract.

So, Beberman was actually teaching the children about equivalence relations: that they are symmetric and transitive, and reflexive. He was building the integers by partitioning the set of natural members and getting your ordered pairs of integers from the equivalence relations. I actually taught a course at the 9th grade level. I was moon lighting those texts, so I took Beberman's text out and had the 9th graders doing equivalence relations of all kinds, and this business about partitioning the set in constructing the integers. So, that's one approach.

The Algebra Project is looking at a different approach. It is not the simple and the abstract and consistency, but complex and concrete and truth. Science has these things which they call observation sentences. Physics may have these big theories and so forth, but it's got to have some kind of observation sentence against which you test these theories. My thinking is: How do we get the kids to develop a set of observation sentences which are the grounding for more abstract mathematics? Every Algebra textbook in the first few chapters has this little sentence "A minus B = A plus the opposite of B." The book somewhere says, in algebra, subtraction is the same as addition, more or less. The student, of course, has been spending years learning that that's not true. The student learned that subtraction is one operation and its metaphor is take away, while addition is another operation and its metaphor is piling stuff on. The book never says how these two metaphors are related. All the book says is that this syntax over here can be changed to this syntax over there. So that book is, in my mind, in the Hilbert/Church tradition, of the simple, abstract, consistency. It's not in the tradition of Frege, Quine and looking at truth. It doesn't give the kids anything. That little sentence I look at as a high level theoretical sentence in the Algebra I textbook. It's not an observation sentence. There's nothing that the kid can take from that and go test it against any experience and see whether or not it's true.

My problem is: How do we provide for the students a whole complex set of observations sentences out of which they could abstract this high level theoretical sentence which is "A minus B = A plus the opposite of B."? The kids need truth and experience. They need some level of complexity which they can use to test against events. That's how I view the problem of constructing curriculum in mathematics. Viewing it that way gets us into these other areas of knowledge and human endeavor.

Issues Surrounding Algebra

Elizabeth Phillips
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I come to this conference as chair of the NCTM Task Force on Algebra; but I wear many other hats concerning algebra. One of my responsibilities at Michigan State University is to upervise and coordinate the teaching of the Intermediate Algebra course. This fall (1993) we have 64 sections (~35 students each) of Intermediate Algebra; these students come to us with 3 and 4 years of high school mathematics. A greater part of my responsibilities are in the area of mathematics education, and many of my efforts in this area are related to the issue of why students are entering the university with 3 and 4 years of mathematics and yet have to start all over. Over the years this has led me to search for answers to the following questions:

- What is Algebra?
- What are the important ideas and processes in algebra?
- What does it mean to understand these ideas and processes?
- How do we teach for understanding?
- How do we assess these understandings?
- How do we implement change into the mathematics classroom?

These questions have played an important part of mathematics and mathematics education courses that I teach for preservice teachers and have played an important role in shaping several professional development projects: the Algebra Workshop (codirected with Glenda Lappan, 1986-88) was a 2-year project designed to develop a cadre of leaders to help implement a vision of algebra that was based on conceptual understandings; the Making Mathematics Accessible to All (directed by Chris Hirsch, 1990-present) is designed to help teams of teachers, counselors, and administrators from Michigan high schools implement a core curriculum in mathematics. Finally, I have been involved in several middle school projects; currently the Connected Mathematics Project, which is an NSF-funded project to develop a complete mathematics curriculum for the middle grades of which algebra is an important strand.

I continue to seek answers to the above questions as I wrestle with how to teach algebra to college students who come with many misconceptions and little understanding, how to educate teachers (both preservice and inservice); and how to develop a curriculum that provides a solid foundation for algebra, for middle school students and teachers.

Defining Algebra

Except for a brief sojourn into developing mathematics as a structure during the "New Math" era, algebra has traditionally been a symbol manipulation course designed to give students the manipulative tools necessary for the eventual study of calculus. This view of algebra and of math in general has not been sufficient and is one of the issues that provided the impetus for the NCTM *Standards* for Curriculum and Evaluation. However, as the NCTM *Standards* are being implemented, the issue of algebra quickly emerges: What is algebra? Why is it necessary? How should it be taught? assessed? Instead of presenting a vision or broad definition of algebra, the *Standards* describe algebraic processes that students should have mastered upon completion of a core curriculum. But the broad, general goals of the *Standards* need elaboration in order to be operationalized by a teacher in the classroom. Such a definition or vision of algebra will be difficult to create because curriculum leaders, mathematics educators, mathematicians and mathematics teachers (kindergarten through college) view algebra differently. Various definitions of algebra include:

- A course with some "mythical" body of knowledge;
- A part of mathematics;
- A generalized arithmetic;
- A symbolic language;
- A language that uses verbal, tabular, graphical, and symbolic forms;
- A study of relationships, patterns, and functions;
- A series of problem-solving strategies;
- A modeling process;
- A way of reasoning; and
- A formal structure.

Redefining Algebra

We need to define a vision of algebra that will serve a wide population and yet be specific enough to allow teachers to interpret the curriculum and make decisions. In developing this vision, some of the issues to be addressed include: How does technology affect the vision of algebra? What is the role of symbols and/or symbolic thinking in a technology-intensive society?

Symbol Manipulation

The issue of how much symbol manipulation students need is central to the debate about algebra. Many teachers claim that very little work with symbolic manipulation is needed; others, including many university mathematicians, are fearful that students will not have the necessary symbolic reasoning skills needed to progress in mathematics and science.

The *Standards* suggest topics that should receive decreased and increased attention (by teachers). On the list of decreased attention is "the use of factoring to solve equations and to simplify rational expressions." Many people will not argue with this statement; graphing calı ulators or computers are readily available for solving equations and for finding the factors oᶠ ı expression, if needed. Yet for many teachers this statement is still quite vague is it proposing that factoring not be taught? Students need to model problems using ılgebraic expressions and to show that different expressions for the same problem are equivalent statements. Students may also need to transform equations into a form that can be entered on a computer or graphing calculator. The distributive property plays an important role in being able to express problems algebraically, to show that two expressions are equivalent and to express an equation in a form suitable for the computer. Factoring is part of the distributive property.

Students who are college-intending need to be able to express statements in equivalent forms. Reasoning ·ith equivalent symbolic forms can very often reveal information that is not apparent in graphs or tables. The *Standards* suggests that we decrease the time spent in symbolic manipulations and increase the time spent on symbolic reasoning. The role of symbols and what we have come to know as symbol manipulations needs careful discussion. How much symbolic transformations or manipulations are needed to reason effectively with symbols? Consider the following example:

Example 1.

Problem: The cube pattern below is made from unit cubes. If the pattern continues, how many cubes will it take to build the next two buildings? How many cubes will it take to build the Nth building?

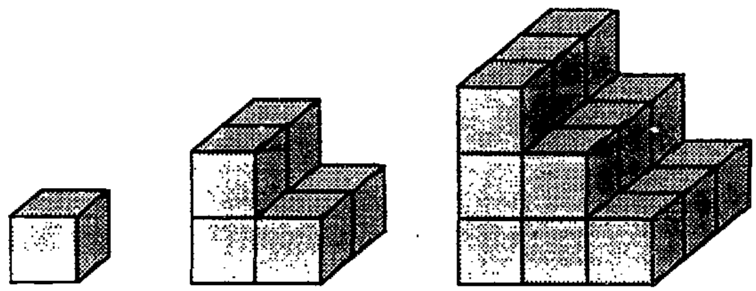


Figure 1

Solution: A class of preservice elementary teachers were given this cube pattern and asked to explain the reasoning they used to generalize the problem. The following is a sample of how students thought about the problem:

Students' Generalizations

1) 1 6 18 40 75... $N^2 ((N + 1))/2$

This student looked for patterns in the sequence of numbers which correspond to the number of cubes in each building.

2) 1x1 2x3 3x6 4x10 5x15... N ([N(N + 1)])/2

This student recognized that the number of cubes of each building was a multiple of a triangular number (or the sum of the first N counting numbers).

3) 1(1) 2(1 + 2) 3(1 + 2 + 3)... N(1 + 2 + 3 + 4 + ... + N)

This student recognized the number of cubes in the front face of each building was the sum of the counting numbers 1 to N and that there were N faces (layers).

4) (1 x 1 x 2)/2 (2 x 2 x 3)/2 (3 x 3 x 4, 2)... (N x N x (N + 1))/2

This student built a rectangular prism with dimensions N, N, and N + 1. The volume of the prism is N x N x N + 1, but this number is twice the number of blocks as in the original building, so he divided by 2.

5) (1)(1) (1 + 2)(2) (1 + 2 + 3)(3)... (1 + 2 + 3 +...+N)(N)

The number of rows x the number of blocks in each row

This student focused on the number of rows in the Nth building which is (1 + 2 +...+N) and noticed that there are N cubes in each row.

6) 1^3 - 0 2^3 - 2 3^3 - 9... N^3 - ((N[N(N - 1)])/2

This student built a cube of dimension N and then subtracted off the extra rectangular prism.

7) Kelly's conjecture: N^2 + ((N - 1)[(N - 1) + 1])/2 + (N- 1) [(N - 1) + 1](N - 1)/2

This student thought of the base as a square (rectangular prism with a square base and a height of 1) and then the rest of the building as two more prisms.

Discussion: How much symbol manipulation is needed to model a problem? Which symbol representation should we use? The first question is an open question, particularly if the pattern one observes is more complex as in Kelly's conjecture. While all the generalizations above are equivalent, several of them are quite different in form. The table and graph of each are identical, but each symbolic form represents a slightly different aspect of the cube pattern. So which representation one chooses depends on the particular aspect of the pattern that one wishes to observe.

On a simpler note, the two expressions, 1 - 1/n and (n - 1)/n are equivalent expressions, and they have the same table and graph. However, symbolically each expression emphasizes a slightly different mathematical application.

When asking students to find the number of handshakes that will occur between N people, some students will argue that each of the N people will shake hands with N - 1 people, but that each handshake is counting twice. Thus, the number of handshakes is (N(N - 1))/2. Other students might observe that the first person will shake hands with N - 1 people, the second person will shake hands with N - 2 people, etc.—so that the total number of handshakes is (N - 1) + (N - 2) +...+1. Thus, we have ((N(N - 1))/2 = (N - 1) + (N - 2) +...+1—equivalent expressions, but each represents a different way of thinking about the problem.

The question "How much symbol manipulation does a student need?" is tied to the broader questions of "How can we develop symbol sense?" "What does it mean to have symbol sense?"

Representations in Patterns and Modeling

With the availability of small, hand-held computers, it is equally important that a broader view of the role of representations in studying patterns and modeling be included in a vision statement of algebra. The role of communicating ideas about patterns needs to include not only symbols but also tables, graphs, and words. The advantages and power of each need to be appreciated. Consider the following example:

Example 2.

The following problem is easy to translate into symbols. In fact, students can probably do most of the problem with only arithmetic. For these reasons this problem (or some other similar setting—walking, running and bicycling rates or the cost of renting rollerblades) provides an appropriate first entry into developing the characteristics of a linear situation. If we explore the problem by using tables, graphs, and an equation, we see that each representation provides insights into the rate of change (slope) in slightly different ways.

Problem: The We Try Harder (WTH) car rental agency charges $23 a day plus 20 cents per mile (plan 1). How much does it cost to drive the car for one day?

Solution: The cost depends on (or is a function of) the number of miles we drive.

Using a table: We can set up a table (figure 2) for the number of miles driven in one day and the corresponding costs.

Miles Driven in one day	Cost WTH in dollars
0	23
100	43
200	63
300	83
400	103
500	123

Figure 2

The table is easy to read. By observing patterns in the table we notice that as the number of miles increases by 100, the costs increase by $20. This constant difference or rate of change between the two variables is the main characteristic of a linear situation.

Using a Graph: If we graph the set of data in the table, we obtain a straight line. The graph is more complete than the table. It is easy to read either the cost or the number of miles. The rate of change between the two variables or slope is seen in the steepness of the line and in the difference between corresponding coordinates of two points.

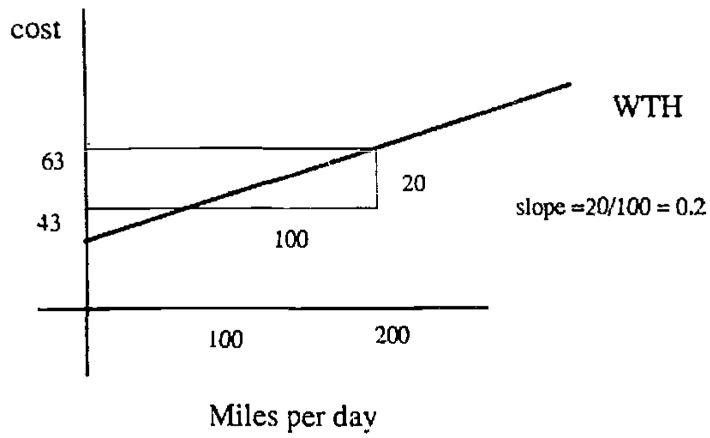


Figure 3

Using an Algebraic Equation: The relationship between the cost and miles can be expressed as an equation. If we let Cost(m) represent the cost to drive m miles in one day, then

$$\text{Cost}(m) = 23 + 0.20\,m$$

The rate of change or slope shows up in the equation as the coefficient of the variable m. The constant, 23, in the equation can be related back to the graph as the y-intercept or the point (0 miles, $23) and in the table as the fixed per day cost of $23.

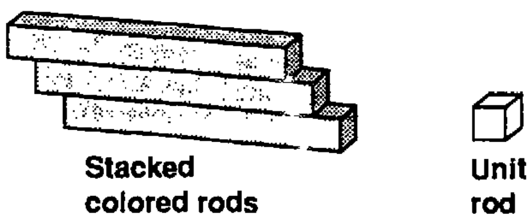
Discussion: The slope is represented quite differently in each representation. Each representation helps to shape an understanding of slope, y-intercept and the relationship between the two variables. If we change the parameters of the problem and look for patterns, we continue to develop an understanding of linearity. For example, if the WTH Agency drops the fixed charge of $23 a day (plan 2), then entries in the cost column in the table (figure 1) shift down by 23. But the rate or the slope stays the same. For every $20 we can drive 100 miles or $20 per every 100 miles or ($20)/(100 miles) = (20 cents)/mile.

If we graph the data for this new plan, we obtain a line which is parallel to the original line, but goes through the origin. This line has the same steepness or slope as the line obtained for the data in the first plan. The two lines are parallel. Similarly, we can change the cost per mile and observe the changes in the three representations.

An important step in understanding linear functions is to be able to recognize the linear relationship in a problem; the rate of change between the two variables is constant in various representations. An equally important step in recognizing linear situations is to study examples that are not linear. Comparing linear growth and exponential growth are interesting situations to use for this purpose.

The power in doing mathematics comes in recognizing a familiar pattern. However, in Example 2 it is easy to translate from the words directly to symbols. Students need the opportunity to investigate situations that can not be directly translated from words into symbols. As an example, consider the problem of investigating the surface area of a stack of Cuisenaire rods of the same color (figure 4).

Example 3



Each rod is 4 units long, one unit high and one unit wide.

Figure 4

Discussion: Some of the symbolic representations that students have used are:

$$[2 + 4(n - 1)](1) + [4 + 2(n - 1)](4)$$

$$18 + 12(n - 1)$$

$$2[4(n) + 1(n) + 4(1)] + 2(n - 1)$$

$$4[2n + 2] + 2(2n - 1)(1)$$

$$12n + 6$$

The pattern is linear, but it is not observed from the statement of the problem. Students either generate a table of values and recognize that it is linear or they observe some features of the patterns and translate it into algebra—similar to the processes used by students for the cube pattern in Example 1.

Technology Manipulation

The availability of graphing calculators has caught the attention of mathematics teachers in ways that no other piece of technology has done in the past 30 years. NCTM has made three important statements concerning the impact of technology on the mathematics curriculum:

- Some mathematics becomes more important because technology requires it.
- Some mathematics becomes less important because technology replaces it.
- Some mathematics becomes possible because technology allows it.

We need to address these statements in more specific ways. What mathematics is more important? What mathematics is less important? We also need examples of curriculum that use technology in ways that provide deeper understandings of important concepts. It is crucial that we address these issues soon. Some curricula are being implemented that claim to use technology, but in fact the focus is on "how to use the graphing calculator." There is a danger that we could be replacing abstract symbol manipulation with equally abstract algorithmic techniques on how to use the graphing calculator (or computer).

The role of technology needs careful discussion; "What does it mean to have graphical sense?" "How do we develop representational sense?"

Toward a Teaching Strategy

There is no area in which the study of patterns is as fundamental as it is in mathematics. Mathematicians observe patterns; they conjecture, test, discuss, verbalize, and generalize these patterns. Through this process they discover the salient features of the pattern, construct understandings of concepts and relationships, develop a language to talk about the pattern, integrate, and discriminate between the pattern and other patterns. When relationships between quantities (variables) in a pattern are studied, knowledge about important mathematical relationships and functions emerges.

To help students learn the important concepts and processes of algebra, it is important to start with problems that embody an important concept or strategy. As the students seek answers to the problems they observe patterns and they reason and communicate about these patterns to develop understandings. The Car Rental Problem (Example 1) is an example of starting with a problem to introduce the idea of linearity.

When students recognize this pattern in another situation, they very often will remark "this is the Car Rental Problem!"

Problems embody the concept or the process. Solving a problem leads to more than an answer—it leads to knowledge and understanding.

Defining Algebra for All Students

Algebra has played the role of "gate keeper" for far too long. The focus should be on what all students will know at the end of high school. A vision statement must stress the knowledge of the basic concepts and processes of algebra that are essential for all students.

- What core of algebraic knowledge will serve students graduating from high school who will be going directly to the work force, or to some technical college or to a university, including liberal arts and science majors?
- What teaching strategies are needed to ensure that all students have access to this knowledge? How will students be assessed?
- Does the vision of algebra allow for the development of algebraic knowledge throughout the K-12 curriculum?

The following working definition of algebra was developed by the NCTM Algebra Task Force. We welcome your reactions.

Toward a Working Definition of Algebra

After a lengthy discussion, the NCTM Algebra Task Force committee agreed that there is a need for a well-developed and clearly articulated national vision of algebra which establishes the necessity of algebra for all students, and an elaboration of that vision with specific discussions of the important ideas, strategies, and processes it includes. The Task Force recommends that NCTM sponsor a series of conferences to present a national vision of algebra and develop strategies for its implementation. The following definition was drafted by the NCTM Algebra Task Force:

A. **The Vision:** Algebra is a study of patterns/relationships and functions which uses a variety of representations including verbal, tabular, graphical, and symbolic.

B. These representations make it possible to:

- Use technological tools effectively;
- Communicate, analyze, and interpret information;

- Formulate and solve problems by collecting, organizing, and modeling data;
- Describe important patterns of behavior of families of functions; and
- Recognize, interpret, and use discrete and continuous relationships.

C. The vision statement must include the following non-negotiable principles:

- A national goal that every student will graduate from high school with the algebraic skills and knowledge needed to function in our technological society.
- Algebra will become a K-12 strand.
- Teachers must use a variety of teaching and assessment strategies and tools to teach and assess algebraic ideas/concepts.

This vision of algebra should be followed by an elaboration of the vision. For example, the elaboration should demonstrate how problem-solving, reasoning, modeling, and structure fit within the vision. The elaborations should also suggest what functions and relationships are considered basic; why they are important; and what is important to know about these relationships. The Task Force has suggested that linear and exponential functions (growth) be part of the core curriculum and that quadratic and simple rational functions be studied more informally. Any investigation of important features of functions should emphasize rates of change, optimization, and important local and global behaviors.

Addendum
Issues Surrounding the Topic of Algebra

The following issues were identified by the NCTM Algebra Task Force:

General

* Why is there such a large population of remedial algebra students at the college level, students with 3 and 4 years of high school algebra?
* Why is there such a large need for math remediation at the business level?
* Why are many states mandating algebra for all students—what algebra?

What is Algebra?

* What is algebra?
* What are the big ideas? concepts?
* What is the core curriculum?
* What mathematics (algebra) should students know and be able to do when they leave high school?
  * College-intending
  * Technical 2-year colleges
  * Industry and business
* Is algebra a course or a strand in the K - 12 curriculum?
* What is the role of technology, symbols?
* What is the role of concepts, strategies, and processes in mathematics (in algebra)?

Assessing Algebraic Knowledge

* How do we assess students' knowledge of algebra?
  * What do we do with the consistent low test scores?
* Can every student do algebra?
  * How do we overcome tracking—or is some tracking acceptable in the transition period providing that students at all levels can truly cross tracks and are held to the same standards?
* How can we help students to appreciate that mathematics requires work and perseverance?

Teaching Algebra

* How do we package algebra?
* How do we deliver algebra?
  * What are effective pedagogies?
  * Are there different learning styles?
  * What are they and what instructional systems do we need to accommodate these learning styles and the needs of a diverse population?
* How do we bring excitement to the teaching and learning of algebra?
* Who is teaching first year algebra?
* How do we help teachers overcome the "comfort level" they have achieved in teaching a "skill-and-drill" curriculum.
* How will we provide the extensive professional development that is needed to help teachers adapt to a curriculum based on the NCTM *Standards*?

Educating the Public

* How do we educate the public—administrators, parents, politicians, businesses, industry?
* What is happening in other countries—is this useful information?

Is Thinking About "Algebra" a Misdirection?

Alan H. Schoenfeld
University of California at Berkeley

Many people, including I imagine a quorum of the people at this conference, believe that much of the problem we face in mathematics instruction has been caused by focusing on algebra as the narrow band of symbol-manipulation techniques that have traditionally constituted ninth-grade mathematics courses. It can be argued that focusing on algebra qua algebra is a misdirection, and that we should be thinking instead of the broad set of symbolic understandings and uses of them that students should develop in their experiences with mathematics. So, my main issue is: What should we be thinking about within the convex hull of algebra-related mathematical stuff? My goal is to raise a few basic questions related to that issue.

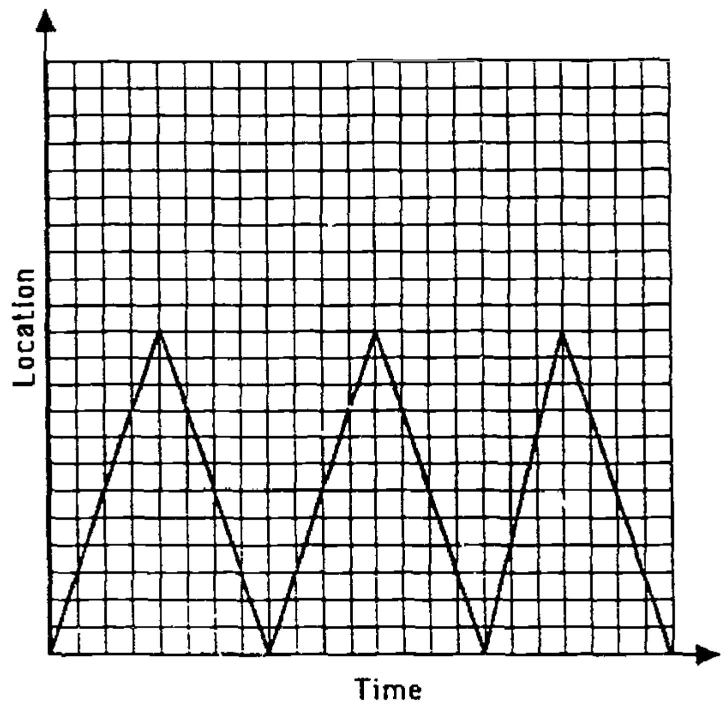
Rather than speaking in abstractions, let me give some concrete examples of problems that I see as being appropriate and algebra-related for our students. Problem 1 comes from an experimental 6-week unit my research group built as an introduction to (mostly linear) functions. Problem 2 is pretty standard, and the source of some controversy related to an innovative high school mathematics curriculum. Problem 3 is one I use in my undergraduate problem solving course, but there's no reason it couldn't be a mainstay at ninth grade. I'll give a student's solution to Problem 1, because you should see what ordinary 9th graders can do. What one can do with Problems 2 and 3 is pretty obvious.

Problem 1.

This was a homework problem from the fifth week of our unit, so the students had only a brief introduction to graphing prior to working it. They had worked with various real world interpretations of linear and nearly linear functions, and had drawn graphs of various real-world phenomena (e.g., a distance-versus-time graph of a mechanical toy walking on a more or less straight path).

The problem statement and a solution produced by a student whose grades were more or less average to this unit follow:

Write a story that could describe this graph.



"The Sea Hog (Militaries latest lard powered submarine) is prepared to go. The Admiral who will navigate the vessel is explaining to the crew why lard is a better fuel in terms of defense cuts. The Sea Hog sets off. By the time it is 12 miles out to sea the Admiral realizes that he has carelessly left the weapons back at the port. He sets back at the same speed he set out at, about 3 MPH (lard is not a very good submarine fuel as far as speed is concerned.)

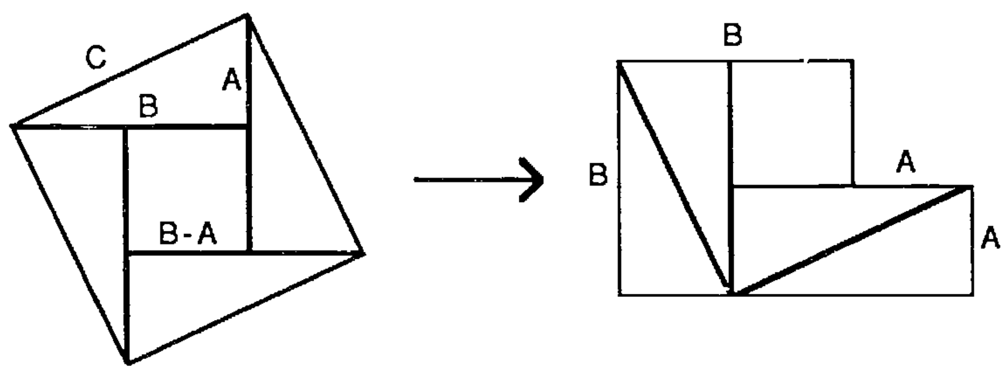
"The admiral had called ahead and asked for the weapons to be ready on port by the time he returned to avoid delay. When he had reached port he didn't even need to stop or slow down. They just threw the weapons in.

"The sea hog sets out again at about 3 MPH, until again about 3 miles from shore. Then it was to the Admiral's surprise that this was not really his crew. His was back at the dock. When the Admiral had returned the submarine didn't even stop or slow down. The wrong crew jumped off and the right crew jumped on. After 3 hours it was 12 miles out that it was discovered that there wasn't enough lard to make the trip. (I can't tell you where the trip was. TOP SECRET you know.) They head back and decided to try again tomorrow."

Problem 2.

Consider the standard figures used to give a purely physical proof of the Pythagorean theorem, such as the one that Bronowsky used for the "Ascent of Man." One figure shows a square (B-A) on a side surrounded by four (A,B,C) right triangles ($A < B < C$), in such a way that the arrangement gives a square of side C. The second is a rearrangement of the same pieces, where the figure can now be seen to comprise two squares, one A on a side, and one B on a side. This is a "physical" proof that for this one figure (and presumably all like it)

$A^2 + B^2 = C^2$, for which you need only know that the area of a square of side S is S^2, and some subtraction.



There is also a lovely symbol-manipulative proof requiring the first figure only. Produce it, and explain the connection.

Problem 3.

Take any three-digit number and write it down twice, to make a six-digit number. (For example, the three-digit number 789 gives us the six-digit number 789,789.) I'll bet you \$1.00 that the six-digit number you've just written down can be divided by 7, without leaving a remainder.

OK, so I was lucky. Here's a chance to make your money back, and then some. Take the quotient that resulted from the division you just performed. I'll bet you \$5.00 that quotient can be divided by 11, without leaving a remainder.

OK, OK, so I was very lucky. Now you can clean up. I'll bet you $25.00 that the quotient of the division by 11 can be divided by 13, without leaving a remainder? Well, you can't win'em all. But you don't have to pay me if you can explain why this works.

Now, I'd argue that our kids should be able to do all of these (and many more of considerably more diversity), and that they should pick up the skills to do so somewhere in the K-10 curriculum. The problems are all, for me, deeply connected to some of the symbolic operations we call algebra. If our kids had the skills to do these—not as replicates of things they'd seen, but as problems they could think their way through—I'd argue that they have some reasonable understanding, and that they've developed have some of the analytical and representational tools that would serve them well later on. If that's the case, how and where should they pick up such skills? How do we provide the instructional contexts, and support structures (for teachers as well as students) to enable them to develop such understandings?

Thoughts Preceding the Algebra Colloquium

Zalman Usiskin
University of Chicago

Betty Phillips began her paper with a description of the many hats she wears when thinking of algebra. It may help people understand what is said here if I indicate not my hats but experiences. First, I went to the University of Illinois as an undergraduate. There I learned the UICSM approach to algebra, which was rather different than the approach I had been taught in school, which was careful but skill-driven. I learned that being able to *do* mathematics was not the same as understanding it; in UICSM, understanding meant to be able to give reasons for what one was doing and to use precise language. Second, I taught algebra in high schools in the sixties out of what was for two decades the canonical school algebra: the Dolciani series. They were very well-written texts that embodied much of new math. Third, when Art Coxford and I worked in the late sixties to develop a school geometry course utilizing transformations, we saw immediate applications to school algebra. As a direct result, in the early 1970s I wrote a second-year algebra text (*Advanced Algebra with Transformations and Applications*) that utilized transformations, matrices, and groups throughout. Fourth, it was clear that this group-theoretic approach was not the approach to take to first-year algebra. Following the beliefs of my colleague Max Bell, in the midseventies with the help of NSF funding, I developed a first-year algebra course (*Algebra Through Applications with Probability and Statistics*) in which applications were used to develop the algebra. Understanding now took on a different meaning, namely, the ability to apply mathematics to situations in the real world. Fifth, from 1979 to 1982, Max and I worked on materials for teachers on applying arithmetic (*Applying Arithmetic: A Handbook for Teachers*), because we realized that a major reason for the difficulty students had in applying algebra was that they could not apply arithmetic. There is a chapter in that book on uses of variables because we realized that one cannot adequately discuss arithmetic without variables. Sixth, I was just about to embark on materials for students for the year before algebra (and geometry) when we learned of the interest of the Amoco Foundation in funding a large-scale project to improve school mathematics in all the grades K-12. So began UCSMP, which has afforded me (and many others) the opportunity to put together ideas learned from our earlier ventures. Algebra is a force in all six of the UCSMP secondary (grades 7-12) texts, and the main subject matter of two of them: *Algebra* and *Advanced Algebra*. For students who are at or above grade level in mathematics, these are designed to be taken in 8th and 10th grade. (Judging from teachers who attend our conferences, about half of UCSMP *Algebra* students in non-state adoption states are 8th graders The percentage is much lower in state adoption states.) We recommend that better-prepared students take these courses earlier; more poorly prepared students take them later. And seventh, we are now working on the second editions of UCSMP texts, which we hope are influenced by the latest in curriculum thought and by the experiences of teachers and students in the thousands and thousands of schools using our first editions.

Levels of Curriculum

Certain conceptions may be useful to us. One conception is the levels of curriculum from SIMS (Second International Mathematics Study):

- Ideal curriculum—as represented in documents like the Standards;
- Implemented curriculum—what is taught; and
- Achieved curriculum—what is learned.

We may wish to add to this:

- Available curriculum—what is found in textbooks and other materials; and
- Tested curriculum—what is found on various evaluation instruments.

It is dangerous to use the worst of the implemented and achieved curricula in our country to justify changing the ideal curriculum unless we compare it with what we think might be the worst of the implemented and achieved curricula under a new ideal curriculum.

Conceptions of Algebra

What is (are) our conception(s) of algebra? Betty Phillips has given 10 conceptions in her paper, but then suggests a vision that I believe is far too encompassing. "Algebra" in it could be replaced by "analysis" or even by "geometry!" I prefer to say that there are many ways to describe patterns. I have difficulty calling all of these algebra: to me tables are arithmetic; graphs are geometric; formulas are algebraic. I am reminded that many students think of an ellipse only as an algebraic figure because they are introduced to it in a formal way by an equation, and they think symmetry requires that one have a coordinate system. Thus, sometimes algebra demeans the visual world as the origin of many concepts. Algebra is fundamental but it is not all of mathematics.

I think four of Betty's conceptions are major:

- Algebra as generalized arithmetic;
- Algebra as a means to solve problems;
- Algebra as a study of relationships; and
- Algebra as the study of mathematical structures.

Underlying all these is algebra as a symbolic language that describes both real and hypothetical patterns.

Algebra as a Language

My experiences have led me to a large number of beliefs about school algebra, too many to list here. The most relevant, however, may come from the notion that algebra is a written language dealing with the use of letters or other symbols (not words) to stand for elements of sets. The following then follow because it is a language:

- Algebra is best learned in context.
- Almost any human being can learn it.
- Familiarity is more easily acquired when one is younger than when one is older.

Should We Worry Whether Algebra is a Course or Not?

Some may wish us to discuss the question whether algebra should be a course or not. This to me is a red herring. Algebra is today found in all mathematics books above grade 6 and in many elementary books as well, whether or not they have "algebra" in their title. In many states it is state law that there be an algebra course (or two), and one will change the curriculum faster by keeping that course and changing i. than by attempting to remove the course. The question should be: What algebra do we wish students to learn? Let the dozen NSF-sponsored projects, other curriculum developers, and state and local educators work on the packaging.

Where Might We Begin?

It used to be that we thought of algebra as everything associated with variables that stand for numbers or operations. In other settings, I have written about conceptions of algebra and related uses of variables. But I think to get into that level of detail at this conference might lead us off track. Variables are in geometry and every other branch of mathematics. I think it would be better for us to begin with the traditional conception(s) of algebra, to begin where most of our readers are at. And then, if we wish to change the conception(s), we will at least have a point of departure. In this regard, I would say that Betty has used the word mythical in the definition of algebra to which it least applies! In my experience, the body of knowledge that has been traditionally called "algebra" is about as well-defined as any body of knowledge we have in mathematics.

What Issues Might We Address?

Betty's enunciation of issues is most helpful. I would hope we also could address some of the myths I mentioned in my earlier notes: (1) that algebra is only useful for future scientists or engineers, so it needs not be taught to all; (2) that algebra is inherently difficult, that it cannot be learned before 9th grade except by the gifted; and (3) that "true" algebra is the manipulation of symbols. There is a fundamental down-to-earth question that we should at least raise, if not try to come to consensus. What algebra should everyone have an opportunity to learn? If not everyone, then who should have an opportunity to learn: factoring of quadratic trinomials; manipulation of rational algebraic expressions; complex numbers: matrices; vectors; manipulations with radical expressions. The starting point for this could be the lists in the NCTM Standards.

Where Might We End?

Judging from the conceptual framework provided by Carole Lacampagne and enunciated in the list of invited participants, our charge in Working Group 1 overlaps with that of Working Group 3. We might quickly try to come to some boundary conditions that would keep the groups from discussing the same things.

We also necessarily overlap with Working Group 4, and not just in the ways that Carole has mentioned. The boundaries between algebra and calculus are fuzzy to anyone who has graphing technology. Some of the new curriculum projects ask students in their first-year course in algebra to answer questions that we did not consider until calculus. Furthermore, the needs of the calculus curriculum are a powerful force in determining what is taught in school algebra. Without a change in the thinking of college-level mathematicians toward school algebra, it is unlikely that any changes in school algebra will be sustained.

What Experiences Can We Bring to Bear?

I assume that Betty's last question was rhetorical—what happens in other countries is very useful information. The Soviets introduce variables in the early elementary grades with a great amount of success and, like the countries of the Far East, give a concentrated algebra experience to all students beginning in grade 7. Let us realize that some countries are ahead of us when it comes to teaching more mathematics to more students, and capitalize on their experience.

One thing we might want to recognize is that, at least in Japan and Taiwan and China, students are sorted by exams into schools at grade 7, and the algebra that is taught in some schools in grades 7-12 is more advanced than the algebra taught in other schools at the same grades, even though all schools follow the same national curriculum. That happens also in the United States, though with us it is unofficial. What goes for "honors" algebra in some U.S. schools would be a typical course in others. When we speak of "Algebra for All," would we really be content with the same algebra for all?

Thoughts Following the Algebra Colloquium

There were really three levels and two types of algebra being discussed at this conference. The two types of algebra were wonderfully explicated by Lynn Steen in his final remarks. One type leads from functional thinking to functions to calculus to analysis. The other type leads from patterns to manipulation (but this is really in both—Lynn should have replaced it with "properties") and then (through an intermediate step felt by Lynn to be missing) to linear and abstract algebra. This intermediate step is the introduction of transformations and matrices and vectors (for linear algebra) and properties and examples of structures (for algebra). I am pleased that we do both of these in the UCSMP curriculum.

Jim Kaput's paper for Working Group 1, and Alba Thompson's remarks at the colloquium stressed only the first type, that involved with functions. In fact, Alba distorted the situation by showing a transparency purportedly summarizing the NCTM Standards statements regarding algebra, but in fact deleting all but the statements concerning functions. On the other hand, Victor Katz's presentation emphasized only the second type of algebra.

I myself believe that it is the second type of algebra with which we should have been primarily concerned, and that the other is properly called analysis, but it is obvious that there is overlap both in school mathematics and in mathematics itself. The three levels were:

1. The introduction of algebraic ideas and algebra itself, roughly covering the ideas that are now encompassed in first-year algebra but with many thoughts for major change both in content and in delivery;
2. The honing of skills (can I say the word?–it was noticeably absent from the discussion) and the elaboration of mathematical relationships and applications that is now covered in the later years of high school or in remedial courses at the college level;
3. The linear algebra and abstract algebra that are taught as post-calculus courses (usually) at the college level.

Of these three levels, I heard very little discussion concerning level (2), though this may have occurred in Working Groups 3 and 4.

The significance of this gap is that, though we spoke of "algebra for all," we spoke almost entirely of access to the ideas, not of success within them. There was much talk about how, if we just changed our approaches, students would be better served, but there was little talk about the next step, and that is, somehow identifying what students who leave high school would be able to do with this "new algebra" that they could not do before, and the implications that would have for colleges. In particular, there was almost no discussion about what these students might be able to do outside of the realm of functions (i.e., in the second type of algebra I referred to above). For many, it will be hollow success if we have more students who pass through some sort of algebra filter if, once they have passed through, there is no tangible increase in their understanding or ability to do anything.

WORKING GROUP 2

Educating Teachers, Including K-8 Teachers, to Provide These Algebra Experiences

Members

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Clare Gifford-Banwart, U.S. Department of Education
Suzanne K. Damarin, Ohio State University
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James Pratt, Johnson Space Center
Tina Straley, National Science Foundation
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Report of Working Group 2

Alphonse Buccino
University of Georgia

Vision of the Achievable

What is the meaning of "Algebra for All" and why should there be a colloquium focused on algebra? For America and many other nations, the characteristic feature of formal education in this century is the provision of increased access: over the last hundred years, increasingly larger proportions of our population have spent more and more time in schools and colleges. This is reflected in the expansion to near universal access to high school education in the early part of the century, followed by a great increase in access to higher education in mid-century. The phenomenon of increased access appears to have leveled off over the last 25 years or so and, as an issue, has been replaced by concern over outcomes and results—specifically, how to help larger proportions of our citizens attain more from access to education, how to bring them to higher levels of achievement in knowledge and skill.

This shift in focus from access to outcomes is part of the contextual scene that led to the Algebra Initiative. The conceptual framework and the discussion at the conference implicitly and explicitly underscore the focus on algebra as a barrier, a stumbling block, to higher achievement. Thus, a major implicit focus of the colloquium is the idea of elevating for all education professionals their *vision of the achievable** and strengthening their professional commitment to Algebra for All.

Two major barriers are associated with these ideals: (a) the intrinsic difficulty of teaching and learning mathematics; and (b) diversity and multicultural issues, barriers to particular groups marked by race, gender, class, first language, or other societal categories. Both these barriers are addressed by the NCTM Standards. However, the standards and subsequent reports must be supplemented by guidance for identifying the nature of these barriers and eliminating them from classroom settings.

*The use of this phrase here is suggested by observations of Philip W. Jackson in Chapter II of *Fundamental Research and the Process of Education*, a report published by the National Academy of Sciences in 1977.

Redefining Algebra

It is in the foregoing context that across the state and nation the call for credibility in school mathematics is focused on a thing called *algebra*—emphasized by the call of algebra for every student. First, this is a response to the recognized need for increased mathematical reasoning and application by all members of society. No longer can the work force function with only a few individuals with mathematical sophistication beyond arithmetic! Second, it is the recognition of the inappropriate use of mathematics, specifically algebra and calculus, as gate keepers in our society. Tracking in mathematics is a major force in effectively excluding individuals from many academic and vocational pursuits.

We must involve all of our constituencies—students, parents, leaders from business, and educators at all levels—in addressing the challenge of algebra for all students. The task will be a difficult, time-consuming one—for every one involved—but we cannot ignore the need. The task will involve major changes of beliefs about what algebra is, how one does algebra, and who can learn algebra. Technology has opened up mathematics to individuals in just about every line of work. We must challenge the widely held American belief that mathematical ability is hereditary.

School algebra evolved as performance of symbolic algebraic procedures, such as: solve, simplify, factor—typically as isolated, symbolic manipulations. We must do something different! The vision of algebra for all students must be clarified and communicated openly within the education community and beyond. First, what is the algebra deemed necessary for all students? Second, how do we change expectations about what algebraic performance is: for teachers of algebra? for students? for parents? for post-secondary gate keepers? for employers? for district administrators? Third, how do teachers, schools, districts, and states monitor student performance in using this algebra? In the fast changing world of mathematics, science and technology, there is no single answer to these questions. Lynn Steen, director of the Mathematical Sciences Education Board and former president of the Mathematical Association of America, states: "For most students the current school approach to algebra is an unmitigated disaster. One out of every four students never takes algebra....And half the students who do...leave the course with a lifelong distaste for mathematics."

Algebra as a Human Endeavor

Numerous groups and projects are making recommendations on algebra for all students. Most call for the immersion of algebra skills in an applied and conceptual curriculum. They suggest the building of an algebra curriculum extended over many years, from the primary grades on, where students search for and describe mathematical patterns. In such a program, algebraic reasoning, descriptions, and symbols are used to describe the world around us.

Many of the projects recommended the concept of function as the unifying theme. Special attention is given to building tables of data and their graphs from observed patterns and experimental data. Experimental data collection from the sciences, social studies, and real world call for comparison between two traits or characteristics in contextual problems. How does one trait vary as we change the other? This is a powerful meaning of the term, variable, which is narrowly used in traditional algebra symbol manipulation. This is the algebra that is used by most adults in everyday life. The reading and interpreting of graphs or tables of data that may show relations between factors. Major decisions are made on the interpretations of these relations!

Even before symbol manipulations skills are learned, students should see and discuss meaningful linear, quadratic, and exponential data. Such data exists in middle science and social studies material. Well-chosen examples introduce students to a major reason for using algebra—modeling real world situations. Attention to this view reinforces mathematical connections within mathematics to measurement, statistics, and geometry. Additionally, there are connections among algebraic representations—graphic, numeric (often tables), and symbolic.

The Tools of Algebra

We must publicize the set of tools that are meaningful in the learning and doing of algebra. Technology (graphing calculators and spreadsheets) and manipulatives (e.g., algebra tiles, integer chips, geoboards) are showing positive value in students who approach algebra with a weak or uneven background and low self-esteem in mathematics. In the past, the use of technology has been delayed until students could do symbolic manipulations by hand—just as was once required with long division and messy fraction calculations. Now there are several algebra curricula which require, or strongly encourage, the use of graphing calculators on a regular basis. At today's prices, a classroom set of 30 calculators is approximately $2,000, the price of one computer workstation.

Algebra for All in a Variety of Ways

As districts move to consensus on algebra for all students, the implementation plans vary. In some cases, all students will be in algebra by the ninth grade. In other districts, a challenging middle grades mathematics program builds algebraic reasoning and multiple algebraic representations of relationships across the curriculum. Many schools use state guidelines which include algebra in an integrated program for traditionally noncollege bound students.

Changes in Postsecondary Mathematics

While often slow to change, college requirements and examinations are beginning to change. New courses are being offered as alternatives to meeting competencies required for graduation. These courses are quite varied, but many involve such topics as management sciences (networks, scheduling, and linear programming), statistics (sampling, surveys, statistical inference using confidence intervals), social decision making (weighted voting systems, fair division and apportionment, game theory) and exponential growth and applications. Students who have been slow and poor in algebraic manipulation have been seen to develop algebraic reasoning and representation for significant mathematical settings they would have never seen in more traditional courses.

A change in view on what algebra is will require thorough, on-going discussions about expectations with staff, school administrators, parents, and community members. First reactions often will be similar to those encountered when the inclusion of calculators in the elementary grades was first introduced. We must be ready to carry on the dialog in-depth and over time.

Educators at all levels of responsibility must be ready to respond to parents, the public, and colleagues on expanding mathematical knowledge arithmetic for all students. This is a challenge consistent with the NCTM Standards and the national concern for improved mathematical power.

Teacher Preparation Programs

We turn now from the foregoing view of redefining algebra to the following observations and recommendations regarding the education of teachers, including K-8 teachers, to provide this new algebra experience for all K-12 grades. The critical elements of teacher preparation programs are: admissions standards, teaching content, pedagogy, and field experiences. Thus, teacher preparation programs must involve the active collaboration of mathematics departments and departments of mathematics education. In light of our observations regarding a vision of the achievable, another element we refer to as affective development also needs to be considered. Because admission standards to teacher preparation programs play a significant role in determining who become teachers, program improvement and reform should use more stringent admission standards from both the academic and professional point of view, especially for programs for the preparation of middle school and elementary school teachers. We turn now to addressing the other elements, adding discussions of faculty enhancement to facilitate implementation of our recommendations. In another section, we link the field experiences element to a discussion of the World of Practice where teachers practicing in schools are identified as co-equal with college and university faculty in addressing reform of algebra. An addendum by one of the Working Group members is also included which provides elaboration of several of the ideas discussed.

Affective Development

The reform effort in algebra teaching should include attention to affective development. There should be the infusion of a multicultural background into the algebra reform effort that includes:

- Studies both to synthesize existing knowledge and to create new knowledge concerning the ways in which the teaching of mathematics encodes biases which are often not recognized by teachers;
- Developing mathematics examples which can be included in multicultural education courses required of prospective teachers;
- Providing specific information and guidance to teachers in mathematics methods courses and professional development programs; and

- Developing a syllabus for a course on the multicultural history of mathematics which should be required of all mathematics education students.

Curriculum Content

Model teacher preparation curricula need to be developed in line with the ideas of a strands approach for the teaching of algebra for all and in light of new technologies. These curricula models should include pedagogy, content, epistemology, and experiences appropriate for the education level to be taught. This will require reconceptualization of the college curriculum for the preparation of K-5, 5-8, and 9-12 teachers.

Elementary and middle school teacher preparation programs should pay particular attention to the mathematics of quantity and change where opportunities to learn algebra occur. An emphasis on representation, generality, and structure should permeate that curriculum. The application of technology and the development of skills in the use of symbolic manipulators and graphing utilities should be acquired by their use in courses throughout the program and should not be tagged on to be acquired in a separate course.

Pedagogy

First and foremost, those who offer teacher preparation programs should themselves model good instruction. Moreover, the quality of instruction in these programs must be examined and modified as necessary to:

- Incorporate a cognitive orientation (take account of what students learn and what they make of our teaching); use on-line assessment; sensitivity to students' concept images;
- Maintain open lines of communication with students;
- Emphasize conceptual development of math ideas; reduce emphasis on technique building; build on algebraic intuition; and
- Organize curriculum of courses in terms of cognitive objectives as opposed to content objectives.

The work in pedagogy should be content-specific. There should be differentiation by education level (e.g., K-8 versus early childhood and middle school). Moreover, there should be a pairing of content and pedagogy work such as shadow seminars (seminars on teaching issues associated with specific content courses).

Field Experience

The field experience component of a teacher preparation program is significant. Accordingly, the selection of supervising teachers is critical. Overall, the quality of practicum or field experiences for all prospective teachers is a problem in need of attention. This is discussed further in the section below headed "The World of Practice."

Faculty Enhancement

Development of new teacher preparation programs requires faculty enhancement so that faculty model exemplary teaching methodology associated with algebra (mathematics) reform. Moreover, doctoral programs in mathematics and mathematics education should incorporate pedagogy and epistemology of mathematics. There is a need to expand faculty enhancement programs which give mathematics and mathematics education faculty (preferably jointly) opportunities to:

- Develop and experiment with a broad repertoire of teaching strategies to enhance the conceptual understanding of their students;
- Become more knowledgeable about how students learn and how to assess the learning of their students;
- Explore curricular issues related to learning of algebra and preparing K-12 teachers to implement new curricula; and
- Develop and/or become familiar with exemplary curriculum materials which enhance understanding of algebra.

The World of Practice

Effective teacher preparation programs must have robust and active relations with the world of practice. Relationships with the world of practice in the schools should be strengthened through the implementation of the concept of co-reform: reform of teacher preparation programs jointly with the reform of practice in the schools. This will require the establishment of partnerships between teacher preparation programs and the schools. Colleges and universities should review and strengthen their invitational/outreach efforts, in the light of co-reform, to establish and maintain lines of communication among mathematicians, practicing teachers, and mathematics educators and work to create a climate that is more teacher friendly. For example, colleges and universities can be more accommodating in their scheduling of courses to meet the needs of practicing teachers.

These invitational/outreach efforts should involve a more cognitive stance in teaching while developing greater sensitivity to issues of cultural diversity with a multicultural perspective.

Moreover, an algebra reform movement requires change in current practices of teaching algebra in schools. The school or school district is the locus of such change and the success of such a movement is dependent on changing the practices of current teachers, since obviously the time line of change must be shorter than the attrition of practitioners. In fact, if practices of current teachers are not changed, even newly graduated teachers who have been prepared for implementing new methods will find it difficult to do so in an environment that does not support new ideas. Few first- and second-year teachers are leaders among the faculty of their school; rather, they look to the experienced teachers for guidance and, indeed, survival. If that experienced faculty does not participate in the reform movement, then most new teachers will gradually adopt the ideas of their senior colleagues who convince them older methods are "what works."

Crucial Concepts for Teachers of Pre-college Algebra

An understanding, concept definition, and concept image of the dimension of vector spaces, for example, is probably crucial for teachers of third and fourth-year high school mathematics courses. It is useful for any teacher of high school-level algebra. It is, however, beyond any feasible goal as a necessity (or even ideal) for every elementary and middle school teacher who is responsible for the mathematics education of students. The vast majority of teachers who influence the algebra-readiness and hence the algebra success of pre-college students are not teaching higher level high school mathematics courses; and for the near future, they are not the products of reformed preserves university teacher education departments. The majority are elementary and middle-school teachers who completed their degree programs in traditional programs. While they do not need in-depth understanding of linear algebra, they do need such things as:

- A working knowledge of using technology in instruction;
- Conceptualization of relationships among quantities;
- A commitment to development of algebraic thinking for all students;
- An understanding of the role of algebra as a gateway to academic development and full participation in citizenship· and
- Access to real-world examples of the uses of algebra.

What must these teachers do? What must they learn? How can they learn to prepare young people for success in algebra? The answer lies in both curriculum and professional growth—concepts that are interdependent. While curriculum is not the topic of this paper, the approach to professional growth will make the assumption that a new curriculum for pre-college algebra will treat it not as a ninth-grade course (taken also by advanced eighth graders and precocious seventh graders), but as the study and utilization of quantitative relationships, appropriate for all, from pre-school through a lifetime. This is consistent with the National Council of Teachers of Mathematics (NCTM) curriculum standards.

Many currently practicing elementary and middle school teachers and many secondary mathematics teachers are not prepared to implement this approach to algebra, nor are they receptive to top-down mandated change. Therefore, the key to change is that practicing teachers must take responsibility for their own professional growth. They will also need assistance from many sources—professional organizations, university teacher education departments, university mathematics departments, their own school administrators, their state departments of education, and their peers. Indeed they do need such assistance. The critical attribute is that the practicing teachers determine and seek fulfillment of their own needs. These external sources must facilitate, rather than control.

Next Steps

Model programs should be identified and developed in which practicing teachers take the initiative in identifying and satisfying their needs for professional development in teaching and encouraging algebraic thinking K-10. These should be characterized by:

- A team approach to the identification of needs;
- Support of the school-district administrators;
- A variety of possible sources of the professional development, including universities, professional organizations, school-district instructional administrative staff, and other practicing teachers; and
- Mutual respect and parity among all the players—the practicing teachers, providers of instruction, school administrators.

In closing, we observe that there is a crisis in content in the elementary school. Current ideas about knowledge and teaching at the early grade levels have acquired increasing sophistication in virtually every field, but especially for mathematics and science. This crisis in content makes it increasingly difficult for any one person to teach all subjects, as is still the expectation in most elementary schools. What is needed are new models of school structure and organization to experiment with new approaches to content specialization or differential roles for teachers in the lower grades. Serious efforts along these lines will require changes in such policy areas as the regulation of education and teaching: certification, licensure, professional standards, and accreditation.

Addendum on Teacher Education

by Guershon Harel, Purdue University (in print): On teacher education programs in mathematics, *International Journal for Mathematics Education in Science and Technology*.

> Current teacher education programs suffer from a lack of attention to the three crucial components of teachers' knowledge: *mathematics content, epistemology, and pedagogy.* As a result, they cannot achieve the desired quality in teachers as was envisioned by the current mathematics education leadership. Teachers' mathematics knowledge is far from being satisfactory even in terms of the standards for high-school mathematics. The work on epistemology and pedagogy is detached from a personal experiential basis of teaching, and thus it is in conflict with the well established principle that knowledge construction (and this includes mathematics knowledge as well as knowledge of mathematics epistemology and pedagogy) is a product of personal, experiential problem solving activity. The effort of teacher education programs must center on these three components of teachers' knowledge base. In particular, teachers' knowledge of mathematics should be promoted and evaluated in terms of mathematics values, not specific skills, concepts, and symbol manipulations.

Mathematics content, which refers to the breadth and, more importantly, the depth of the mathematics knowledge possessed by the teachers, is a crucial component because it "affects both what [the teachers] teach and how they teach it" (NCTM 1991, p. 132). Moreover, a solid mathematics background is indispensable "to understand the complexities of the mathematics embodied in the *[S&E] Standards*, complexities that may have an effect on their implementations" (RAC 1988, p. 294).

Knowledge of epistemology includes the teachers' understanding of how students learn mathematics. Teachers must understand fundamental psychological principles of learning: that students construct their own meaning as a result of a disequilibration while they encounter new knowledge, that the source of any knowledge construction is an experiential problem solving activity (e.g., Balacheff 1990; Harel 1985; Piaget 1983; and Thompson 1985) and that mathematics is a social construct students establish through a negotiation process (Brousseau 1986).

Knowledge of pedagogy refers to teachers' ability to implement these psychological principles; that is, their skills to teach in accordance with the nature of mathematics learning. Mathematics teachers should be able to make their classes a place of "collaborative practice involving social construction of knowledge and socially distributed problem solving, ... [a place] where students, working collaboratively and under the tutelage of their mathematics teacher, engage in substantive mathematical activity" (RAC 1988, p. 342).

While these general characterizations are, I believe, in line with the theoretical perspective taken by the *Standards*, they are in sharp contrast to the current state of learning and teaching (see, for example, NAEP 1986). This contrast points to a major effort that is needed to help teacher educators "interpret the *Standards* and take steps toward implementation" (RAC 1988, p. 340). In particular, it points to a need for a collaborative effort in developing adequate teacher education programs which will prepare our next generation of mathematics teachers in the spirit of the *Standards*. The development of such programs is *not* an easy task and involves financial, organizational, political, and curricular problems. I will focus on three of these problems: learning how to teach via learning how to learn, learning how to teach via teaching how to learn, and teaching only if knowing what to teach.

Learning How to Teach via Learning How to Learn

Currently, teacher education programs include content courses taught by mathematicians and "methods" courses (usually only one) taught by mathematics educators. The responsibility in the education of teachers for the learning and teaching of mathematics is placed in the methods courses only. The problem with this pattern of teacher preparation is that teachers' beliefs of what mathematics is and, in particular, how it should be taught are tacitly formed by the way they are taught mathematics in their precollege and college mathematics education. Unfortunately, as the recent report of a joint task force of the Mathematical Association of America and the Association of American Colleges, MAA-AAC (1990), put it, "most college students don't know how to learn mathematics, and most college faculty don't know how students do learn mathematics. It is a tribute to the efforts of individual students and teachers that any learning takes place at all" (p. 8). "The way mathematics is taught at most colleges—by lectures—has changed little over the past 300 years, despite mounting evidence that the lecture-recitation method works well only for a relatively small proportion of students" (NRC 1991, p. 17). Thus, mathematics teacher education as currently practiced is inappropriate to carry out the mission of teacher preparation. It is not possible for prospective teachers to change their beliefs and conceptions about mathematics they have formed during a long period of time in one methods course. The goal of revitalizing the mathematics instruction in schools will be achieved only if the ways preservice teachers are taught mathematics were compatible with the principles of learning underlying the recommendations of the *Standards*.

Learning How to Teach via Teaching How to Learn

The second problem also arises from the way teacher education programs are currently conducted. In these programs, prospective teachers have very little contact with students, and the contact they do have—what's called student-teaching experience—is not integrated in, and it is remote from, their study of the teaching and learning of mathematics. Moreover, these methods courses are separated from prospective teachers' own experience of learning mathematics. In other words, prospective teachers learn about mathematics epistemology and mathematics instruction, in isolation from an experiential basis of teaching—thus in only a hypothetical context of students' learning—and without reflection on their own learning. Personal experience and self-reflection are, I believe, indispensable for understanding and appreciating the problems that are inherent in learning and teaching situations. The current teacher education programs are, therefore, in direct conflict with the epistemological principle that knowledge construction is a product of personal, experiential problem solving activity, because this principle applies to the problems that mathematics teaching and learning present as well as to learning mathematics. Without resolving the conflict, the methods course as they are offered today contain elements of preaching, not just teaching.

Teaching Only if Knowing What to Teach

The third problem concerns teachers' knowledge of mathematics. The *Professional Standards* (NCTM 1991), and a recent document of recommendations for the mathematical preparation of teachers of mathematics by the Mathematical Association of America, MAA

(1991), address this component of teachers: knowledge. To mathematics educators who do believe that mathematics teachers need to have a strong mathematics background, the picture drawn from the current research on teachers' knowledge of mathematics is very depressing. Extensive surveys of the state of mathematics teaching and learning is readily available, and there seems to be little point in repeating it here (see, for example, Romberg and Carpenter 1986; Suydam and Osborne 1977). Rather, I will discuss some findings from my own research with teachers, which indicates that teachers' mathematics knowledge is below even the level expressed in the recommendations of the *S&E Standards* (NCTM 1989), for high-school students. For example, the recommendations concerning the idea of mathematical proof in high school state that:

> In grades 9-12, the mathematics curriculum should include principles of inductive and deductive reasoning so that ALL students can: make and test conjectures, formulate counter examples; follow logical arguments; judge the validity of arguments; construct simple valid arguments, and so that, in addition, college-intending students can: construct formal proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction (p. 143).

The results presented below show that what prospective teachers acquire from their mathematics classes in high school and college is no more than a distorted notion of mathematical proof. Some of the research questions posed were (see the two publications of Martin and Harel 1989): Do prospective teachers understand that inductive arguments are not proofs for mathematical statements? Are they able to judge the validity of mathematical arguments? To what extent are they influenced by the ritualistic aspect of proof? It was found that:

> Many students accepted inductive arguments as proof of mathematical statements...and that acceptance of inductive and deductive arguments as mathematical proof was not found to be mutually exclusive..Many students who correctly accepted a general-proof verification did not reject a false-proof verification; they were influenced by the appearance of the argument—the ritualistic aspects of the proof—rather than the correctness of the argument..Such students appear to rely on a syntactic-level... in which a verification of a statement is evaluated according to ritualistic, surface features. Alternatively, relatively few students have a conceptual-level deductive scheme in which a judgment is made

according to causality and purpose of the argument (Martin and Harel 1989, pp. 48-49).

These results are consistent with the conclusion made in Even (1985) that "we can no longer assume that our preservice secondary mathematics teachers have an adequate grasp of the mathematics that they will be expected to teach" (cited in Norman 1992).

The state of the elementary mathematics teacher is no better. For example, work with preservice elementary school teachers (Graeber, Terosh, and Glover 1989; Mangan 1986) and 4-6 grade teachers (Harel, Behr, Post, and Lesh, in press) in the domain of multiplication and division shows that these teachers possess the very same misconceptions that have been identified with children. Like children's knowledge, preservice and inservice teachers' knowledge includes beliefs that are incongruent with the multiplicative operation of rational numbers. These beliefs thus block their attempts to solve many multiplicative problems correctly. Examples of such misconceptions include "multiplication makes bigger" and "division makes smaller" (Bell, Fishbein, and Greer 1984; Bell, Swan, and Taylor 1981; Fishbern, Deri, Nello, and Marino 1985; Vergnaud 1983).

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Educating Teachers to Provide Appropriate Algebra Experiences: Practicing Elementary and Secondary Teachers— Part of the Problem or Part of the Solution?

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Change in Methodology Needed

An algebra reform movement requires change in current practices of teaching algebra in pre-college schooling. Among the needs are the following:

- Integration of algebraic thinking skills throughout the K-12 mathematics curriculum;
- Transition from students as objects of instruction to participants in learning; transition of teachers from dispensers of knowledge to facilitators of learning;
- Transition from algebra for the elite to algebra for all;
- Transition from focus on algorithms to focus on conceptualization and problem solving; and
- Utilization of technology.

School as the Locus of Change

The school or school district is the locus of such change and the success of such a movement is dependent on changing the practices of current teachers, since obviously the timeline of change must be shorter than the attrition of practitioners. In fact, if practices of current teachers are not changed, even newly graduated teachers who have been prepared for implementing new methods will find it difficult to do so in an environment that does not support new ideas. Few first- and second-year teachers are leaders among the faculty of their school; rather, they look to the experienced teachers for guidance and, indeed, survival. If that experienced faculty does not participate in the reform movement, then most new teachers will gradually adopt the ideas of their senior colleagues who convince them older methods are "what works."

Crucial Concepts for Teachers of Pre-college Algebra

It is not surprising that university faculty who teach mathematics courses such as abstract algebra and linear algebra see the rudiments of these courses as essential for any algebra teacher. Indeed, an understanding, concept definition, and concept image of the dimension of vector spaces (for example) is probably crucial for teachers of third- and fourth-year high school mathematics courses. It is useful for any teacher of high-school level algebra. It is, however, beyond any feasible goal as a necessity (or even ideal) for every elementary and middle school teacher who is responsible for the mathematics education of students.

The vast majority of teachers who influence the algebra readiness and hence the algebra success of pre-college students are not teaching higher level high school mathematics courses; and for the near future, they are not the products of reformed pre-service university teacher education departments. The majority are elementary and middle-school teachers who completed their degree programs in traditional programs and may likely not have a major in mathematics. While they do not need in-depth understanding of linear algebra, they do need such things as:

- Numeracy—comfort and confidence in handling numerical data;
- A belief in the value of mathematics;
- Conceptualization of relationships among quantities;
- A working knowledge of using technology in instruction;
- A commitment to the development of algebraic thinking for all students;
- An understanding of the role of algebra as a gateway to academic development and full participation in citizenship; and
- Access to real-world examples of the uses of algebra.

What must these teachers do? How can they prepare young people for success in algebra? How can they change the filter to a pump?

The answer lies in both curriculum and professional growth—concepts that are interdependent. While curriculum is not a topic of this paper, the approach to professional growth will make the assumption that a new curriculum for pre-college algebra will treat it not as a ninth-grade course (taken also by advanced eighth graders and precocious seventh graders), but as the study and utilization of quantitative relationships, appropriate for all, from pre-school through a lifetime. This is consistent with the National Council of Teachers of Mathematics curriculum standards as set forth in *Curriculum and Evaluation Standards for School Mathematics*.

Most currently practicing elementary and middle school teachers and many secondary mathematics teachers are not prepared to implement this approach to algebra, nor are they receptive to top-down mandated changes. But they *are* dedicated, capable, creative professionals who, given the proper opportunity, can and will learn and make changes that they believe benefit their students.

Therefore, the key to change is that practicing teachers must take responsibility for their own professional growth. That does not mean that they do not need assistance from many sources—professional organizations, university teacher education departments, university mathematics departments, their own school administrators, their state departments of education, and their peers. Indeed they do need such assistance. The critical attribute is that the practicing teachers determine and seek fulfillment of their own needs. These external sources must facilitate, rather than control.

A major question at this time is how are teachers stimulated to do this? What motivates a school-based movement of change and improvement? What outside forces can plant the seed and what kind of seed will grow? How can agencies/organizations provide awareness without a top-down approach? What barriers do schools and school districts face? What support is needed? How is funding appropriately provided? Those are topics for further exploration.

Next Steps

Model programs should be identified and/or developed in which practicing teachers take the initiative in identifying and satisfying their needs for professional development in teaching and encouraging algebraic thinking, K-10. These should be characterized by:

- A team approach to needs identification
- Support of school district administrators
- A variety of sources of professional development
  - Universities
  - Professional organizations
  - School district administrators/staff
  - Other practicing teachers
  - Independent consultants

- A variety of forms of learning
  - University pedagogy courses
  - University mathematics courses
  - Staff development workshops and institutes
  - Professional conferences
  - Technology-based
  - Job-embedded learning (peer observation, study/discussion groups, reading, journal writing, self-analysis of teaching action research projects)
- Mutual respect and parity among all the players
  - Practicing teachers
  - School administrators
  - University faculty
  - Other providers of instruction (math supervisors, private consultants and others)

When elementary and secondary teachers take the responsibility for their own continuing professional development and when the system allows them the resources to carry out that responsibility, then the changes in teaching methodology required for an algebra reform movement can become a reality.

Educating Teachers for Algebra

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The Challenge: Algebra for All

What is the meaning of "Algebra for All," and why should there be a colloquium focused on algebra? For America and many other nations, the characteristic feature of formal education in this century is the provision of increased access: over the last hundred years, increasingly larger proportions of our population have spent more and more time in schools and colleges. This is reflected in the expansion to near universal access to high school education in the early part of the century, followed by a great increase in access to higher education in mid-century. The phenomenon of increased access appears to have leveled off over the last 25 years or so and, as an issue, has been replaced by concern over outcomes and results—specifically, how to help larger proportions of our citizens attain more from access to education, how to bring them to higher levels of achievement in knowledge and skill.

This shift in focus from access to outcomes is part of the contextual scene that led to the Algebra Initiative. The conceptual framework provided prior to the conference implicitly and explicitly underscores the focus on algebra as a barrier, a stumbling block, to higher achievement. It is a relatively new idea that algebra is necessary for everyone for economic competitiveness and quality of life. While algebra has always been a staple of the college preparatory curriculum, it is only recently that new programs such as "tech prep," designed for those not bound for the university, have a strong mathematics component recognizing the need for more powerful mathematics preparation for the job market.

Two major barriers are associated with attempts to establish algebra and success in learning algebra as basic components of the general education of all students: (a) the intrinsic difficulty of teaching and learning algebra; and (b) diversity and multicultural issues, barriers to particular groups marked by race, gender, class, first language, or other societal categories. Because algebra has functioned as a critical filter separating many students from further study of mathematics, any reform of the teaching of algebra must address these barriers.

The Paradox of Teaching

There are several models that have recently been developed regarding the education of teachers. One published by the Mathematical Association of America (Committee on the Mathematical Education of Teachers of Mathematics, 1991) is cited in the paper providing the conceptual framework for this colloquium. All the models are based on or derive a set of core propositions for what teachers should know or be able to do (National Board for Professional Teaching Standards, 1989). All include a core proposition like the NBPTS Proposition 1: Teachers Are Committed to Students and Their Learning.

Despite what appears to be a consensus regarding the ideal of algebra for all students, there is by no means a universal commitment to the ideal on the part of mathematicians and mathematics educators at any level, from elementary grades through college. This is based on doubts about whether the ideal is realistic and feasible. In fact, the ideal has not been accepted by many algebra teachers in the schools who retain the conviction that some students just cannot learn algebra because they cannot deal with abstraction.

Some of this doubt, in turn, appears to arise from skepticism about even the possibility of teaching. Westheimer (1987) has asserted that scientists do not really like to teach the "unwashed." Certainly, he meant his observations to apply to mathematicians, as well. On the basis of personal experience, I am becoming more and more completely persuaded that many of my university colleagues, especially in science and mathematics, do not believe in teaching. Conversations I have had over the years reveal that while they may believe in such a thing as learning, they do not believe they can do much as teachers to help people learn science or mathematics.

This phenomenon is reminiscent of the *Meno*, where Socrates asserts the impossibility of teaching. This, of course, is based on Socrates' idea on the nature of knowledge, also expressed in the *Meno*: knowledge is innate and instilled in the human soul at birth. The paradox about teaching is that Socrates goes on to demonstrate his revolutionary new approach to teaching as a way of bringing this innate knowledge forth in the student (see Gardener 1985). Freud also asserted the impossibility of teaching, while—like Socrates—he introduced a radically new contribution to the art of pedagogy

(Felman 1987). This paradox, of instituting a revolutionary pedagogy while asserting the impossibility of teaching, is relevant today. What model of teaching do our mathematicians associate with failure? What model of teaching allows our best students to flourish? What policies can be adopted to mediate this paradox?

One consequence of the absence of commitment to the possibility of teaching is the reinforcement of the elitist perspective I mentioned earlier. Thus, the educational process does not provide instruction, but functions as a sorting machine, separating the talented from the untalented. The very title of a recent report on the problems of calculus teaching and learning (Steen 1988) underscores this phenomenon.

At the elementary school level, two related characteristics stand out: the significance of innate ability relative to effort, and the American tendency for categorizing students (Stevenson and Stigler 1992). For Americans, learning mathematics and science is frequently seen as a process of rapid insight rather than lengthy struggle. Sort of the light bulb going off in the head of the cartoon character to signify "aha." As a consequence of this view, American children are prone to give up too soon in problem solving. They often give up before reaching genuine understanding. In Japan, this belief is not shared. The Japanese believe that success in problem solving and understanding indeed involve lengthy struggle. Those who make the effort in this struggle are rewarded with success.

The other characteristic, the pervasive American tendency to categorize students, is reflected in the veritable explosion in special education classes and the growth of categories of learning disability. Individual students are labeled as having good innate ability for mathematics or science, or not so good innate ability, and they are classified accordingly. Even when there is not formal tracking, the practice in American schools and classrooms is to use ability groupings. In Japan, at least in the elementary school, there is much less categorizing. All, or almost all, children are expected to learn what the curriculum requires and they exert effort until they do.

If we are seriously to address the improvement of teaching, we must first resolve our ambivalence about its very possibility. Moreover, we must develop a genuine commitment to the ideal of algebra for all students. The resolution and commitment demand that we look not only at teaching and learning but also at curriculum—what we propose to teach and what we expect to be learned.

Responses to the Challenge and Paradox

Mimetic and Transformative Teaching Traditions

This section presents two ideas suggesting responses that might be devised for responding to the aforementioned challenge of abstraction and paradox of teaching. The first idea is provided in an analysis by Philip W. Jackson (1986). In a monograph, Jackson distinguishes between two traditions in teaching and learning: one that he calls the mimetic, which sees education's purpose in terms of the transmission of factual knowledge; and the other that he calls the transformative, which sees it in terms of changes in the learner in areas such as character, morals, attitudes, or interests.

In considering the issue of whether teachers can have an enduring effect on their students, Jackson reports on a survey that produced many reports from individuals who remembered most vividly those teachers who succeeded in "transforming," profoundly and enduringly, at least some of the students in their charge.

The mimetic and the transformative are not necessarily mutually exclusive nor exact opposites. Both may be intertwined in a given teaching situation. Nevertheless, the dichotomy is significant and highly valuable in thinking about the problem of teaching algebra. This is so because algebra is seen widely in terms of the transmission of formulas and facts—a mimetic effort. Yet algebra can play a critical role in the transformative development of students; it helps to shape their self-image and even their world view. Fortunately, many mathematicians claim to have distinctly transformative aspirations in their teaching, such as wanting their students to gain an appreciation for the power and beauty of their subject, quite beyond facts and concepts. This aspiration is clearly an incentive that policymakers and administrators will want to use in encouraging good mathematics education—helping faculty members to do what they claim they want to do. Clearly, there are lots of references to the notion of mathematical power in the NCTM Standards. We must give scope and expression to this notion in curriculum and teaching.

Thus, the central challenge to algebra teaching at all levels, whether addressed to the education of all students or to specialized education, is to shift the balance from the mimetic to the transformative. A great irony here is that advances in science and mathematics themselves have had a transformative impact. In discussing the Copernican Revolution, Kuhn (1957) describes this impact as follows:

> Though the Revolution's name is singular, the event was plural. its core was a transformation of mathematical astronomy, but it embraced conceptual changes in cosmology, physics, philosophy, and religion as well.

Clearly, the Wiles proof of Fermat's Last Theorem has this transformative character as is indicated by Ribet (1993):

> Wile's proof of Taniyama's conjecture represents an enormous milestone for modern mathematics. On the one hand, it illustrates dramatically the power of the abstract "machinery" we have amassed for dealing with concrete Diophantine problems. On the other, it brings us significantly closer to the goal of tying together automorphic representations of algebraic varieties.

We must make the teaching of algebra as interesting, important, and dynamic as the subject itself.

Content Pedagogy

I turn now to the second idea for meeting the challenge and paradox of teaching algebra. To begin with, I propose that we couple with the perspective of the transformative teaching tradition, Shulman's answer to the question of what one needs to know about a subject in order to teach it. Shulman (1986, 1987) calls this kind of knowledge pedagogical knowledge of content, which refers to:

> ...for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice.

> Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If these preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners, because those learners are unlikely to appear before them as blank slates.

There are several precursors, in mathematics generally and algebra particularly, to Shulman's identification of pedagogical knowledge of content. There are also several program models that exist, but I will not discuss them here, except to cite two: the program model for science education at the University of Chicago articulated by Schwab (see Westbury and Wilkof 1978) and a similar model developed at Harvard by Conant (see Nash 1950). My favorite example in algebra is Weyl's monograph on symmetry (1952).

Weyl presents many interesting examples including a very provocative one about the swastika. Weyl notes that the swastika is "...one of the most primeval symbols of mankind, common possession of a number of apparently independent civilizations." In the book, Weyl recalls an observation he made in 1937, regarding Hitler's occupation of Austria. He said then concerning the swastika: "In our days it has become a symbol of a terror far more terrible than the snake-girdled Medusa's head." In the book he goes on to observe: "It seems that the origin of the magic power ascribed to these patterns lies in their startling incomplete symmetry—rotations without reflections." (p. 67) Relating aesthetic and emotional sensibilities to the formal structure of symmetry illustrates an extraordinary way to integrate the transformative perspective and content pedagogy in the teaching of algebra.

I suggest that our discussion at the colloquium might spend some time identifying these precursors and models. Most especially, we must recognize that pedagogical knowledge of content (or content pedagogy) is a special form of content knowledge and is, theretore, subject-specific. Once it is identified, it is immediately clear that content pedagogy is quite important for teachers.

Unfortunately, despite the preoccupation with subject matter in current concerns about education, and despite the evident significance of content pedagogy for the teaching and learning of subject matter, the emphasis in teacher education—at least for early childhood teachers and perhaps even early adolescence teachers—is on the generic and not the subject-specific. Further, this situation is not based on conviction, but convenience, as many teacher education programs are so small that subject-specific pedagogy is not considered an efficient option, regardless of its intrinsic merit. One major policy focus must be aimed at greater articulation between teaching and content at all levels. For one thing, it is not clear whether responsibility for pedagogical knowledge of content lies in the arts and sciences college or the school or college of education. However, on both the arts and sciences side and the education side, this kind of knowledge is often recognized. Again, our colloquium discussions might usefully address this issue.

In addition to such general ideas as transformative teaching and content pedagogy for responding to the challenge of implementing the ideal of algebra for all students, there are other resources that are more immediate a pragmatic. Although space does not permit a fuller discussion, I make an observation and cite two references here, regarding such resources. I suggest that the colloquium may wish to focus on them.

The observation is that technology applied to teaching and learning of algebra presents a powerful new tool to address the questions and issues. These new tools help make algebra an experimental science. Visualization in mathematics, empowered by technological tools, permits a new attack on the tyranny of abstraction. Additionally, there are two timely and significant references that should be central to the work of this colloquium: Wagner and Kieran (1989) and Wilson (1993).

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Experience, Abstraction, and "Algebra for All": Some Thoughts on Situations, Algebra, and Feminist Research

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Recent research related to mathematics education points our attention toward the constructedness of knowledge, the importance of representations (both cognitive and sensory) of mathematical ideas to knowledge construction, the idea that cognitive obstacles can impede the construction of appropriate knowledge, and the idea that all learning is situated in the sense that knowledge is constructed in response to the moment-to-moment contingencies both within the instructional (or life) settings of students and in relation to the habits, goals, biases, and beliefs they bring to an instructional or work setting. These theoretical and empirical developments are important in and of themselves to the development of "Algebra for All" because they provide new grounding and a "fresh approach" to the development of new curricula. Moreover, because they are consistent with the more general developments in the sociologies of science and of knowledge, epistemologies and theories of knowledge, these developments in mathematics education allow and invite new conceptual relations with "strange bed-fellows," that is, with theories and practices developed in the fields other than physical science and psychology, the traditional affiliates of mathematics education, and which are as remote as cultural studies, feminist studies, black studies, and others.

Of particular importance to this paper is the potential for exchange between theories and research grounded in feminism and the field of mathematics education. Like mathematics education, feminist theory views knowledge as constructed, acknowledges the importance of images and representations to the construction of knowledge, understands the function of cognitive and affective obstacles to the construction of valid knowledge, and argues that all knowledge is situated. As Donna Haraway states, "feminist objectivity means quite simply *situated knowledge*" (Haraway 1991, p. 188, emphasis in original).

If we are to take seriously both the goals and the responsibilities implicit in the phrase "Algebra for All," we must attend not only to the nature of algebra but also to the nature of the "All" to whom the curriculum is (or, will be) addressed. Our history of success and failure with the curriculum and instruction of "arithmetic for all" as entailed in the universal requirement of schooling through grade eight cautions against the idea that a didactic curriculum of essential facts and procedures will reach all students.

At the Algebra Initiative Conference, Marjorie Enneking described one example of a student whose experience with "3 * 0 = 0" so affected her that she could not engage with mathematics in subsequent years. Although there is no empirical research base to identify and verify the frequency of similar occurrences, those of us who regularly collect mathematical autobiographies or discuss affect toward mathematics in our classes can testify to the prevalence of such anecdotal evidence of the failure of arithmetic and algebra instruction to meet the contingencies of student knowledge building, and to the resultant development, among many students, of obstacles to learning which are both cognitive and affective. Perhaps because these obstacles occur more frequently among girls and women, or perhaps because evidence of their occurrence is most often forthcoming in collegiate courses dominated by women (e.g., methods courses for elementary teachers, courses on women and science), such obstacles seem to affect the mathematical development of many women, in particular.

Many of the stories which reveal the initial occurrence of cognitive-affective obstacles to further learning of mathematics are stories of perceived (and unrelieved) mismatch between the "facts of mathematics" and the "facts of life." At least in my hearing, these stories tend to focus on the introduction of particular abstract objects and procedures: zero (and the empty set), irrational square roots, quotients of fractions, products of negative numbers, and various phenomena of algebra.[1] In all cases, the objects named are remote from direct experience, or at least from experience which has been articulated in relation to the meanings of mathematics. In the case of zero, for example, every child who has played *Go Fish!* (or a similar game) has enjoyed experiences with the empty set; but these experiences are typically remote from the salient activities of arithmetic class. What seems to have been problematic for Enneking's student, and for many of my own, is the relative salience, within arithmetic classes of the experience of three-ness (and its naming as 3) when compared with the experience of none-ness (and its naming as 0). This is an issue of the situatedness of knowledge; and it points to some very real questions about the relations between experiences and abstractions.

Abstraction and Experience

The theme that many examples should precede abstraction is one that runs through the discussions of algebra curriculum reform. This theme also occurs in the conclusions of *Women's Ways of Knowing*, a study of women's epistemology by Mary Belenkey and colleagues:[2]

> Most of these women were not opposed to abstraction as such. They found concepts useful in making sense of their experiences, but they balked when the abstractions preceded the experiences or pushed them out entirely. Even the women who were extraordinarily adept at abstract reasoning preferred to start from personal experience. (Belenkey et al. 1986, pp. 201-202)

These findings suggest that a serious attempt to teach the abstractions of algebra and mathematical modeling to all will require either the development of activities that students are willing to count as "personal experience," or a better understanding of the extracurricular personal experiences of young women and how these experiences might be transported into mathematics classes.

Feminist philosopher Maria Lugones (1987) discusses "travel" of an individual from one situation to another. In her construction of situated knowledge, each person is viewed not as a single unified knower but as a confederation of knowers; as a person moves from one situation to another, she or he becomes a "different knower" whose actions and understandings are predicated upon knowledge particular to the new situation. In order for a person to function comfortably and appropriately in a situation it is important for her or him to share in the language, norms, local history, and human bonds of that site. Mathematics educators recognize the importance of the latter understandings when they argue that instruction must engage students in the "culture" of mathematics.

For the woman cited by Marge Enneking and for many others, by the time the student reaches grade eight or nine, mathematics class is already a different culture from personal experience and requires a "different knower" in the sense of Lugones. The problem for the teacher (and for the curriculum developer) is to recognize the need for travel between the cultures of algebra and personal experience and to facilitate that movement. This entails an understanding of not only the ways in which mathematics class differs from the personal experiences of young women but also the links that students might construct in an effort to bridge those differences. In the following paragraphs, I offer examples and discussion related to some of these differences.

The ways in which women personally experience mathematics outside of mathematics classrooms include not only the use of numerical and geometric reasoning in the pursuit of personal tasks but also the receipt of numerous messages concerning the "maleness of mathematics." Newspaper articles, parents, guidance counselors, TV sit-coms, and even Barbie dolls all deliver messages regarding the difficulty of math for females in particular, the "inferiority" of women's performance in mathematics, and (in many cases) the lesser importance of mathematical achievement to women than to men. Regardless of the (un)truth of these messages, they are a "fact of life" salient in the personal experiences of young women; efforts to bridge personal experience and mathematics classes must recognize these messages explicitly and deal with them (Damarin 1990). Bringing these messages (and analogous messages related to race) into mathematics class for the purposes of countering them is critical to making algebra accessible to all.

Other concerns are more directly related to the setting of mathematical problems and the ways in which these are related to personal experience. The way in which a problem is posed can have ramifications for the ways in which students take it up and/or seek relations to personal experience. In the next few paragraphs, I discuss many issues of experiences which surround the following interesting problem which originates in the Quantitative Reasoning Project and which was presented by Alba Thompson at the Algebra Initiative conference.

> Fred and Frank are fitness fanatics. They run at the same speed and walk at the same speed. Fred runs half the distance and walks half the distance. Frank runs half the time and walks half the time. Who wins when they run a race together?

As reported, the first response of most students to this question is that the race will culminate in a tie. Upon being told that someone wins outright, the question becomes how to think about this problem. One can begin by thinking about Fred or about Frank. Thinking about Fred's behavior is a relatively simple matter. Invocation of the distance formula is straightforward in part because experience with races suggests a fixed distance and brings to mind halfway markers, mileposts, a known number of laps around a track, or other means of conceptualizing Fred's ability to divide the total distance. There is no real mystery as to how he can manage to split the distance easily for walking and running.

But Frank's behavior, based on the division of time, is a different matter. Experience with races suggests that total time will not be fixed until the race is over; moreover, the passage of time spent on a racecourse is not marked with the same neat divisions as distance. How, then, can a student extrapolate from personal experience to think about Frank's behavior? Attempts to model Frank's behavior lead to confusion and questions: how does he allot exactly half of his time to each of two activities, especially if he doesn't know what the total time required will be?

Some would be quick to point out that a model for Frank's behavior is not necessary to the solution of this problem. For students (such as the young woman cited by Marge Enneking) whose relation with mathematics has been made tenuous by repeated assertions that "you don't need to understand, you just need to know/do," this observation is not helpful. For them, it shifts the domain of relevant experience from that of races and division of activity along various dimensions to that of mathematics classes and failures to understand. In the sense of Lugones, this comment is an invitation to travel from being a "knower" of races to being a "knower" of mathematics as "an alien world designed by and for people different from us (Turkle, p. 119)."

Interestingly, in a generalized way, Frank's behavior is connected with the life experiences of young women. The division of time (between family and career, between role of wife and role of mother, between care of the self and care of others) is a primary life issue for women. Indeed, sociologist William Maines (in Fennema 1985) has argued from his data that the need of pubertal young women to prepare for multiple roles of career, wife, mother explains the inability of (some) young women to give to the study of mathematics the time and attention it requires. Whether these life conditions make female students more likely to focus on Frank rather than Fred is a researchable question. However, in this life context of time-sharing among multiple activities, the "news" that Frank actually wins the race might be useful and encouraging information to young women. Extrapolating mathematical findings to female personal experience through class discussion might help bridge gaps between women and mathematics. Although the problem setting involves named people and familiar events, it is exceedingly abstract; it demands a suspension of disbelief concerning the behavior of people in races. As stated, the problem projects behavior and projects it a way that few would find plausible. (Why on Earth would they walk so much? Wouldn't the trailing racer switch to running more when he saw he was behind?) A change in setting might alleviate the plausibility issue; consider

Racing Fans Disappointed

Courtesy of racing network news

Fans at the recent race between Fred and Frank were counting on the time trials in which the contestants consistently walked at the same speed and ran at the same speed. In the actual race, Fred walked half the distance and ran half the distance. Frank walked half the time and ran half the time. Fans who came out expecting a photo finish went home disappointed.

This shift of the problem into the past tense and the presentation of it as a news clipping change the relations to experience in several important ways. First, implausibly rigid adherence to strategy on the part of Fred and Frank need not be assumed (indeed, the reader can assume that the loser would likely change strategy for the next race). What we have now is a report on a past event, but a report with critical missing information. Such reports, and the strategies for "filling in the blanks" form the experience base to which students might resort. With the simultaneous explosion of information on the one hand, and growth of cryptic reporting of information (in the style of *USA Today*) on the other, the ability to read beyond the "givens" may be a mathematically based skill of increasing social importance.

As a final note on Fred and Frank, I would note that neither the maleness of both racers, nor the setting in athletics is likely to affect women's ability to relate to this particular problem (although problem sets in which all actors were male and/or all problems concerned athletics might have such an effect). Indeed, by naming both racers as male, the problem writer has avoided attaching sex to winning or losing.

The problem of Fred, Frank, and their fitness routines is interesting mathematically, more interesting than most problems at this level. I hope I have amply demonstrated that it is also interesting as a domain for exploring the relations between abstraction and experience. I would hypothesize that, across many problems, the complexity of the abstraction-experience relationship is directly related to the mathematical interest of a problem under consideration.

As we move toward the creation of a curriculum of "Algebra for All" which is both relevant to the lives of students and productive toward preparing all students for the 21st century, it is important to incorporate a fuller understanding of the experiences and situations of all students. This will entail acquiring a better understanding of the particularities of those situations and the creation of new ways of promoting student travel between experiential and mathematical knowledge. Hypertext may provide a valuable resource for delivering instruction and problem solving assistance (scaffolding) in ways that reflect and honor student experience. However, before we can construct appropriate hypertexts, we must gain better insights into the lives and experiences of all students. In this regard, the fields of sociology, women's studies, black studies, and other ethnic studies may be as valuable to the construction of the mathematics curricula of the future as psychology and physics have been to the curricula of the past.

Notes

1. There is another class of stories about negative experiences in mathematics classes: these stories concern instructional practices which students (again, primarily females) found threatening. Practices such as relay races in which students write answers on the board and other activities which call for competition and/or public display of skill are cited by many women as debilitative to their mathematics achievement and/or attitudes. (Also see Isaacson 1991).

2. Belenkey, Clinchy, Goldberger, and Tarule studies 135 women from 5 educational settings ranging from an elite women's college to a center in which welfare mothers were taught skills of parenting, homemaking, and employability.

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Educating Teachers, Including K-8 Teachers, to Provide Appropriate Algebra Experiences

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The first section of this paper leaves in place my thoughts and musings before the conference. In the second section, I share some of my reactions to the discussions as they relate to my original concerns.

Introductory Thoughts

The Algebra as Language Metaphor

The opening line of the conceptual framework for the conference, "Algebra is the language of mathematics," invites speculation on the uses of language in general and how these uses may be interpreted for the role of algebra. Some of the uses of language are suggested below for the purpose of considering how teacher education may enable teachers to use and teach algebra as a language. But the metaphor of algebra as language also begs the question of whether or not the metaphor is itself adequate or proper as laying the groundwork for disquisition.

Certainly, one level of language that algebra serves admirably is to put information into succinct and unambiguous form which, in turn, allows easy manipulation of the symbols to reveal basic relationships. Solving equations or systems of equations would seem to be the prototype of school algebra. "Solving for x" has a mystique that transcends our classes. This is likely the level of language that is being dealt with when we talk about algebraic skills. I would like to pose the question (and perhaps introduce some expressions here), "What level of language would we be addressing if we were to talk about developing algebraic insight or algebraic intuition?" Since mathematical insight and intuition are much more about thinking mathematically than is proficiency in manipulative skills, intuition and insight should be essential to our educational program. Based on my intuition about learning, these levels of learning should not be taught sequentially as hierarchical abilities, but somehow be braided together in instruction.

Do we study language for its own sake or as a means of communicating knowledge? We can talk about the structure of language itself, questions of grammar or rules. We can look at the pieces, vocabulary development. Perhaps some of the way out of this dilemma is to push the algebra as language analogy further, language as a means to shape, critique and disseminate cultural knowledge. Language is related to comprehension. Language reaches its pinnacle in literature. What are the "literary" aspects of algebra that are or should be in our K-8 curriculum?

Preparing Teachers for Teaching the Levels of Algebraic Language

What do teachers need to know if we are thinking of algebra in a cultural context? As we are taking on the challenge to educate a diverse population, including such categories as gender, learning styles, and cultural background, we are in the midst of a paradox. While it would seem that different contextual settings are appropriate for each group, we are trying to achieve some basic, universal education which would equip all these factions of our population to understand one another and contribute to our society.

Teachers would likely benefit from knowing about the development of numbers and algebraic structures in different cultures with two purposes in mind: recognizing that there is diversity in representation and in popular use of algorithms, and seeking to identify the abstract concepts underlying each system. It is also possible that knowledge of the development, the intellectual and historical influences, of algebra as we know it would enhance the recognition of cultural diversity, on the one hand, but provide awareness of the need to find a common form of expression on the other hand. Further, the study of the development of arithmetic and algebra in different cultural settings should highlight the cultural bases for the development of mathematics (i.e., what motivated civilizations to start making mathematics).

In educating teachers, it is especially critical that connections with other fields of mathematics, especially geometry, and the real world be made as an essential part of the subject matter. Overwhelmingly, the population can be characterized as users of mathematics (unfortunately, the reality in this country may better be described as avoiders of mathematics) rather than creators of mathematics.

A question that arises is at what point do we want to separate college students who are studying mathematics into tracks according to which they will be users of mathematics? Some breakdowns according to future careers/professions are: teachers of primary school mathematics, teachers of elementary school mathematics, teachers of high school mathematics, teachers of college mathematics, research mathematicians, engineers, statisticians, computer scientists, and applied mathematicians. Ideally, we would be able to devise a core curriculum that allowed these groups to associate with one another for some of their mathematics education. For teachers, it is especially important that they be aware of and, indeed, know a variety of people who are serious users of mathematics. The often heard student refrain of "what's it good for" would best be answered before it is stated. Teaching mathematics with a regular, nonartificial infusion of anecdotes and information of its place in people's lives would be a strong position for teaching mathematics rather than the usual defensive posture. The ultimate avoidance of a contextual mathematics, the worn admonishment of "you'll need this later," will become archaic, one hopes.

Post Conference Remarks

The algebra as language metaphor really took on a new aspect in light of the keynote address of Victor Katz on the learning and teaching of algebra in the last 4,000 years. The first shock was that our stock algebra word problems have been around from ancient times. The difference between then and now is that problems were solved by verbal algorithms without the benefit of algebraic symbols (i.e., there were no equations). Ancient algebra texts sounded astoundingly contemporary but they looked weird by our standards. What the ancients did have as a resource was the power of geometric models to inform and give structure to their methods.

In my consideration about language and algebra, I would like to pose the possibility that for learning algebra, geometry is a potent and eloquent visual language for teaching. Continuing in an exploratory mode, considering ways to truly talk about algebraic relationships may be an important link for preparing students for the symbolic representation of algebraic relationships. Within the discussions at the conference, there did seem to be agreement on having students experience algebraic situations before abstractly presenting or codifying the material. My assertion is based on collecting rather diverse statements—the computer experiences advocated by James Kaput, the importance of students looking at examples before proving results stressed by Michael Artin, and the algebraic reasoning examples of children's experiences presented by Alba Thompson, to mention a few—which did show the variety of contexts in which it should be possible to let students "muck around" before settling down into formal notation or proof. This may be the avenue for developing "algebraic intuition" about which I expressed concern in my first notes.

A Last Thought

The call for "algebra for everyone" has the ring of a political, not an educational, statement. The historical perspective we gained at the conference underscored that algebra has served political ends in the past, so we are reinterpreting an ancient tradition. The consistency in the new movement, as so strongly presented by Robert Moses, is that we are recognizing that algebraic know-how gives individuals power. But, whereas in the past, the policy was to restrict algebraic power to an elite class, we now are challenged to see that that knowledge is available to the many.

On the Learning and Teaching of Linear Algebra*

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Jacob, a prospective mathematics teacher at the end of his senior year, was given the problem:

> V is a 40 dimensional vector-space. U and W are subspaces of V, dim U = 2, dim W = 3, Can dim $(U \cap W) = 0$?

Jacob responded: "No, because there must be an overlapping between the two." From a further conversation with him on why should U and W overlap, it turned out that he simply viewed $U \cap W$, and V as sets consisting of 2, 3, and 4 items, respectively, from which he concluded that $U \cap W$ must be non-empty; the data concerning vector-space, subspace, and dimension was totally ignored by Jacob. When the instructor asked "Is $U \cap W$ a vector-space?," Jacob was unable to respond, nor was he able to respond to the question, "What is the definition of vector-space?" And when the instructor asked Jacob to give an example of a vector-space, he responded:

> A polynomial ... The set of all polynomials of degree 10 ... The set of all polynomials of degree 0. No, the first is not, because when you multiply you get a higher degree so it is not closed.

Responses like Jacob's are not uncommon; they reflect the confusion students have with linear algebra concepts (Harel 1985).

Vinner (1985) would say that students like Jacob have failed to build adequate concept images for the concept definitions we present to them. A concept definition, according to Vinner, is a verbal definition, appearing in a textbook or written by the instructor on the blackboard, that accurately describes the concept in a noncircular way. A concept image, on the other hand, is a mental scheme, a network, consisting of (a) what has been associated with the concept in the person's mind and (b) what the person

*This short paper is part of a paper entitled "The Linear Algebra Curriculum Study Group Recommendations; Moving Beyond Concept Definition," to be published in the *College Mathematical Journal*.

can do in regard to the concept. It may include, for instance, analogies and relationships to other concepts, propositions on or relating to the concept, examples and nonexamples, ways of solving certain problems. It is worth mentioning that concept images do not necessarily include spatial visualizations, as the term image may suggest. In fact, it was found that some people possess effective concept images, and yet their mode of thinking is purely analytic, not spatially visual (Eisenberg and Dreyfus 1986).

The most important indicator for understanding a concept is the ability to solve problems related to the concept, where by solving a problem, it is meant knowing **both** what to do and why. This indicator, however, is too general, since problems can be of different levels. In my research on the concept of proof (see below), I have defined four other indicators for understanding a concept:

- The ability to remember, not just memorize;
- The ability to think in general terms;
- The ability to communicate ideas in one's own words; and
- The ability to connect ideas.

A student with an effective concept image is one who has these abilities relative to the concept. In order for linear algebra students to develop effective concept images, they must learn to not just memorize concept definitions but must construct rich and effective concept images that will enable them to remember what they learn, think in general terms, communicate, and connect mathematical ideas.

For most students, the construction of an effective concept image is a long and painstaking process. It is not always easy for us, as teachers, to realize this fact, for, as Piaget (1960) pointed out, a concept is deceptively simple when it has reached its final equilibrium (i.e., has become part of the concept image), but its genesis is much more complex. The building of an effective concept image in linear algebra requires a major effort and sufficient time on the part of the students as well as their teachers. Yet, we allocate only one course in the entire undergraduate mathematics curriculum to linear algebra. In comparison, as Alan Tucker (1993) has pointed out, we devote an entire year-and-half of the lower division core mathematics to calculus. Even with this amount of time, calculus is still difficult for students, a fact which raises doubts on the sufficiency of the time allocated to linear algebra.

In the case of calculus, we understand that students must build solid concept images for on-variable calculus concepts and, rightly so, we devote two courses to this goal, before we introduce multivariable calculus. For example, we understand that students must gradually abstract the idea of derivative by first dealing with it extensively in the case of one-variable functions, then abstract it into higher, yet spatially imaginable, cases (i.e., real-valued functions $f: R^2 \to R$ and $R^3 \to R$), and only then move to general functions $f: R^n \to R^m$. On the other hand, we do not seem to have the same patience for the abstraction process in linear algebra. Nor do we take into consideration the obvious fact that linear algebra concepts are indispensable for understanding many multivariable calculus ideas. In most cases, ideas that require linear algebra background are shuffled under separate sections of exercises labeled "Optional."

The imbalance between the time allocated to calculus and that allocated to linear algebra is, in fact, even greater because high school mathematics is geared toward calculus needs more than linear algebra needs. This argument may not be true if examined solely from the viewpoint of content. High school curricula does include topics such as systems of linear equations, analytic geometry, and Euclidean space; all are part of linear algebra. But these topics are taught in high school in ways that have little to do with the basic ideas of linear algebra. High school students are not prepared for the objects, language, ideas, and ways of thinking that are unique to linear algebra.

From the students' point of view, calculus is a natural continuation of the mathematics they learned in high school. After all, students deal with real numbers and functions of real numbers in high school, and continue to deal with these objects in calculus. Also, they are often impressed by the power of calculus tools to help them solve problems in familiar domains, such as finding the area of nonstandard figures, or modeling projectile motion. In contrast, students make little or no connection between the ideas they learn in linear algebra and the mathematics they learn in high school. In the current situation, the only connection that potentially exists between high school mathematics and linear algebra is the studv of systems of linear equations. But even this connection is superficial. High school students' involvement with systems of linear equations amounts to learning a solution procedure for 2×2 and 3×3 systems. They do not deal with matrix representations of these systems, questions about existence and uniqueness of solutions, relations to analytic geometry of lines and planes in space, geometric transformations, matrix algebra and determinants. Evidence that students place a low value on the relevance of linear algebra for high school mathematics can be derived from a recent survey of mathematics education graduates of one major university in the United States. In this survey, 45 percent of respondents believed that the value of linear algebra to their profession is marginal or useless, in contrast to an average of only 13 percent who thought so about calculus.

Clearly, there is a need to prepare students for the unique environment of linear algebra prior to their first exposure to this topic in college. Below I will discuss two ideas briefly: one deals with the need for and feasibility of incorporating linear algebra in high school. The other, apart from this, suggests how, prior to their first course in linear algebra, students can be acquainted with the environment of linear algebra.

I believe that there should be an introduction to linear algebra in high school. The idea may seem too ambitious to some. But if we believe in the pedagogical importance of and the need for a continuity between high-school mathei tics and college mathematics and we recognize the problem of the insufficient time allocated to linear algebra in college, then it should be clear that an introductory treatment of linear algebra in high school is a necessity. Introducing linear algebra in high school does more than prepare students to do matrix algebra and compute determinants. It will lay the grounds for building rich and effective concept images for linear independence, spanning set, vector-space, and linear transformation. Students who would go through a linear algebra program in high school will become motivated and more cognitively prepared to abstract these and other ideas in their first college course in linear algebra. This is a worthy long-term investment which requires a restructuring of the existing high school mathematics curricula. I suggest that this restructuring take place in two areas: First, the traditional high school topics—such as systems of linear equations, analytic geometry, and Euclidean space—be taught from a linear algebra viewpoint. Second, given the imbalance currently existing between the time allocated to linear algebra to that allocated to calculus, I suggest to reduce the high school program in calculus.

The following is a very brief description of a linear algebra program which I have developed for and implemented with upper level high school students. Results from several teaching experiments showed that the program was highly successful (for more details, see Harel 1985 and 1989). The program consists of three phases: the first phase deals with the geometric spaces of directed line-segments. The objective of this phase was to begin building effective, visually oriented concept images for important linear algebra concepts. Further, the construction processes of these concepts were chosen to be analogous to their construction in the general case. For example, when dealing with the 3-dimensional geometric model, a basis can be defined as three noncoplanar line segments. But such a definition is restrictive and model-dependent, since it does not transfer to vector-spaces in general. In this program, this process is fully explored by showing how the concept of basis in a geometric model is developed from a spanning set.

After the central concepts of dependence, independence, linear combination, basis, and dimension are thoroughly studied in the line, plane, and space, I introduced the second phase in which the algebraic spaces R^1, R^2, and R^3 along with these central concepts are built through the idea of vector coordinates. In fact, the unit dealing with these algebraic spaces was mostly constructed by the students themselves through individual and team projects. One of the pedagogical values of this stage is that students could see in a concrete environment how one mathematical system can be transformed into another system which is more amenable to computational techniques. The resulting system is, of course, isomorphic to the original. Finally, in the third phase, these central concepts are defined a third time in R^n, with the introduction of linear transformation and their matrix representations in R^2 and R^3. More specifically, this phase dealt with: analytic geometry of the plane and space; systems of linear equations (algebraic and geometric investigations); linear transformations in the plane and their matrix representations; and matrix algebra.

The LACSG recommendations have set forth the standard for the first course in linear algebra to be an intellectually challenging course, with careful definitions and statements of theorems, and proofs that enhance understanding. From a cognitive and pedagogical viewpoint, a linear algebra course that stresses proofs is both a necessity and a challenge. It is a necessity because the emphasis on proofs is indispensable for the development of rich and effective concept images in linear algebra. Without understanding the reasoning behind the construction of concepts and the justification of arguments, students will end up memorizing algorithms and reciting definitions. It is a challenge because, as we all know, proofs are a stumbling block for many students. Research has shown that many students carry serious misconceptions about proofs. For example, students do not understand that inductive arguments are not proofs in mathematics (Martin and Harel 1989); they do not see the need for deductive verifications (Martin and Harel 1989; Venner 1977); they are influenced by the ritualistic aspect of proof (Martin and Harel 1989); and they do not understand that a proof confers on it a universal validity, excluding the need for any further checking (Fischbein and Kedem 1982). This situation requires, therefore, careful consideration and special attention to the teaching and learning of mathematical proof.

In the current situation, the first course in linear algebra, if it emphasizes proofs, would be students' first experience with algebraic proofs, because calculus often is being taught proof-free and, traditionally, the idea of proof, as a deductive process, where hypotheses lead to conclusions, is stressed in the teaching of geometry but not in the teaching of algebra. Philip Davis and Reuben Hersh (1982) pointed out that "as late as the 1950s one heard statements from secondary school teachers, reeling under the impact of the 'new math,' to the effect that they had always thought geometry had 'proof' while arithmetic and algebra did not." The death of the "new math" put an end to algebra proofs in school mathematics.

In the last few years, I have been working on the epistemology of the concept of mathematical proof with students at various levels. One of the conclusions coming from this work is that a major reason that students have serious difficulties producing, understanding, and even appreciating the need for proofs is that we, their teachers, take for granted what constitutes justification and evidence in their eyes (Harel, in preparation). Rather than gradually refine students' conception of what constitutes evidence in mathematics, we impose on them proof methods and implication rules that in many cases are extraneous to what convinces them. This begins when the notion of proof is first introduced in high school geometry. We demand, for example, that proofs be written in a two-column format, with formal "justifications" whose need is not always understood by a beginning student (e.g., Statement: AB = AB. Reason: Reflexive property). Also, we present proofs of well-stated, and in many cases obvious, propositions, rather than ask for explorations and conjecturing. As a consequence, students do not learn that proofs are first and foremost convincing arguments, that proofs (and theorems) are a product of human activity, in which they can and should participate, and that it is their responsibility to read proofs and understand the motivation behind them.

No one can expect to remedy students' misconceptions and "fill in" other missing conceptions about proofs in one single course. To meet the challenge to teach a linear algebra course that emphasizes proof, we must succeed in educating our students throughout the mathematics curriculum in school and college to appreciate, understand, and produce proofs. The movement toward this important goal cannot start in the first course in linear algebra; it must begin in the high school years and continue into the calculus courses. In fact, with a careful approach and a suitable level, we should begin educating students about the value of justification (not mathematical proof, of course) in the elementary school years. Despite this, I believe that an emphasis on proof in the first course in linear algebra, as was recommended by the LACSG, is vital.

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Algebra: The Next Public Stand for the Vision of Mathematics for All Students*

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Across the state and nation, the call for credibility in school mathematics is focused on a thing called "algebra"—emphasized by the call of algebra for every student. First, this is a response to the recognized need for increased mathematical reasoning and application by all members of society. No longer can the work force function with only a few individuals with mathematical sophistication beyond arithmetic! Second, it is the recognition of the inappropriate use of mathematics, specifically algebra and calculus, as gate keepers in our society. Tracking in mathematics is a major force in effectively excluding individuals from many academic and vocational pursuits.

We must involve all of our constituencies—students, parents, leaders from business, and educators at all levels—in addressing the challenge of algebra for all students. The task will be a difficult, time-consuming one—for everyone involved—but we cannot ignore the need. The task will involve major changes of beliefs about what algebra is, how one does algebra, and who can learn algebra. Technology has opened up mathematics to individuals in just about every line of work. We must challenge the widely held American belief that mathematical ability is hereditary.

Redefining Algebra

School algebra evolved as performance of symbolic algebraic procedures, such as—solve, simplify, factor—typically as isolated, symbolic manipulations. Recently, an Algebra I student reported: "Of course it doesn't make sense. This is algebra; it's not supposed to make sense." We must do something different! The vision of algebra for all students must be clarified and communicated openly within the education community and beyond. First, what is the algebra deemed necessary for all students? Second, how do we change expectations about what algebraic performance is—for teachers of algebra? for students? for parents? for postsecondary gate keepers? for employers? for district administrators?

*This article appeared in *Educational Forward*, Wisconsin Department of Public Instruction, May 1993.

Third, how do teachers, schools, districts, and states monitor student performance in using this algebra? In the fast changing world of mathematics, science and technology, there is no single answer to these questions.

Lynn Steen (1992), director of the Mathematical Sciences Education Board and former president of the Mathematical Association of America, states, "For most students the current school approach to algebra is an unmitigated disaster. One out of every four students never takes algebra... And half the students who do...leave the course with a lifelong distaste for mathematics."

Algebra as a Human Endeavor

Numerous groups and projects are making recommendations on algebra for all students. Most call for the immersion of algebra skills in an applied and conceptual curriculum. They suggest the building of an algebra curriculum extended over many years, from the primary grades on, where students search for and describe mathematical patterns. In such a program, algebraic reasoning, descriptions, and symbols are used to describe the world around us!

Many of the projects recommend the concept of function as the unifying theme. Special attention is given to building tables of data and their graphs from observed patterns and experimental data. Experimental data collection from the sciences, social studies, and real world call for comparison between two traits or characteristics in contextual problems. "How does one trait vary as we change the other?" This is a powerful meaning of the term variable, which is narrowly used in traditional algebra of symbol manipulation.

This is the algebra that is used by most adults in everyday life. The reading and interpreting of graphs or tables of data that may show relations between factors. Major decisions are made on the interpretations of these relations!

Even before symbol manipulation skills are learned, students should see and discuss meaningful linear, quadratic and exponential data. Such data exists in middle school science and social studies material. Well-chosen examples introduce students to a major reason for using "algebra"—modeling real-world situations. Attention to this view reinforces mathematical connections, not only a nod to applications outside of mathematics but to connections within mathematics to measurement, statistics, and geometry. Additionally, there are connections among algebraic representations—graphic, numeric (often tables), and symbolic.

The Tools of Algebra

We must publicize the set of tools that are meaningful in the learning and doing of algebra. Technology (graphing calculators and spreadsheets) and manipulatives (e.g., algebra tiles, integer chips, geoboards) are showing positive value in students' conceptual base and use of algebra. This is particularly important for students who approach algebra with a weak or uneven background and low self-esteem in mathematics. In the past, the use of technology has been delayed until students could do symbolic manipulations by hand—just as was once required with long division and messy fraction calculations. Now there are several algebra curricula which require, or strongly encourage, the use of graphing calculators on a regular basis. At today's prices, a classroom set of 30 calculators is approximately $2,000, the price of one computer workstation.

Algebra for All in a Variety of Ways

As districts move to consensus on algebra for all students, the implementation plans vary. In some cases, such as Milwaukee Public Schools, all students will be in algebra by the ninth grade. In other districts, a challenging middle grades mathematics program builds algebraic reasoning and multiple algebraic representations of relationships across the curriculum. Many schools use the Wisconsin Applied Mathematics Guidelines, which includes algebra in an integrated program for traditionally noncollege-bound students.

Changes in Postsecondary Mathematics

While often slow to change, college requirements and examinations are beginning to change. At UWM, the course, Contemporary Applications of Mathematics, is offered as an alternative to Intermediate Algebra as mathematics competency for university graduation. I taught this course in the Mathematics Department in fall 1992, and I will teach it again next fall. All my students had struggled with traditional algebra several times in high school and postsecondary courses. While their math backgrounds had many holes, I found these students to be good at mathematical reasoning when done in contextual settings. They were involved, and often excited about the topics which included management sciences (networks, scheduling, and linear programming), statistics (sampling, surveys, and statistical inference using confidence intervals), social decisionmaking (weighted voting systems, fair division and apportionment, and game theory) and exponential growth and applications. Many of these students were slow and poor in algebraic manipulation, but they developed algebraic reasoning and representations for significant mathematical settings they would have never seen in Intermediate Algebra.

A change of view on what algebra is will require thorough, on-going discussions about expectations with staff, school administrators, parents, and community members. First reactions often will be similar to those encountered when the inclusion of calculators in the elementary grades was first introduced. We must be ready to carry on the dialog in-depth and over time.

As education leaders, no matter our levels of responsibility, we must be ready to respond to parents, the public, and colleagues on expanding mathematical knowledge beyond arithmetic for all students. This is a challenge consistent with the NCTM Standards and the national concern for improved mathematical power.

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WORKING GROUP 3

Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce

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Report of Working Group 3

Susan S. Forman
Mathematical Sciences Education Board

During the early stages of their discussion, the group agreed on the following premises:

- The technical work force comprises all employees in technical positions, including those with high school, associate, bachelors, or advanced degrees.
- Students need to master two things to be able to succeed in the technical work force: appropriate mathematics content and skills for independent learning.

Further discussion focused on the need to identify the "big ideas" that students ought to know; organize the algebra curriculum so that students learn to use mathematics as a resource to solve problems and model situations; and create a mathematics program that values diverse ways to succeed.

Areas of consensus:

- There is a common core of mathematics that all students should study through grade 11.
- Algebra should be a significant part of that curriculum, although not necessarily a discrete course.
- Two-year college faculty need to develop courses for technical education based on concepts and skills developed through the study of algebra.
- The mathematics community needs to gather better information about the kinds of mathematics and reasoning skills that are needed in the work force.
- The education community needs better information about what schools should be doing to prepare students more effectively for the school-to-work transition.
- A broad spectrum of people (mathematicians, mathematics educators, and users of mathematics) need to be involved in the effort to change the curriculum and improve the ways in which algebra is learned and taught.

- All mathematics courses (K-16) should integrate preparation for the technical work force into the curriculum; pedagogy in those courses should prepare students to become independent learners of mathematics and other technical subjects.
- The mathematics we are teaching to 7th and 8th graders is .it of date. These courses were developed prior to World War II for students who did not expect to continue their formal education beyond those years and were intended to help students develop "shopkeeping" skills in mathematics. The courses need to be revisited and redesigned.

There were no real areas of disagreement.

Recommendations

- **All students should study the same mathematics curriculum through grade 11, but not the curriculum that currently exists.** Mathematics courses in grade 12 should offer many alternatives. The mathematics curriculum should be grounded in problemsolving that reflects real-world situations and offers a variety of methods of solution. Assessment should be an integrated part of every course. Students should have opportunities to participate in group work, make appropriate use of technology, and develop their communication skills, including reading and writing technical material.
- **Community college mathematics faculty must be involved in curriculum development in technical areas—health, human services, business and information management, agriculture and agribusiness, and engineering and industry.** They also must take an active role in needs assessment and articulation with secondary schools and 4-year colleges and universities. The 2-year college mathematics curriculum should provide students with wide options, including transfer to 4-year colleges. In some cases, intervention programs may be needed to prepare students for transfer.
- **Research is needed into the kinds of mathematics students will need to be successful employees in a technical work force.** Discovering how mathematics is used in the workplace is a subtle research problem that should be carried out by mathematicians who can recognize when math is being used in situations that do not look like textbook examples.

Algebra, Jobs, and Motivation

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For all types of students, the first exposure to algebra is the beginning of a transition from numbers to letters, from concrete to abstract language. The importance of learning algebra as a language rather than as a set of memorized rituals will not be apparent until students recognize its power as a problem solving tool. Important questions of "Why?" and "What if?" can be posed and answered in this new language, a key feature too often concealed from students.

A core set of algebra skills can be defined by identifying the kinds of problems the target audience needs to solve. Besides defining the core, those problems must motivate the study of algebra. Indeed, one could argue that no concept of algebra should be taught unless it can be motivated by a problem that is likely to be part of the students' experience in the near future. Unfortunately, the mathematics itself may be hiding in a spread sheet, for example. Or its real value may appear in the need for careful quantitative reasoning rather than in explicit manipulation of symbols. Ideally, students can be motivated both by the need to acquire quantitative thinking skills and by the demands of specific technical challenges. In any case, the problems of uncovering mathematics at use in the workplace and conveying those experiences to students are formidable challenges that must be overcome. (The beginnings of one attempt at the graduate level in mathematics are described by P.W. Davis 1991, a SIAM Report.)

More simply, one can identify some specific skills students need at the pre-college level through widely available licensing examinations for many trades and professions. Students who aspire to become plumbers, electricians, or health care professionals, for example, will be asked to find areas of plane figures and volumes of common solids on their licensing examinations. They will have to manipulate relationships involving rates and slopes to pass those exams and to cope with their daily work.

Identifying these explicit challenges and conveying them to students can provide motivation of a very concrete sort. Of course, an algebra curriculum needs to go far beyond the need for formulas if it aims to develop flexible quantitative thinking skills.

Can algebra be presented as the key that unlocks those doors rather than the bar that blocks them? Can algebra become the language of the relationships among dimensions, areas, and volumes? Among rise, run, slope, and rate? Can it become the language of success, opportunity, and access?

Mastery of the language of algebra also lays a foundation for mathematical maturity. Familiarity with mathematical ideas enables an auto mechanic to manage comfortably the business affairs of an independent garage, to make informed decisions about loans and equipment depreciation. Good algebra skills set the stage for a trained secretary to undertake basic accounting and advance to a position as manager of a small business. Those same skills enhance the computer literacy of an electronic technician with the ability to implement spreadsheet calculations that speed and record a new procedure for testing equipment.

Necessity builds ownership, and genuine applications are evidence of the necessity of algebra in the professional life of the technical work force. Students will take ownership if their algebra courses incorporate authentic applications like those they will encounter in their professions. The challenge is identifying those applications and using them as instructional vehicles.

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To Strengthen Technical Education Systemically

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We should undertake to develop a system of educational development, research, and practice that enables technical education to evolve progressively, increasing in its productivity and value for American workers, businesses, and the society. Components of such a system would include:

- Ongoing research, development, and practices of workplace activity to understand and improve mathematical reasoning and understanding that are integral in productive work and continuing learning in technical occupations;
- Ongoing research, development, and practices of technical education in mathematics to understand and improve preparation of students to become productive workers and learners in technical occupations;
- Ongoing interaction among the communities concerned with technical work and technical education with communities concerned with mathematics education in pre-technical schooling, to understand and improve the inclusion of mathematical learning activities that prepare students well for technical education and work in the general curriculum.

Mathematical Reasoning, Understanding, and Learning in Work

The ways in which mathematical reasoning and understanding are involved in productive work are poorly understood, but scientific resources are available for developing a strong knowledge base about these processes. As Richard Lesh pointed out in our workshop discussions, most job analyses of the kind that are done currently take an inappropriately limited view of mathematics that tends to recognize only standard symbolic operations as mathematical reasoning. A more realistic view would recognize the need for many workers to represent and reason about quantities and complex quantitative relations in ways that are crucial for job success, but that often do not use standard forms of mathematical representation in job settings.

Researchers in cognitive science have developed methods to investigate and characterize knowledge of procedures and concepts by individuals, and researchers in social science have developed methods to investigate and characterize patterns of social and individual activity in technical work. There is a strong potential for interdisciplinary work that would merge these lines of scientific accomplishment in ways that would provide important basic understanding of the requirements for mathematical understanding and reasoning in successful job performance.

Technologies for the support of technical work are often designed to off-load cognitive activity, rather than to support individual and social processes of reasoning, understanding, and learning. Rather than trying to minimize cognitive involvement in work, we could design information systems that would engage workers significantly and support learning so that their knowledge, understanding, and abilities for reasoning in work would grow productively in their work activities. Development of technologies designed for learning by individuals and groups in their work and research about the individual and social aspects of cognition and learning in work should be two aspects of a single activity, in which better technologies are suggested by results of research, and research examines the functioning of new technologies in order to support further improvements.

An important goal would be to organize technical work in ways that supported the development of careers, building into the work of any job resources for learning that would qualify successful workers for meaningful advancement. Such reorganization would need supporting analyses not only of the cognitive and behavioral aspects of success in specific jobs but of requirements for success in more advanced jobs, and how work experience can provide the basis for performance and learning in those more advanced positions.

Meaningful Mathematics Curriculum for Technical Work

Studies of the processes of mathematical reasoning, understanding, and learning in work should be integral components of projects that also include development of mathematics curriculum designed to prepare students for those kinds of reasoning, understanding, and learning. These curriculum efforts would be focused particularly in community colleges, technical training schools, and mathematics coursework in the last year or two of high school for students who are not intending to enter 4-year colleges and universities.

The development of such curricula would be consistent with general efforts being made in mathematics education reform. These efforts are developing environments and teaching methods for meaningful mathematics learning, involving understanding of mathematical concepts and methods both in contexts of mathematical inquiry and in contexts of use of mathematics to reason and understand in other domains. It seems likely that the mathematics curriculum for technical work might emphasize understanding of mathematical ideas and methods as resources for reasoning in other domains. Development of that curriculum requires progress in fundamental knowledge of what constitutes generality of mathematical understanding. Such general understanding is different from knowing how to manipulate abstract symbols, but probably includes symbolic fluency. One crucial question is how mathematics can be 'earned so that its symbols can function effectively as a language for formulating and solving problems in other domains with understanding and communicating effectively about alternative conjectures and courses of action.

Mathematical Preparation for Technical Education and Work

Concerns about technical education should play a major role in shaping the general curriculum in mathematics that students will have through most of their elementary and secondary school education. Achieving the major general goal of reform efforts, to foster practices of mathematical reasoning and sense-making, can contribute to the value of mathematics education for all students, including those who eventually enter the technical work force. For mathematics education to benefit all students, however, we need to fundamentally restructure the social organization of mathematics learning away from its present emphasis on selection of students who are qualified to continue in serious mathematical study, to an activity in which all students are expected to succeed and are supported in developing their abilities to participate in meaningful mathematical practices, as Robert Moses emphasizes.

An Agenda for Developing a National Competence for Technical Mathematics Education

Goals such as these will not be achieved by simply instituting a program. Instead, we need to build a basic competence for the educational system to evolve in positive directions. This competence should include structures of support for collaborative research, development, and practice.

The effort will require the participation of educational practitioners, educational researchers, developers of curriculum and learning environments, and members of professional communities that practice the kinds of technical work that students are preparing for. Members of these communities have knowledge and abilities that could support productive participation in these efforts, but a substantial investment would be needed to support development of practices of collaboration in which the required multiple resources can be utilized for the benefit of technical mathematics education.

Development of the research agenda for studying mathematical reasoning and understanding in work can be accomplished by constructing collaborations of researchers who study the social and individual aspects of cognition, along with mathematics education researchers and members of technical professions. These studies would investigate processes of social interaction and individual reasoning in work settings, focusing on ways in which understanding of mathematical concepts and principles functions in effective work activity. They also should include efforts to design resources to support learning in technical work, including technologies that engage workers in collaborative reflection about their methods of recognizing and solving problems in their work activity.

Such studies need to be coupled with efforts to develop curriculum materials, learning environments, and teaching practices that prepare students for the kinds of individual and social activity that are important for success and advancement in technical occupations. This will require the collaboration, in addition to the groups discussed above, of educational practitioners in community colleges, technical education programs, and secondary schools, along with professional mathematicians who will contribute insight into significant aspects of the subject matter. Such collaborations need to provide for engagement of all the participants in the various functions of the activity, including analyses that contribute to research products as well as development of materials and practices that contribute to the improvement of educational activities.

Thoughts About Reshaping Algebra to Serve the Evolving Needs of a Technical Workforce

Richard Lesh
Educational Testing Service

At each grade level in school, and in each mathematics course that is offered, the justification for what is taught and how it is taught is mainly based on conceptions (which are often naive and misinformed) about what is needed as preparation for the next mathematics course in the program. Therefore, in a perverse and distorted way, the goals of the entire mathematics curriculum are aimed to serve the fraction of 1 percent of the students who never quit taking more courses (i.e., doctoral students in mathematics); and this is the case in spite of the fact that potential Ph.D.'s in mathematics are as poorly served as any other subgroup of students by the poor quality of mathematics that is offered.

What would the algebra curriculum look like if an attempt actually was made to teach mathematics so as to be useful to meet the needs of a technical work force? The following fact is clear. If today's responses to this question are the same as they were (or could have been) 10 years ago, then the chances are high that we are seriously naive and misguided. Furthermore, *what* is taught may be far less significant than *how* it is taught.

During the past decade, a revolution has occurred in the core curriculum areas—reading, writing, and mathematics. Powerful conceptual technologies are used on a daily basis in fields ranging from the sciences to the arts and the humanities, and in professions ranging from agriculture to business and engineering, and in employment positions ranging from entry level to the highest levels of leadership; and these tools have radically expanded the kinds of knowledge and abilities which are needed for success in a technology-based society, and the kinds of problem-solving/decision-making situations which are priorities to address in instruction and assessment. For example, when a business manager uses a graphing calculator (or a graphics-based spreadsheet) to make predictions about maximizing cost-benefit trends, these tools not only amplify the manager's conceptual and procedural capabilities when dealing with old decision-making issues, they also enable the manager to create completely new types of business systems which did not exist before the tools were available; and completely new types of problems and issues arise as priorities to address.

As an appendix to this brief paper, I have attached a comparison of mechanistic and organic views of mathematics, learning, and problem solving. My claim is that, as long as mathematics educators cling to the kind of mechanistic views that are described, it will do little good to tinker with the topics that are taught, or the problems that are presented, in algebra instruction. However, shifting to organic views involves much more than simply making a vow to do so. Just as at other levels of the mathematics curriculum, such a shift will only be accomplished effectively when reforms are based on a solid body of research investigating the nature of students' knowledge (e.g., in topic areas such as whole number arithmetic, rational numbers, or proportional reasoning). Yet, in the area of algebraic knowledge and abilities, the research base is only beginning to develop.

In the absence of the preceding research base, it seems likely that the scenario that plays itself out will be much the same for algebra as it was for other levels and types of mathematics instruction. That is, in the history of mathematics and science education R&D, every decade or so, the pendulum of curriculum reform swings from emphasizing behavioral objectives (BOs: strings of factual or procedural rules) to emphasizing process objectives (POs: content & content-independent processes and strategies), or vice versa. Also, periodically, some attention is given to affective objectives (AOs: feelings, values). Yet, in all of the preceding cases, cognitive objectives (COs: models for describing patterns and regularities in the world) tend to be almost completely neglected. To a large extent, for curriculum reform in algebra, the rhetoric that I hear about algebra instruction sounds like little more than another swing from BOs to POs and AOs; and, if this is the case, then I see little more chance of success this time than at similar times in the past.

Each time another curriculum reform is attempted without success, post hoc assessments tend to notice that—in the euphoria of developing exciting new instructional materials (e.g., books, software)—two factors were once again neglected: teacher development and assessment.

At the level of high school/college mathematics, as much as at any other level of the curriculum, instructors notions about the nature of students' knowledge is extremely naive; and the assessment instruments that are used reflect only narrow, shallow, and biased views about what it means to understand the domain in question.

In the past, if a narrow conception of mathematical ability was correlated with a more representative interpretation, testing specialists tended to treat modest (.5 to .6) correlations as though they were sufficient—that is, sufficient for selecting small-but-adequate numbers of students for access to scarce-but-adequate resources. But today, even beyond concerns about fairness, national assessment priorities have changed. At a national level, our foremost problem is not to screen talent; it is to identify and nurture capable students. The pool of students receiving adequate preparation in mathematics is no longer adequate; and far too many capable students are being "shut out" or "turned off" by textbooks, teaching, and tests which give excessive attention to views of mathematics and problem solving which are fundamentally inconsistent with national curriculum and assessment standards for school mathematics.

My own research on real-life problem solving suggests that most students, including those who have been identified as unlikely to succeed, are able to invent more powerful mathematical ideas than their teachers are trying to teach (Lesh & Lamon 1992). Yet these abilities seldom are recognized or rewarded because of the impoverished conceptions of mathematical ability that are built into most textbooks, tests, and teaching—especially at the level of introductory college-level mathematics.

How can a broader range of mathematically capable students be identified and encouraged? Our research suggests that the key is to focus on the kind of models and modeling processes that are needed when elementary mathematical systems are used to describe, explain, manipulate, predict, or control everyday problem solving situations (Lesh, Hoover, & Kelly 1993). On the other hand, our research also suggests that new students are not likely to emerge if modeling is treated as another attempt to teach content-independent Polya-style heuristics, strategies, and processes (POs), or if the applications are used mainly as devices to increase motivation and interest (AOs).

As a result of the conference at which this brief paper was presented, there were many reasons to be hopeful about the chances of success for curriculum reform in algebra. For example, mathematicians have always been well represented among those who are willing to devote extraordinary amounts of time and energy to the improvement of education for all students. On the other hand, the community of university-based mathematicians sometimes tend to be a rather arrogant and impatient lot, and they are often unwilling to question the validity of the implicit theories about learning and problem solving—or their prejudices about what should be taught and how it should be taught in schools.

Simplistic solutions and piecemeal approaches to curriculum reform will be no more likely to succeed for algebra instruction than for any other level or strand of the mathematics curriculum; and many basic questions remain unanswered which will be critical for long-term progress. For example, even in this brief paper, my comments have touched on issues ranging from equity to technology to psychology and to competency in real-life situations. Clearly, some immediate actions can be taken which are likely to yield positive effects. But, at the same time that action-oriented initiatives are taken, similar resources should be devoted to basic research as well as to the translation of practical problems into researchable issues, and to the translation of research results into practical implications.

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Appendix
A Comparison of Mechanistic and Organic Views of Mathematics, Learning, and Problem Solving

The Nature of Mathematics

Mechanistic Perspectives

The objectives of instruction are stated in this form: *Given...the student will...* That is, knowledge is described using a list of mechanistic condition-action rules (definitions, facts, skills) some of which are higher order metacognitive rules for making decisions about: (1) which lower level rules should be used in a particular situation, and (2) how lower level rules should be stored and retrieved when needed.

Organic Perspectives

Knowledge is likened not to a machine but to a living organism. Many of the most important cognitive objectives of mathematics instruction are descriptive or explanatory systems (i.e., mathematical models) which are used to generate predictions, constructions, or manipulations in real life problem solving situations ... or whose underlying patterns can be explored for their own sakes.

According to the Mathematical Sciences Education Board's *Reshaping School Mathematics*, two outdated assumptions are that: (i) mathematics is a fixed and unchanging body of facts and procedures, and (ii) to do mathematics is to calculate answers to set problems using a specific catalog of rehearsed techniques. (p. 4) As biology is a science of living organisms and physics is a science of matter and energy, so mathematics is a science of patterns. ... Facts, formulas, and information have value only to the extent that they support effective mathematical activity. (p. 12)

The Nature of Problem Solving

Mechanistic Perspectives

In general, problem solving is described as *getting from givens to goals when the path is not obvious*. But, in mathematics classrooms, problem solving is generally restricted to answering questions which are posed by others, within situations that are described by others, to get from givens to goals which are specified by others, using strings of facts and rules which are restricted in ways that are artificial and unrealistic. In this way, students' responses can be evaluated by making simple comparisons to the responses expected by the authority (the tutor).

Organic Perspectives

Many of the most important aspects of real-life problem solving involve developing useful ways to "think about" the nature of givens, goals, and possible solution paths. Solutions typically involve several "modeling cycles" in which descriptions, explanations, and predictions are gradually refined and elaborated. Therefore, several levels and types of responses are nearly always acceptable (depending on purposes and circumstances); and students themselves must be able to judge the relative usefulness of alternative models.

Problems in textbooks and tests tend to emphasize the ability to create meanings to explain symbolic descriptions; but real problems more often emphasize the ability to create symbolic descriptions to explain (manipulate, predict, or control) meaningful situations. For example, for a mountain climber, the main problem is to understand the terrain of a given mountain or cliff; and, once the terrain is understood, the activity of getting from the bottom to the top is simply a (strenuous, complex) exercise.

The Nature of Experts

Mechanistic Perspectives

Humans are characterized as information processors; and outstanding students (teachers, experts) are those who flawlessly remember and execute factual and procedural rules ... and who are clever at assembling these facts and rules in ritualized settings.

Organic Perspectives

Experts are people who have developed powerful models for constructing, manipulating, and making sense of structurally interesting systems; and they are people who are proficient at adapting, and extending, or refining their models to fit new situations.

The essence of an *age of information* is that many of the most important "things" that influence peoples' daily lives are communication systems, social systems, economic systems, education systems, and other systems which are created by humans—as a direct result of internal structural metaphors which structure the world at the same time they structure humans' interpretations of that world. Therefore: (i) there is no fixed and final state of evolution, even in the context of elementary mathematical ideas, and (ii) reducing the definition of an expert to a single static list of condition-action rules is impossible (in principle) ... not just difficult (in practice).

The Nature of Learning

Mechanistic Perspectives

Learning is viewed as a process of gradually adding, deleting, and de-bugging mechanistic condition—action rules (definitions, facts, or skills).

Organic Perspectives

Humans are model builders, theory builders, and system builders; and the models that are constructed develop along dimensions such as concrete-to-abstract, particular-to-general, undifferentiated-to-refined, intuitive-to-analytic-to-axiomatic, situated-to-decontextualized, and fragmented-to-integrated. Therefore, development often involves discontinuities and conceptual reorganizations ... such as those which occur when students go beyond thinking WITH a given model to also think **ABOUT** it. Experts not only know more than novices, they also know differently.

If the precise state of knowledge is known for an expert (E) and for a given novice (N), then the difference between these two states is portrayed as the subtracted difference (E-N).

The Nature of Teaching

Mechanistic Perspectives

Teaching is considered to involve mainly: (i) demonstrating relevant facts, rules, skills, and processes; (ii) monitoring activities in which students repeat and practice the preceding items; and (iii) correcting errors that occur.

Teaching focuses on providing carefully structured experiences for students ... where they confront the need for mathematically significant models, and where responses involve constructing, refining, integrating, or extending relevant models.

Organic Perspectives

According to the Mathematical Sciences Education Board's publication *Everyone Counts*: The teaching of ma hematics is shifting from an authoritarian model based on "transmission of knowledge" to a student-centered practice featuring stimulation of learning." (p. 5) ... Teachers should be catalysts who help students learn to think for themselves. They should not act solely as trainers whose role is to show the "right way" to solve problems. ... The aim of education is to wean students from their teachers. (p. 40)

Algebra for the Technical Workforce of the 21st Century

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C.D. Searle & Company

Like everything else, the algebraic needs of the technical work force are continually changing over time. What are the impediments to changing the curriculum in algebra? What are some of the algebraic concepts, educational issues, and possible approaches which may prove useful in educating students in algebra? This essay considers these ideas from the perspective of the educational needs of students planning on entering the technical work force in the 21st century.

Many people feel that U.S. students are already entering the technical work force with inadequate preparation in mathematical problem solving, planning and optimizing, and mathematical modeling. With the increasing mathematization of human affairs in the social, cultural, commercial, and scientific spheres we are currently experiencing and can expect to continue, the preparation of people destined for the technical work force will only get worse and not better if no attempts are made to reform instruction in mathematics in general and algebra in particular. (See Glimm 1991; Friedman, Glimm, and Lavery 1992.)

This essay is based on the perspective of one individual's career in industry which has spanned more than two decades. This career has consisted of a series of positions in many applied computer departments in a pharmaceutical company. The departments ranged from scientific and statistical computer support, centralized corporate management information systems, main frame system software, and back again. The work involved provided exposure to many facets of the algebraic plight of people in the technical work force and the application of mathematical sciences to the pharmaceutical industry, from undergraduate student interns to doctoral level research scientists.

Impediments to Curricular Change

One of the biggest impediments to change within many businesses seems to be the fact that the management structure usually does not perceive mathematics in general and algebra in particular as being relevant to the day-to-day affairs of the company. This has been one of the common understandings brought to light by the Mathematical Association of America's Committee on Mathematicians Outside Academia in a continuing series of panel sessions on mathematical life outside academia. Another impediment to change is the low achievement level in algebra which is currently being experienced. It is hard to improve to a high level of performance, if not seemingly impossible, if one has to start essentially from zero.

Some of the managers of technical workers have come to realize this in Illinois and have adopted strategies to cope. Instead of expecting technical workers to accurately perform routine ratio and proportion calculations, they provide such workers with premixed or known amounts of materials and COLOR CODE them. The worker is not expected to be colorblind. They are told to take the RED bottle and add its' contents to the YELLOW bottle and pour the resulting solution into the BLUE container. How relevant do such people feel algebra to be? If they do feel that algebra is important, they often cope by sending their employees back to the community colleges or, worse yet, they cope by developing their own mathematics courses, entirely bypassing the educational system.

This is not an isolated phenomena. The widespread infusion of the microchip into every and any device goes hand in hand with a consumerism which presents this increasing automation and ease of use as highly desirable, if not necessary. Cars. programmable VCRs. Photography: automatic exposure, automatic flash, just point-and-shoot! Nobody has any need to carry out an actual calculation anymore!

Yet again another impediment is the perceived lack of utility of algebra. One of the problems continually being faced in industry is the confrontation between people with a classical, school mathematics understanding of mathematics and the role of algebra and practical, real-life business problems. Oftentimes the assumptions upon which the algebraic methods are based cannot be justified in practice, with the net result that abstract methods such as algebra are not being considered even when they are applicable, relevant, and useful.

This does not bode well for reforming the algebra curriculum because of widely held beliefs that algebra has no real role to play in our society after secondary school. "When was the last time you had to use the quadratic formula in everyday life?" is a question that is oft repeated to justify the lack of utility of algebra. Even the microchips are not seen as doing this kind of algebra for you.

Fundamental Algebraic Concepts

What concepts in algebra help prepare students for entry into the technical work force? A deeper and more comprehensive understanding of the basic concepts in algebra would go a long way. For example, one of the situations which seems rather common is failure to grasp the patterns in a situation. Each problem is seen as unique, needing to be addressed with specific methods if at all. What is missing is the facility that algebra provides, especially, for example, in 'story problems,' for providing abstract models of problem structure and solution. The impact in business is, given such an abstract understanding of a business process, the ability to create general purpose software tools which solve families of problems. When this process does occur it is perceived as a very powerful and effective one. When it does not, there is a tremendous lost opportunity cost which is paid by business.

Fundamental Educational Issues

Sometimes the connections between algebraic problem solving techniques at the primary and secondary school level are lost by students as they go through college, possibly graduate school, especially by people with less, as opposed to more, mathematics. In 'story problems' the idea of 'symbols for unknown quantities is introduced. One of the reasons is to provide a method, expressed in terms of the symbols, which is easy to comprehend and use. Usually the 'quantities' are numerical. What happens to many people in the technical work force is the loss of the idea of 'representation' and why it was (is, could be) so effective. Many a graduate level scientist or statistician has failed to take advantage of the simplifications and power 'let x = ...' provides when complex (symbolic) quantities have to be manipulated. The advantage apparently does not go away when mathematical technology is employed. On the contrary, such technology becomes more effective the more its work is lessened through appropriate use of algebraic concepts.

This fall off in retention of algebraic concepts as time goes by can be specifically addressed in the K through graduate school mathematics curriculum by the use of 'strands' of algebraic ideas to pull these ideas together and infuse them with relevance and life as the student progresses through the curriculum, to the point that algebraic modes of thought are seen as natural and intuitive.

Instructional Input From the Technical Work Force

Students and faculty at all levels from K through graduate school should be exposed to problems actually being faced by industry. This can be done through a program of student and faculty interns. In the schools, career counseling departments could be charged with the mission to actively identify those employers in their area with mathematical (algebraic) needs and facilitate the interaction between businesses and those in the school responsible for curriculum development.

Mathematics depa ·ments could request businesses to supply speakers for talks to students and faculty. If those talks were focused on algebra and the role that mathematical sciences play in technology, such as computer algebra systems, then there could be an infusion of algebraic methods actually, currently being practiced into the curriculum development process. The real problem here is to take what is already happening locally on a small scale in a sporadic way and incorporate it into the educational process of delivery of algebraic ideas via systemic change.

Summary

How do algebra concepts prepare people for employment and lifelong learning in mathematics? One invariant in modern life is the universal presence of change. In the 20th century, technological change has been increasing in an exponential fashion, no where more so than with technology for machine computation.

By providing the reform of curriculum in algebra with input on current and next generation mathematical sciences technology, students would be better prepared to step into an industrial problem solving environment with algebraic tools and, more importantly, algebraic power which are both relevant and effective in terms of facing this anticipated demand for continual personal growth and change.

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Some Thoughts on Algebra for the Evolving Work Force

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Everything would be perfect if...algebra made sense.—Amstel Beer TV advertisement

In real life, I assure you, there is no such thing as algebra.—Fran Lebowitz, December 1, 1993, *Wisconsin State Journal*

These sentiments—suggesting that algebra does not make sense or is just something one does in school—which are so common in the public at large, are at the root of the problem of providing an appropriate algebra curriculum for the evolving work force in this country. In this paper, the roots and role of this perception are addressed first. This is followed with a suggestion for a shift in emphasis in what and how algebra in school mathematics might be taught. The paper concludes with a description of a set of units in the middle school designed to teach algebra based on this shift. Our argument throughout this paper is that all students, as members of tomorrow's work force, need to see algebra as important and useful, and to regard it as making sense.

School Algebra

The Algebra for the Twenty-First Century Conference, in August 1992, summarized the current view of school algebra as a collection of isolated topics along with a set of prescribed skills that are quickly forgotten and with a limited view of variable and function. This view is undoubtedly based on how algebra is organized and taught in schools. For example, Weller (1991) described a typical algebra class. The first 5 to 10 minutes of each class usually began with the correction of the previous day's work. The second segment of the class, which lasted from 10 to 17 minutes, was devoted to the presentation of new material. Example problems that had been placed on the chalkboard prior to the beginning of school were used throughout the day. After each class period, the answers were erased so that the problems would not have to be rewritten for the next class. The remaining third of the class was study time, during which students were to work on the day's assignment, which was always to be completed by the following day. During this study period, the teacher would either look at individual students' work and assist those who had questions or would sit at her desk at the back of the room evaluating notebooks and/or calling students to her desk to ask about late assignments.

Emphasis was placed upon determining the one correct answer, and the teacher provided a step-by-step explanation of her thinking processes as exercises were computed. Students who had skipped steps, but who arrived at the same answer as the teacher were immediately corrected. Students were expected to use the prescribed computational thinking process presented in the textbook. Practice problems were prototypes of those in the assignment that students would be eventually completing. Students rarely raised questions or sought explanations regarding computation.

The following student-teacher exchange illustrates the importance of following the teacher's process. After taking a test, a student asked about the wording of one of the test questions, which he had answered incorrectly. The teacher's reply was, "That's not the way you should be thinking about it, if you are thinking right." Embarrassed by the teacher's comment, the student looked down at his test. No further comment was made by the student, nor was further explanation suggested by the teacher. After the teacher's remark, all hands went down.

The teacher had transferred authority to an unnamed "they." The teacher was the mediator between the source of knowledge and those who were expected to acquire the prescribed information. The knowledge of the teacher was deliberately suppressed in favor of that in the textbook. The study of algebraic content was justified on the ground that it was necessary for college entrance. Transmission of abstract knowledge appeared to be the primary focus of mathematics instruction. On several occasions, the teacher acknowledged to students the fact that the information being presented would soon be forgotten. The only justification for studying the material presented was, in fact, to pass subsequent tests so that they might meet the requirements of college entrance examinations. Thus, for both the teacher and the student, the purpose of the course was not to learn the big ideas of algebra or provide students with employable skills. Its sole purpose was access to further courses and to college.

The textbook was the primary factor in determining the curriculum. Algebra was presented as a process of learning methods needed to solve standard problems. The teacher relied on surface features of algebra and taught manipulation skills and the routine application of standard algorithmic techniques. This teacher's perspective of algebra fits the dualistic conception of mathematics in that an external authority, in the form of the textbook, was the source of truth, and algebra consisted of a collection of true propositions apart from the context in which they were developed.

Wenger (1987) found that students could use the various techniques they had studied but when problems were presented out of context and the students had to select, as well as apply the methods of solving them, they had great difficulty. Students sometimes perform the manipulations correctly, but they make "strategy errors," which are poor choices of what to do next (Wenger 1987). This observation suggests that students' difficulties result not so much from the content of their mathematical knowledge base, but from its organization. The ways in which knowledge is organized, accessed, and used are important determinants of intellectual performance. The packaging of subject matter into single chunks in a one-rule-per-section format may leave students with the impression that solutions should be straightforward, requiring only the use of one rule or procedure.

The following assumptions stem from Wenger's work (1987) with high school and college students:

- Students learned algebra primarily from examples and practice tasks found in textbooks. They did not usually learn by understanding the explanations of procedures and using those explanations. Rather, they figured out what the procedures were about by working through them.
- An important force driving student behavior was the need to make sense of things by creating simple, straightforward procedures that work. As students worked problems, they invented rules for the procedures that seemed to fit the expected answers. Often those rules were as simple as possible and were incorrect.
- The rules and procedures that students invented became part of their problem-solving approach. Students often continued to use one of these inferred procedures until encountering a task that revealed it to be incorrect (p. 221).

An instructional consequence stemming from these assumptions is that if a list of textbook tasks can be completed successfully with the right answers using a sequence of procedures inferred by the student, then the correctness or the validity of the strategy is confirmed and the student believes he or she "understands" the algebraic topic.

The underlying basis for these descriptions of traditional instruction in algebra rest with common beliefs about mathematics. Many nonmathematicians such as sociologists, psychologists, school administrators, and even curriculum generalists see mathematics as a static, bounded discipline. Indeed, according to Edward Barbeau (1989), "Most of the population perceive mathematics as a fixed body of knowledge long set into final form. Its subject matter is the manipulation of numbers and the proving of geometrical deductions. It is a cold and austere discipline which provides no scope for judgment or creativity" (p. 1). These views reflect an absolutist perspective about mathematics and are undoubtedly a reflection of the mathematics studied in school rather than an insight into the discipline itself.

During the past two decades, there has been a growing awareness of the need to represent better what mathematics is about, to illustrate what mathematicians do, and to attempt to popularize the discipline. This is not easy because the subject has many facets. One can define mathematics "as a language, as a particular kind of logical structure, as a body of knowledge about numbers and space, as a series of methods for deriving conclusions, as the essence of our knowledge of the physical world, or as an amusing intellectual activity" (Kline 1962, p. 2). Understanding these variations is important because different features have been emphasized in school mathematics programs at different times and by different authors. Furthermore, proponents of the current reform movement argue for a particular perspective that is different from that held by diverse individuals, including the perspective of many (if not most) working mathematicians (Wheeler 1991).

In summary, the roots for the current perspectives about algebra are from an absolutist perspective about mathematics in general—"a body of truth far removed from the affairs and values of humanity" (Romberg 1992, p. 751). According to this view, to know algebra is to master a large collection of concepts and procedures. In addition, there is a second problem with this perspective in American schools—the idea that the successful completion of algebra courses gains one access to the future (e.g., college entrance, programs, jobs) (Secada, in press). Thus, by making the mastery of a collection of poorly understood procedures the goal of such course taking, the courses function as filters for denying access to large numbers of students. The fact is that the mathematics course, algebra, creates a major barrier to current reform efforts. Students often find algebra one of the most alienating parts of their school curriculum. Based on whether they are permitted to enter this course or on their experiences with it, they come to view themselves as having little potential for further involvement with mathematics (Wisconsin Center for Educational Research 1993).

A Shift in Emphasis

A growing number of philosophers of mathematics (e.g., Davis & Hersh 1981; Kitcher 1988; Lakatos 1976; Tymoczko 1986) argue that mathematics is "fallible, changing, and like any other body of knowledge, the product of human inventiveness" (Ernest 1991, p. xi). Mathematics must be considered as a process of inquiry and coming to know, a continually expanding field of human creation and invention, not a finished product. Such a dynamic view of mathematics has powerful educational consequences. The aims of teaching mathematics need to include the empowerment of learners to create their own mathematical knowledge; mathematics can be reshaped, at least in school, to give all groups more access to its concepts and to the wealth and power its knowledge brings; and the social contexts of the uses and practices of mathematics can no longer be legitimately pushed aside—the implicit values of mathematics need to be squarely faced. When mathematics is seen in this way, it needs to be studied in living contexts that are meaningful and relevant to the learners, including their languages, cultures, and everyday lives, as well as their school-based experiences.

This philosophy of mathematics has been labeled by Ernest as "social constructivism" which, he argues, is based on three premises:

- The basis of mathematical knowledge is linguistic knowledge, conventions, and rules, and language is a social construction.
- Interpersonal social processes are required to turn an individual's subjective mathematical knowledge, after publication, into accepted objective mathematical knowledge.
- Objectivity itself will be understood to be social. (p. 42)

In particular, from this perspective, views of the nature of algebra are changing. An important factor underlying current changes in the view of the algebra curriculum is the move from an emphasis on manipulative skills to an emphasis on conceptual understanding and problem solving. This is a move from doing algebra to using algebra. One reason for this is the introduction into the classroom of increasingly more powerful calculators and computers that are changing what students are able to do in this subject.

Attention needs to be given not only to what is being represented in terms of underlying structures and relationships in problems (the semantic aspects of algebra) but also to how these are represented (the syntactic aspects of algebra). An important feature of algebraic thinking is the development of flexibility with regard to mode of representation. Students need to be able to recognize different forms of representation, know what advantages each has to offer, and be able to translate freely among them. Both the involvement of technology, and research evidence as to the kinds of procedures that students use naturally, point to a greater use of recursive techniques and a possible abandoning of some of the deductive procedures that form a large part of the current algebra curriculum (Booth 1989). The current increase in attention on graphing and functions and the corresponding decrease in attention on factoring, powers, and roots are continuations of a gradual process that has been developing for some time.

Shifting the emphases in school algebra raises some difficult epistemological and philosophical questions. If it is a generalization of arithmetic, then its behavior is tied to the properties of quantitative arithmetic. However, if it is a forerunner of abstract algebra, it is better understood as being determined by a set of rules for the manipulation of the symbols in the system in which it is written. School algebra originates in the boundary between formalized arithmetical algebra and the development of arbitrary algebraic systems. As a consequence, it presents learners with validation problems. Do you determine the truth of an algebraic statement, or justify the steps of an algebraic transformation by the appeal to the behavior of numbers, or according to a set of formal rules? Wheeler (1989) suggests it is not surprising that so many high school students and teachers are confused and uncertain about the reasons underlying the procedures in school algebra (Wheeler 1989).

Mathematics' educators should be involved in exploration of existing school algebra and serious examination of plausible alternatives. We need to know more about the role of routines in algebra, whether the practice of routines should follow or accompany conceptual understanding, what students need to know in order to manage and direct algorithmic behavior, to what extent algebraic behavior can be deduced from arithmetic or from geometry, how to abstract patterns, how to generalize and how to justify generalizations, and how to apply algebra to problem situations.

> "The winds of change are whipping around the algebra curriculum for a variety of reasons. Some concerns are generated by research findings about what people do and do not learn well. A curriculum that has been with us for quite some time is in a state of flux and under close examination."
>
> (Ed Silver, quoted in Wenger 1987, p. 244)

Kaput (e-mail, April 6, 1993) identifies the following as contexts within which algebra reform is situated:

- Understanding the nature of the domain of algebra;
- Understanding algebraic reasoning and how it is different from other forms of reasoning;
- Understanding how algebra is best learned, by whom, with what tools and resources, and at what developmental levels; and
- Understanding what kinds of classrooms foster understanding and competence, what are teacher-preparation implications, and how to bridge the gap between middle school and high school.

The NCTM *Standards* (1989) emphasize the importance of models, data, graphs, and other mathematical representations to facilitate the learning of concepts and structure in algebra. Students should understand the concepts of variable, expression, and equation, and should explore the interrelationships between number patterns and tables, graphs, verbal rules, and equations. They should informally investigate inequalities and nonlinear equations and develop confidence in solving linear equations by both formal and informal methods. They should be able to use algebraic methods to solve a variety of real-world and mathematical problems. The goals articulated in the *Standards* stress two aspects of algebra: algebra as a language or structure and the concept of function.

Kaput (cited in Wenger 1987) argues that it is a serious error to treat algebra as if it were a self-contained domain. The basic problem with learning algebra is that it is a language without much semantic content. The semantic content that it does have has developed among experts through their manipulation of sophisticated algebraic expressions over long periods of time. Therefore, much of its substantive structure has a sort of invented semantic content. In Kaput's view, the primary source of meaning for algebraic expressions is in their capturing of numerical patterns from arithmetic, from science, or from anywhere you find them.

For students, algebra should be a way to express real-world phenomena in mathematical language. Their experience of algebra should include many and varied problems from the real world so they will gain understanding of the power and usefulness of algebraic notations and conventions. Algebra should be an extension of the study of patterns, functions, and ratios. There should be an emphasis on understanding concepts in realistic contexts as opposed to manipulating symbols and memorizing procedures void of any meaning. Students should develop an understanding of variables, expression, equations, and properties through the use of tables, graphs, and physical models. Manipulating expressions can be introduced as the need arises in problem contexts. Although by the end of eighth grade, students should be able to formally solve linear equations, they should have arrived at this method only after using concrete and informal techniques. In addition, they should have had a wide range of experiences with inequalities and nonlinear equations.

Algebra in a New Curriculum

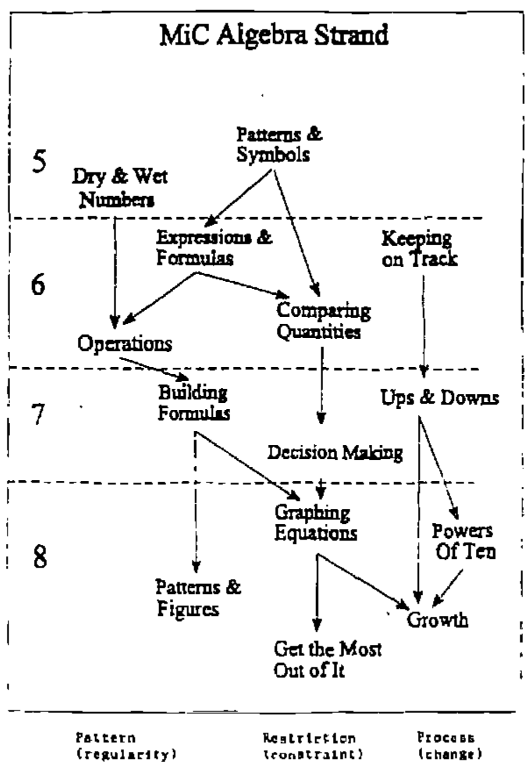
The following example comes from *Mathematics in Context* (Romberg et al., in press), a middle school curriculum project funded by the National Science Foundation and being developed jointly by the National Center for Research in Mathematical Sciences Education at the University of Wisconsin-Madison and the Freudenthal Institute at the University of Utrecht.

The belief underlying the *Mathematics in Context* curriculum is that mathematics is a process of inquiry and coming to know that includes creating and inventing. In presenting mathematics as a human activity, the concept of "meaning" is central. Especially important is the question of how to involve students so that problems will have a sense of reality for them. In what Treffers (1987) calls a "domain theory of realistic mathematics education," realistic refers to reality in the sense of what is real for students. This does not necessarily imply real life situations, but may also include fantasy and mathematical objects. Thus, mathematics is viewed as a human activity that starts in a reality that makes sense to students.

The idea of "mathematizing" (Freudenthal 1983) is central to this view of mathematics education. According to Freudenthal, mathematizing is the process by which mathematicians streamline or simplify reality to meet their needs and preferences. Mathematizing involves representing relationships within a complex situation in such a way as to make it possible to put them into a quantitative relationship with each other. The mathematician must first decide which variables and relationships between variables are important and which do not matter when confronting a complex problem situation. Then, a mathematical model is made, numbers reassigned to variables, and numerical procedures used to make predictions. Finally, the results are examined. Mathematizing is an organizing activity. It includes mathematical content, expression, and more intuitive lived experience, expressed in everyday language. Mathematics is seen as a process, as a "doing" discipline, as a practical skill that involves both art and techniques.

Algebra in *Mathematics in Context* (Romberg et al., in press) as diagrammed in figure 1 reflects three important ideas: regularity, restriction, and process. The titles of the algebra units are placed in the diagram according to which of these ideas they are designed to develop.

Figure 1—Logic of the algebra strand (Roodhardt as cited in Romberg et al., in press)



Regularity or pattern involves looking at and expressing relations among two or more quantities. This is a representation of quantitative information, and then, reasoning about this information. Key mathematical ideas required to reason about regularities are the core concepts of beginning algebra: variables, functions, relations, equations, and inequalities. This involves the mathematical process of generalizing or of expressing generality. Patterns and regularity are part of the process of generalizing. Generalizing is characterized by looking for similarities, analogies, and classifying.

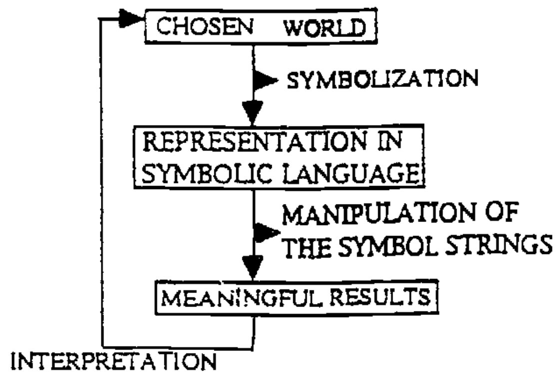
Restrictions involve restrictions on the domain of a function and comparisons that function as constraints on equations and inequalities. It involves maximizing and minimizing a quantity while at the same time making sure that constraints on other quantities are satisfied. Limiting interpretations and determining validity are part of this theme. Equations describe the constraints between quantities in situations.

The traditional approach to the mathematics of process or change is calculus. The changing system is modeled by a special equation that describes the relations between the rates of change of different variables. Process includes this, but it also deals with recognizing, describing, and using functional relationships; with recognizing variation in situations. Commonly occurring types of variation in relationships between variables including linear, reciprocal, exponential, and quadratic are explored and related to graphical, symbolic, and numeric representations of these types of relationships.

The belief underlying the algebra strand in *Mathematics in Context*, then, is that algebra is a tool for making sense of the world—for making predictions and for making inferences about things that you cannot measure or count. Algebra is manipulation with substitutes temporally free from the original meanings, but whose meanings are recoverable, whenever necessary. The substitutes are the symbols Among them are symbols for number, variable, actions, and relations.

Figure 2 depicts the process of building meaning for working with symbols. The symbolization cycle is one of the important ideas in algebra and in topics that use algebra. The chosen world can be part of reality, but it can also be fantasy or a mathematical abstraction. The emphasis placed on the process is in order to prevent early automization as a substitute for student reasoning. The thoughtful use of an algorithm for special purposes without memorizing the algorithm is an important and essential step toward learning formal algebraic operations.

Figure 2—Symbolization cycle (Roodhardt as cited in Romberg et al., in press)



Lins (1990) states that the belief that algebraic thinking can only happen in the context of algebraic symbolism is misleading. He distinguishes between two fields of reference: the numerical and the analogical. The numerical frame of reference means that only the arithmetical environment is relevant to the process of manipulating or exploring a situation. The analogical frame of reference means that a situation is manipulated or explored by manipulating features of the situation itself. An example of an analogical frame of reference is when a situation or algebraic problem is modeled with a geometrical shape or pattern such as an array of dots.

This analogical frame of reference has been called referential algebra. Referential algebra as described by Kaput (e-mail, April 6, 1993) is the algebra of quantitative modeling; of functions and relations that refer to elements and relations in realistic situations. As a conceptual structure, it has its foundation in knowledge of quantification and quantities. Formal algebra is the algebra of symbolic systems, of formal elements, whose foundation may be in quantitative activity and situations, but where attention is on the form of the representation itself, not on what it might be representing.

Kaput uses the metaphor of a person looking through a lens at a shape to distinguish formal algebra from referential algebra. Think of a person looking through a tall vertical lens to an amorphous, vertically elongated blob with Aristotelian sight-lines passing through configurations on the lens to configurations of the blob. The lens stands for a symbol system, and the blob stands for a situation or referent. Symbols can be used transparently to reason about the situation by using knowledge of the situation. However, you can also think of the sight-lines being generated and wiggled by their ends in the blob, with the intersections on the lens being derivative of the reasoning that is tied to the blob. Here, there is a semantically guided two-way interaction between symbols and situation. The main source of guidance and inference again is in your knowledge of the situation. On the other hand, you could truncate the sight-lines at the lens and do syntactically guided maneuvers of the symbol system, treating the lens as opaque. This truncated activity is analogous to formal algebraic reasoning. Mason (1992) refers to the most productive reasoning as "shifts of attention" between the symbolic lens and the situation. The wiggling of the sight-lines is controlled sometimes by the configuration of the lens, sometimes by those in the situation-blob, and sometimes by the comparative shifting between the two. Supports or constraints for your reasoning come from the symbol system, from the field of reference, or from an interaction between the two (Kaput, e-mail, April 6, 1993).

According to Kaput, the standard, rules-based algebraic symbolic lens has its syntactic properties inherited from the quantitative properties of the standard number system. Configurations on a character-string lens are constructed by acts of abstraction or generalization, or both, from action on the reference field of numbers. This is algebra as generalized arithmetic. It is a limited form of algebra, and it is not very helpful as a resource for reasoning, especially about general quantitative relationships (Lee & Wheeler, 1989). One of the strengths of a referential, or modeling, approach is that it helps to remove the ambiguity in the two levels of structure between that of objects such as functions and operations on them. You can concentrate first on building functions as expressions of quantitative patterns before going on to manipulate them for purpose of comparison or combination (Kaput, e-mail, April 6, 1993).

To illustrate this approach, "Patterns and Triangles" is a unit in the algebra strand in *Mathematics in Context* that is concerned with regularity. This is algebra of quantitative modeling or referential algebra. Referential algebra contains models, descriptions, concepts, procedures, and strategies that refer to concrete or archetypical situations. Models and strategies refer to the situation in problems.

The algebra in this unit is classified in a typical algebra textbook under the topic of sequence and series. The way that the algebra is presented here, however, is not as it is usually introduced in beginning algebra textbooks. This algebra fits the description of referential algebra. The big idea underlying the unit is regularity. A situation is manipulated or explored by manipulating features of the situation itself. Attention is on the forms of representation and on what is being represented. The focus of the unit is on the representation of algebraic expressions and functions, including symbolic, numeric, and graphical representations.

Regularity is situated in the context of patterns and geometrical shapes and in sequences of numbers. Pattern is a metaphor that is used for regularity in numbers. Special attention is paid to triangular numbers as regularity becomes situated in triangular numbers. Mathematical content includes establishing patterns in data, creating both linear and quadratic formulas, dynamics in patterns, functions, and series and sequences.

Through recursive techniques, the use of tables, and visually by tessellating black and white tiles in equilateral triangles, students discover that the sum of the odd integers, $(2k-1) = 1 + 3 + 5 + \ldots + (2n-1) = n^2$; "n" is a row number in the referent triangle. Triangles become symbolic shapes with the same pattern. Students learn that if they square the row number "n" of the triangle, they will get the total number of black and white tiles. This formula is written as $S_t = n^2$ for the sum of the total number of tiles.

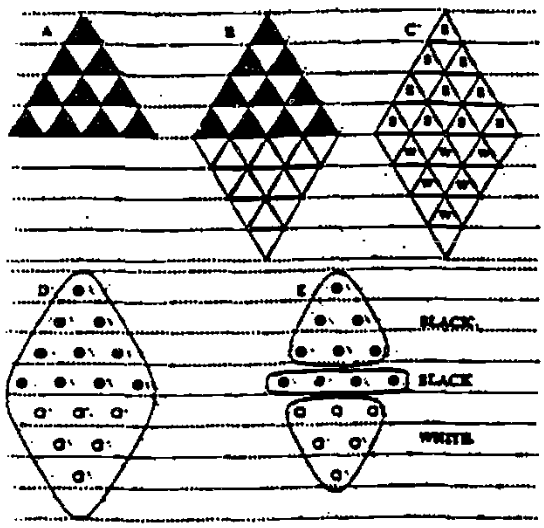
Students also build an understanding of the sum of integers, $k = 1 + 2 + 3 + \ldots + n = n(n-1)/2$ from exploring patterns of black and white tiles. The formulas $S_b = 1/2n^2 + 1/2n$ and $S_w = 1/2n^2 - 1/2n$ are derived visually. S_b stands for the sum of the black tiles, and S_w stands for the sum of the white tiles. The referent or situation (elongated blob) for the symbolic expression in the unit is a particular grouping and splitting of a rhombus (diamond).

Figure 3—Referent for formulas in "Patterns and Triangles" (Romberg et al., in press)

A: the floor
B. the white tiles reflection in the baseline
C: a simplification of the pattern
D: a further simplification. For counting, triangles are re with costs
E: the new -- : -- and regroup the pattern

We still need to find formulas for the total number of black tiles (S^b) and of white tiles (S^w). Let's look at the reasoning for $n = 4$.

Triangular Numbers



Students are able to use this referent throughout the rest of the unit to give meaning to formulas used to solve a variety of problems. Here the representation becomes an object of study itself. It is a source of new abstractions that serve as useful models of patterns in concrete situations (Fey 1990).

Conclusion

An important reason that so many students and teachers have difficulty with this subject is the approach taken to the subject. As it is currently organized and taught, school algebra is a barrier that many people are not able to overcome. Therefore, a critical examination of the content, place, and purpose of school algebra is long overdue. The shift in emphasis and the approach we have proposed here should provide all students with a better understanding of the important and useful mathematical ideas of algebra. Thus, it should provide the mathematical foundation for the work force of tomorrow.

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Algebra: A Vision for the Future

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As currently taught, algebra can be characterized as a certain body of mathematics, largely skill-based, with which students must be moderately successful to move on to mathematical or other endeavors. The algebra taught in middle and secondary schools and in developmental programs at many community colleges is largely a set of skills. Frequently, algebra courses are organized around sets of skills clustered by topic, such as factoring, linear equations, polynomials, rational expressions, and radicals. These skills are learned temporarily and hence are forgotten easily because they are taught as an end in themselves. Such skills must be relearned when needed in higher level mathematics courses such as precalculus, calculus, or discrete mathematics or for applications such as chemistry, physics, or economics.

I propose instead that algebra be taught as a way of thinking, a conceptual and symbolic framework which supports exploratory thinking in mathematics. A student challenged with an interesting and relevant problem situation analyzes it initially using mathematics within his or her current knowledge base. Appropriate algebra, that which will enhance the understanding, exploration, and solution of the problem, is introduced *at the point of need* to enable deeper analysis. Symbolic manipulation such as simplifying expressions, laws of exponents, factoring, and the like, should be taught only as necessary and when fitting the context of the interesting problem setting. Skills will be remembered as they are connected to the context in which they are learned. The algebra underpinning provides the vehicle to move a problem solution to deeper levels and better results.

The traditional secondary school or lower division collegiate algebra classroom of the present, with an omniscient lecturer at the helm and too many unsuccessful students, shows few similarities to a new vision for the study of algebra. First, no separate algebra course will be offered. Rather, algebra will be integrated into a context-based "mathematics-plus" course where algebra is presented as an underpinning to an applications-driven mathematics curriculum. The teacher will become facilitator and guide, directing the students as they together explore, conjecture, hypothesize, react, test, and conclude. Students will learn actively in collaborative settings, with continual development of *each* student's ability to plan, organize, speak, listen, and write. The curriculum must be responsive to the ever-changing needs of the workplace, to new information about the learning and teaching process, to new technologies, and to the community the educational system serves. The natural institution to handle the teaching of algebra to prepare for an ever-changing technical work force is the 2-year college, with its diversity of programs and strong connections to business and industry. Assessment, conducted by students themselves as well as educators, is ongoing and monitors student learning, the relevance of the curriculum, and the quality of pedagogy.

The teaching of algebra should focus on several major ideas, among them equation-solving (in an appropriate context). Equation-solving is a common thread running through our skill-based courses as an end in itself, but can truly illustrate the power of mathematics to find answers to meaningful questions. One or more equations containing one or more variables resulting from data analysis or some other situation creates a challenge for the solver. Technology support and pencil-and-paper techniques enable varied means of finding exact or approximate solutions including numeric, graphical, and symbolic methods. Another important concept which should remain in a renewed algebra curriculum is the concept of function. Important function ideas include the study of relationships between and among variables, the study of the effects of varying parameters on families of functions, and the use of functions to enable students to observe patterns and model physical phenomena.

The SCANS Report, *Learning a Living*, calls for full implementation of the teaching of "workplace know-how" from kindergarten through college by the year 2000. This know-how encompasses such skills as the ability to learn, reason, think creatively, make decisions, and solve problems. Competencies such as collecting, interpreting, and evaluating data, using appropriate technology, and working effectively in a team are included. The algebra curriculum can be a strong advocate for encouraging SCANS know-how and a powerful mechanism for its teaching. Various initiatives such as Tech Prep, JTPA, and Perkins programs enable the tailoring and coordination of educational programs to meet the needs of the workplace.

Changing the nation's understanding of the role of algebra in its educational programs will not be easy. As the NCTM *Standards* have influenced widely courses and curricula, so must the larger educational community understand a redefinition of what it means to study algebra. Educators, administrators, parents, students, and citizens must be convinced that the certification formerly offered by completing an algebra course is being replaced by increased access to a strong, sound curriculum producing successful students. Concerns about teacher retraining and professional development will attempt to roadblock reform in the algebra curriculum. The development and dissemination of high-quality instructional materials which model effective curriculum and pedagogy are critical to assisting schools and colleges in their efforts to make changes. Many other issues, equity and diversity among them, surround appropriate algebra for the 21st century. Essential is the elimination of courses in algebra as screening devices for students wishing to pursue a variety of fields or other mathematics-related topics such as statistics. Retention of students from underrepresented groups and students with varying learning styles is a must.

A new draft document, the "Curriculum and Pedagogy Reform for Two-year College and Lower Division Mathematics Standards," under preparation by the American Mathematical Association of Two-Year Colleges, calls for the natural and routine incorporation of technology into the learning and teaching of mathematics. In mathematics in general and algebra in particular, students should be able to choose the appropriate technology, use it with confidence, and understand the nature of the results obtained. However, this same wording summarizes my thoughts about the direction which algebra should take: **ALGEBRA**, with a revitalized definition, should be incorporated naturally and routinely into the student's mathematics experience, seamlessly integrated and connected with other mathematics and mathematics-related topics which are learned in context through an active, collaborative approach.

WORKING GROUP 4

Renewing Algebra at the College Level to Serve the Future Mathematician, Scientist, and Engineer

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Report of Working Group 4

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Although many issues were discussed, our group came to conclusions on only a handful of them.

Algebra Across the Grades

There should be more dialog between K-12 educators and 13-16 educators. We discussed several benefits from such a dialog:

- It might be possible to lay the basis in high school for some important ideas from linear and abstract algebra. These ideas, including various uses of proof and explanation, reasoning about binary operations, and working with symbol systems, are important in the algebra experience of all students, not just those who are college bound. With college-bound students, it might even be possible to introduce some of the actual topics from advanced algebra (especially from linear algebra).
- New ways must be found to connect with and engage students in linear and abstract algebra courses. The algebra experiences of university students should build on and be informed by the various reforms and changes that are taking place at the secondary level. These include mixing deduction with experiment, using cooperative teams of learners, developing project-based activities, and incorporating technology into courses.
- Part of the dialog between K-12 and 13-16 educators might be a search for algebraic themes that could run throughout a student's algebra experience.

Technology in College Courses

Educators ought to consider the use of technology in linear and abstract algebra courses. We discussed several issues that must be addressed in order to make this happen. Two important ones are:

- Faculty need to devise effective uses for technology in their courses. There are computational environments that seem well suited for use in linear and abstract algebra (*MATLAB, ISETL*, and systems like *Geometer's Sketchpad,* *Maple*, and *Mathematica*, for example). There is also a great deal that is known about the uses of technology for learning mathematics at the pre-college level, and some of this can be transported to the university setting. But there are challenges in teaching linear and abstract algebra with technology, challenges that call for new ideas and solutions. Innovative work is going on at some colleges and universities, but significant research is needed to find effective ways to use technology to help students learn algebra.

- If technology is used in a course it ought to be integrated rather than appended. Integration of technology into algebra requires a significant re-thinking of the algebra curriculum. Different uses of technology will lead to different kinds of curricula. For example, a linear algebra course that uses *MATLAB* to explore applications of matrix algebra to topics in engineering is likely to be quite different than a course that asks students to use, say, *Maple* to write programs that test vectors for linear independence.

The Role of Applications

It is preferable to integrate applications into a course rather than offer separate applied courses. Applications in elementary courses should not be so over-powering as to detract from the mathematics. Applications can be used for motivation or to enhance understanding. The applications can be from practical situations or to other parts of mathematics.

Abstraction and Proof

For many students there is a barrier between the courses that stress mechanical skills and the courses that emphasize abstraction and proof.

What can we do to remove this barrier? Several strategies were discussed:

- Abstractions should come only after concrete examples have been introduced. Some of these examples might come from previous courses and experiences; others might be developed within the linear or abstract algebra course itself. In order for students to see the utility of abstraction in algebra, they need to have a concrete basis from which abstractions can be made.

- Linear algebra is a good course for developing some proof techniques. Most proofs of results in elementary linear algebra are transparent and constructive enough so that students can build several important skills.
- Students should use proof as a research technique throughout their mathematics experiences. The de-emphasis on two-column proof in geometry and the reduced emphasis of proof in calculus and other elementary courses should not signal a completely proof-free experience for students before they get to linear algebra. Activities should be developed in lower level courses that illustrate the value of proof as a mechanism for communicating, establishing, and even discovering new results.

Some Thoughts on Teaching Undergraduate Algebra

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Abstract Algebra and Introductory Analysis are usually the first undergraduate courses where students must come to grips with the idea of proof. The students may have some familiarity with the abstract concepts of vector space and linear transformation, and they may have had to write out a few short proofs, but they are not usually prepared for the depth of these courses, where almost everything is proved carefully, and where the students are expected to devise their own proofs. In the abstract algebra course the students are not only learning the subject matter but also the whole culture of pure mathematics. Not only must the students understand the definitions and theorems, they must learn to construct their own examples and counterexamples. There was considerable discussion at the Colloquium about the "wall" students encounter when they face the transition from the lower division courses to the abstract upper division courses.

Learning and teaching abstract algebra is labor intensive. There is no way around this. The only way the students will learn to write proofs is by trying to do so. The students' attempts must be read carefully, critiqued, and then returned along with helpful comments and suggestions. As anyone who has graded student exercises knows, this is the course of the long, wrong argument. Whenever I teach this course I assign quite a few exercises every week, grade them, and return them, usually at the next class meeting. The exercise grade is a significant part of the final course grade. The students are encouraged to work together in groups, but this is not always feasible for some students. I suggest that they think about the exercises before meeting with their group, and I require them to write up their own solutions rather than appointing a scribe for the group and having everyone else copy the scribe's work. One thing that I have found to be a good investment of time is allowing the students to rewrite their first few exercise sets after they get their papers back with my comments. By doing this a student can recover up to one half of the points lost on the first try. This successive approximation method seems to help students improve faster than any other technique I've tried. Once students begin to understand what constitutes proof, the papers become much easier to read. This method requires that the comments on the students' papers be extensive. Another advantage gained from having assigned lots of work at the beginning of the term is that students tend to allocate the same time intervals each week for studying each of their courses. Staking an early claim to a major part of the students' study time allots to the algebra course the time it deserves and needs. Even if you are fortunate enough to have a paper grader assigned to your course I would encourage you to grade some of the students' work yourself—it's a great eye-opener.

Applications and historical comments are often suggested as important for motivation. It has been my experience that students almost always enjoy historical or biographical comments. Recently, there has been a tendency to yield to demands of "relevancy," and to include applications in the course. It is my feeling that such inclusions often tend to be superficial. In order to make room for the inclusion of applications, some important mathematical concepts have to be sacrificed. It is clear that one must have substantial experience with abstract algebra before any genuine applications can be treated. For this reason, I feel that the most honest introduction concentrates on the algebra. I prefer to motivate the subject with concrete problems from areas of mathematics that the students have previously encountered, namely, the integers and polynomials over the real numbers. Geometric considerations, such as symmetry groups and the impossibility of trisecting the angle or duplicating the cube with straightedge and compass, offer real motivation. In the end, the ideas of abstract algebra are beautiful in their own right, and once the students begin to have success in the course, the subject's intrinsic attractiveness is more than ample motivation. The trick is to supply the help necessary for the student to experience some success early on. It is hard work for the student and the teacher, but it is worth the effort.

A technique, with which one of my colleagues has had success, is to give a five-minute quiz at the end of every class. The quiz always consists of a question on that day's material, usually an easy question, and a second part in which the student asks the instructor a question. After an initial shakedown period during which the class learns that questions of the sort "will this be on the test?" are not what the exercise is all about, these student questions become an assessment device by which the instructor learns about the class's progress and is able to make daily adjustments as needed.

Finally, I would like to mention an idea that I believe is worth trying. In addition to the regular three credit-hour per semester abstract algebra course, I would like to see some optional one credit-hour modules or seminars developed. These optional classes could be taken concurrently with the algebra course. One such module might be devoted to applications and would be an opportunity to delve deeply into several applications. This module would make the most sense in the second semester. During the first semester, there might be a module which deals with elementary logic and proof techniques. Another module might be developed which uses *Maple* or some similar computational package. Professor H. Montgomery has, I believe, had considerable success along these lines with the undergraduate number theory course at the University of Michigan.

Toward One Meaning for Algebra

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Introduction

There are currently two meanings attached to algebra in American education. There is the algebra that students take in middle and high school (many students, it seems, are taking it in college as well), and there is the algebra in courses with titles like "abstract algebra." Most people see absolutely no connection between the two algebras; school algebra and abstract algebra are simply different subjects with the same name. One reasonable response to this last comment is "so what?" It may be silly to use the same name for two different mathematics courses, but does it really cause any harm?

I think it does. The lack of communication between algebraists and the people who teach algebra in school has had several bad consequences:

- School algebra has become a very strange creature over the last two decades, and it is now a course with very little algebraic content. To oversimplify, algebra in school has become a mix of drill in meaningless manipulations of polynomials and rational expressions together with a collection of topics that most algebraists would not recognize. For example, students spend a great deal of time converting among the "point-slope," "two-point," "linear," "slope-intercept," and "$y = mx + b$" forms for the equation of a line (some of these maybe the same—I'm never sure). The courses hardly ever distinguish between polynomials and polynomial functions, so that statements like $(x-1)(x+1) = x^2-1$ have at least two distinct meanings. And students spend a great deal of time in algebra developing something called "function sense." This isn't a well-defined idea, and it has very little to do with functions (and almost nothing to do with algebra), but it seems to center around the ability to predict the change that will occur in the graph of a polynomial function when one of its coefficients changes. Like a culture cut off from the outside world, school algebra has evolved ar ntity and an aesthetic of its own, quite remote from its roots.

- Stude- , manage to survive in mathematics long enough to *get* to abstract algebra have no experiential bases from which they can abstract notions about algebraic structures. So, they think of the course as something that is closer to botany than to mathematics. It all seems like a game in classification; there are groups, rings, and fields, each having its own collection of sub-species, and the objects of the game are to be able to define each of the sub-species and to be able to derive simple consequences from these definitions. Even among the students who enjoy abstract algebra, many see it as an intellectual exercise. having little to do with real mathematics (like differential equations) and almost nothing to do with that "other algebra" they learned in high school. For too many students, abstract algebra seems like so much abstract nonsense, and the immense utility of the structural approach to algebra never surfaces.
- Imagine the schizophrenia that must be experienced by people studying to be high school mathematics teachers. They know that they'll be expected to teach the $y = mx + b$ algebra after graduation, but, as preparation for this task, they are given the group-ring-field algebra. In conversations over the last two decades with my colleagues who teach high school mathematics, I've found that many practicing teachers who *took* abstract algebra in college will tell you that they never had an algebra course beyond what they studied in high school; when reminded of their abstract algebra course, the common response is that it was a complete waste of time.

It doesn't have to be this way, because it hasn't always been this way. School algebra in the early part of this century was a recognizable preparation for and precursor to university algebra. There were wonderful books (like Hall and Knight's classic *Higher Algebra*) that connected techniques for solving linear and quadratic equations to more subtle topics from the theory of equations. This study was then continued in college, leading up to some elementary Galois theory (that's where group theory usually first appeared). But as research in algebra became more abstract, and as algebra broadened to include more than the theory of equations, the thread was broken, and algebra during and after high school went their separate ways.

It would be a good idea, then, to put some effort into solving the dilemma of the two algebras, looking for ways to connect pre-college algebra with algebra as it is used by mathematicians, scientists, and engineers. This isn't a call for teaching group theory in middle school, nor is it asking that the content of abstract algebra be watered down. It is simply a plea that we do some hard thinking about linking the algebra experiences of students at all levels. This would add some much-needed substance to school algebra, and it would provide a context from which abstractions could emerge as useful constructions for high school graduates who plan to make, use, and teach mathematics. Put another way, I'd like to see each working group think about the *whole* picture and to look for algebraic threads that can be pulled from kindergarten through graduate school.

Some of these threads might not be about specific content at all; in EDC's *Connected Geometry* curriculum development project, we're trying to build activities that will help students develop what we're calling mathematical "habits of mind." Examples include using proof as a research technique and using proportional reasoning. It should certainly be possible to identify some algebraic habits of mind, like looking for algorithms in repeated calculations, developing and using algebraic notation, and looking for structural similarities in sets equipped with binary operations. Perhaps we might spend some time at the meeting identifying a small set of such habits, thinking about how students might develop them over their entire education.

Other threads might be more closely tied to specific content, to topics that already are (or could be) in the curriculum. In the rest of this paper, I'll describe one such thread that could begin very early in school and be woven throughout a student's algebra experience, right through college.

Algebra as a Theory of Calculation

Most algebraists enjoy a good calculation. The calculation can be in any kind of system; algebraists calculate with numbers, polynomials, functions, permutations, sets, propositions, even calculations themselves. Sometimes the calculation can just be for fun, as in the famous

$$\left(\frac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x^2}-\frac{1}{y^2}} - \frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x^2}+\frac{1}{y^2}} \right) \div \left(\frac{8}{\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}-2\right)} \right)$$

but most times calculations are carried out and analyzed for a purpose. Looking into the properties of calculations could be a central thread in algebra. Initially, this might mean carefully analyzing calculations to see what information they give you or how they can be generalized. As students gain more experience with analyzing calculations, they can be on the lookout for systems that "calculate the same," and that can lead to the notion of algebraic structure.

Thinking about calculations, designing efficient algorithms for performing calculations, and predicting the outcomes of calculations without performing them are all important facets of algebraic thinking that find utility in a variety of contexts, some of which seem quite remote from mathematics. Users of spreadsheets in business worry about calculations all the time (the reference manual for Excel™, for example, shows how a culture, somewhat disjoint from mathematics, has developed its own vocabulary and metaphors for supporting thought experiments about calculations). The symbol manipulation used by people who process text would be quite familiar to algebraists (think about what you do when you get an e-mail that has paragraph returns at the end of each line and you want to get rid of these but keep the double returns that separate paragraphs).

Technology can help support students at every stage of their investigations into calculations. Computers can be tools for investigating algebra in several ways:

- Computers give students the expressive power needed to describe the essential features of calculations.
- Computers provide students with environments in which they can build mechanical (computational) models of algebraic structures.
- Students can experiment with their models, studying their behavior and making conjectures about what they observe.
- When students construct the models for themselves, they have a basis from which they might establish their conjectures; logical connections among the properties of the algebraic calculations or structures are often mirrored in the ways that the computational models are built.

Maybe a concrete example would help make these points clear. The next section outlines an investigation using *Mathematica* that gets at the arithmetic similarities between Z and **F[x]** where F = Q or R. It could be started in high school (maybe earlier), but it also leads one to develop some ideas in abstract algebra that are important to the charge for Working Group 4. A more detailed account of how this material was used with a group of high school teachers is available as a preprint (Cuo 1992).

An Example: Arithmetic to Algebra

Start with an analysis of Euclid's algorithm for finding the greatest common divisor for two integers:

```
          14
     ________
216 ) 3162
      3024      1
      ----  ______
       138 ) 216
             138     1
             ---  ______
              78 ) 138
                    78     1
                    --  ______
                    60 ) 78
                         60     3
                         --  ______
                         18 ) 60
                              54    3
                              --  ______
                               6 ) 18
                                   18
                                   --
                                    0
```

This works because remainders are smaller than divisors and because the gcd function has two properties:

Euclid's Theorem. If a and b are integers

$$\gcd[a,b] = \gcd(\mathrm{mod}[b,a],a)$$

if $a \neq 0$,

$$\gcd[0,b] = b$$

Mathematica allows you to abstract off the essential features of a calculation and to express them as functional equations; the following two lines build a *Mathematica* model for gcd:

gcd[a_,b_] := gcd[mod[b,a],a]

gcd[0,b_] := b

This process of "nailing down" a function that captures a process by constraining it with functional equations is already a powerful technique for abstraction. It takes a major cognitive step to be able to think about describing processes with functional equations, but my experience is that many high school students can do it.

Analyzing the calculation described by Euclid's algorithm shows that the gcd of two integers is a linear combination of them. Indeed, the divisor-remainder pairs in our example are:

$$(216,3162) \to (138,216) \to (78,138) \to (60,78) \to (18,60) \to (6,18) \to (0,6)$$

which all look like this:

$$\ldots \to (a,b) \to (\mathrm{mod}(b,a),a) \to \ldots$$

These all have the same gcd, say d, and a little experimenting and calculating shows that if d is a combination of any pair in the stream, it is also a combination of the pair that points to it. The essence of this calculation is that if fcoeff and scoeff are modeled in like this:

```
fcoeff[0,b_] := 0
scoeff[0,b_] = 1
fcoeff[a_,b_] := scoeff[mod[b,a],a] - quot[b,a]* fcoeff[mod[b,a],a]
scoeff[a_,b_] := fcoeff[mod[b,a],a]
```

(the first equation is somewhat arbitrary), then

$$\mathrm{gcd}(a,b) = a \cdot \mathrm{fcoeff}(a,b) + b \cdot \mathrm{scoeff}(a,b)$$

This construction has several benefits. For example, on a concrete level, it allows students to classify all the functions that behave like fcoeff and scoeff (this amounts to changing the initial condition for fcoeff), and on an abstract level, it provides a computational proof that Z is a PID.

Students can then use these basic functions to construct models for quotient rings of Z. One way to do this is to represent a congruence class as a pair whose first element is a distinguished representative of the class (say, the least positive residue) and whose second element is the modulus. The constructors and selectors look like this:

```
class[a_,m_] := {mod[a,m],m}
rep := First
modulus := Last
```

and the operations on the classes can be modeled in a way that is quite close to the mathematics:

```
add[c_,d_] := class [rep [c] + rep [d], modulus[c]]
mult[c_,d_] := class [rep [c] * rep [d], modulus[c]]
```

Here's an example of how one can use this package as a Z/mZ calculator:

```
In [1]:= class [51,7] == class [16,7]; Out [1]= True
In [2]:= add [class [5,7]], class [8,7]]; Out [2]= {6, 7}
In [3]:= mult [class [4,10], class [3,10]]; Out [3]= {2, 10}
In [4]:= mult [class [4,10], class [15,10]]; Out [4]= {0, 10}
```

From here, students can look into the structure of Z/mZ, asking, for example, when every non-zero element is a unit. Standard arguments show that the multiplicative inverse of unit in Z/mZ can be calculated as the first coefficient needed to write 1 as a combination of the unit and m:

```
recip[c_] := class[ fcoeff[ rep[c],modulus[c] ],modulus[c] ]
```

For example,

```
In [1]:= recip [ class [15,71] ]; Out [1]= {19,71}
In [2]:= mult [class [15,71], class [19,71 ] ]; Out [2]= {1,71}
```

An interesting investigation is to explain the behavior of recip when it is given a non-unit in Z/mZ.

What students have here is a small number theory laboratory: they have *Mathematica* models for some basic functions in arithmetic and a working model for Z/mZ. They can use this laboratory to investigate topics in arithmetic, building, for example, a function that solves simultaneous linear congruences (and hence providing a computational proof of the Chinese remainder theorem). But the real payoff is that (with minor modifications) this *same* laboratory can be used to investigate other algebraic structures.

For example, one can calculate the greatest common divisor of two polynomials (in one variable x over, say, Q) by Euclid's algorithm:

$$
\begin{array}{r|l}
 & \quad 3 \\
2x^2 - x - 1 & 6x^2 + x - 1 \\
 & \underline{6x^2 - 3x - 3} \\
\end{array}
\qquad
\begin{array}{r|l}
 & \tfrac{1}{2}x - \tfrac{1}{2} \\
4x + 2 & 2x^2 - x - 1 \\
 & \underline{2x^2 - x - 1} \\
 & \qquad 0
\end{array}
$$

and, as before, the greatest common divisor of two polynomials is a linear combination of the two polynomials. This process can be captured by the same *Mathematica* models:

```
gcd[a_,b_] := gcd[ mod[b,a],a ]
gcd[0,b_] := b
fcoeff[0,b_] := 0
scoeff[0,b_] = 1
fcoeff[a_,b_] := scoeff[ mod[b,a],a ] - quot[b,a]* fcoeff[ mod[b,a],a ]
scoeff[a_,b_] := fcoeff[ mod[b,a],a ]
```

provided we make changes to mod and quot so that they take *polynomial* (rather than integer) remainders and quotients (this is easy to do in *Mathematica* and in most programmable symbol manipulators).

Here's a sample *Mathematica* session that shows how this set-up might work:

In [1]:= gcd[$x^4 + x^2 + 1, x^3 + 1$]; Out [1]= $1 - x + x^2$

In [2]:= fcoeff [$x^4 + x^2 + 1, x^3 + 1$]; Out [2]= 1;

In [3]:= scoeff [$x^4 + x^2 + 1, x^3 + 1$]; Out [3]= -x;

In [4]:= Simplify [$x^4 + x^2 + 1 + -x (x^3 + 1)$]; Out [4]= $1 - x + x^2$

The fact that the same computational models for arithmetic algorithms work for integers and polynomials suggests that Z and, say, Q[x] have some structural similarities, and looking into these similarities can lead students to some important abstractions. This approach has been taken in several books (Sawyer 1959 and, more recently, Ireland and Rosen 1982 and Dubinsky and Leron 1993, for example).

There are also important differences between the integers and polynomials over a field. Notice that the greatest common divisor of two polynomials as calculated by Euclid's algorithm may not be identical to the result obtained by using the typical high school method (factoring into primes), and the analysis of this fact leads to a consideration of the units in Q[x]. It also points to needed modification in the recip function: We want the product of a class modulo f and its reciprocal to be class(1,f), but our definition of recip:

```
recip[c_] := class[ fcoeff[ rep[c],modulus[c] ], modulus[c] ]
```

only guarantees that the product of a class c and its reciprocal will be the class

class (gcd(rep(c), modulus(c)), modulus(c))

To remedy the situation, we redefine recip like this:

```
recip[c_] := class[(1/gcd[rep[c],modulus[c]]) *
    fcoeff[ rep[c],modulus[c] ],modulus[c] ]
```

Once this is done, students can investigate calculations in quotient rings of Q[x]; this gives students a rich and varied collection of algebraic structures in which they can make conjectures and establish results. Two short examples suggest the possibilities:

Units in rings. What are the units in Q[x]/f, Q[x]? The theory tells us that they are precisely the elements $g \in Q[x]$ so that the reduction of g modulo f (and hence g itself) is relatively prime to f in Q[x]. Seeing this as a calculation with polynomials adds a different insight; suppose we ask our polynomial calculator for a generic formula for reciprocals modulo x^2-1:

In [1]: = Simplify [recip[class[a+bx, x^2-1]]]; Out [1] = {(a - bx)/(a^2-b^2) , -1 + x^2}

So, and element a + bx of $Q[x]/(x^2-1)Q[x]$ has a multiplicative inverse precisely when $a^2 \neq b^2$. Lifting this result back to Q[x] yields the more traditional classification of units modulo $x^2 - 1$, and it also opens up many opportunities for a great deal of sense-making.

Extensions of fields. Suppose we reduce modulo a prime p in Q[x]. Then every non-zero class has a multiplicative inverse, so the quotient ring is, in fact, a field. Indeed, it's a field in which the class of p is 0. Kronecker used this construction to build root fields for polynomials. Students can use their *Mathematica* models to go through exactly the same process, with the added benefit that the rules for arithmetic in these splitting fields can be found by performing generic calculations. Here's a *Mathematica* session that shows how the rules for arithmetic in C can be found by calculating in $R[x]/(x^2+1)R[x]$:

In [4]:= add[class[a + bx, x^2 + 1], class[c + dx, x^2 + 1]];

Out [4] = {a + c + (b + d)x, 1 + x^2}

In [5]:= mult [class [a + bx, x^2 + 1], class [c + dx, x^2 + 1]];

Out [5] = { ac - bd + (bc + ad)x, 1 + x^2}

In [6]:= Simplify [recip [class [a + bx, x^2 + 1]]];

Out [6] = {(a - bx^2)/(a^2 + b^2) ,1 + x^2}

In [7]:= class [x^2, x^2 + 1]; Out [7] = {-1, 1 + x^2}

This example can be investigated without a very deep background, but it leads directly to important ideas from field theory. As a bonus, students get to experience a genuinely big idea in mathematics: reducing modulo a troublesome element or set is a general purpose mechanism that can be used to force desirable properties on a structure.

This is Just a Suggestion

The preceding section is only meant to give an example of one thread that can be used to tie together high school algebra and abstract algebra. The thread is algebra as a theory of calculation, the theme is arithmetic in Euclidean rings, and the major computational medium is a programmable symbol manipulator. The example has the essential features that most students can jump onto the investigation (say, by studying Euclid's algorithm and its consequences in arithmetic), there are several logical places to jump off (depending on students' interests and inclinations), and, if taken far enough, the investigation leads to substantial topics in abstract algebra.

There are certainly other examples. Another thread might be algebra as a theory of number systems and their extensions. Students might look at the problem of generating Pythagorean triples (certainly a high school topic) by studying the properties of the norm from Z[i] to Z. This could generalize to a search for triangles with integral sides and some specified angle (a 60° angle connects to the ring of integers in the field of cube roots of unity). And *this* could lead to a more general study of cyclothymic fields (an area that is well suited to computational investigations and that re-visits the algebra as calculation thread), ruler and compass constructions, and special cases of Galois theory.

Organizing courses around examples like these would certainly require changes in the precollege algebra curriculum, but it would also require changes in the approach to abstract algebra in college. The major message in abstract algebra should be that abstraction is an important and useful habit of mind. In order for this message to get to students, they need an experiential basis from which abstractions can be constructed, the abstract algebra course has to draw on this basis, and the power of the abstractions have to be an explicit part of the course.

Not everyone who studies algebra in high school goes to college, and not everyone who studies algebra in college becomes an algebraist. As I read the charges for each of the four working groups, I see a danger that "algebra for everyone" might become "different algebras for different people." Maybe that's inevitable, but it just might be that there is a small set of big ideas that gets at the essence of algebraic thinking. If that's the case, then courses designed around these ideas, at appropriate levels of abstraction, and with sensible support from computational media, would benefit students from elementary through graduate school. Maybe there *is* one meaning for algebra. I think we should look for it.

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Some Thoughts on Abstract Algebra

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The paradox we currently face in teaching abstract algebra is that, even though algebra is being applied in more ways in science and mathematics, and thus more people need to learn algebra, our students are not as well prepared as before to handle proofs and abstractions. For example, some recent applications include: the use of number theory in cryptography, and of group theory, finite fields, lattice theory, linear algebra, and even algebraic geometry in coding theory. Note the article in *Science,* October 29, 1993, on the relationship between nonlinear codes and linear codes using Z/4Z. Of course groups and Lie algebras have been important in physics for a long time.

However, our students now have even less experience with proofs than in previous years. Once we did proofs in calculus; now only the bravest attempt this in most universities. After that, it was still possible to remind the students that they had seen (and, one hoped, even done) proofs in plane geometry in high school. Now, however, proofs have all but been abandoned in high school. I heard recently that one of the most popular current high school geometry texts contains only two proofs in the entire book. Some reasons for this which I have heard is that students didn't understand what a proof was anyway—they think it means a line down the middle of a page with statements on the left and reasons on the right—and that it is more important to develop intuition. If a fact is "obviously" true, why should it need a proof? The result of these new developments is that it is entirely possible for a math major in your abstract algebra class never to have had to prove anything before.

One consequence of this is that we spend so much time convincing the students what a proof is that we don't get very far into the actual content of algebra. Some students even struggle with the idea of a definition and how to verify that something satisfies it. They end up feeling that the course is only about abstractions. And the instructor, in desperation, chooses topics because the students are able to understand them, or compute examples easily, rather than because these topics will be important in the long run (in future science or math courses) or because they are part of the fundamental methods or structure of algebra. I'm sure most of us are all too familiar with this situation.

A related problem is the generally poor preparation of American students who do go on to graduate school; they are typically a year or two behind the foreign students and have to take remedial courses in algebra and analysis to catch up, if they ever do (there was an article in *UME Trends* last year about such courses being instituted at Rutgers). Given the trends described above, it is hard to believe that this will improve in the near future. Thus, our problem is how to present the important ideas and techniques in algebra, along with their abstract structure, and with a long-range view as to why these topics are taught, to students who are less and less prepared to deal with the formalism.

I don't really have a solution for this, except to mention some things that have worked for me. We have had a very successful course at USC in applied algebra for a number of years, probably due to our strong groups in Computer Science doing cryptography and in Electrical Engineering doing coding theory; in fact, their very best students take our graduate algebra sequence. I've been using the book by Gilbert, which seems like a very nice combination of algebra with applications. For example, the group theory emphasizes group actions, including symmetry groups, the regular polyhedra, orbits in general, and Polya-Burnside enumeration in particular (including applications to switching circuits and chemistry); the methods are also applied to the usual group theory topics.

The population for this course has mostly been beginning graduate students in engineering; the only prerequisite is advanced calculus, though in practice most students have had linear algebra. The students seem to like the material and do fairly well, though they are much more motivated than our typical undergraduate student.

This year, I experimented with some of these topics in the undergraduate course, using the new book by M. Artin. I used to think that group actions were too sophisticated for this course; however, this turned out not o be true. I started out by considering actions of the finite groups C_n and D_n as rotations and reflections of the plane. After asking the students to draw the orbits of various points in the plane under these actions and then asking them to find the stabilizers, they were then ready to believe that the product of the number of elements in the orbit times the order of the stabilizer equaled the order of the group. Then when I used this fact to prove more abstract things like Cauchy's theorem, they seemed to think it was a natural way to proceed. A miracle!

Thus, this is one possible approach: introduce more concrete topics, including some geometric reasoning, but with the end of giving them some intuition as to why tha more general theory is natural.

Although I've been concerned here with abstract algebra, I'd also like to comment on the importance of linear algebra. We have two levels of linear algebra at USC. The first is the sophomore-level course, with applications to systems of linear differential equations, which is really part of the calculus sequence, and the second is the advanced course which usually follows a semester of abstract algebra. Although most of the students in the first course are science and engineering majors, and consequently proofs are not emphasized (though we do manage to put a few in!), never-the-less a student's performance in linear algebra is the best predictor of success in abstract algebra, much more so than how well they do in the rest of the calculus courses. I don't completely understand why this is so, except that there is some abstract structure in the course, which forces the students to organize the material in a new way. This suggests that more linear algebra in high school might help the student to deal with abstractions later, and help compensate for the removal of formalism in geometry.

Suggestions for the Teaching of Algebra

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As I looked over the material that was sent to me, I decided to focus in on three different topics.

- Sociology of the classroom,
- Use of technology in the classroom, and
- Course content.

Sociology of the Classroom

The notion that we have to have more cultural diversity in mathematics based careers is a topic that has been given a great deal of press recently. It is an absolute necessity that all individuals in this country have access to these careers. But how is this lofty goal to be achieved? That is the real question.

I am not going to emphasize this question in this paper. I think that we have a more basic problem in this country, namely, how can we attract *all* of our own children into the study of mathematics? I have to admit that I do not have clear ideas as to how to encourage our own children to avoid the culture of drugs and alcohol that is so pervasive in this country and instead choose a more reflective lifestyle. Even as I write this sentence, I am overcome by the ridiculousness of this statement. Our children are being taken away from us by a culture of violence, and we appear almost powerless to entice them from this.

It is this point of view that flavors my comments. I am not going to focus on how to attract those very bright children into the study of algebra. Instead, I want to think of how we can attract all of our children into being mathematically literate, so that they will have the opportunity to participate in the present technological revolution. I believe that this task requires not only a rethinking of what it is that we teach but also how it is that we present that material.

As I attempt to encourage the Chicano students in the Southwest to consider mathematics—based fields, I am struck by one fact. Whatever it is that I do to entice these students into these studies has little to do with the fact that they share the same cultural heritage as I, but rather it has everything to do with a simple dictum that my mother used to tell me, *Hablando se entiende la gente,* that is, "By speaking we make ourselves understood." I do not think that the mathematics community has made a very concerted effort to inform our populace that mathematics is not only an exciting field of study, but it also leads to interesting careers that carry with them attractive monetary and intellectual rewards. In order to emphasize this point, I would like to make some personal remarks.

I grew up in that poverty that was part of the culture of being a Chicano in the Southwest. This poverty and a Mexican-American up-bringing created culture shock as I was forced to function in the American school system. There are differences in the cultures, differences which sometimes make living together difficult. With this in mind, allow me to recount the following incidents. On two separate occasions the following occurred. I had called a colleague at his home in order to pick up a book or something. It was agreed that I could come over and pick it up. When I arrived the colleague opened the door, told me that he would get the object for me, then closed the door in my face as he retreated into his home to get the object. This person had invited me over and left me standing outside. I was shocked by this behavior, and I have often told myself that no Hispanic would behave in such an uneducated fashion as to leave a guest standing on his/her doorstep. These two incidents have always stood out in mind as I have thought of the interactions of the Mexican-American and the Anglo cultures. In the Mexican culture, the term *educado* carries with it not the implication of a formal education but rather the connotation of a person who is cultured enough to treat others with dignity and respect. From my own cultural perspective, I often felt that Anglo academics were not *educados* in spite of their "education."

I mention these incidents for the following reasons. I believe that the community of mathematics professors invites students into their classrooms and then closes the door and leaves these students on the doorstep, mostly in shock and feeling demoralized. I am not saying that Anglo professors leave minority students in the cold, though that may be true. I am saying that mathematics professors leave most students there.

One of the reasons for this behavior is cultural. The mathematics profession is a culture inhabited by mostly very bright people, or so we want the rest of the world to think. In order to maintain this perception, we must behave in ways that would encourage the population to believe this. We are rational human beings whose professional goals are to think about this very difficult subject, mathematics. We teach our classes with a certain aloofness so that students won't bother us. There is no question that there is a certain facade to our profession, a facade that makes communication difficult.

But it is also true that we are abstract thinkers, and pretty good at it. In fact, those who end up teaching mathematics at universities are the survivors of the training process for the profession; they are the ones who are good at it. So how do we communicate this to our students? Communicate what? This abstract thinking is such a natural part of our being that it does not occur to us most of the time to even think of this aspect of teaching. Yet this is also a great cultural difference between ourselves, as professors, and the students who must not only struggle with the new concepts but also with this new way of thinking.

In summary, I would make the following recommendations:

- Communication is the key to education. Besides teaching our students we should take time to speak with them about mathematics and the career opportunities for mathematics-based fields.
- Our main job as teachers of mathematics is not to pass on to the students a technique, but rather a point of view.
- We should recognize that mathematicians are abstract thinkers, and good at it. However, the student population is not as adept at it as we are, and they must be trained.
- We should attempt to teach the students who show up in our classrooms. It is pointless to compare these students to the past or to wish for the students of the future.

Technology in the Classroom

In the fall semester of 1992, I had the good fortune to teach first semester calculus using the Harvard Calculus Consortium notes. The leading philosophy of the course is the Rule of Three, that all ideas should be presented numerically, geometrically, and analytically. This philosophy has made me think about how it is that I present material, and it also forces me to think how it is that I could use technology to better present the material. The profession has barely begun to scratch the surface here. I think that we are at the beginning of a real revolution.

At the same time that I began to teach this course, I also began using the IBM Personal Science Laboratory (PSL). The PSL connects to a computer in the classroom and allows the instructor to run experiments in the classroom. The data from these experiments is projected onto a screen as the experiment is running. This provides an immediate connection between the experiment and the data. The PSL comes with probes for measuring distance, temperature, and light intensity, among others. The distance probe allows one to measure the distance of a moving object. This piece of technology allows the instructor to generate data immediately, display the data geometrically, and then fit a curve to the data. This is a wonderful example of the Rule of Three.

If I am to address the issue of technology in the classroom, then I cannot address this issue in abstract terms, but rather I must discuss my own very limited experiences.

Knowing that I had the computer technology available for my first semester Calculus class, I gave the following assignment to my students. I asked the students to devise two experiments which I would be able to run using the PSL. One of the experiments was to produce data that was linear and the other was to have data that was quadratic. I fully expected that the quadratic data was to be very easy, simply drop a ball. I was quite surprised to find that many of the students did not know that a falling object should have a quadratic path. This assignment had been given more than halfway into the course, and we had had a couple of homework problems dealing with a quadratic equation modeling a falling body. Yet many of the students still had not internalized this fact. In discussions that I had with the students, I found out that some thought that the data should be exponential. But even here there was disagreement; some said that it should be e^(kt), while others thought it should have the form a*(1-e^(-kt)), where k is positive. I was shocked, yet here was reality.

I corrected the students' thinking on this topic and pointed out that falling bodies should have quadratic paths. So now students gave me all kinds of experiments, drop a ball or bounce a ball off a wall. I pointed out that I had to be able to run the experiments in the classroom, and dropping a ball would take less than three seconds, not enough time to gather decent data. Besides, I didn't want to break the probe by dropping a ball on it. Finally, we discussed rolling objects down an inclined plane. I told the students that they had to bring in things to roll down. It was wonderful. Students brought in skateboards, bowling balls, basketballs, and pairs of rolling skates tied together, but the results of the experiments were disappointing. The data did not look quadratic enough. As a ball was let go down an inclined plane, the data looked almost linear. I was panicking as I had the students looking at me as I ran these disappointing experiments in front of them, and then it occurred to me! Why not roll the ball up the plane and let it come down. The data was dramatic, a perfect parabola.

Though these experiments were in a Calculus class, I think it appropriate to discuss them here, as we address the issue of presenting algebra. After all, a polynomial, one of the basic ingredients of algebra, is being used to model a very real situation.

Though I have taught calculus and linear algebra courses that used either a graphics calculator or used some specific software, I have not had enough experience to make solid recommendations. I am convinced that the intelligent use of technology will serve to make mathematics instruction more effective. I would hope that in the future all of our mathematics courses have a laboratory associated to them.

Summarizing, I would make the following recommendations:

- The mathematics community should aggressively seek new ways of presenting the material. The use of technology in the classroom should be encouraged at all levels of instruction.
- Change does not necessarily occur spontaneously. Our faculty have to be encouraged to seek new ways of instruction.

Content of Our Algebra Courses

Algebra is the study of structure, yet this is probably news to the undergraduate student. To most of our students, algebra is equated with calculation, and this calculation is the servant of science, not a subject of interest in itself. This same attitude is reinforced in our Calculus classes.

We no longer teach structure in high school or in Calculus. Perhaps this is one of the reasons that the transition from lower division mathematics to our abstract upper division mathematics courses is so difficult. We emphasize calculations at the beginning of the training of our mathematics undergraduates, and then change the rules on these students when they make the transition to the upper division mathematics courses. These lower division students think that they are doing well in their studies, only to find out that the major that they chose has suddenly become almost unrecognizable to them as they begin studying abstract and linear algebra and advanced calculus. We have tried to soften this transition by introducing a proof course, but these courses have had little success.

Perhaps the first instance in a student's career where structure cannot be ignored is the study of differential equations. In many of these sophomore level courses, ideas from linear algebra are introduced in order to be able to explicitly obtain solutions to classes of differential equations. This infusion of linear algebra is not done because of the love of algebra, but rather because the structure of the solutions can no longer be ignored.

It is my own belief that linear algebra is the most important upper division course that a student can take. The methods and results of this subject pervade all aspects of mathematics and its applications. The subject introduces itself early in the high school curriculum, makes itself indispensable at the sophomore level, in a context completely different form its nascency, and then, phoenix-like, rises again to motivate so much graduate mathematics and applications. If we are looking for an algebra thread to weave through a student's education, from high school through graduate studies, what better than the subject of linear algebra?

One problem with linear algebra, at present, is that the subject almost dies through three semesters of Calculus. Now here is a challenge. Is it possible to bring this subject of linear algebra into our Calculus course in a meaningful way?

The other mainstay of algebra at the upper division level is, of course, modern algebra. Many universities offer a two semester course in this subject. The first semester is usually the standard course covering the topics of groups, rings, and fields. The second semester can be more of the same abstract development or it can be a course in applied algebra. I would prefer the second course for the following reason. For those students taking a second course in abstract algebra, which would cover theoretical topics such as Galois theory, I would assume that these students are going on to graduate school. If so, the students will cover this material in greater depth at a higher level, so why have two semesters of this course? I would prefer a second course to cover a variety of applications, for example, coding theory, cryptography, crystallographic groups, or finite fields. Perhaps a second course could focus on algorithmic aspects of algebra, covering Grobner bases, for example. Such an applied course has the benefit that it can draw in a wider audience and attract students from engineering and computer science.

In summary, I would make the following recommendations:

- College algebra should reflect the transition from our knowledge in high school to a higher level of sophistication.

- Linear algebra should be the thread that goes through mathematics, from high school through graduate school.
- A course on the applications of algebra should be offered instead of a second course in abstract algebra.
- In all of these subjects, the use of technology should be pursued in order to better present these abstract ideas.
- It is laudable that we have attempted to make calculus more intuitive. Proofs have been replaced by an appeal to geometric intuition or numerical calculations. Students leave calculus with a better understanding of the basic ideas of calculus. Since most of our calculus students will not go on to become mathematics majors, I think that these students leave with a better understanding of mathematics. However, some of these students will go on to take upper division mathematics courses. In these upper division courses, proof still reigns supreme. Proof is what makes our science. Proof is our tool for sharpening our intuition and for carrying out our investigations. Proof cannot be ignored. The challenge is then to find a way of bridging the gap that exists between the way we present lower division mathematics to the way we present upper division mathematics.

Appendices

UNITED STATES DEPARTMENT OF EDUCATION

OFFICE OF THE ASSISTANT SECRETARY
FOR EDUCATIONAL RESEARCH AND IMPROVEMENT

THE ALGEBRA INITIATIVE COLLOQUIUM
The Xerox Document University, Leesburg Virginia
December 9 through 12, 1993

Thursday evening, December 9, 1993

3:00 - 6:30	Registration
6:30 - 7:00	Cash Bar (Dining Room, Center Section)
7:00 - 9:00	Tracing the Development of Algebra and of Algebra Education in the Schools
	Dinner Session
	Welcome & Introduction: Joseph Conaty, Director, Office of Research, OERI
	Speaker: Victor Katz, University of the District of Columbia
	Discussion
	Charge to Participants: Carole Lacampagne, Office of Research, OERI
9:00-9:30	Meeting of Working Group Chairs and At-Large Participants

Friday, December 10, 1993

7:00 - 8:00	Breakfast
8:30 - 10:00	Creating an Appropriate Algebra Experience for all K-12 Students (Room 3475 Red)
	Speaker: James Kaput, University of Massachusetts at Dartmouth
	Responders: Gail Burrill, Whitnall High School, Greenfield, Wisconsin
	James Fey, University of Maryland
	Discussion

10:00 - 10:30 Break

10:30 - 12:00 Working Group Sessions

Participants will meet with their assigned working groups for discussion of key issues

(Working Group 1, Room 3263 Yellow)
(Working Group 2, Room 3461 Red)
(Working Group 3, Room 3465 Red)
(Working Group 4, Room 3463 Red)

12:00 - 1:00 Lunch

1:30 - 3:00 Renewing Algebra at the College Level to Serve the Future Mathematician, Scientist, and Engineer (Room 3475 Red)

Speaker: Michael Artin, Massachusetts Institute of Technology

Responder: Vera Pless, University of Illinois at Chicago

Discussion

3:00 - 3:30 Break

3:30 - 5:00 Working Group Sessions

Participants will meet with their assigned working groups for discussion of key issues

(Working Group 1, Room 3263 Yellow)
(Working Group 2, Room 3461 Red)
(Working Group 3, Room 3465 Red)
(Working Group 4, Room 3463 Red)

6:00 - 7:00 Dinner

7:00 - 8:30 Cross-cutting Panel and Discussion (Room 3475 Red)

Two representatives from each Working Group will outline issues discussed in their Working Groups that cut across Working Group boundaries. Whole group discussion will follow.

8:30 - 9:00 Meeting of Working Group Chairs and At-Large Participants

Saturday, December 11, 1993

8:00 - 8:45	Breakfast
8:45 - 10:15	Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce (Room 3475 Red) Speaker: Henry Pollak, Teachers College, Columbia University Responder: Solomon Garfunkel, COMAP, Inc. Discussion
10:15 - 10:45	Break
10:45 - 11:45	Working Group Sessions Participants will meet with their assigned working groups for discussion of key issues (Working Group 1, Room 3263 Yellow) (Working Group 2, Room 3461 Red) (Working Group 3, Room 3465 Red) (Working Group 4, Room 3463 Red)
11:45 - 12:30	Lunch
1:00 - 2:30	Educating Teachers, Including K-8 Teachers, to Provide Appropriate Algebra Experiences for Their Students (Room 3475 Red) Speaker: Alba Gonzalez Thompson, San Diego State University Responder: Marjorie Enneking, Portland State University Discussion
2:30 - 3:00	Break
3:00 - 5:00	Working Group Sessions Participants will meet with their assigned working groups for discussion of key issues (Working Group 1, Room 3263 Yellow) (Working Group 2, Room 3461 Red) (Working Group 3, Room 3465 Red) (Working Group 4, Room 3463 Red)

5:30 - 6:30 Dinner

7:00 - 8:30 Cross-cutting Groups

Representatives from each Working Group will meet in four Cross-cutting Groups to outline issues discussed in their respective Working Groups that cut across Working Group boundaries.

(Cross-cutting Group 1, Room 3263 Yellow)
(Cross-cutting Group 2, Room 3461 Red)
(Cross-cutting Group 3, Room 3465 Red)
(Cross-cutting Group 4, Room 3463 Red)

8:30 - 9:00 Meeting of Working Group Chairs and At-Large Participants

Sunday, December 12, 1993

8:00 - 8:45 Breakfast

8:45 - 9:45 Cross-cutting Working Groups Panel and Discussion (Room 3475 Red)

Two representatives from each Cross-cutting Working Group will present the conclusions of those working groups. Whole group discussion will follow.

9:45 - 10:15 Break

10:15 - 11:45 Wrap up and Look to the Future (Room 3475 Red)

Speaker: Lynn Steen, Mathematical Sciences Education Board

Discussion

Where Do We Go From Here?

Discussant: Carole Lacampagne

11:45 - 12:30 Lunch

1:00 Bus departs for Dulles Airport from Plaza B

Please leave your room key at the registration desk or at the key drop box provided at Plaza B.

Appendix B
Conceptual Framework for the Algebra Initiative of the National Institute on Student Achievement, Curriculum, and Assessment

Carole B. Lacampagne

Algebra is the language of mathematics. It opens doors to more advanced mathematical topics for those who master basic algebraic concepts. It closes doors to college and to technology-based careers for those who do not. Those most seriously affected by lack of algebraic skills are students from minority groups. Moreover, advances in the field of algebra, technology, in the needs of the work force, and in research on the teaching and learning of algebra should profoundly affect what algebra is taught and how it is taught and learned. Unfortunately, changes in the algebra curriculum and in its teaching and learning do not manifest themselves frequently in the classrooms of America. The proposed National Institute on Student Achievement, Curriculum, and Assessment initiative will confront this current crisis in the learning and teaching of algebra at all levels, kindergarten through graduate school.

Problem and Significance of the Initiative

Algebra is central to continued learning in mathematics. The position of the National Council of Teachers of Mathematics and of the ensuing reform in mathematics recognizes the need for restructuring algebra to make it part of the curriculum for all students. Moreover, reform is just under way in college curriculum and teaching of mathematics, including algebra. However, there is no coordinating mechanism to link all the groups concerned with the content, teaching and learning, and research in algebra. It is the aim of the proposed Algebra Initiative to provide this coordinating mechanism.

Conceptual Framework

Four key constructs provide a conceptual framework for the National Institute's Algebra Initiative:

- Creating an appropriate algebra experience for all grades K-12 students;

- Educating teachers, including K-8 teachers, to provide these algebra experiences;
- Reshaping algebra to serve the evolving needs of the technical work force; and
- Renewing algebra at the college level to serve the future mathematician, scientist, and engineer.

These constructs will be addressed in light of current and proposed advances in computer technology that should seriously affect what algebra is learned in the 21st century, and how it is taught.

Moreover, the uniqueness of the National Institute on Student Achievement, Curriculum, and Assessment's Algebra Initiative is that these constructs will be dealt with in toto; that is, the continuum of and branching of algebra across the gamut of potential users—those whose formal experience with algebra ends in high school through the research mathematician.

Algebra for All Students

In its *Curriculum and Evaluation Standards for School Mathematics*, the National Council of Teachers of Mathematics recommends algebra for all students, beginning in the middle school years as a bridge between the concrete mathematics curriculum of the elementary school and the more formal curriculum of the high school. Moreover, algebra is to be part of a core curriculum in high school mathematics, to be taken by all students.

The National Center for Research in Mathematical Sciences Education's (NCRMSE) Working Group on Learning/Teaching of Algebra and Quantitative Analysis is also re-examining the place of algebra in a core quantitative mathematics curriculum. They are looking to reform the algebra curriculum along such coherent mathematical ideas as function and structure and to study the effect of such a curriculum on how teachers teach algebra. They are looking at the reformed curriculum in light of advances in computer software and technology which allow one to do many of the algorithmic and graphing processes of algebra on a computer or sophisticated calculator.

Besides supporting the NCRMSE Working Group on Learning/Teaching of Algebra and Quantitative Analysis, the U.S. Department of Education supports other algebra initiatives including Robert Moses' Algebra Project, a pre-algebra project designed to bridge the gap between arithmetic and algebra for minority students.

The National Science Foundation is funding a variety of projects to develop instructional materials in mathematics, K-12. Many of these materials include algebra strands. One such materials development project addresses directly a core high school mathematics curriculum which includes algebra. Several others have algebra strands cutting across several grade levels. Still others stress algebraic applications to other disciplines such as science and business.

Questions that need to be addressed include:

- What algebra concepts should be part of the core mathematics curriculum for all students?
- How should they be taught and learned?
- What is the role of technology in this curriculum and pedagogy?
- How will these algebra concepts be tied into the needs of an educated citizen and a technical worker?
- How will they be related to the continuum of algebra in college and beyond?
- How will they prepare students for lifelong learning in mathematics?

Algebra for Future School Teachers of Mathematics

With the teaching and learning of algebra moved down to the middle school years, elementary school teachers now need a thorough understanding of the underlying concepts of algebra, of how children understand/misunderstand basic algebra concepts, and of pedagogical approaches suitable to helping children develop early algebra concepts. High school teachers of mathematics need a new and different knowledge of algebra and appropriate pedagogy. They need to understand the links between algebra and other fields of mathematics as well as between algebra and its applications to the social, natural, and physical sciences.

Thus, mathematics curriculum and pedagogy for future teachers of mathematics need to be rethought in light of the changing role of algebra in the school curriculum. The Mathematical Association of America recognizes the need for a different experience in algebra for future school teachers of mathematics in its document *A Call for Change: Recommendations for the Mathematical Preparation of Teachers*. However, much work is needed to effect change in the algebraic education of future teachers.

Questions that need to be addressed include:

- What algebra concepts and pedagogy should be fundamental to mathematics courses required of future elementary school teachers? middle school teachers? high school teachers?

- How can future teachers best acquire such knowledge and skills?
- How can we enhance the algebraic and pedagogical skills of current teachers?

Algebra for the Technical Work Force

The majority of U.S. students are leaving school ill-prepared in mathematical problem solving, planning and optimizing, and mathematical modeling—mathematical tools needed in forward-thinking business and industry. All of the above areas are based on the language and operations of algebra and all should play an important role in curricula designed for those entering the technical work force—be they products of high school vocational education, 2-plus-2 plans, or 4-year college technical education curricula. Emphasis here will be on developing curricula and pedagogy to equip students to meet the demands of the 21st century work force and to provide the base for lifelong learning in mathematics as technology shifts and careers change.

Questions to be addressed include:

- What algebra concepts, beyond the core, do students preparing to enter the technical work force need?
- How should these concepts be taught and learned?
- What is the role of technology in such a curriculum and pedagogy?
- How will these algebra concepts prepare students for continuing learning in mathematics?
- Can we circumvent the dreary sequence of remedial courses in arithmetic and algebra that many community college students must take?

Algebra for the Future Mathematician, Scientist, and Engineer

Algebra is the language of mathematics. Just as school algebra allows students to communicate mathematically, to model problems and to solve them, its extensions, linear and abstract algebra form a basis for many areas of mathematics: group theory, ring theory, algebraic topology, and algebraic number theory, to name a few. Basic knowledge of linear and abstract algebra concepts are required of all college mathematics majors, and a deeper understanding of these concepts are required of all serious users of mathematics.

Recent developments in computer software have made complex matrix manipulation, graphing, and object manipulation quicker and have enabled further developments in several areas of algebra.

Key questions to be considered include:

- How should the transition from school algebra to the advanced algebra concepts taught in college be accomplished?
- How can we help students develop mathematical maturity and smooth their transition into a first proof course?
- What advanced algebra concepts should form the core of the collegiate mathematics major? for the graduate degree in mathematics?
- How should these concepts be taught and learned?
- What is the role of technology in such a curriculum and pedagogy?
- What research opportunities in algebra can be afforded students in their undergraduate and early graduate experience in mathematics?
- How will these algebra concepts prepare students for employment outside of academia?

Action Plan

The action plan for the National Institute on Student Achievement, Curriculum, and Assessment's Algebra Initiative includes:

- A 3 1/2 day invitational colloquium;
- A widely distributed summary document for policymakers and teachers on the major issues discussed at the colloquium;
- A complete *Proceedings* of the colloquium to be distributed to the mathematics and mathematics education communities;
- A 2-year discretionary grant solicitation for a Linking Algebra project; and
- The linking of other initiatives, both within the Department of Education and across other federal agencies, to what is learned from the Algebra Initiative.

Invitational Colloquium

The National Institute on Student Achievement, Curriculum, and Assessment's Algebra Initiative will begin with a 3 1/2 day invitational colloquium on algebra involving about 40 key players from NCRMSE, the Algebra Project, principal investigators from NSF's algebra materials development and research projects, college teachers of algebra, and mathematics and mathematics education researchers plus additional mathematics experts from federal agencies. Carole Lacampagne will run the colloquium in coordination with William Blair, James Kaput, and Richard Lesh.

Summary Document

A report writer with experience in writing articles about mathematics will be hired to write a short document for policymakers and teachers based on issues raised at the colloquium. This document will be distributed widely within 4 months of the colloquium.

Proceedings

All speakers, discussants, and participants in the colloquium will be asked to submit papers to be published in a *Proceedings*. This document will be distributed to the mathematics community.

Further dissemination of knowledge gained through the colloquium will occur at workshops, minicourses, and talks on algebra given at annual meetings of the National Council of Teachers of Mathematics, the Society for Industrial and Applied Mathematics, and at semiannual joint meetings of the Mathematical Association of America, and the American Mathematical Society.

Two-year Discretionary Grant Solicitation for a Linking Algebra Project

It is anticipated that the Department of Education, perhaps in collaboration with other federal agencies, will fund a Linking Algebra project organized around the four key constructs of the colloquium to extend the concepts of, embark on research in, and prepare for implementation in schools and colleges knowledge gained in the colloquium.

It is anticipated that the Algebra Initiative through taking the broad view of algebra, from the teaching and learning of algebra in the elementary school through breakthroughs in research, will spur teachers of mathematics, teachers of teachers of mathematics, and the mathematics and mathematics education communities to coordinate and extend the role of algebra for all learners and users of mathematics.

Appendix C
Participant List

Working Group 1: Creating an Appropriate Algebra Experience For All K-12 Students

Invited Participants

Diane Briars
Pittsburgh Public Schools
Pittsburgh, PA

Gail Burrill, WG1 Reactor
Whitnall High School
Greenfield, WI

Daniel Chazan
Michigan State University
East Lansing, MI

Robert Davis
Rutgers University
New Brunswick, NJ

James Fey, WG1 Reactor
University of Maryland
College Park, MD

Rogers Hall
University of California
Berkeley, CA

James Kaput, WG1 Speaker
University of Massachusetts at Dartmouth
North Dartmouth, MA

Robert Moses
The Algebra Project, Inc.
Cambridge, MA

Betty Phillips
Chair, NCTM Algebra Task Force
East Lansing, MI

Alan Schoenfeld, WG1 Chair
University of California, Berkeley
Berkeley, CA

Zalman Usiskin
University of Chicago
Chicago, IL

Federal Participants

Eric Robinson
National Science Foundation
Arlington, VA

Charles Stalford
U.S. Department of Education
Washington, DC

Working Group 2: Educating Teachers, Including K-8 Teachers, to Provide These Algebra Experiences

Invited Participants

Alphonse Buccino, WG2 Chair
University of Georgia
Athens, GA

Suzanne Damarin
The Ohio State University
Columbus, OH

Marjorie Enneking, WG2 Reactor
Portland State University
Portland, OR

Naomi Fisher
University of Illinois at Chicago
Chicago, IL

Guershon Harel
Purdue University
West Lafayette, IN

Alba Gonzalez Thompson, WG2 Speaker
San Diego State University
San Diego, CA

Federal Participants

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U.S. Department of Education
Washington, DC

Henry Kepner
National Science Foundation
Arlington, VA

James Pratt
Johnson Space Center
Houston, TX

Cindy Musick
U.S. Department of Energy
Washington, DC

Tina Straley
National Science Foundation
Arlington, VA

Working Group 3: Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce

Invited Participants

Paul Davis
Worcester Polytechnic Institute
Worcester, MA

Susan Forman, WG3 Chair
Mathematical Sciences Education Board
Washington, DC

Solomon Garfunkel, WG3 Reactor
COMAP, Inc.
Lexington, MA

James Greeno
Stanford University
Stanford, CA

Richard Lesh
Educational Testing Service
Princeton, NJ

Patrick McCray
G D Searle & Co.
Evanston; IL

Henry Pollak, WG3 Speaker
Columbia University
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Thomas Romberg
University of Wisconsin-Madison
Madison, WI

Susan S. Wood
J. Sargeant Reynolds Community College
Richmond, VA

Federal Participants

Elizabeth Teles
National Science Foundation
Arlington, VA

Donna Walker
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Working Group 4: Renewing Algebra at the College Level to Serve the Future Mathematician, Scientist, and Engineer

Invited Participants

Michael Artin, WG4 Speaker
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Cambridge, MA

William Blair
Northern Illinois University
DeKalb, IL

John Conway
Princeton University
Princeton, NJ

Al Cuoco
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Charles F. Osgood
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Ft. George G. Meade, MD

Joan Straumanis
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At-Large Participants

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EDRS

Lynn Steen, Wrap-up Speaker
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ISBN 0-16-048075-2
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SAI 95–3030–2